Hida's paper

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1 Quadratic forms

Let F be a field of characteristic $\neq 2$. Let (V, Q) be a quadratic space over F of dimenson m, S the associated symmetric bilinear form and suppose that it is non-degenerate.
(V, Q) is called anisotropic if $Q(x) \Leftrightarrow x = 0$, otherwise it is called isotropic.

For a proof of the statements in this section, consult [Shi10].

Theorem 1.1 (Witt)**.** *Suppose that* $(V, Q) = (V_1, Q_1) \oplus (V_2, Q_2) = (V_1', Q_1') \oplus (V_2', Q_2')$*.*
If $(V_1, Q_1) \cong (V_1', Q_1')$*, then* $(V_2, Q_2) \cong (V_2', Q_2')$*.*

Theorem 1.2. *Given* (V, Q) *with* Q *non-degenerate, there exist a decomposition called the Witt decomposition:*

$$V = X \oplus \sum_{i=1}^{s} (Fe_i \oplus Ff_i)$$

such that

- $Q(e_i) = Q(f_i) = 0$ *and* $S(e_i, f_j) = \delta_{ij}$*.*
- $X = \left(\sum_{i=1}^{s}(Fe_i \oplus Ff_i)\right)^{\perp}$ *and* $(X, Q_{|X})$ *is anisotropic.*

Moreover, s *and the isomorphism class of* $(X, Q_{|X})$ *are completely determined by the isomorphism class of* (V, Q)*.*

Let M be a symmetric matrix representing the bilinear form S in a chosen basis of V. Then M is well defined up to conjugation $\alpha M {}^t\alpha$ by an element $\alpha \in \mathrm{Aut}(V)$. Thus the coset $(-1)^{n(n-1)/2}\det(M)F^{\times 2}$ in $F^{\times}/F^{\times 2}$ is completely determined by Q. We call this coset the discriminant of Q and denote it by $\delta_0(Q)$. Consider the field $K_0 = F[\delta_0(Q)^{1/2}]$, and define the discriminant algebra K of Q to be:

$$K = \begin{cases} K_0 & \text{if } K_0 \neq F \\ F \oplus F & \text{if } K_0 = F \end{cases}$$

which we equip with the canonical involution fixing F.

1.1 Clifford Algebra

There exist a unique pair (up to isomorphism) (A, p) consisting of a unital F-algebra $A = A(V)$ and an F-linear map $p : V \to A$ such that:

(1) As an F-algebra, A is generated by $p(V)$.

(2) $p(v)^2 = Q(v)1_A$ for all $v \in V$.

(3) If (A_1, p_1) is another pair satisfying (2), then there is an F-algebra homomorphism $f : A \to A_1$ such that $p_1 = f \circ p$.

We call $A(V)$ the Clifford algebra associated to (V, Q).

Applying (3) to $A_1 = A$ and $p_1(v) = -p(v)$, we get an endomorphism $f : A \to A, a \mapsto a'$ satisfying $v' = -v$ for every $v \in V$. We also let $* : A \to A$ be the canonical involution.
We put:

$$A^+(V) = \{a \in A(V) \mid a' = a\}$$
$$A^-(V) = \{a \in A(V) \mid a' = -a\}$$

$A^+(V)$ is a subalgebra called the even Clifford algebra.

Lemma 1.3. *Let $e_1, \cdots, e_m$ be a basis of V, then the elements*

$$e_{i_1} \cdots e_{i_s} \quad \text{with } i_1 < \cdots < i_s,\ 0 \le s \le m$$

form a basis of A. In particular, $\dim_F(A) = 2^m$.

Theorem 1.4. *Let $V = X \oplus \sum_{i=1}^{s}(Fe_i \oplus Ff_i)$ be a weak Witt decomposition, and let $n = 2^s$, then $A(V) \cong M_n(A(X))$. Moreover, we have that $A^+(V) \cong \begin{cases} M_n(A^+(X)) & \text{if } X \neq 0 \\ M_{n/2}(F) \oplus M_{n/2}(F) & \text{if } X = 0 \end{cases}$*

Proof. The general case follows from the case $s = 1$ by induction. So suppose that $s = 1$ and define an F-linear map:

$$\Psi : V \to M_2(A(X))$$
$$x + re + tf \mapsto \begin{pmatrix} x & r \\ t & -x \end{pmatrix}$$

We have $\Psi(x + re + tf) = (x^2 + rt)\,\mathrm{id} = Q(x + re + tf)\,\mathrm{id}$. Hence by the universal property of the Clifford algebra, we can extend this to an F-algebra homomorphism

$$\Psi : A(V) \to M_2(A(X))$$

Observing that $\Psi(e) = E_{12}$, $\Psi(f) = E_{21}$, $\Psi(ef) = E_{11}$, and $\Psi(fe) = E_{22}$ where E_{ij} are the elementary matrices, we easily see that $\Psi(A(V))$ generates all of $M_2(A(X))$. Since $A(V)$ and $M_2(A(X))$ have the same dimension, we conclude that they are isomorphic.

