Congruences between automorphic representations

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1 Introduction

Recall from last time that for an automorphic representation $\pi \in \mathrm{Coh}(G, \lambda, K_f)$ and an admissible signature ϵ for π, we have defined

$$L^{\mathrm{alg}}(1, \mathrm{Ad}^0, \pi, \epsilon) := \frac{L(1, \mathrm{Ad}^0, \pi)}{\omega_F \cdot \mathfrak{p}_{\mathrm{ram}} \cdot \mathfrak{p}_\infty(\pi) \cdot \mathfrak{p}^\epsilon(\pi) \cdot \mathfrak{q}^{\tilde{\epsilon}}(\tilde{\pi})}$$

We showed that this is actually an algebraic number contained in $\mathbb{Q}(\pi)$.

For this talk, we will consider a finite extension E of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\kappa = \mathcal{O}/\wp$. Our goal will be to explain and prove the following theorem:

Theorem 1.1. *[BR14] There exist finite sets S_1, S_2 and S_3 consisting of rational primes such that if*

$$v_\wp\big(L^{alg}(1, \mathrm{Ad}^0, \pi, \epsilon)\big) > 0$$

(1) and if $p \notin S_1$, then there exists π' congruent to π modulo $\wp$ and $\pi' \ncong \pi$,

(2) and if $p \notin S_2$ and if π is of parallel weight, then there exists π' congruent to π modulo $\wp$ and $\pi' \ncong {}^\sigma\pi$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,

(3) and if $p \notin S_3$ and if π is of parallel weight, then there exists $\sigma \in \mathrm{Aut}(\mathbb{C})$ with $\pi' = {}^\sigma\pi$ congruent to π modulo $\wp$ and $\pi' \ncong \pi$.

2 Congruence module

Let V and $\tilde{V}$ be finitely generated vector spaces over E, and let L and $\tilde{L}$ be $\mathcal{O}$-lattices in V and $\tilde{V}$ respectively.
Suppose that we have a non-degenerate bilinear form $\langle\cdot,\cdot\rangle : V \times \tilde{V} \to E$ with a decomposition $V = V_1 \oplus V_2$ and $\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2$ that respects this pairing. We also suppose that this pairing restricts to a perfect pairing $L \times \tilde{L} \to \mathcal{O}$.

We have exact sequences:

$$0 \to L_2 \to L \xrightarrow{\pi_1} \Lambda_1 \to 0$$
$$0 \to L_1 \to L \xrightarrow{\pi_2} \Lambda_2 \to 0$$

where π_i is the projection of V onto V_i, $L_i = L \cap V_i$, and $\Lambda_i = \pi_i(L)$. Similarly we define $\tilde{L}_i$ and $\tilde{\Lambda}_i$. The congruence module of L with respect to this decomposition is defined by

$$\mathcal{C}(L, V_1, V_2) := \frac{L}{L_1 \oplus L_2} \cong \frac{\Lambda_i}{L_i} \cong \frac{\Lambda_1 \oplus \Lambda_2}{L}$$

where the first isomorphism is induced by the projection maps and the second by the natural inclusion $\Lambda_i \hookrightarrow \Lambda_1 \oplus \Lambda_2$. Note that by calculating the ranks of these modules, we get that $\mathcal{C}(L, V_1, V_2)$ is a finite torsion module over $\mathcal{O}$ (since it is a PID).

If $(e_i)_i$ is a basis of L_1 and $(\tilde{e}_j)_j$ is a basis of $\tilde{L}_1$, then we define the discriminant (up to a unit of $\mathcal{O}$) by:

$$\mathrm{disc}(L_1 \times \tilde{L}_1) =_{\mathcal{O}^\times} \det\big(\langle e_i, \tilde{e}_j\rangle\big)_{i,j}$$

If we let $\tilde{L}_1^* = \{v \in V_1 \mid \langle v, w\rangle \in \mathcal{O},\ \forall w \in \tilde{L}_1\}$, then we have

$$\mathrm{disc}(L_1 \times \tilde{L}_1) =_{\mathcal{O}^\times} |\tilde{L}_1^*/L_1|$$

