Mémoire

en vue de l'obtention du diplôme de

Master de Sorbone Université

Discipline : **Mathématiques**

Laboratoire Analyse, Géométrie et Applications

Présenté par

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The Taylor-Wiles-Kisin method

Sous la direction du Professeur Stefano Morra

Soutenu publiquement le 10 Septembre 2020 devant le jury composé de :

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Acknowledgements

I would like to express my deep and sincere gratitude to my supervisor, Professor Stefano Morra, for giving me the opportunity to work with him during this internship. I am extremely thankful to him for providing invaluable guidance and motivation to work, and for always finding the time to check on my progress and answer my questions despite the uncertainties brought by this period of crisis.

I would also like to thank Mme. Ariane Mézard and M. Benoît Stroh for accepting to take part in the jury of this thesis.

I am grateful to my friends and family for standing by my side and supporting me. Their help and encouragement, especially during the period of quarantine, kept me going. Special thanks go to Ahmed, Amine, Elise, Fadhel, Malek, Mohamed, and Syrine for being like a second family to me here where I am far from home.

I would also like to thank Arnab, Tongmu, and Vincent for the interesting mathematical discussions we have had this year, and for the study group I had the chance to take part in.

Finally, I would like to thank A.Abbes for the advice and guidance he provided me with throughout my mathematical studies.

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Chapter 1

Introduction

In 1994, when A. Wiles presented the proof of the Taniyama-Shimura conjecture, the whole mathematical community was in surprise and admiration. He had not only solved a problem that, for centuries, withstood the attacks of some of the most brilliant, but he also had introduced an arsenal of tools and methods that will certainly be useful for generations to come. This thesis is an attempt to expose some of these ideas and see how they are applied in various settings. Indeed, given a number field F and a Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ such that its reduction $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is known to be automorphic, one could ask as in the work of A. Wiles, what conditions should ρ satisfy for it to also be automorphic. Results that go in this direction are called modularity lifting theorems.

In the first part of this thesis, we will present and prove an instance of these theorems. The general strategy for the proof is to introduce a universal deformation ring R which in some sense encodes the lifts of the residual representation $\overline{\rho}$, and also a Hecke algebra $\mathbf{T}$ acting on a space of automorphic forms S together with a representation $\rho^{\mathrm{mod}} : G_F \to \mathrm{GL}_2(\mathbf{T})$ that includes all the automorphic Galois representations reducing to $\overline{\rho}$. Universality of R implies that we have a surjective morphism $R \twoheadrightarrow \mathbf{T}$ which turns out to be an isomorphism, thereby giving us a morphism $\varphi : \mathbf{T} \to \overline{\mathbb{Q}}_p$ such that $\rho = \varphi \circ \rho^{\mathrm{mod}}$. This is exactly saying that our representation is automorphic, and so the whole difficulty lies in showing the equality "$R = \mathbf{T}$" which is done using the "Taylor-Wiles-Kisin method". The way this method works is that we carefully choose different sets Q of places of F, called sets of Taylor-Wiles places, in a way that if we relax the conditions on the deformations of $\overline{\rho}$ at these primes, we obtain a larger universal ring R_Q which fits into a diagram

$$\begin{tikzcd} \mathcal{J}[\Delta_Q] \arrow[r, two heads] & R_Q \arrow[r] \arrow[d, two heads] & \mathrm{End}(S_Q) \arrow[d] \\ & R \arrow[r] & \mathrm{End}(S) \end{tikzcd}$$

where $\mathcal{J}[\Delta_Q]$ is a group algebra with an augmentation ideal $\mathfrak{a}_Q$, and S_Q is a space of automorphic forms which is finite free over $\mathcal{J}[\Delta_Q]$ with $S_Q/\mathfrak{a}_Q = S$ and $R_Q/\mathfrak{a}_Q = R$. In a process of taking the limit of this diagram over various Taylor-Wiles sets, the algebra $\mathcal{J}[\Delta_Q]$ successively approximates a power series algebra $\mathcal{J}_\infty$ with augmentation ideal $\mathfrak{a}_\infty$, and we get a diagram

$$\begin{tikzcd} \mathcal{J}_\infty \arrow[r, two heads] & R_\infty \arrow[r] \arrow[d, two heads] & \mathrm{End}(S_\infty) \arrow[d] \\ & R \arrow[r] & \mathrm{End}(S) \end{tikzcd}$$

with the same properties as the one above. Only this time, by careful considerations of the dimension of R_∞ and using tools from commutative algebra, we are able to deduce that S_∞ is finite free over R_∞. Reducing modulo $\mathfrak{a}_\infty$, we get that S is finite free over R. But given that the action of R on S factors through $\mathbf{T}$, which acts faithfully on S, we deduce the equality $R = \mathbf{T}$.
The subtleties that arise in using this method, notably in the choice of the Taylor-Wiles places, makes it technically very difficult to work with reducible $\overline{\rho}$, so one usually imposes many restrictions on the residual representation.

In their recent work [NT20], J. Newton and J. Thorne used the Taylor-Wiles-Kisin method to prove, under very weak conditions, the vanishing of the adjoint Bloch-Kato Selmer group for automorphic Galois representations. In fact in their treatment, they were able to avoid imposing restrictions on the residual representation. The content of their article will be the focus of the second part, which is the core of this thesis. We will be interested in showing how they managed to surpass the technical difficulties that arise when working with a reducible residual representation, in the hope of being able to transpose the techniques they used to other problems and situations in the future.

Part I

Some modularity lifting theorems

Chapter 2

Deformation of Galois representations

One of the main tools in the proof of modularity lifting theorems is the use the deformation theory of Galois representations introduced and developed by B. Mazur. The idea is that given a residual representation $\overline{\rho} : \Gamma \to \mathrm{GL}_n(k)$ of a group Γ with coefficients in a field k, one can consider the functor that associates to a local of all deformations of ρ. The universal deformation ring obtained is far too big to be useful. However we will be able to cut it down by adding conditions on the deformations of our residual representation. In particular, if F is a number field and $\overline{\rho} : G_F \to \mathrm{GL}_n(k)$, then we can these conditions take place for the restriction of the lifts to each local Galois group associated to a place.

2.1 Basic definitions and results

Let Γ be a profinite group, and $\overline{\rho} : \Gamma \to \mathrm{GL}_n(k)$ be a continuous representation with coefficients in a finite field k of characteristic p. Let Λ be a complete discrete valuation ring with residue field k (e.g., $W(k)$). We let $\mathcal{C}_\Lambda^0$ to be the category of Artinian local Λ-algebras of residue field k, with local morphisms that induce the identity on k. Let $\mathcal{C}_\Lambda$ be the category which consists of complete Noetherian local Λ-algebras with residue field k.
If $A \in \mathcal{C}_\Lambda$, we say that a continuous homomorphism

$$\rho : \Gamma \to \mathrm{GL}_n(A)$$

is a lift of $\overline{\rho}$ if $\pi \circ \rho = \overline{\rho}$ where $\pi : A \to k$ is the canonical quotient. We say that two lifts of $\overline{\rho}$

$$\rho_1, \rho_2 : \Gamma \to \mathrm{GL}_n(A)$$

are strictly equivalent if there exists a matrix $M \in \Gamma_n(A) = \ker\big(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k)\big)$ such that $M\rho_1 = \rho_2 M$. This is an equivalence relation, and we define a deformation of $\overline{\rho}$ to be a strict equivalence class of lifts of $\overline{\rho}$. This way, we can define two functors:

$$\begin{aligned} \mathcal{D} : \mathcal{C}_\Lambda &\to \mathbf{Sets} \\ A &\mapsto \{ \text{ deformations of } \overline{\rho} \text{ with coefficients in } A\} \end{aligned}$$

which is called the deformation functor of $\overline{\rho}$, and

$$\begin{aligned} \mathcal{D}^\square : \mathcal{C}_\Lambda &\to \mathbf{Sets} \\ A &\mapsto \{ \text{ lifts of } \overline{\rho} \text{ with coefficients in } A\} \end{aligned}$$

which is called the lifting functor of $\overline{\rho}$.

$\mathcal{D}$ and $\mathcal{D}^\square$ are continuous functors, so they are completely determined by their values in the subcategory $\mathcal{C}^0_\Lambda$.

Suppose that Γ satisfies Mazur's Φ_p-condition:

Condition Φ_p : For each open subgroup $\Gamma' \subseteq \Gamma$, there are only finitely many continuous homomorphisms $\Gamma' \to \mathbb{F}_p$

Then, the functor $\mathcal{D}^\square$ is representable, i.e., there exists a ring $R^\square_{\text{univ}} \in \mathcal{C}_\Lambda$ and a continuous representation $\rho_{\text{univ}} : \Gamma \to \mathrm{GL}_n(R^\square_{\text{univ}})$ such that for all $A \in \mathcal{C}_\Lambda$, we have a natural bijection

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_\Lambda}(R^\square_{\text{univ}}, A) &\xrightarrow{\sim} \mathcal{D}^\square(A) \\ f &\mapsto f \circ \rho_{\text{univ}} \end{aligned}$$

For example this is the case if $\Gamma = G_K$ where K is a local field , or if $\Gamma = G_{F,S}$ for F a number field and S a finite set of places.

If moreover, $\overline{\rho}$ has trivial endomorphisms, which is the case if $\overline{\rho}$ is absolutely irreducible, then by [Gou95, Theorem 3.10], we get that $\mathcal{D}$ is representable. So there exists a ring $R_{\text{univ}} \in \mathcal{C}_\Lambda$ and a deformation of $\overline{\rho}$, $\rho_{\text{univ}} : \Gamma \to \mathrm{GL}_n(R_{\text{univ}})$ with $\mathrm{Hom}_{\mathcal{C}_\lambda}(R_{\text{univ}}, -) \cong \mathcal{D}(-)$.

2.1.1 The tangent space

We now define the tangent space of the universal deformation ring which can be naturally identified with certain cohomology groups. Using results from Galois cohomology, this allows us to find bounds on the number of generators and relations of the universal deformation ring.

Let $A \in \mathcal{C}_\Lambda$. We define the cotangent space of A to be

$$\mathfrak{t}^\vee_A = \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_\Lambda)$$

Note that this has the structure of a $\Lambda/\mathfrak{m}_\Lambda = k$-vector space which is finite dimensional since A is Noetherian. We define the Zariski tangent space of A to be the dual of the cotangent space, i.e.,

$$\mathfrak{t}_A = \mathrm{Hom}_k\left(\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_\Lambda), k\right)$$

Proposition 2.1.1. $\mathfrak{t}_A$ *is naturally isomorphic to* $\mathrm{Hom}_\Lambda(A, k[\epsilon])$.

Proof. A morphism of Λ-algebras $A \to k[\epsilon]$ is given by $x \mapsto \overline{x} + \phi(x)\epsilon$ where $\phi : A \to k$ is a Λ-algebra morphism. It is not hard to see that ϕ is determined by the image of $\mathfrak{m}_A$ and that it kills $\mathfrak{m}_\Lambda$ and $\mathfrak{m}_A^2$. □

Lemma 2.1.2. $\mathcal{D}^\square(k[\epsilon])$ *is canonically isomorphic to* $Z^1(\Gamma, \mathrm{ad}\,\overline{\rho})$*, and* $\mathcal{D}(k[\epsilon])$ *is canonically isomorphic to* $H^1(G, \mathrm{ad}\,\overline{\rho})$.

Proof. Let ρ be a lift of $\overline{\rho}$ to $k[\epsilon]$, then we can write $\rho(\gamma) = \overline{\rho}(\gamma) + c(\gamma)\overline{\rho}(\gamma)\epsilon$ with $c(\gamma) \in \mathrm{M}_n(k)$. One can verify that $\gamma \mapsto c(\gamma)$ defines an element of $Z^1(\Gamma, \mathrm{ad}\,\overline{\rho})$. Conversely, for such a c, the formula gives a lift of $\overline{\rho}$. A strict equivalence in $\mathcal{D}^\square(k[\epsilon])$ corresponds to adding a coboundary in $Z^1(\Gamma, \mathrm{ad}\,\overline{\rho})$. □

Proposition 2.1.3. *We have* $\dim_k \mathcal{D}^\square(k[\epsilon]) = \dim_k \mathcal{D}(k[\epsilon]) + n^2 - \dim_k H^0(\Gamma, \mathrm{ad}\,\overline{\rho})$.

Proof. The map $\mathcal{D}^\square(k[\epsilon]) \to \mathcal{D}(k[\epsilon])$ corresponds to $Z^1(G, \mathrm{ad}\,\overline{\rho}) \to H^1(G, \mathrm{ad}\,\overline{\rho})$. Its kernel $B^1(\Gamma, \mathrm{ad}\,\overline{\rho})$ corresponds to the image of $\mathrm{ad}\,\overline{\rho}$ in $Z^1(\Gamma, \mathrm{ad}\,\overline{\rho})$ via the map $M \mapsto (\gamma \mapsto \overline{\rho}(\gamma)M\overline{\rho}(\gamma)^{-1} - M)$. So it is $\mathrm{ad}\,\overline{\rho}/\mathrm{ad}\,\overline{\rho}^\Gamma$. □

Note that $H^0(\Gamma, \mathrm{ad}\,\overline{\rho}) = \{M \in \mathrm{M}_n(k) \mid \overline{\rho}(\gamma)M = M\overline{\rho}(\gamma)\ \forall\gamma \in \Gamma\}$, so if $\overline{\rho}$ is irreducible, then $\dim_k H^0(\Gamma, \mathrm{ad}\,\overline{\rho}) = 1$.

2.2 Deformation conditions

In practice, the universal deformation ring is too big to be useful. That is why we need to cut it down by imposing several conditions on the deformation functor such as fixing the determinant of the representations considered. In order for the obtained subfunctor to be represented in $\mathcal{C}_\Lambda$, it needs to satisfy certain properties which are given in the following definition:

Definition 2.2.1. A lifting problem $\mathcal{P}$ is the data, for all $A \subseteq \mathcal{C}_\Lambda^0$, of a subset $\mathcal{D}_\mathcal{P}^\square(A)$ of $\mathcal{D}^\square(A)$ satisfying the following properties:

1. $\overline{\rho} \in \mathcal{D}_\mathcal{P}^\square(k)$.
2. Let $f : A \to B$ be a map in $\mathcal{C}_\Lambda^0$. If $\rho \in \mathcal{D}_\mathcal{P}^\square(A)$ then $f \circ \rho \in \mathcal{D}_\mathcal{P}^\square(B)$.
3. $\mathcal{D}_\mathcal{P}^\square$ is a continuous subfunctor of $\mathcal{D}^\square$.
4. Let $A \to C$ and $B \to C$ be maps in $\mathcal{C}_\Lambda^0$, and let $D = A \times_C B$ with maps $p : D \to A$ and $q : D \to B$. Then, $\rho \in \mathcal{D}_\mathcal{P}^\square(D)$ if and only if $p \circ \rho \in \mathcal{D}_\mathcal{P}^\square(A)$ and $q \circ \rho \in \mathcal{D}_\mathcal{P}^\square(B)$.
5. Let $f : A \to B$ be an injective map in $\mathcal{C}_\Lambda^0$. Then, $\rho \in \mathcal{D}_\mathcal{P}^\square(A)$ if and only if $f \circ \rho \in \mathcal{D}_\mathcal{P}^\square(B)$.

A deformation problem is a lifting problem satisfying moreover

6. If $\rho \in \mathcal{D}_\mathcal{P}^\square(A)$ and $g \in \Gamma_n(A)$, then $g\rho g^{-1} \in \mathcal{D}_\mathcal{P}^\square(A)$.

This gives a functor

$$\mathcal{D}_\mathcal{P}^\square : \mathcal{C}_\Lambda^0 \to \mathbf{Sets}$$

which we extend to $\mathcal{C}_\Lambda$ by continuity.

Remark 2.2.2. Let $\mathcal{D}'$ be a subfunctor of $\mathcal{D}$ and assume they are both representable by R' and R, then there is a natural map $R \to R'$. This map is surjective. Indeed, it suffices to check that the map on the cotangent spaces is surjective. But the map on the cotangent spaces is dual to the map on the tangent spaces which is injective since $\mathcal{D}'(k[\epsilon]) \subseteq \mathcal{D}(k[\epsilon])$.
We want for a deformation condition corresponds to a closed subspace of the space of all deformations $\operatorname{Spec} R$. This is given by the existence of an ideal I such that for any $f : R \to A$, $f \circ \rho^{\mathrm{univ}} A$ satisfies the deformation condition if and only if f factors through R/I.

This idea is illustrated by the following proposition.

Proposition 2.2.3. *Assume that $\mathcal{D}^\square$ is represented by $R^\square$.*

1. *Let $\mathcal{P}$ be a lifiting problem. There exists a closed ideal $I(\mathcal{P})$ of $R^\square$ such that: for any object A of $\mathcal{C}_\Lambda$, $\rho \in \mathcal{D}_\mathcal{P}^\square(A)$ if and only if the map $R^\square \to A$ corresponding to ρ factors through $R^\square/I(\mathcal{P})$.*
2. *If I be a closed ideal of $R^\square$, we define $\mathcal{P}(I)$ by letting, for each object A of $\mathcal{C}_\Lambda^0$, $\mathcal{D}_{\mathcal{P}(I)}^\square(A)$ to be the set of $\rho \in D^\square(A)$ such that the corresponding map $R^\square \to A$ factors through I. Then, $\mathcal{P}(I)$ is a lifting problem.*
3. *$I\big(\mathcal{P}(I)\big) = I$ and $\mathcal{P}\big(I(\mathcal{P})\big) = \mathcal{P}$.*
4. *If $\mathcal{P}$ is a lifting condition, then $\mathcal{D}_\mathcal{P}^\square$ is represented by $R^\square/I(\mathcal{P})$.*

Proof. Let $\mathcal{E}$ be the set of open ideals I of $R^\square$ such that the representation given by the map $R^\square \to R^\square/I$ lies in $\mathcal{D}^\square_{\mathcal{P}}(R^\square/I)$. Then, $\mathcal{E}$ is non-empty since it contains the maximal ideal by 1). Moreover, given two ideals I_1 and I_2 in $\mathcal{E}$, we have an injection

$$R^\square/(I_1 \cap I_2) \hookrightarrow R^\square/I_1 \times_{R^\square/(I_1+I_2)} R^\square/I_2$$

so by conditions 4) and 5), we get that $I_1 \cap I_2 \in \mathcal{E}$ (The fiber product exists in $\mathcal{C}$ since all the rings are Artinian).
Now, given a descending chain of ideals in $\mathcal{E}$ $I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$, we have $R/\bigcap_i I_i = \varprojlim R/I_i$. Hence, condition 3) gives $\bigcap_i I_i \in \mathcal{E}$. We define $I(\mathcal{P})$ to be the intersection of all the ideals in $\mathcal{E}$. It is a closed ideal since here, open ideals are closed. And by the above properties, $I(\mathcal{P}) \in \mathcal{E}$.
Consider now an object A of $\mathcal{C}_\Lambda$, and $\rho \in \mathcal{D}^\square(A)$. Let $u : R^\square \to A$ be the associated morphism. If $I(\mathcal{P}) \subseteq \ker(u)$, so that u factors through $R^\square/I(\mathcal{P})$ and $\rho \in \mathcal{D}^\square_{\mathcal{P}}(A)$ by property 2).
Conversely, suppose that $\rho \in \mathcal{D}^\square_{\mathcal{P}}(A)$. We have an injection $R^\square/\ker u \to A$. So by property 5), the representation corresponding to $R^\square \to R^\square/\ker u$ lies in $\mathcal{D}^\square_{\mathcal{P}}(R^\square/\ker u)$. Hence, $I(\mathcal{P}) \subseteq \ker(u)$. ☐

Remark 2.2.4. The same theorem hold if we assume that $\mathcal{P}$ is a deformation problem. Although we additionally require that I is radical and $1 + \mathrm{M}_n(\mathfrak{m}_{R^\square})$-stable (see [BLGHT11, Lemma 3.2]).

2.3 Representations with values in a subring

In this section, we present a theorem due to Carayol which states that a deformation of an absolutely irreducible representation is uniquely determined by its trace. Before stating and proving this theorem, we need to introduce some other preliminary results.

Theorem 2.3.1 (Jacobson Density)**.**
Let E be a simple R-module, and let $D = \mathrm{End}_R(E)$. If $f \in \mathrm{End}_D(E)$ and $x_1, \dots, x_n \in E$, then there exists $r \in R$ such that $f(x_i) = r \cdot x_i$ for all $1 \le i \le n$. Consequently, if E is finitely generated over R, then the natural map $R \to \mathrm{End}_D(E)$ is surjective.

Proof. Consider the following morphism

$$f^{(n)} : E^n \to E^n$$
$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \mapsto \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_n) \end{pmatrix}$$

Let $D^{(n)} = \mathrm{End}_R(E^n)$ which identifies with the ring $\mathrm{M}_n(D)$ of matrices with coefficients in D. Since f commutes with the elements of D, then $f^{(n)} \in \mathrm{End}_{D^{(n)}}(E^n)$. Now let $x = {}^t(x_1, \dots, x_n)$, by semi-simplicity of E^n, we can write $E^n = R \cdot x \oplus W$. So let us denote by π the projection map of E onto $R \cdot x$. Therefore, $\pi \in D^{(n)}$, and

$$f^{(n)}(x) = f^{(n)}(\pi(x)) = \pi(f^{(n)}(x))$$

So $f^{(n)}(x) \in R \cdot x$, from which the theorem follows. ☐

Corollary 2.3.2 (Burnside)**.** *If $\overline{\rho} : \Gamma \to \mathrm{GL}_n(k)$ is absolutely irreducible, then the map $k[\Gamma] \to \mathrm{M}_n(k)$ is surjective.*

Proof. We apply Theorem 2.3.1 to $R = k[\Gamma]$ and $E = k^n$. Since $\overline{\rho}$ is absolutely irreducible, we have that $D = k$ from which the corollary follows. ☐

Theorem 2.3.3. *Let $R \in \mathcal{C}_\Lambda$, and let $\rho, \rho' : \Gamma \to \mathrm{GL}_n(R)$ be two representations with coefficients in R. Suppose that $\overline{\rho}$ is absolutely irreducible and that $\operatorname{Tr}\rho(\gamma) = \operatorname{Tr}\rho'(\gamma)$ for all $\gamma \in \Gamma$. Then, ρ and ρ' are equivalent.*

Proof. The two representations ρ, ρ' extend to R-algebra homomorphisms $u, u' : R[\Gamma] \to \mathrm{M}_n(R)$. We will show that these two morphisms are conjugated by an element of $\mathrm{GL}_n(R)$. Given that it is enough to show this modulo $\mathfrak{m}_R^i$ for $i \geq 1$, we can suppose that R is Artinian.
We proceed by induction on the length of R, so let us first show the result for $R = k$ a field. Consider the linear map $t : R[\Gamma] \to k, x \mapsto \operatorname{Tr} u(x)$. Then, by the fact that the trace map is a perfect pairing on matrices and by surjectivity of u (Corollary 2.3.2), we have that $\ker u = \{x \in R[\Gamma] \mid t(xy) = 0\ \forall y \in R[\Gamma]\}$. This description shows that $\ker u' \subseteq \ker u$, so that we have a natural surjective map $R[\Gamma]/\ker u' \twoheadrightarrow R[\Gamma]/\ker u \cong \mathrm{M}_n(k)$. But since $\dim_k R[\Gamma]/\ker u' = \dim_k \operatorname{im}(u') \leq n^2$, this map is actually an isomorphism which implies that $\ker u = \ker u'$ and $\operatorname{im}(u') = \mathrm{M}_n(R)$. Now every automorphism of $\mathrm{M}_n(k)$ is given by conjugation by an element of $\mathrm{GL}_n(k)$, applying this to the above isomorphism gives the conjugation between u and u' as desired.
Now assume that R is not a field, and let $I = (a)$ be an ideal of R such that $\mathfrak{m}_R I = 0$. By the induction hypothesis, we can suppose that ρ and ρ' coincide modulo I. So let us write $\rho = \rho' + \delta$ with $\delta : R[\Gamma] \to \mathrm{M}_n^0(I)$ (trace 0 matrices) is an R-linear map. It is not hard to verify that $\delta(xy) = \overline{\rho}(x)\delta(y) + \delta(x)\overline{\rho}(y)\ \forall x, y \in R[\Gamma]$. If $y \in \ker\overline{\rho}$, then $\operatorname{Tr}\big(\overline{\rho}(x)\delta(y)\big) = \operatorname{Tr}\big(\delta(xy)\big) = 0\ \forall x \in R[\Gamma]$. Since $\overline{u}$ is surjective (Corollary 2.3.2), we have that $\delta(y) = 0$. Therefore, δ factors through $\ker\overline{\rho}$ so it induces a k-linear map $\mathrm{M}_n(k) \to \mathrm{M}_n(k) \cong \mathrm{M}_n(I)$ (I is principal) which is a derivation. But all derivations on $\mathrm{M}_n(k)$ are given by conjugation by a matrix (in other words the first lie algebra cohomology group $H^1(\mathfrak{gl}_n, \operatorname{ad}\mathfrak{gl}_n)$ vanishes). So there exists $M \in \mathrm{M}_n(k)$ such that $\delta(x) = \overline{\rho}(x)M - M\overline{\rho}(x)\ \forall x \in R[\Gamma]$. Therefore, $(1 + M)\rho = \rho'(1 + M)$ as desired.

□

Let $R \in \mathcal{C}_\Lambda$ and suppose that $k = R/\mathfrak{m}_R$ is finite. Let R' be an R-algebra which is finite as an R-module. Then, R' is semi-local, and we can write $R' = \prod_{i=0}^r R'_i$ where R'_i is local R-algebra with maximal ideal m'_i and residual field $k'_i \supseteq k$.
Suppose that we have representations $\rho'_i : \Gamma \to \mathrm{GL}_n(R'_i)$, which give rise to a representation $\rho' : \Gamma \to \mathrm{GL}_n(R') = \prod_{i=0}^r \mathrm{GL}_n(R'_i)$. Furthermore, suppose that $\forall \gamma \in \Gamma$, $\operatorname{Tr}\rho'(\gamma) \in R$. In particular, we have for all $\gamma \in \Gamma$, $\operatorname{Tr}\big(\overline{\rho}'_i(\gamma)\big) \in k$ and is independent of i. Finally, suppose that for some i, the residual representation $\overline{\rho}'_i$ is absolutely irreducible.

Theorem 2.3.4. *Under these hypotheses, ρ' is equivalent to a representation coming from a representation $\rho : \Gamma \to \mathrm{GL}_n(R)$. Moreover, ρ is unique up to strict equivalence.*

Proof. Uniqueness follows from Theorem 2.3.3, so we only need to show existence.
First, suppose that $R = k$ and $R' = k'$ is a finite extension of k. The representation $\rho' : \Gamma \to \mathrm{GL}_n(k')$ induces a morphism of k'-algebras $u : k'[\Gamma] \to \mathrm{M}_n(k')$. Let $e_1, \dots, e_{n^2} \in \rho'(\Gamma)$ a basis of $\mathrm{M}_n(k')$ as a k'-vector space (which is possible by Corollary 2.3.2), and let $A = u(k[\Gamma])$ so that $\forall a \in A$, $\operatorname{Tr}(a) \in k$. We want to show that $\dim_k A = n^2$. Writing $a \in A$ as $a = \alpha_1 e_1 + \dots + \alpha_{n^2} e_{n^2}$, this amounts to saying that $\alpha_i \in k$. But we have $\operatorname{Tr}(ye_j) = \sum_i \alpha_i \operatorname{Tr}(e_i e_j)$, and the matrix

$$\big(\operatorname{Tr}(e_i e_j)\big)_{i,j} \in \mathrm{M}_{n^2}(k)$$

is invertible in $\mathrm{M}_n(k')$, hence in $\mathrm{M}_n(k)$, since the trace pairing is a perfect pairing. So we get that $\alpha_i \in k$ as desired.
Therefore, we have that $A \otimes_k k' \cong \mathrm{M}_n(k')$ which implies that A is a central simple algebra. Indeed, $Z(A) \otimes_k k' \subseteq Z(A \otimes_k k') = k' \cdot \mathrm{id}$, and comparing dimensions gives us $Z(A) = k \cdot \mathrm{id}$. Moreover, if I is a two sided left ideal of A, then $I \otimes_k k'$ is a two sided left ideal of $\mathrm{M}_n(k')$, so $I \otimes_k k' = 0$ which

implies that $I = 0$. However, since k is finite, $\mathrm{Br}(k'/k) = 0$, so $A \cong \mathrm{M}_n(k)$. Therefore, given that any automorphism of $\mathrm{M}_n(k')$ is inner, ρ' is conjugate to a representation with values in $\mathrm{GL}_n(k)$.
Back to the general setting, the previous case tells us that the $\overline{\rho}'_i$ are all conjugate to the extension to k'_i of some representation $\overline{\rho} : \Gamma \to \mathrm{GL}_n(k)$. So up to conjugation, we can assume that all the $\overline{\rho}'_i$ have image in $\mathrm{GL}_n(k)$ and are equal, in which case ρ' takes values in $\mathrm{GL}_n(R'')$ where $R'' = \{x = (x_i) \in R' \mid \overline{x}_i = \overline{x}_j \in k \ \forall i, j\}$ is local with residue field k. Therefore, we can suppose that R' is local with residue field k, and by arguments of continuity we can also suppose that R' is Artinian.
Now to prove the theorem, we proceed by induction on the length of R'. The case $\mathrm{length}(R') = 0$ i.e. $R' = k$ has already been dealt with. So let us assume that R' is not a field and let I be a non-zero principal ideal such that $\mathfrak{m}_{R'} I = 0$. We have an injection $R/(R \cap I) \hookrightarrow R'/I$, so by induction hypothesis, we can assume that $\rho' : \Gamma \to \mathrm{GL}_n(R'/I)$ has coefficients in $R/(R \cap I)$. Up to changing R' by the subring $R'' = \{x = (x_i) \in R' \mid x_i \equiv x_j \in R/(R \cap I) \mod I\}$, we can assume that $R'/I = R/(R \cap I)$. If $I \subseteq R$, then we get $R = R'$, and the theorem is trivial. Otherwise, $R \cap I = \{0\}$ ($\mathrm{length}(I) = 1$). So $R' = R \oplus I$ with multiplication given by $(r, i)(r', i') = (rr', r'i + ri')$, and we can write $\rho'(\gamma) = \rho_0(\gamma) + \rho_1(\gamma)$ with $\rho_0 : \Gamma \to \mathrm{GL}_n(R)$ a representation and $\rho_1(\gamma) \in \mathrm{M}_n(I)$. As $\mathrm{Tr}\,\rho'(\gamma) \in R$, we have $\mathrm{Tr}\,\rho' = \mathrm{Tr}\,\rho_0$, so by Theorem 2.3.4, they are equivalent which is what we want to prove. $\square$

2.4 Global deformation problems

Let F be a number field, S a finite set of finite places of F containing all places above p, and T a subset of S. We fix an algebraic closure $\overline{F}$ of F, and we let F_S be the maximal subextension of $\overline{F}$ which is unramified outside of S. We write $G_F = \mathrm{Gal}(\overline{F}/F)$ and $G_{F,S} = \mathrm{Gal}(F_S/F)$. We consider an absolutely irreducible representation $\overline{\rho} : G_{F,S} \to \mathrm{GL}_n(k)$ and we let $\Lambda = W(k)$.

The Galois representations that we will study have local properties which we want to single out (for example properties related to ramification and p-adic Hodge theory which are satisfied by automorphic Galois representations). Our goal in this section will be to set the right framework to do so by defining a global deformation functor which encodes deformation conditions at different places of F. The same approach in a slightly more general setting can be found in [Tho16, §5.2 and §5.3].

Let $\psi : G_{F,S} \to \Lambda^\times$ be a lift of $\det \overline{\rho}$, $\Gamma_n(A) = \ker\big(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k)\big)$, and suppose that $(p, n) = 1$. The last condition implies that $\mathrm{ad}^0\overline{\rho}$ is a direct summand of $\mathrm{ad}\overline{\rho}$.
For $v \in S$, we write $\mathcal{D}_v^\square : \mathcal{C}_\Lambda \to \mathrm{Sets}$ for the lifting functor of $\overline{\rho}_{|G_v}$ with fixed determinant ψ which is represented by an object $R_v^\square \in \mathcal{C}_\Lambda$. As a standard notation, we will write $h^i(\cdots)$ for $\dim_k H^i(\cdots)$.

Definition 2.4.1. A global deformation problem is a tuple

$$\mathcal{S} = \big(\overline{\rho}, \psi, S, \{\mathcal{D}_v\}_{v \in S}\big)$$

where:

- $\overline{\rho}$,ψ and S are defined as above;
- For each $v \in S$, $\mathcal{D}_v \subseteq \mathcal{D}_v^\square$ is a deformation problem for $\overline{\rho}_{|G_{F_v}}$, called a local deformation problem.

Definition 2.4.2. Consider a global deformation problem $\mathcal{S} = \big(\overline{\rho}, \psi, S, \{\mathcal{D}_v\}_{v \in S}\big)$. Let $A \in \mathcal{C}_\Lambda$, and let $\rho : G_{F,S} \to \mathrm{GL}_n(A)$ be a lift of $\overline{\rho}$. We say that ρ is of type $\mathcal{S}$ if it satisfies the following conditions:

- $\det \rho = \psi$, i.e, $\det \rho : G_{F,S} \to A^\times$ agree with the composite of $\psi : G_{F,S} \to \Lambda^\times$ with the structural morphism $\Lambda^\times \to A^\times$;
- For each $v \in S$, the restriction $\rho_{|G_{F_v}}$ lies in $\mathcal{D}_v(A)$.

We write $\mathcal{D}_{\mathcal{S}}^{\square}$ for the functor $\mathcal{C}_\Lambda \to \text{Sets}$ that associates to $A \in \mathcal{C}_\Lambda$ the set of liftings $\rho : G_{F,S} \to \mathrm{GL}_n(A)$ of $\overline{\rho}$ that are of type $\mathcal{S}$.

Definition 2.4.3. If $A \in \mathcal{C}_\Lambda$, we define a T-framed lifting of $\overline{\rho}$ to A to be a tuple $(\rho, \{\alpha_v\}_{v\in T})$, where $\rho : G_{F,S} \to \mathrm{GL}_n(A)$ is a lifting of $\overline{\rho}$ and $\alpha_v \in \mathrm{GL}_n(A)$. Two T-framed liftings $(\rho, \{\alpha_v\}_{v\in T})$ and $(\rho', \{\alpha'_v\}_{v\in T})$ are said to be strictly equivalent if there exists $\beta \in \Gamma_n(A)$ such that $\rho' = \beta\rho\beta^{-1}$ and $\alpha'_v = \beta\alpha_v$ for all $v \in T$.

We write $\mathcal{D}_{\mathcal{S}}^{T}$ for the functor $\mathcal{C}_\Lambda \to \text{Sets}$ that associates to $A \in \mathcal{C}_\Lambda$ the set of strict equivalence classes of T-framed liftings $(\rho, \{\alpha_v\}_{v\in T})$ such that ρ is of type $\mathcal{S}$.

Theorem 2.4.4. *Let $\mathcal{S} = \big(\overline{\rho}, \psi, S, \{\mathcal{D}_v\}_{v\in S}\big)$ be a global deformation problem. The functors $\mathcal{D}_{\mathcal{S}}^{\square}$ and $\mathcal{D}_{\mathcal{S}}^{T}$ are represented by objects $R_{\mathcal{S}}^{\square}$ and $R_{\mathcal{S}}^{T}$ of $\mathcal{C}_\Lambda$.*

If $T = \emptyset$, we write $R_{\mathcal{S}}^{\text{univ}}$ for $R_{\mathcal{S}}^{T}$. Let $\mathcal{S} = \big(\overline{\rho}, \psi, S, \{\mathcal{D}_v\}_{v\in S}\big)$ be a global deformation problem, then $\mathcal{D}_v \subseteq \mathcal{D}_v^{\square}$ is represented by a ring R_v which is a quotient of $R_v^{\square}$. There is a natural transformation $\mathcal{D}_{\mathcal{S}}^{T} \to \prod_{v\in T} \mathcal{D}_v$, that sends $\big(\rho, (\alpha_v)_{v\in T}\big)$ to $(\alpha_v^{-1}\rho_{|G_v}\alpha_v)_{v\in T}$. But the functor $\prod_{v\in T}\mathcal{D}_v$ is represented by $R_{\text{loc}} := \widehat{\otimes}_{v\in T} R_v$ (where the tensor product is taken over Λ). So we get a map of rings $R_{\text{loc}} \to R_{\mathcal{S}}^{T}$.

Remark 2.4.5. Fix some $v_0 \in T$, and let $T' = T - \{v_0\}$. We define a functor $\mathcal{D}'$ by letting $\mathcal{D}'(A) = \mathcal{D}_{\mathcal{S}}^{\square}(A) \times \prod_{v\in T'} \Gamma_n(A)$, for $A \in \mathcal{C}_\Lambda$. Then, we have a natural transformation $\mathcal{D}' \to \mathcal{D}_{\mathcal{S}}^{T}$ sending $\big(\rho, (\alpha_v)_{v\in T'}\big)$ to $\big(\rho, (\alpha_v)_{v\in T}\big)$ with $\alpha_{v_0} = 1$, which is actually an isomorphism. Since the functor $A \mapsto \Gamma_n(A)$ is represented by $\Lambda[[X_1, \dots, X_{n^2}]]$, we get that the ring $R_{\mathcal{S}}^{T}$ is a power series ring over $R_{\mathcal{S}}^{\square}$ in $n^2(\#T - 1)$ variables. On the other hand, the functor $D_{\mathcal{S}}^{\square} \to D_{\mathcal{S}}^{\emptyset}$ is formally smooth. So looking at the description of the tangent spaces as $k[\epsilon]$-points, we get that $R_{\mathcal{S}}^{\square}$ is a power series ring over $R_{\mathcal{S}}^{\text{univ}}$ in $n^2 - 1$ variables ($\overline{\rho}$ has trivial endomorphisms). In conclusion, $R_{\mathcal{S}}^{T}$ is a power series ring over $R_{\mathcal{S}}^{\text{univ}}$ in $n^2\#T - 1$ variables.

Presentation of the deformation ring

We want to compute the tangent space of the deformation ring $R_{\mathcal{S}}^{T}$ as an R_{loc}-algebra. That is to compute $\mathfrak{m}_{R_{\mathcal{S}}^{T}}/(\mathfrak{m}_{R_{\mathcal{S}}^{T}}^2, \mathfrak{m}_{R_{\text{loc}}})$. Recall that we have canonical isomorphisms:

$$Z^1(G_v, \mathrm{ad}^0\overline{\rho}) \cong \mathrm{Hom}_k\left(\mathfrak{m}_{R_v^{\square}}/(\mathfrak{m}_{R_v^{\square}}^2, \mathfrak{m}_\Lambda), k\right) \cong \mathrm{Hom}_{\mathcal{C}_\Lambda}(R_v^{\square}, k[\epsilon])$$

We let $\widetilde{\mathcal{L}}_v \subseteq Z^1(G_v, \mathrm{ad}^0\overline{\rho})$ be the the preimage of the subspace $\mathrm{Hom}_k\left(\mathfrak{m}_{R_v}/(\mathfrak{m}_{R_v}^2, \mathfrak{m}_\Lambda), k\right)$ under the above isomorphism. Since $\mathcal{D}_v$ is a deformation problem, $\widetilde{\mathcal{L}}_v$ is the preimage of a subspace $\mathcal{L}_v \subseteq H^1(G_v, \mathrm{ad}^0\overline{\rho})$.
For our purpose, we introduce some cohomology groups. So let us consider the following complexes:

$$C^i_{\text{loc}}(\mathrm{ad}^0\overline{\rho}) = \begin{cases} \bigoplus\limits_{v\in T} C^0(G_v, \mathrm{ad}\overline{\rho}) & \text{if } i = 0 \\ \oplus_{v\in T} C^1(G_v, \mathrm{ad}^0\overline{\rho}) \bigoplus \oplus_{v\in S-T} C^1(G_v, \mathrm{ad}^0\overline{\rho})/\widetilde{\mathcal{L}}_v & \text{if } i = 1 \\ \bigoplus\limits_{v\in S} C^i(G_v, \mathrm{ad}^0\overline{\rho}) & \text{otherwise} \end{cases}$$

where $C^\bullet(G_v, \mathrm{ad}^0\overline{\rho})$ is the cochain complex of continuous inhomogeneous cochains.
Denote by $\widetilde{C}^\bullet(G_{F,S}, \mathrm{ad}^0\overline{\rho})$ the complex given by

$$\widetilde{C}^i(G_{F,S}, \mathrm{ad}^0\overline{\rho}) = \begin{cases} C^0(G_{F,S}, \mathrm{ad}\overline{\rho}) & \text{if } i = 0 \\ C^i(G_{F,S}, \mathrm{ad}^0\overline{\rho}) & \text{if } i > 0 \end{cases}$$

The inclusion $G_v \hookrightarrow G_{F,S}$ gives rise to a map of cochain complexes $\widetilde{C}^\bullet(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \to C^\bullet_{\mathrm{loc}}(\mathrm{ad}^0\overline{\rho})$. Taking the cone of this map, we obtain a cochain complex

$$C^\bullet_{\mathcal{S},T}(\mathrm{ad}^0\overline{\rho}) = \widetilde{C}^\bullet(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \oplus C^{\bullet-1}_{\mathrm{loc}}(\mathrm{ad}^0\overline{\rho})$$

where the boundary map is given by

$$\begin{aligned} C^i_{\mathcal{S},T}(\mathrm{ad}^0\overline{\rho}) &\to C^{i+1}_{\mathcal{S},T}(\mathrm{ad}^0\overline{\rho}) \\ \big(\phi, (\varphi_v)_v\big) &\mapsto \big(\partial\phi, (\phi_{|G_v} - \partial\varphi_v)_v\big) \end{aligned}$$

Later we will identify the tangent space of $R^T_{\mathcal{S}}$ over R_{loc} with $H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho})$. Hence, computing its dimension over k tells us the number of generators of $R^T_{\mathcal{S}}$ over R_{loc}, which is what we will do next. So consider the short exact sequence:

$$0 \to C^{\bullet-1}_{\mathrm{loc}}(\mathrm{ad}^0\overline{\rho}) \to C^\bullet_{\mathcal{S},T}(\mathrm{ad}^0\overline{\rho}) \to \widetilde{C}^\bullet(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \to 0$$

which induces a long exact sequence of cohomology groups

$$\begin{tikzcd} 0 \arrow[r] & H^0_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \arrow[r] & H^0(G_{F,S}, \mathrm{ad}\overline{\rho}) \arrow[r] & \bigoplus_{v\in T} H^0(G_v, \mathrm{ad}\overline{\rho}) \arrow[dll] \\ & H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \arrow[r] & H^1(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \arrow[r] & \bigoplus_{v\in T} H^1(G_v, \mathrm{ad}^0\overline{\rho}) \oplus \bigoplus_{v\in S-T} H^1(F_v, \mathrm{ad}^0\overline{\rho})/\mathcal{L}_v \arrow[dll] \\ & H^2_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \arrow[r] & H^2(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \arrow[r] & \bigoplus_{v\in S} H^2(G_v, \mathrm{ad}^0\overline{\rho}) \arrow[r] & \cdots \end{tikzcd}$$

Note that all the cohomology groups involved are finite. Moreover if $i \geq 3$, for each finite place v we have $H^i(G_v, \mathrm{ad}^0\overline{\rho}) = 0$; and we have $H^i(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \cong \bigoplus_{v\in S_\infty} H^i(G_v, \mathrm{ad}^0\overline{\rho})$. But given that $p > 2$, for an infinite place v, the orders of G_v and $\mathrm{ad}^0\overline{\rho}$ are coprime, so we have $H^i(G_v, \mathrm{ad}^0\overline{\rho}) = 0$ for $i > 0$. In particular, $H^i(G_{F,S}, \mathrm{ad}^0\overline{\rho}) = 0$ for $i \geq 3$. As a consequence, for $i > 3$, we have $H^i_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) = 0$. Taking the Euler characteristic of the above long exact sequence, we get a formula:

$$\chi_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) = \chi(G_{F,S}, \mathrm{ad}^0\overline{\rho}) - \sum_{v\in S} \chi(G_{F_v}, \mathrm{ad}^0\overline{\rho}) - \sum_{v\in S-T} \big(\dim_k \mathcal{L}_v - h^0(G_v, \mathrm{ad}^0\overline{\rho})\big) + 1 - \#T \quad (2.1)$$

since $h^0(G_v, \mathrm{ad}\overline{\rho}) = h^0(G_v, \mathrm{ad}^0\overline{\rho}) + 1$ as well as $h^0(G_{F,S}, \mathrm{ad}\overline{\rho}) = h^0(G_{F,S}, \mathrm{ad}^0\overline{\rho}) + 1$ (the identity matrix is fixed by $\mathrm{ad}\overline{\rho}$). By Tate's global Euler characteristic formula [Mil06, Ch. I,Thm. 5.1] we have

$$\chi(G_{F,S}, \mathrm{ad}^0\overline{\rho}) = \sum_{v\in S_\infty} h^0(G_v, \mathrm{ad}^0\overline{\rho}) - [F:\mathbb{Q}](n^2-1)$$

and by Tate's local Euler characteristic formula [Mil06, Ch. I,Thm. 2.8], for a non-archimedean place v, we have:

$$\chi(G_v, \mathrm{ad}^0\overline{\rho}) = -\dim_k \big([\mathcal{O}_v : |\mathrm{ad}^0\overline{\rho}|\mathcal{O}_v]\big)$$

which equals to 0 if $p \nmid v$, so we have :

$$\sum_{v\in S} \chi(G_v, \mathrm{ad}^0\overline{\rho}) = -\sum_{p|v} (n^2-1)[F_v : \mathbb{Q}_p] = -(n^2-1)[F:\mathbb{Q}]$$

Putting this in equation (2.1), we get:

$$\chi_{\mathcal{S},T}(G_{F,S},\mathrm{ad}^0\overline{\rho}) = \sum_{v\in S_\infty} h^0(G_v,\mathrm{ad}^0\overline{\rho}) - \sum_{v\in S-T}\left(\dim_k \mathcal{L}_v - h^0(G_v,\mathrm{ad}^0\overline{\rho})\right) + 1 - \#T$$

We will need to use Tate's local Duality, for which we recall the statement:

Theorem 2.4.6. *Let v be a finite place of F, $\overline{F_v}$ an algebraic closure of F_v, and μ_∞ the G_v-module of all roots of unity in $\overline{F_v}$. If M a finite G_v-module, let $M^* = \mathrm{Hom}_{\mathbb{Z}}(M,\mu_\infty)$. Then, we have:*

(1) For $i = 0,1,2$, the cup-product induced a perfect pairing:

$$H^i(G_v,M)\times H^{2-i}(G_v,M^*) \to H^2(G_v,\mu_\infty) = \mathbb{Q}/\mathbb{Z}$$

(2) If char$(k(v))$ does not divide the order of M, then the unramified classes:

$$H^1(G_v/I_v,M^{I_v}) \quad \text{and} \quad H^1(G_v/I_v,M^{*I_v})$$

are the exact annihilators of each other under the pairing above.

We will also need The Poitou-Tate exact sequence given as follows (see [Mil06, Ch. 1, Thm. 4.10] for a more general statement):

Theorem 2.4.7. *Let M be a finite $G_{F,S}$-module. We let $P^i_S(F,M) = \prod_{v\in S_\infty}\widehat{H}^i(G_v,M)\prod_{v\in S}H^i(G_v,M)$ such that for an archimedean place v, $\widehat{H}^i(G_v,M)$ denotes the i-th Tate cohomology group. Then, we have a nine term exact sequence:*

$$\begin{tikzcd}
0 \arrow[r] & H^0(G_{F,S},M) \arrow[r] & P^0_S(F,M) \arrow[r] & H^2(G_{F,S},M^*)^\vee \arrow[d] \\
& H^1(G_{F,S},M^*)^\vee \arrow[d] & P^1_S(F,M) \arrow[l] & H^1(G_{F,S},M) \arrow[l] \\
& H^2(G_{F,S},M) \arrow[r] & P^2_S(F,M) \arrow[r] & H^0(G_{F,S},M^*)^\vee \arrow[r] & 0
\end{tikzcd}$$

where for an abelian group A, $A^\vee = \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$.

Now given that $p\nmid n$, we have a perfect pairing of Galois modules:

$$\begin{aligned} \mathrm{ad}^0\overline{\rho}\times\mathrm{ad}^0\overline{\rho}(1) &\to k(1) \\ (X,Y) &\mapsto \mathrm{Tr}(XY) \end{aligned}$$

Since $k(1)\cong \mu_{\#k}$ as Galois modules, we get that $\mathrm{ad}^0\overline{\rho}(1)\cong(\mathrm{ad}^0\overline{\rho})^*$. So by Tate's local duality, for each finite place v of F, we have a perfect pairing between $H^1(G_v,\mathrm{ad}^0\overline{\rho})$ and $H^1(G_v,\mathrm{ad}^0\overline{\rho}(1))$. We write $\mathcal{L}_v^\perp \subseteq H^1(F_v,\mathrm{ad}^0\overline{\rho}(1))$ for the annihilator of $\mathcal{L}_v$ under this pairing, and we define

$$H^1_{\mathcal{S},T}(G_{F,S},\mathrm{ad}^0\overline{\rho}(1)) = \ker\left(H^1(G_{F,S},\mathrm{ad}^0\overline{\rho}(1)) \to \prod_{v\in S-T} H^1(F_v,\mathrm{ad}^0\overline{\rho}(1))/\mathcal{L}_v^\perp\right) \tag{2.2}$$

so that upon dualising, we get an exact sequence:

$$\bigoplus_{v\in S-T}\mathcal{L}_v \to H^1(G_{F,S},\mathrm{ad}^0\overline{\rho})^\vee \to H^1_{\mathcal{S},T}(G_{F,S},\mathrm{ad}^0\overline{\rho}(1))^\vee \to 0$$

By the Poitou-Tate exact sequence, we get an exact sequence:

$$
\begin{tikzcd}
H^1(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \arrow[r] & \bigoplus_{v\in T} H^1(G_v, \mathrm{ad}^0\overline{\rho}) \oplus \bigoplus_{v\in S-T} H^1(F_v, \mathrm{ad}^0\overline{\rho})/\mathcal{L}_v \arrow[dll, rounded corners] \\
H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1))^\vee \arrow[r] & H^2(G_{F,S}, \mathrm{ad}^0\overline{\rho}) \arrow[r] & \bigoplus_{v\in S} H^2(G_v, \mathrm{ad}^0\overline{\rho}) \arrow[dll, rounded corners] \\
H^0(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1))^\vee \arrow[r] & 0
\end{tikzcd}
$$

where we ignored the infinite places thanks to the condition $p > 2$ (as mentioned before, the higher cohomology groups at the infinite places with coefficients in $\mathrm{ad}^0\overline{\rho}$ vanish). If we compare this exact sequence with the long exact sequence of the cohomology groups, we find that:

$$
\begin{aligned}
H^3_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) &\cong H^0(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1))^\vee \\
H^2_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) &\cong H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1))^\vee
\end{aligned}
$$

Finally, note that since $T \neq \emptyset$, we have $h^0_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) = 0$, so putting everything together, we get:

$$
\begin{aligned}
h^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}) =& h^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)) - h^0(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)) - \sum_{v|\infty} h^0(G_v, \mathrm{ad}^0\overline{\rho}) \\
&+ \sum_{v\in S-T} \big(\dim_k \mathcal{L}_v - h^0(G_v, \mathrm{ad}^0\overline{\rho})\big) + \#T - 1
\end{aligned} \tag{2.3}
$$

Proposition 2.4.8. *There is a canonical isomorphism*

$$
\mathrm{Hom}_k\left(\mathfrak{m}_{R^T_{\mathcal{S}}}/(\mathfrak{m}^2_{R^T_{\mathcal{S}}}, \mathfrak{m}_{R_{loc}}), k\right) \xrightarrow{\simeq} H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho})
$$

in particular, $R^T_{\mathcal{S}}$ is a quotient of a power series ring over R_{loc} in $h^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho})$ variables.

Proof. By Proposition 2.1.1, we have an isomorphism between $\mathrm{Hom}_k\left(\mathfrak{m}_{R^T_{\mathcal{S}}}/(\mathfrak{m}^2_{R^T_{\mathcal{S}}}, \mathfrak{m}_{R_{\mathrm{loc}}}), k\right)$ and the subgroup of morphisms $f : R^T_{\mathcal{S}} \to k[\epsilon]$ that send $\mathfrak{m}_{R_{\mathrm{loc}}}$ to zero, i.e., when restricted to R_{loc}, f factor through k. Having that in mind, it is not hard to see that $\mathrm{Hom}_k\left(\mathfrak{m}_{R^T_{\mathcal{S}}}/(\mathfrak{m}^2_{R^T_{\mathcal{S}}}, \mathfrak{m}_{R_{\mathrm{loc}}}), k\right)$ is in bijection with the subset of $\mathcal{D}^T_{\mathcal{S}}(k[\epsilon])$ of lifts that map to trivial lifts when restricted to G_v for $v \in T$.
An element of the set $\mathcal{D}^T_{\mathcal{S}}(k[\epsilon])$ corresponds to an equivalence class $\big((1+\epsilon c)\overline{\rho}, (1+\epsilon\beta_v)_{v\in T}\big)$ with $c \in Z^1\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}\big)$ and $\beta_v \in \mathrm{ad}\overline{\rho}$. The condition that it gives a trivial lifting at $v \in T$ is equivalent to the condition

$$
(1-\epsilon\beta_v)(1+\epsilon c_{|G_v})\overline{\rho}_{|G_v}(1+\epsilon\beta_v) = \overline{\rho}_{|G_v}
$$

which is equivalent to $c(g) = \beta_v - \overline{\rho}(g)\beta_v\overline{\rho}(g)^{-1} = \beta_v - \mathrm{ad}\overline{\rho}(g)(\beta_v)$ for all $g \in G_v$.
Two pairs $\big((1+\epsilon c)\overline{\rho}, (1+\epsilon\beta_v)_{v\in T}\big)$ and $\big((1+\epsilon c')\overline{\rho}, (1+\epsilon\gamma_v)_{v\in T}\big)$ are equivalent if and only if there exists $m \in \mathrm{ad}\overline{\rho}$ such that

$$
\begin{aligned}
c'(g) &= c(g) + \big(1 - \mathrm{ad}^0\overline{\rho}(g)\big)m' \\
\beta_v &= \alpha_v + m
\end{aligned}
$$

where m' the projection of m in $\mathrm{ad}^0\overline{\rho}$. Indeed, since $(p,n) = 1$, we have that $(1 - \epsilon\frac{\mathrm{Tr}(m)}{n})(1+\epsilon m) = (1+\epsilon m')$ Hence, conjugating by $(1+\epsilon m)$ is the same as conjugating by $(1+\epsilon m')$.
Therefore, the tuple (c, β_v) up to equivalence describes exactly an element of $H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho})$. □

2.5 Taylor-Wiles primes

In this section we will present the notion of sets of Taylor-Wiles places and prove their existence. Their interesting feature is that we will be able to have control over the size of the tangent space of the global deformation functor as we relax the conditions at these places. We will work in the GL_2 case, but one can find in [Tho12, §4] the GL_n case under stronger assumptions on the residual representation. We are keeping the same notation and hypothesis as in section §2.4 but with $n = 2$.

Definition 2.5.1. A place v of F is called a Taylor-Wiles place of level N if it satisfies the following conditions

(1) $v \notin S$,

(2) $\#k(v) \equiv 1 \mod p^N$,

(3) The eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$ are distinct and belong to k.

where $k(v)$ is the residual field of F_v.

Lemma 2.5.2. *Let $R \in \mathcal{C}_\Lambda$. Then,* $\ker\big(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k)\big)$ *is a pro-p group.*

Proof. Since

$$\ker\big(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k)\big) = \varprojlim_k \ker\big(\mathrm{GL}_n(R/\mathfrak{m}^k) \to \mathrm{GL}_n(k)\big)$$

it suffices to show that $\ker\big(\mathrm{GL}_n(R/\mathfrak{m}^k) \to \mathrm{GL}_n(k)\big)$ is a p-group. So suppose that R is an Artinian ring with $\mathfrak{m}^k = 0$ for some $k \geq 0$. We have $\ker\big(\mathrm{GL}_n(R) \to \mathrm{GL}_n(k)\big) = \mathrm{id} + \mathfrak{m}\mathrm{M}_n(R)$, and

$$(\mathrm{id} + \mathfrak{m}\mathrm{M}_n(R))^{p^l} = \sum_{i=0}^{p^l} \binom{p^l}{i} \mathfrak{m}^i \mathrm{M}_n(R)$$

By some theorem of Kummer, we have $v_p(\binom{p^l}{i}) \geq l - v_p(i)$. So if we choose $l = 2k$, then for $i = 1, \ldots, p^k - 1$ $v_p(\binom{p^l}{i}) \geq k$. And since $p \in \mathfrak{m}$, we have for $i = 1, \ldots, p^{2k}$

$$\binom{p^l}{i} \mathfrak{m}^i \mathrm{M}_n(R) \subseteq \mathfrak{m}^k \mathrm{M}_n(R) = 0$$

Therefore, $(\mathrm{id} + \mathfrak{m}\mathrm{M}_n(R))^{p^l} = \mathrm{id}$. Since R is Artinian and k is finite, then so is R, and subsequently $\mathrm{M}_n(R)$. This shows that $\ker\big(\mathrm{GL}_n(R/\mathfrak{m}^k) \to \mathrm{GL}_n(k)\big)$ is a p-group. $\square$

The following lemma can be proved by a calculation trick and using Hensel's lemma.

