A new perspective on Teichmüller theory

joint work with Jean-Marc Schlenker

Mingkun LIU

University of Luxembourg

December 20, 2022

Classical Teichmüller theory

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The *classical Teichmüller theory* studies *nice* maps

$$\pi_1(S) \longrightarrow \mathrm{PSL}(2, \mathbb{R})$$

where $\pi_1(S)$ is the fundamental group of S, and

$$\mathrm{PSL}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\} \Big/ \{\pm \mathrm{Id}\}$$

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Athanase Papadopoulos: *Institut de Recherche Mathématique Avancée, Strasbourg, France*

A publication of European Mathematical Society

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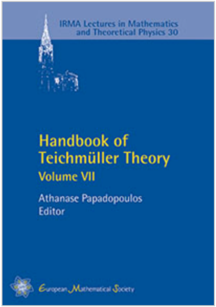

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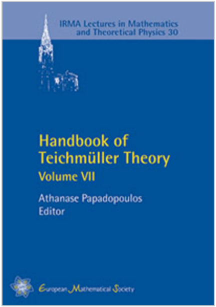

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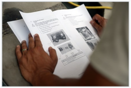
Handbook

From Wikipedia, the free encyclopedia

This article is about the type of reference work. For the subnotebook computer, see Gateway Handbook.
"Pocket reference" redirects here. For the all-purpose reference work by Thomas J. Glover, see Pocket Ref.
"Vademecum" redirects here. For other uses, see Vademecum (disambiguation).

A **handbook** is a type of reference work, or other collection of instructions, that is intended to provide ready reference. The term originally applied to a small or portable book containing information useful for its owner, but the *Oxford English Dictionary* defines the current sense as "any book ... giving information such as facts on a particular subject, guidance in some art or occupation, instructions for operating a machine, or information for tourists."[1]



A technician referring to an instructional handbook for the operation of a machine

A handbook is sometimes referred to as a ***vade mecum*** (Latin, "go with me") or **pocket reference**. It may also be referred to as an enchiridion.

Handbooks may deal with any topic, and are generally compendiums of information in a particular field or about a particular technique. They are designed to be easily consulted and provide quick answers in a certain area. For example, the *MLA Handbook for Writers of Research Papers* is a reference for how to cite works in MLA style, among other things. Examples of engineering handbooks include *Perry's Chemical Engineers' Handbook*, *Marks' Standard Handbook for Mechanical Engineers*, and the *CRC Handbook of Chemistry and Physics*.

See also [edit]

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Generality is an idol before whom pure mathematicians torture themselves. Arthur Eddington

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Figure: Jean-Marc Schlenker

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Lower Teichmüller theory

Lower Teichmüller theory, or Schlenker theory

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rather than studying things like

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that we shall call *Schlenkerian representations*.

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Therefore, we obtain the following complete classification of Schlenkerian representations.

Theorem (Riemann–Schlenker uniformization theorem)

All Schlenkerian representations are isomorphic.

Work in progress

Work in progress: Lowest Teichmüller theory

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And if you gaze long into an abyss, the abyss also gazes into you.

Friedrich Nietzsche

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$$\pi_1(S) \longrightarrow \mathrm{PSL}(☺, \mathbb{R}).$$

We encounter immediately fundamental difficulties:

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- What is the determinant of a $☺ \times ☺$ matrix?
- What is a morphism $\pi_1(S) \to \mathrm{PSL}(☺, \mathbb{R})$?

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Voilà voilà ☺