Lemma 2.1. *The module of congruence $\mathcal{C}(L, V_1, V_2)$ is non-zero if and only if*

$$v_\wp\big(\,\mathrm{disc}(L_1 \times \tilde{L}_1^*)\big) > 0$$

Proof. By the above formula, it suffices to show that $\Lambda_1 = \tilde{L}_1^*$.
First, it is straightforward that $\Lambda_1 \subseteq \tilde{L}_1^*$.
Second, by the structure theorem of finitely generated modules over a PID, we can find a basis $u_1, \ldots, u_t$ of $\tilde{L}$ such that $l_1 u_1, \ldots, l_s u_s$ is a basis of $\tilde{L}_1$ with $l_1, \ldots, l_s \in \mathcal{O}$. But by definition, if $l_i u_i \in \tilde{L}_1$ we must have $u_i \in \tilde{L}_1$. Thus we can take the l_i to be equal to 1.
Let $u_1^*, \ldots, u_t^*$ be the corresponding dual basis for the dual lattice L. Since $\tilde{L}_1$ generate $\tilde{V}_1$, we get that $\pi_1(u_i^*) = 0$ for $i > s$ so that $\pi_1(u_1^*), \ldots, \pi_1(u_s^*)$ generate the lattice Λ_1 over $\mathcal{O}$.
Now if $v \in \tilde{L}_1^*$, we can write $v = \sum_i a_i \pi_1(u_i^*)$ with $a_i \in E$. But then

$$\mathcal{O} \ni \langle v, u_i\rangle = \sum_j a_j \langle \pi_1(u_j^*), u_i\rangle = a_i$$

which proves the other inclusion. □

Now let $\mathcal{H}^\circ \subseteq \mathrm{End}_{\mathcal{O}}(L)$ be a reduced subalgebra generated by a pairwise commuting operators acting on L, and let $\mathcal{H}_E^\circ = \mathcal{H}^\circ \otimes E \subseteq \mathrm{End}_E(V)$ which is semi-simple.
Suppose that the action of $\mathcal{H}_E^\circ$ preserves a subspace $V_1 \subseteq V$, then taking the kernel of the restriction to this subspace, we get an ideal of $\mathcal{H}_E^\circ$ which by semisimplcity gives a decomposition:

$$\mathcal{H}_E^\circ = e_1 \mathcal{H}_E^\circ \times e_2 \mathcal{H}_E^\circ$$

where e_1, e_2 are orthogonal idempotents satisfying $e_1 + e_2 = \mathrm{id}$. Letting $V_2 = \ker e_1$, we have a decomposition $V = V_1 \oplus V_2$ stable under $\mathcal{H}^\circ_E$. (Note that if $\mathcal{H}^\circ$ is equivariant with respect to a non degenerate bilinear form, then V_2 is just the orthogonal of V_1).

We let $\mathcal{H}^\circ_i = \mathcal{H}^\circ \cap e_i \mathcal{H}^\circ_E$ and we define the congruence module

$$Q(\mathcal{H}^\circ; e_1, e_2) = \frac{\mathcal{H}^\circ}{\mathcal{H}^\circ_1 \oplus \mathcal{H}^\circ_2} \cong \frac{e_i \mathcal{H}^\circ}{\mathcal{H}^\circ_i}$$

There is a natural map $Q(\mathcal{H}^\circ; e_1, e_2) \to \mathrm{End}_{\mathcal{O}}(\mathcal{C}(L; V_1, V_2))$ induced by the inclusion $\mathcal{H}^\circ \subset \mathrm{End}_{\mathcal{O}}(L)$.

Suppose that $\mathcal{C}(L; V_1, V_2) \neq 0$ and let us see what it means. Since it is a torsion module, there exists an element $\overline{x} \in \mathcal{C}(L; V_1, V_2) \setminus \{0\}$ such that $\varpi \overline{x} = 0$. Lifting $\overline{x}$ to an element $x \in L$, this means that there exist $f \in L_1$ and $g \in L_2$ such that $f - g = \varpi x$, or in other words, $f \equiv g \mod \wp$ with $f \notin \wp L$.