Lemma 2.5.3. *Let $R \in \mathcal{C}_\Lambda$. Let $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_2(k)$ be a matrix such that $\beta \neq \alpha$ and are both non-zero. Suppose that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a lift of this matrix to $M_2(R)$. Then, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \simeq \begin{pmatrix} \widetilde{\alpha} & 0 \\ 0 & \widetilde{\beta} \end{pmatrix}$ where $\widetilde{\alpha}, \widetilde{\beta}$ lift α, β.*

Although we will relax the conditions at the Taylor-Wiles places, the

Lemma 2.5.4. *Let v be a Taylor-Wiles place, $R \in \mathcal{C}_\Lambda$, and $\rho : G_F \to \mathrm{GL}_2(R)$ be a lift of $\overline{\rho}$. Then, $\rho_{|G_{F_v}}$ is a sum of two tamely ramified characters $\eta_1 \oplus \eta_2$.*

Proof. Since $\overline{\rho}$ is unramified at v, the image of the inertia group at v I_{F_v} lies in $1 + \mathrm{M}_2(\mathfrak{m}_R)$. By lemma 2.5.2, the latter is a pro-p group, and so ρ factors through the tame inertia subgroup of I_{F_v}. The tame Galois group is generated by $\sigma = \mathrm{Frob}_v$ and the tame inertia group $I^t_{F_v}$. For every $\tau \in I^t_{F_v}$, we have the relationship

$$\sigma\tau\sigma^{-1} = \tau^q \quad (*)$$

By the Taylor-Wiles assumption on the Frobenius, $\overline{\rho}(\sigma)$ has distinct eigenvalues. So by Lemma 2.5.3, we can find a basis of $M_2(R)$ such that $\rho(\sigma) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ for some α, β lifting the eigenvalues of $\overline{\rho}(\sigma)$. With respect to this basis, we can write

$$\rho(\tau) = \mathrm{id} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $\tau \in I^t_v$ and $a, b, c, d \in \mathfrak{m}_Q$ (since ρ is unramified at v). Now applying ρ to $(*)$, we get

$$\mathrm{id} + \begin{pmatrix} a & \alpha\beta^{-1}b \\ \beta\alpha^{-1}c & d \end{pmatrix} = \sum_{k=0}^{q} \begin{pmatrix} q \\ k \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^k$$

For $k \geq 2$, the top right and bottom left entries of the right side summands lie in $\mathfrak{m}_Q I$, where I is the ideal generated by b and c. Therefore, the above equality gives

$$b(\alpha\beta^{-1} - q), c(\beta\alpha^{-1} - q) \in \mathfrak{m}_Q I$$

But by assumption, α and β are residually distinct. And since $q \equiv 1 \mod p$, we get that

$$(\alpha\beta^{-1} - q), (\beta\alpha^{-1} - q) \not\equiv 0 \mod p$$

so they are both units in R. Thus, $b, c \in \mathfrak{m}_Q I$, i.e., $I = \mathfrak{m}_Q I$; which, by Nakayama's lemma, implies that $I = 0$. Hence, $b = c = 0$ and $\rho(\tau)$ is diagonal. Since τ was chosen arbitrarily, we get the desired result. □

Lemma 2.5.5. *Let v be a Taylor-Wiles place. Then, we have*

$$\dim_k H^0(G_{k(v)}, \mathrm{ad}^0\overline{\rho}) = \dim_k H^0\big(G_{k(v)}, \mathrm{ad}^0\overline{\rho}(1)\big) = 1$$

and

$$\dim_k H^1(G_{k(v)}, \mathrm{ad}^0\overline{\rho}) = \dim_k H^1\big(G_{k(v)}, \mathrm{ad}^0\overline{\rho}(1)\big) = 1$$

. *where $G_{k(v)}$ is the absolute Galois group of $k(v)$.*

Proof. Since $\chi_p(\mathrm{Frob}_v) = q \equiv 1 \mod p$, the action of $G_{k(v)}$ on $\mathrm{ad}^0\overline{\rho}(1)$ is the same as that on $\mathrm{ad}^0\overline{\rho}$. So it suffices to show the result on the latter.
By definition of the Taylor-Wiles primes and Lemma 2.5.3, $\overline{\rho}(\mathrm{Frob}_v) \sim \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with $\alpha \neq \beta$. This matrix only commutes with the subspace generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathrm{ad}^0\overline{\rho}$. This gives the first equality. For $H^1(k_v, \mathrm{ad}^0\overline{\rho})$, a cocycle is determined only by the image of Frob_v. Thus, $\dim_k Z^1(k_v, \mathrm{ad}^0\overline{\rho}) = 3$. Since $B^1(k_v, \mathrm{ad}^0\overline{\rho}) = \{(g \mapsto g \cdot m - m) \mid \text{for } m \in \mathrm{ad}^0\overline{\rho}\}$, we have $\dim_k B^1(k_v, \mathrm{ad}^0\overline{\rho}) = 3 - \dim_k(\mathrm{ad}^0\overline{\rho})^{\mathrm{Frob}_v} = 2$. Therefore, $\dim_k H^1(k_v, \mathrm{ad}^0\overline{\rho}) = 1$. □

One important hypothesis for the existence of Taylor-Wiles primes is the condition that

$$\overline{\rho}_{|G_{F(\zeta_p)}} \text{ is absolutely irreducible}$$

This even implies the following stronger statement.

Lemma 2.5.6. *$\overline{\rho}_{|G_{F(\zeta_{p^n})}}$ is absolutely irreducible for all $n \geq 1$.*

Proof. Let $G = G_{F(\zeta_p)}$ and $H = G_{F(\zeta_{p^n})}$. Suppose that $\overline{\rho}_{|H}$ is not irreducible, then there exists a line L in k^2 which is invariant under H. And since k^2 is irreducible as a G-module, there exists $g \in G$ such that $g \cdot L \neq L$. Moreover, $g \cdot L$ is invariant under H (since it is a normal subgroup of G). Therefore, $\overline{\rho}_{|H}$ is the sum of two characters. By irreducibility of $\overline{\rho}$ over G, G/H must permute these characters. But G/H is a p-group so it cannot act transitively on a set with 2 elements. Thus, the two characters are equal.
This means that H stabilizes every line of k^2, and there are $|\mathbb{P}(k)| = k + 1$ of them which is not divisible by p. So the size some orbit of the action of G/H on the set of lines of k^2 must be prime to p. However, the size of each orbit divides $|G/H| = p^k$. So the only way this is possible is that the size of this orbit is 1. This contradicts the irreducibility of k^2 as a G-module. Thus, $\overline{\rho}_{|H}$ is irreducible.
The same argument can be carried out if we first extend the scalars to a finite extension of k. Hence, $\overline{\rho}_{|H}$ is absolutely irreducible. $\square$

Let $H = \ker \mathrm{ad}^0\overline{\rho}$. We set $F_0 = \overline{F}^H$ and $F_n = F_0(\zeta_{p^n})$ for $n \geq 1$.

Lemma 2.5.7. *Let $\psi \in H^1\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\big) - \{0\}$ and $n \geq 1$. Then, $\psi(G_{F_n})$ is non-zero.*

Proof. For $n \geq 1$, there is an inflation-restriction exact sequence

$$0 \to H^1\big(G_{F_n/F}, \mathrm{ad}^0\overline{\rho}(1)\big) \xrightarrow{\mathrm{inf}} H^1\big(G_F, \mathrm{ad}^0\overline{\rho}(1)\big) \xrightarrow{\mathrm{res}} H^1\big(G_{F_n}, \mathrm{ad}^0\overline{\rho}\big)$$

It suffices to show that $H^1\big(G_{F_n/F}, \mathrm{ad}^0\overline{\rho}(1)\big) = 0$. For then ψ would restrict to a non-zero element of $H^1\big(G_{F_n}, \mathrm{ad}^0\overline{\rho}\big)$, and in particular $\psi(G_{F_n}) \neq 0$. So consider another inflation-restriction exact sequence

$$0 \to H^1\big(G_{F_0/F}, (\mathrm{ad}^0\overline{\rho}(1))^{G_{F_0}}\big) \xrightarrow{\mathrm{inf}} H^1\big(G_{F_n/F}, \mathrm{ad}^0\overline{\rho}(1)\big) \xrightarrow{\mathrm{res}} H^1\big(G_{F_n/F_0}, \mathrm{ad}^0\overline{\rho}(1)\big)^{G_{F_n/F}} \qquad (2.4)$$

where the action of $g \in G_{F_n/F}$ on the rightmost term is given by $\eta \mapsto \big(h \mapsto g^{-1}\eta(ghg^{-1})\big)$. This allows us to reduce to showing that the rightmost and the leftmost terms in (2.4) are zero.
We begin by the rightmost term. There is a restriction-corestriction sequence

$$H^1\big(G_{F_n/F_0}, \mathrm{ad}^0\overline{\rho}(1)\big) \xrightarrow{\mathrm{res}} H^1\big(G_{F_n/F_1}, \mathrm{ad}^0\overline{\rho}\big) \xrightarrow{\mathrm{cores}} H^1\big(G_{F_n/F_0}, \mathrm{ad}^0\overline{\rho}(1)\big)$$

for which the composition is the multiplication by $|G_{F_1/F_0}|$ which is prime to p since it is $\leq p-1$. Hence, Res is injective. It also sends $G_{F_n/F}$-invariants to $G_{F_n/F}$-invariants. So it suffices to show that $H^1\big(G_{F_n/F_1}, \mathrm{ad}^0\overline{\rho}(1)\big)^{G_{F_n/F}} = 0$. Looking at the commutative diagram

$$\begin{tikzcd}
G_{F_n/F} \arrow[r, two heads] & G_{F(\zeta_{p^n})/F} \\
G_{F_n/F_1} \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

we know that $G_{F(\zeta_{p^n})/F}$ is commutative so the action by conjugation of $G_{F_n/F}$ on G_{F_n/F_1} is trivial. Moreover, G_{F_n/F_1} acts trivially on $\mathrm{ad}^0\overline{\rho}(1)$ by definition of F_0 and by the fact that $\zeta_p \in F_1$. Combining these two facts, we get

$$H^1\big(G_{F_n/F_1}, \mathrm{ad}^0\overline{\rho}(1)\big)^{G_{F_n/F}} = \mathrm{Hom}\left(G_{F_n/F_1}, \mathrm{ad}^0\overline{\rho}(1)\right)^{G_{F_n/F}} = \mathrm{Hom}\left(G_{F_n/F_1}, \mathrm{ad}^0\overline{\rho}(1)^{G_{F_n/F}}\right)$$

However $\big(\mathrm{ad}^0\overline{\rho}(1)\big)^{G_{F_n/F}} = 0$. Indeed, any $G_{F_n/F}$-invariant element of $\mathrm{ad}^0\overline{\rho}(1)$ is equivalently a trace 0 intertwining operator $V \to V(1)$ (where V is the underlying space of $\overline{\rho}$). But since $\overline{\rho}$ is irreducible, using Schur's lemma we get that $\overline{\rho} \simeq \overline{\rho}(1)$. Thus, $\det \overline{\rho} = \chi_p^2 \det \overline{\rho}$. So the square every element in k is

1 which can't happen if $p > 3$ (which is our assumption). Hence, we just showed that the rightmost term in (2.4) is zero.
For the leftmost term, Note that if $F_0 \not\supseteq F(\zeta_p)$, then $\left(\mathrm{ad}^0\overline{\rho}(1)\right)^{G_{F_0}} = 0$. Indeed, the hypothesis tells us that $\chi_p(G_{F_0}) \neq 1$. But G_{F_0} acts trivially on $\mathrm{ad}^0\overline{\rho}$, so it cannot fix a non-zero element of $\mathrm{ad}^0\overline{\rho}(1)$. Therefore, we can assume that $F_0 \supseteq F(\zeta_p)$.
Since $\left(\mathrm{ad}^0\overline{\rho}(1)\right)^{G_{F_0}}$ has p-power order, we have an injection

$$0 \to H^1\left(G_{F_0/F}, \mathrm{ad}^0\overline{\rho}(1)^{G_{F_0}}\right) \xrightarrow{\text{res}} H^1\left(P, \mathrm{ad}^0\overline{\rho}(1)^{G_{F_0}}\right)$$

where P is the p-Sylow subgroup of $G_{F_0/F}$. Hence, we can assume that P is not trivial, i.e., $p \mid |G_{F_0/F}|$. Moreover, since F_0 is the field cut out by $\mathrm{ad}^0\overline{\rho}$, we have that $\overline{\rho}(G_{F_0}) \subseteq \{\lambda\,\mathrm{id} \mid \lambda \in k^\times\}$. Hence, $G_{F_0/F}$ is isomorphic to the projective image of $\overline{\rho}$. Using this information, we will try to determine this group.

Fact Let $H \subseteq \mathrm{PGL}_2(\overline{\mathbb{F}_p})$ be a finite non trivial subgroup. Then, one of the following assertions is true

1. H is conjugate to a subgroup of the upper triangular matrices.
2. H is conjugate to $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ or $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ for some $r \geq 1$
3. H is isomorphic to A_4,A_5,S_4, or D_{2r} for $r \geq 2$. And if $H \simeq D_{2r} = \langle s, t \mid s^2 = t^r = 1, sts = t^{-1}\rangle$, then it is conjugate to the image of D_{2r} given by

$$s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}$$

where ζ is a primitive r-th root of unity.

Actually, $G_{F_0/F}$ can be none of the above:

– $G_{F_0/F}$ cannot be conjugate to a subgroup of the upper triangular matrices, since $\overline{\rho}_{|G_{F(\zeta_p)}}$ is absolutely irreducible.

– The assumption that $p > 5$ and the fact that $p \mid |G_{F_0/F}|$ elliminates the possibilities A_4,A_5,S_4,$D_{2,r}$ for $p \not| r$.

– $\mathrm{PSL}_2(\mathbb{F}_{p^r})$ is simple for $r > 5$. But $G_{F_0/F}$ has a non-trivial subgroup $G_{F_0/F(\zeta_p)}$ (its order is $\leq p \leq |G_{F_0/F}|$).

– Suppose that $G_{F_0/F} \simeq \mathrm{PGL}_2(\mathbb{F}_{p^r})$. The only non-trivial quotient of $\mathrm{PGL}_2(\mathbb{F}_{p^r})$ is of order 2. But then in the exact sequence

$$0 \to Z \to \mathrm{im}(\overline{\rho}) \to \mathrm{im}(\mathrm{ad}^0\overline{\rho}) \to 0$$

where Z is group of scalar matrices, the order of $\mathrm{im}(\mathrm{ad}^0\overline{\rho})$ is either 1 or 2. Let A be a preimage of the non trivial element of $\mathrm{im}(\mathrm{ad}^0\overline{\rho})$. Then, $\mathrm{im}(\overline{\rho})$ is generated by Z and A. But A has a non trivial invariant subspace (after possibly base changing to a finite extension), then so does $\mathrm{im}(\overline{\rho})$. This contradicts the absolute irreducibility of $\overline{\rho}$. Thus, this possibility is ruled out.

This is a contradiction. Thus, the leftmost term in (2.4) is zero. This implies that the middle term in (2.4) is zero, which in turn, implies that $\psi(G_{F_n})$ is non-zero.

□

Proposition 2.5.8. *Let $r = \dim_k H^1\left(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\right)$. For every $N \geq 1$, we can construct a set Q_N of Taylor-Wiles places of level N, i.e.,*

1. *For each $v \in Q_N$, $\#k(v) \equiv 1 \mod p^N$,*

2. *For each $v \in Q_N$, $\overline{\rho}(Frob_v)$ has two distinct eigenvalues in k,*

3. *$|Q_N| = r$.*

Proof. If v is a Taylor-Wiles place then by Lemma 2.5.5, we have $\dim_k H^1\big(k_v, \mathrm{ad}^0\overline{\rho}(1)\big) = 1$. Thus, it suffices to show that the restriction morphism

$$H^1\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\big) \to \bigoplus_{v \in Q_n} \dim_k H^1\big(k_v, \mathrm{ad}^0\overline{\rho}(1)\big)$$

is an isomorphism, so that equating the dimensions would get us 3).
To prove this, we need to show that for any global cocycle ψ, there exists a place v_ψ satisfying 1) and 2) such that $\mathrm{Res}_{v_\psi}(\psi) \neq 0$. For then, the set of places corresponding to a basis of $H^1\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\big)$ would consitute the desired Taylor-Wiles set.
Actually, we can rephrace the problem as follows: we need to show that we can find $\sigma \in G_{F,S}$ satisfying

(a) $\sigma_{|G_{F(\zeta_{p^N})}} = 1$,

(b) $\mathrm{ad}^0\overline{\rho}(\sigma)$ has an eigenvalue other than one,

(c) $\psi(\sigma) \notin (\sigma - 1)\mathrm{ad}^0\overline{\rho}(1)$

Indeed, all of these conditions are open conditions. So by Chebotarev's density theorem, there exists some Frob_v satisfying them. So we can take $v_\psi = v$. Now by Lemma 2.5.7, $\psi(G_{F_N})$ is not trivial. And we have for all $\tau, \tau' \in G_{F_N}$, $\sigma \in G_{F(\zeta_{p^N})}$:

$$\begin{aligned}\psi(\sigma\tau\sigma^{-1}) &= \psi(\sigma) + \sigma\psi(\tau\sigma^{-1}) \\ &= \psi(\sigma) + \sigma\psi(\tau) + \sigma\tau\psi(\sigma^{-1}) \\ &= \psi(\sigma) + \sigma\psi(\tau) + \sigma\psi(\sigma^{-1}) = \sigma\psi(\tau)\end{aligned}$$

which holds because τ acts trivially on $\mathrm{ad}^0\overline{\rho}$. Also,

$$\psi(\tau\tau') = \psi(\tau) + \tau\psi(\tau') = \psi(\tau) + \psi(\tau')$$

Therefore, $k \cdot \psi(G_{F_N})$ is a non-zero $G_{F_N/F(\zeta_{p^N})}$-submodule of $\mathrm{ad}^0\overline{\rho}$.
We want to find an element $g \in G_{F_N/F(\zeta_{p^N})}$ such that $\overline{\rho}(g)$ has distinct eigenvalues and fixes an element of $k \cdot \psi(G_{F_N})$. In order to do this, we will verify that among the possible candidates H for the projective image of $G_{F_N/F(\zeta_{p^N})}$ by $\overline{\rho}$, there always exists an element of H with distinct eigenvalues which fixes an elements of $G_{F_N/F(\zeta_{p^N})}$. We use the list of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}_p})$ given in the proof of Lemma 2.5.7:

- First note that if the property is true for a subgroup H, then it is also true for any subgroup H' containing H. So it suffices to check the following cases.

- By absolute irreducibility of $\overline{\rho}_{|G_{F(\zeta_{p^N})}}$, H cannot be conjugate to a subgroup of the upper triangular matrices.

- $\mathrm{ad}^0\overline{\rho}$ is a simple $\mathrm{PSL}_2(\mathbb{F}_{p^r})$-module so we have $k \cdot \psi(G_{F_N}) = \mathrm{ad}^0\overline{\rho}$ and $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ fixes $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in k \cdot \psi(G_{F_N})$. Since $p > 5$, we can take $\alpha \neq \alpha^{-1}$.

– For D_4, $\mathrm{ad}^0\overline{\rho}$ decomposes as $V_1 \oplus V_2 \oplus V_3$ where

$$V_1 = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad V_2 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad V_3 = \left\langle \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\rangle$$

with D_4 acting on each subspace by $\pm\,\mathrm{id}$. Since there are three non trivial elements, for each irreducible submodule one of them must act trivially. Note also that they have distinct eigenvalues ± 1.

– For D_{2r} with r odd, $\mathrm{ad}^0\overline{\rho}$ decomposes as $W_1 \oplus W_2$ where

$$W_1 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle, \quad W_2 = \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$$

with $\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}$ fixing W_1 and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ fixing $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in W_2$.

Thus, we get an element $g \in G_{F_N/F(\zeta_{p^N})}$ such that $\overline{\rho}(g)$ has distinct eigenvalues and fixes an element of $k \cdot \psi(G_{F_N})$. Actually $\overline{\rho}(g)$ even fixes an element of $\psi(G_{F_N})$. Indeed, if $k_1, \dots, k_r$ forms a basis of k over $\mathbb{F}_p$, then for a non-zero element $m \in k \cdot \psi(G_{F_N})$ fixed by g, we can express it as $m = k_1\psi(\tau_1) + \cdots + k_r\psi(\tau_r)$ with at least one of the $\psi(\tau_i) \neq 0$. But then we get that

$$gm - m = k_1\big((g-1)\psi(\tau_1)\big) + \cdots k_r\big((g-1)\psi(\tau_r)\big) = 0$$

and by linear independence, we get the desired element.
Now choose a lift $\sigma_0 \in G_{F,S}$ of g. For $\tau \in G_{F_N}$, we have

$$\psi(\tau\sigma_0) = \psi(\tau) + \tau\psi(\sigma_0) = \psi(\tau) + \psi(\sigma_0)$$

If $\psi(\sigma_0) \notin (\sigma_0 - 1)\mathrm{ad}^0\overline{\rho}(1)$, then choose $\tau = 1$.
Otherwise, let $\tau_0 \in G_{F_N}$ be such that $\overline{\rho}(\sigma_0)$ fixes $\psi(\tau_0) \neq 0$ (that we just proved its existence). Then, $\psi(\tau_0) \notin (\sigma_0 - 1)\mathrm{ad}^0\overline{\rho}(1)$.Because otherwise, if we write $(\sigma_0 - 1)x = \psi(\tau_0)$, then

$$(\sigma_0 - 1)^2 x = (\sigma_0 - 1)\psi(\tau_0) = 0$$

but $\overline{\rho}(\sigma_0)$ has distinct eigenvalues so it acts semi-simply on $\mathrm{ad}^0\overline{\rho}$ with the eigenvalue 1 occurring with multiplicity 1. So we must have $(\sigma_0 - 1)x = 0 = \psi(\tau_0)$ which contradicts the choice of τ_0.
Either way, we get that

$$\psi(\tau_0\sigma_0) = \psi(\tau_0) + \psi(\sigma_0) \notin (\sigma_0 - 1)\mathrm{ad}^0\overline{\rho} = (\tau\sigma_0 - 1)\mathrm{ad}^0\overline{\rho}$$

Therefore, the element $\tau_0\sigma_0$ verifies the conditions $(a), (b)$ and (c) as we wanted (remember that $\overline{\rho}(\tau_0)$ is a scalar matrix). This finishes the proof. □

Theorem 2.5.9. *Suppose that F is a totally real number field, let $\mathcal{S} = (\overline{\rho}, \psi, S, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem, and set $g = h^1\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\big) - [F:\mathbb{Q}] + \#T - 1$. For each $N \geq 0$, there exists a finite set of primes Q_N of F, which is disjoint from S such that*

1. *If $v \in Q_N$, then $\#k(v) \equiv 1 \mod p^N$ and $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues.*
2. *$|Q_N| = h^1\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\big)$ and $R^T_{\mathcal{S}_{Q_N}}$ is topologically generated by g elements as an R_{loc}-algebra.*

where $\mathcal{S}_{Q_N} = (\overline{\rho}, \psi, S \cup Q_N, \{\mathcal{D}_v\}_{v\in S})$.

Proof. The first part of the theorem follows from Proposition 2.5.8. For the last assertion, recall that by Proposition 2.4.8 and equation (2.3), $R^T_{\mathcal{S}_{Q_N}}$ is generated by:

$$g = h^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)) - h^0(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)) - \sum_{v|\infty} h^0(G_v, \mathrm{ad}^0\overline{\rho}) + \sum_{v\in Q_N} \big(\dim_k \mathcal{L}_v - h^0(G_v, \mathrm{ad}^0\overline{\rho})\big) + \#T - 1$$

Let us study each term on the right hand side

- The first term:
  By Proposition 2.5.8, we have:

$$H^1_{\mathcal{S},T}(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)) = \ker\left(H^1(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)) \to \bigoplus_{v\in Q_N} H^1(G_v, \mathrm{ad}^0\overline{\rho}(1))\right) = 0$$

- The global term:
  An element of $H^0\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\big)$ corresponds to an intertwining operator $\overline{\rho} \to \overline{\rho}(1)$ between irreducible $G_{F(\zeta_p)}$-modules. Either they are not isomorphic and so the intertwining operator is 0. Or they are isomorphic, and the intertwining operator is scalar. But since $p > 2$, the only trace zero scalar matrix is 0.

- $v \in T$:
  Since $p > 2$, $\mathrm{ad}^0\overline{\rho}$ is a direct summand of $\mathrm{ad}\overline{\rho}$. So the term of the product corresponding for $v \in T$ is $|k|^{1-\delta_v}$ where $\delta_v = \dim_k H^0(G_v, \mathrm{ad}\overline{\rho})$.

- $v \mid \infty$:
  By hypothesis, $\overline{\rho}$ is odd, i.e., for v archimedean with $G_{F_v} = \{\mathrm{id}, c\}$, $\overline{\rho}(c)$ can be represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to some basis. Hence, $\mathrm{ad}^0\overline{\rho}(c)$ can be diagonalized to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. But since G_{F_v} is cyclic of order 2, we have that $Z^1(G_{F_v}, \mathrm{ad}^0\overline{\rho}) = \ker(\mathrm{ad}^0\overline{\rho}(c)+1)$ which is 2-dimensional and $B^1(G_{F_v}, \mathrm{ad}^0\overline{\rho}) = \mathrm{im}(\mathrm{ad}^0\overline{\rho}(c)-1)$ which is also 2-dimensional. Hence, $H^1(G_{F_v}, \mathrm{ad}^0\overline{\rho}) = 0$. Moreover, $H^0(G_{F_v}, \mathrm{ad}^0\overline{\rho})$ corresponds to the eigenspace of $\mathrm{ad}^0\overline{\rho}(c)$ with eigenvalue 1, so it is 1-dimensional.

- $v \in Q_n$:
  First note that $\mathcal{L}_v = H^1(G_v, \mathrm{ad}^0\overline{\rho})$. By definition of the local Euler characteristic, we have that

$$\frac{|H^1(G_{F_v}, \mathrm{ad}^0\overline{\rho})|}{|H^0(G_{F_v}, \mathrm{ad}^0\overline{\rho})|} = |H^2(G_{F_v}, \mathrm{ad}^0\overline{\rho})| \cdot \chi(G_{F_v}, \mathrm{ad}^0\overline{\rho})^{-1}$$

  By Tate duality, we have that $h^2\big(G_{F_v}, \mathrm{ad}^0\overline{\rho}\big) = h^1\big(G_{F_v}, \mathrm{ad}^0\overline{\rho}(1)\big) = 1$ by Lemma 2.5.5 ($\overline{\rho}$ is unramified at v). And by the local Euler characteristic formula [Mil06, Ch. I, Thm. 5.1], $\chi(G_{F_v}, \mathrm{ad}^0\overline{\rho}) = [\mathcal{O}_v : |\mathrm{ad}^0\overline{\rho}|\mathcal{O}_v]^{-1} = 1$ (since the order of $\mathrm{ad}^0\overline{\rho}$ is a power of p, hence prime to v).

Putting everything together, we have:

$$\begin{aligned} g &= 0 - 0 - \sum_{v|\infty} 1 + \sum_{v\in Q_N} 1 + \#T - 1 \\ &= \#Q_N - [F:\mathbb{Q}] + \#T - 1 \end{aligned}$$

where we use the fact that F is totally real. ☐

Chapter 3

A modularity lifting theorem

Let us fix a prime $p > 5$. We let F be a totally real number field and $L/\mathbb{Q}_p$ a finite extension where L has a ring of integers $\mathcal{O}$, a maximal ideal λ and a residue field $\mathbb{F}$. We suppose that L is big enough to contain the images of all embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$. The goal of this chapter is to prove the following modularity lifting theorem.

Theorem 3.0.1. *Let $\rho, \rho_0 : G_F \to \mathrm{GL}_2(\mathcal{O})$ be two continuous representations such that reducing modulo λ we have $\overline{\rho} = \overline{\rho_0}$. Assume that ρ_0 is modular and that ρ is geometric (i.e., it satisfies the Fontaine-Mazur hypothesis). Assume moreover that we have:*

(1) For all $\sigma : F \hookrightarrow L$, $\mathrm{HT}_\sigma(\rho_0) = \mathrm{HT}_\sigma(\rho)$, and contains two distinct elements.

(2)
- *For all $v \mid p$, $\rho_{|G_{F_v}}$ and $\rho_{0|G_{F_v}}$ are crystalline;*
- *p is unramified in F;*
- *For all $\sigma : F \hookrightarrow L$, the two elements of $\mathrm{HT}_\sigma(\rho)$ differ by at most $p-2$;*

(3) $\overline{\rho}_{|G_{F(\zeta_p)}}$ is absolutely irreducible.

Then, ρ is modular.

We will start by introducing the spaces of automorphic forms with which we will work. Then, we will perform base change to reduce the hypotheses on the representation ρ. Finally, we will perform a patching argument using ultrafilters to prove the theorem. The material presented in this chapter will be largely based upon that of [Gee] and [Tay].

3.1 Automorphic forms on Quaternion algebras

We will work with Quaternionic automorphic forms. Although they were not present in Wiles' original work, they allow us to avoid using an argument involving étaleness properties of modular curves, which we substitute by an easy group theoretic argument.

3.1.1 General definition

Let D be a quaternion algebra over F with $S(D)$ being the set of places at which D ramifies. Note that by the fundamental exact sequence of class field theory, the map

$$D \mapsto S(D)$$

gives a bijection

$$\begin{Bmatrix} \text{Quaternion algebras over } F \\ \text{up to isomorphism} \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \text{Finite subsets of places} \\ \text{of } F \text{ of even cardinality} \end{Bmatrix}$$

We can define an algebraic group G_D over $\mathbb{Q}$ by letting $G_D(R) = (D \otimes_{\mathbb{Q}} R)^\times$ for R a $\mathbb{Q}$-algebra. For each real place v, we define a subgroup U_v of $G_D(F_v)$ by letting $U_v = G_D(F_v) \cong \mathbb{H}^\times$ if $v \in S(D)$, and $U_v = \mathbb{R}_{>0}\mathrm{SO}(2) \subseteq \mathrm{GL}_2(\mathbb{R}) \cong G_D(F_v)$ if $v \notin S(D)$. We also fix the weights $(k_v, \eta_v) \in \mathbb{Z}^{\geq 2} \times \mathbb{Z}$, and we require that $w = k_v + 2\eta_v - 1$ is independent of v.

For each real place v, we define a representation (τ_v, W_v) of U_v over $\mathbb{C}$ as follows:

- If $v \in S(D)$, we have $U_v \hookrightarrow \mathrm{GL}_2(\overline{F}_v) \cong \mathrm{GL}_2(\mathbb{C})$ which acts on $\mathbb{C}^2$ in the usual way, we let (τ_v, W_v) be the representation
$$(\mathrm{Sym}^{k_v-2}\,\mathbb{C}^2) \otimes (\wedge^2\mathbb{C}^2)^{\eta_v}$$
- If $v \notin S(D)$, then we have $U_v \cong \mathbb{R}_{>0}\mathrm{SO}(2)$, and we let $W_v = \mathbb{C}$ with the action given by
$$\tau_v(\gamma) = j(\gamma, i)^{k_v}(\det\gamma)^{\eta_v - 1}$$

We write $U_\infty = \prod_{v|\infty} U_v$, $W_\infty = \otimes_{v|\infty} W_v$, $\tau_\infty = \otimes_{v|\infty}\tau_v$.

Finally, we define our space of automorphic forms $S_{D,k,\eta}$ to be the set of function $\varphi : D^\times \setminus G_D(\mathbb{A}_{\mathbb{Q}}) \to W_\infty$ satisfying:

(1) $\varphi(gu_\infty) = \tau_\infty(u_\infty)^{-1}\varphi(g)$ for all $u_\infty \in U_\infty$ and $g \in G_D(\mathbb{A}_{\mathbb{Q}})$.

(2) There is a non-empty open subset $U_\infty \subseteq G_D(\mathbb{A}_{\mathbb{Q}}^\infty)$ such that $\varphi(gu) = \varphi(g)$ for all $u \in U^\infty$ and $g \in G_D(\mathbb{A}_{\mathbb{Q}})$.

(3) Let S_∞ denote the set of finite places, then if $g \in G_D(\mathbb{A}_{\mathbb{Q}}^\infty)$ and $h_\infty \in \mathrm{GL}_2(\mathbb{R})^{S_\infty - S(D)} \subseteq G_D(\mathbb{R})$, then the function
$$\begin{aligned} (\mathcal{H}^\pm)^{S_\infty - S(D)} &\to W_\infty \\ h_\infty(i, \dots, i) &\mapsto \tau_\infty(h_\infty)\varphi(gh_\infty) \end{aligned}$$
which is well defined since $U_{S_\infty - S(D)}$ is the stabilizer of $(i, \dots, i)$, is holomorphic.

(4) If $S(D) = \emptyset$ ($G_D \cong GL_2$), then for all $g \in G_D(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, we have
$$\int_{F\setminus\mathbb{A}_F} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, dx = 0$$

If moreover we have $F = \mathbb{Q}$, then we require that for all $g \in G_D(\mathbb{A}_{\mathbb{Q}}^\infty)$, $h_\infty \in GL_2(\mathbb{R})^+$, the function $\varphi(gh_\infty)|\operatorname{im}\big(h_\infty(i)\big)|^{k/2}$ is bounded on $\mathcal{H}^\pm$.

The group $G_D(\mathbb{A}_{\mathbb{Q}}^\infty)$ acts by right translation on $S_{D,k,\eta}$. In fact, can prove that $S_{D,k,\eta}$ is a semisimple admissible representation of $G_D(\mathbb{A}_{\mathbb{Q}}^\infty)$. Its irreducible constituents are called the *cuspidal automorphic representations of* $G_D(\mathbb{A}_{\mathbb{Q}}^\infty)$ *of weight* (k, η).

3.1.2 The Jacquet-Langlands correspondence

In the case where $S(D) = S_\infty$, the algebraic group G_D is isomorphic to GL_2 which is a more convenient setting to work with. In order to reduce to this case, we will use the Jacquet-Langlands correspondence which relates automorphic representations on G_D to automorphic representations on GL_2.

The local statement

Let v be a place of F such that $v \nmid p$, and suppose that $D_v = D \otimes_F F_v$ is non-split (i.e. $v \in S(D)$). Before stating the theorem, we first give the definition of the Harish-Chandra character, which plays the role of the trace function of a representation of $\mathrm{GL}_2(F_v)$.

If (π, V) is an admissible representation of $G := \mathrm{GL}_2(F_v)$, then for any f in the Hecke algebra $\mathcal{H}(G)$, the operator $\pi(f) : V \to V$ has image contained in the finite dimensional subspace V^K for any compact open subgroup K such that f is left K-invariant. Thus, we can define $\mathrm{Tr}\pi(f) = \mathrm{Tr}\big(\pi(f) \mid \pi(f)V\big) = \mathrm{Tr}\big(\pi(f) \mid V^K\big)$

Theorem 3.1.1. *Let (π, V) be an irreducible smooth representation of G. Then, there is a unique smooth function $\Theta_\pi : G_{rs} \to \mathbb{C}$ called the Harish-Chandra character such that extending Θ_π arbitrarily to G, Θ_π is locally integrable on G, and for any $\mathcal{H}(G)$, we have*

$$\mathrm{Tr}\,\pi(f) = \int_G f(g)\Theta_\pi(g)\ dg$$

where G_{rs} is the set of semi-simple regular elements of G. Moreover, $|D|^{\frac{1}{2}}\Theta_\pi$ is bounded on G_{rs}, where $D(g) = 4 - \det(g)^{-1}\,\mathrm{Tr}(g)^2$ for $g \in G$.

For regular semi-simple elements $\gamma \in \mathrm{GL}_2(F_v)$ and $\gamma' \in D_v^\times$, we write $\gamma \sim \gamma'$ if they have the same trace and determinant.

Theorem 3.1.2 (Local JL correspondence)**.**
Let $\omega : F_v^\times \to \mathbb{C}^\times$ be a smooth character. There is a unique bijection

$$\begin{array}{c}\{\ \textit{irreducible discrete series representation of } \mathrm{GL}_2(F_v) \textit{ with central character } \omega\ \} \\ \updownarrow \\ \{\ \textit{irreducible smooth representations of } D_v^\times \textit{ with central character } \omega\ \}\end{array}$$

such that for any $\pi \leftrightarrow \pi'$ and regular semi-simple elements $\gamma \in \mathrm{GL}_2(F_v)$, $\gamma' \in D_v^\times$ with $\gamma \sim \gamma'$, we have

$$\Theta_\pi(\gamma) = \mathrm{Tr}\,\pi'(\gamma')$$

We have a compatibility with twists: if $\pi \leftrightarrow \pi'$, then $\pi \otimes (\mu \circ \det) \leftrightarrow \pi' \otimes (\mu \circ Nm)$ for any smooth character $\mu : K^\times \to \mathbb{C}^\times$.

Remark 3.1.3. The bijection associates the Steinberg representation to the trivial representation of $D^\times$, or more generally, $\mathrm{Sp}_2(\mu|\cdot|^{-\frac{1}{2}}) \leftrightarrow \mu \circ \mathrm{Nm}$ for $\mu : K^\times \to \mathbb{C}^\times$ a smooth character. Hence, π is supercuspidal if and only if its associated irreducible representation π' of $D^\times$ has dimension > 1.

Global Statement

We have a global version of this correspondence which is compatible with the local one.

Theorem 3.1.4. *Let $\omega : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a smooth character. There is a unique injection*

$$\begin{array}{c}\{\ \textit{irreducible automorphic representations of } (\mathbb{A}_F \otimes_F D)^\times \textit{ of dimension } > 1 \textit{ with central character } \omega\ \} \\ \big\downarrow \\ \{\ \textit{irreducible cuspidal automorphic representations of } \mathrm{GL}_2(\mathbb{A}_F) \textit{ with central character } \omega\ \}\end{array}$$

such that $\pi' \leftrightarrow \pi$ if and only if $\pi'_v \simeq \pi_v$ for all $v \notin S(D)$, and $\pi'_v \leftrightarrow \pi_v$ for all $v \in S(D)$ in the sense of Theorem 3.1.2. We also have compatibility with twists: if $\pi' \leftrightarrow \pi$, then $\pi' \otimes (\mu \circ Nm) \leftrightarrow \pi \otimes (\mu \circ \det)$ for any smooth character $\mu : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{C}^\times$.

3.1.3 Galois representation associated to automorphic representations

Under conditions of algebraicity, we can attach a family of Galois representations to automorphic forms on $\mathrm{GL}_2(\mathbb{A}_F)$. The precise result in the case that we will use is given as follows:

Theorem 3.1.5. *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ of weight (k,η). Then, there exists a CM field L_π such that for each finite place λ of L_π there is a continuous irreducible Galois representation*

$$\rho_\lambda(\pi) : G_F \to \mathrm{GL}_2(\overline{L}_{\pi,\lambda})$$

satisfying:

(1) For each finite place v, we have

$$\mathrm{WD}\left(\rho_\lambda(\pi)_{|G_{F_v}}\right)^{F\text{-}ss} \cong rec_{F_v}(\pi_v \otimes |\cdot|^{\frac{1}{2}} \circ \det)$$

(2) If v divides the residue characteristic of λ, then $\rho_\lambda(\pi)_{|G_{F_v}}$ is deRham, with τ-Hodge-Tate weights $\eta_\tau, \eta_\tau + k_\tau - 1$ where $\tau : F \hookrightarrow \overline{L}_\pi \subseteq \mathbb{C}$ an embedding lying over v. Moreover, if π_v is unramified, then $\rho_\lambda(\pi)_{|G_{F_v}}$ is crystalline.

(3) If c_v is a complex conjugation, then $\det \rho_\lambda(\pi)(c_v) = -1$.

Definition 3.1.6. We say that a continuous Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is modular of weight (k,η) if it is isomorphic to $\rho_\lambda(\pi)$ for some cuspidal automorphic representation π of weight (k,η).

3.2 Integral theory of automorphic forms

In this subsection, suppose that $[F:\mathbb{Q}]$ is even (we will reduce to this case by base change), and that $S(D) = S_\infty$. In particular, we have $G_D(\mathbb{A}_F^\infty) = \mathrm{GL}_2(\mathbb{A}_F^\infty)$.
Let us fix an isomorphism $\iota : \overline{L} \to \mathbb{C}$, and some $k \in \mathbb{Z}_{\geq 2}^{\mathrm{Hom}(F,\mathbb{C})}$, $\eta \in \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}$ with $w = k_\tau + 2\eta_\tau - 1$ independent of $\tau \in \mathrm{Hom}(F,\mathbb{C})$. Let $U = \prod_v U_v \subseteq \mathrm{GL}_2(\mathbb{A}_F^\infty)$ be a compact open subgroups such that if $v \notin S$ then $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$, where S is a finite set of finite places of F not containing any place lying over p. Let $U_S = \prod_{v\in S} U_v$ and $U^S = \prod_{v\notin S} U_v$ so that $U = U_S U^S$.
We consider continuous homomorphism $\psi : U_S \to \mathcal{O}^\times$ and an algebraic grossencharacter $\chi_0 : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ such that

- χ_0 is unramified outside of S.
- For each place $v \mid \infty$, $\chi_{0|(F_v^\times)^\circ}(x) = x^{1-w}$.
- $\chi_{0|(\prod\limits_{v\in S} F_v^\times)\cap U_S} = \psi^{-1}$

By the Langlands correspondence for GL_1, we get a character

$$\chi_{0,\iota} : \mathbb{A}_F^\times / \overline{F^\times (F_\infty^\times)^\circ} \to \overline{L}^\times$$

given by

$$x \mapsto (\prod_{\tau : F \hookrightarrow L} \tau(x_p)^{1-w})\iota^{-1}\left((\prod_{\tau : F \hookrightarrow \mathbb{C}} \tau(x_\infty))^{w-1} \chi_0(x) \right)$$

Let $\Lambda = \bigotimes_{\tau:F\hookrightarrow\mathbb{C}} \mathrm{Sym}^{k_v-2}(\mathcal{O}^2)\otimes(\wedge^2\mathcal{O}^2)^{\eta_v}$, and let $\mathrm{GL}_2(\mathcal{O}_{F,p}) = \prod_{v|p}\mathrm{GL}_2(\mathcal{O}_{F_v})$ which acts on Λ via $\iota^{-1}\tau$ on the τ-factor.
Let A be a finite $\mathcal{O}$-module, we define $S(U,A) = S_{k,\eta,\iota,\psi,\chi_0}(U,A)$ to be the space of functions $\phi : D^\times \setminus \mathrm{GL}_2(\mathbb{A}_F^\infty) \to \Lambda\otimes_{\mathcal{O}} A$ such that for all $g\in\mathrm{GL}_2(\mathbb{A}_F^\infty)$, $u\in U$, $z\in(\mathbb{A}_F^\infty)^\times$, we have

$$\phi(guz) = \chi_{0,\iota}(z)\psi(u_S)^{-1}u_p^{-1}\phi(g)$$

3.2.1 Hecke algebras

We start by recalling basic facts about smooth irreducible representations of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$.

If v is a finite place of F, we define local Hecke algebra $\mathcal{H}(\mathrm{GL}_2(F_v))$ to be the set of locally constant functions $\mathcal{C}_c^\infty(\mathrm{GL}_2(F_v),\mathbb{C})$ equipped with the convolution product

$$(f*f')(g) = \int_{\mathrm{GL}_2(F_v)} f(h)f'(h^{-1}g)\,\mathrm{d}\mu_v \quad \text{for } f,f'\in\mathcal{C}_c^\infty(\mathrm{GL}_2(F_v),\mathbb{C})$$

where μ_v is a Haar measure on $\mathrm{GL}_2(F_v)$, which we will normalize so that $\mu_v(\mathrm{GL}_2(\mathcal{O}_{F_v}))=1$. We can also see $\mathcal{H}(\mathrm{GL}_2(F_v))$ as a convolution algebra of density measures with respect to μ_v.
If (π,V) is a smooth representation of $\mathrm{GL}_2(F_v)$ on a complex vector space V, we can equip V with the structure of a smooth $\mathcal{H}(\mathrm{GL}_n(F_v))$-module by setting

$$\pi(f)\cdot v = \int_{\mathrm{GL}_2(F_v)} f(g)\pi(g)v\mathrm{d}\mu_v \quad \text{for } v\in V, g\in\mathrm{GL}_2(F_v) \text{ and, } f\in\mathcal{H}(\mathrm{GL}_2(F_v))$$

In fact, every smooth $\mathcal{H}(\mathrm{GL}_2(F_v))$-module is of this type, and irreducible smooth representations of $\mathrm{GL}_2(F_v)$ are determined up to isomorphism by their $\mathcal{H}(\mathrm{GL}_2(F_v))$-module structure.

If $K\subset\mathrm{GL}_2(F_v)$ is a compact open subgroup, we have a unipotent element $e_K = \mu(K)^{-1}\mathbb{1}_K \in \mathcal{H}(\mathrm{GL}_2(F_v))$. We define the K-invariant Hecke-algebra $\mathcal{H}(\mathrm{GL}_2(F_v),K)$ to be the subalgebra

$$\mathcal{C}_c^\infty(K\setminus\mathrm{GL}_2(F_v)/K,\mathbb{C}) = e_K*\mathcal{H}(\mathrm{GL}_2(F_v))*e_K$$

The Spherical Hecke algebra

Let us take $K=\mathrm{GL}_2(\mathcal{O}_{F_v})$. In this case, we call $\mathcal{H}(\mathrm{GL}_2(F_v),K)$ the spherical Hecke algebra.
From the Cartan decomposition

$$\mathrm{GL}_2(F_v) = \bigsqcup_{n_1\geq n_2} K\begin{pmatrix}\varpi_v^{n_1} & 0\\ 0 & \varpi_v^{n_2}\end{pmatrix}K \tag{3.1}$$

where ϖ_v is a uniformizer of $\mathcal{O}_{F_v}$, we see that $\mathcal{H}(\mathrm{GL}_2(F_v),K)$ is generated by the characteristic functions $\mathbb{1}_{(n_1,n_2)}$ of the double cosets $U\begin{pmatrix}\varpi_v^{n_1} & 0\\ 0 & \varpi_v^{n_2}\end{pmatrix}U$, for $n_1\geq n_2$.
It is a standard notation to let T_v and S_v denote the function $\mathbb{1}_{(1,0)}$ and $\mathbb{1}_{(1,1)}$ respectively, where S_v is invertible with inverse $S_v^{-1}=\mathbb{1}_{(-1,-1)}$. In fact one can prove the following theorem:

Theorem 3.2.1. *We have an isomorphism of* $\mathbb{C}$*-algebras* $\mathcal{H}(\mathrm{GL}_2(F_v),\mathrm{GL}_2(\mathcal{O}_{F_v}))) \cong \mathbb{C}[T_v,S_v^{\pm1}]$.

An irreducible smooth representation (π,V) of $\mathrm{GL}_2(F_v)$ is called spherical if if $V^K\neq 0$. In this case, V^K has the structure of a non-zero $\mathcal{H}(\mathrm{GL}_2(F_v),K)$-module, which is commutative, so we must have $\dim_{\mathbb{C}} V^K = 1$. Now let us give examples of such representations

– If μ is an unramified character of $F_v^\times$ i.e. it is trivial on $\mathcal{O}_{F_v}^\times$, then $(\mu \circ \det, \mathbb{C})$ is clearly a spherical representation since $\det(K) \subseteq \mathcal{O}_{F_v}^\times$.

– If $\chi = (\chi_1, \chi_2)$ is a character of the diagonal torus T with $\chi_1/\chi_2 \neq |\cdot|_v^{\pm 1}$, we would like to construct a non-trivial K-fixed element of $\mathrm{Ind}_B^{\mathrm{GL}_2(F_v)} \chi$ (where B is the subgroup of upper triangular matrix, and Ind is the normalized induction). For this, consider the Iwasawa decomposition $\mathrm{GL}_2(F_v) = B \cdot K$, so that for all $g \in \mathrm{GL}_2(F_v)$, we can write

$$g = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \cdot k \quad \text{with } k \in K$$

If $f \in \mathrm{Ind}_B^{\mathrm{GL}_2(F_v)} \chi$ is a fixed U-vector, then it satisfies $f(g) = f\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \cdot u\right) = \chi_1(a)\chi_2(b)|\frac{a}{b}|^{\frac{1}{2}} f(\mathrm{id})$. So if $\mathrm{Ind}_B^{\mathrm{GL}_2(F_v)} \chi$ is spherical, the space of K-fixed vectors, which is one dimensional, must be generated by

$$f : g = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \cdot k \mapsto \chi_1(a)\chi_2(b)|\frac{a}{b}|_v^{\frac{1}{2}}$$

But this formula only makes sense if $\chi_1(a)\chi_2(b) = 1$ for all $a, b \in \mathcal{O}_{F_v}^\times$, i.e., if χ_1 and χ_2 are unramified. In this case, we say that f is the normalized spherical vector in $\mathrm{Ind}_B^{\mathrm{GL}_2(F_v)} \chi$. And we see that $\mathrm{Ind}_B^{\mathrm{GL}_2(F_v)} \chi$ is spherical if and only if χ_1 and χ_2 are unramified.

In fact, if (π, V) is an irreducible smooth spherical representation of G, then it is isomorphic to one of the above examples.

Smooth irreducible representations of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$

There exists a unique Haar measure μ on $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ such that $\mu(\prod_v X_v) = \prod_v \mu_v(X_v)$ if $X_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for almost all places v. As before, we define the global Hecke algebra to be $\mathcal{H}(\mathrm{GL}_2(\mathbb{A}_F^\infty) = \mathcal{C}_c^\infty(\mathrm{GL}_2(\mathbb{A}_F^\infty), \mathbb{C})$ equipped with the convolution algebra with respect to μ. Then we have a natural isomorphism

$$\mathcal{H}(\mathrm{GL}_2(\mathbb{A}_F^\infty)) \cong \otimes'_{\{\mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}\}} \mathcal{H}(\mathrm{GL}_2(F_v)) \tag{3.2}$$

where the symbol $\otimes'_{\{\mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}\}}$ denotes the restricted tensor product with respect to the family of idempotents $\{\mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}\}_v$. Therefore by [Bum97, Theorem 3.44], if π is an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$, then there exist unique irreducible smooth representations π_v of $\mathrm{GL}_2(F_v)$ such that for almost all places v of F, there exists a non-zero element $e_v \in \pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$ with

$$\pi \cong \otimes'_{\{e_v\}} \pi_v$$

Moreover, this decomposition is compatible with (3.2).

Action of the Hecke algebra on the space of automorphic forms

Back to our setting, if $v \notin S$ and $v \nmid p$ recall that $U_v = \mathrm{GL}_2(\mathcal{O}_v)$ and that $S(U, A)$ is left invariant by the action of U_v by right translation. So if $u \in \mathrm{GL}_2(F_v)$, we can define an operator $[U_v u U_v]$ on $S(U, A)$ by setting

$$([U_v u U_v]\phi)(g) = \sum_i \phi(g u_i)$$

where the $u_i \in \mathrm{GL}_2(F_v)$ are defined by the decomposition

$$U_v u U_v = \bigsqcup_i u_i U_v$$

For $\phi \in S(U,A)$ and $g \in \mathrm{GL}_2(\mathbb{A}_F^\infty)$. Now let $\mathbf{T}^{\mathrm{univ}} = \mathcal{O}[T_v, S_v \ : \ v \notin S, v \nmid p]$ denote the universal Hecke algebra where T_v acts on $S(U,A)$ via $[U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_v]$ and where S_v acts via $[U_v \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U_v]$. Let $\mathbf{T}_U$ be the image of $\mathbf{T}^{\mathrm{univ}}$ in $\mathrm{End}_{\mathcal{O}}\big(S(U,\mathcal{O})\big)$ so that $\mathbf{T}_U$ is a commutative $\mathcal{O}$-algebra which acts faithfully on $S(U,\mathcal{O})$.

We have a decomposition

$$\mathrm{GL}_2(\mathbb{A}_F^\infty) = \bigsqcup_{i\in I} D^\times g_i U(\mathbb{A}_F^\infty)^\times$$

where I is a finite set. This way we get an injective morphism $S(U,A) \hookrightarrow \oplus_{i\in I}(\Lambda \otimes_{\mathcal{O}} A)$ sending a function ϕ to $\big(\phi(g_i)\big)_{i\in I}$. We would like to determine its image. So let $\omega \in \Lambda \otimes_{\mathcal{O}} A$, a function ϕ sending g_i to ω and satisfying the desired properties is well defined on the double coset $D^\times g_i U(\mathbb{A}_F^\infty)^\times$ if and only if for every $\delta, \delta' \in D^\times$, $u, u' \in U$ and $z, z' \in (\mathbb{A}_F^\infty)^\times$ such that $\delta g_i u z = \delta' g_i u' z'$,

$$\chi_{\iota,0}(z)\psi^{-1}(u_S)u_p^{-1}\omega = \chi_{\iota,0}(z')\psi^{-1}(u'_S){u'_p}^{-1}\omega$$

this amounts to checking that for $\delta g_i u z = g_i$, $\chi_{\iota,0}(z)\psi(u_S)^{-1}u_p^{-1}\omega = \omega$. Therefore, the injection induces an isomorphism:

$$S(U,\mathcal{O}) \xrightarrow{\cong} \bigoplus_{i\in I} (\Lambda \otimes_{\mathcal{O}} A)^{\left(g_i^{-1}D^\times g_i \cap U\cdot(\mathbb{A}_F^\infty)^\times\right)/F^\times} \tag{3.3}$$

The group $\Delta_{g_i,U} := \left(g_i^{-1}D^\times g_i \cap U \cdot (\mathbb{A}_F^\infty)^\times\right)/F^\times$ is both discrete ($D^\times$ is discrete inside $G_D(\mathbb{A}_F^\infty)$) and compact, so it is finite. We say that U is sufficiently small for p if $p \nmid \#\Delta_{g,U}$ for all g.

Lemma 3.2.2. *If $[F(\zeta_p), F] > 2$, then U is sufficiently small.*

Proof. Suppose that $\delta \in D^\times$ such that $g^{-1}\delta g \in \Delta_{g,U}$ and has order p, in other words, $\delta^p \in F^\times$. Then, we have

$$(\frac{\delta^2}{\det \delta})^p = \frac{\delta^{2p}}{\det \delta^p} = 1$$

where the last equality is true because $\det \delta^p = \delta^{2p}$ since $\delta^p \in F^\times$. Therefore, $\frac{\delta^2}{\det \delta}$ is a p-th root of unity. If $\frac{\delta^2}{\det \delta} = 1$, then $\delta \in F^\times$ already. Otherwise, $D^\times$ contain $F(\zeta_p)$, but since D is four dimensional over F, it can only contain a field extension of F of degree 2 which contradicts the hypothesis. ☐

Note that this condition is satisfied if for example p is unramified at F, since in that case, $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$ so $[F(\zeta_p) : F] = p - 1 > 2$ if $p > 3$ (as we are assuming).

Proposition 3.2.3. *If U is sufficiently small, then we have the following properties:*

1) *$S(U,\mathcal{O})$ is a free $\mathcal{O}$-module,*

2) *$S(U,\mathcal{O}) \otimes_{\mathcal{O}} A \xrightarrow{\cong} S(U,A)$,*

3) *If V is an open subgroup of U with $\#(U/V)$ a power of p, then $S(V,\mathcal{O})$ is a free $\mathcal{O}[U/V \cdot \big(U \cap (\mathbb{A}_F^\infty)^\times\big)]$-module.*

Proof. Since $p \nmid \#\Delta_{g,U}$, the latter is a unit in $\mathcal{O}$, so we get a projection

$$\begin{aligned} \Lambda \otimes_{\mathcal{O}} A &\to (\Lambda \otimes_{\mathcal{O}} A)^{\Delta_{g,U}} \\ x \otimes a &\mapsto \frac{1}{\#\Delta_{g,U}} \sum_{\delta \in \Delta_{g,U}} \delta \cdot x \otimes a \end{aligned}$$

So $(\Lambda \otimes_{\mathcal{O}} A)^{\Delta_{g,U}}$ is a direct summand of $\Lambda \otimes_{\mathcal{O}} A$. It follows that $(\Lambda \otimes_{\mathcal{O}} A)^{\Delta_{g,U}} \cong \Lambda^{\Delta_{g,U}} \otimes_{\mathcal{O}} A$ from which we get 2) (using equation (3.3)). Since Λ is a free $\mathcal{O}$-module, and since $\mathcal{O}$ is a PID, we also get that $\Lambda^{\Delta_{g,U}}$ is free. Thus, we also get 1).
To prove 3), let us write $U = \bigsqcup_{j\in J} u_j V \cdot \big(U \cap (\mathbb{A}_F^\infty)^\times\big)$. We claim that $\mathrm{GL}_2(\mathbb{A}_F^\infty) = \bigsqcup_{i\in I, j\in J} g_i u_j V (\mathbb{A}_F^\infty)^\times$ from which the result follows. Indeed, one would have:

$$S(V,\mathcal{O}) \cong \bigoplus_{i\in I} \bigoplus_{j\in J} \Lambda^{\Delta_{g_i u_j, V}}$$

but we have that $\Lambda^{\Delta_{g_i u_j, V}} = u_j^{-1} \Lambda^{\Delta_{g_i, V}}$. Therefore, we get:

$$\begin{aligned} S(V,\mathcal{O}) &\cong \bigoplus_{i\in I} \bigoplus_{j\in J} u_j^{-1} \Lambda^{\Delta_{g_i, V}} \\ &= \bigoplus_{i\in I} \mathcal{O}[U/V \cdot \big(U \cap (\mathbb{A}_F^\infty)^\times\big)] \otimes_{\mathcal{O}} \Lambda^{\Delta_{g_i, V}} \end{aligned}$$

To prove our claim, we need to show that if $g_i u_j = \delta g_{i'} u_{j'} v z$, then $i = i'$ and $j = j'$. The fact that $i = i'$ follows immediately from the decomposition of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ with respect to U. So we get, $g_i^{-1} \delta g_i = u_{j'} v u_j^{-1} z \in \Delta_{g_i, U}$. Since U is sufficiently small, there exists some N coprime to p such that $\delta^N \in F^\times$, so $(u_{j'} v u_j^{-1})^N \in (\mathbb{A}_F^\infty)^\times$. But given that V is normal in U, we can write $(u_{j'} v u_j^{-1})^N = (u_{j'} u_j^{-1})^N v'$ for some $v' \in V$, i.e., $(u_{j'} u_j^{-1})^N \in V \cdot \big(U \cap (\mathbb{A}_F^\infty)^\times\big)$. Given that $\#(U/V)$ is a power of p, we get that $u_{j'} u_j^{-1} \in V \cdot \big(U \cap (\mathbb{A}_F^\infty)^\times\big)$ so $j = j'$ as desired. □

Lemma 3.2.4. *We have an isomorphism*

$$S(U,\mathcal{O}) \otimes_{\mathcal{O},\iota} \mathbb{C} \xrightarrow{\simeq} \mathrm{Hom}_{U_S}\left(\mathbb{C}(\psi^{-1}), S_{D,k,\eta}^{U^S,\chi_0}\right)$$

which is $\mathbf{T}^{univ}$-equivariant.