Now put $l = e - f$, then $l^2 = -1$ and $lxl^{-1} = -x = x'$ for every $x \in X$, and so $lal^{-1} = a'$ for every $a \in A(X)$. So putting $\Lambda = \mathrm{diag}(l, -l)$, we get that

$$\Lambda\Psi(a)\Lambda^{-1} = \Psi(a') \quad \text{for every } a \in A(V) \tag{1.1}$$

this follows from the fact that it is true for $a \in V$. Now we have that:

$$\Psi(A^{\pm}(V)) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in A^{\pm}(X) \text{ and } b, c \in A^{\mp}(X)\} \tag{1.2}$$

Indeed, for $\Psi(\alpha) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $\Psi(\alpha') = \begin{pmatrix} a' & -b' \\ -c' & d' \end{pmatrix}$ by (1.1). Thus $\alpha \in A^{\pm}(V)$ if and only if $a' = \pm a$, $b' = \mp b$, $c' = \mp c$, and $d' = \pm d$.
If $X \neq 0$, let $h \in X$ such that $h^2 \neq 0$. Then we define a new algebra isomorphism:

$$\Xi : A(V) \xrightarrow{\sim} M_2(A(X))$$
$$\alpha \mapsto \Delta^{-1}\Psi(\alpha)\Delta$$

with $\Delta = \mathrm{diag}(h, 1)$. Then since $h^{-1}A^+(X) = A^{-1}(X)h = A^+(X)$ and $h^{-1}A^+(X)h = A^+(X)$, from (1.2) we obtain that

$$\Xi(A^+(V)) = M_2(A^+(X)) \quad \text{if } X \neq 0$$

□

Lemma 1.5. *Given an orthogonal basis $h_1, \dots, h_m$ of V, put $z = h_1 \cdots h_m$. Then the following assertions hold:*

(1) $z^{-1}vz = (-1)^{m-1}v$ for every $v \in V$.

(2) $z^2 = (-1)^{\frac{m(m-1)}{2}} h_1^2 \cdots h_m^2$ and $z^* = (-1)^{\frac{m(m-1)}{2}} z$.

(3) Fz is independent of the choice of the basis $\{h_i\}_i$.

(4) $F \oplus Fz$ is isomorphic to the discriminant algebra of Q.

Proof. Since the basis h_i is orthogonal, we easily see that $h_i z = (-1)^{m-1} z h_i$ for every i. Since the h_i span V, we get the first assertion. The second assertion is a straightforward calculation. Since $z^2 = F$ and $z \notin F$, we see that if $v \in F \oplus Fz$ and $v \notin F$, then $v \in Fz$. Hence (3) is a consequence of the following assertion proved in the next theorem: $F \oplus Fz$ is the center of $A(V)$ or $A^+(V)$ according to whether m is odd or even. Finally, (4) follows easily from (2). □

Theorem 1.6. *Let $\mathfrak{C}$ be the center of $A(V)$, $\mathfrak{C}^+$ is the center of $A^+(V)$ and z be as in the previous lemma.*

(1) Suppose $m = \dim(V)$ is even strictly positive. Then $A(V)$ is a central simple algebra over F and $\mathfrak{C}^+ = F \oplus Fz$. $A^+(V)$ is a central simple algebra over $\mathfrak{C}^+$ if $\mathfrak{C}^+$ is a field; otherwise, $A^+(V)$ is the direct sum of two central simple algebras over F of the same degree.

(2) Suppose $m = \dim(V)$ is odd. Then $A^+(V)$ is a central simple algebra over F, $\mathfrak{C} = F \oplus Fz$, and $A(V) = A^+(V) \otimes_F \mathfrak{C}$.

Proof. Let $\overline{F}$ be the algebraic closure of F. Suppose that $m = 2s+1$, then $V_{\overline{F}}$ has a Witt decomposition $V_{\overline{F}} = \overline{F}g \oplus \sum_{i=1}^{s} (\overline{F}e_i \oplus \overline{F}f_i)$ (recall that there is only one quadratic form up to iso in an algebraically closed field), and $A^+(\overline{F}g) = \overline{F}$. By theorem 1.4, $A^+(V_{\overline{F}})$ is isomorphic to $M_n(\overline{F})$ with $m = 2^s$. Since $A^+(V_{\overline{F}}) = A^+(V) \otimes_F \overline{F}$, we get that $A^+(V)$ is a central simple algebra. Now by lemma 1.5, $zv = vz$ for all $v \in V$ and so $z \in \mathfrak{C}$. Since $z \in A^-(V)$ (given that m is odd) and that z is invertible, we have that $A(V) = A^+(V) \oplus A^+(V)z$. Now $F \oplus Fz$ is a commutative algebra, and so $A(V) = A^+(V) \otimes_F (F \oplus Fz)$. But since the center of $A^+(V)$ is F, we get that $\mathfrak{C} = F \oplus Fz$.
Now suppose that $m = 2s$, then we have a Witt decomposition $V_{\overline{F}} = \sum_{i=1}^{s} (Fe_i \oplus Ff_i)$, and $A(V_{\overline{F}})$ is isomorphic to $M_n(\overline{F})$ with $n = 2^s$. Hence $A(V)$ is central simple over F. By theorem 1.4, $A^+(V_{\overline{F}})$ is isomorphic to $M_{n/2}(\overline{F}) \oplus M_{n/2}(\overline{F})$ whose center is $\overline{F} \oplus \overline{F}$ and so $[\mathfrak{C}^+ : F] = 2$. By lemma 1.5, we have that $z\alpha = \alpha z$ for all $\alpha \in A^+(V)$ and so $\mathfrak{C}^+ = F \oplus Fz$. If $\mathfrak{C}^+$ is a field, then $A^+(V)$ is central simple over $\mathfrak{C}^+$. Otherwise, there is an element c of F such that $z^2 = c^2$. Put $\epsilon = (1 + c^{-1}z)/2$ and $\delta = (1 - c^{-1}z)/2$, then $1 = \epsilon + \delta$, $\epsilon^2 = \epsilon$, $\delta^2 = \delta$, and $\epsilon\delta = 0$. Therefore $\mathfrak{C}^+$ is isomorphic to the algebra $F \oplus F$ and $A^+(V)$ is the direct sum of two central simple algebras $A^+(V)\epsilon$ and $A^+(V)\delta$. □

1.2 Clifford groups

We define:

$$G_V = \{\alpha \in A(V)^\times \mid \alpha^{-1} V \alpha = V\}$$

$$G_V^+ = G_V \cap A^+(V) \qquad G_V^- = G_V \cap A^-(V)$$

We consider the group homomorphism $\tau_V : G_V \to O_V$ given for $\alpha \in G_V$ by:

$$\tau_V(\alpha) : v \mapsto \alpha v \alpha^{-1}$$

This is well defined since $Q(\tau_V(\alpha)v) = (\alpha v \alpha^{-1})^2 = v^2 = Q(v)$.