Now let us also suppose that the algebra $\mathcal{H}^\circ$ acts on V_1 via a character $\chi : \mathcal{H}^\circ \to \mathcal{O}$. Then there exists an element $g \in L_2$ which is an eigenvector modulo $\mathfrak{p}$ for $\mathcal{H}^\circ$ with system of eigenvalues $\overline{\chi}$. This puts us in the setting of Deligne-Serre's lemma:

Lemma 2.2 (Deligne-Serre)**.** *[Bel]*
Assume that there exists an element $g \in L_2$ such that $g \notin \wp L_2$ and such that for every $T \in \mathcal{H}^\circ$, $T \cdot g \equiv \overline{\chi}(T) g \mod \wp L_2$ for some character $\overline{\chi} : \mathcal{H}^\circ \to \kappa$. Then for E large enough, there exists a vector $g' \in L_2$ which is an eigenvector for $\mathcal{H}^\circ$ whose system of eigenvalues χ' satisfies $\chi' \equiv \overline{\chi} \mod \wp$.

Proof. Since $\mathcal{H}^\circ$ is a finite flat $\mathcal{O}$-algebra, the map

$$\mathrm{Spec}\, \mathcal{H}^\circ \to \mathrm{Spec}\, \mathcal{O}$$

satisfies the gowing down and the incomparability property. This implies that it sends the closed points to the special point and the non-closed points (they correspond to minimal primes) to the generic point.

On the other hand, we have a decomposition

$$\mathcal{H}^\circ \cong \prod_i \mathcal{H}^\circ_{\mathfrak{m}_i}$$

where $\mathfrak{m}_i$ range over the maximal ideals of $\mathcal{H}^\circ$. To see this, note that $\mathcal{H}^\circ/\wp$ is artinian over κ so it has such a decomposition and since $\mathcal{O}$ is henselian we can lift the associated idempotents to idempotents in $\mathcal{H}^\circ$ (it can even be taken as a definition for being henselian [aut]).

In particular, every prime ideal lies in exactly one maximal ideal. This gives us a commutative diagram:

$$\begin{array}{ccc} \{\text{ non-closed points of } \mathrm{Spec}\, \mathcal{H}^\circ\} & \xrightarrow{\sim} & \mathrm{Hom}_{E-\mathrm{alg}}\left(\mathcal{H}^\circ_E, \overline{E}\right)/(G_E\text{-action}) \\ \downarrow & & \downarrow \\ \{\text{ closed points of } \mathrm{Spec}\, \mathcal{H}^\circ\} & \xrightarrow{\sim} & \mathrm{Hom}_{\kappa-\mathrm{alg}}\left(\mathcal{H}^\circ_\kappa, \overline{\kappa}\right)/(G_\kappa\text{-action}) \end{array}$$

where the leftmost vertical map sends a character $\chi : \mathcal{H}^\circ_E \to \overline{E}$ to the reduction of the character $\chi_{|\mathcal{H}^\circ} : \mathcal{H}^\circ \to \overline{\mathcal{O}}$.
Now let $\mathfrak{m} = \ker \overline{\chi}$. If $(L_2)_{\mathfrak{m}} = 0$, then there exists an element $h \in \mathcal{H}^\circ \setminus \mathfrak{m}$ such that

$h \cdot g = 0$. Reducing modulo $\wp$, we get that $\overline{\chi}(h)\overline{g} = 0$ and since $\overline{\chi}(h) \neq 0$, we get that $\overline{g} = 0$ which contradicts our assumptions. Therefore, $\mathfrak{m} \in \mathrm{Supp}_{\mathcal{H}^\circ}(L_2/\wp L_2)$.
Now by what we have just seen, the $\mathcal{H}^\circ$-module $(\mathcal{H}^\circ \cdot g)_{\mathfrak{m}}$ is non-zero.
Since $\mathcal{H}^\circ$ is noetherian, there exists a prime $\mathfrak{p} \subseteq \mathfrak{m}$ and an element $\frac{x}{h} \in (\mathcal{H}^\circ \cdot g)_{\mathfrak{m}}$ such that $\mathfrak{p} = \mathrm{Ann}_{\mathcal{H}^\circ}(x)$ (it is a maximal ideal with respect to this property). And since $\mathfrak{p}$ is finitely generated, we can find an element $h' \in \mathcal{H}^\circ$ such that $\mathfrak{p}$ is the annihilator of $g' = h'x \in L_2$. Now by the above commutative diagram, if we suppose E large enough, $\mathfrak{p}$ is the kernel of a character $\chi' : T \to \mathcal{O}$ reducing to $\overline{\chi}$. In particular, for all $h \in \mathcal{H}^\circ$, $h - \chi'(h) \in \mathfrak{p}$ meaning that $h \cdot g' = \chi'(h)g'$. $\square$

Therefore if we assume that E is large enough, then $\mathcal{H}^\circ$ has an eigenvector in L_2 which is congruent to an eigenvector in L_1.