Proof. Applying the definitions, we see that $\mathrm{Hom}_{U_S}\left(\mathbb{C}(\psi^{-1}), S_{D,k,\eta}^{U^S,\chi_0}\right)$ is equal to the set

$$\left\{ \begin{array}{l} \varphi : D^\times \setminus G_D(\mathbb{A}_F) \to \bigotimes_{v|\infty} (\mathrm{Sym}^{k_v-2}\,\mathbb{C}^2) \otimes (\wedge^2\mathbb{C}^2)^{\eta_v} \text{ such that} \\ \bullet\ \varphi(g u_\infty) = \tau_\infty(u_\infty)^{-1}\varphi(g), \quad u_\infty \in D_\infty^\times, g \in G_D(\mathbb{A}_F) \\ \bullet\ \varphi(g u) = \psi^{-1}(u_S)\varphi(g), \quad u \in U, g \in G_D(\mathbb{A}_F) \\ \bullet\ \varphi(g z) = \chi_0(z)\varphi(g), \quad z \in \mathbb{A}_F^\times, g \in G_D(\mathbb{A}_F) \end{array} \right\}$$

For $\phi \in S(U,\mathcal{O})$, the bijection is given by

$$\varphi : g \mapsto \tau_\infty(g_\infty)^{-1} \iota\big(g_p \cdot \phi(g^\infty)\big)$$

which one can verify that it lies in the set above. For example, let us verify the third condition (the other two are easier calculations):

$$\begin{aligned} \varphi(gz) &= \tau_\infty(z_\infty)^{-1}\tau_\infty(g_\infty)^{-1}\iota\big(g_p z_p \cdot \varphi(g^\infty z^\infty)\big) \\ &= z_\infty^{-k_v+2-2\eta_v}\tau_\infty(g_\infty)^{-1}\iota\big(g_p \cdot z_p^{k_v-2+2\eta_v}\varphi(g^\infty z^\infty)\big) \\ &= z_\infty^{1-w}\iota\big(z_p^{w-1}\chi_{0,\iota}(z^\infty)\big)\tau_\infty(g_\infty)^{-1}\iota\big(g_p \cdot \varphi(g^\infty)\big) \\ &= \chi_0(z)\varphi(g) \end{aligned}$$

where we use the expression of χ_0 obtained from $\chi_{0,\iota}$. □

From the description we gave in the last proof, we have that

$$\mathrm{Hom}_{U_S}\left(\mathbb{C}(\psi^{-1}), S_{D,k,\eta}^{U^S,\chi_0}\right) \cong \bigoplus_{\pi} \mathrm{Hom}_{U_S}\left(\mathbb{C}(\psi^{-1}), \pi_S\right) \otimes \otimes'_{v \notin S} \pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})} \tag{3.4}$$

where π ranges over the cuspidal automorphic representations of $G_D(\mathbb{A}_F^\infty)$ of weight (k,η) which have central character χ_0 and are unramified outside of S. This induces an isomorphism

$$\mathbf{T}_U \otimes_{\mathcal{O},\iota} \mathbb{C} \xrightarrow{\cong} \prod_{\pi,\ \mathrm{Hom}_{U_S}\left(\mathbb{C}(\psi^{-1}),\pi_S\right) \neq (0)} \mathbb{C}$$

which sends T_v, S_v to their eigenvalues in $\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$ and where π are the same as above. This map is in fact surjective, because if not, then it would lend in a subalgebra which must be defined by at least two coordinates being equal. This would mean that there are $\pi \neq \pi'$ that have the same T_v-eigenvalues for almost all v which contradicts the strong multiplicity one theorem.

Note that this implies that $\mathbf{T}_U$ is reduced (since $\mathbf{T}_U$ is free over $\mathcal{O}$, so it injects into $\mathbf{T}_U \otimes_{\mathcal{O},\iota} \mathbb{C}$). Moreover, this gives a bijection:

$$\left\{ \begin{array}{c} \text{cuspidal automorphic reps of} \\ G_D(\mathbb{A}_F^\infty) \text{ of weight } (k,\eta),\ \text{of central character } \chi_0 \\ \text{and unramified outside of } S \\ \text{with } \mathrm{Hom}_{U_S}\left(\mathbb{C}(\psi^{-1}),\pi_S\right) \neq (0) \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \mathcal{O}\text{-linear ring homomorphisms} \\ \mathbf{T}_U \to \overline{L} \\ \text{modulo } G_L \text{ action} \end{array} \right\} \tag{3.5}$$

Since $\mathbf{T}_U \otimes_{\mathcal{O}} L$ is finite over L, a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_U \otimes_{\mathcal{O}} L$ is the kernel of a K-algebra homomorphism $\mathbf{T}_U \otimes_{\mathcal{O}} L \to \overline{L}$. And given that composition with the action of G_L does not change the kernel, we get an identification:

$$\{\text{ maximal ideals of } \mathbf{T}_U \otimes_{\mathcal{O}} L\} = \mathrm{Hom}_{L-\mathrm{alg}}\left(\mathbf{T}_U \otimes_{\mathcal{O}} L, \overline{L}\right)/(G_L\text{-action})$$

But the inclusion $\mathbf{T}_U \hookrightarrow \mathbf{T}_U \otimes_{\mathcal{O}} L$ identifies maximal ideals of $\mathbf{T}_U \otimes_{\mathcal{O}} L$ with minimal prime ideals of $\mathbf{T}_U$ thanks to the following lemma:

Lemma 3.2.5. *The minimal prime ideals of* $\mathbf{T}_U$ *are those lying above the ideal* (0) *of* $\mathcal{O}$.

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of $\mathbf{T}_U$. Since $\mathbf{T}_U$ is finite flat over $\mathcal{O}$, it satisfies the going down property. Thus, we get that $\mathfrak{p} \cap \mathcal{O} = (0)$. Conversely, suppose that $\mathfrak{p} \cap \mathcal{O} = (0)$ and that there exists a prime ideal $\mathfrak{p}'$ satisfying $\mathfrak{p}' \subseteq \mathfrak{p}$, since $\mathbf{T}_U$ is an integral extension of $\mathcal{O}$, there are no strict inclusions between prime ideals lying over (0). So $\mathfrak{p} = \mathfrak{p}'$ and $\mathfrak{p}$ is minimal. □

Therefore, if $\mathfrak{p}$ is a minimal prime ideal of $\mathbf{T}_U$, then by what we just proved, there is an injection $\theta_\pi : \mathbf{T}_U/\mathfrak{p} \hookrightarrow \overline{L}$ corresponding to some π as above (it sends T_v and S_v to the inverse image by ι of their corresponding eigenvalues in $\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$) .

Now by finiteness of $\mathbf{T}_U$ as a $\mathcal{O}$-module, it is semilocal and we have a decomposition

$$\mathbf{T}_U = \prod_{\mathfrak{m}} \mathbf{T}_{U,\mathfrak{m}}$$

where $\mathfrak{m}$ ranges over the maximal ideals of $\mathbf{T}_U$ (there are finitely many of them). This shows in particular that any minimal prime ideal $\mathfrak{p}$ sits inside a unique maximal ideal of $\mathbf{T}_U$.

For a fixed maximal ideal $\mathfrak{m}$, we will now construct representation

$$\overline{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbf{T}_U/\mathfrak{m})$$

and if $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, a lift

$$\overline{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbf{T}_{U,\mathfrak{m}})$$

For this, recall that one can associate to each π considered above a Galois representation. So upon taking the product of these, we get a massive Galois representation:

$$\rho^{\mathrm{mod}} : G_F \to \prod_{\pi} \mathrm{GL}_2(\overline{L}) = \mathrm{GL}_2(\mathbf{T}_U \otimes_{\mathcal{O}} \overline{L})$$

which is unramified outside of $S \cup \{v \mid p\}$, and for any $v \notin S$, $v \nmid p$, $\mathrm{Tr}\,\rho^{\mathrm{mod}}(\mathrm{Frob}_v) = T_v$ and $\det \rho^{\mathrm{mod}}(\mathrm{Frob}_v) = (\#k_v)S_v$. If $\mathfrak{p} \subseteq \mathfrak{m}$ is a minimal prime ideal, let π be the corresponding automorphic representation with the inclusion $\theta_\pi : \mathbf{T}_U/\mathfrak{p} \hookrightarrow \overline{L}$. If $\rho_\pi : G_F \to \mathrm{GL}_2(\overline{L})$ is the associated Galois representation, taking the semisimplification of the mod p reduction gives a residual representation $\overline{\rho}_\pi : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}})$ and we have:

$$\mathrm{Tr}(\rho_\pi) \in \mathbf{T}_U/\mathfrak{p} \subseteq \mathcal{O}_{\overline{L}} \Rightarrow \mathrm{Tr}\,\overline{\rho}_\pi \in \mathbf{T}_U/\mathfrak{m} \subseteq \overline{F}$$

By Theorem 2.3.4, $\overline{\rho}_\pi$ can be conjugated to a representation

$$\overline{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbf{T}_U/\mathfrak{m})$$

as desired. Localizing at $\mathfrak{m}$, we obtain a Galois representation:

$$\rho_{\mathfrak{m}}^{\mathrm{mod}} : G_F \to \mathrm{GL}_2(\mathbf{T}_{U,\mathfrak{m}} \otimes_{\mathcal{O}} \overline{L}) = \prod_{\pi} \mathrm{GL}_2(\overline{L})$$

where π ranges over the considered automorphic representations whose corresponding prime ideal lies inside $\mathfrak{m}$ (i.e., such that $\overline{\rho}_\pi \cong \overline{\rho}_{\mathfrak{m}}$). If we moreover suppose that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, then by Theorem 2.3.4 again, we obtain a representation

$$\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbf{T}_{U,\mathfrak{m}})$$

To conclude this discussion, note that for our application, we will need to consider Hecke operators at places in S. For this to work, we need a set of places $T \subseteq S$ such that $\psi_{|U_T} = \mathrm{id}$. In this case, if we choose $g_v \in \mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v \in T$, then we set $W_v = [U_v g_v U_v]$ and define $\mathbb{T}'_U \subseteq \mathrm{End}_{\mathcal{O}}(S(U,\mathcal{O}))$ to be the algebra generated by $\mathbf{T}_U$ and the operators W_v for $v \in T$. Tensoring with $\mathbb{C}$ we get:

$$\mathbf{T}'_U \otimes_{\mathcal{O},\iota} \mathbb{C} \xrightarrow{\sim} \prod_{\pi} \otimes_{v\in T}\{ \text{ subalgebra of } \mathrm{End}_{\mathbb{C}}(\pi_v^{U_v}) \text{ generated by } W_v\}$$

This shows that we have a bijection between ι-linear homomorphisms $\mathbf{T}'_U \to \mathbb{C}$ and tuples $(\pi, \{\alpha_v\}_{v\in T})$ where α_v is an eigenvalue of W_v on $\pi_v^{U_v}$.

3.3 Base change

Using base change, we will be able to simplify the hypotheses of theorem 3.0.1. Let us first give the cyclic base change theorem for GL_2.

Theorem 3.3.1. *Let E/F be a cyclic extension of totally real fields of prime degree. Let* $\mathrm{Gal}(E/F) = \langle\sigma\rangle$ *and* $\mathrm{Gal}(E/F)^\vee = \mathrm{Hom}(\mathrm{Gal}(E/F), \mathbb{Z}) = \langle\delta_{E/F}\rangle$. *Let π be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ *of weight (k,η), then there exists a cuspidal automorphic representation* $\mathrm{BC}_{E/F}(\pi)$ *of* $\mathrm{GL}_2(\mathbb{A}_E^\infty)$ *of weight* $\big(\mathrm{BC}_{E/F}(k), \mathrm{BC}_{E/F}(\eta)\big)$ *such that:*

(1) For all finite places v of F and $w \mid v$ of E, $\mathrm{rec}_{E_w}(\mathrm{BC}_{E/F}(\pi)_w) = (\mathrm{rec}_{F_v}(\pi_v))_{\mathcal{W}_{E_w}}$; in particular, $r_\lambda(\mathrm{BC}_{E/F}(\pi)) \cong r_\lambda(\pi)_{|G_E}$.

(2) $\mathrm{BC}_{E/F}(k)_w = k_v$, $\mathrm{BC}_{E/F}(\eta)_w = \eta_v$.

(3) $\mathrm{BC}_{E/F}(\pi) \cong \mathrm{BC}_{E/F}(\pi')$ if and only if $\pi \cong \pi' \otimes (\delta^i_{E/F} \circ \mathrm{Art} \circ \det)$ for some i.

(4) A cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_E^\infty)$ is in the image of $\mathrm{BC}_{E/F}$ if and only if $\pi \circ \sigma \cong \pi$.

Proposition 3.3.2. *Suppose that $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ is a continuous representation, and that E/F is a finite solvable Galois extension of totally real fields. Then, $\rho_{|G_E}$ is modular if and only if ρ is modular.*

Proof. Using induction, we can reduce to proving the proposition for a cyclic extension E/F of prime degree. So let us write $\mathrm{Gal}(E/F) = \langle \sigma \rangle$ and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E^\infty)$ such that $\rho_{|G_E} \cong \rho_\lambda(\pi)$. We first show that $\pi \cong \pi \circ \sigma$.
Let Σ be a finite set of finite places of E outside of which π is unramified, let us also fix a finite place $w \notin \Sigma$, and write $w' = w \circ \sigma^{-1}$. Then, $\pi_w \cong \mathrm{Ind}(\chi_1, \chi_2)$ (resp. $\pi_{w'} \cong \mathrm{Ind}(\chi_1', \chi_2')$) for unramified characters χ_1, χ_2 (resp. χ_1', χ_2') of $\mathrm{GL}_2(E_w)$ (resp. $\mathrm{GL}_2(E_{w'})$), and $(\pi \circ \sigma)_w$ is the representation whose underlying vector space is that of $\pi_{w'}$ and where $g \in \mathrm{GL}_2(E_w)$ acts on it via $\sigma(g)$. We can explicitly verify that this implies that $(\pi \circ \sigma)_w \cong \mathrm{Ind}(\chi_1' \circ \sigma, \chi_2' \circ \sigma)$. Now looking at the Galois representation attached to π, the characteristic polynomial of $\rho_\lambda(\pi)(\mathrm{Frob}_w)$ is given by

$$P_w(X) = X^2 - t_w X + \#k(w) s_w$$

where $t_w = \#k(w)^{\frac{1}{2}}(\chi_1(\varpi) + \chi_2(\varpi))$ and $s_w = \chi_1(\varpi)\chi_2(\varpi)$, ϖ a uniformizer of E_w. Similarly, the characteristic polynomial of $\rho_\lambda(\pi)(\mathrm{Frob}_w')$ is given by

$$P_{w'}(X) = X^2 - t_{w'} X + \#k(w) s_{w'}$$

where $t_w = \#k(w)^{\frac{1}{2}}(\chi_1'(\sigma(\varpi)) + \chi_2'(\sigma(\varpi))$ and $s_{w'} = \chi_1'(\sigma(\varpi))\chi_2'(\sigma(\varpi))$ ($\#k(w) = \#k(w')$). To relate both, we can write $\mathrm{Frob}_w' = \sigma \,\mathrm{Frob}_w \,\sigma^{-1}$; and since ρ is defined over G_F, we get that $P_w = P_w'$ (this is where we use that hypothesis). Therefore, we have $t_w = t_w'$ and and $s_w = s_w'$. Since all the characters are unramified, this implies that $(\chi_1, \chi_2) = (\chi_1' \circ \sigma, \chi_2' \circ \sigma)$ so that $\mathrm{Ind}(\chi_1, \chi_2) \cong \mathrm{Ind}(\chi_1' \circ \sigma, \chi_2' \circ \sigma)$, i.e., $(\pi \circ \sigma)_w \cong \pi_w$. We conclude by the strong multiplicity theorem that $\pi \circ \sigma \cong \sigma$.

□

The following lemma will be useful and is proved using class field theory.

Lemma 3.3.3. *[Tay03, Lemma 2.2]*
Let F be a number field, and let Σ be a finite set of places of K. For each $v \in \Sigma$, let L_v be a finite Galois extension of E_v. Then, there is a finite solvable Galois extension E/F such that for each place w of E above $v \in \Sigma$, there is an isomorphism $L_v \cong E_w$ as F_v-algebras. Moreover, if F^{avoid}/F is any finite extension, then we can choose E to be linearly disjoint from F^{avoid}.

Using these two results, we can reduce our hypothesis on Theorem 3.0.1. Thereby, upon replacing F by a solvable totally real extension (this is possible by including the infinite primes in the set Σ of the previous lemma) which is unramified at all primes above p, we can assume that:

- $[F : \mathbb{Q}]$ is even.
- $\overline{\rho}$ is unramified outside of p (for each v at which $\overline{\rho}$ is unramified, we let $L_v = \overline{F}_v^{\ker \overline{\rho}}$ in Lemma 3.3.3).

- For all primes $v \nmid p$, both $\rho(I_{F_v})$ and $\rho_0(I_{F_v})$ are unipotent (by Grothendieck's monodromy theorem).
- If ρ or ρ_0 are ramified at some place $v \nmid p$, then $\overline{\rho}_{|G_{F_v}}$ is trivial, and $\#k(v) \equiv 1 \mod p$.
- $\det \rho = \det \rho_0$.

Let us explain how we can realise the last condition. Given that ρ and ρ_0 are both crystalline with the same Hodge-Tate weights for all the places above p, then $\det \rho / \det \rho_0$ is crystalline with Hodge-Tate weight 0, hence unramified for all the places above p (it is well known that crystalline + $\mathbb{C}_p$-admissible = unramified). On the other hand, by the previous conditions, $\rho(I_{F_v})$ and $\rho_0(I_{F_v})$ are both unipotent for $v \nmid p$, so we get that the character $\det \rho / \det \rho_0$ is unramified at all primes. Therefore, it has a finite order (the Hilbert class field of F is finite over it). And since it is residually trivial, it has p-power order, so it is trivial on all complex conjugations. The extension of F cut out by its kernel is thereby finite, abelian, totally real and unramified at all the places above p.

Note that all the hypothesis of Theorem 3.0.1 are still satisfied, except for the conditions on $\overline{\rho}_{|G_{F(\zeta_p)}}$. To remedy this, when we use lemma 3.3.3, we let F^{avoid} to be $\overline{F}^{\ker \overline{\rho}}(\zeta_p)$. By linear disjointedness of E and F^{avoid} in the diagram

$$\begin{tikzcd}
 & EF^{\text{avoid}} \arrow[dl, dash] \arrow[dr, dash] & \\
E(\zeta_p) \arrow[dr, dash] & & F^{\text{avoid}} \arrow[dl, dash] \\
 & F(\zeta_p) &
\end{tikzcd}$$

we get that $\mathrm{Gal}(EF^{\text{avoid}}/E(\zeta_p)) \cong \mathrm{Gal}(F^{\text{avoid}}/F(\zeta_p))$. But by definition of F^{avoid}, $\overline{\rho}_{|G_{F(\zeta_p)}}$ factors through $\mathrm{Gal}(F^{\text{avoid}}/F(\zeta_p))$, hence its image is left unchanged.

In what follows, we will assume that all these conditions hold. We will write $\chi = \det \rho = \det \rho_0$. Moreover, we will assume that L is large enough so that it contains a primitive p-th root of unity and that $\mathbb{F}$ contains the eigenvalues of $\overline{\rho}(g)$ for all $g \in G_F$.

3.4 The Taylor-Wiles-Kisin method

3.4.1 Setup

Recall that we have a finite extension $L/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ whose maximal ideal and residue field are λ and $\mathbb{F}$ respectively. We consider a quaternion algebra D over F ramified exactly at the infinite places (which exists since $[F : \mathbb{Q}]$ is assumed to be even). Let T_p be the set of places of F lying over p, T_r be the set of primes not lying over p at which ρ or ρ_0 ramify, and $T = T_r \cup T_p$. If $v \in T_r$, we fix a topological generator σ_v of I_{F_v}/P_{F_v} (where I_{F_v} and P_{F_v} are respectively the inertia and the wild inertia groups).
For each set of Taylor-Wiles primes Q, consider the Global deformation problem $\mathcal{S}_Q = (\overline{\rho}, \chi, T \cup Q, \{\mathcal{D}_v\})$ and $\mathcal{S}'_Q = (\overline{\rho}, \chi, T \cup Q, \{\mathcal{D}'_v\}, \chi)$ defined by the following conditions:

- If $v \in T_p$, we let $\mathcal{D}_v = \mathcal{D}'_v$ be the local deformation problem consisting of crystalline lifts with the prescribed HT-weights $\{\mathrm{HT}_\sigma(\rho)\}$;
- If $v \in Q$, we do not impose any local conditions;

- If $v \in T_r$, we let $\mathcal{D}_v$ (resp. $\mathcal{D}'_v$) be the local deformation problems consisting of all lifts $\widetilde{\rho}$ of $\overline{\rho}_{G_{F_v}}$ with $\mathrm{char}_{\widetilde{\rho}(\sigma)}(X) = (X-1)^2$ (resp. with $\mathrm{char}_{\widetilde{\rho}(\sigma)}(X) = (X-\zeta_p)(X-\zeta_p^{-1})$);

We will write $\mathcal{S}_\emptyset$, $\mathcal{S}'_\emptyset$ for the similar global deformation problems without accounting for the Taylor-Wiles primes. So that the difference between say $\mathcal{S}_Q$ and $\mathcal{S}_\emptyset$ is that we allow ramification at the primes in Q.
We let R_{loc} and R'_{loc} be the universal rings corresponding to $\mathcal{S}_Q$ and $\mathcal{S}'_Q$ respectively, as described in section 2.4. Given that $\zeta_p \equiv 1 \mod \lambda$, we have that $R_{\mathrm{loc}}/\lambda \cong R'_{\mathrm{loc}}/\lambda$. Moreover, we have the following facts which are highly non-trivial:

- $(R'_{\mathrm{loc}})^{\mathrm{red}}$ is irreducible, $\mathcal{O}$-flat, and has Krull dimension $1 + 3\#T + [F : \mathbb{Q}]$.
- $(R_{\mathrm{loc}})^{\mathrm{red}}$ is $\mathcal{O}$-flat, equidimensional of Krull dimension $1 + 3\#T + [F : \mathbb{Q}]$, and reduction modulo λ gives a bijection between the irreducible components of $\operatorname{Spec} R^{\mathrm{loc}}$ and those of $\operatorname{Spec} R^{\mathrm{loc}}/\lambda$.

The reason why we introduced the global deformation problem $\mathcal{S}'_Q$ is justified by the fact that $(R'_{\mathrm{loc}})^{\mathrm{red}}$ is irreducible. Later, this will allow us to complete the patching argument for $\mathcal{S}'_Q$, and the relation $R_{\mathrm{loc}}/\lambda \cong R'_{\mathrm{loc}}/\lambda$ will serve as a bridge to complete the patching for $\mathcal{S}_Q$.

Now for the sake of reducing notation, we will write $R_Q^{\mathrm{univ}} := R_{\mathcal{S}_Q}^{\mathrm{univ}}$ and $R_Q^T := R_{\mathcal{S}_Q}^T$ (same thing with $'$ and/or $Q = \emptyset$). Note that we have $R_Q^{\mathrm{univ}}/\lambda \cong R_Q^{\mathrm{univ}\prime}/\lambda$ and $R_Q^T/\lambda \cong R_Q^{T\prime}/\lambda$. In addition, the natural maps $R_{\mathrm{loc}} \to R_Q^{\mathrm{univ}}$ and $R'_{\mathrm{loc}} \to R_Q^{\mathrm{univ}\prime}$ agree after reducing modulo λ. We fix universal deformations $\rho_\emptyset^{\mathrm{univ}}$, $\rho_\emptyset^{\mathrm{univ}\prime}$ of $R_\emptyset^{\mathrm{univ}}$ and $R_\emptyset^{\mathrm{univ}\prime}$ respectively, and choose universal deformations ρ_Q^{univ}, $\rho_Q^{\mathrm{univ}\prime}$ of R_Q^{univ} and $R_Q^{\mathrm{univ}\prime}$ respectively, which are compatible with each other modulo λ and compatible with $\rho_\emptyset^{\mathrm{univ}}$, $\rho_\emptyset^{\mathrm{univ}\prime}$ so that we have surjections:

$$R_Q^{\mathrm{univ}} \twoheadrightarrow R_\emptyset^{\mathrm{univ}} \quad \text{and} \quad R_Q^{\mathrm{univ}\prime} \twoheadrightarrow R_\emptyset^{\mathrm{univ}\prime}$$

which are equal modulo λ.

In lemma 2.5.4, we have shown that for $v \in Q$, we have a decomposition $\rho^{\mathrm{univ}}_{Q|G_{F_v}} = \chi_\alpha \oplus \chi_\beta$ for some tamely ramified characters $\chi_\alpha, \chi_\beta : G_{F_v} \to R_Q^{\mathrm{univ}}$, so let us choose one, say χ_α. If we compose $\chi_{\alpha|I_{F_v}^{\mathrm{ab}}}$ with the Artin map $\mathrm{Art} : \mathcal{O}_{F_v}^\times \xrightarrow{\cong} I_{F_v}^{\mathrm{ab}}$, we get a character $\chi'_\alpha : \mathcal{O}_v^\times \to R_Q^{\mathrm{univ}}$. Now given that $\overline{\rho}$ is unramified at v and by lemma 3, χ'_α has pro-p image. But $1 + \mathfrak{m}_v$ is pro-v, so this character factors into a map $\chi_\alpha : k(v)^\times \to R_Q^{\mathrm{univ}}$, where $k(v)$ is the residue field of F_v. The latter also factors through the maximal p-power quotient of $k(v)^\times$ which we denote by Δ_v.
We let $\Delta_Q = \prod\limits_{v \in Q} \Delta_v$, the choice of χ_α for each $v \in Q$ defines a morphism $\mathcal{O}[\Delta_Q] \to R_Q^{\mathrm{univ}}$, and we have the following expected result:

Lemma 3.4.1. *We have a surjective morphism* $\varphi_Q : R_Q^{univ} \twoheadrightarrow R_\emptyset^{univ}$ *whose kernel is* $\langle \delta - 1 \rangle_{\delta \in \Delta_Q} R_Q^{univ}$.

Proof. We prove this by showing that $R_Q^{\mathrm{univ}}/\langle \delta - 1 \rangle_{\delta \in \Delta_Q} R_Q^{\mathrm{univ}}$ satisfies the desired universal property. $\square$

Let $\mathcal{J} = \mathcal{O}[[x_1, \ldots, x_j]]$ where $j = 4T - 1$. By Remark 2.4.5, we have that $R_Q^T = R_Q^{\mathrm{univ}} \widehat{\otimes}_{\mathcal{O}} \mathcal{J}$ and the morphism $\Delta_Q \to R_Q^{\mathrm{univ}}$ induces a morphism $\mathcal{J}[\Delta_Q] \to R_Q^T$. If we denote $\mathfrak{a}_Q = \langle x_1, \ldots, x_j, \delta - 1 \text{ for } \delta \in \Delta_Q \rangle$ for the augmentation ideal of $\mathcal{J}[\Delta_Q]$, then by Lemma 3.4.1 we have that $R_Q T/\mathfrak{a}_Q = R_\emptyset^{\mathrm{univ}}$.

Let us now define the spaces of automorphic forms on which we will perform the patching. We let χ_0 be an algebraic grossencharacter such that $\chi\epsilon = \chi_{0,\iota}$ (ϵ is the p-adic cyclotomic character) and define k, η by $\mathrm{HT}(\rho_0) = \{\eta_{\iota\tau}, \eta_{\iota\tau} + k_{\iota\tau} - 1\}$. For the compact open subgroup $U_Q = \prod_v U_{Q,v}$, we set:

- $U_{Q,v} = \mathrm{GL}_2(\mathcal{O}_{F_v})$ if $v \notin Q \cup T_r$;
- $U_{Q,v} = \mathrm{Iw}_v = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod v \}$ if $v \in T_r$;
- $U_{Q,v} = \mathrm{Iw}_v^1 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iw}_v \mid ad^{-1} \in k(v)^\times \mapsto 1 \in \Delta_v\}$ if $v \in Q$.

And we let $\psi : \prod_{v \in T_r} U_{Q,v} \to \mathcal{O}^\times$ to be the trivial character. We also define a compact open subgroup $U'_Q = U_Q$ but with a character $\psi' : \prod_{v \in T_r} U_{Q,v} \to \mathcal{O}^\times$ defined as follows. For $v \in T_r$, we have a group homomorphism $U_{Q,v} \to k(v)^\times$ sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $ad^{-1} \mod v$, and we compose this with the morphism $k(v)^\times \to \mathcal{O}^\times$ sending the image of σ_v to $\zeta_p \in \mathcal{O}^\times$ (recall that for $v \in T_r$, we assume that $\# k(v) \equiv 1 \mod p$).

This data gives us spaces of modular forms $S(U_Q, \mathcal{O})$ and $S(U'_Q, \mathcal{O})$ with corresponding Hecke algebras $\mathbf{T}_{U_Q}$ and $\mathbf{T}_{U'_Q}$ generated by the operators T_v and S_v, for $v \notin Q \cup T$, and operators U_{ϖ_v} for $v \in Q$ defined by the double coset

$$U_{\varpi_v} = \left[U_{Q,v} \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U_{Q,v} \right]$$

where ϖ_v is a uniformizer of F_v. Note that by the isomorphism in (3.5) the automorphic form associated to ρ_0 induces a morphism $\mathbf{T}_{U_\emptyset} \to \mathcal{O}$ sending T_v to $\mathrm{Tr}(\rho_0(\mathrm{Frob}_v))$ and S_v to $\# k(v)^{-1} \det \rho_0(\mathrm{Frob}_v)$ for $v \notin T$. We let $\mathfrak{m}_\emptyset$ be maximal ideal of $\mathbf{T}_{U_\emptyset}$ given by the kernel of the map $\mathbf{T}_{U_\emptyset} \to \mathcal{O} \twoheadrightarrow \mathbb{F}$, so it is generated by λ, $\mathrm{Tr}\,\overline{\rho}(\mathrm{Frob}_v) - T_v$ and $\det \overline{\rho}(\mathrm{Frob}_v) - \# k(v) S_v$ for $v \notin T$. Recall that since $\overline{\rho}$ is absolutely irreducible, we also have a Galois representation $\rho^{\mathrm{mod}}_{\mathfrak{m}_\emptyset} : G_F \to \mathrm{GL}_2(\mathbf{T}_\emptyset)$, where $\mathbf{T}_\emptyset := (\mathbf{T}_{U_\emptyset})_{\mathfrak{m}_\emptyset}$, which is of type $\mathcal{S}_\emptyset$. This gives a surjective morphism $R^{\mathrm{univ}}_\emptyset \twoheadrightarrow \mathbf{T}_\emptyset$.
Since $\psi \cong \psi' \mod \lambda$, we have $S(U_\emptyset, \mathcal{O})/\lambda \cong S(U'_\emptyset, \mathcal{O})/\lambda$, and $\mathbf{T}_{U_\emptyset}/\lambda \cong \mathbf{T}_{U'_\emptyset}/\lambda$. So similarly, and we have a surjective morphism $R^{\mathrm{univ}\prime}_\emptyset \twoheadrightarrow \mathbf{T}'_\emptyset := (\mathbf{T}_{U'_\emptyset})_{\mathfrak{m}_\emptyset}$. We set $S_\emptyset = S(U_\emptyset, \mathcal{O})_{\mathfrak{m}_\emptyset}$ and $S'_\emptyset = S(U'_\emptyset, \mathcal{O})_{\mathfrak{m}_\emptyset}$ and the isomorphism $S_\emptyset/\lambda \cong S'_\emptyset/\lambda$ is compatible with $R^{\mathrm{univ}}_\emptyset/\lambda \cong R^{\mathrm{univ}\prime}_\emptyset/\lambda$.

Let π be a cuspidal automorphic representation such that the corresponding Galois representation $\rho_{\pi,\iota} : G_F \to \mathrm{GL}_2(\overline{L})$ satisfies $\overline{\rho}_{\pi,\iota} = \overline{\rho}$, and consider the associated ι-linear ring map $\theta_\pi : \mathbf{T}_{U_Q} \to \mathbb{C}$, and such . For each $v \notin Q \cup T$, θ_π sends T_v to $\iota(\mathrm{Tr}\,\rho_{\pi,\iota}(\mathrm{Frob}_v))$ and S_v to $\iota(\det \rho_{\pi,\iota}(\mathrm{Frob}_v))$. It also sends U_{ϖ_v}, for $v \in Q$, to α_v where α_v is one of its eigenvalues on $\pi_v^{U_{Q,v}}$.
Given that for $v \in Q$, $\pi_v^{\mathrm{Iw}_v^1} \neq 0$, we investigate the possibilities of π_v using Langlands reciprocity. By local-global compatibility, we have $\mathrm{rec}(\pi_v \otimes |\cdot|^{-\frac{1}{2}} \circ \det) = \mathrm{WD}(\rho_{\pi,\iota|G_{F_v}})^{\text{F-ss}}$. But given that $\rho_{\pi,\iota|G_{F_v}}$ is the sum of two tamely ramified characters, then by construction of $(\cdot)_{\mathrm{WD}}$, so is $\mathrm{WD}(\rho_{\pi,\iota|G_{F_v}})^{\text{F-ss}}$. Therefore, we can write $\mathrm{WD}(\rho_{\pi,\iota|G_{F_v}})^{\text{F-ss}} = \chi_\alpha \oplus \chi_\beta$ for χ_α, χ_β tamely ramified (we use the same notation as for the characters of the universal representation, because later we will see that the nilpotent endomorphism $N = 0$ so the characters will agree), and we get that π_v is a subquotient of $\chi_1 \times \chi_2$ whith $\chi_1 = (\iota \circ \chi_\alpha \circ \mathrm{Art}_{F_v}) \cdot |\cdot|^{\frac{1}{2}}$ and $\chi_2 = (\iota \circ \chi_\beta \circ \mathrm{Art}_{F_v}) \cdot |\cdot|^{\frac{1}{2}}$.
Using the Bruhat decomposition, we have:

$$\mathrm{GL}_2(F_v) = B(F_v)\,\mathrm{Iw}_v \bigsqcup B(F_v) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{Iw}_v$$

Since $\mathrm{Iw}_v = \bigsqcup_{\delta \in \Delta_v} \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_v^1$, we deduce another decomposition:

$$\mathrm{GL}_2(F_v) = B(F_v)\,\mathrm{Iw}_v^1 \bigsqcup B(F_v) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathrm{Iw}_v^1$$

which shows that we have an injection

$$\begin{aligned} \pi_v^{\mathrm{Iw}_v^1} &\hookrightarrow \mathbb{C}^2 \\ \phi &\mapsto \big(\phi(\mathrm{id}), \phi(w)\big) \end{aligned}$$

where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is not hard to show that if $\phi \in \pi_v^{\mathrm{Iw}_v^1}$, then $\phi \mathbb{1}_{B(F_v)\,\mathrm{Iw}_v^1}$, $\phi \mathbb{1}_{B(F_v)w\,\mathrm{Iw}_v^1}$, and $\phi' : g \mapsto \phi(wg)$ also belong to $\pi_v^{\mathrm{Iw}_v^1}$. From this, we get that the above morphism is actually an isomorphism, and that

$$\pi_v^{\mathrm{Iw}_v^1} = \mathbb{C}\phi_1 \oplus \mathbb{C}\phi_w$$

where $\phi_1(\mathrm{id}) = \phi_w(w) = 1$, $\mathrm{Supp}(\phi_1) = B(F_v)\,\mathrm{Iw}_v^1$ and $\mathrm{Supp}(\phi_w) = B(F_v)w\,\mathrm{Iw}_v^1$.
To compute the action of U_{ϖ_v} on $\pi^{\mathrm{Iw}_v^1}$, we use the following lemma:

Lemma 3.4.2. *There is a partition*

$$\mathrm{Iw}_v^1 \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_v^1 = \bigsqcup_{\alpha \in k(v)} \begin{pmatrix} \varpi_v & \widetilde{\alpha} \\ 0 & 1 \end{pmatrix} \mathrm{Iw}_v^1$$

where $\widetilde{\alpha}$ is a lift of α to $\mathcal{O}_{F_v}$.

Proof. Since $\mathrm{Iw}_v^1 \subseteq \mathrm{GL}_2(\mathcal{O}_{F_v})$, and given that we have a partition

$$\mathrm{GL}_2(\mathcal{O}_{F_v}) \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v}) = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v}) \bigsqcup \bigsqcup_{\alpha \in k(v)} \begin{pmatrix} \varpi_v & \widetilde{\alpha} \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v}) \tag{3.6}$$

an element $U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U'$ with $U, U' \in \mathrm{Iw}_v^1$ must land in one of the above cosest. First, note that it cannot land in the first coset since

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varpi_v a & b \\ c & \varpi_v^{-1} d \end{pmatrix} \notin \mathrm{GL}_2(\mathcal{O}_{F_v})$$

where $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $d \in \mathcal{O}_{F_v}^\times$. Similarly, if $U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U' \in \begin{pmatrix} \varpi_v & \widetilde{\alpha} \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_v})$, then

$$\begin{pmatrix} \varpi_v & \widetilde{\alpha} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} * & * \\ a'\varpi_v c + c'd & * \end{pmatrix} \in \mathrm{Iw}_v^1$$

where $U' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ which shows the lemma. $\square$

Using this partition, we can get a description of the action of U_{ϖ_v}. Concretely, for $\varphi \in \pi_v^{\mathrm{Iw}_v^1}$, we have:

$$\begin{aligned} U_{\varpi_v} \cdot \varphi(\mathrm{id}) &= \sum_{\alpha \in k(v)} \phi\left(\begin{pmatrix} \varpi_v & \widetilde{\alpha} \\ 0 & 1 \end{pmatrix} \right) \\ &= \sum_{\alpha \in k(v)} \chi_1(\varpi_v) \# k(v)^{-\frac{1}{2}} \phi(\mathrm{id}) = \# k(v)^{\frac{1}{2}} \chi_1(\varpi_v) \phi(\mathrm{id}) \end{aligned}$$

and,

$$\begin{aligned}
U_{\varpi_v}\cdot\phi(w) &= \sum_{\alpha\in k(v)}\phi\left(\begin{pmatrix}0 & 1\\ \varpi_v & \widetilde{\alpha}\end{pmatrix}\right)\\
&= \phi\left(\begin{pmatrix}1 & 0\\ 0 & \varpi_v\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) + \sum_{\alpha\in k(v)^\times}\phi\left(\begin{pmatrix}-\widetilde{\alpha}^{-1}\varpi_v & 1\\ 0 & \widetilde{\alpha}\end{pmatrix}\begin{pmatrix}1 & 0\\ \widetilde{\alpha}^{-1}\varpi_v & 1\end{pmatrix}\right)\\
&= \#k(v)^{\frac{1}{2}}\chi_2(\varpi_v)\phi(w) + \sum_{\alpha\in k(v)^\times}\#k(v)^{-\frac{1}{2}}\chi_1(\varpi_v)(\chi_2/\chi_1)(\widetilde{\alpha})\phi(\mathrm{id})
\end{aligned}$$

These calculations show us that

$$U_{\varpi_v}\cdot\phi_1 = \#k(v)^{\frac{1}{2}}\chi_1(\varpi_v)\phi_1 + X\phi_w$$

and,

$$U_{\varpi_v}\cdot\phi_w = \#k(v)^{\frac{1}{2}}\chi_2(\varpi_v)\phi_w$$

Note that if χ_1/χ_2 is ramified (which will be the case), then it induces a non-zero character of $k(v)^\times$ (using the Teichmuller lift), which is a finite group. Hence, $\sum_{\alpha\in k(v)^\times}(\chi_1/\chi_2)(\widetilde{\alpha}) = 0$, and $X = 0$.
The eigenvalues of $\rho_{\pi,\iota}(\mathrm{Frob}_v)$ are $\{\chi_\alpha(\mathrm{Frob}_v), \chi_\beta(\mathrm{Frob}_v)\}$, which are equal to

$$\{\iota^{-1}(\#k(v)^{\frac{1}{2}}\chi_1(\varpi_v)), \iota^{-1}(\#k(v)^{\frac{1}{2}}\chi_2(\varpi_v))\}$$

so they reduce modulo λ to $\overline{\alpha}_v$ and $\overline{\beta}_v$. If $\chi_1/\chi_2 = |\cdot|^{\pm1}$, then we would get

$$\overline{\alpha}_v/\overline{\beta}_v = \overline{\chi_1(\varpi_v)}/\overline{\chi_2(\varpi_v)} = |\varpi_v|^{\pm1} = \#k(v)^{\pm1} \equiv 1 \mod \lambda$$

which contradicts the fact that v is a Taylor-Wiles prime. Therefore, $\chi_1/\chi_2 \neq |\cdot|^{\pm1}$, so $\pi_v = \chi_1\times\chi_2 \cong \chi_1\times\chi_2$, and we can assume that $\overline{\chi_1}(\varpi_v) = \overline{\beta}_v$, $\overline{\chi_2}(\varpi_v) = \overline{\alpha}_v$.
We see finally that U_{ϖ_v} acts on $\pi_v^{U_{Q,v}}$ with eigenvalues that are lifts of $\overline{\alpha}_v$ and $\overline{\beta}_v$. Reducing the morphism $\iota\circ\theta_\pi$ modulo the maximal ideal, we get a maximal ideal $\mathfrak{m}_Q$ of $\mathbf{T}_{U_Q}$ given by:

$$\mathfrak{m}_Q = \langle\lambda\ ;\ T_v - \mathrm{Tr}(\overline{\rho}(\mathrm{Frob}_v), \#k_vS_v - \det\overline{\rho}(\mathrm{Frob}_v) \text{ for } v\notin T\cup Q\ ;\ U_{\varpi_v} - \overline{\alpha}_v \text{ for } v\in Q\rangle$$

Le us write $\mathbf{T}_Q = (\mathbf{T}_{U_Q})_{\mathfrak{m}_Q}$ and $S_Q = S(U_Q,\mathcal{O})_{\mathfrak{m}_Q}$. We have an action of Δ_Q on S_Q where $\delta\in\Delta_v$ acts via $\begin{pmatrix}\widetilde{\delta} & 0\\ 0 & 1\end{pmatrix}\in\mathrm{Iw}_v$ for a lift $\widetilde{\delta}$ of δ. Concretely, note that from equation (3.4), we have $S_Q\otimes_{\mathcal{O},\iota}\mathbb{C} = \oplus_\pi(\otimes'_{v\notin Q}\pi_v)^{U_Q^Q}\otimes_{v\in Q}X_v$ where X_v is the one dimensional space on which U_{ϖ_v} acts via a lift of $\overline{\alpha}_v$. Since X_v is spanned by ϕ_w, and that we have

$$\begin{pmatrix}\widetilde{\delta} & 0\\ 0 & 1\end{pmatrix}\phi_w = \chi_2(\widetilde{\delta})\phi_w$$

we see that Δ_v acts on S_Q via $\chi_2 = \chi_\alpha\circ\mathrm{Art}_{F_v}^{-1}$. On the other hand, we have another action of Δ_Q on S_Q given by

$$\Delta_Q\to R_Q^{\mathrm{univ}}\twoheadrightarrow\mathbf{T}_Q\to\mathrm{End}_{\mathcal{O}}(S_Q)$$

By construction of the map $\Delta_Q\to R_Q^{\mathrm{univ}}$, we get that the two actions that we just defined are equal.

We define a new compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ by setting $U_{Q,0} := \prod_{v\notin Q}U_{Q,v}\prod_{v\in Q}\mathrm{Iw}_v$. Since $\mathrm{Iw}_v/\mathrm{Iw}_v^1 = \Delta_v$, we have that $U_{Q,0}/U_Q = \Delta_Q$. Then, by 3) of Proposition 3.2.3, we get that S_Q is finite free over $\mathcal{O}[\Delta_Q]$.

Now for a place $v\in Q$, given that $\overline{\alpha}_v\neq\overline{\beta}_v$, by Hensel's lemma, the characteristic polynomial of $\rho_{\mathfrak{m}_\emptyset}^{\mathrm{univ}}(\mathrm{Frob}_v)$ is of the form $(X-A_v)(X-B_v)$ where $A_v, B_v\in\mathbf{T}_\emptyset$ are lifts of $\overline{\alpha}_v, \overline{\beta}_v$.

Proposition 3.4.3. *We have an isomorphism* $\prod_{v\in Q}(U_{\varpi_v}-B_v): S_\emptyset \xrightarrow{\sim} S(U_{Q,0},\mathcal{O})_{\mathfrak{m}_Q}$.

Proof. First note that the above morphism is well defined as we can see the source and the target as submodules of $S(U_{Q,0},\mathcal{O})_{\widetilde{\mathfrak{m}}_\emptyset}$ where $\widetilde{\mathfrak{m}}_\emptyset$ is the ideal generated by λ and $T_v - \operatorname{Tr}\overline{\rho}(\mathrm{Frob}_v), \#k(v) - \det\overline{\rho}(\mathrm{Frob}_v)$ for $v \notin T \cup Q$. We will use the following fact from algebra: if X, Y are finite free $\mathcal{O}$-module, and $X \to Y$ is a morphism such that it is an isomorphism after tensoring with $\overline{L}$, and is injective after tensoring with $\mathbb{F}$, then it is an isomorphism.
So let us check that it is an isomorphism after tensoring with $\overline{L}$, or equivalently with $\mathbb{C}$. We have $S(U_{Q,0},\mathcal{O})_{\widetilde{\mathfrak{m}}_\emptyset}\otimes_{\mathcal{O},\iota}\mathbb{C} = \oplus_\pi(\otimes'_{v\notin Q}\pi_v)^{U_Q^Q}\otimes(\otimes_{v\in Q}\pi_v^{\mathrm{Iw}_v})$ where the sum is taken over the cuspidal automorphic representation π such that $\overline{\rho}_\lambda(\pi)\cong\overline{\rho}$. So first things first, we need to investigate when do we have $\pi^{\mathrm{Iw}_v}\neq 0$ for $v\in Q$. Fix such π, by the Langlands correspondence we either have:

$$\operatorname{rec}(\pi_v)\cong\left(\begin{pmatrix}\chi_p\mu & 0\\ 0 & \mu\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right)$$

in which case if $\widetilde{\mathrm{Frob}_v}$ is a lift of the Frobenius to G_{F_v}, then $\rho_\lambda(\pi)(\widetilde{\mathrm{Frob}_v})$ has eigenvalues α and $\#k(v)\alpha$. But that would imply that $\overline{\rho}(\mathrm{Frob}_v)$ has eigenvalues $\overline{\alpha}$ and $\overline{\#k(v)\alpha}=\overline{\alpha}$ which contradicts the fact that v is a Taylor Wiles prime.
Or we have that $\pi_v\cong\operatorname{Ind}(\chi_1,\chi_2)$ with $\chi_1/\chi_2\neq|\cdot|^{\frac{1}{2}}$. Using the Bruhat decomposition

$$\mathrm{GL}_2(F_v)=B(F_v)\,\mathrm{Iw}_v\bigsqcup B(F_v)w\,\mathrm{Iw}_v$$

we get an injective homomorphism:

$$\begin{aligned}\pi_v^{\mathrm{Iw}_v^1}&\hookrightarrow\mathbb{C}^2\\ \phi&\mapsto\big(\phi(\mathrm{id}),\phi(w)\big)\end{aligned}$$

Note that since $T_0:=T\cap\mathrm{GL}_2(\mathcal{O}_{F_v})$ is a subgroup of Iw_v and satisfies $wT_0=T_0w$, for $\pi_v^{\mathrm{Iw}_v}\neq 0$ we must have that $(\chi_{1,v},\chi_{2,v})(T_0)=1$, i.e. that $\chi_{1,v}$ and $\chi_{2,v}$ are unramified. And in that case, by the same computations done for Iw_v^1 earlier, $\pi_v^{\mathrm{Iw}_v}=\mathbb{C}\phi_1\oplus\mathbb{C}\phi_w$ where $\phi_1(\mathrm{id})=\phi_w(w)=1$, $\mathrm{Supp}(\phi_1)=B(F_v)\,\mathrm{Iw}_v$ and $\mathrm{Supp}(\phi_w)=B(F_v)w\,\mathrm{Iw}_v$. In particular, we have shown that if $\pi_v^{\mathrm{Iw}_v}\neq 0$, then π_v is spherical.

Now back to the proof, the spaces we are considering are:

$$S_\emptyset\otimes_{\mathcal{O},\iota}\mathbb{C}=\oplus_\pi(\otimes'_{v\notin Q}\pi_v)^{U_Q^Q}\otimes(\otimes_{v\in Q}\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})})\ \text{ and }\ S(U_{Q,0},\mathcal{O})_{\mathfrak{m}_Q}\otimes_{\mathcal{O},\iota}\mathbb{C}=\oplus_\pi(\otimes'_{v\notin Q}\pi_v)^{U_Q^Q}\otimes(\otimes_{v\in Q}M_v)$$

where M_v is the subspace of $\pi_v^{\mathrm{Iw}_v}$ on which U_{ϖ_v} acts by a lift of $\overline{\alpha}_v$, so it is the one dimensional space generated by ϕ_w (this is seen using the same calculations as in the case of Iw_v^1).
For $v\in Q$, we let ϕ_0 be the generator of $\pi_v^{\mathrm{GL}_2(\mathcal{O}_{F_v})}$ with $\phi_0(1)=1$. By definition of B_v, we have:

$$B_v^2\phi_0-B_vT_v\phi_0+\#k(v)S_v\phi_0=0$$

but as seen in Section 3.2.1, we have $T_v\phi_0=\#k(v)^{\frac{1}{2}}(\chi_{1,v}(\varpi_v)+\chi_{2,v}(\varpi_v))\phi_0$ and $S_v\phi_0=\chi_{1,v}(\varpi_v)\chi_{2,v}(\varpi_v)\phi_0$, so the equation above becomes:

$$(B_v-\#k(v)^{\frac{1}{2}}\chi_{2,v}(\varpi_v))(B_v-\#k(v)^{\frac{1}{2}}\chi_{1,v}(\varpi_v))\phi_0=0$$

but we have $B_v-\#k(v)^{\frac{1}{2}}\chi_{2,v}(\varpi_v)\notin\mathfrak{m}_\emptyset$, so inverting it we get $B_v\phi_0=\#k(v)^{\frac{1}{2}}\chi_{1,v}(\varpi_v)\phi_0$.
Now since $w\in\mathrm{GL}_2(\mathcal{O}_{F_v})$, we get $\phi_0(w)=1$, so $\phi_0=\phi_1+\phi_w$, and we have :

$$\begin{aligned}(U_{\varpi_v}-B_v)\phi_0&=U_{\varpi_v}\phi_0+U_{\varpi_v}\phi_w)-B_v\phi_0\\ &=\#k(v)^{\frac{1}{2}}\chi_{1,v}(\varpi_v)\phi_1+\#k(v)^{\frac{1}{2}}\chi_{1,v}(\varpi_v)\phi_w+\#k(v)^{\frac{1}{2}}\chi_{2,v}(\varpi_v)\phi_w-\#k(v)^{\frac{1}{2}}\chi_{1,v}(\varpi_v)(\phi_1+\phi_2)\\ &=\#k(v)^{\frac{1}{2}}\chi_{2,v}(\varpi_v)\phi_w\end{aligned}$$

which shows that we have the desired isomorphism $S_\emptyset \otimes_{\mathcal{O},\iota} \mathbb{C} \xrightarrow{\sim} S(U_{Q,0},\mathcal{O})_{\mathfrak{m}_Q} \otimes_{\mathcal{O},\iota} \mathbb{C}$.