Suppose that we have an orthogonal decomposition $V = X \oplus_\perp Y$. Then we can easily verify that $\alpha\beta = \beta\alpha$ if $\alpha \in A^+(X)$ or $\beta \in A^+(Y)$ and $\alpha\beta = -\beta\alpha$ if $\alpha \in A^-(X)$ and $\beta \in A^-(Y)$. In particular, if $\alpha \in A^+(X)^\times$, then $\alpha^{-1} y \alpha = y$ for all $y \in Y$, and $\alpha^{-1} X \alpha = X$ if and only if $\alpha^{-1} V \alpha = V$. Therefore, we can view elements of G_X^+ as elements in G_V^+ (This is not the case for G_X).

Lemma 1.7. *For $v, u \in V$, we have $vuv \in V$. Moreover, if $v^2 \neq 0$, then $v \in G_V^-$ and $v^{-1} \in V$.*

Proof. Both v^2 and $vu + uv$ belong to F, hence $vuv = (vu + uv)v - uv^2 \in V$.
Suppose $v^2 \neq 0$, then v is invertible, $v^{-1} = (v^2)^{-1}v \in V$, and $v^{-1}Vv = v^{-2}vVv \subseteq V$ so that $v \in G_V$. Since $x \in A^-(V)$, we get that $v \in G_V^-$. □

If $v_1, \dots, v_s$ are invertible elements of V, then the previous lemma shows that the product $v_1 \cdots v_s$ belong to G_V^+ or G_V^- according to whether s is even or odd.
To describe the action of these elements more precisely, let $v \in V$ such that $v^2 \neq 0$, and consider the hyperplane $H = (Fv)^\perp$. Then we have a decomposition $V = Fv \oplus H$ and the element $-\tau_V(v) \in O_V$ is the orthogonal symmetry of V with respect to H, i.e,

$$(-\tau_V(v))(v) = -v \quad \text{and} \quad (-\tau_V(v))(h) = h \quad \forall h \in H$$

Lemma 1.8. *Every element of O_V is a product of orthogonal symmetries as described above.*

Now let us put

$$G_V^{\cdot} = G_V^+ \cup G^-(V)$$

Theorem 1.9. *(1) If m is odd, then $\tau_V(G_V^+) = \tau_V(G_V) = SO_V$ and $G_V = \mathfrak{C}^\times G_V^+$.*

(2) If $m > 0$ is even, then $G_V = G_V^{\cdot}$, $[G_V : G_V^+] = 2$, $\tau_V(G_V) = O_V$, $\tau_V(G_V^+) = SO_V$, and $\tau_V(G_V^-) = \{g \in O_V \mid \det(g) = -1\}$. Moreover,

$$\mathfrak{C}^+ \cap G_V^+ = \begin{cases} \mathfrak{C}^{+,\times} = G_V^+ & \text{if } m = 2 \\ F^\times \cup F^\times z & \text{if } m > 2 \end{cases}$$

(3) For both m even and odd, τ_V gives an isomorphism of $G^+/F^\times$ onto SO_V. For even m τ_V gives an isomorphism of $G_V/F^\times$ onto O_V.

(4) If $V = X \oplus_\perp Y$ is an orthogonal decomposition, then

$$G^+(X) = \{\alpha \in G_V^+ \mid \tau_V(\alpha)y = y \text{ for all } y \in Y\}$$

Proof. Let $g \in O_V$. By lemma 1.8, $g = (-\tau_V(v_1)) \cdots (-\tau_V(v_k))$ for invertible $v_i \in V$. Since each orthogonal symmetry $(-\tau_V(v_1))$ has determinant -1, we have that $\det(g) = (-1)^k$. If $g \in SO_V$, then k is even, which shows that $SO_V \subseteq \tau_V(G_V^+)$.

Suppose $m > 0$ is even, then for every invertible $v \in V$, $\det(\tau_V(v)) = -1$. Since $v \in G_V$ and $[O_V : SO_V] = 2$, we see that $\tau_V(G_V) = O_V$. Suppose that $\tau_V(v) = \tau_V(\alpha)$ with $\alpha \in G_V^+$. Then $\alpha^{-1}v$ commutes with every element of V and hence is is the center of $A(V)$ which is F by theorem 1.6. Therefore $v = c\alpha$ with $c \in F$, which is a contradiction, since $c\alpha \in A^+(V)$ and $0 \neq v \in A^-(V)$. Thus $\tau_V(v) \notin \tau_V(G_V^+)$, and so $\tau_V(G_V^+) = SO_V$. From the fact that $G_V^- = vG_V^+$, we get that $\tau_V(G_V^-) = \{g \in O_V \mid \det(g) = -1\}$.
Now let $\gamma \in G_V$. For $\det(\tau_V(\gamma)) = \pm 1$, we have that $\tau_V(\gamma) = \tau_V(\beta)$ for $\beta \in G_V^\pm$. Then $\beta^{-1}\gamma \in F^\times$ and so $\gamma \in G_V^\pm$. Thus $G_V = G_V^{\cdot}$ and $[G_V : G_V^+] = 2$.