3 Application to the proof of the theorem

Let λ be a strongly-pure dominant integral weight (necessary condition for the non-vanishing of cuspidal cohomology), and $\pi \in \mathrm{Coh}(G, K_f, \lambda)$ as considered in the statement of the theorem.

3.1 Case I

We will work with the following finite dimensional vector spaces over E:

$$V = H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \quad \text{and} \quad \tilde{V} = H^t_!(S^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},E})$$

We also consider the following subspaces of V and $\tilde{V}$, respectively:

$$V_1 = \bigoplus_{\epsilon} H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})(\pi_f \times \epsilon) \quad \text{and} \quad \tilde{V}_1 = \bigoplus_{\epsilon} H^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},E})(\tilde{\pi}_f \times \tilde{\epsilon})$$

where ϵ runs over the characters of K_∞/K°_∞ that are admissible for π.

Let V_2 (resp. $\tilde{V}_2$) be the the complement of V_1 (resp. $\tilde{V}_1$) given as before with respect to the Hecke algebra. We consider the following lattices in V and $\tilde{V}$, respectively:

$$L = \bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \quad \text{and} \quad \tilde{L} = \bar{H}^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}})$$

Then we see that

$$L_1 = L \cap V_1 = \bigoplus_{\epsilon} \bar{H}^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})(\pi_f \times \epsilon)$$

and,

$$\tilde{L}_1 = \tilde{L} \cap \tilde{V}_1 = \bigoplus_{\epsilon} \bar{H}^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}})(\tilde{\pi}_f \times \tilde{\epsilon})$$

The cup product induces the pairings

$$\langle \cdot, \cdot \rangle : V \times \tilde{V} \to E \quad \text{and} \quad \langle \cdot, \cdot \rangle : L \times \tilde{L} \to \mathcal{O}$$

where the first pairing is non-degenerate by Poincaré duality. Later we will show that outside a finite set of rational primes S_1, the second pairing is also perfect. Therefore, we can apply the results of the previous section.

In the previous talk, we have chosen an $\mathcal{O}$-basis $\{\vartheta^{\circ}_{b,\epsilon}(\pi)\}_{\epsilon}$ and $\{\tilde{\vartheta}^{\circ}_{t,\tilde{\epsilon}}(\tilde{\pi})\}_{\epsilon}$ for L_1 and $\tilde{L}_1$ respectively, such that $\langle\vartheta^{\circ}_{b,\epsilon}(\pi), \tilde{\vartheta}^{\circ}_{t,\eta}(\tilde{\pi})\rangle$ if $\eta \neq \tilde{\epsilon}$. Hence we get that

$$\mathrm{disc}(L_1 \times \tilde{L}_1) = \prod_{\epsilon} \langle\vartheta^{\circ}_{b,\epsilon}(\pi), \tilde{\vartheta}^{\circ}_{t,\tilde{\epsilon}}(\tilde{\pi})\rangle = \prod_{\epsilon} L^{\mathrm{alg}}(1, \mathrm{Ad}^0, \pi, \epsilon)$$

Therefore, if for some ϵ we have $v_{\wp}(L^{\mathrm{alg}}(1, \mathrm{Ad}^0, \pi, \epsilon)) > 0$, then $v_{\wp}(\mathrm{disc}(L_1 \times \tilde{L}_1)) > 0$ which means that $\mathcal{C}(L; V_1, V_2) \neq 0$. Hence, there exists a automorphic representation π'_f contributing to V_2 which is congruent to π_f. So we get an automorphic representation π' whose finite part contributes to inner cohomology such that $\pi' \equiv \pi(\mod \wp)$ and $\pi' \not\cong \pi$ (by definition of V_1).