Finally, we need to check that the morphism is injective after tensoring with $\mathbb{F}$. The kernel would be a finite module for the Artinian local ring $\mathbf{T}_\emptyset/\lambda$ so for it to be zero, it suffices to prove that it does not have nonzero $\mathfrak{m}_\emptyset$-torsion. Thus, it suffices to show that the map:

$$\prod_{v\in Q}(U_{\varpi_v} - B_v) : (S_\emptyset \otimes \mathbb{F})[\mathfrak{m}_\emptyset] \to S(U_{Q,0},\mathcal{O})_{\mathfrak{m}_Q} \otimes \mathbb{F}$$

is injective. Arguing by induction on the size of Q, we can suppose that $Q = \{v\}$. Suppose that there exists a non-zero $x \in (S_\emptyset \otimes \mathbb{F})[\mathfrak{m}_\emptyset]$ such that $(U_{\varpi_v} - B_v)x = 0$. On the other hand, we have $T_v x = (A_v + B_v)x$, we will show that these two equations lead to a contradiction.
Lemma 3.4.2 and equation (3.6) give us the explicit description of the action of U_{ϖ_v} and T_v from which we get that $\begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} x = T_v x - U_{\varpi_v} x = \overline{\alpha}_v x$ (here we use that x is $\mathfrak{m}_\emptyset$-torsion). And since $w \in \mathrm{GL}_2(\mathcal{O}_{F_v})$ we have that:

$$\begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} x = w \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} wx = \overline{\alpha}_v x$$

and $U_{\varpi_v} x = \sum_{\alpha\in k(v)} \begin{pmatrix} \varpi_v & a \\ 0 & 1 \end{pmatrix} x = \sum_{\alpha\in k(v)} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} x = \#k(v)\overline{\alpha}_v x = \overline{\alpha}_v x$. But we have $U_{\varpi_v} x = \overline{\beta}_v x$ implying that $\overline{\alpha}_v = \overline{\beta}_v$, which is a contradiction. $\square$

3.4.2 The patching argument

We set $S_Q^T = S_Q \otimes_{R_Q^{\mathrm{univ}}} R_Q^T$ which is finite free over $\mathcal{J}[\Delta_Q]$, and we have

$$S_Q^T/\mathfrak{a}_Q = S_Q \otimes_{R_Q^{\mathrm{univ}}} R_Q^T/\mathfrak{a}_Q = S_Q/\otimes_{R_Q^{\mathrm{univ}}} R_\emptyset^{\mathrm{univ}} = S_Q^{\Delta_Q} = S(U_{Q,0},\mathcal{O})_{\mathfrak{m}_Q} \xrightarrow{\simeq} S_\emptyset$$

Recall that for all $N \geq 1$, there exists a set of Taylor-Wiles primes Q_N of order $r = h^1\big(G_{F,S}, \mathrm{ad}^0\overline{\rho}(1)\big)$ such that there is a surjective morphism

$$R_\infty := R_{\mathrm{loc}}[[x_1,\dots,x_g]] \twoheadrightarrow R_{Q_N}^T$$

where $g = r + \#T - 1 - [F,\mathbb{Q}]$. Moreover, if we write $Q_N = \{v_1,\dots,v_r\}$, we get a map:

$$\begin{aligned} f : \mathcal{O}[y_1,\dots,y_r] &\twoheadrightarrow \mathcal{O}[\Delta_{Q_N}] \\ y_i &\mapsto \delta_i - 1 \end{aligned}$$

where δ_i is a generator of Δ_{v_i}. This morphism is continuous if we equip $\mathcal{O}[\Delta_{Q_N}]$ with the p-adic topology (for which it is complete). Indeed, if α_i is the order of δ_i, then $f(y_i^{\alpha_i}) = (1-\delta_i)^{\alpha_i} \in (p)$, so $f^{-1}\big((p)\big) \supseteq \mathfrak{m}^{\sum \alpha_i}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}[y_1,\dots,y_r]$. Therefore, we can extend f to a surjective morphism:

$$\mathcal{O}[[y_1,\dots,y_r]] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}] \tag{3.7}$$

If we let $\mathcal{J}_\infty = \mathcal{J}[[y_1,\dots,y_r]]$, this induces a surjective morphism $\mathcal{J}_\infty \twoheadrightarrow \mathcal{J}[\Delta_{Q_N}]$, which fits in the following commutative diagram:

$$\begin{tikzcd} \mathcal{J}_\infty \arrow[r, two heads] \arrow[d, dashed] & \mathcal{J}[\Delta_{Q_N}] \arrow[d] \\ R_\infty \arrow[r, two heads] & R_{Q_N}^T \end{tikzcd}$$

where the dashed arrow exists since $\mathcal{J}_\infty$ is a power-series ring. Let us also write $\mathfrak{a}_\infty = \langle x_1, \dots, x_j, y_1, \dots, y_r \rangle \triangleleft \mathcal{J}_\infty$, then $S^T_{Q_N}/\mathfrak{a}_\infty = S_\emptyset$, $R^T_{Q_N}/\mathfrak{a}_\infty = R^{\mathrm{univ}}_\emptyset$. We define the ideals $I_N = \ker(\mathcal{J}_\infty \to \mathcal{J}[\Delta_{Q_N}])$. Given that for $v \in Q_N$, $\#k(v)$ is congruent to 1 modulo p^N, we have $I_N \subseteq \langle (y_1-1)^{p^N}-1, \dots, (y_r-1)^{p^N}-1 \rangle$ which shows that $\bigcap_N I_N = 0$. Thus, we can see $\mathcal{J}[\Delta_{Q_N}]$ as successive approximations of $\mathcal{J}_\infty$, and knowing that S_{Q_N} is finite free over $\mathcal{J}_\infty/I_N$, our goal is to construct a module "S_∞" which is finite free over $\mathcal{J}_\infty$ where the action is compatible with that on the ground level $S_\emptyset$. The whole picture can be summarized in the following diagram:

$$\begin{tikzcd}
\mathcal{J}_\infty \arrow[r, two heads] \arrow[d] & \mathcal{J}[\Delta_{Q_N}] \arrow[d, two heads] & \\
R_\infty \arrow[r, two heads] & R^T_{Q_N} \arrow[d, two heads] & \circlearrowright S_{Q_N} \arrow[d, two heads] \\
& R^{\mathrm{univ}}_\emptyset & \circlearrowright S_\emptyset
\end{tikzcd}$$

From which we only need to retain the following for the patching

$$\begin{tikzcd}
\mathcal{J}_\infty \arrow[r] & R_\infty \arrow[d, two heads] & \circlearrowright S_{Q_N} \arrow[d, two heads] \\
& R^{\mathrm{univ}}_\emptyset & \circlearrowright S_\emptyset
\end{tikzcd}$$

Note that the map $R_\infty \twoheadrightarrow R^{\mathrm{univ}}_\emptyset$ depends on N, and in general, the diagrams considered are not compatible for varying N, which is why we use the ultraproduct formalism to find a way to connect them. In that setting, we work with finite rings, so let us consider an open ideal $J \triangleleft \mathcal{J}_\infty$, which implies that $\mathcal{J}_\infty/J$ is finite. We will need the following lemma:

Lemma 3.4.4. *For $N \gg 0$, we have $I_N \subseteq J$.*

Proof. Since $1 + \mathfrak{m}_{J_\infty}$ is pro-p, its image in $\mathcal{J}_\infty/J$ (which is finite) is a finite p-group. So given that $1 + y_i \in 1 + \mathfrak{m}_{J_\infty}$, there exist some $n \geq 0$ such that $(1+y_i)^{p^n} \equiv 1 \mod J$ for all i. Therefore,

$$I_N \subseteq \langle (y_1-1)^{p^n} - 1, \dots, (y_r-1)^{p^n} - 1 \rangle \subseteq J$$

for all $N \geq n$. $\square$

Now let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$; using Proposition A.0.9, we get:

$$\Big(\prod_N \mathcal{J}_\infty/J\Big)_{\mathfrak{p}(\mathcal{F})} \cong \mathcal{J}_\infty/J \tag{3.8}$$

The ring $\prod_N \mathcal{J}_\infty/J$ acts componentwise on $\prod_N S_{Q_N}/J$, so using the isomorphism (3.8), we get an action of $\mathcal{J}_\infty/J$ on

$$S_{\infty,J} := \Big(\prod_N S_{Q_N}/J\Big)_{\mathfrak{p}(\mathcal{F})}$$

Proposition 3.4.5. *We have the following properties:*

(1) $S_{\infty,J}$ is finite free over $\mathcal{J}_\infty/J$;

(2) $S_{\infty,J}/\mathfrak{a}_\infty \cong S_\emptyset/J$;

(3) If $J \subseteq J'$ are open ideals, then the diagram

$$\begin{tikzcd}
S_{\infty,J'}/J \arrow[r, "\simeq"] \arrow[d] & S_{\infty,J} \arrow[d] \\
S_{\emptyset}/J \arrow[r, equal] & S_{\emptyset}/J
\end{tikzcd}$$

commutes;

(4) R_∞ acts on $S_{\infty,J}$ $\mathcal{J}_\infty/J$-linearly, and the induced action of $\mathcal{J}_\infty$ factors through a map $\mathcal{J}_\infty \to R_\infty$. Moreover, for $J \supseteq J'$ open ideals, the diagram

$$\begin{tikzcd}
R_\infty \arrow[r, phantom, "\circlearrowright"] \arrow[d, equal] & S_{\infty,J'} \arrow[d, two heads] \\
R_\infty \arrow[r, phantom, "\circlearrowright"] & S_{\infty,J}
\end{tikzcd}$$

commutes;

(5) The action of R_∞ is compatible with the change of level, i.e., we have a commuting diagram

$$\begin{tikzcd}
R_\infty \arrow[r, phantom, "\circlearrowright"] \arrow[d, two heads] & S_{\infty,J} \arrow[d, two heads] \\
R_{\emptyset}^{univ} \arrow[r, phantom, "\circlearrowright"] & S_{\emptyset}/J
\end{tikzcd}$$

Proof. **1)** Let $d = \mathrm{rank}_{\mathcal{O}} S_\emptyset$ and pick an isomorphism $\mathcal{J}[\Delta_{Q_N}]^d \xrightarrow{\simeq} S_{Q_N}$. Modding out by J, we get:

$$(\mathcal{J}_\infty/J)^d \twoheadrightarrow S_{Q_N}/J$$

inducing a surjective morphism

$$\Big(\prod_N J_\infty/J\Big)^{\oplus d}_{\mathfrak{p}(\mathcal{F})} \twoheadrightarrow \Big(\prod_N S_{Q_N}/J\Big)_{\mathfrak{p}(\mathcal{F})}$$

which is actually an isomorphism. Indeed, for injectivity, suppose that we have a tuple of the form

$$(x^1, \dots, x^d) \cdot y^{-1} \mapsto 0$$

where $y \notin \mathfrak{p}(\mathcal{F})$. Then, there exists $z \notin \mathfrak{p}(\mathcal{F})$ such that $(z_N x^1_N, \dots, z_N x^d_N) \mapsto 0$ for all $N \in \mathbb{N}$. Lemma 3.4.4 implies that for $N \gg 0$, $\mathcal{J}_\infty/J = \mathcal{J}[\Delta_{Q_N}]/J$, on which S_{Q_N}/J is free. So in that case the maps are levelwise injective, and we get $z_N x^i_N = 0$ for all i and $N \gg 0$. Thereby we define

$$z'_N = \begin{cases} z_N & \text{for } N \gg 0 \\ 0 & \text{otherwise.} \end{cases}$$

so that $z'_N x^i_N = 0$ for all $N \in \mathbb{N}$. But $Z(z') \notin \mathcal{F}$ since it differs from $Z(z)$ by a finite set, so $z' \notin \mathfrak{p}(\mathcal{F})$, which implies that $x^1 = \dots = x^r = 0$.
2)&**3)** Follows immediately from 1).
4) Let $\mathfrak{m}_\infty$ be the maximal ideal of R_∞. Since for $N \gg 0$, S_{Q_N}/J is finite free over $\mathcal{J}_\infty/J$ of rank

$\mathrm{rank}_{\mathcal{O}} S_\emptyset$, then the sequence $\left(\#(S_{Q_N}/J)\right)_{N\in\mathbb{N}^*}$ is stationary, so it is bounded by some integer $k \in \mathbb{N}$. By Nakayama's lemma, we get that $\mathfrak{m}_\infty^k(S_{Q_N}/J) = 0$ for all $N \in \mathbb{N}^*$. Now consider the ring

$$R = \prod_N (R_\infty/\mathfrak{m}_\infty^k)$$

For the same ultrafilter $\mathcal{F}$ as considered before, we have a prime $\mathfrak{p}'(\mathcal{F}) \subseteq R$ and an action

$$R_{\mathfrak{p}'(\mathcal{F})} \circlearrowright S_{\infty,J}$$

given componentwise. So by Proposition A.0.9, we get the desired action

$$R_\infty \twoheadrightarrow R_\infty/\mathfrak{m}_\infty \cong R_{\mathfrak{p}'(\mathcal{F})} \circlearrowright S_{\infty,J}$$

On each component N, we have a map $\mathcal{J}_\infty \to R_\infty/\mathfrak{m}_\infty$ which is compatible with the action on S_{Q_N}/J. This induces a map $J_\infty \to R_\infty/\mathfrak{m}_\infty$ which is compatible with the action on $S_{\infty,J}$. Since J_∞ is a powerseries ring, we lift it to a map $J_\infty \to R_\infty$. ☐

Now we define $S_\infty = \varprojlim_J S_{\infty,J}$ where the limit is take over the open ideals of $\mathcal{J}_\infty$. This is a finite free $\mathcal{J}_\infty$-module, and by (2) of Proposition 3.4.5, we have $S_\infty/\mathfrak{a}_\infty \cong S_\emptyset$. Moreover, R_∞ acts on S_∞ J_∞-linearly, and we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{J}_\infty \arrow[r] \arrow[d, two heads] & R_\infty \arrow[d, two heads] & \circlearrowright S_\infty \arrow[d, two heads] \\
\mathcal{J}_\infty/\mathfrak{a}_\infty = \mathcal{O} \arrow[r] & R_\emptyset^{\mathrm{univ}} & \circlearrowright S_\emptyset
\end{tikzcd}$$

It is time we finished the proof, and thankfully only a bit of commutative algebra is left. First note that the whole work can be done in the setting where we add a $'$ to everything, and in a way that is compatible with what we have done modulo λ. Secondly, we have the following equalities:

$$\begin{aligned}
\dim R_\infty &= \dim R_{\mathrm{loc}} + g \\
&= 3\#T + 1 + [F:\mathbb{Q}] + r + \#T - 1 - [F:\mathbb{Q}] \\
&= \#4T + r
\end{aligned}$$

Similarly we have $\dim R'_\infty = 4\#T + r$, and $\dim \mathcal{J}_\infty = 1 + j + r = 4\#T + r$. Since $\mathcal{J}_\infty$ is a power series ring over $\mathcal{O}$, it is Cohen-Macaulay, and given that S_∞ finite free over $\mathcal{J}_\infty$, we have:

$$\mathrm{depth}_{\mathcal{J}_\infty} S_\infty = \mathrm{depth}_{\mathcal{J}_\infty} \mathcal{J}_\infty = \dim \mathcal{J}_\infty = 4\#T + r$$

But the action of $\mathcal{J}_\infty$ on S_∞ factors through R_∞, so we must have $\mathrm{depth}_{R_\infty} S_\infty \geq 4\#T + r$. If $I = \mathrm{Ann}_{R_\infty}(S_\infty)$, then $\mathrm{Spec}\, R_\infty/I = \mathrm{Supp}_{R_\infty} S_\infty$, and we have:

$$4\#T + r = \dim R_\infty \geq \dim R_\infty/I \geq \mathrm{depth}_{R_\infty} S_\infty \geq 4\#T + r$$

then all the inequalities are equalities. In particular, we get that $\dim R_\infty/I = \dim R_\infty$ from which we see that $\mathrm{Supp}_{R_\infty} S_\infty$ is the union of irreducible components of $\mathrm{Spec}\, R_\infty$. Using the same argument, we also get that $\mathrm{Supp}_{R'_\infty} S'_\infty$ is the union of irreducible components of $\mathrm{Spec}\, R'_\infty$. But $\mathrm{Spec}\, R'_{\mathrm{loc}}$ is irreducible, hence so is $\mathrm{Spec}\, R'_\infty$ and we must have $\mathrm{Supp}_{R'_\infty} S'_\infty = \mathrm{Spec}\, R'_\infty$. In particular, we get that $\mathrm{Supp}_{R'_\infty/\lambda} S'_\infty/\lambda = \mathrm{Spec}\, R'_\infty/\lambda$, and by compatibility of the two pictures, we have:

$$\mathrm{Supp}_{R_\infty/\lambda} S_\infty/\lambda = \mathrm{Spec}\, R_\infty/\lambda$$

To sum up, $\mathrm{Supp}_{R_\infty} S_\infty$ is a union of irreducible components of $\mathrm{Spec}\, R_\infty$ and contains $\mathrm{Spec}\, R_\infty/\lambda$. But there is a bijection between irreducible components of $\mathrm{Spec}\, R_\infty/\lambda$ and irreducible components of $\mathrm{Spec}\, R_\infty$ (by a property of R_{loc} seen before) , thus we have $\mathrm{Supp}_{R_\infty} S_\infty = \mathrm{Spec}\, R_\infty$. Then, we have that $\mathrm{Supp}_{R_\infty/\mathfrak{a}_\infty} S_\infty/\mathfrak{a}_\infty = \mathrm{Spec}\, R_\infty/\mathfrak{a}_\infty$, in other words:

$$\mathrm{Supp}_{R_\emptyset^{\mathrm{univ}}} S_\emptyset = \mathrm{Spec}\, R_\emptyset^{\mathrm{univ}}$$

But the action of $R_\emptyset^{\mathrm{univ}}$ on $S_\emptyset$ factors through $\mathbf{T}_\emptyset$, and $S_\emptyset$ is a faithful $\mathbf{T}_\emptyset$-module. Thus, $\ker(R_\emptyset^{\mathrm{univ}} \twoheadrightarrow \mathbf{T}_\emptyset)$ is nilpotent, which means that $(R_\emptyset^{\mathrm{univ}})^{\mathrm{red}} \cong \mathbf{T}_\emptyset$. Note that ρ corresponds to a morphism $R_\emptyset^{\mathrm{univ}} \to \mathcal{O}$, which factors through $(R_\emptyset^{\mathrm{univ}})^{\mathrm{red}}$ and thus gives a morphism $\mathbf{T}_\emptyset \to \mathcal{O}$. Composing this with $\iota : \mathcal{O} \hookrightarrow \mathbb{C}$, we see by (3.5) that it corresponds to an automorphic representation π of weight (k, η) with $\rho \cong \rho_\lambda(\pi)$. This finishes the proof of Theorem 3.0.1.

Part II

Vanishing of the adjoint Bloch-Kato Selmer group of automorphic Galois representations

Introduction

In this part, we study the paper *Adjoint Selmer groups of automorphic Galois representations of unitary type* [NT20] by J. Newton and J. Thorne, adapting their exposition by giving additional details and explanations depending on my knowledge and understanding of the subject. The goal is to prove the vanishing of the adjoint Bloch-Kato Selmer group attached to an automorphic Galois representation, a result which is used by the same authors in their paper *Symmetric power functoriality for holomorphic modular forms* [NT19] to embed an "eigencurve" in a certain trianguline variety. This process allows them to prove that the symmetric power of an automorphic representation is still automorphic as predicted by the Langlands program.

To put things in context, consider a non-archimedean local field K of characteristic 0 with residue characteristic l, and V a continuous p-adic representation of G_K. We want to define the Bloch-Kato Selmer group as a subspace $H^1_f(G_K, V)$ of $H^1(G_K, V)$ capturing some good "geometric conditions". In the case $l \neq p$, the only reasonable condition seems to let $H^1_f(G_K, V) = H^1_{\mathrm{ur}}(G_K, V)$. Indeed, If $V = H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is the étale cohomology of a smooth proper K-scheme, then it is unramified whenever X has good reduction, i.e., if it is the generic fiber of a smooth proper $\mathcal{O}_K$-scheme.
If $l = p$, the story is different. In fact, it is known that if $V = H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, and if X is :

(i) a smooth proper K-scheme,

(ii) the generic fiber of a smooth proper $\mathcal{O}_K$-scheme,

(iii) a smooth proper K-scheme admitting a semistable proper flat $\mathcal{O}_K$-model.

then V is (i) B_{dR}-admissible, (ii) B_{crys}-admissible, (iii) B_{st}-admissible, where we recall that V $B_\dagger$-admissible ($\dagger$ is either ét,crys or st) if

$$\dim_{B_\dagger^{G_K}} D_\dagger(V) = \dim_{\mathbb{Q}_p} V$$

with $D_\dagger(V) = (V \otimes_{\mathbb{Q}_p} B_\dagger)^{G_K}$.
Under this point of view, the crystalline condition is the p-adic analogue of the unramified condition in the case $l \neq p$. Consequently, we define

$$H^1_f(G_K, V) = \ker\left(H^1(G_K, V) \to H^1(G_K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})\right)$$

Alternatively, an element $x \in H^1(G_K, V)$, which corresponds to an extension

$$0 \to V \to W \to \mathbb{Q}_p \to 0$$

is in $H^1_f(G_K, V)$ if and only if the sequence

$$0 \to D_{\mathrm{crys}}(V) \to D_{\mathrm{crys}}(W) \to D_{\mathrm{crys}}(\mathbb{Q}_p) = \mathbb{Q}_p \to 0$$

is exact. In particular, if V is crystalline, then x is in $H^1_f(G_K, V)$ if and only if W is crystalline.

Similarly, we define the geometric Bloch Selmer group $H^1_g(G_K, V)$ to be:

- $H^1(G_K, V)$ if $l \neq p$,
- $\ker(H^1(G_K, V) \to H^1(G_K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}))$ if $l = p$.

If $l = p$ and V is deRham, then elements of $H^1_g(G_K, V)$ correspond to de Rham extensions of V by $\mathbb{Q}_p$.

Now, if F^+ is a number field with S a finite set of places of F^+ containing those above p, and if V is a p-adic Galois representation of $G_{F^+,S}$, the global Bloch-Kato Selmer groups are defined by

$$H^1_f(F^+, V) = \ker\left(H^1(F^+_S/F^+, V) \to \prod_{v \in S} H^1(F^+_v, V)/H^1_f(F^+_v, V)\right)$$
$$H^1_{g,S}(F^+, V) = \ker\left(H^1(F^+_S/F^+, V) \to \prod_{v \in S} H^1(F^+_v, V)/H^1_g(F^+_v, V)\right)$$

where we denote by $H^1(F^+_S/F^+, *)$ and $H^1(F^+_v, *)$ for the continuous group cohomology of $G_{F^+,S}$ and $G_{F^+_v}$ respectively.
In [BK07], Bloch and Kato defined these Selmer groups and conjectured that if V is geometric, then we have the following equality:

$$\mathrm{ord}_{s=0} L(V, s) = \dim_{\mathbb{Q}_p} H^1_f(F^+, V^\vee(1)) - \dim_{\mathbb{Q}_p} H^0(F^+_S/F^+, V^\vee(1))$$

Here $L(V, s)$ is the L-function associated to V which is defined by:

$$L(V, s) = \prod_v L_v(V, s)$$

where if we let q_v be the size of the residue field of F^+_v, then

$$L_v(V, s) = \det\left(\mathrm{id} - (\mathrm{Frob}_v^{-1} q_v^{-s})_{|V^{I_{F^+_v}}}\right)^{-1}$$

if $v \nmid p$ and,

$$L_v(V, s) = \det\left(\mathrm{id} - (\varphi^{-f_v} q_v^{-s})_{|D_{\mathrm{crys}}(V_{|G_{F^+_v}})}\right)^{-1}$$

if $v \mid p$, with φ the crystalline Frobenius and $q_v = p^{f_v}$.
Note that it is only conjectured in general that $L(V, s)$ is well defined at $s = 1$. In fact, if V is moreover pure of weight w, $L(V, s)$ is expected to admit a meromorphic continuation to all of the complex plane and has no zeros on the domain $\mathfrak{Res} \geq w/2 + 1$.

Let us now suppose that F^+ is totally real, F/F^+ a CM extension, and π is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then for any isomorphism $\iota : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{C}$, we can attach a Galois representation

$$r_{\pi,\iota} : G_{F,S} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

which is known to be Geometric and pure. Applying the Bloch-Kato conjecture for $\mathrm{ad}\, r_{\pi,\iota}(1)$, which is pure of weight -2, we would expect that

$$H^1_f(F^+, \mathrm{ad}\, r_{\pi,\iota}) = 0$$

In fact, under a weak hypothesis on the image of the representation $r_{\pi,\iota}$, J.Newton and J.Thorne proved:

Theorem 3.4.6. *[NT20, Theorem A] Let F/F^+ be a CM extension, and π a regular algebraic conjugate self-dual cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$*. Let p be a prime and $\iota : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. If $r_{\pi,\iota}(G_{F(\zeta_{p^\infty})})$ is enormous in the sense of definition 4.4.11, then* $H^1_f(F^+, \mathrm{ad}\, r_{\pi,\iota}) = 0$.

By identifying $H^1_f(F_+, \mathrm{ad} r_{\iota,\pi})$ with the tangent space of a universal pseudodeformation ring of $r_{\pi,\iota}$, we will be able to prove the theorem using an equality of the form "$R = \mathbf{T}$". However, since we do not impose any condition on the residual Galois representation, we can no longer work with the theory of deformations of Galois representations. Instead, we will use pseudorepresentations which are more adapted to this setting.

Chapter 4

Pseudocharacters and Pseudorepresentations

Pseudorepresentations form a particular case of a more general notion which is that of polynomial laws. In this chapter, we introduce both of these objects and study some of their properties. In particular, we will study the theory of deformations of pseudorepresentations and see how one can impose conditions just like in the classical setting of representations. Using these results, we will be able to find sets of Taylor-Wiles places with good properties with which we will do the patching.
To get more details about the subject, one can consult C.W.Erickson's Ph.D thesis [Eri13] or the original source in [Che08].

4.1 Polynomial laws

Let A be a commutative ring. We denote by $\mathbf{Alg}_A$ the category of commutative A-algebras. Each A-module M gives rise to a functor $\underline{M} : \mathbf{Alg}_A \to \mathbf{Ens}, B \mapsto M \otimes_A B$. Given two A-modules M and N, a polynomial law $P : M \to N$ is simply a natural transformation $\underline{M} \to \underline{N}$. In other words, for each $B, B' \in \mathbf{Alg}_A$ and $u \in \mathrm{Hom}_{\mathbf{Alg}_A}(B, B')$, we have a commutative diagram

$$\begin{array}{ccc} M \otimes_A B & \xrightarrow{P_B} & N \otimes_A B \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes u} & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes u} \\ M \otimes_A B' & \xrightarrow{P_{B'}} & N \otimes_A B' \end{array}$$

The set of all polynomial laws from M to N is denoted $\mathcal{P}_A(M, N)$. For $d \geq 1$, a polynomial law $P : M \to N$ is homogenious of degree d if for all $B \in \mathbf{Alg}_A$, $z \in M \otimes_A B$, and $b \in B$, we have

$$P_B(bz) = b^d P_B(z)$$

We denote by $\mathcal{P}_A^d(M, N)$ the A-module of homogeneous of degree d A-polynomial laws from M to N. If M and N are A-algebras, we say that a polynomial law $P : M \to N$ is multiplicative if for all $B \in \mathbf{Alg}_A$, $P_B(1) = 1$ and for all $x, y \in M \otimes_A B$,

$$P_B(xy) = P_B(x)P_B(y)$$

We write $\mathcal{M}_A^d(M, N)$ for the set of all multiplicative, homogenious of degree d A-polynomial laws.

Note that by functoriality, a polynomial law $P : M \to N$ is determined by the maps

$$P_{A[T_1,\ldots,T_n]} : M[T_1, \ldots T_n] \to N[T_1, \ldots, T_n]$$

for all $n \geq 1$. In particular, if P is homogenious of degree d, then it is uniquely determined by the map $P_{A[T_1,\dots,T_d]}$. Indeed, let $I_d = \{(\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d \mid \alpha_1 + \cdots + \alpha_d = d\}$, then for all $x_1, \dots, x_d \in M$, we have:

$$\begin{aligned} P_{A[T_1,\dots,T_d]} : M[T_1, \dots, T_d] &\to N[T_1, \dots, T_d] \\ x_1T_1 + \cdots + x_dT_d &\mapsto \sum_{\alpha \in I_d} P^{[\alpha]}(x_i \mid \alpha_i \neq 0)T^\alpha \end{aligned} \tag{4.1}$$

To see that the polynomial is homogenious, we apply the functoriality of P to the map $A[T_1, \dots, T_d] \to A[T_1, \dots, T_d, X]$ sending T_i to XT_i. And to see the map only depends on $P_{A[T_1,\dots,T_d]}$, note that the coefficient $P^{[\alpha]}(x_i \mid \alpha_i \neq 0)$ can be recovered using the functoriality of P to the map $A[T_1, \dots, T_d] \to A[T_1, \dots, T_d]$ sending T_i to 0 if $\alpha_i = 0$.
Therefore, if $X \subseteq M$ generates M as an A-module, then P is uniquely determined by the finite set of functions $P^{[\alpha]} : X^d \to N$ for $\alpha \in I_d$.

Definition 4.1.1. If M, N are two A-modules and $P \in \mathcal{P}_A(M, N)$ we define $\ker(P) \subseteq M$ as the subset whose elements are the $x \in M$ such that for every commutative A-algebra B,

$$\forall b \in B, \forall m \in M \otimes_A B, P_B(x \otimes b + m) = P_B(m)$$

From the definition, we see that $\ker(P)$ is an A-submodule of M. We say that P is faithful if $\ker(P) = 0$.

Lemma 4.1.2. *With the same notation as before*

(1) $\ker(P)$ *is the biggest A-submodule $K \subseteq M$ such that P admits a factorisation $P = \widetilde{P} \circ \pi$ where $\pi : M \to M/K$ is the canonical projection and $\widetilde{P} \in \mathcal{P}_A(M/K, N)$.*

(2) $\widetilde{P} : M/\ker(P) \to N$ *is faithful.*

(3) If B is a commutative A-algebra, then

$$\operatorname{im}(\ker(P) \otimes_A B \to M \otimes_A B) \subseteq \ker(P \otimes_A B)$$

Proof. **(3)** is clear from the transitivity of the tensor product.
(1) It is immediate that if $P = \widetilde{P} \circ \pi$ for $K \subseteq M$ as in the definition, then $K \subseteq \ker(P)$.
On the other hand, let $K \subseteq \ker(P)$ and define for B a commutative A-algebra $K_B = \operatorname{im}(K \otimes_A B \to M \otimes_A B)$. Then, $\pi \otimes_A B : (M \otimes_A B)/K_B \cong (M/K) \otimes_A B$, and by (3) $K_B \subseteq \ker(P \otimes_A B)$. In particular, the map $P_B : M \otimes_A B \to N \otimes_A B$ satisfies $P_B(k+m) = P_B(m)$ for $k \in K_B$ and $m \in M \otimes_A B$. Therefore, we can define a map $\widetilde{P}_B : (M/K) \otimes_A B \to N \otimes_A B$ by setting

$$\widetilde{P}_B\big((\pi \otimes_A B)(m)\big) = P_B(m), \ \forall m \in M \otimes_A B \tag{4.2}$$

The collection of maps $\widetilde{P}_B$ define a polynomial law $\widetilde{P} \in \mathcal{P}_A(M/K, N)$.
(2) From (4.2), we see that $\ker(\widetilde{P}) = \ker(P)/K$ which gives (2). $\square$

Lemma 4.1.3. *Let R, S be two A-algebras and $P \in \mathcal{M}^d_A(R, S)$.*

(1) $\ker(P) = \{r \in R, \forall B, \forall r' \in R \otimes_A B, \ P(1+rr') = 1\} = \{r \in R, \forall B, \ \forall r' \in R \otimes_A B, P(1+r r') = 1\}$.

(2) $\ker(P)$ *is a two sided ideal of R, it is proper if $d > 0$ and $R \neq 0$. It is the biggest two sided ideal $K \subseteq R$ such that P admits a factorisation $P = \widetilde{P} \circ \pi$ with π is the canonical surjection $R \to R/K$ and $\widetilde{P} \in \mathcal{M}^d_A(R/K, S)$.*

Proof. **(1)** Let $r \in \ker(P)$, B a commutative A-algebra and $m = 1 + h \in R \otimes_A B$. We want to show that

$$P_B(1 + r(1 + th)) = P_B(1 + (1 + th)r) = 1$$

in $S \otimes_A B[t]$. Since they are polynomials of degree d, it is enough to check this in $S \otimes_A B[t]/(t^{d+1})$. Since $(1 + th) \in R \otimes_A B[t]/(t^{d+1})$ is invertible, we have

$$P(1 + r(1 + th)) = P((1 + th)^{-1} + r)P(1 + th) = P((1 + th)^{-1})P(1 + th) = P(1) = 1$$

and similarly we see that $P(1 + (1 + th)r) = 1$. From this we also see that an element in the sets on the right hand side of (1) lie in $\ker(P)$, so we get the desired equalities.
(2) By (1), $\ker(P)$ is a two sided ideal of R. Since $P(1) = 1$, we have $P(1 - t) = (1 - t)^d$ and so $1 \notin \ker(P)$ if $d > 0$. The rest follows similarly to (2) of the previous lemma from (4.2). □

Remark 4.1.4. Note that by the previous lemma, $r \in \ker(P)$ if and only if for any $r_1, \dots, r_n \in R$, we have

$$P(1 + r(t_1 r_1 + \cdots + t_n r_n)) = 1$$

So if $A = S$ is an infinite domain, we have $\ker(P) = \{r \in R, \forall r' \in R,\ P(1 + rr') = 1\}$. Indeed, in this case the polynomial $P(1 + r(t_1 r_1 + \cdots + t_n r_n)) - 1 \in A[t_1, \dots, t_n]$ would have infinitely many roots (by functoriality of P) so it must be zero.

4.1.1 Representability

Definition 4.1.5. The A-algebra of divided powers on M, denoted $\Gamma_A(M)$ is the A-algebra generated by the symbols $m^{[i]}$ for $m \in M$, $i \in \mathbb{N}$ which is subject to the following relations:

- $m^{[0]} = 1$ for all $m \in M$.
- $(am)^{[i]} = a^i m^{[i]}$ for all $a \in A$, $m \in M$.
- $m^{[i]} m^{[j]} = \frac{(i+j)!}{i!j!} m^{[i+j]}$ for all $i, j \in \mathbb{N}$, $m \in M$.
- $(m + m')^{[i]} = \sum_{p+q=i} m^{[p]} m'^{[q]}$ for all $i \in \mathbb{N}$, $m, m' \in M$.

Definition 4.1.6. Let B be a commutative A-algebra. We define the B-module $\exp(B)$ to be the subgroup of the power series algebra over B consisting of elements $f \in B[[t]]^\times$ satisfying

- $f(0) = 1$
- $f(t_1 + t_2) = f(t_1)f(t_2)$ for free commutative variables t_1, t_2.

with the b-module structure given by $(b \cdot f)(t) = f(bt)$.

Proposition 4.1.7. *The functors $\Gamma_A : \boldsymbol{Mod}_A \leftrightarrows \boldsymbol{Alg}_A : \exp$ are adjoint to each other. In other words, we have a natural bijection*

$$\mathrm{Hom}_{\boldsymbol{Alg}_A}(\Gamma_A(M), B) \xrightarrow{\simeq} \mathrm{Hom}_{\boldsymbol{Mod}_A}(M, \exp(B))$$

sending $f : \Gamma_A(M) \to B$ to $g : m \mapsto \sum_{i=0}^{\infty} f(m^{[i]}) t^n$.

Corollary 4.1.8. *Let A be a commutative ring. Then, the algebra of divided powers over A satisfies the following properties:*

(1) If B is a commutative A-algebra, and M an A-modules, then we have a canonical isomorphism of B-algebras

$$\begin{aligned}\Gamma_A(M) \otimes_A B &\xrightarrow{\simeq} \Gamma_B(M \otimes_A B)\\ m^{[i]} \otimes 1 &\mapsto (m \otimes 1)^{[i]}\end{aligned}$$

(2) If $M = \varinjlim_i M_i$ is a colimit of A-modules, then we have an isomorphism of A-algebras

$$\varinjlim_i \Gamma_A(M_i) \cong \Gamma_A(\varinjlim_i M_i)$$

(3) If M_1 and M_2 are two A-modules, then there is a canonical isomorphism

$$\begin{aligned}\Gamma_A(M_1 \oplus M_2) &\xrightarrow{\simeq} \Gamma_A(M_1) \otimes_A \Gamma_A(M_2)\\ (m_1, m_2)^{[i]} &\mapsto \sum_{p+q=i} m_1^{[p]} \otimes m_2^{[q]}\end{aligned}$$

which respects grading, i.e., for each $d \geq 0$, it induces an isomorphism

$$\Gamma^d(M_1 \oplus M_2) \xrightarrow{\simeq} \bigoplus_{p+q=d} \Gamma_A^p(M_1) \otimes \Gamma_A^q(M_2)$$

Given two A-modules M, N, a polynomial law $P \in \mathcal{P}_A(M, N)$, and an A-algebra B, we define the Taylor expansion of P at a point $x \in M \otimes_A B$ with respect to $m \in M$ to be

$$S_m(P)_B(x) := P_{B[t]}(m \otimes t + x) \in N \otimes_A B[t]$$

For any morphism of A-algebras $u : B \to B'$, by functoriality of P, we have a commutative diagram:

$$\begin{tikzcd} m \otimes t + x \arrow[r] \arrow[d] & P_{B[t]}(m \otimes t + x) \arrow[d] \\ m \otimes t + u(x) \arrow[r] & P_{B'[t]}(m \otimes t + u(x)) \end{tikzcd}$$

which shows that $S_m(P) \in \mathcal{P}_A(M, N \otimes_A A[t])$. Therefore, the composition of the polynomial law $S_m(P)$ with the linear polynomial law $N \otimes_A B[t] \to N \otimes_A B, \sum_i n_i t^i \mapsto n_i$ gives a polynomial law which we denote $\partial_m^i(P) \in \mathcal{P}_A(M, N)$. It is straightforward to check that $\delta_m^i : \mathcal{P}(M, N) \to \mathcal{P}(M, N)$ is an A-linear map. We denote by $\mathcal{D}$ the A-subalgebra of $\mathrm{End}_A(\mathcal{P}(M, N))$ generated by the δ_m^i for $m \in M$ and $i \in \mathbb{N}$. Now given an A-algebra B, $m_1, m_2 \in M$ and $P \in \mathcal{P}_A(M, N)$, we have

$$\begin{aligned}P_{B[t_1,t_2]}(m_1 \otimes t_1 + m_2 \otimes t_2 + x) &= P_{B[t_1,t_2]}(m_2 \otimes t_2 + m_1 \otimes t_1 + x)\\ S_{m_1}(P)_{B[t_2]}(m_2 \otimes t_2 + x) &= S_{m_2}(P)_{B[t_1]}(m_1 \otimes t_1 + x)\\ S_{m_2}(S_{m_1}(P))_B(x) &= S_{m_1}(S_{m_2}(P))_B(x)\end{aligned}$$

comparing the coefficients in t_1, t_2 we find that $\partial_{m_2}^i \partial_{m_1}^j t_2^i t_1^j = \partial_{m_1}^j \partial_{m_2}^i t_1^j t_2^i$ for all $i, j \geq 0$ which shows that $\mathcal{D}$ is commutative.

Lemma 4.1.9. *The map $S_{(\cdot)} : M \to \mathcal{D}[[t]]$ defines an A-linear map $S : M \to \exp(\mathcal{D})$.*

Proof. We need to show that for $m \in M$, $S_m = \sum_{i=0}^{\infty} \partial_m^i t^i$ lies in $\exp(\mathcal{D})$. First, note that for an A-algebra B, by applying the functoriality of P for the map $B[t] \to B : t \mapsto 0$, we get that $\partial_m^0(P)(x) = P_B(x)$ for $x \in M \otimes_A B$. So $\partial_m^0(P) = P$, and $\partial_m^0 = 1 \in \mathcal{D}$.
Now for $m \in M$, B an A-algebra and $P \in \mathcal{P}_A(M, N)$, the functoriality of P applied for the map $B[t] \to B[t_1, t_2] : t \mapsto t_1 + t_2$ gives a commutative diagram

$$\begin{array}{ccc} m\otimes t+x & \longrightarrow & P_{B[t]}(m\otimes t+x) \\ \downarrow & & \downarrow \\ m\otimes t_1+m\otimes t_2+x & \longrightarrow & P_{B[t_1,t_2]}(m\otimes t_1+m\otimes t_2+x) \end{array}$$

which shows that $S_m(t_1+t_2)(P)_B(x) = P_{B[t_1,t_2]}(m\otimes t_1+m\otimes t_2+x) = \big(S_m(t_1)S_m(t_2)\big)(P)_B(x)$. Thus, $S_m(t_1+t_2) = S_m(t_1)\cdot S_m(t_2)$ as desired. It remains to show that S is an A-linear map which follows the same ideas. $\square$

Definition 4.1.10. We define the universal homogenious of degree d polynomial law $P_d^{\text{univ}} : M \to \Gamma_A^d(M)$ by setting for each A-algebra B,

$$\begin{gathered} P_{d,B}^{\text{univ}} : M\otimes_A B \to \Gamma_B(M\otimes_A B) \cong \Gamma_A(M)\otimes_A B \\ m\otimes b \mapsto (m\otimes b)^{[d]} \cong m^{[d]}\otimes b^d \end{gathered}$$

Theorem 4.1.11. *[Rob80, Theorem III.1] Let M be an A-module and $d\geq 1$, then $\Gamma_A^d(M)$ represents the functor $N\mapsto \mathcal{P}_A^d(M,N)$. In other words, we have a bijection*

$$\begin{gathered} \operatorname{Hom}_A(\Gamma_A^d(M),N) \xrightarrow{\simeq} \mathcal{P}_A^d(M,N) \\ f\mapsto f\circ P_d^{univ} \end{gathered}$$

Proof. Let us first show the injectivity of the defined map. So let $P\in\mathcal{P}_A^d(M,N)$ such that $P = f\circ P_d^{\text{univ}}$ for some $f\in\operatorname{Hom}_A(\Gamma_A^d(M),N)$. Since $\Gamma_A^d(M\otimes_A A[t_1,\dots,t_d]) \cong \Gamma_A^d(M)\otimes_A A[t_1,\dots,t_d]$, for all $\alpha\in I_d^d$ and $m_1,\dots,m_d\in M$, we have

$$\begin{aligned} P_d^{\text{univ}}(m_1\otimes t_1+\dots+m_d\otimes t_d) &= (m_1\otimes t_1+\dots+m_d\otimes t_d)^{[d]} \\ &= \sum_{\alpha\in I_d^d}\prod_{1\leq i\leq d} m_i^{[\alpha_i]}t_i^{\alpha_i} \end{aligned}$$

then by definition of the $P^{[\alpha]}$ (see equation (4.1)), we get that $f(m_1^{[\alpha_1]}\cdots m_d^{[\alpha_d]}) = P^{[\alpha]}(m_1,\dots,m_d)$. But $\Gamma_A^d(M)$ is generated as an A-module by the $\prod_{1\leq i\leq d} m_i^{[\alpha_i]}$ for $\alpha\in I_d^d$ and $m_1,\dots,m_d\in M$ which shows that P determines f, i.e., the map is injective.
To prove surjectivity, let $P\in\mathcal{P}_A^d(M,N)$, we need to show that there exists an A-linear map $f : \Gamma_A^d(M)\to N$ such that

$$f(m_1^{[\alpha_1]}\cdots m_d^{[\alpha_d]}) = P^{[\alpha]}(m_1,\dots,m_d)$$

for all $\alpha\in I_d^d$ and $m_1,\dots,m_d\in M$. From now on we will write $\underline{m} = (m_1,\dots,m_d)\in M^d$ for briefty. By the previous lemma, we have an A-linear map $S : M\to\exp(\mathcal{D})$ which by adjointness gives us a map of A-algebras $\widetilde{S} : \Gamma_A(M)\to\mathcal{D}$. One can check that $\widetilde{S}(\underline{m}^{[i]}) = \partial_{\underline{m}}^i$. On the other hand, the data of P induces an A-algebra morphism

$$\begin{gathered} \operatorname{ev}_P : \mathcal{D}\to N \\ \partial\mapsto\partial(P)_A(0) \end{gathered}$$

evaluating the differential operators at 0. Setting $f = \operatorname{ev}_P\circ\widetilde{S}$, we get that $f(\underline{m}^{[\alpha]}) = \partial_{\underline{m}}^\alpha(P)_A(0)$ But for an A-algebra B and $x\in M\otimes_A B$, we have

$$P_{B[t_1,\dots,t_d]}(m_1\otimes t_1+\dots+m_d\otimes t_d+x) = \sum_\alpha \partial_{\underline{m}}^\alpha(P)_B(x)t^\alpha$$

evaluating at 0, we get that:

$$P_{B[t_1,\dots,t_d]}(m_1\otimes t_1+\dots+m_d\otimes t_d) = \sum_\alpha \partial_{\underline{m}}^\alpha(P)_B(0)t^\alpha$$

by definition of $P^{[\alpha]}$, we get that $P^{[\alpha]}(\underline{m}) = f(\underline{m}^{[\alpha]})$ as desired. $\square$

4.2 Pseudorepresentations

Informally, a pseudorepresentation $D : G \to A$ of a group G with coefficients in A is the data for each $g \in G$ of a characteristic polynomial $\chi_D(g,t) \in A[t]$ subject to conditions making these polynomials behave as the characteristic polynomials of a representation $\rho : G \to \mathrm{GL}_n(A)$. Note that in [Che08], this notion is called group determinant.

Definition 4.2.1. Let R be an A-algebra, G be a group, and $d \geq 1$.

(1) A pseudorepresentation of dimension d, denoted $D : R \to A$, or (R, D) is an element of $\mathcal{M}_A^d(R, A)$.

(2) A pseudorepresentation of G, denoted $D : G \to A$ is a pseudo-representation of $A[G]$.

(3) If $D : R \to A$ is a pseudo-representation, and $x \in R$, we define the characteristic polynomial $\chi_D(x,t) \in A[t]$ by $\chi_D(x,t) = D_{A[t]}(t - x)$

Let $D : R \to A$ and $D' : R' \to A'$ be pseudorepresentations. A morphism of pseudorepresentations $\rho : (R, D) \to (R', D')$ is the data of a pair (f, g) where $f : A \to A'$ is a ring homomorphism, and $g : R \otimes_A A' \to A'$ is an A'-algebra homomorphism such that $f \circ D = D' \circ g$.

Note that $\chi_D(x,t)$ is the image under $R[t,t'] \to R[t]$, sending t' to x, of the polynomial $D_{A[t,t']}(t-t') = \sum_{i=0}^d D^{[i]}(t')(-t')^i t^{d-i}$ where $D^{[0]}(t') = 1$ (as seen via the map $R[T,t'] \to R[T]$ sending t' to 0 and by multiplicativity of D). Therefore, we can write:

$$\chi_D(x,t) = \sum_{i=0}^{d} (-1)^i \Lambda_i(x) t^{d-i}$$

This defines A-polynomial laws $\Lambda_i : R \to A$ of degree i, for $0 \leq i \leq d$, where $\Lambda_0 = 1$ and $\Lambda_d = D$. We define the trace of D by $\operatorname{Tr} D = \Lambda_1$.

We let $\det_A(R,d) : \mathbf{Alg}_A \to \mathbf{Sets}$ be the covariant functor associating to any commutative A-algebra B, the set of B-valued pseudorepresentations $R \otimes_A B \to B$ of dimension d. By Corollary 4.1.8, we have that $\mathcal{M}_A^d(R,B) \cong \mathcal{M}_B^d(R \otimes_A B, B)$. So actually, this functor sends B to the set of homogenious multiplicative A-polynomial laws $R \to B$ of dimension d. But thanks to Theorem 4.1.11, this is equivalent to giving an A-algebra homomorphism $\Gamma_A^d(R) \to B$, which factors through $\Gamma_A^d(R)^{\mathrm{ab}}$ by commutativity of B. Hence, we get that:

Proposition 4.2.2. *The functor $\det_A(R,d)$ is represented by the A-algebra $\Gamma_A^d(R)^{ab}$.*

If G is a group, we denote $\Gamma_{\mathbb{Z}}^d(\mathbb{Z}[G])^{\mathrm{ab}}$ by $\mathbb{Z}(G,d)$.

4.2.1 Universal polynomial identities

Let X be a totally ordered finite set, and let X^+ be the monoid of words with letters in this set equipped with the induced total lexicographic order. We say that a word $w \in X^+$ is a Lyndon word if w is less or equal any of its rotations (or equivalently if $w = xw'$ with $x \in X^+$, then $w \leq w'$). We denote by $\mathcal{L}_X$ the set of Lyndon words on X. By the Chen-Fox-Lyndon theorem, any word $w \in X^+$ can be uniquely factored into a Lyndon decomposition

$$w = w_1^{l_1} \dots w_s^{l_s}, \quad w_i \in \mathcal{L}_X \text{ with } w_1 > w_2 > \dots > w_s$$

There is a unique function $\epsilon : X^+ \to \{\pm 1\}$ which is defined by $\epsilon(w) = (-1)^{\ell(w)-1}$ if $w \in \mathcal{L}_X$, and by $\epsilon(w) = \prod_{i=1}^s \epsilon(w_i)^{l_i}$ if $w \in X^+$ and $w = w_1^{l_1} \dots w_s^{l_s}$ is its Lyndon factorisation.

Proposition 4.2.3. *[Che08, 1.12] Let A be a ring, R be an A-algebra, and $d \geq 1$. Consider $D : R \to B$ a homogenious of degree d polynomial law into a commutative A-algebra B, and let $\Lambda_{i,B} : R \to B$ be the induced characteristic polynomial coefficient polynomial law. Then, we have the following polynomial identities*

(1) For all $r, r' \in R$, $D(1 + rr') = D(1 + r'r)$.

(2) The Λ_i satisfy Amistur's formula, i.e., for any finite subset $X = \{r_1, \ldots, r_n\} \subseteq R$, totally ordered by the indices, we have

$$\Lambda_{i,A}(r_1 + \cdots + r_n) = \sum_{\ell(w)=i} \epsilon(w)\Lambda(w)$$

where $\Lambda(w) := \Lambda_{l_s}(w_s) \cdots \Lambda_{l_1}(w_1)$, with $w = w_1^{l_1} \ldots w_s^{l_s}$ is the Lyndon decomposition of $w \in X^+$.

(3) Tr *satisfies the d-dimensional pseudocharacter identity.*

Proof. **1)** First note that if r is invertible, then by commutativity of B and multiplicativity of D we get the result, since

$$D(1 + rr') = D(r)D(r^{-1} + r') = D(r^{-1} + r')D(r) = D(1 + r'r)$$

Now for the general case, let us work in $R[t]$, and set $r = 1 + u$. If we show that

$$D(1 + r(1 + ut)) = D(1 + (1 + ut)r) \in B[t]$$

then specializing to $t = 1$ gives us the result. Now since both polynomials are of degree $\leq d$, it suffices to show the equality in $B[t]/(t^{d+1})$. But $(1 + tu)$ is invertible in $R[t]/(t^{d+1})$ so we can apply the previous case.
2) Let $r_1, \ldots, r_n \in R$, and consider the A-algebra $A_m = A[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^m$. We have the following equality in $R \otimes_A A_m$:

$$\begin{aligned}
\frac{1}{1 - (t_1 r_1, \ldots, t_n r_n)} &= 1 + (t_1 r_1, \ldots, t_n r_n) + \cdots + (t_1 r_1, \ldots t_n r_n)^{m-1} \\
&= \sum_{w \in X^+,\ \ell(w) < m} w \\
&= \prod_{w \in \mathcal{L}_X,\ \ell(w) < m} (1 + w + \cdots + w^{m-1}) = \prod_{w \in \mathcal{L}_X,\ \ell(w) < m} \frac{1}{1 - w}
\end{aligned}$$

where $X = \{t_1 r_1, \ldots, t_n r_n\}$, and the last product is taken in the decreasing lexicographic order. Note that the third equality follows from the existence and unicity of the Lyndon decomposition. Applying D and inverting, we get

$$D(1 + \sum_{i=1}^{n} t_i r_i) = \prod_{w \in \mathcal{L}_X} \chi_D(w, 1) = \prod_{w \in \mathcal{L}_X} \big(\sum_{i=0}^{d} (-1)^i \Lambda_i(w)\big) \tag{4.3}$$

where we now take the product over all Lyndon words, (we previously restricted ourselves to words of length $< m$ because the determinant commutes only to *finite* products). This equation does not depend on m, hence it holds in $B[[t_1, \ldots, t_n]]$. If we take an integer $i \geq 0$, the homogenious part of the equality of degree i writes as

$$\Lambda_i(t_1 r_1 + \cdots + t_n r_n) = \sum_{\ell(w)=i} \epsilon(w)\Lambda(w) \tag{4.4}$$

Indeed for each word w in the sum, we have $\ell(w) = i = \sum_{k=1}^{s} l_k \ell(w_k)$, where $w = w_1^{l_1} \cdots w_s^{l_s}$ is the Lyndon decomposition of w. Hence, $\epsilon(w) = (-1)^{\sum_k l_k - i}$. The equality in (4.4) holds a priori in $B[[t_1, \ldots, t_n]]$, but both sides live inside $B[t_1, \ldots, t_n]$, so it is also an equality in $B[t_1, \ldots, t_n]$. Sending each t_i to 1 gives us the desired formula in B.
3) Applying equation (4.4) with $i = n = d + 1$, we get

$$\prod_{w \in \mathcal{L}_X} \big(\sum_{i=0}^{d} (-1)^i \Lambda_i(w) \big) = 0$$

If we consider the component which is homogenious of degree 1 in each t_j, then we are actually taking the sum over the words of length $d+1$ that are written with distinct letters, so each w of these words correspond to a permutation $\sigma \in \mathfrak{S}_{d+1}$ with $\epsilon(w) = \epsilon(\sigma)$. The equation that we obtain this way correspond to the d-dimensional character identity for $\Lambda_1 = \mathrm{Tr}$. $\square$

Corollary 4.2.4. *Let D be an A-valued determinant on a group Γ of dimension d, and $B \subseteq A$ the subgring generated by the coefficients $\Lambda_i(\gamma)$ of $\chi_D(\gamma, t)$ for all $\gamma \in \Gamma$. Then, D factors through a unique B-valued determinant on Γ of dimension d.*

Proof. We need to show that for all $\gamma_1, \ldots, \gamma_n \in \Gamma$, $D(\gamma_1 t_1 + \cdots + \gamma_n t_n) \in B[t_1, \ldots, t_n]$. But by Amitsur's formula (4.4), such a determinant is a signed sum of monomials in $\Lambda_i(w)$ where w is a word in $\gamma_1, \ldots, \gamma_n$, in particular $w \in \Gamma$, hence the result. $\square$

Remark 4.2.5. One interesting and useful fact is that the polynomial identities between the $\lambda_i(w)$ (where w is a word in elements of R) which hold for the determinant of a matrix algebra, also hold for a general pseudorepresentation. To see this, let X be a set, and consider the $\mathbb{Z}$-algebra $F_X(d) = \mathbb{Z}[x_{i,j} \ : \ 1 \leq i, j \leq d, x \in X]$. We have a universal representation:

$$\rho^{\mathrm{univ}} : \mathbb{Z}\{X\} \to F_X(d)$$

defined by $x \mapsto (x_{i,j})_{i,j}$. By Corollary 4.2.4, we get a pseudorepresentation:

$$\det \circ \rho^{\mathrm{univ}} : \mathbb{Z}\{X\} \to E_X(d)$$

where $E_X(d)$ is the subring of $F_X(d)$ generated by the coefficients of the polynomials of the $\rho^{\mathrm{univ}}(w)$ for $w \in \mathbb{Z}\{X\}$. This induces an isomorphism $\Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\})^{\mathrm{ab}} \cong E_X(d)$ ([Che08, Theorem 1.15]).

It is also a fact that if X is finite, then $E_X(d)$ is actually a finite type $\mathbb{Z}$-algebra (see [Che08, §2.7]). But if G is a finitely generated group, with set of generators $X \subseteq G$, and A is a commutative ring, we have a surjective map

$$A \otimes_{\mathbb{Z}} \Gamma^d_{\mathbb{Z}}(\mathbb{Z}\{X\}) \twoheadrightarrow \Gamma^d_A(A[G])$$

so we get that $\Gamma^d_A(A[G])$ is a finite type A-algebra.

4.2.2 Deformation theory of pseudorepresentations

Before we dig into the deformation theory of pseudorepresentations, we need to be able to say when a pseudorepresentation is continuous. For this, we have the following definition:

Definition 4.2.6. Let G be a topological group, and A a topological ring. A pseudorepresentation $D : A[G] \to A$ of dimension d is said to be continuous if for each $\alpha \in I_d$, the map $D^{[\alpha]} : G^d \to A$ is continuous.

Note that by Amitsur's formula, D is continuous if and only if $\Lambda_i : G \to A$ is continuous for all $i \leq d$.

For our purposes, we suppose that G is profinite. So in the case where A is equipped with the discrete topology, D is continuous if and only if the characteristic polynomial map

$$\begin{aligned} G &\to A[t] \\ g &\mapsto D(1+tg) \end{aligned}$$

factors through $G \to G/H$ for some normal open subgroup H. For such a subgroup, we define the ideal

$$J(H) := \ker(A[G] \to A[G/H])$$

and we equip $A[G]$ with the topology generated by this set of ideals.

Lemma 4.2.7. *A B-valued determinant D on G, viewed as an element $P \in \mathcal{D}_A^d(A[G], B)$, is continuous if and only if, $\ker(P) \subseteq A[G]$ is open for the topology defined as above. In this case, the natural representation*

$$G \to (B[G]/\ker(D))^{\times}$$

factors through a finite quotient G/H of G for some normal open subgroup H.

Proof. If $J(H) \subseteq \ker(P)$, then by Lemma 4.1.3, P factors through $\widetilde{P} \in \mathcal{M}_A^d(A[G/H], B)$ and D is obviously continuous. Conversely, suppose that D is continuous. Given that B is discrete and G is profinite, there is a normal open subgroup H of G such that $\Lambda_i : G \to B$ factor through G/H. Hence, by Amitsur's formula, we get for $g \in G, h \in H$:

$$D(t(g-gh) + \sum_i t_i g_i) = D(\sum_i t_i g_i)$$

which means that $g - gh \in \ker(P)$, and $J(H) = \sum_{g \in G, h \in H} Ag(h-1) \subseteq \ker(P)$. $\square$

Let us fix a profinite group G, a prime number p, and a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Recall that $\mathcal{C}_{\mathcal{O}}^0$ is the category of Artinian local $\mathcal{O}$-algebras with residue field k, and we let $\widehat{\mathcal{C}_{\mathcal{O}}}$ be the category of pro-Artinian local $\mathcal{O}$-algebras with residue field k whose morphisms are local $\mathcal{O}$-algebra homomorphisms. We fix a continuous pseudorepresentation of dimension d

$$\overline{D} : k[G] \to k$$

and we denote $\mathrm{Def}_{\overline{D}} : \widehat{\mathcal{C}_{\mathcal{O}}} \to \mathbf{Sets}$ for the functor which associates to $A \in \widehat{\mathcal{C}_{\mathcal{O}}}$ the set of continuous pseudorepresentations D of G over A such that $D \otimes_A k = \overline{D}$.

Lemma 4.2.8. *If $A = \varprojlim_i A_i$ for $A_i \in \widehat{\mathcal{C}_{\mathcal{O}}}$, then the natural map $\mathrm{Def}_{\overline{D}}(A) \to \varprojlim_i \mathrm{Def}_{\overline{D}}(A_i)$ is an isomorphism.*

Proof. Since the functor $\mathcal{M}_{\mathcal{O}}^d(\mathcal{O}[G], -)$ from $\mathcal{O}$-algebras to sets is representable, it commutes with any projective limit. Hence, the lemma follows from the fact that a map $G \to \varprojlim_i A_i$ is continuous if and only if $G \to A_i$ is continuous for each i. $\square$

Proposition 4.2.9. *The functor $\mathrm{Def}_{\overline{D}}$ is representable by a ring $R_{\overline{D}} \in \widehat{\mathcal{C}_{\mathcal{O}}}$.*

Proof. We let $B = (\Gamma^d_{\mathcal{O}}(\mathcal{O}[G]))^{\mathrm{ab}}$ representing the functor $\det_{\mathcal{O}}(\mathcal{O}[G], d)$, and we consider the universal multiplicative polynomial law $P^u : \mathcal{O}[G] \to B$ (see Proposition 4.2.2). Let $\psi : B \to k$ be the $\mathcal{O}$-algebra morphism corresponding to the pseudorepresentation $\overline{D}$.
We say that an ideal $I \subseteq B$ is open if $I \subseteq \ker\psi$, B/I is a finite local ring, and the induced multiplicative law $P_I : \mathcal{O}[G] \to B/I$ is continuous. Note that if I, J are two open ideals, then so is $I \cap J$. Indeed, we have an injection $B/(I \cap J) \hookrightarrow B/I \times B/J$ which is a homeomorphism (since everything is discrete). Therefore, these ideals define a topology for which they form a basis. We consider the completion of B for this topology:

$$R_{\overline{D}} = \varprojlim_{I \text{ open}} B/I$$

and the pseudorepresentation $P(\overline{D}) = \iota \circ P^u : \mathcal{O}[G] \to R_{\overline{D}}$ where $\iota : B \to R_{\overline{D}}$ is the natural map. Then, $R_{\overline{D}} \in \widehat{\mathcal{C}}_{\mathcal{O}}$, and by Lemma 4.2.8 we have:

$$P(\overline{D}) = (P_I)_I \in \mathrm{Def}_{\overline{D}}(R_{\overline{D}}) = \varprojlim_{I \text{ open}} \mathrm{Def}_{\overline{D}}(B/I)$$

Now if $A \in \mathcal{C}^0_{\mathcal{O}}$, and $P \in \mathrm{Def}_{\overline{D}}(A)$, then there is a unique $\mathcal{O}$-algebra morphism $\psi_A : B \to A$ such that $\psi_A \mod \mathfrak{m}_A = \psi$ and $P = \psi_A \circ P^u$. Hence, $\ker\psi_A \subseteq \ker\psi$, and $B/\ker\psi_A \subseteq A$ is necessarily finite local. By continuity of P, we get that $\ker\psi_A$ is open, hence the result. ☐

Remark 4.2.10. Suppose that G is topologically finitely generated, and let $H \subseteq G$ be a finitely generated dense subgroup. By definition of the continuity of pseudorepresentations, the natural transformation:

$$\mathrm{Def}_{\overline{D}} \to \mathrm{Def}_{\overline{D}_{|H}}$$

is injective. In particular, we have $\mathrm{Def}_{\overline{D}}(k[\epsilon]) \subseteq \mathrm{Def}_{\overline{D}_{|H}}(k[\epsilon])$, which implies that

$$\dim_k(\mathfrak{m}_{R_{\overline{D}}}/\mathfrak{m}^2_{R_{\overline{D}}}) \leq \dim_k(\mathfrak{m}_{R_{\overline{D}|H}}/\mathfrak{m}^2_{R_{\overline{D}|H}})$$

But by construction, $R_{\overline{D}|H}$ is topologically generated by $\Gamma^d_{\mathcal{O}}(\mathcal{O}[H])$ which is a finite $\mathcal{O}$-algebra by Remark 4.2.5. Therefore, $\dim_k(\mathfrak{m}_{R_{\overline{D}|H}}/\mathfrak{m}^2_{R_{\overline{D}|H}}) < \infty$, which implies that $R_{\overline{D}}$ is topologically finitely generated.