Now suppose that m is odd. Suppose that $\tau_V(\alpha) = -\mathrm{id}$ for some $\alpha \in G_V$. Then $\alpha^{-1}v\alpha = -v$ for every $v \in V$, so that $\alpha^{-1}y\alpha = y'$ for every $y \in A(V)$. Let z be as in lemma 1.5, Then $z' = -z$ and z belongs to the center $\mathfrak{C}$ of $A(V)$. Thus $z = \alpha^{-1}z\alpha = z' = -z$ contradiction. Thus $-\mathrm{id} \notin \tau_V(G_V)$, so that $\tau_V(G_V) = \tau_V(G_V^+) = SO_V$. Take any $\gamma \in G_V$, then $\tau_V(\gamma) = \tau_V(\beta)$ with $\beta \in G_V^+$, and so $\beta^{-1}\gamma \in \mathfrak{C}$ so that $\gamma \in \mathfrak{C}^\times G_V^+$. Clearly $\mathfrak{C}^\times \subset G_V$, hence $G_V = \mathfrak{C}^\times G_V^+$.

As for (3), if $\alpha \in G_V^+$ and $\tau_V(\alpha) = \mathrm{id}$, then $\alpha \in F^\times$ as an immediate corollary of theorem 1.6 ($\mathfrak{C} \cap \mathfrak{C}^+ = F$). If n is even, the same is true since $\mathfrak{C} = F$.
Now for (4), we saw that G_V^+ is contained in the RHS of the equality. For the other inclusion, let $\alpha \in G_V^+$ such that $\tau_V(\alpha)$ fixes Y. Then by applying (1) and (2) to V and X, we get that $\alpha \in SO_X$ and there exist $\beta \in G_X^+$ such that $\tau_V(\alpha) = \tau_V(\beta)$. Thus $\alpha = c\beta$ with $c \in F^\times$ by (3) and so $\alpha \in G_V^+$. □

Corollary 1.10. *For $\alpha \in G^{\cdot}$, put $\nu(\alpha) = \alpha\alpha^*$.*

(1) G_V^+ (resp. G_V^-) consists of all the products of even (resp. odd) number of elements of V that are invertible in $A(V)$.

(2) $G_V^{\cdot}$ is a subgroup of G_V, $[G^{\cdot} : G_V] = 2$, and we have a homomorphism $\nu : G^{\cdot} \to F^\times$. Moreover, $\nu(\alpha) = \nu(\alpha^*) = \nu(\alpha')$ for every $\alpha \in G_V^{\cdot}$.

Proof. We have $G_V^{\cdot} = vG_V^+$ for any invertible $v \in V$. Therefore, it suffices to prove (1) for G_V^+. If $w_1, \dots, w_k$ are invertible elements of V, then each v_i belongs to G_V by lemma 1.7 and so $w_1 \cdots w_k \in G_V^+$ if k is even. To prove the converse, let $\alpha \in G_V^+$. Then $\tau_V(\alpha) = \tau_V(v_1, \cdots, v_k)$ with $v_1, \cdots, v_k \in V \cap A(V)^\times$ and even k as shown in the proof of theorem 1.9. By (3) of the same theorem, $\alpha = cv_1, \cdots v_k$ with $c \in F^\times$ which proves (1). The first part of (2) is then clear, and if $\alpha = v_1 \cdots v_k$ with $v_i \in V \cap A(V)^\times$, then

$$\nu(\alpha) = \alpha\alpha^* = x_1 \cdots x_k x_k \cdots x_1 = x_1^2 \cdots x_k^2$$

from which the remaining part of (2) follows. ☐

1.3 Lower dimensional cases

One can prove that a quadratic form on a space of dimension > 4 over a local field is always isotropic. Thus over a local field, we have a Witt decomposition with an anisotropic space Z of dimension ≤ 4. Therefore it is important to investigate the Clifford algebra of such a Z. Here we will only consider the cases $m = 2, 3$.

First, let us give a few examples:

Take a couple (K, ι) consisting of an F-algebra K of rank 2 and an F-linear automorphism of K belonging to the following two types:

(I) K is a quadratic extension of F and ι is the generator of $\mathrm{Gal}(K/F)$.

(II) $K = F \oplus F$ and $\iota(a, b) = (b, a)$.

In both cases, we obtain a quadratic space (K, κ) of dimension two by putting $\kappa(x) = N_{K/F}(x) = xx^\iota$ for $x \in K$, and we have $2S(x, y) = \mathrm{Tr}_{K/F}(xy^\iota)$. Clearly κ is anisotropic if and only if K is a field.

Now take a quaternion algebra D over F and consider the main involution ι. We have a direct sum decomposition

$$D = F \oplus D^\circ \quad \text{with } D^\circ = \{x \in D \mid x^\iota = -x\}$$

Putting $N(x) = N_{D/K}(x) = xx^\iota$ for $x \in X$, we get quadratic spaces (D, N) and $(D^\circ, N^\circ = N_{|D^\circ})$ of dimension 4 and 3. We clearly have that D is a division algebra if and only if N is anisotropic, if and only if N° is anisotropic. We also see that $2S(x, y) = \mathrm{Tr}_{D/K}(xy^\iota)$ for $x, y \in D$. If $D = K \oplus Kw$ for an element w such that $w = \gamma^2 \in F^\times$, then $N(x + yw) = N_{K/F}(x) - \gamma N_{K/F}(y)$ for $x, y \in K$. Thus:

$$(D, N) \cong (K, \kappa) \oplus (K, -\gamma\kappa)$$

Lemma 1.11. *Let K be the discriminant algebra of Q which we view as a subalgebra of $A(Q)$ by lemma 1.5. Then the following assertions hold:*

(1) If $m = 2$, then (V, Q) is isomorphic to $(K, c\kappa)$ for some $c \in F^\times$. Moreover $A(V)$ is the quaternion algebra $\left(\frac{\delta_0(Q), c}{F}\right)$, $A^+(V) = K$, $SO_V = \{x \in K^\times \mid xx^\iota = 1\}$, $G_V^+ = K^\times$, $G_V = K^\times \cup K^\times h$ for any $h \in V \setminus \{0\}$.