3.2 Case II

Now assume moreover that the weight λ is parallel (it is invariant under the action of $\mathrm{Aut}(\mathbb{C})$). It follows by a result of Clozel, that for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the representation ${}^{\sigma}\pi$ is also cohomological with respect to the weight λ.
As in case I, we will consider the finite dimensional E-vector spaces:

$$V = H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \quad \text{and} \quad \tilde{V} = H^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},E})$$

But we will work with the following subspaces of V and $\tilde{V}$:

$$V_1 = \bigoplus_{\sigma,\epsilon} H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})({}^{\sigma}\pi_f \times \epsilon) \quad \text{and} \quad \tilde{V}_1 = \bigoplus_{\sigma,\epsilon} H^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},E})({}^{\sigma}\tilde{\pi}_f \times \tilde{\epsilon})$$

where the direct sum is taken over characters ϵ on $K_{\infty}/K^{\circ}_{\infty}$ that are permissible for π and over the finite set of embeddings $\sigma : \mathbb{Q}(\pi) \hookrightarrow \mathbb{C}$. As before we have complements V_2 and $\tilde{V}_2$ with respect to the Hecke action.
We consider the lattices:

$$L = \bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \quad \text{and} \quad \tilde{L} = \bar{H}^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}})$$

Then we have:

$$L_1 = L \cap V_1 \supseteq L(\pi) := \bigoplus_{\sigma,\epsilon} \overline{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})({}^{\sigma}\pi_f \times \epsilon)$$

and,

$$\tilde{L}_1 = \tilde{L} \cap \tilde{V}_1 \supseteq \tilde{L}(\tilde{\pi}) := \bigoplus_{\sigma,\epsilon} \overline{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}})({}^{\sigma}\tilde{\pi}_f \times \tilde{\epsilon})$$

I think that some of these summands might be zero since the character through which the Hecke algebra acts might not take values in $\mathcal{O}$. Anyways, after excluding a finite set of rational primes (that figure in the denominator of theser characters), we can assume that the above inclusions are equalities.

Now the cup product induces the following pairings:

$$\langle\cdot,\cdot\rangle : V \times \tilde{V} \to E \quad \text{and} \quad \langle\cdot,\cdot\rangle : L \times \tilde{L} \to \mathcal{O}$$

where the first pairing is non-degenerate and the second pairing is perfect outside the finite set of primes S_1. So in this case the set of primes which we exclude is

$$S_2 = S_1 \cup \{p \,|L(\pi) \subsetneq L_1\}$$

Considering the $\mathcal{O}$-bases $\{\vartheta^{\circ}_{b,\epsilon}({}^{\sigma}\pi)\}_{\sigma,\epsilon}$ and $\{\tilde{\vartheta}^{\circ}_{t,\tilde{\epsilon}}({}^{\sigma}\tilde{\pi})\}_{\sigma,\epsilon}$ for L_1 and $\tilde{L}_1$ respectively. Given that the $({}^{\sigma}\pi \times \epsilon)$-component of V pairs non-trivially with the $({}^{\tau}\tilde{\pi} \times \tilde{\epsilon})$-component of $\tilde{V}$ if and only if $\sigma = \tau$. We get that :

$$\mathrm{disc}(L_1 \times \tilde{L}_1) = \prod_{\epsilon,\sigma} \langle \vartheta^{\circ}_{b,\epsilon}({}^{\sigma}\pi), \tilde{\vartheta}^{\circ}_{t,\tilde{\epsilon}}({}^{\sigma}\tilde{\pi}) \rangle$$

So for any ϵ, we have:

$$\begin{aligned} v_{\wp}(L^{\mathrm{alg}}(1, \mathrm{Ad}^0, \pi, \epsilon)) > 0 &\Rightarrow v_{\wp}(\prod_{\epsilon} L^{\mathrm{alg}}(1, \mathrm{Ad}^0, {}^{\sigma}\pi, \epsilon)) > 0 \\ &\Leftrightarrow v_{\wp}(\mathrm{disc}(L_1 \times \tilde{L}_1)) > 0 \\ &\Leftrightarrow \mathcal{C}(L; V_1, V_2) \neq 0 \end{aligned}$$