Lemma 4.2.11. *Let $A \in \widehat{\mathcal{C}}_{\mathcal{O}}$, and let $D : G \to k$ be a continuous pseudorepresentation deforming $\overline{D}$. Then, D factors through $A[G] \to A[G/H]$ where $H \subseteq J := \ker(\overline{\rho} : G \to \mathrm{GL}_d(\overline{k}))$ is the smallest closed normal subgroup such that J/H is pro-p.*

Proof. We need to verify that $D(T - gh) = D(T - g)$ for all $g \in G$, $h \in H$. This can be checked on the finite quotients of A, so we can assume that A is finite. By Lemma 4.2.7, we can assume that D factors through a finite quotient G'. The ring $B := A[G']/\ker(D) = A[G]/\ker(D)$ is finite, and we have an induced pseudorepresentation

$$\overline{D}' : B \otimes_A k/\ker(\overline{D}) \to k$$

such that $\overline{D} = \overline{D}' \circ (k[G] \to B \otimes_A k/\ker(\overline{D}))$. By [Che08, Theorem 2.16], $B \otimes_A k/\ker(\overline{D})$ is a finite semisimple k-algebra, so extending the scalars to $\overline{k}$, we get by the unicity in [BC09, Theorem 2.12] the following commutative diagram

$$\begin{tikzcd} k[G], \arrow[r] \arrow[d, equal] & B \otimes_A k/\ker(\overline{D}) \arrow[d, hook] \\ k[G] \arrow[r, "\overline{\rho}"] & \mathrm{M}_d(\overline{k}) \end{tikzcd}$$

This shows that the image of $J - 1$ in B lies in $\ker\big(B \to B \otimes_A k/\ker(\overline{D})\big)$, which is equal to the Jacobson radical $\mathrm{Rad}(B)$ of B by [Che08, Lemma 2.10]. Hence, the image of $J \to B^\times$ lies in the p-group $1 + \mathrm{Rad}(B)$ which gives us our result. (Note that the difficulty was in the fact that we don't necessarily have $\ker(\overline{\rho}) \subseteq \ker(D)$). □

Proposition 4.2.12. *If G satisfies Mazur's Φ_p condition, then $R_{\overline{D}} \in \mathcal{C}_{\mathcal{O}}$.*

Proof. If G is topologically finitely generated, then the result was already established in Remark 4.2.10. The general case follows from this by considering Lemma 4.2.11. Indeed, if $\mathrm{Def}^*_{\overline{D}} : \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$ is the deformation functor of the determinant of $\overline{\rho}$ seen as a G/H representation, then as a consequence of this lemma, the natural functor $\mathrm{Def}^*_{\overline{D}} \to \mathrm{Def}_{\overline{D}}$ is an equivalence. Moreover, the condition Φ_p exactly implies that G/H is topologically finitely generated. □

4.2.3 Cayley-Hamilton representations

If G is a group and $D : G \to A$ is a pseudorepresentation, it is not always true that it can be written of the form $D = \det \circ \rho$ where $\rho : G \to \mathrm{GL}_n(A)$ is a representation of G, or equivalently that there is a morphism of pseudorepresentations $(A[G], D) \to (\mathrm{M}_n(A), \det)$. However, one can always find a morphism of pseudorepresentations $(A[G], D) \to (E, D')$ where (E, D') is a Cayley-Hamilton pseudorepresentation. This is good enough for us since a Cayley-Hamilton pseudorepresentation behaves well under many operations, and as the name suggests, satisfies the Cayley-Hamilton theorem. In some sense, it can be though of as a generalisation of the pseudorepresentation $(\mathrm{M}_n(A), \det)$.

Definition 4.2.13. We call a pseudorepresentation $D : E \to A$ Cayley-Hamilton when E is finitely generated as an A-algebra, and, for every commutative A-algebra B, and every $x \in E \otimes_A B$, the element x satisfies the characteristic polynomial $\chi_D(x,t) \in B[t]$. In this case, we call the pair (E, D) a Cayley-Hamilton A-algebra.

Remark 4.2.14. If $D : E \to A$ is a pseudorepresentation, we denote by $\mathrm{CH}(D) \subseteq E$ the two sided ideal of E generated by the coefficients of the polynomial

$$\chi_D(t_1 r_1 + \cdots + t_n r_n) \in R[t_1, \ldots, t_n]$$

for $r_1, \ldots, r_n \in R$, $n \geq 1$. Then, we see that (E, D) is Cayley-Hamilton if and only if $\mathrm{CH}(D) = 0$. Also by [Che08, Lemma 1.21], we have that $\mathrm{CH}(D) \subseteq \ker(D)$. In particular, $(E/\mathrm{CH}(D), \widetilde{D})$ is a Cayley-Hamilton A-algebra.

Remark 4.2.15. If (E, D) is a Cayley-Hamilton A-algebra, then E is finitely generated as an A-module (see [WWE19, Proposition 2.1.7]).

If G is a group, we define a Cayley-Hamilton representation of G of dimension d to be triple $(A, (E, D), \rho)$ where A is a commutative ring, (E, D) is a Cayley-Hamilton A-algebra of dimension d and $\rho : G \to E^\times$ is a group homomorphism.
A morphism $(A, (E, D), \rho) \to (A', (E', D'), \rho')$ of Cayley-Hamilton representations is a morphism of pseudorepresentations $(E, D) \to (E', D')$ such that $\rho' = (E \to E') \circ \rho$.
We let $\mathcal{CH}_d(G)$ be the category of Cayley-Hamilton representations of G of dimension d with morphisms as we just defined.

Recall that by Proposition 4.2.2, we have a universal determinant of dimension d:

$$D^u : \mathbb{Z}(G, d)[G] \to \mathbb{Z}(G, d)$$

We define the universal Cayley-Hamilton algebra to be

$$R(G,d) = \mathbb{Z}(G,d)[G]/\ker(D^u)$$

which is equipped with a natural group homomorphism $\rho^u : G \to R(G,d)^\times$.
Using the universality of $\mathbb{Z}(G,d)$, it is not hard to see that:

Proposition 4.2.16. *The Cayley-Hamilton representation $(\mathbb{Z}(G,d), (R(G,d), D^u), \rho^u)$ is the initial object in $\mathcal{CH}_d(G)$.*

Deformation of Cayley-Hamilton representations

We keep the notations of subsection 4.2.2, and assume moreover that G satisfies Mazur's Φ_p-condition. Recall that this implies that the universal deformation ring $R_{\overline{D}}$ lies in $\mathcal{C}_{\mathcal{O}}$.

A Cayley-Hamilton pseudorepresentation $(A, (E, D), \rho)$ of G over $A \in \mathcal{C}_{\mathcal{O}}$ has residual representation $\overline{D}$ if its induced pseudorepresentation $D \circ \rho : G \to A$ has residual representation $\overline{D}$. We let $\mathcal{CH}_d(G, \overline{D})$ be the full subcategory of $\mathcal{CH}_d(G)$ whose objects have residual representation $\overline{D}$.

The universal continuous pseudodeformation of $\overline{D}$:

$$D^u_{\overline{D}} : \mathcal{O}[[G]] \otimes_{\mathcal{O}} R_{\overline{D}} \to R_{\overline{D}}$$

induces the universal Cayley-Hamilton algebra

$$E_{\overline{D}} := \big(\mathcal{O}[[G]] \otimes_{\mathcal{O}} R_{\overline{D}}\big)/\mathrm{CH}(D^u_{\overline{D}})$$

And as in Proposition 4.2.16, we have the following fact (see [WWE19, 2.2.10]):

Proposition 4.2.17. *The Cayley-Hamilton representation $(R_{\overline{D}}, (E_{\overline{D}}, D^u_{E_{\overline{D}}}), \rho^u)$ is an initial object in $\mathcal{CH}_d(G, \overline{D})$. In particular, $E_{\overline{D}}$ is a finitely generated $R_{\overline{D}}$-module. The map $\rho^u : R_{\overline{D}}[G] \to E_{\overline{D}}$ is surjective, and $D^u_{E_{\overline{D}}} \to R_{\overline{D}}$ is a factorisation of the universal pseudorepresentation $D^u_{\overline{D}} : G \to R_{\overline{D}}$ through $E_{\overline{D}}$.*

Deformation conditions

As in the case of deformations of Galois representations, we would like to impose certain deformation conditions. For instance in [Ram93], the author observed that on the Artinian level, a deformation defines an element belonging to the category of continuous $\mathbb{Z}_p[G]$-modules whose objects are finite as sets, which we denote by $\mathbf{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$. And if $\mathcal{P} \subset \mathbf{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$ is a full subcategory which preserved under isomorphisms, subquotients and finite direct sums in $\mathbf{Mod}^{\mathrm{fin}}_{\mathbb{Z}_p[G]}$, he was able to prove that the functor of deformations which belong to this subcategory is representable. We say that $\mathcal{P}$ is a stable condition.
In [WWE19], the authors were able to extend this result to the case of pseudorepresentations. The main task is to find a nice way to attach a finite $\mathbb{Z}_p[G]$-module to a pseudorepresentation. Here is where Cayley-Hamilton representations come into play as we will now see.

For $A \in \mathcal{C}_{\mathcal{O}}$, we extend the definition of a stable condition $\mathcal{P}$ to $A[G]$-modules which may not be finite sets as follows: for an $A[G]$-module which is finitely generated as an A-module, we say that M satisfies condition $\mathcal{P}$ if $M/\mathfrak{m}_A^i M$ satisfies $\mathcal{P}$ for all $i \geq 1$.
This way, we can impose a deformation condition on Cayley-Hamilton representations by setting:

Definition 4.2.18. If $A \in \mathcal{C}_{\mathcal{O}}$, and $(A,(E,D),\rho)$ is a Cayley-Hamilton representation of G over A with residual representation $\overline{D}$, we say that $(A,(E,D),\rho)$ satisfies the condition $\mathcal{P}$ if E satisfies $\mathcal{P}$ as an $A[G]$-module (it is a finitely generated A-module by Remark 4.2.15).

We let $\mathcal{CH}_d^{\mathcal{P}}(G,\overline{D})$ be the full subcategory of $\mathcal{CH}_d(G,\overline{D})$ whose objects satisfy condition $\mathcal{P}$. As explained in [WWE19, §2.4 & §2.5], we have the following result:

Proposition 4.2.19. *There is a Cayley-Hamilton representation $(R_{\overline{D}}^{\mathcal{P}},(E_{\overline{D}}^{\mathcal{P}},D_{E_{\overline{D}}^{\mathcal{P}}}),\rho^{\mathcal{P}})$ which is universal in $\mathcal{CH}_d^{\mathcal{P}}(G,\overline{D})$. In other words, a Cayley-Hamilton algebra $(A,(E,D),\rho)$ satisfies condition $\mathcal{P}$ if and only if there exists a morphism of Cayley-Hamilton algebras*

$$(R_{\overline{D}}^{\mathcal{P}},(E_{\overline{D}}^{\mathcal{P}},D_{E_{\overline{D}}^{\mathcal{P}}}),\rho^{\mathcal{P}}) \to (A,(E,D),\rho)$$

Now we are able to define the condition $\mathcal{P}$ on pseudorepresentations.

Definition 4.2.20. Let $A \in \mathcal{C}_{\mathcal{O}}$, and $D : G \to A$ a pseudorepresentation with residual pseudorepresentation $\overline{D}$. Then, we say that D satisfies condition $\mathcal{P}$ if there exists a Cayley-Hamilton representation $(A,(E,D'),\rho)$ satisfying $\mathcal{P}$ such that $D = D' \circ \rho$.

We define the $\mathcal{P}$-pseudodeformation functor $\mathrm{Def}_{\overline{D}}^{\mathcal{P}} : \mathcal{C}_{\overline{O}} \to$ **Sets** by sending $A \in \mathcal{C}_{\mathcal{O}}$ to the set of pseudodeformations $D : G \to A$ of $\overline{D}$ satisfying $\mathcal{P}$.

Theorem 4.2.21. *The functor $\mathrm{Def}_{\overline{D}}^{\mathcal{P}}$ is represented by a ring $R_{\overline{D}}^{\mathcal{P}} \in \mathcal{C}_{\mathcal{O}}$.*

Proof. Let $A \in \mathcal{C}_{\mathcal{O}}$ and $D \in \mathrm{Def}_{\overline{D}}(A)$. If $\varphi_D : R_{\overline{D}} \to A$ is the morphism induced by D, we need to show that D satisfies condition $\mathcal{P}$ if and only if φ_D factors through $R_{\overline{D}} \twoheadrightarrow R_{\overline{D}}^{\mathcal{P}}$.
If φ_D factors through $R_{\overline{D}} \twoheadrightarrow R_{\overline{D}}^{\mathcal{P}}$, then the Cayley-Hamilton algebra

$$\big(A,(E_{\overline{D}}^{\mathcal{P}} \otimes_{R_{\overline{D}}^{\mathcal{P}}} A, D_{E_{\overline{D}}^{\mathcal{P}}} \otimes_{R_{\overline{D}}^{\mathcal{P}}} A), \rho^{\mathcal{P}} \otimes_{R_{\overline{D}}^{\mathcal{P}}} A\big)$$

satisfies condition $\mathcal{P}$ by Proposition 4.2.19, and induced D via $(R_{\overline{D}}^{\mathcal{P}} \to A) \circ D_{E_{\overline{D}}^{\mathcal{P}}} \circ \rho^{\mathcal{P}}$.
On the other hand, assume that D satisfies condition $\mathcal{P}$, i.e., there exists a Cayley-Hamilton representation $(A,(E,D'),\rho)$ such that $D = D' \circ \rho$. By Proposition 4.2.19, there exists a morphism of pseudorepresentations $(E_{\overline{D}}^{\mathcal{P}}, D_{E_{\overline{D}}^{\mathcal{P}}}) \to (E,D)$ such that $\rho = (E_{\overline{D}}^{\mathcal{P}} \to E) \circ \rho^{\mathcal{P}}$. In particular, the implicit morphism of rings $R_{\overline{D}}^{\mathcal{P}} \to A$ factors φ_D. $\square$

4.3 Pseudocharacters

Pseudocharacters were first introduced by A.Wiles for GL_2, and later were generalized to GL_n by R.Taylor, in order to construct some Galois representations with certain properties. Given a group Γ, a pseudocharacter of Γ with coefficients in a ring A is a function $T : \Gamma \to A$ that satisfies certain conditions making it behave similarly to the trace function of a representation $\rho : \Gamma \to \mathrm{GL}_n(A)$. The exact conditions are the following:

- $T(1) = n$;
- For all $\gamma_1, \gamma_2 \in \Gamma$, $T(\gamma_1\gamma_2) = T(\gamma_2\gamma_1)$;
- For $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$,
$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \epsilon(\sigma) T^{\sigma}(\gamma_1, \dots, \gamma_{n+1}) = 0$$
where $T^{\sigma}(\gamma_1, \dots, \gamma_{n+1}) = T(\gamma_{i_1} \cdots \gamma_{i_{n+1}})$ if σ is the cycle $(i_1, \dots, i_{n+1})$, and in general $T^{\sigma} = \prod_i T^{c_i}$ if $\sigma = c_1 \cdots c_r$ is the cycle decomposition of σ.

As one might expect, the trace function of a representation $\rho : \Gamma \to A$ is a pseudocharacter, and the converse holds in various situations such as when A is an algebraically closed field and $n! \in A^\times$.
In [Laf18, §11], the author introduced a new notion of pseudocharacters adapted for reductive groups. We give the definition in the special case of the general linear group.

Definition 4.3.1. A pseudocharacter of Γ of dimension n over a ring A is a collection $\Theta = (\Theta_m)_{m\geq 1}$ of algebra homomorphisms $\Theta_m : \mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n} \to \mathrm{Map}(\Gamma^m, A)$ satisfying the following conditions:

(1) For all $k, l \geq 1$, and each map $\zeta : \{1, \ldots, k\} \to \{1, \ldots, l\}$, and each $f \in \mathbb{Z}[\mathrm{GL}_n^k]^{\mathrm{GL}_n}$, and each $\gamma_1, \ldots, \gamma_l \in \Gamma$, we have

$$\Theta_l(f^\zeta)(\gamma_1, \ldots, \gamma_l) = \Theta_k(f)(\gamma_{\zeta(1)}, \ldots, \gamma_{\zeta(k)})$$

where $f^\zeta(g_1, \ldots, g_l) = f(g_{\zeta(1)}, \ldots, g_{\zeta(k)})$.

(2) For each $k \geq 1$, for each $\gamma_1, \ldots, \gamma_{k+1} \in \Gamma$, and for each $f \in \mathbb{Z}[\mathrm{GL}_n^k]^{\mathrm{GL}_n}$, we have

$$\Theta_{k+1}(\widehat{f})(\gamma_1, \ldots, \gamma_{k+1}) = \Theta_k(f)(\gamma_1, \ldots, \gamma_k\gamma_{k+1})$$

where $\widehat{f}(g_1, \ldots, g_{k+1}) = f(g_1, \ldots, g_k g_{k+1})$.

Just to clarify things, $\mathbb{Z}[\mathrm{GL}_n^m]$ is the algebra of regular function on the $\mathbb{Z}$-group scheme GL_n^m, on which GL_n acts by conjugation on each coordinate.
For each $1 \leq i \leq n$, let $\lambda_i \in \mathbb{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n}$ be the function defined by the equation

$$\det(X - g) = \sum_{i=0}^{n} (-1)^i \lambda_i(g) X^{n-i}$$

We have that $\lambda_i(g) = \mathrm{Tr}(\bigwedge^i g)$ where $\bigwedge^i g$ is the i-th exterior power of g. By [Don92, §3.1], for any $m \geq 1$, $\mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ is generated as a ring by the functions

$$(g_1, \ldots, g_m) \mapsto \lambda_i(g_{i_1} \cdots g_{i_r})$$

for $r \in \mathbb{N}$, $1 \leq i_1, \ldots, i_r \leq m$, together with $(g_1, \ldots, g_m) \mapsto \det^{-1}(g_1, \ldots, g_m)$. So if t is a pseudocharacter, the functions

$$t^{[i]} := t_1(\lambda_i) : \Gamma \to A \tag{4.5}$$

for $0 \leq i \leq n$ determine t. Indeed, by the axioms defining a pseudocharacter, we have

$$t_m\big(\lambda_i(g_{i_1} \cdots g_{i_r})\big)(\gamma_1, \ldots, \gamma_n) = t_1\big(\lambda_i(g)\big)(\gamma_{i_1} \cdots \gamma_{i_r}) \tag{4.6}$$

By a result of Procesi [Pro76, Theorem 1.3], the algebra $\mathbb{Q}[\mathrm{GL}_n]^{\mathrm{GL}_n} \cong \mathbb{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n} \otimes_{\mathbb{Z}} \mathbb{Q}$ (by [Don92, §3.1]) is generated by the functions

$$T_k : g \mapsto \mathrm{Tr}(g^k)$$

together with the inverse of the determinant.So if A is $\mathbb{Z}$-torsion free, a pseudocharacter t is determined by the functions

$$t_1(T_k) : \gamma \to A$$

or even just $t_1(\mathrm{Tr})$. Indeed, by the axioms defining a pseudocharacter, we have

$$t_1(T_k)(\gamma) = t_1(T_1)(\gamma^k) \tag{4.7}$$

On the other hance, since $\bigwedge^{n+1}\mathrm{St} = 0$ where St is the standard representation of GL_n, we get that the following function

$$(g_1, \dots, g_{n+1}) \mapsto \mathrm{Tr}_{\mathrm{St}^{\otimes n+1}}\left(\big(\sum_{\mathfrak{S}_{n+1}} \epsilon(\sigma)\sigma\big)(g_1 \otimes \cdots \otimes g_{n+1}) \right) \tag{4.8}$$

is zero (the linear map considered is zero on the symmetric tensors). But one can check that

$$\mathrm{Tr}_{\mathrm{St}^{\otimes n+1}}\left(\sigma \cdot (g_1 \otimes \cdots \otimes g_{n+1})\right) = \prod_{(i_1,\dots,i_k) \text{ cycle of } \sigma} \mathrm{Tr}(g_{i_k} \cdots g_{i_1})$$

So developing in (4.8) and applying t_1, we get that

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \epsilon(\sigma) t_1(\mathrm{Tr})^{\sigma}(\gamma_1, \dots, \gamma_{n+1}) = 0$$

which is exactly the relation for the definition of a pseudocharacter by R. Taylor. In fact, if A is a $\mathbb{Q}$-algebra, both definitions of R. Taylor and V. Lafforgue give the same thing.

Remark 4.3.2. We can define operations of twisting and duality on the pseudocharacters which are compatible with the usual operations on representations. For instance, consider the involution $\iota : \mathrm{GL}_n \to \mathrm{GL}_n$ defined by ${}^{\iota}g := {}^{t}g^{-1}$, then for a pseudocharacter t, we define its dual $t^{\vee}$ by the formula:

$$t_m^{\vee}(f)(\gamma_1, \dots, \gamma_n) = t_m(f')(\gamma_1, \dots, \gamma_n)$$

where $f'(g_1, \dots, g_m) = f(\iota g_1, \dots, \iota g_n)$. Moreover, if $\chi : \Gamma \to A^{\times}$ is a character, we can define the twist $t \otimes \chi$ of a pseudocharacter t by the formula:

$$(t \otimes \chi)_m(f)(\gamma_1, \dots, \gamma_n) = f'(\gamma_1, \dots, \gamma_m)$$

where $f' \in A[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ is defined by $f'(g_1, \dots, g_n) = f(\chi(\gamma_1)g_1, \dots, \chi(\gamma_n)g_n)$.

If we work with a topological group Γ and a topological ring A, we have a notion of continuity for pseudocharacters.

Definition 4.3.3. Let $t = (t_m)_{\geq 1}$ be a pseudocharacter. We say that t is continuous if for each $m \geq 1$, t_m takes values in the set $\mathrm{Maps}_{\mathrm{cont}}(\Gamma^m, A)$ of continuous functions $\Gamma^m \to A$.

The following theorem proved in [Eme18] says that the notions of pseudocharacter and pseudorepresentations we defined are equivalent. Therefore, we will use them throughout the thesis interchangeably.

Theorem 4.3.4. *For any group Γ and ring A, the pseudocharacters t of dimension n are in canonical bijection with the group determinants D of dimension n. Under this bijection, t is associated to D if and only if $t^{[i]} = D^{[i]}$ for each $0 \leq i \leq n$, and t is continuous if and only if D is.*

The purpose of introducing Lafforgue's pseudocharacters is that in [NT20, §2], the authors are able to prove that if $\rho : \Gamma \to \mathrm{GL}_n(\mathcal{O})$ is a continuous homomorphism which is absolutely irreducible over E, and $t = (t_m)_{m \geq 1} = \mathrm{Tr}\,\rho$ is the pseudocharacter associated to ρ, then deforming ρ infinitesimally is not far from deforming t. More concretely, let $A = \mathcal{O} \oplus \epsilon E/\mathcal{O}$ and let $\alpha_k : A \to A$ be the $\mathcal{O}$-algebra homomorphism sending ϵ to $p^k\epsilon$, then we have the following result:

Proposition 4.3.5. *[NT20, 2.9] There exists an integer $k_0 \geq 0$, depending only on $\rho(\Gamma)$ such that:*

(1) For any lifting t' of t to A, there exists a homomorphism $\rho' : \Gamma \to \mathrm{GL}_n(A)$ lifting ρ such that $\mathrm{Tr}\,\rho' = \alpha_{k_0} \circ t'$. Moreover, if t' is continuous, we can take ρ' to be continuous as well.

(2) If $\rho'_1, \rho'_2 : \Gamma \to \mathrm{GL}_n(A)$ are two liftings of ρ with $\mathrm{Tr}\,\rho'_1 = \mathrm{Tr}\,\rho'_2$, then $\alpha_{k_0} \circ \rho'_1$ and $\alpha_{k_0} \circ \rho'_2$ are conjugate under the action of the group $1 + \epsilon M_n(E/\mathcal{O}) \subset \mathrm{GL}_n(A)$; and if $X \in M_n(E/\mathcal{O})$ is such that $1 + \epsilon X$ centralizes ρ'_1, then $p^{k_0}X$ is a scalar matrix.

4.4 Galois deformation theory of pseudocharacters

We fix again a prime number p, and a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. We also consider a CM extension F/F^+ of a totally real field.
We let S be a finite set of places of F containing S_p (the set of places above p), and we assume that each place of S splits in F^+. So we fix for each $v \in S$, a choice of a place $\widetilde{v}$ of F above it, and we let $\widetilde{S} = \{\widetilde{v} \mid v \in S\}$ and $\widetilde{S}_p = \{\widetilde{v} \mid v \in S_p\}$.
We consider a Galois representation $\rho : G_{F,S} \to \mathrm{GL}_n(\mathcal{O})$ satisfying the following properties:

- $\rho \otimes_{\mathcal{O}} E$ is absolutely irreducible,
- For each $\widetilde{v} \in \widetilde{S}_p$, $\rho_{|G_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} E$ is semistable with Hodge-Tate weights in the interval $[a, b]$.

As in [CHT08, §2.1], we let $\mathcal{G}_n$ be the group scheme over $\mathbb{Z}$ defined as the semi-direct product of $\mathcal{G}_n^0 = \mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, \jmath\}$ acting on $\mathcal{G}_n^0$ by

$$\jmath(g, \mu)\jmath^{-1} = (\mu \cdot {}^t g^{-1}, \mu)$$

The adjoint action of $\mathcal{G}_n$ on $\mathfrak{g} = \mathrm{Lie}\,\mathrm{GL}_n$ is given by

$$(\mathrm{ad}(g, \mu))(x) = gxg^{-1}$$

and

$$(\mathrm{ad}(\jmath))(x) = -{}^t x$$

We let $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ be the homomorphism sending (g, μ) to μ and $\jmath$ to -1.
If R is a ring and $r : G_{F^+} \to \mathcal{G}_n(R)$ is a homomorphism with $r^{-1}(\mathcal{G}_n^0(R)) = G_F$, by abuse of notation we denote $r_{|G_F}$ the composition of $r_{|G_F}$ with the projection $\mathcal{G}_n^0(R) \to \mathrm{GL}_n(R)$.

We suppose that there exits a character $\chi : G_{F,S} \to \mathcal{O}^\times$ such that $\rho^c \cong \rho^\vee \otimes \chi$, then by [CHT08, Lemma 2.1.4], ρ extends to a continuous representation:

$$r : G_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$$

such that $\chi = \nu \circ r$. Note that $\chi_{|F_{\widetilde{v}}}$ is semistable, and there exists $w \in \mathbb{Z}$ such that $\chi\epsilon^w$ has finite order (where ϵ is the cyclotomic character). We assume that $w = a + b$.
We write $W = \mathrm{ad}\, r$, $W_E = W \otimes_{\mathcal{O}} E$, $W_{E/\mathcal{O}} = W_E/W$, and $W_m = W \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$.

Let $\overline{D} : G_{F,S} \to \mathcal{O}$ be the pseudorepresentation associated to $\overline{\rho}$, and we consider the deformation functor $\mathrm{Def}_{\overline{D},S} : \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$ which is represented by a ring $R_{\overline{D},S} \in \mathcal{C}_{\mathcal{O}}$ (see Proposition 4.2.12). The following lemma gives us control over the size of the deformation rings when we will later add various Taylor-Wiles sets.

Lemma 4.4.1. *Fix an integer $q \geq 0$. There exists an integer $g_0 = g_0(S, \overline{D}, q)$ such that for any set of finite places Q of F outside of S with $|Q| = q$, there exists a surjection*

$$\mathcal{O}[[X_1, \ldots, X_{g_0}]] \twoheadrightarrow R_{\overline{D}, S\cup Q}$$

Proof. Let L/F be the extension cut out by $\overline{\rho}$ (it is finite), and let $M_{S\cup Q}$ be the maximal pro-p extension of L unramified outside $S \cup Q$. We want to show that $\mathrm{Gal}(M_{S\cup Q}/L)$ is topologically finitely generated by g_1 elements, where g_1 only depends on q. This amounts to showing (see for example [Gou95, Lemma 2.1]) that

$$\mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(M_{S\cup Q}/L), k) \cong \mathrm{Hom}_{\mathrm{cont}}(G_{L,S\cup Q}, k) \cong H^1(G_{L,S\cup Q}, k)$$

is bounded dependently only on q. We get this by noticing that in the exact sequence

$$0 \to H^1(G_{L,S\cup Q}, k) \to H^1(G_{L,S}, k) \to \bigoplus_{\widetilde{v}|v\in Q} H^1(I_{L_{\widetilde{v}}}, k)$$

the size of $H^1(I_{L_{\widetilde{v}}}, k)$ does not depend on $\widetilde{v}$, since $\widetilde{v} \nmid p$ so that each morphism $I_{L_{\widetilde{v}}} \to k$ factors through the tame quotient $I_{L_{\widetilde{v}}} \twoheadrightarrow \mathbb{Z}_p$.
Now by Proposition 4.2.12, any deformation of $\overline{D}$ to $G_{F,S\cup Q}$ factors through $\mathrm{Gal}(M_{S\cup Q}/L)$. We conclude using the argument in Remark 4.2.10. $\square$

Let us now fix integers $a \leq b$, and let $\mathcal{E}_{F,S}^{[a,b]}$ be the category of finite cardinality $\mathbb{Z}_p[G_{F,S}]$-modules M such that for each place $\widetilde{v} \in \widetilde{S}_p$, M is isomorphic as a $\mathbb{Z}_p[G_{F_{\widetilde{v}}}]$-module to a subquotient of a lattice in a semistable Galois representation of $G_{F_{\widetilde{v}}}$ with Hodge-Tate weights in $[a, b]$. This defines a stable condition, in the sense of (4.2.3), and the corresponding subfunctor $\mathrm{Def}_{\overline{D},S}^{[a,b]} \subseteq \mathrm{Def}_{\overline{D},S}$ is represented by an object $R_{\overline{D},S}^{[a,b]} \in \mathcal{C}_{\mathcal{O}}$ (see Theorem 4.2.21).
If $D : G_{F,S} \to \mathcal{O}$ is the pseudorepresentation associated to ρ, then by hypothesis on ρ, D determines a homomorphism $R_{F,S}^{[a,b]} \to \mathcal{O}$, we write $\mathfrak{q}$ for its kernel. We define Selmer group $H^1_{\mathcal{E}_{F,S}^{[a,b]}}(F, W_m)$ by considering the following local conditions:

- If $\widetilde{v} \in \widetilde{S}_p$, we take the subspace of $H^1(F_{\widetilde{v}}, W_m)$ corresponding to self-extensions of $\rho_{|G_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$ which are subquotients of lattices in semistable representations with Hodge-Tate weights in the interval $[a, b]$.
- if $\widetilde{v} \notin \widetilde{S}_p$ we do not impose any condition.

The following proposition is a consequence of Proposition 4.3.5.

Proposition 4.4.2. *There exists a canonical homomorphism*

$$\mathrm{tr}_m : H^1_{\mathcal{E}_{F,S}^{[a,b]}}(F, W_m) \to \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}/\mathfrak{q}^2, \mathcal{O}/\varpi^m) \tag{4.9}$$

Moreover, there exists a constant $c \geq 1$ depending only on ρ such that for any $m \geq 1$, the kernel and cokernel of tr_m are both annihilated by p^c.

Proof. Let $A_m = \mathcal{O} \oplus \epsilon\varpi^{-m}\mathcal{O}/\mathcal{O} \subseteq A$ and let $\alpha_k : A_m \to A_m$ as in Proposition 4.3.5. As in Lemma 2.1.2, a class $[\phi] \in H^1_{\mathcal{E}_{F,S}^{[a,b]}}(F, W_m)$ corresponds to an equivalence class of liftings

$$\rho_\phi : G_{F,S} \to \mathrm{GL}_n(A_m)$$

such that $\rho_\phi \mod \epsilon = \rho$ and for each $N \geq 1$, $\rho_\phi \mod \varpi^N \in \mathcal{E}_{F,S}^{[a,b]}$. On the other hand, $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}/\mathfrak{q}^2, \mathcal{O}/\varpi^m)$ identifies with the preimage in

$$\mathrm{Hom}_{\mathcal{O}}(R_{\overline{D},S}^{[a,b]}, A_m) \to \mathrm{Hom}_{\mathcal{O}}(R_{\overline{D},S}^{[a,b]}, \mathcal{O})$$

of the classifying morphism of D. We define tr_m to be the map sending $[\phi]$ to the classifying map of the pseudocharacter $\mathrm{tr}\,\rho_\phi$ in $\mathrm{Hom}_{\mathcal{O}}(R_{\overline{D},S}^{[a,b]}, A_m)$ which lies in $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}/\mathfrak{q}^2, \mathcal{O}/\varpi^m)$ under the above identification. Note that $\rho_{p^k\phi} = \alpha_k \circ \rho_\phi$ and that if $\varphi_f \in \mathrm{Hom}_{\mathcal{O}}(R_{\overline{D},S}^{[a,b]}, A_m)$ is the image of $f \in \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}/\mathfrak{q}^2, \mathcal{O}/\varpi^m)$, then $\varphi_{p^k f} = \alpha_k \circ \varphi_f$.

We want to find a power of p which kills $\ker \operatorname{tr}_m$ and which only depends on ρ. So let $[\phi]$ be in the kernel of tr_m, which means that $\operatorname{tr} \rho_\phi = \operatorname{tr} \rho$. By (2) of Proposition 4.3.5, there exists a constant k_0 which depends only on $\rho(G_{F,S}) \subseteq \mathrm{GL}_n(\mathcal{O})$, and an element $X \in \mathrm{M}_n(E/\mathcal{O})$ such that

$$(1+\epsilon X)\rho_{p^{k_0}\phi}(1-\epsilon X) = \rho$$

this means that $p^{k_0}\phi$ is a coboundary in $H^1(F, W_{E/\mathcal{O}})$. But from the short exact sequence

$$0 \to W_m \to W_{E/\mathcal{O}} \xrightarrow{\times \varpi^m} W_{E/\mathcal{O}} \to 0 \tag{4.10}$$

we get that the kernel of $H^1(F, W_m) \to H^1(F, W_{E/\mathcal{O}})$ is isomorphic to $H^0(F, W_{E/\mathcal{O}}) \otimes \mathcal{O}/\varpi^m$. The latter is killed by some p^{c_0} where $c_0 \in \mathbb{N}$ is independent of m (all the ϖ-divisible elements are killed after tensoring with $\mathcal{O}/\varpi^m$). Therefore, $\ker \operatorname{tr}_m$ is killed by $p^{k_0+c_0}$. We may assume without loss of generality that r is surjective

Now we want to do the same thing for $\operatorname{coker} \operatorname{tr}_m$. So let D' be an element in the right hand side of (4.9). By hypothesis, there exists a Cayley-Hamilton representation (B, D'', r) of $G_{F,S}$ such that $D' = D'' \circ r$ and that the finite quotients of B lie in $\mathcal{E}_{F,S}^{[a,b]}$.
By (1) of Proposition 4.3.5, there exists a homomorphism $\rho_\phi : G_{F,S} \to \mathrm{GL}_n(A)$ such that $\alpha_{k_0} \circ D' = \operatorname{tr} \rho_\phi$. But since $\operatorname{tr} \rho_{\varpi^m \phi} = \beta_m \circ D' = \operatorname{tr} \rho$ where $\beta_m : A \to A$ is the ring homomorphism sending ϵ to $\varpi^m \epsilon$, we get from (2) of the same proposition that $[\phi] \in H^1(F, W_{E/\mathcal{O}})$ is killed by multiplication by $\varpi^m p^{k_0}$. So by the exact sequence in (4.10), the element $p^{k_0}[\phi]$ lies in the image of $H^1(F, W_m)$ in $H^1(F, W_{E/\mathcal{O}})$. Therefore, we can assume that there exists a representation $\rho_\phi : G_{F,S} \to \mathrm{GL}_n(A_m)$ such that $\operatorname{tr} \rho_\phi = \alpha_{2k_0} \circ D'$.

It remains to show that there exists a constant $c_1 \in \mathbb{N}$ such that $\alpha_{c_1} \circ \rho_\phi \mod \varpi^N$ lies in $\mathcal{E}_{F,S}^{[a,b]}$ (in this case $p^{c_1}[\phi]$ is in the preimage under (4.9) of the morphism associated to $\alpha_{2k_0+c_1} \circ D'$).
So let $\mathcal{A}_\phi = \rho_\phi(A_m[G_{F,S}]) \subseteq \mathrm{M}_n(A_m)$. By Burnside's lemma (Corollary 2.3.2), since $\rho \otimes_{\mathcal{O}} E$ is absolutely irreducible, $\rho(E[G_{F,S}]) = \mathrm{M}_n(E)$, so $\rho(\mathcal{O}[G_{F,S}])$ is a lattice inside $\mathrm{M}_n(E)$ which means that there exists an integer $k_1 \geq 0$ such that $p^{k_1}\mathrm{M}_n(\mathcal{O}) \subseteq \rho(\mathcal{O}[G_{F,S}])$. Let us show that $\mathcal{A}_{p^{k_1}\phi}$ contains $p^{k_1}\mathrm{M}_n(A_m)$: since $p^{k_1}E_{i,j} \in \rho(\mathcal{O}[G_{F,S}])$ ($E_{i,j}$ is the usual elementary matrix in $\mathrm{M}_n(\mathcal{O})$), there exists a matrix $X_{i,j} \in \mathrm{M}_n(\mathcal{O}/\varpi^m)$ such that $p^{k_1}E_{i,j} + \epsilon X_{i,j} \in \mathcal{A}_\phi$, then applying α_{k_1}, we get that $p^{k_1}E_{i,j} + \epsilon p^{k_1} X_{i,j} \in \mathcal{A}_{p^{k_1}\phi}$. Multiplying by ϵ, we get that $\epsilon p^{k_1} E_{i,j} \in \mathcal{A}_{p^{k_1}\phi}$ for all (i,j), in particular, $\epsilon p^{k_1} X_{i,j} \in \mathcal{A}_{p^{k_1}\phi}$ so that $p^{k_1}E_{i,j} \in \mathcal{A}_{p^{k_1}\phi}$ as desired.
Let $D''' : \mathcal{A}_{p^{k_1}\phi} \to A_m$ be the determinant induced by the inclusion $\mathcal{A}_{p^{k_1}\phi} \hookrightarrow \mathrm{M}_n(A_m)$. If $x \in \ker D'''$, then by (1) of Lemma 4.1.3, $\mathrm{Tr}(xp^{k_1}E_{i,j}) = 0$ where Tr is the usual trace on matrices. Thus, $\ker D'''$ is contained in $\mathrm{M}_n(A_m)[p^{k_1}]$ and so is annihilated by the homomorphism $\alpha_{k_1} : \mathrm{M}_n(A_m) \to \mathrm{M}_n(A_m)$. It follows that there exists a commutative diagram of A_m-algebras:

$$\begin{tikzcd}
A_m[G_{F,S}] \arrow[r, two heads, "\rho_{p^{k_1}\phi}"] \arrow[d, two heads, "r"] & \mathcal{A}_{p^{k_1}\phi} \arrow[r, hook] \arrow[d] & \mathrm{M}_n(A_m) \arrow[d, "\alpha_{k_1}"] \\
B \arrow[r, two heads] & A_m[G_{F,S}]/\ker(\alpha_{2k_0+k_1} \circ D') \arrow[r] & \mathrm{M}_n(A_m)
\end{tikzcd}$$

where the quotient map $A_m[G_{F,S}] \to A_m[G_{F,S}]/\ker(\alpha_{2k_0+k_1} \circ D')$ factors through B since $\ker(r) \subseteq \ker(D') \subseteq \ker(\alpha_{2k_0+k_1} \circ D')$ (by (1) of Lemma 4.1.3) and the bottom right arrow exists because $\alpha_{p^{k_1+2k_0}} \circ D = D''' \circ \rho_{p^{k_1}\phi}$. From this diagram, we get that $\mathrm{M}_n(A_m)$ equipped with the action of $A_m[G_{F,S}]$ induced by $\rho_{p^{2k_1}\phi}$ has finite quotients which lie in $\mathcal{E}_{F,S}^{[a,b]}$ (since this hold for the finite quotients of B). Therefore, we get that $\alpha_{2k_1} \circ \rho_\phi \mod \varpi^N \in \mathcal{E}_{F,S}^{[a,b]}$. In conclusion, we get that the cokernel of tr_m is annihilated by $p^{2k_1+2k_0}$. $\square$

Let R_S be the quotient of $R^{[a,b]}_{\overline{D},S}$ corresponding to pseudocharacters D' such that $(D')^c = (D')^\vee \otimes \chi_{|G_{F,S}}$. Then, ρ determines a morphism $R_S \to \mathcal{O}$, and we write $\mathfrak{q}_S$ for its kernel.

We define local Selmer conditions $\mathcal{L}_S = \{\mathcal{L}_v\} = \{\mathcal{L}_{v,m}\}$ for W_m by setting:

- If $v \notin S$, then $\mathcal{L}_{v,m}$ is the unramified subgroup of $H^1(F_v^+, W_m)$.
- If $v \in S - S_p$, then $\mathcal{L}_{v,m}$ is the whole space $H^1(F_v^+, W_m)$.
- If $v \in S_p$, then $\mathcal{L}_{v,m}$ is the subspace of $H^1(F_v^+, W_m)$ corresponding to self-extensions of $\rho_{G_{F_{\tilde{v}}}} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^m$ which are subquotients of lattices in semistable representations with Hodge-Tate weights in $[a,b]$.

The dual Selmer conditions $\mathcal{L}_S^\perp = \{\mathcal{L}_v^\perp\} = \{\mathcal{L}_{v,m}^\perp\}$ are defined to be the duals of $\mathcal{L}_S$ under Tate's duality. Thus, we have the following Selmer groups:

$$H^1_{\mathcal{L}_S}(F^+, W_m) = \ker\left(H^1(F^+, W_m) \to \prod_v H^1(F_v^+, W_m)/\mathcal{L}_{v,m}\right)$$
$$H^1_{\mathcal{L}_S^\perp}(F^+, W_m(1)) = \ker\left(H^1(F^+, W_m(1)) \to \prod_v H^1(F_v^+, W_m(1))/\mathcal{L}_{v,m}^\perp\right)$$

All the finiteness results that we will implicitly assume on the Galois cohomology groups follow from the following proposition, which is due to J.Tate, and whose proof can be found in [Bel09].

Proposition 4.4.3. *Let G be a profinite group which satisfies Mazur's Φ_p-condition, V be a continuous representation of G, and $\Lambda \subseteq V$ be a G-stable $\mathcal{O}$-lattice in V. Then, we have the following:*

(1) The continuous cohomology group $H^i(G, \Lambda)$ (with Λ given its ϖ-adic topology) is a finite $\mathcal{O}$-module, and we have a canonical isomorphism

$$H^i(G, V) \cong H^i(G, \Lambda) \otimes_{\mathcal{O}} E$$

(2) We have a canonical isomorphism $H^i(G, \Lambda) = \varprojlim H^i(G, \Lambda/\varpi^n\Lambda)$.

In particular, the Selmer groups defined are finite length $\mathcal{O}$-modules. We denote their respective lengths by $h^1_{\mathcal{L}_S}(F^+, W_m)$ and $h^1_{\mathcal{L}_S^\perp}(F^+, W_m(1))$.
We let
$$H^1_{\mathcal{L}_S}(F^+, W_E) = \left(\varprojlim_m H^1_{\mathcal{L}_S}(F^+, W_m)\right) \otimes_{\mathcal{O}} E$$
where the inverse limit is taken with respect to the projections $W_{m+1} \to W_m$, and

$$H^1_{\mathcal{L}_S}(F^+, W_{E/\mathcal{O}}) = \varinjlim_m H^1_{\mathcal{L}_S}(F^+, W_m)$$

where the direct limit is taken with respect to the injections $W_m \cong \varpi W_{m+1} \subset W_{m+1}$.

Proposition 4.4.4. *For each $m \geq 1$, there is a canonical morphism:*

$$\mathrm{tr}_{m,S} : H^1_{\mathcal{L}_S}(F^+, W_m) \to \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_S/\mathfrak{q}_S^2, \mathcal{O}/\varpi^m) \tag{4.11}$$

Moreover, there exists a constant $d \geq 0$ depending only on r such that p^d annihilates the kernel and cokernel of $\mathrm{tr}_{m,S}$.

Proof. The non-trivial element $c \in \mathrm{Gal}(F/F^+)$ acts on $H^1_{\mathcal{E}^{[a,b]}_{F,S}}(F, W_m)$ via its action on W_m. On the other hand, we also have an action of c on $R^{[a,b]}_{\overline{D},S}$ given by sending a pseudorepresentation D' to $(D')^{c,\vee} \otimes \chi_{|G_{F,S}}$ (note that the condition $w = a+b$ ensures that this pseudorepresentation satisfies the condition $\mathcal{E}^{[a,b]}_{F,S}$). This gives an action of c on the right-hand-side of (4.9) and we have that

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_S/\mathfrak{q}_S^2, \mathcal{O}/\varpi^m) = \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}/\mathfrak{q}^2, \mathcal{O}/\varpi^m)^c$$

The map tr_m is c-equivariant, and we get the map $\mathrm{tr}_{m,S}$ by composing $f : H^1_{\mathcal{L}_S}(F^+, W_m) \to H^1_{\mathcal{E}^{[a,b]}_{F,S}}(F, W_m)^c$ with tr_m. By the inflation-restriction exact sequence, the kernel and cokernel of the map f lie respectively in $H^1(\mathrm{Gal}(F/F^+), W_m^{G_{F,S}})$ and $H^2(\mathrm{Gal}(F/F^+), W_m^{G_{F,S}})$ which sizes are bounded independently of m by absolute irreducibility of ρ. □

Proposition 4.4.5. *(1) There is an isomorphism* $\mathrm{tr}_{E,S} : H^1_{\mathcal{L}_S}(F^+, W_E) \to \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_S/\mathfrak{q}_S^2, E)$*,*

(2) The natural map $H^1_{\mathcal{L}_S}(F^+, W_E) \to H^1(F_S/F^+, W_E)$ *identifies* $H^1_{\mathcal{L}_S}(F^+, W_E)$ *with the geometric Selmer group* $H^1_{g,S}(F^+, W_E)$*.*

(3) Assume that for each $v \in S$*,* $\rho_{|G_{F_{\tilde{v}}}}$ *is generic. Then* $H^1_{g,S}(F^+, W_E) = H^1_f(F^+, W_E)$*.*

Proof. **1)** This follows by taking the inverse limit of (4.11) over m and then inverting p.
2) By the main result in [Liu07], $H^1_{\mathcal{L}_S}(F^+, W_E)$ classifies semistable self-extensions of ρ_E. But by [Nek93, Corollary 1.27], a de Rham self-extension of $\rho \otimes_{\mathcal{O}} E$ is semistable. Hence the result.
3) This follows from [All14, Lemma 1.1.7] and the equality

$$\dim_{\mathbb{Q}_p} H^1_{g,S}(F^+, W_E) = \dim_{\mathbb{Q}_p} H^1_f(F^+, W_E) + \dim_{\mathbb{Q}_p} D_{\mathrm{crys}}(W_E(1))^{\varphi=1}$$

□

Lemma 4.4.6. *For* $m' \geq m$*, the inverse image of* $\mathcal{L}_{v,m'}$ *in* $H^1(F_v^+, W_m)$ *under the map*

$$H^1(F_v^+, W_m) \to H^1(F_v^+, W_{m'})$$

induced by the injection $W_m \hookrightarrow W_{m'}$ *is* $\mathcal{L}_{v,m}$*. Consequently, the natural map*

$$H^1_{\mathcal{L}_S}(F^+, W_m) \to H^1_{\mathcal{L}_S}(F^+, W_{E/\mathcal{O}})[\varpi^m]$$

is surjective.

Proof. The natural map $H^1(F_v^+, W_m) \to H^1(F_v^+, W_{m'})$ corresponds to the push-forward of a lift with values in $\mathrm{GL}_n(A_m)$ to a lift with values in $\mathrm{GL}_n(A_{m'})$ via the map $A_m \to A_{m'}$ sending ϵ to $\varpi^{m'-m}\epsilon$. Now $A^n_{m'}$ is finitely generated as an A^n_m-module, so there is a surjective map of $\mathcal{O}[G_{F_{\tilde{v}}}]$-modules $(A^n_m)^k \twoheadrightarrow A_{m'}$. So in case $v \in S_p$, by the fact that $\mathcal{E}^{[a,b]}_{F,S}$ is stable under direct sums and quotients, we see that the semi-stability condition is satisfied.

Now we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1_{\mathcal{L}_S}(F^+, W_m) & \longrightarrow & H^1(F_S/F^+, W_m) & \longrightarrow & \bigoplus_{v \in S_p} H^1(F_v^+, W_m)/\mathcal{L}_{v,m} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1_{\mathcal{L}_S}(F^+, W_{E/\mathcal{O}})[\varpi^m] & \longrightarrow & H^1(F_S/F^+, W_{E/\mathcal{O}})[\varpi^m] & \longrightarrow & \bigoplus_{v \in S_p} \varinjlim_{m'} H^1(F_v^+, W'_m)/\mathcal{L}_{v,m'}
\end{array}$$

where the top row is exact by definition of the local Selmer conditions, and the bottom row is also exact since it comes from taking the colimit of the top row for varying m (cohomology commutes with colimits). Looking at the long exact sequence of cohomology, we see that the middle vertical arrow is surjective. Moreover, by what we proved above, the rightmost vertical arrow is surjective. Therefore, by a diagram chase, we get that the leftmost vertical arrow is surjective as desired. $\square$

4.4.1 Taylor-Wiles primes

Since we do not assume that $\overline{\rho}$ is absolutely irreducible, we need to introduce an adapted notion of a Taylor-Wiles place.

Definition 4.4.7. If Q is a finite place of F^+ and $N \geq 1$, we say that Q is a set of Taylor-Wiles places of level N if it satisfies the following conditions:

- $Q \cap S = \emptyset$
- For each $v \in Q$, $v = ww^c$ splits in F, and $\rho(\mathrm{Frob}_w)$ has n distinct eigenvalues $\alpha_{w,1}, \dots, \alpha_{w,n} \in \mathcal{O}$.
- For each $v \in Q$, $q_v \equiv 1 \mod p^N$.

We say that a tuple $(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \dots, \alpha_{\widetilde{v},n}))$ is a Taylor-Wiles datum of level N if Q is a set Taylor-Wiles places of level N, $\widetilde{Q}$ is a set consisting of a choice, for each $v \in Q$, of a place $\widetilde{v}$ of F above v, and $(\alpha_{\widetilde{v},1}, \dots, \alpha_{\widetilde{v},n})$ is a choice of ordering of the eigenvalues of $\rho(\mathrm{Frob}_{\widetilde{v}})$.

Lemma 4.4.8. *Suppose that the following conditions are satisfied:*

(1) For each $v \in S$, $\rho_{|G_{F_{\widetilde{v}}}}$ is generic.

(2) For each place $v \nmid \infty$, $\chi(c_v) = -1$.

then there exists $d \geq 0$ with the following property: for every $N \geq 1$, every Taylor-Wiles datum $(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \dots, \alpha_{\widetilde{v},n}))$, and every $1 \leq m \leq N$, we have

$$h^1_{\mathcal{L}_{S\cup Q}}(F^+, W_m) \leq d + h^1_{\mathcal{L}^{\perp}_{S\cup Q}}(F^+, W_m(1)) + mn|Q| + \sum_{v\in Q}\sum_{i\neq j} \mathrm{ord}_{\varpi}(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j})$$

Proof. By the Greenberg-Wiles formula (which is the equivalent of (2.3) without the simplifications in that case), we have

$$\begin{aligned} h^1_{\mathcal{L}_{S\cup Q}}(F^+, W_m) =& h^1_{\mathcal{L}^{\perp}_{S\cup Q}}(F^+, W_m(1)) + h^0(F^+, W_m) - h^0(F^+, W_m(1)) \\ &+ \sum_{v\in S\cup Q} \big(\ell_{v,m} - h^0(F_{\widetilde{v}}, W_m)\big) - \sum_{v|\infty} \ell((1+c_v)W_m) \end{aligned} \tag{4.12}$$

where $\ell_{v,m} = \ell(\mathcal{L}_{v,m})$ with ℓ denoting the length of an $\mathcal{O}$-module. For an infinite place v, the description of the cohomology of a cyclic group shows that

$$\ell((1+c_v)W_m) = \ell(W_m^{c_v}) - h^2(F_v^+, W_m)$$

Looking at the action of c_v, we see that $W_m^{c_v}$ consists of anti-symmetric matrices (up to an inner automorphism); moreover, $h^2(F_v^+, W_m)$ can be bounded independently of m. Thus, the contribution of the infinite places to (4.12) equals to $[F^+ : \mathbb{Q}]m\frac{n(n-1)}{2}$ up to a uniformly bounded error.
Now since ρ is absolutely irreducible, we have that $H^0(F^+, W) = 0$, so we get that $H^0(F^+, W_m) = H^1(F^+, W)[\varpi^m]$. But $H^1(F^+, W)$ is a finitely generated module, so $h^0(F^+, W_m)$ can be bounded uniformly. Similarly, since ρ is generic, $H^0(F^+, W(1)) = 0$ ([All14, Lemma 1.1.5]) and we can bound

$h^0(F^+, W_m)$ uniformly.
Now if $v \in Q$, then by the formula for the Euler characteristic and the local Tate duality, we have that

$$\ell_{v,m} - h^0(F_{\widetilde{v}}, W_m) = h^2(F_{\widetilde{v}}, W_m) = h^0(F_{\widetilde{v}}, W_m(1))$$

But since $q_v \equiv 1 \mod p^N$ and $N \geq m$, the action of $\mathrm{Frob}_{\widetilde{v}}$ on W_m via the cyclotomic character is trivial, so we have $h^0(F_{\widetilde{v}}, W_m(1)) = h^0(F_{\widetilde{v}}, W_m)$. The latter is bounded from above by $mn + \sum_{i \neq j} \mathrm{ord}_{\varpi}(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j})$.

Finally, suppose that $v \in S_p$. We consider the following lifting functor

$$\mathcal{D}_v^{\square,[a,b]} : \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$$

sending $A \in \mathcal{C}_{\mathcal{O}}$ to the set of lifts of $\overline{\rho}_{|G_{F_{\widetilde{v}}}}$ to $\mathrm{GL}_n(A)$ whose projections to Artinian quotients are lattices in semistable representations with Hodge-Tate weights in $[a, b]$. This functor is represented by $R_v^{\square,[a,b]} \in \mathcal{C}_{\mathcal{O}}$, and the representation $\rho_{|G_{F_{\widetilde{v}}}}$ determines a morphism $R_v^{\square,[a,b]} \to \mathcal{O}$. If $\mathfrak{q}_v$ denotes the kernel of this homomorphism, then as in the proof of Proposition 4.9, we have that $\mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_v/\mathfrak{q}_v^2, \mathcal{O}/\varpi^m) \cong (\mathfrak{q}_v/\mathfrak{q}_v)^\vee \otimes \mathcal{O}/\varpi^m$ equals to the preimage of $\mathcal{L}_{m,v}$ under the map $Z^1(F_{\widetilde{v}}, W_m) \to H^1(F_{\widetilde{v}}, W_m)$ (since we are considering the *lifting* functor). Therefore, we get that

$$\ell_{v,m} - h^0(F_{\widetilde{v}}, W_m) = \ell(\mathfrak{q}_v/\mathfrak{q}_v^2 \otimes \mathcal{O}/\varpi^m) - mn^2$$

To finish the proof, we want to show that $\ell_{v,m} - h^0(F_{\widetilde{v}}, W_m) - [F^+ : Q]m\frac{n(n-1)}{2}$ can be bounded independently of m. This will be achieved if we can show that $\dim_E(\mathfrak{q}_v/\mathfrak{q}_v^2 \otimes_{\mathcal{O}} E) = n^2 + [F_v^+ : \mathbb{Q}_p]\frac{n(n-1)}{2}$ which we will do now.