(2) If $m = 3$, then there exists a quaternion algebra D over F such that (V, Q) is isomorphic to $(D^\circ, -\delta N^\circ)$ with $\delta \in \delta_0(Q)$. Moreover $A(V) \cong A^+(V) \otimes_F K$, $A^+(V) \cong D$, $G_V^+ \cong D^\times$, $\tau_V(d)x = d^{-1}xd$ for $x \in D^\circ$ and $d \in D^\times$ and the canonical involution of $A(V)$ restricted to $A^+(V)$ correspond to the main involution of D.

Proof. Suppose $m = 2$ and let $V = Fg \oplus Fh$ with elements g, h such that $S(g, h) = 0$. Put $b = g^2$, $c = h^2$, and as in lemma 1.5 $z = gh$. Then $Q(xg + yh) = bx^2 + cy^2$ for $x, y \in F$, $z^2 = -bc$, and $V = Kh$. By a dimension argument, we have that $A^+(V) = K = F \oplus Fz$ and so $A(V) = K \oplus Kh$. Since $z^* = -z$, we see that $\alpha^* = \alpha^\iota$ for $\alpha \in K$. We have by direct calculation that $Q(kh) = cN_{K/F}(k)$ for $k \in K$ and so $k \mapsto kh$ gives an isomorphism of $(K, c\kappa)$ onto (V, Q). Since $hk = k^\iota h$ for $k \in K$, we see that $A(V) = \left(\frac{-bc,c}{F}\right)$. We easily see that $K^\times = G_V^+ \subseteq G_V$ and $h \in G_V^-$ and so by (2) of theorem 1.9, $K^\times = G_V^+$ and $G_V = K^\times \cup K^\times h$.
Next, let $\alpha \in K^\times = G_V^+$ and $v = kh \in V$ for $k \in K$. Then $\tau_V(\alpha)(v) = \alpha kh\alpha^{-1} = \alpha\alpha^{\iota,-1}v$. Thus $\tau_V(\alpha)$ as an element of $\mathrm{End}_F(Kh)$ is multiplication by $\alpha\alpha^{\iota,-1}$. Therefore, $SO_V = \tau_V(G_V^+) = \{k/k^\iota \mid k \in K^\times\} = \{k \in K^\times \mid kk^\iota = 1\}$ (the last equality is an easy lemma).

Now suppose that $m = 3$. Let h_1, h_2, h_3 be an F-basis of V such that $S(h_i, h_j) = c_i\delta_{ij}$. We put:

$$g_1 = h_2h_3 \qquad g_2 = h_3h_1 \qquad g_3 = h_1h_2 \qquad z = h_1h_2h_3$$

$$c = c_1c_2c_3 \qquad T = Fg_1 \oplus Fg_2 \oplus Fg_3 \qquad B = F \oplus T$$

Then $z^2 = -d \in \delta_0(Q)$, $A^+(V) = D$ is a quaternion algebra, $k = F \oplus Fz$, and $A(V) = D \otimes_F K$ by theorem 1.6. Since $g_i^* = -g_i$, and $A^+(V) = F \oplus T$, we see that the involution $*$ coincides with the main involution of D and that $T = D^\circ$. Since $V = Dz$, and $Q(dz) = cdd^*$ for $d \in D^\circ$, (V, Q) is isomorphic to (D°, cN°). We have $d^{-1}D^\circ d = D^\circ$ for every $d \in D^\times$ and so $G_V^+ = D^\times$. □

Now let us suppose that $\dim(V) = 4$, then we have the following facts:

$$V = \{a \in A^-(V) \mid a^* = a\} \tag{1.3}$$

$$F \oplus Fz = \{a \in A^+(V) \mid a^* = a\} \tag{1.4}$$

$$G_V^\pm = \{a \in A^\pm(V) \mid aa^* \in F^\times\} \tag{1.5}$$

where z is as in lemma 1.5. We easily derive the first equality from the fact that $A^-(V) = V \oplus \sum_{i<j<k} Fe_ie_je_k$ with an orthogonal basis $e_1, \ldots, e_4$ of V and the second equality follows similarly. Now for the third equality, we have an obvious inclusion. Conversely, if $a \in A(V)^\times \cap A^\pm(V)$, then for $v \in V$ we have $a^*va \in A^-(V)$ and $(a^*va)^* = a^*va$ so that $a^*va \in V$ by the first equality. If $aa^* \in F^\times$, then $a^{-1}va \in V$ so that $a \in G_V$.