Hence we get an automorphic representation π' whose finite part contributes to V_2 and such that $\pi' \equiv \pi(\mod \wp)$. By definition of V_1, we have that $\pi' \not\simeq {}^{\sigma}\pi$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

3.3 Case III

Here we also suppose that λ is parallel, and we consider the following finite dimensional vector spaces over E:

$$V = \bigoplus_{\sigma,\epsilon} H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})({}^{\sigma}\pi_f \times \epsilon) \quad \text{and} \quad \tilde{V} = \bigoplus_{\sigma,\epsilon} H^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},E})({}^{\sigma}\tilde{\pi}_f \times \tilde{\epsilon})$$

Let V_1 and $\tilde{V}_1$ be the following subspaces of V and $\tilde{V}$ respectively:

$$V_1 = \bigoplus_{\epsilon} H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E})(\pi_f \times \epsilon) \quad \text{and} \quad \tilde{V}_1 = \bigoplus_{\epsilon} H^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},E})(\tilde{\pi}_f \times \tilde{\epsilon})$$

Same as before, we have Hecke complements V_2 and $\tilde{V}_2$ and we define the following lattices in V and $\tilde{V}$ respectively:

$$L = \bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \cap V \quad \text{and} \quad \tilde{L} = \bar{H}^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}}) \cap \tilde{V}$$

Then we have:

$$L_1 = L \cap V_1 = \bigoplus_{\epsilon} \bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})(\pi_f \times \epsilon) \quad \text{and} \quad \tilde{L}_1 = \tilde{L} \cap \tilde{V}_1 = \bigoplus_{\epsilon} \bar{H}^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})(\tilde{\pi}_f \times \tilde{\epsilon})$$

And we have pairings induced by the cup product:

$$\langle \cdot, \cdot \rangle : V \times \tilde{V} \to E \quad \text{and} \quad \langle \cdot, \cdot \rangle : L \times \tilde{L} \to \mathcal{O}$$

where the first pairing is non-degenerate and we assume that the second pairing is perfect after excluding a finite set S_3 of rational primes.
Calculating the discriminant with respect to the bases $\{\vartheta^{\circ}_{b,\epsilon}(\pi)\}_{\epsilon}$ and $\{\tilde{\vartheta}^{\circ}_{b,\tilde{\epsilon}}(\tilde{\pi})\}_{\epsilon}$ of L_1 and $\tilde{L}_1$ respectively, we have:

$$\mathrm{disc}(L_1 \times \tilde{L}_1) = \prod_{\epsilon} \langle \vartheta^{\circ}_{b,\epsilon}(\pi), \tilde{\vartheta}^{\circ}_{b,\tilde{\epsilon}}(\tilde{\pi}) \rangle$$

Thus for any ϵ, $v_{\wp}(L^{\mathrm{alg}}(1, \mathrm{Ad}^0, \pi, \epsilon)) > 0$ implies that $\mathcal{C}(L, V_1, V_2) \neq 0$.

Note that the lattices L and $\tilde{L}$ in this case correspond to the sublattices L_1 and $\tilde{L}_1$ in the previous case. Therefore if $p \notin S_2$ the paring between L and $\tilde{L}$ is non-degenerate if and only if there are no congruences in the second case. Hence we have :

$$S_3 = S_1 \cup \{\text{the congruence primes from Case II}\}$$

So In the case where p is a congruence prime, we get an automorphic representation whose finite part contributes to V_2, and by definition of V_1, this means that there is an embedding $\sigma : \mathbb{Q}(\pi) \hookrightarrow \mathbb{C}$ such that $\pi' = {}^{\sigma}\pi \not\simeq \pi$ and such that $\pi' \equiv \pi (\mod \wp)$.