So let $\mathcal{C}_E$ be the category of local Noetherian E-algebras with residue field E. We follow [Kis09, (2.3)] and define for $B \in \mathcal{C}_E$ the category Int_B whose objects are finitely generated $\mathcal{O}$-subalgebras $A \subseteq E$ such that $A \otimes_{\mathcal{O}} E = B$ such that $b(A) = \mathcal{O}$, where $b : B \to E$ is the canonical projection. The morphisms in this category are given by the natural inclusion, and we note that Int_B is ordered by inclusion. We define $\mathcal{C}'_{\mathcal{O}}$ to be the category consisting of $\mathcal{O}$-algebras A in $\mathcal{C}_{\mathcal{O}}$ equipped with a map of $\mathcal{O}$-algebras $A \to \mathcal{O}$. In particular, Int_B is a subcategory of $\mathcal{C}'_{\mathcal{O}}$.
We have a functor

$$\mathcal{D}_{v,(\rho)}^{\square,[a,b]} : \mathcal{C}'_{\mathcal{O}} \to \mathbf{Sets}$$

sending a couple $(A, A \to \mathcal{O})$ to the set of representations $\rho_A \in \mathcal{D}_v^{\square,[a,b]}(A)$ such that $\rho = (A \to \mathcal{O}) \circ \rho_A$. This functor allows us to define another functor:

$$\mathcal{D}_{v,(\rho)}^{\square,[a,b]} : \mathcal{C}_E \to \mathbf{Sets}$$

by setting for $B \in \mathcal{C}_E$:

$$\mathcal{D}_{v,(\rho)}^{\square,[a,b]}(B) = \varinjlim_{A \in \mathrm{Int}_B} \mathcal{D}_{v,(\rho)}^{\square,[a,b]}(A, b : A \to \mathcal{O})$$

Now any representation $\rho' \in \mathcal{D}_{v,(\rho)}^{\square,[a,b]}(B)$ is induced by a map $\varphi_{\rho'} : R_v^{\square,[a,b]} \to A$ for some $A \in \mathrm{Int}_B$ with $b \circ \varphi_{\rho'} = \varphi_\rho$. Thus, by localisation and then completion, $\varphi_{\rho'}$ extends to a continuous morphism $(R_v^{\square,[a,b]})^\wedge_{\mathfrak{q}_v} \to B$. Conversely, any continuous $(R_v^{\square,[a,b]})^\wedge_{\mathfrak{q}_v} \to B$ sends $(R_v^{\square,[a,b]})^\wedge_{\mathfrak{q}_v}$ to a compact subring A of B, so that $A \in \mathrm{Int}_B$. Therefore, we see that $(R_v^{\square,[a,b]})^\wedge_{\mathfrak{q}_v}$ represents the functor $\mathcal{D}_{v,(\rho)}^{\square,[a,b]}$. On the other hand, we can define another functor:

$$\mathcal{D}_{v,\rho}^{\square,[a,b]} : \mathcal{C}_E \to \mathbf{Sets}$$

sending $B \in \mathcal{C}_E$ to the set of lifts of $\rho_{|G_{F_{\widetilde{v}}}} \otimes_{\mathcal{O}} E$ to $\mathrm{GL}_n(B)$ whose Artinian quotients are semistable with all Hodge-Tate weights in $[a,b]$. Thanks to [Liu07, Conjecture 1.0.1], we have a natural transformation:

$$\mathcal{D}_{v,(\rho)}^{\square,[a,b]} \to \mathcal{D}_{v,\rho}^{\square,[a,b]}$$

which is actually a natural isomorphism. Indeed, let $B \in \mathcal{C}_E$ and $A^\circ \in \mathrm{Int}_B$. Set $\mathfrak{n} = \ker(b : B \to E)$ and $\mathfrak{n}^\circ = \mathfrak{n} \cap A^\circ$ and consider for each $n \geq 1$ the algebra:

$$A_n^\circ = \sum_{j=1}^{\infty} p^{-nj} (\mathfrak{n}^\circ)^j + A^\circ$$

which lies in Int_B. We have that $b^{-1}(\mathcal{O}) = \bigcup_{n \geq 1} A_n^\circ$, and since a representation $\rho_B \in \mathcal{D}_{v,\rho}^{\square,[a,b]}(B)$ factors by definitions through $b^{-1}(\mathcal{O})$, it must factor through A_n° for some sufficiently large n (by compactness of $G_{F_{\widetilde{v}}}$).

In conclusion, we get that $(R_v^{\square,[a,b]})^\wedge_{\mathfrak{q}_v}$ represents the functor $\mathcal{D}_{v,\rho}^{\square,[a,b]}$; which by [All14, Theorem 1.2.7], implies that $(R_v^{\square,[a,b]})^\wedge_{\mathfrak{q}_v}$ is formally smooth of dimension $n^2 + [F_v^+ : \mathbb{Q}_p]\frac{n(n-1)}{2}$ which gives us the result. $\square$

Lemma 4.4.9. *Consider a finitely generated $\mathcal{O}$-module M, and let $N \geq 1$ and $d, g \geq 0$ be integers. Suppose that for all $m \leq N$, we have:*

$$\ell(M/\varpi^m) \leq gm + d$$

then there is a map $\mathcal{O}^g \to M/\varpi^N$ with cokernel of length $\leq d$.

Proof. We use induction on the number of generators of M. First if M is cyclic, then the lemma is trivial. Next, for a general M, note that nothing changes if we replace M by M/ϖ^N, so we do that so that M has finite length. Let C be a cyclic submodule of M of maximal length, and let $N' \leq N$ be the maximal length of a cyclic submodule of $M' = M/C$. For all $m \leq N'$, we have by additivity of the length that $\ell(M'/\varpi^m) = \ell(M/\varpi^m) - m \leq (g-1)m + d$. By the induction hypothesis, we get a map $\mathcal{O}^{g-1} \to M'/\varpi^{N'} = M'/\varpi^N$ with cokernel of length $\leq d$. This map extends to a map $\mathcal{O}^g \to M/\varpi^N$ with the same cokernel. $\square$

Corollary 4.4.10. *With the same hypotheses as in Lemma 4.4.8, there exists an integer $d \in \mathbb{N}$ such that for all $N \geq 1$ and every Taylor-Wiles datum of level N, there is a map*

$$\mathcal{O}^{n|Q|} \to H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_N)$$

with cokernel of length $\leq d + h^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_N(1)) + \sum_{v\in Q}\sum_{i\neq j} \mathrm{ord}_\varpi(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j})$

Proof. Thanks to Lemma 4.4.9 and (4.4.8), we see that to prove this statement, it suffices to find two integers $d_0, d_1 \geq 0$ such that for any $1 \leq m \leq N$ we have:

$$\ell\big(H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_N)/\varpi^m\big) \leq d_0 + h^1_{\mathcal{L}_{S\cup Q}}(F^+, W_m) \tag{4.13}$$

and,

$$h^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_m(1)) \leq d_1 + h^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_N(1)) \tag{4.14}$$

Let us first treat (4.13). From the exact sequence

$$0 \to H^0(F^+, W_{E/\mathcal{O}})/\varpi^m \to H^1(F^+, W_m) \to H^1(F^+, W_{E/\mathcal{O}})[\varpi^m] \to 0 \tag{4.15}$$

and the diagram in Lemma 4.4.6, we see that the map

$$H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_m) \to H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^m]$$

is surjective, with kernel a subquotient of $H^0(F^+, W_{E/\mathcal{O}})$. Applying this for $m = N$, we get a surjective morphism

$$H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_N)/\varpi^m \to H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^N]/\varpi^m$$

But we have that $H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^N]/\varpi^m \cong H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_{E/\mathcal{O}})[\varpi^m]$, so (4.13) holds for $d_0 = h^0(F^+, W_{E/\mathcal{O}})$ (which is finite since $H^0(F^+, W_{E/\mathcal{O}})$ is torsion and embeds into $H^1(F^+, W)$).

For the second inequality, note that since inclusion $W_m(1) \hookrightarrow W_{E/\mathcal{O}}(1)$ factors through $W_N(1)$, the kernel of the map

$$H^1(F_S/F^+, W_m(1)) \to H^1(F_S/F^+, W_N(1)) \tag{4.16}$$

is contained in the kernel of the map

$$H^1(F_S/F^+, W_m(1)) \to H^1(F_S/F^+, W_{E/\mathcal{O}}(1))$$

which is subquotient of $H^0(F^+, W_{E/\mathcal{O}}(1))$ (by the exact sequence similar to (4.15)). But $H^0(F^+, W_{E/\mathcal{O}}(1))$ is torsion and embeds into $H^1(F^+, W(1))$ (by the same argument as in Lemma 4.4.8). Therefore, (4.14) will hold with $d_1 = h^0(F^+, W_{E/\mathcal{O}}(1))$ if we can show that the map (4.16) sends $H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_m(1))$ to $H^1_{\mathcal{L}_{S\cup Q}}(F^+, W_N(1))$. This means that for $v \in S_p$, the map $H^1(F_v^+, W_m(1)) \to H^1(F_v^+, W_N(1))$ should send $\mathcal{L}_{v,m}^\perp$ to $\mathcal{L}_{v,N}^\perp$. But by duality we see that this is immediate from the definitions. $\square$

4.4.2 Enormous subgroups

In order to find a set of Taylor-Wiles places with prescribed properties, one usually puts some technical restrictions on the residual image of ρ. Namely one requires $\overline{\rho}(G_{F(\zeta_p)})$ to act absolutely irreducibly as in the first part of the thesis, or for it to be "big" in the sense of [CHT08, §2.5], or "enormous" in the sense of [KT17, Definition 4.10]. We adapt the last notion to the characteristic zero case, and give the following definition:

Definition 4.4.11. A subgroup $H \subseteq \mathrm{GL}_n(\mathcal{O})$ is said to be enormous if for all simple $E[H]$-submodule $V \subseteq W_E$, we can find $h \in H$ with n distinct eigenvalues in E and an eigenvalue $\alpha \in E$ of h such that $\operatorname{tr} e_{h,\alpha} V \neq 0$, where $e_{h,\alpha} \in W_E$ is the h-equivariant projection to the α-eigenspace.

Note that unlike [KT17, Definition 4.10], we do not require the vanishing of the zeroth and first cohomology groups of H with the adjoint action. In fact this will be a consequence of the purity of the considered Galois representation (see Corollary 4.4.15), and the following lemma:

Lemma 4.4.12. *If $H \subseteq \mathrm{GL}_n(\mathcal{O})$ is an enormous subgroup, then H acts absolutely irreducibly on E^n. In particular, we have $H^0(H, W_E^0) = 0$.*

Proof. Since E^n has no stable subspace under the action of W_E, it suffices to show that H spans W_E as an E-vector space. So for the sake of contradiction, suppose that

$$U = \{u \in W_E,\ \operatorname{tr}(hu) = 0\ \forall h \in H\}$$

is non-zero, and let $V \subseteq U$ be a simple $E[H]$-submodule. Since H is enormous, there exists $h \in H$, and $\alpha \in E$ an eigenvalue of H such that $\operatorname{tr} e_{h,\alpha} V \neq 0$. This is a contradiction since $e_{h,\alpha}$ is a polynomial in h. $\square$

The following lemmas give various reformulations of our definition which we will use, and also an interesting condition for a subgroup of $\mathrm{GL}_n(\mathcal{O})$ to be enormous.

Lemma 4.4.13. *Let $H \subseteq \mathrm{GL}_n(\mathcal{O})$ be a compact subgroup, and suppose that the characteristic polynomial of every element in H splits over E. Then, the following conditions are equivalent:*

(1) H is enormous,

(2) For all simple $E[H]$-submodules $V \subseteq W_E^0 = \mathrm{ad}^0\rho \otimes E$, we can find $h \in H$ with n-distinct eigenvalues and $\alpha \in E$ such that α is an eigenvalue of h and $\mathrm{tr}\, e_{h,\alpha} V \neq 0$.

(3) For all non-zero $E[H]$-submodules $V \subseteq W_E$, there exists $h \in H$ with n distinct eigenvalues such that $V \not\subseteq (h-1)W_E$.

(4) For all non-divisible $\mathcal{O}[H]$-submodules $V \subseteq W_{E/\mathcal{O}}$, there exists $h \in H$ with n-distinct eigenvalues such that $V \not\subseteq (h-1)W_{E/\mathcal{O}}$.

Proof. First note that (1) and (2) are equivalent since the subspace of scalar matrices $Z_E \subseteq W_E$ form a complement to W_E^0 inside W_E, and clearly $\mathrm{tr}\, e_{h,\alpha} z \neq 0$ for all $z \in Z_E$ and any $h \in H$ with eigenvalue $\alpha \in E$.
Now if $h \in H$ has n distinct eigenvalues, then it acts semisimply on W_E. Hence, there is a unique h-equivariant direct sum decomposition $W_E = W_E^h \oplus (h-1)W_E$ ($W_E^h \cap (h-1)W_E = 0$ since $\ker(h-1)^2 = \ker(h-1)$). If $V \subseteq W_E$ is an h-invariant subspace, then we also have a decomposition $V = V^h \oplus (h-1)V$. Since $e_{h,\alpha}$ for an eigenvalue $\alpha \in E$ commutes with h, we have that $\mathrm{tr}\, e_{h,\alpha}(h-1)V = 0$. On the other hand, if $v \in V^h$, then v commutes with h, so it stabilizes the eigenspaces of h. In particular, if $v \neq 0$, then there exists an eigenvalue $\alpha \in E$ such that $\mathrm{tr}\, e_{h,\alpha} v \neq 0$. Therefore, we get that $\mathrm{tr}\, e_{h,\alpha} V \neq 0$ for some $\alpha \in E$ if and only if V^h, which in turn, is equivalent to $V \not\subseteq (h-1)W_E$. This shows that (1) and (3) are equivalent.

It remains to show that (3) and (4) are equivalent. To do this, note that we have a $\mathrm{GL}_n(\mathcal{O})$-equivariant bijection between the E-subspaces of W_E and the divisible submodules of $W_{E/\mathcal{O}}$. This bijection sends $V \subseteq E$ to $V + W/W$, and $V' \in W_{E/\mathcal{O}}$ to

$$V = \{v \in W_E \mid \varpi^{-n} v \mod W \in V', \ \forall n \geq 0\}$$

In particular, it sends $(h-1)W_E$ to $(h-1)W_{E/\mathcal{O}}$ which gives the desired equivalence. □

Lemma 4.4.14. *Let $H \subseteq \mathrm{GL}_n(\mathcal{O})$ be a compact subgroup such that for each $h \in H$, the characteristic polynomial of h splits in E.*

(1) If $H' \subseteq H$ is closed subgroup such that H' is enormous, then so is H.

(2) Let $G \subseteq \mathrm{GL}_n$ be the Zariski closure of H. If G° (the connected component containing the identity) contains regular semisimple elements and acts absolutely irreducibly on E^n, then H is enormous.

Proof. **(2)** We can assume that $G = G^\circ$, in particular G is irreducible (by [Mil18, 2.6]). Let $H^{\mathrm{reg}} \subseteq H$ be the set of regular semisimple elements of H, and similarly for $G^{\mathrm{reg}} \subseteq G$. Then, by hypothesis, G^{reg} is a non-empty Zariski open subset of G, and the Zariski closure of H is contained in the union of the Zariski closure of H^{reg} and $G - G^{\mathrm{reg}}$. By irreducibility of G, this forces H^{reg} to be Zariski dense in G. Now let $v \in W_E$, and suppose that for each $h \in H^{\mathrm{reg}}$, $\mathrm{tr}\, hv = 0$, then by Zariski density of H^{reg}, we must have $\mathrm{tr}\, gv = 0$ for all $g \in G$. But G acts absolutely irreducibly, so by Lemma 2.3.2, $G(E)$ spans W_E which implies that $v = 0$. □

Some vanishing of the cohomology result

The goal of this subsection is to prove Corollary 4.4.15 which we will be using for the existence of the Taylor-Wiles primes. The proof is based on the article [Ser71] where the author proves the result for Galois representations associated to the p-divisible group of an Abelian Variety.

If $\mathfrak{g}$ is a Lie algebra over a field k and M is a $\mathfrak{g}$-module, we define the n-th cohomology group of $\mathfrak{g}$ with coefficients in M to be the n-th left derived functor of the functor of invariants $M \mapsto M^{\mathfrak{g}}$. In other words, it is given by:

$$H^n(\mathfrak{g}, M) = \mathrm{Ext}^n_{U_{\mathfrak{g}}}(k, M)$$

Concretely, we define the space of n-cochains on $\mathfrak{g}$ with coefficients in M to be $C^n(\mathfrak{g}, M) = \mathrm{Hom}_k(\Lambda^n \mathfrak{g}, M)$ which is the space of n-linear alternating forms on M and we set $C^0(\mathfrak{g}, M) = M$. This defines a cochain complex whose differential map $d : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$ is given by

$$\begin{aligned} df(g_1, \ldots, g_{n+1}) = & \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1}) \\ & + \sum_{i=1}^{n+1} (-1)^{i+1} x_i \cdot f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}) \end{aligned}$$

for $f \in C^n(\mathfrak{g}, M)$. This cochain complex is called the Chevalley-Eilenberg complex and taking its cohomology we recover the $H^n(\mathfrak{g}, M)$'s. For each $x \in \mathfrak{g}$, we can also define two maps of complexes:

- The interior product: $(i_x)_n : C^n(\mathfrak{g}, M) \to C^{n-1}(\mathfrak{g}, M)$ given by $(i_x)_n f(g_1, \ldots, g_{n-1}) = f(x, g_1, \ldots, g_{n-1})$, for $f \in C^n(\mathfrak{g}, M)$ and $g_i \in \mathfrak{g}$.
- The lie derivative: $(\theta_x)_n : C^n(\mathfrak{g}, M) \to C^n(\mathfrak{g}, M)$ given by

  $$(\theta_x)_n f(g_1, \ldots, g_n) = x \cdot f(g_1, \ldots, g_n) - \sum_{i=1}^{n} f(g_1, \ldots, [x, g_i], \ldots, g_n)$$

  for $f \in C^n(\mathfrak{g}, M)$ and $g_i \in \mathfrak{g}$.

Note that both maps are related by the Cartan magic formula

$$di_x + i_x d = \theta_x$$

which says that θ_x is null homotopic.
In what follows, we give a criterion, due to Serre [Ser71], for the vanishing of the cohomology groups of a lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ with coefficients in a k-vector space V. If $x \in \mathfrak{g}$, we let L_x be the set of eigenvalues of x acting on V (taken inside the algebraic closure $\overline{k}$ of k). We say that L_x satisfies Serre's condition for $N \geq 1$, which we write (SC_N) if for each tuple $(\lambda_1, \ldots, \lambda_{N+1}, \mu_1, \ldots, \mu_N) \in L_x^{2N+1}$, we have

$$\lambda_1 + \cdots + \lambda_{N+1} \neq \mu_1 + \cdots + \mu_N$$

It is clear that if $n \leq N$, (SC_N) implies (SC_n).

Theorem 4.4.15. *Let $N \geq 1$, if $\mathfrak{g}$ contains an element x such that L_x satisfies (SC_N), then $H^n(\mathfrak{g}, V) = 0$ for all $n \leq N$.*

Proof. Since $\theta_x : C^\bullet(\mathfrak{g}, V) \to C^\bullet(\mathfrak{g}, V)$ is null homotopic, it suffices to show that each $(\theta_x)_n$ is an isomorphism for $n \leq N$.
Seeing $C^n(\mathfrak{g}, V)$ as a subspace of $T^n(\mathfrak{g}') \otimes V$ ($\mathfrak{g}' = \mathrm{Hom}_k(\mathfrak{g}, k)$) identifies $(\theta_x)_n$ with the diagonal action of x such that for $\phi \in \mathfrak{g}'$, $(x \cdot \phi)(y) = -\phi([x, y])$ (the lie dual of the adjoint action). On the other

hand, $\mathfrak{g}$ can be identified with a subspace of $V \otimes V'$ where the adjoint action of x on $\mathfrak{g}$ is compatible with the action of x on $V \otimes V'$ given by

$$x \cdot (a \otimes \phi) = x \cdot a \otimes \phi - a \otimes \phi \circ x$$

Hence, $\mathfrak{g}'$ is a quotient of $V' \otimes V$ compatibly with the actions defined above. In total, $C^n(\mathfrak{g}, V)$ can be seen as a subquotient of $T^{n+1}(V) \otimes T^n(V')$ which implies that the eigenvalues of $(\theta_x)_n$ on $C^n(\mathfrak{g}, V)$ are of the form

$$(\lambda_1 + \cdots + \lambda_{n+1}) - (\mu_1 + \cdots + \mu_n), \quad \lambda_i, \mu_i \in L_x$$

but L_x satisfies (SC_n) by hypothesis, so the eigenvalues of $(\theta_x)_n$ are non-zero, which means that it is an isomorphism as desired. □

This theorem has the following consequence which will be of interest to us. But before stating it, recall that a Galois representation $\rho : G_F \to \mathrm{GL}(V)$ (V is a finite dimensional $\mathbb{Q}_p$-vector space) is pure of weight w if there exists a finite set of places S of F such that for each place $v \notin S$, ρ is unramified at v and each eigenvalue α of $\rho(\mathrm{Frob}_v)$ (the geometric Frobenius) is a Weil number, i.e., for each embedding $\iota : \overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}$, we have

$$|\iota\alpha|_\infty^2 = (\mathbf{N}v)^w$$

(in particular α is algebraic).

Corollary 4.4.16. *[Kis04, Lemma 6.2]*
Let $\rho : G_F \to \mathrm{GL}(V)$ be a pure Galois representation over a finite dimensional $\mathbb{Q}_p$-vector space V. If we let G be the image of G_F in $\mathrm{GL}(V)$, then $H^n(G, V) = 0$ for all $n \geq 0$.

Proof. We let v be a place of F such that $p \nmid v$ and $v \notin S$. If $\alpha_1, \ldots, \alpha_n, \beta_1 \ldots, \beta_m$ are eigenvalues of $\rho(\mathrm{Frob}_v)$ with $n \neq m$, then $\alpha_1 \cdots \alpha_n \beta_1^{-1} \cdots \beta_m^{-1}$ is a Weil number of weight $(n-m)w \neq 0$, hence it is not a root of unity. But G is a compact p-adic lie group with lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, so the p-adic logarithm

$$\log : G \to \mathfrak{g}$$

is well defined over G. We consider $x = \log(\rho(\mathrm{Frob}_v)) \in \mathfrak{g}$ whose set of eigenvalues L_x is formed by the $\log(\alpha)$ for eigenvalues α of $\rho(\mathrm{Frob}_v)$. Then, by the above considerations, we see that L_x satisfies (SC_n) for all $n \geq 0$. Therefore, by Theorem 4.4.15, we get that $H^n(\mathfrak{g}, V) = 0$ for all $n \geq 0$. But by a theorem due to Lazard [Laz65, Theorem V.2.4.10(iii)], $H^n(G, V)$ is a sub-$\mathbb{Q}_p$-vector space of $H^n(\mathfrak{g}, V)$ hence the result. □

4.4.3 Existence of the Taylor-Wiles primes

We now have all the ingredients to find a set of Taylor-Wiles places with nices properties.

Lemma 4.4.17. *Let $q \geq \mathrm{corank}_{\mathcal{O}} H^1(F_S/F^+, W_{E/\mathcal{O}}(1))$ $(= \mathrm{rank}_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}\left(H^1(F_S/F^+, W_{E/\mathcal{O}}(1)), E/\mathcal{O}\right))$ and suppose that ρ satisfies the following conditions*

(1) ρ is pure of some weight.

(2) $\rho(G_{F(\zeta_{p^\infty})})$ is enormous.

then there exists $d \in \mathbb{N}$, such that for any $N \in \mathbb{N}$ we can find a Taylor-Wiles datum $\left(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \ldots, \alpha_{\widetilde{v},n})_{\widetilde{v} \in \widetilde{Q}}\right)$ of level N with $|Q| = q$ and

(i) for all $v \in Q$ and $i \neq j$, we have $\mathrm{ord}_\varpi(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j}) \leq d$

(ii) $h^1_{\mathcal{L}^\perp_{S\cup Q}}\big(F^+, W_N(1)\big) \le d$

Proof. By definition of the Selmer groups, for a set of Taylor-Wiles places Q we have an exact sequence

$$0 \to H^1_{\mathcal{L}^\perp_{S\cup Q}}(F^+, W_N(1)) \to H^1_{\mathcal{L}^\perp_S}(F^+, W_N(1)) \to \bigoplus_{v\in Q} H^1(k(v), W_N(1))$$

If we show that there exists $\sigma_1, \dots \sigma_q \in G_{F(\zeta_{p^\infty})}$ such that

- for each $1 \le i \le q$, $\rho(\sigma_i)$ has n distinct eigenvalues in E,
- the kernel of the map $H^1\big(F_S/F^+, W_{E/\mathcal{O}}(1)\big) \to \bigoplus_{i=1}^q H^1\big(\langle\rho(\sigma_i)\rangle, W_{E/\mathcal{O}}(1)\big) \cong \bigoplus_{i=1}^q W_{E/\mathcal{O}}(1)/(\sigma_i - 1)W_{E/\mathcal{O}}$ is a finite length $\mathcal{O}$-module, where $\langle\rho(\sigma_i)\rangle \subseteq \mathrm{GL}_n(\mathcal{O})$ is the procyclic group topologically generated by $\rho(\sigma_i)$.

then from the long exact sequence of cohomology coming from the short exact sequence

$$0 \to W_N(1) \to W_{E/\mathcal{O}}(1) \xrightarrow{\times\varpi^N} W_{E/\mathcal{O}}(1) \to 0$$

we get the following Cartesian diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0\big(F_S/F^+, W_{E/\mathcal{O}}(1)\big)/\varpi^N & \longrightarrow & H^1\big(F_S/F^+, W_N(1)\big) & \longrightarrow & H^1\big(F_S/F^+, W_{E/\mathcal{O}}(1)\big)[\varpi^N] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{i=1}^q H^0\big(\langle\rho(\sigma_i)\rangle, W_{E/\mathcal{O}}(1)\big)/\varpi^N & \longrightarrow & \bigoplus_{i=1}^q H^1\big(\langle\rho(\sigma_i)\rangle, W_N(1)\big) & \longrightarrow & \bigoplus_{i=1}^q H^1\big(\langle\rho(\sigma_i)\rangle, W_{E/\mathcal{O}}(1)\big)[\varpi^N] & \longrightarrow & 0 \end{array}$$

Note that since $W_E(1)$ is pure of weight -2, we have by Corollary 4.4.16 that $H^0\big(F_S/F^+, W_E(1)\big) = 0$, hence $H^0\big(F_S/F^+, W_{E/\mathcal{O}}\big)$ is a finite length $\mathcal{O}$-module. Using the snake lemma in the above diagram, we conclude that the kernel of

$$H^1\big(F_S/F^+, W_N(1)\big) \to \bigoplus_{i=1}^q H^1\big(\langle\rho(\sigma_i)\rangle, W_N(1)\big)$$

has length which is bounded independently of N. Now by Chebotarev's density theorem, for each $N \ge 1$, we can find places $v_1, \dots v_q$ of F^+ such that $\mathrm{Frob}_{v_1}, \dots, \mathrm{Frob}_{v_q} \in G_{F(\zeta_{p^N})}$ and for each $1 \le i \le q$, Frob_{v_i} is sufficiently close to σ_i so that $\rho(\mathrm{Frob}_{v_i})$ has n distinct eigenvalues, and $(\sigma_i - 1)W_{E/\mathcal{O}}(1) = (\mathrm{Frob}_{v_i} - 1)W_{E/\mathcal{O}}(1)$ ($W_{E/\mathcal{O}}(1)$ has the discrete topology).
It remains to show the existence of the σ_i with the mentioned properties. Noting that a divisible $\mathcal{O}$-module of corank d can be written in the form $(E/\mathcal{O})^d \times N$ where N is a finite length $\mathcal{O}$-module, it suffices to show that for each nonzero morphism $f : E/\mathcal{O} \to H^1\big(F_S/F^+, W_N(1)\big)$, we can find a $\sigma \in G_{F(\zeta_{p^\infty})}$ such that $\rho(\sigma)$ has distinct eigenvalues and $\mathrm{Res}^{G_{F^+,S}}_{\langle\rho(\sigma)\rangle} \circ f : E/\mathcal{O} \to W_{E/\mathcal{O}}(1)/(\sigma - 1)W_{E/\mathcal{O}}(1)$ is still non-zero.
Now let L'_∞/F^+ be the extension cut out by $W_E(1)$ and let $L_\infty = L'_\infty \cdot F(\zeta_{p^\infty})$. By Corollary 4.4.16, we have $H^1\big(F_S/L'_\infty, W_E(1)\big) = 0$. But since the extension cut out by the cyclotomic character is $F^+(\zeta_{p^\infty})$, we have that $F^+(\zeta_{p^\infty}) \subseteq L'_\infty$ and the extension L_∞/L'_∞ is finite. Since $W_E(1)$ is $\mathbb{Z}$-divisible, we have that $H^1(L_\infty/L'_\infty, W_E(1)) = 0$ and by the inflation-restriction exact sequence, we get that $H^1(L_\infty/F^+, W_E(1)) = 0$. Thus, from the long exact sequence of cohomology, we see that $H^1\big(L_\infty/F^+, W_{E/\mathcal{O}}(1)\big)$ is a finite length $\mathcal{O}$-module, and in particular it is killed by p^d for some $d \ge 1$. From the inflation-restriction exact sequence

$$0 \to H^1\big(L_\infty/F^+, W_{E/\mathcal{O}}(1)\big) \to H^1\big(F_S/F^+, W_{E/\mathcal{O}}(1)\big) \to H^1\big(F_S/L_\infty, W_{E/\mathcal{O}}(1)\big)^{G_{F^+,S}}$$

and the fact that $H^1\big(F_S/L_\infty, W_{E/\mathcal{O}}(1)\big)^{G_{F^+,S}} \cong \mathrm{Hom}_{G_{F^+,S}}\big(G_{L_\infty,S_{L_\infty}}, W_{E/\mathcal{O}}(1)\big)$ ($W_{E/\mathcal{O}}(1)$ is a trivial $G_{L_\infty,S_{L_\infty}}$-module) we see that :

$$\mathrm{Res}^{G_{F^+,S}}_{G_{L_\infty,S_{L_\infty}}} \circ f : E/\mathcal{O} \to \mathrm{Hom}_{G_{F^+,S}}\big(G_{L_\infty,S_{L_\infty}}, W_{E/\mathcal{O}}(1)\big)$$

is still non-zero since $E/\mathcal{O}$ is p^d-divisible. We let $M \subseteq W_{E/\mathcal{O}}(1)$ be the $\mathcal{O}$-submodule generated by the $f(x)(\sigma)$ for $x \in E/\mathcal{O}$ and $\sigma \in G_{L_\infty}$. It is non-zero by what we have just proved, and it is a divisible $\mathcal{O}[G_{F(\zeta_{p^\infty})}]$-submodule of $W_{E/\mathcal{O}}(1)$. Then, since $\rho(G_{F(\zeta_{p^\infty})})$ is enormous, there exists $\sigma \in G_{F(\zeta_{p^\infty})}$ such that $\rho(\sigma)$ has n-distinct eigenvalues in E and $M \not\subset (\sigma-1)W_{E/\mathcal{O}}(1)$. In other words, there exists $m \geq 0$, $\tau \in G_{L_\infty}$ such that $f(1/\varpi^m)(\tau) \notin (\sigma-1)W_{E/\mathcal{O}}(1)$. If $f(1/\varpi^m)(\sigma) \notin (\sigma-1)W_{E/\mathcal{O}}(1)$, then $\mathrm{Res}^{G_{F^+,S}}_{\langle\sigma\rangle} \circ f$ is non-zero and we are done. Otherwise, $f(1/\varpi^m)(\tau\sigma) \notin (\sigma-1)W_{E/\mathcal{O}}(1) = (\tau\sigma-1)W_{E/\mathcal{O}}(1)$ (since τ acts trivially) and $\mathrm{Res}^{G_{F^+,S}}_{\langle\tau\sigma\rangle} \circ f$ is non-zero. □

Finally, we have the following result which summarized all the work we have done up until now :

Theorem 4.4.18. *[NT20, 2.31]*
Let $q \geq \mathrm{corank}_{\mathcal{O}} H^1(F_S/F^+, W_{E/\mathcal{O}}(1))$, and suppose that ρ satisfies the following conditions:

(1) ρ is pure of some weight.

(2) For each $v \in S$, $\rho_{|G_{F_{\tilde{v}}}}$ is generic.

(3) For each place $v \mid \infty$ of F^+, $\chi(c_v) = -1$.

(4) $\rho(G_{F(\zeta_{p^\infty})})$ is enormous.

then there exists $d \in \mathbb{N}$ such that for each $N \in \mathbb{N}$, we can find a Taylor-Wiles datum Q_N of level N, with $|Q_N| = q$ and a map

$$\mathcal{O}[[x_1, \ldots, x_{nq}]] \to R_{S\cup Q_N}$$

such that the images of the x_i are in $\mathfrak{q}_{S\cup Q_N}$, and

$$\mathfrak{q}_{S\cup Q_N}/(\mathfrak{q}^2_{S\cup Q_N}, x_1, \cdots, x_{qn})$$

is a quotient of $(\mathcal{O}/\varpi^d)^{g_0}$, where $g_0 = g_0(S, \overline{\rho}, q)$ as in Lemma 4.4.1.

Proof. Recall that by Corollary 4.4.10 that for all $N \geq 1$ there is a map

$$\mathcal{O}^{nq} \to H^1_{\mathcal{L}_{S\cup Q_N}}(F^+, W_N)$$

whose cokernel is of length $\leq d_1$ for some $d_1 \in \mathbb{N}$ which is independent of N thanks to Lemma 4.4.17. Hence, by Proposition 4.4.4, composing this map with $\mathrm{tr}_{m,S\cup Q_N}$ and using the $\mathcal{O}$-module isomorphism $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N \cong \mathrm{Hom}_{\mathcal{O}}(\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N}, \mathcal{O}/\varpi^N)$ we get a map

$$\mathcal{O}^{nq} \to \mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N$$

whose cokernel is killed by ϖ^d for some $d \in \mathbb{N}$ independent of N. Therefore, we can define a map $\mathcal{O}[[x_1, \ldots, x_{nq}]] \to R_{S\cup Q_N}$ sending the x_i to the images of the generators of $\mathcal{O}^{nq}$ in $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N$ such that

$$\mathfrak{q}_{S\cup Q_N}/(\mathfrak{q}^2_{S\cup Q_N}, x_1, \ldots, x_{nq}) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^N$$

is killed by ϖ^d. We claim that $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N}$ is a quotient of $\mathcal{O}^{g_0}$. Indeed, it is a finitely generated $\mathcal{O}$-module, and $\mathfrak{q}_{S\cup Q_N}/\mathfrak{q}^2_{S\cup Q_N} \otimes \mathcal{O}/\varpi \cong \mathfrak{m}_{R_{S\cup Q_N}}/(\mathfrak{m}_{R_{S\cup Q_N}}, \varpi)$ which is generated by g_0 elements by

Lemma 4.4.1. So applying Nakayama's lemma, we get our claim.
Therefore, we have an $\mathcal{O}$-module $M = \mathfrak{q}_{S\cup Q_N}/(\mathfrak{q}^2_{S\cup Q_N}, x_1, \ldots, x_{nq})$ which is a quotient of $\mathcal{O}^{g_0}$ and such that $M/\varpi^N M$ is killed by ϖ^d. Up to shifting the Taylor-Wiles sets, we can assume that $N > d$. In this case, the fact that M/ϖ^N is killed by ϖ^d can only happen if M itself is killed by ϖ^d which gives the result. □

Chapter 5

The Iwahori-Hecke algebra

5.1 The Bernstein presentation

The goal of this section is to give a presentation, due to Bernstein, of the Iwahori Hecke algebra of GL_n (which easily generalized to a split p-adic group). This will later be useful in our treatment of the automorphic theory.
First, we will recall some facts about split semisimple reductive groups. Then, we will introduce the Iwahori-Matsumoto presentation of the Iwahori Hecke algebra from which we will derive our desired presentation.
To write this section, we mainly used the references [Lus89, Bum10, Kir97].

We fix once and for all a field K which is a finite extension of $\mathbb{Q}_l$ for a prime l, with ring of integers $\mathcal{O}_K$ and residue field k. We let $q_k = \#k$, and we fix a uniformizer π.

5.1.1 The root system of a split reductive group

Let (G, T) be a split reductive group over K with lie algebra $\mathfrak{g} = \ker\big(G(K[\epsilon]) \to G(K)\big)$, and let $\mathrm{ad} : G \to \mathrm{GL}_{\mathfrak{g}}$ be the adjoint representation. Since T is diagonalizable its action on $\mathfrak{g}$ induces a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0 = \mathfrak{g}^T = \mathrm{Lie}(G^T)$ and $\mathfrak{g}_\alpha$ is the subspace on which T acts via a non-trivial character α. The characters occurring in this decomposition are called the roots of (G, T) and form a finite set $\Phi(G, T) \subset X^*(T)$. Since $G^T = C_G(T) = T$ (its centralizer) so that $\mathfrak{g}_0 = \mathrm{Lie}(T) = \mathfrak{t}$, we can write

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha$$

The Weyl group $W(G, T)$ is defined to be the quotient $N_G(T)/C_G(T) = N_G(T)/T$. For an element $\sigma \in W$, represented by $n \in N_G(T)$, we have a morphism $T \to T$ given by conjugation by n which only depends on σ. Therefore, we have an action of $W(G, T)$ on $X^*(T)$ given by $(\sigma\alpha)(t) = \alpha(n^{-1}tn)$ for $\alpha \in X^*(T)$, which can be seen to preserve Φ. Similarly, we also have an action of $W(G, T)$ on $X_*(T)$ given by $(\sigma\lambda)(x) = n\lambda(x)n^{-1}$.
If α is a root of (G, T), we let $T_\alpha = \ker(\alpha)$ and $G_\alpha = C_G(T_\alpha)$. The pair (G_α, T) is a split reductive group of semisimple rank 1 with Lie algebra

$$\mathrm{Lie}(G_\alpha) = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

where $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$. Moreover, there exists a unique algebraic subgroup U_α of G, called the root group, which is normalized by T and isomorphic to $\mathbb{G}_a$. For every isomorphism $u : \mathbb{G}_a \to U_\alpha$, we have

$$t \cdot u(a) \cdot t^{-1} = u(\alpha(t)a), \quad \forall t \in T(R), a \in \mathbb{G}_m(R),\ R\ K\text{-algebra}$$

The Weyl group $W(G_\alpha, T)$ contains only one non-trivial element s_α, and there exists a unique cocharacter $\alpha^\vee \in X_*(T)$ such that

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \cdot \alpha \quad \forall x \in X^*(T)$$

The set of coroots $\Phi^\vee(G,T)$ is the subset of $X_*(T)$ consisting of the cocharacters $\alpha^\vee$ for each $\alpha \in \Phi(G,T)$. In fact, the tuple $\mathcal{R}(G,T) = (X_*(T), \Phi(G,T), X^*(T), \Phi^\vee(T))$ is a reduced root datum attached to (G,T), and the above observation identifies the abstract Weyl group attached to $\mathcal{R}(G,T)$ with $W(G,T)$.
The proof of these statements can be found in [Mil15, 22.43].

Example: Root datum associated to GL_n:
Let T be the diagonal torus consisting of diagonal entries in $G = \mathrm{GL}_n$. The character group $X^*(T)$ identifies with $\mathbb{Z}^n$ via the map sending the character $\chi_i : \mathrm{diag}(x_1, \dots, x_n) \mapsto x_i$ to the tuple $e_i = (0, \cdots, 1, \cdots, 0)$ ($e_1, \dots, e_n$ is the standard basis if $\mathbb{Z}^n$). Similarly, we identify the cocharacter grpup $X_*(T)$ with $\mathbb{Z}^n$ by sending the cocharacter $\lambda_i : t \mapsto \mathrm{diag}(1, \dots, t, \dots, 1)$ to the tuple $e_i = (0, \dots, 1, \dots, 0)$. The Lie algebra $\mathfrak{g} = M_n(E)$ decomposes as a direct sum

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{\alpha_{i,j}}$$

where $\mathfrak{t}$ is the vector space generated by the $E_{i,i}$ for $1 \leq i \leq n$, and $\mathfrak{g}_{\alpha_{i,j}}$ is the vector space generated by the matrix $E_{i,j}$ on which T acts via $\alpha_{i,j} = \chi_i - \chi_j$. Thus, the set of roots is

$$\Phi(G,T) = \{\chi_i - \chi_j \mid 1 \leq i,j \leq n,\ i \neq j\}$$

With respect to the Borel subgroup B consisting of upper triangular matrices, the set of positive roots is equal to

$$\Phi^+ = \{\chi_i - \chi_j \mid 1 \leq i < j \leq n\}$$

and $\Delta = \{\alpha_i = \chi_i - \chi_{i+1} \mid 1 \leq i \leq n-1\} \subset \Phi^+$ is a set of simple roots. Therefore, a weight $x = x_1\chi_1 + \cdots + x_n\chi_n$ is dominant if and only if $x_1 \geq \cdots \geq x_n$.
For $1 \leq i \leq n-1$, we have that

$$T_\alpha = \{\mathrm{diag}(x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_n) \mid x_1 \cdots x \cdot x \cdots x_n \neq 0\}$$

and,

$$G_{\alpha_i} = \begin{pmatrix} * & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & * & * & & \vdots \\ \vdots & & * & * & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & * \end{pmatrix} \quad \text{with} \quad n_{\alpha_i} = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & 0 & 1 & & \vdots \\ \vdots & & 1 & 0 & & \\ & & & & \ddots & 0 \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}$$

The action of n_{α_i} on T consists of switching the i-th and the $(i+1)$-th coordinate. One easily verifies that if $x \in X^*(T)$, $s_\alpha x = x - \langle \lambda_i - \lambda_{i+1}, x \rangle \cdot (\chi_i - \chi_{i+1})$. In general, the coroot of $\alpha_{i,j} \in \Phi$ is $\alpha_{i,j}^\vee = \lambda_i - \lambda_j$.

5.1.2 The extended affine Weyl group

Let T be an abstract torus and write $X = X^*(T)$, $X^\vee = X_*(T)$ with the perfect paring

$$X \times X^\vee : \langle \cdot, \cdot \rangle \to \mathbb{Z}$$

We let $(X, \Phi, \alpha \mapsto \alpha^\vee)$ be a reduced root datum as in [Mil18, 19]. We decompose Φ into positive and negative roots Φ^+, Φ^- and we fix a set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \Phi^+$. We will use the element $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ which satisfies $\langle \alpha_i^\vee, \rho \rangle = 1$ for $1 \leq i \leq r$. We let $Q^\vee = \bigoplus_i \mathbb{Z}\alpha_i \subseteq X^\vee$ be the coroot lattice. For reasons which will become apparent later, we will assume that for any $\alpha \in \Phi$, $\alpha \notin 2X$. We will also assume that Φ is irreducible, which implies the existence of a (unique) highest root $\theta \in \Phi$.
Let $\widehat{X} = X \oplus \mathbb{Z}\delta$ whose elements are interpreted as function on $X^\vee$ via $(x, k)(x') = \langle x, x' \rangle + k$. We define the affine root system $\widehat{\Phi} = \Phi \times \mathbb{Z}\delta$, its subset of positive affine roots $\widehat{\Phi}^+ = \{\alpha + k\delta \in \widehat{\Phi} \mid k > 0 \text{ or } k = 0, \alpha \in \Phi^+\}$, and its subset of simple affine roots $\widehat{\Delta} = \{\alpha_0 = (-\theta, 1), (\alpha_1, 0), \dots (\alpha_r, 0)\}$.
For each $\widehat{\alpha} \in \widehat{\Phi}$, we define the reflections $r_{\widehat{\alpha}} : \widehat{X} \to \widehat{X}$ by

$$r_{\widehat{\alpha}} : \widehat{x} \mapsto \widehat{x} - \langle x, \alpha^\vee \rangle \cdot \widehat{\alpha}$$

where $\widehat{x} = x + m\delta$ and $\widehat{\alpha} = \alpha + k\delta$. We will write $s_0, s_1, \dots, s_r$ for $r_{\alpha_0}, r_{\alpha_1}, \dots, r_{\alpha_r}$. The affine Weyl group W_{aff} is defined to be the subgroup of $\mathrm{GL}(\widehat{X})$ generated by the reflections $r_{\widehat{\alpha}}$ for $\widehat{\alpha} \in \widehat{\Phi}$. The Weyl group W associated to Φ identifies as the subgroup of W_{aff} generated by $s_1, \dots, s_r$. We have the following standard facts about the affine Weyl group:

Proposition 5.1.1. *1. $W_{aff} = W \ltimes \tau(Q^\vee)$, where the action of $\lambda \in Q^\vee$ over $\widehat{X}$ is given by*

$$\tau(\lambda) : \widehat{x} \mapsto \widehat{x} - \langle x, \lambda \rangle \cdot \delta$$

Concretely, we have $w\tau(\lambda) \cdot w'\tau(\lambda') = ww'\tau(w'^{-1}(\lambda) + \lambda')$, for $w \in W$ and $\lambda, \lambda' \in Q^\vee$.

2. *W_{aff} is a Coxeter group with generators $\{s_0, \dots, s_r\}$.*

3. *For every $w\tau(\lambda) \in W_{aff}$, its length $\ell(w\tau(\lambda))$ with respect to the generators $s_0, \dots, s_r$ is equal to*

$$\begin{aligned}\ell(w\tau(\lambda)) &= |\widehat{\Phi}^+ \cap (w\tau(\lambda))^{-1}\widehat{\Phi}^-| \\ &= \sum_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Phi^-}} |\langle \lambda, \alpha^\vee \rangle + 1| + \sum_{\substack{\alpha \in \Phi^+ \\ w(\alpha) \in \Phi^+}} |\langle \lambda, \alpha^\vee \rangle|\end{aligned} \tag{5.1}$$

Now we define the extended affine Weyl group $\widetilde{W}$ to be the semi-direct product $\widetilde{W} = W \ltimes \tau(X^\vee)$ where the action of $X^\vee$ on $\widehat{X}$ is given by the same formula as in 1. of the previous proposition. The action of $\widetilde{W}$ on $\widehat{X}$ preserves $\widehat{\Phi}$, and W_{aff} is a normal subgroup of $\widetilde{W}$ with $\widetilde{W}/W_{\text{aff}} = X^\vee/Q^\vee$. Although $\widetilde{W}$ is not a Coxeter group, we can extend the definition of the length ℓ to all of $\widetilde{W}$ using the formula (5.1). To ease the notation, we will write $\ell(\lambda)$ for $\ell(\tau(\lambda))$ if $\lambda \in X^\vee$.
The element of length 0 :

$$\Omega = \{\widetilde{w} \in \widetilde{W} \mid \ell(\widetilde{w}) = 0\} = \{\widetilde{w} \in \widetilde{W} \mid \widetilde{w}(\widehat{\Delta}) = \widehat{\Delta}\}$$

form a subgroup of $\widetilde{W}$, and we have that $\widetilde{W} = W_{\text{aff}} \rtimes \Omega$ so that $\Omega \cong X^\vee/Q^\vee$. We have the following properties of the length function:

- $\ell(\pi\widetilde{w}) = \ell(\widetilde{w})$ for $\pi \in \Omega, \widetilde{w} \in \widetilde{W}$.

- $l(\widetilde{w}s_i) = \begin{cases} \ell(\widetilde{w})+1, & \widetilde{w}(\alpha_i) \in \widehat{\Phi}^+ \\ \ell(\widetilde{w})-1, & \widetilde{w}(\alpha_i) \in \widehat{\Phi}^- \end{cases}$

The Braid group B is defined to be the group generated by the symbols $T_{\widetilde{w}}$ for $\widetilde{w} \in \widetilde{W}$ subject to the relations

$$T_{\widetilde{w}}T_{\widetilde{w}'} = T_{\widetilde{w}\widetilde{w}'} \quad \text{if } \ell(\widetilde{w}\widetilde{w}') = \ell(\widetilde{w}) + \ell(\widetilde{w}')$$

In particular, the elements T_π, for $\pi \in \Omega$, form a subgroup of B isomorphic to Ω. We will write T_i instead of T_{s_i} for $0 \leq i \leq r$, and T_λ instead of $T_{\tau(\lambda)}$ for $\lambda \in X^\vee$.

Let q be an indeterminate. We define the affine Hecke algebra $\mathbf{H}_{\text{aff}}$ to be the quotient of the group algebra of B over $\mathcal{L} := \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by the two-sided ideal generated by the elements

$$(T_i + 1)(T_i - q), \quad 0 \leq i \leq r$$

It is a fact that the elements $T_{\widetilde{w}}$, $\widetilde{w} \in \widetilde{W}$ form a basis of $\mathbf{H}_{\text{aff}}$ over $\mathcal{L}$.

Remark 5.1.2. Note that in the special case where $q = 1$, we get that $T_i^2 = 1$ for all $1 \leq i \leq r$. So in this case, $\mathbf{H}_{\text{aff}}$ identifies with the group algebra of $\widetilde{W}$ with coefficients in $\mathcal{L}$.

Letting $X_+^\vee = \{x \in X^\vee \mid \langle x, \alpha_i \rangle \geq 0,\ 1 \leq i \leq r\}$ be the set of dominant coweights, we see from the formula (5.1) that for $\mu \in X_+^\vee$,

$$\ell(\mu) = \sum_{\alpha \in \Phi^+} \langle \mu, \alpha^\vee \rangle = 2\langle \rho, \mu \rangle$$

so that if we let $\mu' \in X_+^\vee$ and $w \in W$, then

$$\ell(\mu + \mu') = \ell(\mu) + \ell(\mu') \quad \text{and} \quad \ell(w\tau(\mu)) = \ell(w) + \ell(\mu)$$

This allows us to define elements $e^\lambda \in \mathbf{H}_{\text{aff}}$ for $\lambda \in X^\vee$ by setting

$$e^\lambda := q^{(\frac{\ell(\nu)-\ell(\mu)}{2})} T_\mu (T_\nu)^{-1} \tag{5.2}$$

where $\lambda = \mu - \nu$ with $\mu, \nu \in X_+^\vee$. In particular, if $\mu \in X_+^\vee$, then $e^\mu = q^{-\frac{\ell(\mu)}{2}} T_\mu$. The above formulas also show that $e^\lambda e^{\lambda'} = e^{\lambda+\lambda'}$ for $\lambda, \lambda' \in X^\vee$, and that for $w \in W$ and $\mu \in X_+^\vee$, $e^\mu \cdot \widetilde{T}_w = \widetilde{T}_{\tau(\mu)w}$ where $\widetilde{T}_w := q^{-\frac{\ell(w)}{2}} T_w$ (renormalization).
It is shown in [Lus89, 2.8] that $\mathbf{H}_{\text{aff}}$ is generated as an algebra by the T_μ for $\mu \in X_+^\vee$ and the T_w for $w \in W$.

Lemma 5.1.3. *Let $\lambda \in X^\vee$ and $\alpha_i \in \Delta$.*

1. *If $\langle \lambda, \alpha_i \rangle = 0$, then $T_i e^\lambda = e^\lambda T_i$.*
2. *If $\langle \lambda, \alpha_i \rangle = 1$, then $q e^\lambda = T_i e^{s_i(\lambda)} T_i$.*

Proof. **1)** First note that we can write $\lambda = \mu - \nu$ with $\mu, \nu \in X_+^\vee$ and $\langle \mu, \alpha_i \rangle = \langle \nu, \alpha_i \rangle = 0$, so we can suppose that $\lambda \in P_+^\vee$. Next, from the formula (5.1), we see that $\ell(s_i \tau(\lambda)) = \ell(\lambda) + 1$ and we also get from the properties of the length function we mentioned that $\ell(\tau(\lambda)s_i) = \ell(\lambda) + 1$. So by the braid relations, we get that $T_{s_i\tau(\lambda)} = T_i T_\lambda = T_\lambda T_i$. We conclude by noting that since we supposed that λ is dominant, $e^\lambda = q^{\ell(\lambda)/2} T_\lambda$.
2) Same as before, we can write $\lambda = \mu - \nu$ with $\mu, \nu \in X_+^\vee$, $\langle \mu, \alpha_i \rangle = 1$ and $\langle \nu, \alpha_i \rangle = 0$. So we can

suppose that $\lambda \in X_+^\vee$. Let $\epsilon = \lambda + s_i(\lambda) = 2\lambda - \alpha_i^\vee$, then given that if $\beta \in \Delta$ with $\beta \neq \alpha_i$ we have $\langle \beta, \alpha_i \rangle \leq 0$, we see that $\epsilon \in X_+^\vee$. So we have:

$$\ell(\epsilon) = 2\langle \epsilon, \rho \rangle = 4\langle \lambda, \rho \rangle - 2\langle \alpha_i^\vee, \rho \rangle = 2\ell(\lambda) - 2$$

Moreover, since $\tau(\lambda)(\alpha_i) \in \widehat{\Phi}^-$, we have $\ell(\tau(\lambda)s_i) = \ell(\lambda) - 1$, and since $\langle \epsilon, \alpha_i \rangle = 0$ we have from the above argument that $\ell(s_i\tau(\epsilon)) = \ell(\epsilon) + 1$. Thus, from the identity $s_i\tau(\epsilon) = (\tau(\lambda)s_i)(\tau(\lambda))$ (which one can easily verify), we get by the Braid relations that

$$T_iT_\epsilon = T_{\tau(\lambda)s_i}T_\lambda = T_\lambda T_i^{-1}T_\lambda$$

where the second equality follows from the equality $\ell(\tau(\lambda)s_i) = \ell(\lambda) - 1$. Since ϵ and λ are dominant, $e^{s_i(\lambda)} = q^{1-\ell(\lambda)/2}T_\epsilon T_\lambda^{-1}$. So the above equality gives us our desired formula. $\square$

Lemma 5.1.4. *The elements $T_w \cdot e^\lambda$ (resp. $e^\lambda \cdot T_w$), for $w \in W$ and $\lambda \in X^\vee$, are linearly independent over $\mathcal{L}$.*

Proof. Suppose that we have a relation $\sum_{i=1}^n f_iT_{w_i} \cdot e^{\lambda_i} = 0$ where $(w_1, \lambda_1), \ldots, (w_n, \lambda_n)$ are distinct elements of $W \times P^\vee$ and $f_1, \ldots, f_n \in \mathcal{L}$. We can find an element $\mu \in X_+^\vee$ such that for all i, $\lambda_i + \mu \in X_+^\vee$. Multiplying the relation by e^μ on the right, we get

$$0 = \sum_{i=1}^n q^{-\frac{\tau(\lambda_i+\mu)}{2}} f_iT_{w_i} \cdot e^{\lambda_i+\mu} = \sum_{i=1}^n q^{-\frac{\ell(\lambda_i+\mu)}{2}} f_iT_{w_i\tau(\lambda_i+\mu)}$$

but the $T_{\widetilde{w}}$ for $\widetilde{w} \in \widetilde{W}$ are linearly independent, so we must have $f_1 = \cdots = f_n = 0$. This shows that the family of elements $T_w \cdot e^\lambda$ are linearly independent (the argument for the second family is similar). $\square$

Let Θ be the $\mathcal{L}$-submodule of $\mathbf{H}_{\text{aff}}$ generated by the elements e^λ for $\lambda \in X^\vee$. This is a subalgebra of $\mathbf{H}_{\text{aff}}$ isomorphic to $\mathcal{L}[X^\vee]$.

Proposition 5.1.5. *Let $\lambda \in X^\vee$ and $\alpha_i \in \Delta$. Then, $e^\lambda - e^{s_i(\lambda)}$ is divisible by $1 - e^{-\alpha_i^\vee}$ inside Θ and*

$$e^\lambda T_i - T_ie^{s_i(\lambda)} = T_ie^\lambda - e^{s_i(\lambda)}T_i = (q-1)\frac{e^\lambda - e^{s_i(\lambda)}}{1 - e^{-\alpha_i^\vee}}$$

Proof. It is enough to show the equality

$$e^\lambda T_i - T_ie^{s_i(\lambda)} = (q-1)\frac{e^\lambda - e^{s_i(\lambda)}}{1 - e^{-\alpha_i^\vee}}$$

since the other one follows from it by substituting λ with $s_i(\lambda)$ and multiplying by -1.
Now suppose that the formula is true for a fixed $\alpha_i \in \Delta$ and for $\lambda, \lambda' \in X^\vee$, then using the formula in the equality

$$e^{\lambda+\lambda'}T_i - T_ie^{s_i(\lambda+\lambda')} = e^\lambda[e^{\lambda'}T_i - T_ie^{s_i(\lambda')}] + [e^\lambda T_i - T_ie^{s_i(\lambda)}]e^{s_i(\lambda')}$$

and simplifying, we see that it is also true for $\lambda + \lambda'$. Similarly we show that if it is true for $\lambda \in X^\vee$, then it is also true for $-\lambda$. Therefore, it suffices to show the equality for a set of generators of $X^\vee$. By our assumption that $\alpha_i \notin 2X$, there exists $\lambda_1 \in X^\vee$ such that $\langle \alpha_i, \lambda_1 \rangle = 1$ and $X^\vee$ is generated by λ_1 and the elements $\lambda' \in X^\vee$ such that $\langle \alpha_i, \lambda' \rangle = 0$.
If $\langle \alpha_i, \lambda \rangle = 0$, then $s_i(\lambda) = \lambda$ and the formula reduces to $e^\lambda T_i = T_ie^\lambda$ which follows from Lemma 5.1.3. Similarly, if $\langle \alpha_i, \lambda \rangle = 1$, then $s_i(\lambda) = \lambda - \alpha_i^\vee$ and the equality reduces to $e^\lambda T_i - T_ie^{s_i(\lambda)} = (q-1)e^\lambda$. This follows from Lemma 5.1.3 and the identity $T_i^{-1} = q^{-1}T_i + (q^{-1} - 1)$. $\square$

Proposition 5.1.6. *The elements $T_w \cdot e^\lambda$ (resp. $e^\lambda \cdot T_w$), for $w \in W$ and $\lambda \in X^\vee$, form a basis of $\mathbf{H}_{aff}$ over $\mathcal{L}$.*

Proof. It remains to show that they are a spanning family. So let us consider H_1 (resp. H_2) to be the $\mathcal{L}$-submodule of $\mathbf{H}_{\text{aff}}$ generated by the $T_w e^\lambda$ (resp. $e^\lambda T_w$). Using Proposition 5.1.5, we prove by induction on $\ell(w)$ that $T_w e^\lambda \in H_2$ and $e^\lambda T_w \in H_1$ for any $w \in W$, $\lambda \in X^\vee$. Hence, $H_1 = H_2$, but H_1 is stable by left multiplication by T_w and H_2 is stable by left multiplication by e^λ. But since $\mathbf{H}_{\text{aff}}$ is generated as an algebra by these elements and $1 \in H_1 = H_2$, we get that $H_1 = H_2 = \mathbf{H}_{\text{aff}}$ as desired. $\square$

From this, we see that we have an isomorphism of $\mathcal{L}$-modules $\mathbf{H}_{\text{aff}} \cong \Theta \otimes_{\mathcal{L}} H(W)$, where H_W is the Hecke algebra associated to the Coxeter group W with coefficients in $\mathcal{L}$. Note that this is not a $\mathcal{L}$-algebra homomorphism.