Now let us consider the case where Q is isotropic. We have a weak Witt decomposition $V = X \oplus (Fe \oplus Ff)$ for U a subspace of dimension 2. Then by theorem 1.4, $A(V)$ (resp. $A^+(V)$) is isomorphic to $M_2(A(X))$ (resp. $M_2(A^+(X))$). Let K be the discriminant algebra of (V, Q) which is also the discriminant algebra of $(X, Q_{|X})$ (since the discriminant of a hyperbolic space is 1). By (1) of lemma 1.11 and its proof, we can put $A^+(X) = K$ and $X = Kh$ with an element h such that $hk = k^*h$ for every $k \in K$. Define $\Xi : A(V) \xrightarrow{\sim} M_2(A(X))$ as in the proof of theorem 1.4, then we can easily verify that:

$$\Xi(\alpha^*) = J^{-1t}\Xi(\alpha')J \quad \forall \alpha \in A(V) \tag{1.6}$$

by verifying this for $\alpha \in V$. The map $M \mapsto J^tMJ^{-1}$ with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the main involution of $M_2(K)$. Therefore if we identify $A^+(V)$ with $M_2(K)$, then (1.6) shows that $\alpha \mapsto \alpha^*$ is the main involution of $M_2(K)$. Thus we get fro (1.5) that

$$\Xi(G_V^+) = \{M \in \mathrm{GL}_2(K) \mid \det(M) \in F^\times\}$$

and $\nu(\alpha) = \det(\Xi(\alpha))$ for $\alpha \in G_V^+$.

Let D be a quaternion algebra over F and put $(V,Q) = (D, cN)$ with $c \in F^\times$. We consider the linear map:

$$\begin{aligned} p : D &\to M_2(D) \\ x &\mapsto \begin{pmatrix} 0 & cx \\ x^\iota & 0 \end{pmatrix} \end{aligned} \tag{1.7}$$

observe that $p(x)^2 = cxx^\iota \,\mathrm{id} = Q(x) \cdot \mathrm{id}$. Now $p(D)$ generates $M_2(D)$ as an F-algebra. Indeed, take $x, y \in D$ such that $xy = -yx \in D^\times$. Then $p(xy)p(1) = \begin{pmatrix} xy & 0 \\ 0 & y^\iota x^\iota \end{pmatrix}$ and $p(x)p(y^\iota) = c\begin{pmatrix} xy & 0 \\ 0 & -y^\iota x^\iota \end{pmatrix}$ so that:

$$p(xy)p(1) + p(x)p(y^\iota) = \begin{pmatrix} 2cxy & 0 \\ 0 & 0 \end{pmatrix}$$

This way we can easily verify the claim. Since $\dim_F M_2(D) = 2^4$, we get by the above that $M_2(D) = A(V)$ with the identification $V = p(D) = \{\begin{pmatrix} 0 & cx \\ x^\iota & 0 \end{pmatrix} \mid x \in D\}$. Then $A^+(V) = \{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in D\} \cong D \times D$. For $\alpha = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M_2(B) = A(V)$, we have:

$$\alpha^* = \begin{pmatrix} p^\iota & cr^\iota \\ c^{-1}q^\iota & s^\iota \end{pmatrix} \tag{1.8}$$

since this is true for $\alpha \in V$. Then if we identify $A^+(V)$ with $D \times D$, then

$$G_V^+ = \{(x,y) \in D \times D \mid xx^\iota = yy^\iota \in \mathbb{Q}^\times\}$$

and $\nu((x,y)) = xx^\iota$. We have that $\tau_V(\alpha)p(d) = p(xdy^{-1})$ for $d \in D$, $\alpha = (x,y) \in G_V^+$. Furthermore, $G_V = G_V^+ \cup G_V^+\eta$ with $\eta = \begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix}$ and $p(d)\tau_V(\eta) = p(x^\eta)$ for all $d \in V$.
The main involution ι of D belongs to O_V and has determinant -1. Thus O_V is generated by SO_V and ι.

2 The Weil representation

Let (V,Q) be a quadratic space over $\mathbb{Q}$ with dimension m. The quadratic form produces a $\mathbb{Q}$-bilinear pairing $S(x,y) = Q(x+y) - Q(x) - Q(y)$ which we suppose to be non-degenerate.
Let $\mathbb{T}$ be the multiplicative group of complex numbers of absolute value 1, which we also identify with $\mathbb{R}/\mathbb{Z}$ by $x \mapsto \exp(2i\pi x)$. Then for $? = p, \infty$, or $\mathbb{A}$, we identify the Pontryagin dual $V_?^* = \mathrm{Hom}_{\mathrm{cont}}(V_?, \mathbb{T})$ of $V_?$ with itself via the symmetric bilinear pairing:

$$\begin{aligned} \langle \cdot, \cdot \rangle : V_? \times V_? &\to \mathbb{T} \\ (x,y) &\mapsto \mathbf{e}_?(S(x,y)) \end{aligned}$$

where:

- For $x \in \mathbb{R}$, $\mathbf{e}_\infty = \exp(2i\pi x)$.
- For $x \in \mathbb{Q}_p$, write $x = \sum_{n \gg -\infty} c_n p^n$ with $0 \leq c_p < p$. We let $[x]_p = \sum_{n<0} c_p p^n \in \mathbb{Q}$ and $\mathbf{e}_p(-2i\pi[x]_p)$.
- For $x = (x_v) \in \mathbb{A}$, we let $\mathbf{e}_{\mathbb{A}}(x) = \prod_v e_v(x_v)$ which induces a character $\mathbf{e}_{\mathbb{A}} : \mathbb{A}/\mathbb{Q} \to \mathbb{T}$.

Let $\mathrm{d}v$ be a Haar measure on V. For $\phi : V \to \mathbb{C}$ an integrable function, we define its Fourier transform:

$$\widehat{\phi}(x) = \int_V \phi(y)\langle y, x\rangle \mathrm{d}y$$

and we normalize $\mathrm{d}v$ so that $\widehat{\widehat{\phi}}(x) = \phi(-x)$ (it is the unique Haar measure satisfying this).