4 The sets of excluded primes

We will now proceed to describe the set S_1 of excluded primes. Our goal is to find a finite set of rational primes outside of which the Poincaré pairing between the following cohomology groups

$$\bar{H}^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \quad \text{and} \quad \bar{H}^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}})$$

is perfect. In other words, we want to show that the following maps

$$\bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \to \mathrm{Hom}_{\mathcal{O}}(\bar{H}^t_!(\widetilde{\mathcal{S}}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}}), \mathcal{O})$$
$$\bar{H}^t_!(\widetilde{\mathcal{S}}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},\mathcal{O}}) \to \mathrm{Hom}_{\mathcal{O}}(\bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}), \mathcal{O})$$

are isomorphisms. Since these modules are finite free over $\mathcal{O}$, it suffices to show that the first map is an isomorphism. Now injectivity follows from the injectivity of the corresponding map with rational coefficients:

$$H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,E}) \to \mathrm{Hom}_E(H^t_!(\widetilde{\mathcal{S}}^G_{K_f}, \widetilde{\mathcal{M}}_{\tilde{\lambda},E}), E)$$

which is an isomorphism by Poincaré duality.

Again by Poincaré duality, we have a commutative diagram where the top row is an isomorphism:

$$\begin{array}{ccc}
\bar{H}^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{O}}(\bar{H}^t_c(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}), \mathcal{O}) \\
\uparrow & & \uparrow{\scriptstyle i^*} \\
\bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) & \hookrightarrow & \mathrm{Hom}_{\mathcal{O}}(\bar{H}^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}), \mathcal{O})
\end{array}$$

We need to show that the bottom row is surjective. So let $\alpha \in \mathrm{Hom}_{\mathcal{O}}(\bar{H}^t_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}), \mathcal{O})$, then there exists $x \in \bar{H}^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})$ such that

$$\alpha \circ i(y) = \langle x, y \rangle, \quad \forall y \in \bar{H}^t_c(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})$$

But since the bottom row is an isomorphism after tensoring with E, we get that

$$\overline{x \otimes 1} = 0 \quad \text{inside } \frac{\bar{H}^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \otimes E}{\bar{H}^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \otimes E} = \frac{H^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})}{H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})} \otimes E$$

In other words, x is a torsion element of

$$\frac{H^b(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})}{H^b_!(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})}$$

By the exact sequence in the next subsection, we get that this quotient lies inside the boundary cohomology $H^b(\partial\overline{\mathcal{S}}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}})$. Therefore, we can take the set of excluded primes to be

$$S_1 = \{p \mid H^b(\partial\overline{\mathcal{S}}^G_{K_f}, \widetilde{\mathcal{M}}_{\lambda,\mathcal{O}}) \text{ has } p \text{ torsion}\}$$

4.1 Borel Serre compactification

Consider the inclusion into the Borel-Serre compactification $i : \mathcal{S}^G_{K_f} \hookrightarrow \overline{\mathcal{S}}^G_{K_f}$ which is a homotopy equivalence. There are two natural ways to extend the sheaf $\widetilde{\mathcal{M}}_\lambda$ to a sheaf on $\overline{\mathcal{S}}^G_{K_f}$.

First, we can take the sheaf

$$i_*(\widetilde{\mathcal{M}_\lambda}) \text{ on } \overline{\mathcal{S}}^G_{K_f}$$

In fact the functor i_* is exact, so we get from Leray's spectral sequence that:

$$H^\bullet(\overline{\mathcal{S}}^G_{K_f}, i_*(\widetilde{\mathcal{M}}_\lambda)) \cong H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda)$$

On the other hand, we can also consider the extension by zero $i_! \widetilde{\mathcal{M}}_\lambda$. Then we have:

$$H^\bullet_c(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \cong H^\bullet(\overline{\mathcal{S}}^G_{K_f}, i_!(\widetilde{\mathcal{M}}_\lambda))$$

From the exact sequence

$$0 \to i_!(\widetilde{\mathcal{M}}_\lambda) \to i_*(\widetilde{\mathcal{M}}_\lambda) \to i_*(\widetilde{\mathcal{M}}_\lambda)/i_!(\widetilde{\mathcal{M}}_\lambda) \to 0$$

and noting that $i_*(\widetilde{\mathcal{M}}_\lambda)/i_!(\widetilde{\mathcal{M}}_\lambda) \to$ is just the extension by 0 of the restriction of the sheaf $i_*(\widetilde{\mathcal{M}}_\lambda)$ to $\partial\overline{\mathcal{S}}^G_{K_f}$, we get a long exact sequence:

$$\cdots \to H^q_c(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^q(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^q(\partial\overline{\mathcal{S}}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to H^{q+1}_c(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}}_\lambda) \to \cdots$$

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