Proposition 5.1.7. *The center of $\boldsymbol{H}_{aff}$ is equal to Θ^W.*

Proof. Note that Θ^W is generated by the elements $z_M = \sum_{\lambda \in M} e^\lambda$ where M is a W-orbit in $X^\vee$. From the formula in Proposition 5.1.5, we see that $e^\lambda + e^{s_i(\lambda)}$ commutes with T_i for $1 \leq i \leq r$. This shows that z_M commutes with the T_w for $w \in W$, and consequently that Θ^W lies in the center of $\mathbf{H}_{\text{aff}}$. Using the specialisation $q \mapsto 1$, $\mathbf{H}_{\text{aff}}$ identifies with $\mathcal{L}[B]$. In this case it is not hard to see that $\mathcal{L}[B]^W = \mathcal{L}[X^\vee]^W$. A fortiori, the same must be true for $\mathbf{H}_{\text{aff}}$. $\square$

5.1.3 The Iwahori Hecke algebra for GL_n

Let $I \subseteq G = \mathrm{GL}_n$ be the Iwahori subgroup, and $U = \mathrm{GL}_n(\mathcal{O}_K)$ be the maximal compact subgroup of G. We consider $\mathcal{H}(G, I)$ to be the convolution $\mathbb{Z}$-algebra of compactly supported, I-biinvariant functions $f : G \to \mathbb{Z}$, where the Haar measure μ on G is normalized so that $\mu(I) = 1$.
The goal of this subsection is to identify this algebra with the affine Hecke algebra $\mathbf{H}_{\text{aff}}$ associated to the root datum of G, after extending the scalars to $\mathcal{L}$.

First, note that we can make an identification

$$\widetilde{W} \cong N_G(T)(K)/T(\mathcal{O}_K) \tag{5.3}$$

Indeed there is an isomorphism $X_*(T) \cong T(K)/T(\mathcal{O}_K)$ sending a cocharacter λ to $\lambda(\pi^{-1})$ (which does not depend on the choice of the uniformizer π since we quotient by $T(\mathcal{O}_K)$). So to prove (5.3) it suffices to show that the action of $W = N_G(T)(K)/T(K)$ on both sides is compatible which is straightforward.
Since $s_0 = r_{-\theta}\tau(-\theta^\vee)$ where $\theta = \chi_1 - \chi_n$ is the longest root, it is identified via (5.3) with

$$s_0 = \begin{pmatrix} 0 & & & & \pi^{-1} \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \pi & & & & 0 \end{pmatrix}$$

We also choose the matrix

$$t = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ \pi & & & & 0 \end{pmatrix}$$

which corresponds to the element $s_{n-1}\cdots s_1\tau(\lambda_1)$ where λ_1 is a representative of the generator of $X^\vee/Q^\vee \cong \mathbb{Z}$. t is chosen so that it normalizes I, and we have that $\widetilde{W}$ is generated by $s_0,\dots,s_n,t$ with $ts_it^{-1}=s_i$ for all $1\le i\le n-1$.
By the example in [Iwa66, 2], we have a Bruhat decomposition

$$G=\bigsqcup_{\widetilde{w}\in\widetilde{W}} I\widetilde{w}I \tag{5.4}$$

Therefore, the Iwahori-Hecke algebra $\mathcal{H}(G,I)$ is freely generated as a $\mathbb{Z}$-module by the characteristic functions $f_{\widetilde{w}}$ of double cosets $[I\widetilde{w}I]$ for $\widetilde{w}\in\widetilde{W}$. Using the Iwahori factorisation, calculations show that $\int_G f_{\widetilde{w}}\,\mathrm{d}\mu = q_k^{\ell(w)}$. As a consequence, we have the following relations:

Lemma 5.1.8. *(1) If $\ell(\widetilde{w}\widetilde{w}')=\ell(\widetilde{w})+\ell(\widetilde{w}')$, then $f_{\widetilde{w}}*f_{\widetilde{w}'}=f_{\widetilde{w}\widetilde{w}'}$.*

(2) $f_{s_i}*f_{s_i}=(q_k-1)f_{s_i}+q_kf_{\mathrm{id}}$.

From these relations, we see that $\mathcal{H}(G,I)\otimes_{\mathbb{Z}}\mathcal{L}$ is isomorphic to $\mathbf{H}_{\mathrm{aff}}$ under the specialisation $q\mapsto q_k$. In this setting, the subalgebra $\mathcal{H}(U,I)\otimes\mathcal{L}$ is sent to $H(W)$.

5.2 A result about the Iwahori Hecke algebras

Let $p\neq l$ be a prime number. We will work with a coefficient field E which is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and a chosen uniformizer ϖ. Suppose that $\mathcal{O}$ contains a square root of q_k so that by the work we did previously, the Iwahori Hecke algebra $\mathcal{H}_I=\mathcal{H}(G,I)\otimes_{\mathbb{Z}}\mathcal{O}$ has the following presentation

$$\mathcal{H}_I\cong\mathcal{O}[X_*(T)]\otimes_{\mathcal{O}}\mathcal{O}[I\setminus U/I]$$

which we recall is not an $\mathcal{O}$-algebra isomorphism.
Using the identification $\mathcal{S}:=\mathcal{O}[X_*(T)]=\mathcal{O}[x_1^{\pm},\dots,x_n^{\pm}]$, by Proposition 5.1.7 the center of $\mathcal{H}_I$ identifies with $\mathcal{R}:=\mathcal{O}[X_*(T)]^{\mathfrak{S}_n}=\mathcal{O}[e_1,\dots,e_n,e_n^{-1}]$ where $e_1,\dots,e_n$ are the elementary symmetric polynomials in $x_1,\dots,x_n$. The ring $\mathcal{S}$ is a free $\mathcal{R}$-module of rank $n!$, with a basis given by the monomials $x_{\mathbf{a}}=x_1^{a_1}\cdots x_n^{a_n}$ for $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n$ satisfying $0\le a_i\le i-1$ (we will write A for the set of these tuples). Indeed, the minimal polynomial of x_i in $\mathcal{O}[e_1,\dots,e_n,x_1,\dots,x_{i-1}]$ is $f_i(X)=(X-x_i)\cdots(X-x_n)$.

Denoting $\mathcal{H}_U=\mathcal{H}(G,U)\otimes_{\mathbb{Z}}\mathcal{O}$, by [HKP10, 4.6] there is a canonical isomorphism $\mathcal{H}_U\cong Z(\mathcal{H}_I)\cong\mathcal{R}$ with $z\in Z(\mathcal{H}_I)$ corresponding to $h=\mathbb{1}_U*z$. So if M is a $\mathcal{O}[G(K)]$-module, M^U can be seen as an $\mathcal{R}$-submodule of M^I, and there is a canonical morphism

$$M^U\otimes_{\mathcal{R}}\mathcal{S}\to M^I \tag{5.5}$$

given by the formula $m\otimes s\mapsto s\cdot m$. Since $\mathcal{S}$ is free over $\mathcal{R}$ with basis $x_{\mathbf{a}},\mathbf{a}\in A$, we have an isomorphism $M^U\otimes_{\mathcal{R}}\mathcal{S}\cong\oplus_{\mathbf{a}\in A}M^U$, and the above map sends $(m_{\mathbf{a}})_{\mathbf{a}\in A}$ to $\sum_{\mathbf{a}\in A}x_{\mathbf{a}}\cdot m_{\mathbf{a}}$.

Lemma 5.2.1. *Consider the $n!\times n!$ matrix $P=(P_{\sigma,\mathbf{a}})_{\sigma,\mathbf{a}}$ for $\sigma\in\mathfrak{S}_n$, $\boldsymbol{a}\in A$, with $P_{\sigma,\mathbf{a}}=\sigma(x_{\mathbf{a}})$. Then, there exists a unique matrix $Q=(Q_{\mathbf{a},\sigma})_{\mathbf{a},\sigma}$ with coefficients in $\mathbb{Z}[x_1,\dots,x_n]$ such that $PQ=QP=\Delta^{n!}$.*

Proof. The uniqueness follows from considering P and Q as matrices with coefficients in the field $\mathbb{Q}(x_1,\dots,x_n)$, so it suffices to prove existence. Now note that the square of the determinant of P is equal to the determinant of the finite ring extension $R=\mathbb{Z}[e_1,\dots,e_n]\to R'=\mathbb{Z}[x_1,\dots,x_n]$. We have the following presentation $R'=R[X_1,\dots,X_n]/\big(f_1(X_1),\dots,f_n(X_n)\big)$ where $f_i(X)=(X-x_i)\cdots(X-x_n)$, then by [Sta18, Tag 0BVZ] and [Sta18, Tag 0BWG], we see that the different of R' over R equals to Δ. But by [Sta18, Tag 0C17], we see that the determinant of this ring extension is $\Delta^{n!}$. Hence, the determinant of P equals to $\pm\Delta^{\frac{n!}{2}}$. Therefore, there exists a matrix Q' with coefficients in $\mathbb{Z}[x_1,\dots,x_n]$ (which is the adjugate matrix up to a sign) such that $PQ'=\Delta^{\frac{n!}{2}}$, so we take $Q=\Delta^{\frac{n!}{2}}Q'$. □

Proposition 5.2.2. *Let $N \geq 1$, and let M be an $\mathcal{O}/\varpi^N[\mathrm{GL}_n(K)]$-module. If $q_k \equiv 1 \mod \varpi^N$, then the morphism $f : M^U \otimes_{\mathcal{R}} \mathcal{S} \to M^I$ has kernel and cokernel annihilated by $\Delta^{n!}$.*

Proof. By Remark 5.1.2, given that $q_k \equiv 1 \mod \varpi^N$, we can identify $\mathcal{H}_I \otimes \mathcal{O}/\varpi^N$ with the group algebra $\mathcal{O}/\varpi^N[\widetilde{W}]$. In particular, $\mathcal{O}/\varpi^N[I \setminus U/I]$ identifies with $\mathcal{O}/\varpi^N[\mathfrak{S}_n]$. We let $e = \sum_{\sigma \in \mathfrak{S}_n} \sigma \in \mathcal{O}/\varpi^N[\mathfrak{S}_n]$, then $e = \mathbb{1}_U$ (note that e is not necessarily an idempotent since μ is normalized with respect to I).
Now define a morphism $g : M^I \to \oplus_{\mathbf{a} \in A} M^U \cong M^U \otimes_{\mathcal{R}} \mathcal{S}$ by the formula $g(m) = (eQ_{\mathbf{a},1}m)_{\mathbf{a} \in A}$, then by the description of f given before, we have

$$f(g(m)) = \sum_{\mathfrak{a} \in A} x_{\mathbf{a}} e Q_{\mathbf{a},1} m = \sum_{\mathbf{a} \in A} \sum_{\sigma \in \mathfrak{S}_n} x_{\mathbf{a}} \sigma(Q_{\mathbf{a},1}) \sigma(m)$$

From the identity $\sigma(P)\sigma(Q) = \Delta^{n!}$ and the uniqueness of the inverse, we get that $\sigma(Q_{\mathbf{a},1}) = Q_{\mathbf{a},\sigma}$. So we can write

$$f(g(m)) = \sum_{\sigma \in \mathfrak{S}_n} \sum_{\mathbf{a} \in A} P_{1,\mathbf{a}} Q_{\mathbf{a},\sigma} \sigma(m) = \Delta^{n!} m$$

From this, we see that the cokernel of f is killed by $\Delta^{n!}$. On the other hand, we have for $m = (m_{\mathbf{a}})_{\mathbf{a} \in A} \in M^U \otimes_{\mathcal{R}} \mathcal{S}$:

$$(g(f(m)))_{\mathbf{a}} = eQ_{\mathbf{a},1} \sum_{\mathbf{b} \in A} x_{\mathbf{b}} \cdot m_{\mathbf{b}} = \sum_{\sigma \in \mathfrak{S}_n} \sum_{\mathbf{b} \in A} Q_{\mathbf{a},\sigma} P_{\sigma,\mathbf{b}} \sigma(m)$$

Since $\mathfrak{S}_n \subseteq \mathcal{O}[I \setminus U/I]$ acts trivially on M^U, we get that:

$$(g(f(m)))_{\mathbf{a}} = \sum_{\mathbf{b} \in A} \sum_{\sigma \in \mathfrak{S}_n} Q_{\mathbf{a},\sigma} P_{\sigma,\mathbf{b}} m = \Delta^{n!} m_{\mathbf{a}}$$

This shows that the kernel of f is also killed by $\Delta^{n!}$. □

5.3 The Tame Hecke algebra

The tame subgroup I_t consists of the matrices in I which reduce to unipotent upper triangular matrices modulo π. One can ask whether the Hecke algebra $\mathcal{H}(G, I_t)$ has a similar presentation to $\mathcal{H}(G, I)$. The answer to this question is found in the paper [Fli11] where the author gives a presentation of this algebra in terms of generators and relations, building on ideas used in [HKP10] to prove the Bernstein presentation. We will summarized the results of this paper, which we will need later on.

To make the notation less cumbersome, we will write in this exposition G,T,… for $G(K)$, $T(K)$…
From the Bruhat decomposition in (5.4), we have:

$$G = I \cdot N_G(T) \cdot I = I_t \cdot N_G(T) \cdot I_t$$

So to obtain a decompositon similar to (5.4), we define the tame affine Weyl group W_t to be $W_t = N_G(T)/T_t(\mathcal{O}_K)$ where $T_t(\mathcal{O}_K) = T(\mathcal{O}_K) \cap I_t$, and we get:

$$G = \bigsqcup_{w \in W_t} I_t w I_t$$

We let $\mathcal{H}_t = \mathcal{H}(G, I_t) \otimes_{\mathbb{Z}} \mathbb{C}$ to be the tame Hecke algebra. By the above decomposition, it is a free $\mathcal{O}$-module with basis given by the T_w, the characteristic function of $I_t w I_t$ divided by $\mu(I_t)$, for $w \in W_t$.

To ease the notation, we will assume that $\mu(I_t) = 1$.
With respect to these generators, the relations are given by:

$$T_w T_{w'} = T_{ww'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w'),\ w, w' \in W_t$$

where we extend the length function to W_t by setting for $w \in W_t$, $\ell(w) = \ell(\overline{w})$ with $\overline{w}$ being the image of w in $\widetilde{W}$, and also by:

$$T_{s_i}^2 = q_k T_{\mathrm{id}} + \sum_{x \in k^\times} T_{\alpha_{i,x} s_i} \quad \text{for } 1 \le i < n$$

where $\alpha_{i,x} = \mathrm{diag}(1, \dots, 1, -[x]^{-1}, [x], 1, \dots, 1)$ with x^{-1} being in the i-th position and $[\cdot] : k^\times \to \mathcal{O}_K^\times$ being the Teichmuller lift.

Let us now give presentation of $\mathcal{H}_t$ rather with respect to the decomposition $W_t = T/T_t(\mathcal{O}) \rtimes W$. For this, N be the group of unipotent upper triangular matrices and consider the universal tame principal series module M_t which is defined by $M_t = \mathcal{C}_c(T_t(\mathcal{O}_K)N \setminus G/I_t)$. It is the set of I_t-fixed vectors in the smooth G-module $\mathcal{C}_c^\infty(T_t(\mathcal{O}_K)N \setminus G)$ (where G acts by left translation). Consequently, M_t can be equipped with a right $\mathcal{H}_t$-action. Moreover, the natural map

$$W_t \to T_t(\mathcal{O}_K) \setminus G/I_t$$

is actually an isomorphism, a basis of M_t as a $\mathbb{C}$-vector space is given by the characteristic functions $v_w = \mathbb{1}_{T_t(\mathcal{O}_K)NwI_t}$ for $w \in W_t$.
The group algebra $R_t = \mathcal{C}_c(T/T(\mathcal{O}_K)) = \mathbb{C}[T/T_t(\mathcal{O})] \cong \mathbb{C}[T/T(\mathcal{O}_K) \times T(k)]$ (via the non-canonical factorisation $K^\times \cong k^\times \times \mathbb{Z}$) has a left action on M_t given by the following: for $a \in T/T_t(\mathcal{O})$, let λ_a to be the unique cocharacter such that $a \mapsto \lambda_a(\pi)$ under $T/T_t(\mathcal{O}) \to T/T(\mathcal{O}) \cong X_*(T)$, we then set $a \cdot v_w = q_k^{-\langle \rho', \lambda_a \rangle} v_{aw}$ where ρ' is the half sum of the roots of T in $\mathrm{Lie}(N)$ (note that we have $q_k^{-\langle \rho', \lambda_a \rangle} = \delta_B^{1/2}$). The actions of R_t and $\mathcal{H}_t$ commute, so we get a structure of $R_t \otimes_{R_{f,t}} \mathcal{H}_t$-module on M_t, where the group algebra $R_{f,t} = \mathbb{C}[T(k)]$ is contained in both R_t and $\mathcal{H}_t$.

There is also another interpretation of the module M_t which is given follows. The representation $\mathcal{C}_c^\infty(T_t(\mathcal{O}_K)N \setminus G)$ is compactly induced from the trivial representation of $T_t(\mathcal{O}_K)N$. Inducing in stages, we have:

$$\mathcal{C}_c^\infty(T_t(\mathcal{O}_K)N \setminus G) = \mathrm{Ind}_{T_t(\mathcal{O}_K)N}^G(\mathrm{id}) = \mathrm{Ind}_B^G \circ \mathrm{Ind}_{T_t(\mathcal{O}_K)N}^B(\mathrm{id}) = \mathrm{Ind}_B^G(R_t)$$

where R_t is viewed as a T-module via $\chi_{\mathrm{univ}}^{-1} : T/T_t(\mathcal{O}_K) \to R_t^\times, a \mapsto a$. Concretely, $\mathrm{Ind}_B^G(R_t)$ consists of functions $\phi : G \to R_t$ such that $\phi(aug) = \delta_B^{-1/2} \cdot a^{-1} \cdot \phi(g)$ for $a \in T, u \in N, g \in G$ with the action of G by right translation. We also have an R_t-module structure on $\mathrm{Ind}_B^G(R_t)$ given by $(r\phi)(g) = r \cdot \phi(g)$. In fact, the isomorphism $\mathcal{C}_c^\infty(T_t(\mathcal{O}_K N \setminus G) = \mathrm{Ind}_B^G(R_t)$ induces an isomorphism of $R_t \otimes_{R_{f,t}} H_t$-modules $M_t \cong \mathrm{Ind}_B^G(R_t)^{I_t}$.

Note that if we have a character $\chi : T/T_t(\mathcal{O}_K) \to \mathbb{C}^\times$ which extends to a $\mathbb{C}$-algebra morphism $R_t \to \mathbb{C}^\times$, we have an isomorphism of $\mathcal{H}_t$-modules:

$$\mathbb{C} \otimes_{R_t, \chi} M_t \cong \mathbb{C} \otimes_{R_t, \chi} \mathrm{Ind}_B^G(R_t)^{I_t} = \mathrm{Ind}_B^G(\chi^{-1})^{I_t}$$

Proposition 5.3.1. *The map $\mathcal{H}_t \to M_t : h \mapsto v_1 h$ is an isomorphism of right $\mathcal{H}_t$-modules. In particular, we have $\mathcal{H}_t = \mathrm{End}_{\mathcal{H}_t}(M_t)$ identifying $\eta \in \mathcal{H}_t$ with the morphism $\varphi_\eta : v_1 h \mapsto v_1 \eta h$.*

This way, we can embed R_t inside $\mathcal{H}_t$ by viewing it as a subalgebra of $\mathcal{H}_t$-equivariant endomorphisms of M_t. More precisely we have:

$$\begin{aligned} R_t &\hookrightarrow \mathcal{H}_t \\ a &\mapsto e^a := q^{\frac{\langle \rho', \lambda_{a_2} - \lambda_{a_1} \rangle}{2}} T_{a_1} T_{a_2}^{-1} \end{aligned}$$

where $a = a_1 a_2^{-1} \in T(K)/T_t(\mathcal{O})$ with λ_{a_1} and λ_{a_2} being dominant cocharacters.
Now we are able to describe our presentation, which is given by the isomorphism of $\mathbb{C}$-modules

$$\mathcal{H}_t \cong R_t \otimes_{R_{f,t}} \mathbb{C}[I_t \setminus U/I_t]$$

induced by multiplication inside $\mathcal{H}_t$. Note that $\mathbb{C}[I_t \setminus U/I_t]$ is generated by the T_w for $w \in W_{f,t} = N_U(T(\mathcal{O}))/T_t(\mathcal{O})$. But since we are taking the tensor product with respect to $R_{f,t}$, to finish giving the presentation we only need to describe the relations between the generators e^a for $a \in T(K)/T_t(\mathcal{O})$ of R_t and the T_i for $1 \leq i < n$. This is given by

$$T_{s_i} e^a - e^{s_i(a)} T_{s_i} = (e^{s_i(a)} - e^a) \frac{\sum_{x \in k^\times} e^{\alpha_i^\vee(x\pi)}}{1 - e^{\alpha_i^\vee(\pi)}}$$

Moreover, the center $Z(\mathcal{H}_t)$ of $\mathcal{H}_t$ equals to $R_t^{W_{f,t}}$.

Finally, let us give a characterization of tamely ramified representations. So consider an admissible irreducible representation Π of $\mathrm{GL}_n(K)$ over $\mathbb{C}$ which is tamely ramified (i.e. $\Pi^{I_t} \neq 0$). Since I_t is normal in I, the finite abelian group I/I_t acts on Π^{I_t}, so the latter splits into a sum of eigenspaces spaces

$$\Pi^{I_t,\chi} = \{v \in \Pi^{I_t} \mid gv = \chi(g)v \ \forall g \in I\}$$

indexed by the characters of $I/I_t = T(\mathcal{O})/T_t(\mathcal{O}) = T(k)$.

Theorem 5.3.2. *The space $\Pi^{I_t,\chi}$ is non-zero if and only if Π embeds in $\mathrm{Ind}_B^G(\chi_T)$ for some character χ_T of $T(F)$ whose restriction to $T(\mathcal{O})$ is χ. In particular Π is tamely ramified if and only if it is a constituent of $\mathrm{Ind}_B^G(\chi_T)$ for some tamely ramified χ_T.*

Chapter 6

The Automorphic theory

Let F be CM number field with a maximal totally real subfield F^+ such that F/F^+ is everywhere unramified and $[F^+:\mathbb{Q}]$ is even. We denote by δ_{F/F^+} the non trivial character of $\mathrm{Gal}(F/F^+)$ valued in $\{\pm 1\}$, which we also see as a character of $F^\times \setminus \mathbb{A}_{F^+}$ via the composition with Art_{F^+}.

We fix a prime number p, an isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$, and a finite extension E over $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ such that E contains every embedding $F \hookrightarrow \overline{\mathbb{Q}}_p$. We consider a finite set S of finite places of F^+ containing the set S_p of places above p. We assume that each place $v \in S$ splits in F, and we choose a place $\widetilde{v}$ of F lying above v. We set $\widetilde{S} = \{\widetilde{v} \mid v \in S\}$, $\widetilde{S}_p = \{\widetilde{v} \mid v \in S_p\}$, and we let $\widetilde{I}_p$ be the set of embeddings $F \hookrightarrow E$ inducing the places in $\widetilde{S}_p$.

We let $\mathbb{Z}^n_+ = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 \geq \cdots \geq a_n\}$ be the set of dominant weights of GL_n. If π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we say that π is regular algebraic of weight $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\mathbb{C})}$ if π^∞ has the same infinitesimal character as the dual of the algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_{n/F}) \times_{\mathbb{Q}} \mathbb{C} = \prod_{\mathrm{Hom}(F,\mathbb{C})}(\mathrm{GL}_{n/\mathbb{C}})$ of highest weight λ. A pair (π, χ) is said to be a polarized automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ if:

- π is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$,
- $\chi : \mathbb{A}^\times_{F^+}/(F^+)^\times \to \mathbb{C}$ is a continuous character such that $\chi_v(-1) = (-1)^n$,
- $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$.

where π^c denotes the composition of π with the complex conjugation on $\mathrm{GL}_n(\mathbb{A}_F)$.
From now on, we fix a cuspidal polarized regular algebraic automorphic representation $(\pi, \delta^n_{F/F^+})$ of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\lambda \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,\mathbb{C})}$ (we say that π is RACSDC: regular algebraic conjugate self dual cuspidal [All14, 2.1]). We assume that for each place $w \mid p$ of F, π_w has an Iwahori-fixed vector and that S contains all the places above which π is ramified.
We have the following instance of a theorem which we used before in the particular case $n = 2$ and F totally real (Theorem 3.1.5).

Theorem 6.0.1. *[BLGGT14, Theorem 2.1.1] There exists a continuous semi-simple representation*

$$r_{p,\iota}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

and and integer m with the following properties:

(1) For each finite place w, we have

$$\iota WD(r_{p,\iota}(\pi)_{|G_{F_w}})^{F\text{-}ss} \cong \mathrm{rec}(\pi_w \otimes |\cdot|_w \circ det^{\frac{1-n}{2}})$$

and the Weil-Deligne representations are pure of weight m.

(2) If $w \mid p$, then $r_{p,\iota}(\pi)_{|G_w}$ is de Rham such that for $\tau : F \to \overline{\mathbb{Q}}_p$, we have

$$\mathrm{HT}_\tau(r_{p,\iota}(\pi)) = \{\lambda_{\iota\tau,j} + n - j\}_{1\leq j\leq n}$$

Moreover, if π_w is unramified, then $r_{p,\iota}(\pi)_{|G_{F_w}}$ is crystalline.

After possibly enlarging E, we can suppose that there exists a model $\rho : G_F \to \mathrm{GL}_n(\mathcal{O})$ of $r_{p,\iota}(\pi)$. By conjugate self-duality, ρ extends to a homomorphism $r : G_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$ such that $\nu \circ r = \epsilon^{1-n}\delta^n_{F/F^+}$. By the condition imposed on the image of $r_{p,\iota}(\pi)$ in theorem 3.4.6, we can suppose by lemma 4.4.12 that $r_{p,\iota}(\pi)$ is absolutely irreducible. Consequently, we can apply results of section 4.4. In particular, if we denote by $\overline{D}$ the group determinant of $\overline{\rho}$, then we have pseudodeformation rings $R_{\overline{D},S}$ and R_S, with ρ inducing a morphism $R_S \to \mathcal{O}$.

We define additional deformation rings as follows: for a Taylor-Wiles datum $\big(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \dots, \alpha_{\widetilde{v},n})_{v\in Q}\big)$, we have a deformation ring $R_{S\cup Q}$. We let $R_{S\cup Q,\mathrm{ab}}$ be the maximal quotient of $R_{S\cup Q}$ such that for each $v \in Q$, the restriction of the universal pseudocharacter in $R_{S\cup Q,\mathrm{ab}}$ to the Weil group $W_{F_{\widetilde{v}}}$ factors through $W^{\mathrm{ab}}_{F_{\widetilde{v}}}$. Since $Q \cap S = \emptyset$ by definition, we get a composition of surjections

$$R_{S\cup Q} \to R_{S\cup Q,\mathrm{ab}} \to R_S$$

6.1 Definite unitary groups

Let A denote the matrix algebra $\mathrm{M}_n(F)$. An involution $\ddagger$ of the second kind on A (i.e., which restricts to c on F) gives rise to a reductive algebraic group $G_\ddagger$ over F^+ by setting

$$G_\ddagger(R) = \{g \in A \otimes_{F^+} R \mid g^\ddagger g = 1\}$$

for any F^+-algebra R. It is called the unitary group attached to $(F^+, F, A, \ddagger)$.
We have the following classification theorem for unitary groups over F^+.

Theorem 6.1.1. *[Bel, Theorem 1.1]*

1. *If G and G' are two unitary groups such that $G_v \cong G'_v$ for all place v of F^+, then $G \cong G'$.*
2. *Let $(G_v)_v$ be a family of unitary groups such that G_v is attached to the extension $F \otimes F^+_v/F^+_v$. Suppose that G_v is quasi-split for almost all places v. Then, if n is odd, there exists a unitary group $\widetilde{G}$ attached to F/F^+ such that for every place of F^+, $\widetilde{G}_v = G_v$. And if n is even, then the same is true if and only if we have $\prod_v \epsilon_v = 1$, where for a finite place v $\epsilon_v = 1$ if G_v is quasi-split and $\epsilon_v = -1$ otherwise, and $\epsilon_v = p_v - n/2$ if v is real with $G_v = U(p_v, n-p_v)$.*

Following this theorem, and our hypothesis that $[F^+ : \mathbb{Q}]$ is even, we may choose an involution $\ddagger$ satisfying:

- For every infinite place v of F^+, we have $G_\ddagger(F^+_v) \cong U_n(\mathbb{R})$,
- For every finite place v of F^+, $G_\ddagger$ is quasi-split at v.

We choose an order $\mathcal{O}_A$ of A such that $\mathcal{O}_A^\ddagger = \mathcal{O}_A$ and $\mathcal{O}_{A,w}$ is a maximal order of B_w for every place $w \in F$ which is split over F^+. This allows us to view $G_\ddagger$ as an algebraic group over $\mathcal{O}_{F^+}$, which we denote from now on by G.
For every finite place v of F^+ which splits as ww^c in F, we let us an isomorphism

$$\iota_v : \mathcal{O}_{A,v} \xrightarrow{\sim} \mathrm{M}_n(\mathcal{O}_{F,v}) = \mathrm{M}_n(\mathcal{O}_{F,w}) \oplus \mathrm{M}_n(\mathcal{O}_{F,w^c})$$

such that $\iota_v(x^\ddagger) = {}^t\iota_v(x)^c$. This induces an isomorphism

$$\iota_w : G(\mathcal{O}_{F_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_w})$$

sending $\iota_v^{-1}(x, {}^t x^{-c})$ to x and which extends to an isomorphism $G(F_v^+) \xrightarrow{\sim} \mathrm{GL}_n(F_w)$.
Now if v is a real place of F^+, we let $\kappa : F^+ \hookrightarrow \mathbb{R}$ be an embedding inducing v. For each $\widetilde{\kappa} : F \hookrightarrow \mathbb{C}$ extending κ, we choose an isomorphism

$$\iota_{\widetilde{\kappa}} : A \otimes_{F^+,\kappa} \mathbb{R} \xrightarrow{\sim} A \otimes_{F,\widetilde{\kappa}} \mathbb{C} = \mathrm{M}_n(\mathbb{C})$$

such that $\iota_{\widetilde{\kappa}}(x^\ddagger) = {}^t(\iota_{\widetilde{\kappa}}(x)^c)$. Then, $\widetilde{\kappa}$ identifies $G(F_v^+)$ with $U_n(\mathbb{R})$.

Let $T, B \subset \mathrm{GL}_n$ be respectively the maximal torus consisting of diagonal matrices, and the Borel subgroup of upper triangular matrices. If $\lambda \in X_*(T)$ is a dominant character of T, we can form the induced representation

$$\mathcal{V}_\lambda := \mathrm{Ind}_B^{\mathrm{GL}_n}(w_0\lambda)_{/\mathcal{O}} = \{f \in \mathcal{O}[\mathrm{GL}_n] \mid f(bg) = (w_0\lambda)(b)f(g), \quad \forall \mathcal{O} \to R, g \in \mathrm{GL}_n(R), b \in B(R)\}$$

with GL_n acting by right translation and where w_0 is the longest element in the Weyl group. This is an algebraic representation of $\mathrm{GL}_{n/\mathcal{O}}$. Since E is a flat $\mathcal{O}$-module, we have $\mathcal{V}_\lambda \otimes_\mathcal{O} E = \mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/E}$ which is the irreducible representation of $\mathrm{GL}_{n/E}$ of highest weight λ.
We let M_λ be the finite free $\mathcal{O}$-module obtained by evaluating $\mathcal{V}_\lambda$ on $\mathcal{O}$. It carries and action of $\mathrm{GL}_n(\mathcal{O})$, and $W_\lambda := M_\lambda \otimes_\mathcal{O} K$ carries an action of $\mathrm{GL}_n(K)$.

A dominant weight for G is a tuple $\lambda \in (\mathbb{Z}_+^n)^{\widetilde{I}_p}$, and for such a tuple, we let

$$M_\lambda = \otimes_{\tau \in \widetilde{I}_p, \mathcal{O}} M_{\lambda_\tau} \quad \text{and} \quad W_\tau = \otimes_{\tau \in \widetilde{I}_p, \mathcal{O}} W_{\lambda_\tau} = M_\lambda \otimes_\mathcal{O} E$$

and we define the representation

$$\begin{aligned} \mathcal{V}_\lambda : G(F_p^+) &\to \mathrm{GL}(W_\lambda) \\ g &\mapsto \otimes_{\tau \in \widetilde{I}_p} \mathcal{V}_{\lambda_\tau}\big(\tau(\iota_{\widetilde{v}(\tau)} g)\big) \end{aligned}$$

where $\widetilde{v}(\tau) \in \widetilde{S}_p$ is the place induced by τ. This restricts to a representation

$$\mathcal{V}_\lambda : G(\mathcal{O}_{F^+,p}) \to \mathrm{GL}(M_\lambda)$$

6.2 Automorphic data

Up until now, we fixed an RACSDC automorphic representation π of weight λ, and chose a unitary group G. We now define the space of algebraic automorphic forms with which we will work: for any $\mathcal{O}$-algebra, we let $S_\lambda(A)$ be the space of functions

$$f : G(F^+) \setminus G(\mathbb{A}_{F^+}^{\infty,p}) \to M_\lambda \otimes_\mathcal{O} A$$

such that there exists a compact open subgroup (depending on f)

$$U \subseteq G(\mathbb{A}_{F^+}^{\infty,p}) \times G(\mathcal{O}_{F^+,p})$$

with $u_p f(gu) = f(g)$ for all $u \in U, g \in G(\mathbb{A}_{F^+}^\infty)$.
The group $G(\mathbb{A}_{F^+}^{\infty,p}) \times G(\mathcal{O}_{F^+,p})$ acts on $S_\lambda(A)$ via the formula

$$(g \cdot f)(h) = g_p f(hg)$$

If U is a compact open subgroup of $G(\mathbb{A}_{F^+}^{\infty,p})\times G(\mathcal{O}_{F^+,p})$, we define $S_\lambda(U,A)$ to be the space of invariants $S_\lambda(A)^U$. If we take two of such groups U_1, U_2 and $g \in G(\mathbb{A}_{F^+}^{\infty,p}) \times G(\mathcal{O}_{F^+,p})$, then $\#U_1gU_2/U_2 < \infty$ (it is compact and discrete) and we can define a linear map

$$\begin{aligned}[U_1gU_2] : S_\lambda(U_2, A) &\to S_\lambda(U_1, A)\\ f &\mapsto \big(h \mapsto \sum_i (g_i)_p f(hg_i)\big)\end{aligned}$$

where $U_1gU_2 = \bigsqcup_i g_iU_2$.
In this setting, we say that compact open subgroup $U \subseteq G(\mathbb{A}_{F^+}^\infty)$ is sufficiently small if for all $g \in G(\mathbb{A}_{F^+}^\infty)$, $gUg^{-1} \cap G(F^+) = \{1\}$. This condition allows us to prove the following:

Lemma 6.2.1. *[CHT08, 3.3.1] Let $U \subseteq G(\mathbb{A}_{F^+}^{p,\infty}) \times G(\mathcal{O}_{F^+,p})$ be a sufficiently small open compact subgroup, $V \subseteq U$ be a normal open subgroup, and A an $\mathcal{O}$-algebra. Then, $S_\lambda(V,A)$ is a finite free $A[U/V]$-module and the trace map $\mathrm{Tr}_{U/V}$ induces an isomorphism $S_\lambda(V,A)_{U/V} \cong S_\lambda(U,A)$.*

For each embedding $\kappa : F^+ \hookrightarrow \mathbb{R}$, there is a unique complex embedding $\widetilde{\kappa} : F \hookrightarrow \mathbb{C}$ such that $\iota^{-1}\widetilde{\kappa} \in \widetilde{I}_p$, inducing a map $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{C}$. Then, $W_\lambda \otimes_{E,\iota} \mathbb{C}$ can be equipped with the following continuous $G(F_\infty^+)$-action

$$g \mapsto \otimes_\kappa \mathcal{V}_{\lambda_{\iota^{-1}\widetilde{\kappa}}}\big(\iota_{\widetilde{\kappa}}(\kappa(g))\big)$$

where $\mathcal{V}_{\iota^{-1}\kappa}$ is regarded as an algebraic representation of $\mathrm{GL}_n(\mathbb{C})$. We denote this representation by $\mathcal{V}_{\lambda,\iota}$. Let $\mathcal{A}$ be the space of automorphic forms on $G(F^+) \setminus G(\mathbb{A}_{F^+})$. We have an isomorphism of $G(\mathbb{A}_{F^+}^{\infty,p})$-modules

$$c_\iota : S_\lambda(\overline{\mathbb{Q}}_p) \otimes_{\overline{\mathbb{Q}}_p,\iota} \mathbb{C} \xrightarrow{\sim} \mathrm{Hom}_{G(F_\infty^+)}\big(\mathcal{V}_{\lambda,\iota}^\vee, \mathcal{A}\big) \tag{6.1}$$

given by

$$c_\iota(f)(\alpha)(g) = \alpha\big(\mathcal{V}_{\lambda,\iota}(g_\infty)^{-1}\big(g_p f(g^\infty)\big)\big)$$

There exists an automorphic representation of $G(\mathbb{A}_{F^+})$ with the following properties:

- For each finite inert place v of F^+, $\sigma_v^{G(\mathcal{O}_{F_v^+})} \neq 0$.
- For each split place $v = ww^c$ of F^+, $\sigma_v \cong \pi_w \circ \iota_w$.
- For each embedding $\kappa : F^+ \hookrightarrow \mathbb{R}$ inducing a place v of F^+ and each $\widetilde{\kappa} : F \hookrightarrow \mathbb{C}$ extending κ, we have $\sigma_v \cong \mathcal{V}^\vee_{\lambda_{\iota^{-1}\widetilde{\kappa}}} \circ \iota_{\widetilde{\kappa}} \circ \kappa$ (where $\mathcal{V}_{\lambda_{\iota\widetilde{\kappa}}}$ is a representation of $\mathrm{GL}_n(\mathbb{C})$).

Taking the U-invariants in the isomorphism (6.1), we get an isomorphism of $G(\mathbb{A}_{F^+}^{\infty,p})$-modules

$$S_\lambda(U,\mathcal{O}) \otimes_{\mathcal{O},\iota} \mathbb{C} \cong \oplus_\mu (\mu^\infty)^U \tag{6.2}$$

where the sum is over automorphic representations of $G(\mathbb{A}_{F^+})$ with multiplicity, such that $\mu_\infty \cong \sigma_\infty$.

6.3 Setup for patching

From now on we will work with a fixed open compact subgroup $U = \prod_v U_v$ of $G(\mathbb{A}_{F^+}^{p,\infty}) \times G(\mathcal{O}_{F^+,p})$ which satisfies:

- For each place v of S_p, $U_v = \iota_{\widetilde{v}}^{-1}(\mathrm{Iw}_{\widetilde{v}})$,
- For each inert place v of F^+, $U_v = G(\mathcal{O}_{F_v^+})$,

- $(\sigma^\infty)^U \neq 0$,
- U is sufficiently small.

Let $V = \prod_v V_v$ is a compact open subgroup of U with T is a finite set of places of F^+ containing all the places v such that $V_v \neq G(\mathcal{O}_{F_v^+})$, and let w be a place of F split over F^+ and lying over a place $v \notin T$ of F^+. For each $i = 1, \dots, n$ we let $T_w^{(i)}$ denote the endomorphism

$$\left[\iota_w^{-1}\left(\mathrm{GL}_n(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w 1_i & 0\\ 0 & 1_{n-i}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_w})\right) \times V^v\right]$$

of $S_\lambda(V, A)$, where ϖ_w is a uniformizer of $\mathcal{O}_{F_w}$. The operators $T_w^{(i)}$ for varying w and i all commute with each other and we write $\mathbf{T}_\lambda^T(V, A)$ for the A-subalgebra of $\mathrm{End}_A(S_\lambda(V, A))$ generated by these operators.

Recall that since π_w is an irreducible unramified representation $\mathrm{GL}_n(F_w)$, then there exists unramified characters $\chi_1, \dots, \chi_n$ of $F_w^\times$ such that

$$\pi_w \cong \chi_1 \boxplus \cdots \boxplus \chi_n$$

where $\chi_1 \boxplus \cdots \boxplus \chi_n$ is the unique unramified constituent of $\chi_1 \times \cdots \times \chi_n$. By the Iwasawa decomposition $\mathrm{GL}_n(F_w) = B(F_w)\,\mathrm{GL}_n(\mathcal{O}_{F_w})$, we have that $\dim_{\mathbb{C}} \pi_w^{\mathrm{GL}_n(\mathcal{O}_{F_w})} = 1$, hence we get a character

$$\psi_{\pi_w} : \mathcal{H}(G(F_v^+), V_v) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_p$$

sending $T_w^{(i)}$ to its eigenvalue which equals to $q_v^{\frac{i(n-i)}{2}} e_{i,v}$ where $e_{i,v}$ is the i-th symmetric polynomial in $\iota^{-1}(\chi_1(\varpi_w)), \dots, \iota^{-1}(\chi_n(\varpi_w))$. So by Langlands reciprocity, the characteristic polynomial of $r_{p,\iota}(\pi)(\mathrm{Frob}_w)$ is given by

$$X^n - \psi_{\pi_w}(T_w^{(1)})X^{n-1} + \cdots + (-1)^n \psi_{\pi_w}(T_w^{(n)})$$

Therefore, after possibly enlarging E, we get a homomorphism

$$h_{V,\sigma} : \mathbf{T}_\lambda^T(V, \mathcal{O}) \to \mathcal{O}$$

We let $\mathfrak{m}_V$ be the unique maximal ideal containing $\ker h_{V,\sigma}$. By the isomorphism (6.2), we get that $\mathbf{T}_\lambda^T(V, \mathcal{O})_{\mathfrak{m}_V} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p = \prod_\mu E_\mu$ is a product of fields indexed by automorphic representations μ of $G(\mathbb{A}_{F^+})$ with $\mu_{\mathfrak{m}_V}^V \neq 0$ and $\mu_\infty \cong \sigma_\infty$. By [All14, Corollary 2.2.4] for each of these μ, the induced morphism $\psi_\mu : \mathbf{T}_\lambda^T(V, \mathcal{O})_{\mathfrak{m}_V}[1/p] \to E_\mu$ gives rise to a Galois representation

$$r_\mu : G_{F,S} \to \mathrm{GL}_n(E_\mu) \tag{6.3}$$

such that for every finite place w of F which is split over $v \notin S$ in F^+, the characteristic polynomial of $r_\mu(\mathrm{Frob}_w)$ is

$$X^n - \psi_\mu(T_w^{(i)})X^{n-1} + \cdots + (-1)^n \psi_\mu(T_w^{(n)}) \tag{6.4}$$

Therefore, we get a huge Galois representation

$$r^{\mathrm{mod}} : G_{F,S} \to \prod_\mu \mathrm{GL}_n(E_\mu) = \mathrm{GL}_n(\mathbf{T}_\lambda^R(V, \mathcal{O})_{\mathfrak{m}_V} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p)$$

such that the coefficients of the characteristic polynomial of $r^{\mathrm{mod}}(\mathrm{Frob}_w)$ lie in $\mathbf{T}_\lambda^T(V, \mathcal{O})_{\mathfrak{m}_V} \hookrightarrow \prod_\mu E_\mu$ (the embedding is given by the ψ_μ) for almost all finite places w of F. Since the residual representation is not absolutely irreducible, Carayol's result does not apply and we do not necessarily get a Galois representation with coefficients in $\mathbf{T}_\lambda^T(V, \mathcal{O})_{\mathfrak{m}_V}$. However, by Chebotarev's density theorem and Corollary 4.2.4, the group determinant $\det \circ r^{\mathrm{mod}}$ factors through a group determinant of $G_{F,S}$ with values in $\mathbf{T}_\lambda^T(V, \mathcal{O})_{\mathfrak{m}_V}$ which we denote $D_{V,\lambda}$. Consequently, there is a surjective morphism $R_{\overline{D},S} \to \mathbf{T}_\lambda^T(V, \mathcal{O})_{\mathfrak{m}_V}$ that classifies $D_{V,\lambda}$.

Lemma 6.3.1. *The map $R_{\overline{D},S} \to \mathbf{T}^T_\lambda(V,\mathcal{O})_{\mathfrak{m}_V}$ factors through the quotient R_S.*

Proof. As seen above, $\mathbf{T}^T_\lambda(V,\mathcal{O})_{\mathfrak{m}_V} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p = \prod_\mu E_\mu$ is a product of fields indexed by automorphic representations μ. So it suffices to show that each induced map $R_{\overline{D},S} \to E_\mu$ factors through R_S. But this map classifies the determinant of the Galois representation r_μ associated to μ which satisfies the conjugate-self duality and the semi-stability conditions. □

Let us now consider a set Q of Taylor-Wiles places, and define the open compact subgroups $U_0(Q) = \prod_v U_0(Q)_v$ and $U_1(Q) = \prod_v U_1(Q)_v$ given by:

- If $v \notin Q$, let $U_0(Q) = U_1(Q) = U_v$,
- if $v \in Q$, let $U_0(Q)_v = \iota_{\widetilde{v}}^{-1}(\mathrm{Iw}_{\widetilde{v}})$ and let $U_1(Q)_v$ be the smallest open subgroup of $U_0(Q)_v$ such that $U_0(Q)_v/U_1(Q)_v$ is a p-group.

We set $\Delta_Q = U_0(Q)/U_1(Q)$ which is naturally isomorphic to $\prod_{v\in Q} k(v)^\times(p)^n$, where $k(v)^\times(p)$ is the maximal p-quotient of $k(v)^\times$.
To ease the notation, we set

$$S_\emptyset = S_\lambda(U,\mathcal{O})_{\mathfrak{m}_U} \quad \text{and} \quad \mathbf{T}_\emptyset = \mathbf{T}^S_\lambda(U,\mathcal{O})_{\mathfrak{m}_U}$$

We also let $\mathfrak{m}_Q = \mathfrak{m}_U \cap \mathbf{T}^{S\cup Q}_\lambda(U,\mathcal{O})$, we denote by $\mathfrak{m}_{0,Q}$ the pre-image of $\mathfrak{m}_Q$ in $\mathbf{T}^{S\cup Q}_\lambda(U_0(Q),\mathcal{O})$ and by $\mathfrak{m}_{1,Q}$ the pre-image of $\mathfrak{m}_{0,Q}$ in $\mathbf{T}^{S\cup Q}_\lambda(U_1(Q),\mathcal{O})$. We define:

$$\mathbf{T}_{0,Q} = \mathbf{T}^{S\cup Q}_\lambda(U_0(Q),\mathcal{O})_{\mathfrak{m}_{0,Q}} \quad \text{and} \quad \mathbf{T}_Q = \mathbf{T}^{S\cup Q}_\lambda(U_1(Q),\mathcal{O})_{\mathfrak{m}_{1},Q}$$

For a Taylor-Wiles place v, let Ξ_v be the quotient of $(F^\times_{\widetilde{v}})^n$ corresponding to $(k(v)(p)^\times)^n \times \mathbb{Z}^n$ under the noncanonical isomorphism $(F^\times_{\widetilde{v}})^n \cong (k(v)^\times)^n \times \mathbb{Z}^n$. We also let $\Xi^+_v \subset \Xi_v$ be the submonoid corresponding to $(k(v)(p)^\times)^n \times \mathbb{Z}^n_+$. We can define a multiplicative map $\mathcal{O}[\Xi^+_v] \to \mathcal{H}(G(F^+_v), U_1(Q)_v) \otimes_{\mathbb{Z}} \mathcal{O}$ sending $\lambda \in \Xi^+_v$ to $q_v^{\langle \lambda', \rho + (n-1)/2 \det\rangle}[\iota_{\widetilde{v}}^{-1}(I_t \lambda I_t)]$ where λ' is the image of λ under the projection $\Xi^+_v \to \mathbb{Z}^n_+ \cong X^+_*(T)$ and ρ is the halft sum of the positive roots. By the results in Section 5.3 and the fact that q_v is a unit in $\mathcal{O}$, these elements are invertible, thus we can uniquely extend this map to an $\mathcal{O}$-algebra morphism:

$$\mathcal{O}[\Xi_v] \to \mathcal{H}(G(F^+_v), U_1(Q)_v) \otimes_{\mathbb{Z}} \mathcal{O} \tag{6.5}$$

Given $\alpha \in F^\times_{\widetilde{v}}$, we let $t_{v,i}(\alpha) \in \mathcal{H}(G(F^+_v), U_1(Q)_v) \otimes_{\mathbb{Z}} \mathcal{O}$ be the image under this isomorphism of the element $(1,\dots,1,\alpha,1,\dots,1)$ where α is in the i-th position. We also let $e_{v,i}(\alpha) \in \mathcal{H}(G(F^+_v), U_1(Q)_v) \otimes_{\mathbb{Z}} \mathcal{O}$ be the term corresponding to $(-1)^i X^{n-i}$ in the polynomial $\prod_{i=1}^n (X - t_{v,i}(\alpha))$. The results of Section 5.3 give us the following proposition:

Proposition 6.3.2. *[ACC+ 18] Let π_v be an irreducible admissible $\overline{\mathbb{Q}}_p[G(F^+_v)]$-module.*

(1) We have $\pi_v^{U_1(Q)_v} \neq 0$ if and only if $\pi_v \circ \iota_{\widetilde{v}}^{-1}$ is isomorphic to a subquotient of a representation $\mathrm{Ind}^{\mathrm{GL}_n(F_{\widetilde{v}})}_{B(F_{\widetilde{v}})} \chi_1 \otimes \cdots \otimes \chi_n$, where $\chi : \chi_1 \otimes \cdots \otimes \chi_n : (F^\times_{\widetilde{v}})^n \to \mathbb{C}$ is a smooth character which factors through the quotient $(F^\times_{\widetilde{v}})^n \to \Xi_v$.

(2) Suppose that $\pi_v^{U_1(Q)_v} \neq 0$, then for any $\alpha \in F^\times_{\widetilde{v}}$, $e_{v,i}(\alpha)$ acts on $\pi_v^{U_1(Q)_v}$ as a scalar $e_{v,i}(\alpha,\pi_v) \in \overline{\mathbb{Q}}_p$.

(3) Suppose that $\pi_v^{U_1(Q)_v} \neq 0$, and let $(r_{\tilde{v}}, N_{\tilde{v}}) = \mathrm{rec}^T_{F_{\tilde{v}}}(\pi_v \circ \iota_{\tilde{v}}^{-1})$. Then, for any $\sigma \in W_{F_{\tilde{v}}}$, the characteristic polynomial of σ in $r_{\tilde{v}}$ is

$$X^n - e_{1,v}(\alpha, \pi_v)X^{n-1} + \cdots + (-1)^n e_{n,v}(\alpha, \pi_v)$$

where $\alpha = \mathrm{Art}^{-1}_{F_{\tilde{v}}}(\sigma_{|W^{ab}_{F_{\tilde{v}}}})$.

We define $\mathbf{T}^Q_{0,Q} \subseteq \mathrm{End}\left(S_\lambda(U_0(Q), \mathcal{O})_{\mathfrak{m}_{0,Q}}\right)$ to be the subalgebra generated by $\mathbf{T}_{0,Q}$ and the elements $t_{v,i}(\alpha)$ for all $v \in Q$, $1 \le i \le n$, and $\alpha \in F^\times_{\tilde{v}}$. Similarly, we define $\mathbf{T}^Q_Q \subseteq \mathrm{End}\left(S_\lambda(U_1(Q), \mathcal{O})_{\mathfrak{m}_{1,Q}}\right)$.

For each $v \in Q$, the universal pseudocharacter over $R_{S\cup Q,\mathrm{ab}}$ determines by restriction an n-dimentional pseudocharacter γ_v of $W^{\mathrm{ab}}_{F_{\tilde{v}}}$ with values in $R_{S\cup Q,\mathrm{ab}}$. Each restriction $\gamma_{v|I_{F_{\tilde{v}}}}$ factors through the quotient corresponding to $k(v)^\times(p)$ under $\mathrm{Art}_{F_{\tilde{v}}} : \mathcal{O}_{F_{\tilde{v}}} \xrightarrow{\sim} I_{F_{\tilde{v}}}$.
On the other hand, for each $1 \le i \le n$, there is a character $\alpha_{v,i} : W^{\mathrm{ab}}_{F_{\tilde{v}}} \to (\mathbf{T}^Q_Q)^\times$ sending $\mathrm{Art}_{F_{\tilde{v}}}(\alpha)$ to $t_{v,i}(\alpha)$ for $\alpha \in F^\times_{\tilde{v}}$. We let α_v be the pseudocharacter $\alpha_v = \mathrm{tr}\,\alpha_{v,1} \oplus \cdots \oplus \alpha_{v,n}$. By local-global compatibility, we can relate these two pseudocharacters of $W^{\mathrm{ab}}_{F_{\tilde{v}}}$ via the following lemma:

Lemma 6.3.3. *(1) The map $R_{S\cup Q} \to \mathbf{T}_Q$ factors through the quotient $R_{S\cup Q,ab}$.*

(2) For $v \in Q$, the composite of γ_v with the map $R_{S\cup Q,ab} \to \mathbf{T}_Q \hookrightarrow \mathbf{T}^Q_Q$ equals to α_v.

(3) The image of the map $R_{S\cup Q,ab} \to \mathbf{T}_Q \hookrightarrow \mathbf{T}^Q_Q$ contains the Hecke operators $e_{v,i}(\alpha)$ for each $v \in Q$, $1 \le i \le n$, and $\alpha \in F^\times_{\tilde{v}}$.

Proof. $\mathbf{T}_Q$ embeds into $\mathbf{T}_Q \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \cong \prod_\mu E_\mu$ which is a product of fields indexed by automorphic representations μ such that there exists a Galois representation $r_\mu : G_{F,S} \to E_\mu$ as in (6.3) with $\det \circ r_\mu$ inducing the morphism $R_{S\cup Q} \to \mathbf{T}_Q \to E_\mu$. Therefore, to prove (1), it suffices to show that for each $v \in Q$, the restriction of $\det \circ r_\mu$ to $W_{F_{\tilde{v}}}$ factors through $W^{\mathrm{ab}}_{F_{\tilde{v}}}$. But by (3) of Proposition 6.3.2, the coefficients of the characteristic polynomial of an element $\sigma \in W_{F_{\tilde{v}}}$ in $\mathrm{WD}(r_{\mu|G_{F_{\tilde{v}}}})$ depends only on its restriction to $W^{\mathrm{ab}}_{F_{\tilde{v}}}$, hence the result. (2) and (3) follow similarly, by comparing the characteristic polynomials in (6.4) and in (3) of Proposition 6.3.2. ☐

Using the morphism (6.5), we can give $\mathbf{T}^Q_Q$ the structure of a $\mathcal{O}[\Delta_Q]$-algebra, with the image of $\mathcal{O}[\Delta_Q]$ in $\mathbf{T}^Q_Q$ being generated by the $t_{v,i}(\alpha)$ with $\alpha \in \mathcal{O}^\times_{F_{\tilde{v}}}$. We denote by $\mathbf{a}_Q$ the augmentation ideal of $\mathcal{O}[\Delta_Q]$. Then, by Lemma 6.2.1, $S_\lambda(U_1(Q), \mathcal{O})$ is a finite free $\mathcal{O}[\Delta_Q]$-module and the trace map induces an isomorphism

$$S_\lambda(U_1(Q), \mathcal{O})/\mathbf{a}_Q \cong S_\lambda(U_0(Q), \mathcal{O}) \tag{6.6}$$

Let $A_Q = \otimes_{v\in Q}\mathcal{O}[(t_v^{(1)})^{\pm 1}, \ldots, (t_v^{(n)})^{\pm 1}]$ which, by sending $(t_v^{(i)})$ to the Hecke operator $t_{v,i}(\varpi_v)$, can be identified as in Section 5.2 with a subalgebra of

$$\otimes_{v\in Q}\mathcal{H}(G(F_v^+), U_0(Q)_v) \otimes_{\mathbb{Z}} \mathcal{O}$$

on which we have an action of the group $W_Q = \prod_{v\in Q}\mathfrak{S}_n$. As in (5.5), for every $m \ge 1$, we have a morphism of $\mathbf{T}_Q$-modules

$$\eta_{Q,m} : S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U} \otimes_{A_Q^{W_Q}} A_Q \to S_\lambda(U_0(Q), \mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q}} \tag{6.7}$$

The goal of Section 5.2 is to prove the following result:

Proposition 6.3.4. *Let $d \in \mathbb{N}$, there exists a constant $c \in \mathbb{N}$ such that for each N and each Taylor-Wiles datum $\big(Q, \widetilde{Q}, (\alpha_{\widetilde{v},1}, \dots, \alpha_{\widetilde{v},n})_{\widetilde{v}\in\widetilde{Q}}\big)$ for $r_{\iota,p}(\pi)$ of level N satisfying*

$$\sum_{v\in Q}\sum_{1\le i<j\le n} \mathrm{ord}_{\varpi}(\alpha_{\widetilde{v},i} - \alpha_{\widetilde{v},j}) \le d$$

there is an element $f_Q \in R_{S\cup Q,ab}$ such that

(1) f_Q kills the kernel and cokernel of $\eta_{Q,m}$ for all $m \le N$,

(2) the image $f_{Q,\sigma}$ of f_Q under the composition

$$R_{S\cup Q,ab} \to \mathbf{T}_Q \xrightarrow{h_{U_1(Q),\sigma}} \mathcal{O}$$

satisfies $\mathrm{ord}_{\varpi}(f_{Q,\sigma}) \le c$.