We let $W = V \times V$ which inherits a non-degenerate bilinear pairing $\langle \cdot, \cdot\rangle$ from V given coordinate-wise. We can write an automorphism $\sigma : W_? \to W_?$ as a matrix:

$$(x, y) \mapsto (x, y)\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

with $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in \mathrm{End}(V)$. We then define an alternating bilinear form $J : W_? \times W_? \to \mathbb{T},\ J((x, y),(x', y')) = \langle -y, x'\rangle\langle x, y'\rangle$ which we can write symbolically:

$$(x, y)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \langle -y, x'\rangle\langle x, y'\rangle$$

We then define the group $Sp(W_?) \subset \mathrm{Aut}(W_?)$ to be the stabilizer of J. From this definition, we get that for $\sigma \in \mathrm{Aut}(W_?)$,

$$\sigma^{-1} = \begin{pmatrix} d_\sigma & -b_\sigma \\ -c_\sigma & a_\sigma \end{pmatrix}$$

A continuous function $f : W_? \to \mathbb{T}$ is called a multiplicative quadratic form if the map:

$$(w, w') \mapsto f(w + w')f(w)^{-1}f(w')^{-1}$$

is a bicharacter. In this case, there is a unique symmetric endomorphism $\rho \in \mathrm{End}(W_?)$ such that:

$$f(w + w')f(w)^{-1}f(w')^{-1} = \langle w, \rho(w')\rangle$$

2.1 The Heisenberg group

For each $w = (v_1, v_2) \in W_?$, we define the unitary operator $U(w)$ on $L^2(V_?)$ by:

$$(U(w)\Phi)(v) = \Phi(v + v_1)\langle v, v_2\rangle$$

For $\Phi \in L^2(V_?)$. Then for $w' = (v_1', v_2') \in W_?$, we get by direct computation:

$$U(w')U(w) = \langle v_1, v_2'\rangle U(w + w') = F(w, w')U(w + w')$$

where we set $F(w, w') = \langle v_1, v_2'\rangle$. Thus $H(V_?) = \{tU(w) \mid t \in \mathbb{T}, w \in W_?\}$ is a subgroup of unitary operators acting on $L^2(V_?)$ called the Heisenberg group.
Since $U(w)U(w') = U(w')U(w)$ implies that $\langle v_1, v_2'\rangle = \langle v_2, v_1'\rangle$, if $U(w)$ commutes with all other elements of the Heisenberg group, then $w = 0$. Thus the center is given by $Z(H(V_?)) = \{tU(0) \mid t \in \mathbb{T}\} \cong \mathbb{T}$, and so we have a central extension:

$$1 \to \mathbb{T} \to H(V_?) \to W_? \to 1$$

We write $B(V_?)$ for the automorphism group of $H(V_?)$ which induce the identity on $\mathbb{T}$.
Let $s \in B(V_?)$ and let:

$$s(U(w)) = f(w)U(\sigma w)$$

for $\sigma \in \mathrm{Aut}(W_?)$ and $f(w) \in \mathbb{T}$ and so we write $s = (\sigma, f)$. The composition formula is given by:

$$(\sigma', f'(w)) = (\sigma \circ \sigma', f(w)f'(\sigma w))$$

Note that we have we have:

$$\begin{aligned}f(w')f(w)F(\sigma w',\sigma w)U(\sigma w'+\sigma w) &= f(w')U(\sigma w')f(w)U(\sigma w)\\ &= s(U(w'))s(U(w))\\ &= s(U(w')U(w))\\ &= F(w',w)f(w'+w)U(\sigma w'+\sigma w)\end{aligned}$$

By the composition law for the Heisenberg group, we get that:

$$f(w'+w)f(w')^{-1}f(w)^{-1} = F(\sigma w',\sigma w)F(w',w)^{-1} \tag{2.1}$$

This shows that f is a multiplicative quadratic form on $W_?$. Conversely, one can check that for any function f on $W_?$ satisfying the above formula, the couple (σ, f) defines an element of $B(V_?)$.
Given that the right-hand-side of the equation 2.1 is symmetric with respect to w and w', we get that:

$$F(\sigma w',\sigma w)F(w',w)^{-1} = F(\sigma w,\sigma w')F(w,w')^{-1}$$

Since $J(w,w') = F(w,w')F(w',w)^{-1}$, σ preserves J and so $\sigma \in Sp(V_?)$.
Therefore we have a group homomorphism $\pi : B(V_?) \to Sp(V_?)$ given by the projection to the first coordinate. Its kernel consists of couples $(1, f)$ where f is a character of $W_?$ and so is of the form:

$$f(w) = \langle w, w_f\rangle$$

Calculations show that the automorphism of $H(V_?)$ associated to the couple $(1, f)$ is the conjugation by $U(w_f)$. Hence the kernel of π consists of interior automorphisms of $H(V_?)$, and since its center is $\mathbb{T}$, we get that $\ker(\pi) \cong H(V_?)/\mathbb{T} \cong W_?$.

On the other hand, one can check that by defining

$$f_\sigma((v_1,v_2)) = \langle v_1, 2^{-1}a_\sigma^* b_\sigma v_1\rangle\langle 2^{-1}d_\sigma^* c_\sigma v_2, v_2\rangle\langle c_\sigma v_2, b_\sigma v_1$$

for $\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma\end{pmatrix} \in Sp(V_?)$, we get a section of π given by $\sigma \mapsto (\sigma, f_\sigma)$. Thus we find that $B(V_?) \cong Sp(V_?) \ltimes W_?$.