Proof. Let

$$\overline{f}_Q = \left(\prod_{v\in Q}\prod_{1\le i<j\le n}(t_{v,i}(\varpi_v) - t_{v,i}(\varpi_v))\right)^{n!} \in \mathbf{T}_Q^Q$$

By (3) of Lemma 6.3.3, $\overline{f}_Q$ lies in the image of $R_{S\cup Q,\mathrm{ab}}$, so let f_Q be in its pre-image. By Proposition 5.2.2, f_Q kills both the kernel and cokernel of $\eta_{Q,m}$ for all $n \le M$. If we set $c = n!d$, then the second part of the proposition follows by (2) of Lemma 6.3.3. $\square$

6.4 The patching argument

Let us fix $q = \mathrm{corank}_{\mathcal{O}} H^1(F_S/F^+, \mathrm{ad}\rho(1) \otimes_{\mathcal{O}} E/\mathcal{O})$. Applying theorem 4.4.18, we consider a Taylor-Wiles datum Q_N of level N for each $N \ge 1$, and to simplify the notation, we write $\Delta_N = \Delta_{Q_N}$, $\mathbf{a}_N = \mathbf{a}_{Q_N}$, $\mathbf{T}_N = \mathbf{T}_{Q_N}$ and $R_N = R_{S\cup Q_N,\mathrm{ab}}$ with $R_0 = R_S$. We set $\mathfrak{q}_N = \ker(R_N \xrightarrow{h_{U_1(Q_N),\sigma}} \mathcal{O})$ and $\mathfrak{q}_0 = \ker(R_0 \xrightarrow{h_{U,\sigma}} \mathcal{O})$.
We let $g = nq$, and $R_\infty = \mathcal{O}[[x_1, \dots, x_g]]$ and $\mathfrak{q}_\infty = (x_1, \dots, x_g)$. For each $N \ge 1$, there exists a morphism $R_\infty \to R_N$ such that $\mathfrak{q}_\infty R_N \subseteq \mathfrak{q}_N$ and $\mathfrak{q}_N/(\mathfrak{q}_N^2, \mathfrak{q}_\infty)$ is killed by a power of ϖ which is independent of N.

We fix an ordering on each Q_N and generators of $k(v)^\times(p)$ for all N and all $v \in Q_N$, and we let $S_\infty = \mathcal{O}[[y_1^{(i)}, \dots, y_q^{(i)} \ : \ 1 \le i \le n]]$ so that we have a fixed surjective homomorphism $S_\infty \to \mathcal{O}[\Delta_N]$ for each $N \ge 1$ as in (3.7). We let $\mathbf{a}_\infty = \langle y_j^{(i)} \mid 1 \le j \le q,\ 1 \le i \le n\rangle$ be the augmentation ideal of S_∞ which corresponds to the inverse image of $\mathbf{a}_N$ under each of these morphisms.
Using the fixed ordering on the Q_N, we can identify all the Weyl groups $W_{Q_N} = \prod_{v\in Q_N} \mathfrak{S}_n$, and we will denote them by W. Then, W acts on S_∞ by permutation of the coordinates, and we can write $S_\infty^W = \mathcal{O}[[e_1^{(i)}, \dots, e_q^{(i)} \ : \ 1 \le i \le n]]$ where $e_j^{(i)}$ is the i-th symmetric polynomial in $y_j^{(1)}, \dots, y_j^{(n)}$. This is a regular local $\mathcal{O}$-algebra with S_∞ being a finite free S_∞^W-algebra.

If we fix a uniformizer ϖ_v for every $v \in Q_N$ and every $N \ge 1$, then we can think of the pseudocharacters γ_v as pseudocharacters of $k(v)^\times(p) \times \mathbb{Z}$. And since we have fixed a generator of $k(v)^\times(p)$, which corresponds to a surjection $\mathbb{Z}_p \to k(v)^\times(p)$, we get by pullback for every N a q-tuple $(\gamma_{1,N}, \dots, \gamma_{q,N})$ of pseudocharacters of $(\mathbb{Z}_p \times \mathbb{Z})$ with coefficients in R_N.
Thanks to the following lemma, we get for each $N \ge 1$ a homomorphism $S_\infty^W \to R_N$ classifying the q-tuple $(\gamma_{1,N}, \dots, \gamma_{q,N})$.

Lemma 6.4.1. *The functor of deformation of the pseudocharacter of $\mathbb{Z}_p$ attached to the trivial representation of dimension n is represented by $\mathcal{O}[[X_1, \ldots, X_n]]^{\mathfrak{S}_n}$, with the univeresal characteristic polynomial of $1 \in \mathbb{Z}_p$ is equal to*

$$\chi(1, t) = \prod_{i=1}^{n} \big((t-1) - X_i\big)$$

Proof. Let Θ be a continuous pseudocharacter of $\mathbb{Z}_p$ over $A \in \mathcal{C}_{\mathcal{O}}$, $f \in \mathbb{Z}[\mathrm{GL}_n]^{\mathrm{GL}_n}$ and $k \geq 1$. Recursively applying (2) of Definition 4.3.1, we get that

$$\Theta_1(f)(\gamma_1 \cdots \gamma_k) = \Theta_k(\widehat{f}^k)(\gamma_1, \ldots, \gamma_k)$$

where $\widehat{f}^k(g_1, \ldots, g_k) = f(g_1 \cdots g_k)$. Also applying (1) of the same definition for the function $\zeta : \{1, \ldots, k\} \to \{1\}$ defined in the obvious way, we get

$$\Theta_1((\widehat{f}^k)^{\zeta})(\gamma) = \Theta_k(\widehat{f}^k)(\gamma, \ldots, \gamma)$$

So after combining the two equalities, we get that for each $k \geq 1$,

$$\Theta_1((\widehat{f}^k)^{\zeta})(1) = \Theta_1(f)(k)$$

Since a pseudocharacter is uniquely determined by the morphism Θ_1, we conclude by continuity of Θ and equation (4.6) that it is uniquely determined by the ring homomorphism

$$\begin{aligned} \theta : \mathbb{Z}[\lambda_1, \ldots, \lambda_n] &\to A \\ \lambda_i &\mapsto \Theta_1(\lambda_i)(1) \end{aligned}$$

where the λ_i are algebraically independent (the restriction of λ_i to the diagonal torus is equal to the i-th symmetric function). Therefore, we see that a residually trivial pseudocharacter of $\mathbb{Z}_p$ of dimension n over a ring $A \in \mathcal{C}_{\mathcal{O}}$ corresponds to such a θ with its reduction modulo $\mathfrak{m}_A$ being the morphism

$$\begin{aligned} \mathbb{Z}[\lambda_1, \ldots, \lambda_n] &\to k \\ \lambda_i &\mapsto \lambda_i(\mathrm{id}) \end{aligned}$$

This is equivalent to giving a continuous $\mathcal{O}$-algebra morphism $\mathcal{O}[[X_1, \ldots, X_n]]^{\mathfrak{S}_n} \to A$. $\square$

Using the ordering on Q_N, we obtain an action of $A = \otimes_{j=1}^{q} \mathcal{O}[(t_j^{(1)})^{\pm}, \ldots, (t_j^{(n)})^{\pm}]$ on the spaces $S_\lambda(U_0(Q_N), \mathcal{O})_{\mathfrak{m}_{0,Q_N}}$ and $S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q_N}}$ via the identification of A with A_{Q_N}. We have characters $\alpha_j^{(i)} : \mathbb{Z}_p \times \mathbb{Z} \to (S_\infty \otimes_{\mathcal{O}} A)^\times$ defined in the obvious way for $1 \leq i \leq n$ and $1 \leq j \leq q$. By Lemma 6.3.3, the pushforward of the pseudocharacter $\alpha_j = \mathrm{Tr}\, \alpha_j^{(1)} \oplus \cdots \oplus \alpha_j^{(n)}$ to $\mathrm{End}_{\mathcal{O}}(S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q_N}})$ lies in $\mathbf{T}_N$ and is equal to the pushforward of $\gamma_{N,j}$ there.

The patching techniques that we are going to use where developed in [Lue19] for completed cohomology, but were adapted to our setting in [NT20]. We will work with ultrafilters, so let $\mathcal{F}$ be a non-principal ultrafilter of $\mathbb{N}$, and let $\mathbf{R} = \prod_{N \in \mathbb{N}} \mathcal{O}$. Then, the localisation $\mathbf{R}_{\mathfrak{p}(\mathcal{F})}$ is a quotient of $\mathbf{R}$, and the quotient map $\mathbf{R} \to \mathbf{R}_{\mathfrak{p}(\mathcal{F})}$ factors through the projection $\prod_{N \geq 1} \mathcal{O} \to \prod_{N \geq m} \mathcal{O}$ for all $m \geq 1$.

Lemma 6.4.2. *Suppose for any $i \in \mathbb{N}$, M_i is an $\mathcal{O}$-module with decreasing filtrations of $\mathcal{O}$-modules $M_i \supseteq M_{i,1} \supseteq M_{i,2} \supseteq \cdots$. Then, the natural map*

$$\prod_{i \in \mathbb{N}} M_i \to \varprojlim_n \big((\prod_{i \in \mathbb{N}} M_i / M_{i,n}) \otimes_{\mathbf{R}} \mathbf{R}_{\mathfrak{p}(\mathcal{F})}\big)$$

is surjective. Then, kernel consists of all the elements of the form $(m_i)_{i \in \mathbb{N}}$ such that for any n, there exists $I_n \in \mathcal{F}$ with $m_i \in M_{i,n}$ for any $i \in I_n$.

Proof. An element $m = (m_i)_{i\in\mathbb{N}}$ of $\prod_{i\in\mathbb{N}} M_i$ is in the kernel of the above map if and only if for every $n \geq 1$, m is sent to zero in $\big((\prod_{i\in\mathbb{N}} M_i/M_{i,n}) \otimes_{\mathbf{R}} \mathbf{R}_{\mathfrak{p}(\mathcal{F})}\big)$, i.e., if there exists $I_n \in \mathcal{F}$ such that for all $i \in I_n$, $m_i = 0$ in $M_i/M_{i,n}$, in other words, $m_i \in M_{i,n}$. To prove surjectivity, let $[(m_{i,n})_i]_n$ be an element of the right hand side, with $m_{i,n} \in M_i/M_{i,n}$. By compatibility in the projective limits, for each $n \geq 1$, there exists $I_n \in \mathcal{F}$ such that for all $i \in I_n$, $a_{i,n} \equiv a_{i,n+1} \mod M_{i,n}$. Since $\mathcal{F}$ is stable by intersections and is a non principal ultrafilter, we can assume that $I_n \supseteq I_{n+1}$, and that the intersection of all the I_n is empty (for example we can replace I_n by $I_n \setminus \{n\}$). For any $i \in I_n \setminus I_{n+1}$, let m_i be a lift of $m_{i,n+1}$ to M_i, and for any $i \notin I_1$, set $m_i = 0$. Then, the element $(m_i)_i \in \prod_{i\in\mathbb{N}} M_i$ is sent to $[(m_{i,n})_i]_n$. ☐

Remark 6.4.3. Since $\mathcal{O}/(\varpi^m)$ has finite cardinality, we have an isomorphism induced by the diagonal map

$$\mathcal{O}/(\varpi^m) \cong (\prod_{i\in\mathbb{N}} \mathcal{O}/(\varpi^m) \otimes_{\mathbf{R}} \mathbf{R}_{\mathfrak{p}(\mathcal{F})})$$

Thus, taking the inverse limit of the maps $\prod_{i\in\mathbb{N}} \mathcal{O} \to (\prod_{i\in\mathbb{N}} \mathcal{O}/(\varpi^m) \otimes_{\mathbf{R}} \mathbf{R}_{\mathfrak{p}(\mathcal{F})})$ we get a map

$$\prod_{i\in\mathbb{N}} \mathcal{O} \to \mathcal{O}$$

which is surjective by Lemma 6.4.2, and whose kernel is formed by the tuples $(a_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} \mathcal{O}$ such that for any $m \in \mathbb{N}$, there exists $I_m \in \mathcal{F}$ with $a_i \in (\varpi^m)$ for any $i \in I_m$.

Definition 6.4.4. We let:

- $M_1 = \varprojlim_m \Big(\mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} \big(S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_{1,Q_N}}/\mathfrak{m}^m_{S_\infty}\big)\Big)$,
- $M_0 = \varprojlim_m \Big(\mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} S_\lambda(U_0(Q_N), \mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q_N}}\Big)$,
- $M = \varprojlim_m \Big(\mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U} \otimes_{A^W_{Q_N}} A_{Q_N}\Big)$.

The action of $A^W_{Q_N}$ on $S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U}$ is via the spherical Hecke algebra at the places in Q_N. Via the identification $A = A_{Q_N}$ for each $N \geq 1$, we obtain compatible actions of A on M_1, M_0 and M. Since $S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U}$ is finite, we have an isomorphism

$$S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U} = \mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} S_\lambda(U_1(Q_N), \mathcal{O}/\varpi^m)_{\mathfrak{m}_U}$$

and we get that

$$S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U} \cong \varprojlim_m \left(\mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} S_\lambda(U_1(Q_N), \mathcal{O}/\varpi^m)_{\mathfrak{m}_U}\right)$$

So we can equip $S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U}$ with an A^W-action (on each factor it acts via $A^W_{Q_N}$), and we have an isomorphism $M \cong S_\lambda(U, \mathcal{O})_{\mathfrak{m}} \otimes_{A^W} A$.

We give the following technical lemmas that we will be using.

Lemma 6.4.5. *Let R be a complete Noetherian local ring.*

(1) *Suppose* $\{M_i\}_{i\in\mathbb{N}}$ *is a projective system of flat* R*-modules. Then,* $M = \varprojlim M_i$ *is also flat over* R *and* $M/JM \cong \varprojlim M_i/JM_i$ *for any ideal* J *of* R*.*

(2) *Let* I *be an ideal of* R*, and* $\{N_i\}_{i\in\mathbb{N}}$ *be a projective system of* R*-modules such that* N_i *is a flat* R/I^i*-module and the transition maps induce isomorphisms* $N_i \cong N_{i+1}/I^iN_{i+1}$*. Then,* $N = \varprojlim N_i$ *is a flat* R*-module and* $N_i \cong N/I^iN$*. Moreover, for every ideal* J *of* R*, we have* $N/JN \cong \varprojlim N_i/JN_i$*.*

Lemma 6.4.6. *Let* R *be a Noetherian ring, and* $M = \prod_{i\in I} M_i$ *be a product of* R*-modules* M_i*. Then,* $M/IM \cong \prod_{i\in I} M_i/IM_i$ *for every ideal* I *of* R*.*

Proof. We have a canonical surjective morphism

$$f : M/IM \twoheadrightarrow \prod_{i\in I} M_i/IM_i$$

to show that it is injective, consists of showing that $(a_i)_{i\in I} \in IM$ for $a_i \in IM_i$. But since I is finitely generated, say by $f_1, \ldots, f_r$, we can write $(a_i)_{i\in I} = f_1(a_i^{(1)})_{i\in I} + \cdots f_r(a_i^{(r)})_{i\in I} \in IM$ as desired. $\square$

Lemma 6.4.7. *Let* R *be a Noetherian ring and* $\{M_i\}_{i\in\mathbb{N}} \to \{N_i\}_{i\in\mathbb{N}}$ *be a map between two projective systems of* R*-modules. Suppose that there exists an element* $f \in R$ *that kills the kernel and the cokernel of* $M_i \to N_i$ *for every* $i \in \mathbb{N}$*. Then,* f^2 *kills the kernel and cokernel of* $\varprojlim M_i \to \varprojlim N_i$*.*

Proof. Let $(n_i)_i \in \varprojlim N_i$, then by hypothesis, for every i, there exists $b_i \in M_i$ mapping to $fa_i \in N_i$. Since the maps are compatible and their kernels are killed by f, we get that $fb_{i+1} = fb_i$, so $(fb_i)_i$ is a well defined element of $\varprojlim M_i$ which is sent to $f^2(a_i)_i \in \varprojlim N_i$. The statement on the kernels is straightforward. $\square$

Proposition 6.4.8. *The following properties are true:*

(1) M_1 *is a flat* S_∞*-module.*

(2) *The trace map induces* $M_1/\mathbf{a}_\infty \cong M_0$*.*

(3) *There is a map* $\eta : M \to M_0$ *induced by the* $\eta_{Q_N,m}$*, which has kernel and cokernel killed by* f*, where* $f = (f_{Q_N}^2) \in \prod_{N\in\mathbb{N}} R_N$*,* f_{Q_N} *as in the statement of Proposition 6.3.4.*

Proof. We let for each m,

$$M_{1,m} = \mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} \left(S_\lambda(U_1(Q_N)), \mathcal{O})_{\mathfrak{m},Q_N}/\mathfrak{m}_{S_\infty}^m\right)$$

1) & 2) Since $S_\lambda(U_1(Q_N), \mathcal{O})$ is a free $\mathcal{O}[\Delta_N]$ module, and that $S_\infty/\mathfrak{m}_{S_\infty}^m$ for $m \leq N$ is a quotient of $\mathcal{O}[\Delta_N]$, we get that $S_\lambda(U_1(Q_N), \mathcal{O})_{\mathfrak{m}_1,Q_N}/\mathfrak{m}_{S_\infty}^m$ is a flat $S_\infty/\mathfrak{m}_{S_\infty}^m$-module. Hence, $\prod_{N\geq m} \left(S_\lambda(U_1(Q_N)), \mathcal{O})_{\mathfrak{m},Q_N}/\mathfrak{m}_{S_\infty}^m\right)$ is flat over $S_\infty/\mathfrak{m}_{S_\infty}^m$, and by flatness of $\mathbf{R}_{\mathfrak{p}(\mathcal{F})}$ over $\mathbf{R}$, we see that $M_{1,m}$ is a flat $S_\infty/\mathfrak{m}_{S_\infty}^m$-module. Moreover, by Lemma 6.4.6, we have

$$M_{1,m+1}/\mathfrak{m}_{S_\infty}^m = \mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m+1} \left(S_\lambda(U_1(Q_N)), \mathcal{O})_{\mathfrak{m},Q_N}/\mathfrak{m}_{S_\infty}^m\right) = M_{1,m}$$

and,

$$\begin{aligned} M_{1,m}/\mathbf{a}_\infty &= \mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} \left(S_\lambda(U_1(Q_N)), \mathcal{O})_{\mathfrak{m}_1,Q_N}/(\mathbf{a}_\infty, \mathfrak{m}_{S_\infty}^m)\right) \\ &= \mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq m} \left(S_\lambda(U_1(Q_N)), \mathcal{O}/\varpi^m)_{\mathfrak{m}_0,Q_N}\right) \end{aligned}$$

So using (2) of Lemma 6.4.5, we conclude that M_1 is a flat S_∞-module and that $M_1/\mathbf{a}_\infty \cong M_0$.
3) This follows from (6.4.7). $\square$

Definition 6.4.9 (The patched pseudo-deformation ring)**.**
For $m \geq 1$, we define $R^{\mathrm{p}}_m = \mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod\limits_{N\geq 1} R_N/(\mathfrak{m}_{R_N} f_{Q_N})^m$, and $R^{\mathrm{p}} = \varprojlim\limits_m R^{\mathrm{p}}_m$.

Thanks to the following lemma, we have an action of R^{p} over M_1.

Lemma 6.4.10. *For each $m \geq 1$, there exists an integer $n(m)$ which is independent of N, such that $(\mathfrak{m}_{R_N} f_{Q_N})^{n(m)}$ annihilates $S_\lambda(U_1(Q), \mathcal{O})_{m_1,Q_N}/\mathfrak{m}^m_{S_\infty}$ for all $N \geq m$.*

Proof. Since $\mathbf{a}_\infty \subseteq \mathfrak{m}_\infty$, it suffices to prove that there exists an integer $n(m)$ which is independent of N such that for all $N \geq m$, $S_\lambda(U_0(Q), \mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q_N}} = S_\lambda(U_0(Q), \mathcal{O})_{\mathfrak{m}_{0,Q_N}} \otimes \mathcal{O}/\varpi^m$ is annihilated by $(\mathfrak{m}_{R_N} f_{Q_N})^{n(m)}$.
Now since f_{Q_N} annihilates the cokernel of the map (6.7), the length of $f_{Q_N} S_\lambda(U_0(Q), \mathcal{O}/\varpi^m)_{\mathfrak{m}_{0,Q_N}}$ as an $\mathcal{O}$-module is bounded by $qn!$ times the length of $S_\lambda(U, \mathcal{O}/\varpi^m)_{\mathfrak{m}_U}$. Its length as an R_N-module is bounded independently of N, so it is annihilated by some power of $\mathfrak{m}_{R_N}$ which is independent of N. $\square$

We have a natural map $\prod_{N\geq 1} R_N \to R^{\mathrm{p}}$ which is surjective thanks to Lemma 6.4.2. Moreover, taking the limit of the maps

$$R^{\mathrm{p}}_m \to \mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq 1} R_0/\mathfrak{m}^m_{R_0} = R_0/\mathfrak{m}^m_{R_0}$$

(the last equality holds since $R_0/\mathfrak{m}^m_{R_0}$ is finite), we get a natural map $R^{\mathrm{p}} \to R_0$. This map is surjective, since by Lemma 6.4.2, we have a surjection

$$\prod_{N\geq 1} R_N \to \varprojlim_m \left(\mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq 1} R_0/\mathfrak{m}^m_{R_0} \right)$$

which factors through R^{p}.

For $1 \leq j \leq q$, let $\gamma_{\infty,j}$ be the n-dimensional pseudocharacter of $\mathbb{Z}_p \times \mathbb{Z}$ with coefficients in R^{p} given by the composition of the pseudo-character $(\gamma_{N,j})_{N\geq 1}$ with the map $\prod_{N\geq 1} R_N \to R^{\mathrm{p}}$.

Lemma 6.4.11. *Let $1 \leq j \leq q$.*

(1) Composing $\gamma_{\infty,j}$ with the map $R^{\mathrm{p}} \to R_0$ gives a pseudocharacter which is inflated from the 'unramified quotient' $\mathbb{Z}_p \times \mathbb{Z} \to \mathbb{Z}$ (it is the projection on the second factor).

(2) The module M_1 has a natural structure of $(S_\infty \otimes_{\mathcal{O}} A)$-module, and the composite of $\gamma_{\infty,j}$ with the map $R^{\mathrm{p}} \to \mathrm{End}(M_1)$ equals to the composition of α_j with the map $S_\infty \otimes_{\mathcal{O}} A \to \mathrm{End}(M_1)$. Consequently, the map $R^{\mathrm{p}} \to \mathrm{End}(M_1)$ is a homomorphism of S^W_∞-algebras.

Proof. **1)** This holds thanks to the analogous statement for the $\gamma_{N,j}$ which is true since R_0 classifies pseudorepresentations which are unramified at the places in Q_N.
2) This follows from the definition of the α_j and Lemma 6.3.3. $\square$

We let the ideal $\mathfrak{q}_{\mathrm{p}}$ be the inverse image of $\mathfrak{q}_0$ under the morphism $R^{\mathrm{p}} \to R_0$.

Lemma 6.4.12. *The image of $\prod\limits_{N\geq 1} \mathfrak{q}_N$ under $\prod\limits_{N\geq 1} R_N \to R^{\mathrm{p}}$ is $\mathfrak{q}^{\mathrm{p}}$.*

Proof. We have commutative diagram

$$\begin{tikzcd}
\prod_{N\geq 1} R_N \arrow[rr, two heads] \arrow[d] & & \prod_{N\geq 1} \mathcal{O} \arrow[d, "\phi"] \\
R^{\mathrm{p}} \arrow[r, two heads] & R_0 \arrow[r, two heads] & R_0/\mathfrak{q}_0 \cong \mathcal{O}
\end{tikzcd}$$

where the map ϕ is defined in Remark 6.4.3. We let $I = \ker\left(\prod_{N\geq 1} R_N \to R^{\mathrm{p}}\right)$ and I' be the image of I inside $\prod_{N\geq 1}\mathcal{O}$. Then, by commutativity of the diagram, we have that inside $\prod_{N\geq 1} R_N$, $\left((\prod_{N\geq 1}\mathfrak{q}_N), \ker\phi\right) = \left(I, \mathfrak{q}^{\mathrm{p}}\right)$. So to prove the lemma, it is enough to show that $I' = \ker\phi$.
By Lemma 6.4.2, I is consists of elements $(x_N)_{N\geq 1} \in \prod R_N$ such that for each $m \geq 1$, there exists $I_m \in \mathcal{F}$ with $x_N \in (f_{Q_N}\mathfrak{m}_{R_N})^m$ for all $N \in I_m$. But by definition of $\mathfrak{q}_N$, we have that

$$(f_{Q_N}\mathfrak{m}_{R_N})^m + \mathfrak{q}_N = (f_{Q_N}\varpi)^m + \mathfrak{q}_N \subseteq (\varpi^m) + \mathfrak{q}_N$$

This shows that $I' \subseteq \ker\phi$. For the other inclusion, let $(y_N)_{N\geq 1} \in \ker\phi$. By Proposition 6.3.4, there exists a constant $c \in \mathbb{N}$ such that for each $N \geq 1$, the image of f_{Q_N} inside $R_N/\mathfrak{q}_N = \mathcal{O}$ has ϖ-adic valuation $\leq c$. For $m \geq 1$, we let $I_m = \{N \geq 1 \mid \mathrm{ord}_\varpi(y_N) \geq m(c+1)\}$ so that $I_m \in \mathcal{F}$ (since $(y_N)_{N\geq 1} \in \ker\phi$), $I_1 \supseteq I_2 \subset I_3 \supseteq \cdots$, and $\cap_{m\geq 1} I_m = \emptyset$. Since $(f_{Q_N}\mathfrak{m}_{R_N})^m + \mathfrak{q}_N = (f_{Q_N}\varpi)^m + \mathfrak{q}_N \supseteq (\varpi^{m(1+c)}) + \mathfrak{q}_N$, for each $m \geq 1$, and $N \in I_m$, there exists $x_{N,m} \in (f_{Q_N}\mathfrak{m}_{Q_N})^m$ such that $x_{N,m} \equiv y_N \mod \mathfrak{q}_N$. Define $(x_N)_{N\geq 1} \in \prod R_N$ by letting x_N be any lift of y_N to R_N for $N \notin I_1$ and $x_N = x_{N,m}$ for $N \in I_m - I_{m+1}$. Then, $(x_N)_{N\geq 1}$ lies inside I and maps to $(y_N)_{N\geq 1}$. This shows the second equality $\ker\phi \subseteq I'$, and the lemma. □

Lemma 6.4.13. *Let $m \geq 1$.*

1. *We have $\prod_{N\geq 1}\mathfrak{q}_N^m = (\prod_{N\geq 1}\mathfrak{q}_N)^m$ in $\prod_{N\geq 1} R_N$.*
2. *The image of $\prod_{N\geq 1}\mathfrak{q}_N^m$ inside R^{p} is equal to $(\mathfrak{q}^{\mathrm{p}})^m$.*

Proof. Since the image of $\prod_{N\geq 1}\mathfrak{q}_N$ in R^{p} is $\mathfrak{q}_{\mathrm{p}}$, the first statement implies the second. Now by theorem 4.4.18, there exists an integer g_0 such that for every $N \geq 1$, $\mathfrak{q}_N/\mathfrak{q}_N^2$ is generated by g_0 elements. Thus, for every $N \geq 1$, there exists a surjection $\mathcal{O}[[x_1, \ldots, x_{g_0}]] \twoheadrightarrow R_N$ where the image of the x_i lies in $\mathfrak{q}_N$. So the result follows from the equality

$$\begin{aligned}
\prod_{N\geq 1}(x_1, \ldots, x_{g_0})^m &= (x_1, \ldots, x_{g_0})^m \cdot \prod_{N\geq 1}\mathcal{O}[[x_1, \ldots, x_{g_0}]] \\
&= \left((x_1, \ldots, x_{g_0}) \cdot \prod_{N\geq 1}\mathcal{O}[[x_1, \ldots, x_{g_0}]]\right)^m = \left(\prod_{N\geq 1}(x_1, \ldots, x_{g_0})\right)^m
\end{aligned}$$

where the first and last equality follows from the fact that $(x_1, \ldots, x_{g_0})$ is a finitely generated ideal of $\prod_{N\geq 1}\mathcal{O}[[x_1, \ldots, x_{g_0}]]$. □

Proposition 6.4.14. *(1) The $\mathcal{O}$-module*

$$\mathfrak{q}^{\mathrm{p}}/\left((\mathfrak{q}^{\mathrm{p}})^2, \mathfrak{q}_\infty\right)$$

is killed by ϖ^c, where c is as in theorem 4.4.18.

(2) The natural map of completed local rings

$$(R_\infty)^\wedge_{\mathfrak{q}_\infty} \to (R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$$

is surjective. In particular, $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$ is a complete noetherian local E-algebra with residue field E.

Proof. **1)** From theorem 4.4.18, we have that the cokernel of the map

$$\prod_{N\geq 1} \mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \prod_{N\geq 1} \mathfrak{q}_N/(\mathfrak{q}_N)^2$$

is killed by ϖ^c. So using Lemma 6.4.13, we only need to show that the images of $\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2$ and $\prod_{N\geq 1} \mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2$ coincide inside $\mathfrak{q}^{\mathrm{p}}/(\mathfrak{q}^{\mathrm{p}})^2$. First note that the map $\prod_{N\geq 1} \mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \mathfrak{q}^{\mathrm{p}}/(\mathfrak{q}^{\mathrm{p}})^2$ is a morphism of $\prod_{N\geq 1} \mathcal{O}$-modules. But since we have a composition of $\mathcal{O}$-modules

$$\prod_{N\geq 1} \mathcal{O} \to R^{\mathrm{p}} \to R_0/\mathfrak{q}_0 \to \mathcal{O}$$

(which is equal to the morphism defined in Remark 6.4.3), the two actions of $\prod_{N\geq 1} \mathcal{O}$ on R^{p}, where one factors through $\prod_{N\geq 1} \mathcal{O} \to \mathcal{O}$, coincide modulo $\mathfrak{q}^{\mathrm{p}}$. Hence, the action of $\prod_{N\geq 1} \mathcal{O}$ on $\mathfrak{q}^{\mathrm{p}}/(\mathfrak{q}^{\mathrm{p}})^2$ factors through $\prod_{N\geq 1} \mathcal{O} \to \mathcal{O}$. Therefore, it suffices to show that the composition

$$\mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \prod_{N\geq 1} \mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2 \to \mathcal{O} \otimes_{\prod_{N\geq 1} \mathcal{O}} \prod_{N\geq 1} \mathfrak{q}_\infty/(\mathfrak{q}_\infty)^2$$

is surjective. But this follows from the next Lemma 6.4.15.
2) By (1), for each $i \geq 1$ the $\mathcal{O}$-module $\mathfrak{q}^{\mathrm{p}}/((\mathfrak{q}^{\mathrm{p}})^i, \mathfrak{q}_\infty)$ is killed by a power of ϖ, so it becomes zero after inverting ϖ. This implies that the map

$$g_i : (R_\infty/\mathfrak{q}_\infty^i)_{\mathfrak{q}_\infty} \to (R^{\mathrm{p}}/(\mathfrak{q}^{\mathrm{p}})^i)_{\mathfrak{q}^{\mathrm{p}}}$$

is surjective for all $i \geq 1$. Since the sequence $(\ker g_i)_{i\geq 1}$ consists of finite length ideals (they are contained inside Artinian rings), it satisfies the Mittag-Leffler condition which implies that $\varprojlim_i g_i$ is surjective as desired. $\square$

Lemma 6.4.15. *Let R be a commutative ring, and M a finitely generated R-module. Suppose we have an R-algebra map $\prod_{N\geq 1} R \xrightarrow{\lambda} R$. Then, the composite map*

$$M \to \prod_{N\geq 1} M \to R \otimes_{\prod_{N\geq 1} R} \prod_{N\geq 1} M$$

is surjective.

Proof. If $M \cong R^d$ for some $d \geq 1$, then we have a composition of maps

$$R^d \to \prod_{N\geq 1} R^d = (\prod_{N\geq 1} R)^d \to R \otimes_{\prod_{N\geq 1} R} (\prod_{N\geq 1} R)^d \cong R^d$$

which is just the identity since λ is R-linear. For the general case, M is a quotient of a finite free R-module F. The map $F \to R \otimes_{\prod_{N\geq 1} R} \prod_{N\geq 1} F \to R \otimes_{\prod_{N\geq 1} R} \prod_{N\geq 1} M$ is surjective and factors through M. $\square$

Now let us define the following modules:

- $\mathrm{m}_1 = (M_1/\mathfrak{a}_\infty^2)_{\mathfrak{q}^{\mathrm{p}}}$,
- $\mathrm{m}_0 = (M_0)_{\mathfrak{q}^{\mathrm{p}}}$,
- $\mathrm{m} = M_{\mathfrak{q}^{\mathrm{p}}} = M_{\mathfrak{q}_0}$.

Lemma 6.4.16. *(1)* m_1 *is a finite free* $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$*-module.*

(2) The trace map induces an isomorphism $\mathrm{m}_1/\mathbf{a}_\infty \cong \mathrm{m}_0$.

(3) The map η *induces an isomorphism* $\eta : \mathrm{m} \cong \mathrm{m}_0$.

Proof. **2)** This follows immediately from localizing the isomorphism in Proposition 6.4.8.
3) By Proposition 6.4.8 the kernel and cokernel of the map $\eta : M \to M_0$ are killed by $f \in \prod_{N\geq 1} R_N$ such that the image of f under the map $\prod_{N\geq 1} R_N \to \prod_{N\geq 1} \mathcal{O} \to \mathcal{O}$ is non-zero by Proposition 6.3.4 and Remark 6.4.3. Therefore, the image of f in R^{p} does not lie inside $\mathfrak{q}^{\mathrm{p}}$, which means that localizing the map η at $\mathfrak{q}^{\mathrm{p}}$ induces the desired isomorphism.
1) By definition of the ideal $\mathfrak{q}_{\mathrm{p}}$, its inverse image in S_∞^W is $\mathbf{a}_\infty^W$. So by the consequence of (2) in Lemma 6.4.11, the action of S_∞ on m_1 factors through $S_\infty \otimes_{S_\infty^W} (S_\infty^W)_{\mathbf{a}_\infty^W} = S_{\infty,\mathbf{a}_\infty}$ (this equality is justified by the next Lemma 6.4.17). By Proposition 6.4.8, $M_1/\mathbf{a}_\infty^2$ is a flat $S_\infty/\mathbf{a}_\infty^2$, so its localisation m_1 is a flat $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$-module. Now since M is a finite $\mathcal{O}$-module, $\mathrm{m} \cong \mathrm{m}_1/\mathbf{a}_\infty$ (by (2) and (3)) is a finite dimensional E-vector space, and we get that m_1 is finitely generated over $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$. ☐

Lemma 6.4.17. *Let* A *be a ring and* G *be a finite group acting by ring automorphisms on* A*. Let* A^G *be the subring of invariant elements and* $\pi : \operatorname{Spec} A \to \operatorname{Spec} A^G$ *the induced morphism. Then,* π *is a finite morphism and the fibers of* π *are precisely the* G*-orbits of the natural action of* G *on* $\operatorname{Spec} A$.

Note that since m_1 is a finite dimensional E-vector space, the action of the local E-algebra $R^{\mathrm{p}}_{\mathfrak{q}^{\mathrm{p}}}$ factors through an Artinian quotient. Hence we get an action of $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$ on m_1.

Now we go back to the pseudorepresentations $\gamma_{\infty,j}$ of $\mathbb{Z}_p \times \mathbb{Z}$ with coefficients in R^{p} for $1 \leq j \leq q$. We let $\delta_j \in R^{\mathrm{p}}$ be the discriminant of the characteristic polynomial $\chi_j(t) \in R^{\mathrm{p}}[t]$ of the element $(0,1) \in \mathbb{Z}_p \times \mathbb{Z}$ (corresponding to the Frobenius) under the pseudorepresentation $\gamma_{\infty,j}$.

Lemma 6.4.18. *For* $1 \leq j \leq q$, $\delta_j \notin \mathfrak{q}_{\mathrm{p}}$ *and* $\chi_j(t) \mod \mathfrak{q}_{\mathrm{p}}$ *splits into linear factors in* $E[t]$.

Proof. To show that $\delta_j \notin \mathfrak{q}_{\mathrm{p}}$, it suffices to show that there exists an integer $m \geq 1$ such that the image of δ_j under the map

$$R^{\mathrm{p}} \to R_0 \xrightarrow{h_{U,\sigma}} \mathcal{O} \to \mathcal{O}/\varpi^m$$

is non-zero. By the Lemma 4.4.17, if we choose $m > dn(n-1)$ (where d is as in the statement of the lemma), and if we can identify the image of δ_j in $\mathcal{O}/\varpi^m$ with the discriminant of the characteristic polynomial of a Frobenius element $\mathrm{Frob}_{\tilde{v}}$ for some $v \in Q_N$, then we are done. Let m' be an integer such that the map $R_0 \to \mathcal{O}/\varpi^m$ factors through $R_0/\mathfrak{m}_{R_0}^{m'}$. And we can identify the image of δ_j with the image of $(\delta_{j,N}) \in \prod_{N\geq 1} R_0/\mathfrak{m}_{R_0}^{m'}$ in $\mathbf{R}_{\mathfrak{p}(\mathcal{F})} \otimes_{\mathbf{R}} \prod_{N\geq 1} R_0/\mathfrak{m}_{R_0}^{m'} \cong R_0/\mathfrak{m}_{R_0}^{m'}$, with $\delta_{j,N}$ is the image of the discriminant of the characteristic polynomial for the Frobenius element corresponding to the j-th place of Q_N. Therefore, the image of δ_j in $R_0/\mathfrak{m}_{R_0}^{m'}$ coincides with one of these discriminants.
Moreover, by hypothesis on $\mathcal{O}$, the characetristic polynomial of $\mathrm{Frob}_{\tilde{v}}$ splits in $\mathcal{O}[t]$, thus $\chi_j(t)$ splits in $\mathcal{O}/\varpi^m[t]$. Hence, the second statement follows using Hensel's lemma. ☐

For $1 \leq j \leq q$, let $x_j^{(1)}, \ldots, x_j^{(n)}$ be the pairwise distinct roots of $\chi_j(t) \mod \mathfrak{q}_{\mathrm{p}}$ in E. Then, by [NT20, Lemma 4.28], there is a unique collection of continuous characters

$$\gamma_j^{(i)} : \mathbb{Z}_p \times \mathbb{Z} \to \left((R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}}\right)^\times$$

such that $\gamma_j^{(i)} \mod \mathfrak{q}_{\mathrm{p}}$ is the character $(a,b) \mapsto (x_j^{(i)})^b$ and $(\gamma_{\infty,j})_{\mathfrak{q}_{\mathrm{p}}} = \operatorname{tr} \gamma_j^{(1)} \oplus \cdots \oplus \gamma_j^{(n)}$, where $(\gamma_{\infty,j})_{\mathfrak{q}_{\mathrm{p}}}$ is the composite of $\gamma_{\infty,j}$ with $R^{\mathrm{p}} \to (R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}}$.
The characters ${\gamma_j^{(i)}}_{|\mathbb{Z}_p\times 0} : \mathbb{Z}_p \to (R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}}$ induce a morphism $S_\infty \to (R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}}$ which extends the morphism

$S_\infty^W \to R^{\mathrm{p}}$. Given that $\mathbf{a}_\infty \subseteq \mathfrak{q}_{\mathrm{p}}$, this morphism extends to a map from the formally smooth E-algebra $(S_\infty)^\wedge_{\mathbf{a}_\infty}$, and we can choose a lift of this map through the surjective map in Proposition 6.4.14

$$(R_\infty)^\wedge_{\mathfrak{q}_\infty} \to (R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$$

to get a morphism $(S_\infty)^\wedge_{\mathbf{a}_\infty} \to (R_\infty)^\wedge_{\mathfrak{q}_\infty}$. We let A' be the localisation of A at the prime ideal $(t_j^{(i)} - x_j^{(i)} : 1 \leq j \leq q,\ 1 \leq i \leq n)$ and define

- $\mathrm{m}_1' = \mathrm{m}_1 \otimes_A A'$,
- $\mathrm{m}_0' = \mathrm{m}_0 \otimes_A A'$.

Lemma 6.4.19. *(1) For each $1 \leq i \leq n$ and $1 \leq j \leq q$, the pushforwards of $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$ to $\mathrm{End}(\mathrm{m}_1')$ are equal.*

(2) The natural structure of S_∞-module on m_1' coincides with that induced by morphism $S_\infty \to (R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}}$.

(3) The map $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}} \to (R_0)^\wedge_{\mathfrak{q}_0}$ factors through the quotient $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}}/\mathbf{a}_\infty$.

(4) The trace map induces an isomorphism $\mathrm{m}_1'/\mathbf{a}_\infty \cong \mathrm{m}_0'$.

(5) m_1' is a non-zero finite free $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty^2$-module.

Proof. **1)** Let $X = \{\alpha_j^{(i)}(z), \gamma_j^{(i)}(z) \in \mathrm{End}(\mathrm{m}_1') \mid z \in \mathbb{Z}_p \times \mathbb{Z}, 1 \leq i \leq n,\ 1 \leq j \leq q\}$ and let L be the E-subalgebra of $\mathrm{End}(\mathrm{m}_1')$ generated by the elements of X. Then, L is commutative, and is an Artinian E-algebra; so we can write $L = \prod_k L_k$ where L_k is a local E-algebra. The pushforwards of the characters $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$ take value in L, and the pseudocharacters $\mathrm{tr}\,\alpha_j^{(1)} \oplus \cdots \oplus \alpha_j^{(n)}$ and $\mathrm{tr}\,\gamma_j^{(1)} \oplus \cdots \oplus \gamma_j^{(n)}$ are equal to the same pseudocharacter after pushforward to L, which we call T_j. To prove (1) it suffices to show that the pushforward of $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$ agree on each localisation L_k of L. For this, we use [BC09, Proposition 1.5.1] which, when applied to the pseudocharacter $T_j : \mathbb{Z}_p \times \mathbb{Z} \to L_k$, states that there exists *unique* characters $T_j^{(i)} : \mathbb{Z}_p \times \mathbb{Z} \to L_k$ such that:

(i) $T_j = \mathrm{tr}\, T_j^{(1)} \oplus \cdots \oplus T_j^{(n)}$,

(ii) $T_j^{(i)} \not\equiv T_j^{(i')} \mod \mathfrak{m}_k$ if $i \neq i'$.

where $\mathfrak{m}_k$ is the maximal ideal of L_k (this is because in *loc.cit.* $I_{\mathcal{P}} = 0$, also note that the two notions of pseudocharacters agree since we are working with $\mathbb{Q}$-algebras). Therefore, it suffices to show that $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$ agree after pushforward to each residue field of L. But since m_1' is an Artinian A'-module, the element $\alpha_j^{(i)}(0,1) - x_j^i = t_j^{(i)} - x_j^{(i)} \in \mathfrak{m}_{A'}$ is a nilpotent element of $\mathrm{End}(\mathrm{m}_1')$. Similarly, the action of $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}^{\mathrm{p}}}$ on m_1 factors through an Artinian quotient, so the element $\gamma_j^{(i)}(0,1) - x_j^{(i)} \in \mathfrak{q}^{\mathrm{p}}$ is a nilpotent element of $\mathrm{End}(m_1')$. Therefore, the difference $\alpha_j^{(i)}(0,1) - \gamma_j^{(i)}(0,1)$ lie in the Jacobson radical of L, and this implies (1).

2) Since both S_∞-module structures are induced by the two set of characters $\alpha_j^{(i)}$ and $\gamma_j^{(i)}$, then (2) follows immediately from (1).

3) The statement follows if we show that the pushforward of the character $\gamma^{(i)}_{j|\mathbb{Z}_p \times 0}$ through the map $(R^{\mathrm{p}})^\wedge_{\mathfrak{q}_{\mathrm{p}}} \to (R_0)^\wedge_{\mathfrak{q}_0}$ is trivial. But by (1) of Lemma 6.4.11, the pushforward of the pseudocharacter $\mathrm{tr}\,\gamma_j^{(1)} \oplus \cdots \oplus \gamma_j^{(n)}$ to $(R_0)^\wedge_{\mathfrak{q}_0}$ factors through the projection $\mathbb{Z}_p \times \mathbb{Z} \to \mathbb{Z}$. Therefore, we get the result by

applying the previous argument that uses [BC09, Proposition 1.5.1], to the two families of characters $\gamma_j^{(i)}$ and $\gamma_j'^{(i)} = \gamma_j^{(i)} \circ (\mathbb{Z} \to \mathbb{Z}_p \times \mathbb{Z})$.
4) This follows immediately from (2) of Lemma 6.4.16.
5) Given that m_1' is a direct summand of m_1, we only need to prove that m_1' is non-zero, or even that m_0' is non-zero. But by (2) of Lemma 6.4.11, the characteristic polynomial $\prod_i (t - t_j^{(i)})$ of $(0,1)$ under α_j pushes forward to $\prod_i (t - x_j^{(i)}) \equiv \chi_j(t) \mod \mathfrak{q}^{\mathrm{p}}$ in $\mathrm{End}(\mathrm{m}_0)$. This is because thanks to (3) of Lemma 6.4.16, the action of $(R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}_{\mathrm{p}}}$ on $\mathrm{m}_0 \cong \mathrm{m}$ factors through

$$(R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}_{\mathrm{p}}} \to (R_0)^{\wedge}_{\mathfrak{q}_0} \to (\mathbf{T}_{\emptyset})_{\mathfrak{q}_0} = E$$

Therefore, the action of A^W on m_0 factors through $A^W \to E$ sending $t_j^{(i)}$ to $x_j^{(i)}$. Since we have an isomorphism of A_W-modules $\mathrm{m}_0 \cong S_\lambda(U, \mathcal{O})_{\mathfrak{q}_0} \otimes_{A^W} A$, we get that the localisation m_0' is nonzero. $\square$

In order to complete the patching argument, we will use the following theorem

Theorem 6.4.20. *[Bro17, Theorem 1.1] Let $A \to B$ be a local morphism of noetherian local rings satisfying*

$$\mathrm{edim}(B) \leq \mathrm{edim}(A)$$

where edim *is the embedding dimension, i.e. the minimal number of generators of the maximal ideal. If M is a non-zero A-flat B-module which is finitely generated over B, then M is finite free over B.*

We are now finally able to prove Theorem 3.4.6 announced in the introduction. So let us apply Theorem 6.4.20 for $A = S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty$, $B = (R_\infty)^{\wedge}_{\mathfrak{q}_\infty}/\mathbf{a}_\infty^2$, and $M = \mathrm{m}_1'$. In fact, $S_{\infty,\mathbf{a}_\infty}/\mathbf{a}_\infty$ has embedding dimension nq, and $(R_\infty)^{\wedge}_{\mathfrak{q}_\infty}/\mathbf{a}_\infty^2$ has embedding dimension $\leq nq$ since $(R_\infty)^{\wedge}_{\mathfrak{q}_\infty}$ is a power series ring over E in nq variables. Thus we get that m_1' is a finite free module over $(R_\infty)^{\wedge}_{\mathfrak{q}_\infty}/\mathbf{a}_\infty^2$, and in particular m_0' is finite free over $(R_\infty)^{\wedge}_{\mathfrak{q}_\infty}/\mathbf{a}_\infty$. Since the action of $(R_\infty)^{\wedge}_{\mathfrak{q}_\infty}/\mathbf{a}_\infty$ on m_0' factors through the surjective maps

$$(R_\infty)^{\wedge}_{\mathfrak{q}_\infty}/\mathbf{a}_\infty \twoheadrightarrow (R^{\mathrm{p}})^{\wedge}_{\mathfrak{q}_{\mathrm{p}}}/\mathbf{a}_\infty \twoheadrightarrow (R_0)^{\wedge}_{\mathfrak{q}_0} \twoheadrightarrow (\mathbf{T}_{\emptyset})_{\mathfrak{q}_0} = E$$

we get that each of these maps is an isomorphism. In particular, $(R_0)^{\wedge}_{\mathfrak{q}_0} \cong E$. We deduce our theorem by identifying it with the tangent space of $(R_0)^{\wedge}_{\mathfrak{q}_0}$ via proposition 4.4.5.

Appendices

Appendix A

Ultrafilters

Definition A.0.1. A filter is a set $\mathcal{F}$ of subsets of $\mathbb{N}$ such that:

(1) $\mathbb{N} \in \mathcal{F}$, and $\emptyset \notin \mathcal{F}$;

(2) if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$;

(3) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

An ultrafilter is a filter with the following additional property:

$$\text{If } A \subseteq \mathbb{N}, \text{ then either } A \in \mathcal{F} \text{ or } A^{\mathrm{c}} \in \mathcal{F}$$

Lemma A.0.2. *A filter $\mathcal{F}$ is an ultrafilter if and only if the following property holds:*

$$A \cup B \in \mathcal{F} \Rightarrow A \in \mathcal{F} \text{ or } B \in \mathcal{F}$$

The examples to keep in mind are the following :

- For $n \in \mathbb{N}$, $\mathcal{F}_n = \{A \subseteq \mathbb{N} \mid n \in A\}$ is an ultrafilter called a principal ultrafilter.
- $\mathcal{F}_{\mathrm{cof}} = \{A \subseteq \mathbb{N} \mid A^{\mathrm{c}} \text{ is finite }\}$ is a filter called the cofinite filter.

Lemma A.0.3. *An ultrafilter $\mathcal{F}$ is principal if and only if $\mathcal{F} \not\supseteq \mathcal{F}_{cof}$.*

Proof. If $\mathcal{F} = \mathcal{F}_n$ is principal, then $\mathbb{N} \setminus \{n\} \notin \mathcal{F}$, so $\mathcal{F} \not\supseteq \mathcal{F}_{\mathrm{cof}}$. Conversely, if $\mathcal{F} \not\supseteq \mathcal{F}_{\mathrm{cof}}$, then by the definition of an ultrafilter, there exists a finite subset $A \in \mathcal{F}$. By the previous lemma, this implies that there is some $n \in \mathbb{N}$, such that $\{n\} \in \mathcal{F}$. Therefore, $\mathcal{F}$ is principal. □

Lemma A.0.4. *An ultrafilter is a maximal filter, with the order given by inclusion.*

Proof. Suppose that $\mathcal{F}$ is a maximal filter which is not an ultrafilter. Then, there exists a subset $A \in \mathbb{N}$ such that $A, A^{\mathrm{c}} \notin \mathcal{F}$. The set of subsets $\mathcal{F}' = \{C \supseteq A \cap B \text{ for some } B \in \mathcal{F}\}$ is a filter containing $\mathcal{F}$: contradiction. □

Corollary A.0.5. *There exist non-principal ultrafilters.*

Proof. By Zorn's lemma, and the previous lemma, any filter is contained in an ultrafilter. Applying this to the filter $\mathcal{F}_{\mathrm{cof}}$, we get the result. □

Next, we will show how ultrafilters allow us to study the spectrum of countable products of rings. So consider for all $n \in \mathbb{N}$ a local Artinian ring R_n with maximal ideal $\mathfrak{m}_n$, and let $R = \prod_{n\in\mathbb{N}} R_n$. To an element $x = (x_n)_n \in R$, we associate the set

$$Z(x) = \{n \in \mathbb{N} \mid x_n \in \mathfrak{m}_n\}$$

which satisfy the following properties:

- $Z(x+y) \supseteq Z(x) \cap Z(y)$;
- $Z(xy) = Z(x) \cup Z(y)$.

And to a subset $A \subseteq \mathbb{N}$, we associate the idempotent:

$$(e_A)_n = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

which in turn satisfies the following properties:

- $Z(e_A) = A^c$;
- $e_{A\cap B} = e_A e_B$;
- $e_{A\cup B} = e_A + e_B - e_A e_B$.

Now given a prime ideal, we define

$$\mathcal{F}(\mathfrak{p}) := \{Z(x) \mid x \in \mathfrak{p}\} = \{A^c \mid e_A \in \mathfrak{p}\}$$

Lemma A.0.6. *For any prime ideal $\mathfrak{p} \subseteq R$, $\mathcal{F}(\mathfrak{p})$ is an ultrafilter.*

Proof. Since $0 \in \mathfrak{p}$, we have $\mathbb{N} \in \mathcal{F}$, and since $1 \notin \mathfrak{p}$, $\emptyset \notin \mathcal{F}(\mathfrak{p})$. Now if $A \subseteq B$, then $e_{A^c} e_{B^c} = e_{B^c}$, so $A \in \mathcal{F}(\mathfrak{p}) \Rightarrow B \in \mathcal{F}(\mathfrak{p})$. Given that $e_A e_{A^c} = 0 \in \mathfrak{p}$, we either have $A \in \mathcal{F}(\mathfrak{p})$ or $A^c \in \mathcal{F}(\mathfrak{p})$. Together with $e_{A\cap B} = e_A e_B$, this implies that if $A, B \in \mathcal{F}(\mathfrak{p})$, then $A \cap B \in \mathcal{F}(\mathfrak{p})$. Therefore, $\mathcal{F}(\mathfrak{p})$ is an ultrafilter. $\square$

Conversely, we easily check that given an ultrafilter $\mathcal{F}$, we get a prime ideal defined by:

$$\mathfrak{p}(\mathcal{F}) = \{x \in R \mid Z(x) \in \mathcal{F}\}$$

Theorem A.0.7. *These two constructions are inverse to each other, and induce a bijection between the set of ultrafilters of $\mathbb{N}$ and the set of prime ideals of R.*

Remark A.0.8. Note that two elements $(x_n), (y_n) \in R$ agree in $R_{\mathfrak{p}(\mathcal{F})}$ if and only if $x_n = y_n$ for $\mathcal{F}$-many n.

Proposition A.0.9. *Suppose that $R_n = R_0$ for all but finitely many n, with $\#R_0 < \infty$. If $\mathcal{F}$ is a non-principal ultrafilter, then we have an isomorphism*

$$\begin{aligned} f : R_0 &\xrightarrow{\simeq} R_{\mathfrak{p}(\mathcal{F})} \\ x_0 &\mapsto x = (x_n)_n \end{aligned}$$

where $x_n = x_0$ if $R_n = R_0$ and 0 otherwise.

Proof. Let us start by proving injectivity. Suppose that $x_0 \neq 0 \in R_0$ maps to 0, i.e., that there exists $y = (y_n)_n \notin \mathfrak{p}(\mathcal{F})$ such that $yf(x_0) = (y_n f(x_0)_n) = 0$. If $y_n \notin \mathfrak{m}_n$ for $n \gg 0$, then it is invertible and we get $f(x_0)_n = x_0 = 0$. Therefore, for $n \gg 0$, $y_n \in \mathfrak{m}_n$, so $Z(y)$ is cofinite hence in $\mathcal{F}$, which contradicts the choice of y.
For surjectivity, let us first fix an integer n_0 such that for $n \geq n_0$, $R_n = R_0$. Suppose that we have an element $x/y \in R_{\mathfrak{p}(\mathcal{F})}$ with $y \notin \mathfrak{p}(\mathcal{F})$, we need to show that there exists $r \in R_0$ and $z \notin \mathfrak{p}(\mathcal{F})$ such that $f(r)yz = xz$.
For $r \in R_0$, consider the set $\{n \geq n_0 \mid y_n r = x_n\}$. Since $Z(y) \notin \mathcal{F}$, for $n \in Z(y)^c$ and $n \geq n_0$, y_n is invertible in R_0, and we can find some $r \in R_0$ such that $y_n r = x_n$. Thus, we have:

$$Z(y)^c \cap \{n \in \mathbb{N} \mid n \geq n_0\} \subseteq \bigcup_{r \in R_0} \{n \geq n_0 \mid y_n r = x_n\} \in \mathcal{F}$$

By Lemma A.0.2, since R_0 is finite, there exist some $r \in R_0$ such that $A = \{n \geq n_0 \mid y_n r = x_n\} \in \mathcal{F}$. Letting $z = e_A$, we trivially have $yf(r)z = xz$, and $Z(z) = A^c \notin \mathcal{F}$, so $z \notin \mathfrak{p}(\mathcal{F})$. $\square$

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