Theorem 2.1. *Let $\mathbb{B}(V_?)$ be the normalizer of $H(V_?)$ in $Aut(L^2(V_?))$. Then we have a canonical exact sequence*

$$1 \to \mathbb{T} \to \mathbb{B}(V_?) \xrightarrow{\mu} B(V_?) \to 1$$

We define the metaplectic group $Mp(V_?)$ by:

$$Mp(V_?) = \{u \in \mathbb{B}(V_?) \mid \mu(u) = s = (\sigma, f_s) \text{ for } \sigma \in Sp(V_?) \text{ and } f_s \text{ homogenious multiplicative}\}$$

By definition $Mp(V_?)$ is a central extension of $Sp(V_?)$ and we have a short exact sequence:

$$1 \to \mathbb{T} \to Mp(V_?) \xrightarrow{\pi} Sp(V_?) \to 1$$

In general, this extension is non-trivial. However, over some subset of $Sp(V_?)$, one can define a section $\mathbf{r}$ of π.
Let

$$U(V_?) = \{\begin{pmatrix} 1 & \rho \\ 0 & 1\end{pmatrix} \in Sp(V_?) \mid \rho \in \mathrm{End}(V_?)\}$$

then since it is a subgroup of the symplectic group, ρ is symmetric with respect to $\langle\cdot,\cdot\rangle$ and so we can associate to it a multiplicative quadratic form $f_\rho(v) = \langle v, 2^{-1}\rho v\rangle$. Then we define a section $\mathbf{r} : U(V_?) \to B(V_?)$ by:

$$\mathbf{r}(\begin{pmatrix} 1 & \rho \\ 0 & 1\end{pmatrix}) = (\begin{pmatrix} 1 & \rho \\ 0 & 1\end{pmatrix}, f_\rho)$$

which we extend to $\mathbf{r} : U(V_?) \to Mp(V_?)$ by:

$$\left(\mathbf{r}(\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix})\Phi\right)(v) = \Phi(v) f_\rho(v) \quad \text{for } \Phi \in L^2(V_?)$$

For the subgroup

$$L(V_?) = \{\begin{pmatrix} a & 0 \\ 0 & a^{*,-1} \end{pmatrix} \mid a \in \mathrm{Aut}(V_?)\}$$

of $Sp(V_?)$, we also define a section $\mathbf{r} : L(V_?) \to B(V_?)$

$$\mathbf{r}(\begin{pmatrix} a & 0 \\ 0 & a^{*,-1} \end{pmatrix}) = (\begin{pmatrix} a & 0 \\ 0 & a^{*,-1} \end{pmatrix}, 1)$$

and we extend it to $\mathbf{r} : L(V_?) \to Mp(V_?)$ via:

$$\left(\mathbf{r}(\begin{pmatrix} a & 0 \\ 0 & a^{*,-1} \end{pmatrix})\Phi\right)(v) = \sqrt{|a|}\phi(a^{-1}v) \quad \text{for } \Phi \in L^2(V_?)$$

Finally for $c \in \mathrm{Aut}(V_?)$, we let:

$$\left(\mathbf{r}(\begin{pmatrix} 0 & -c^{*,-1} \\ c & 0 \end{pmatrix})\Phi\right)(v) = \sqrt{|c|}^{-1}\widehat{\Phi}(-c^* v) \quad \text{for } \Phi \in L^2(V_?)$$

where we fix a Haar measure $\mathrm{d}v$ on $V_?$ and $\widehat{\Phi}$ is the Fourier transform on Φ.
Let $\Omega = \Omega(V_?)$ be the collection of all the $\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in Sp(V_?)$ with $c_\sigma \in \mathrm{Aut}(V_?)$. Then using the decomposition:

$$\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} = \begin{pmatrix} 1 & a_\sigma c_\sigma^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -c_\sigma^{-1,*} \\ c_\sigma & 0 \end{pmatrix} \begin{pmatrix} 1 & c_\sigma^{-1} d_\sigma \\ 0 & 1 \end{pmatrix}$$

(the equality $a_\sigma c_\sigma^{*,-1} d_\sigma - c_\sigma^{*,-1} = b_\sigma$ follows from the formula of the inverse of σ), we can extend the section to $\mathbf{r} : \Omega \to Mp(V_?)$. Explicitly, we have:

Lemma 2.2. *[Wei64, Lem. 6] The group $Sp(V_?)$ is the group generated by the elements $\Omega(V_?)$ subject to the relations $\sigma\sigma' = \sigma''$ for $\sigma, \sigma', \sigma'' \in \Omega(V_?)$ if the same equality holds in $Sp(V_?)$.*

2.2 The Siegel-Weil formula

For $\Phi \in \mathcal{S}(V_{\mathbb{A}_F})$, we can form a theta series as a function on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times O_V(F)\backslash O_V(\mathbb{A}_F)$:

$$\theta(g, h, \Phi) = \sum_{(x,u)\in V\times F^\times} \mathbf{r}(g, h) \quad (g, h) \in \widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \times O_V(\mathbb{A}_F)$$

When V has even dimension, we can define the theta series for $\Phi \in \widetilde{S}(V_{\mathbb{A}_F} \times \mathbb{A}_F^\times)$ as an automorphic form on $\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}) \times GO_V(F)\backslash GO_V(\mathbb{A})$:

$$\theta(g, h, \Phi) = \sum_{(x,u)\in V\times F^\times} \mathbf{r}(g, h)$$

Now we introduce the Siegel Eisenstein series. For $\Phi \in \mathcal{S}(V_{\mathbb{A}_F})$ and $s \in \mathbb{C}$, we have a section:

$$g \mapsto \delta(g)^s \mathbf{r}(g)\Phi(0)$$

in

$$\mathrm{Ind}_{P^1(\mathbb{A}_F)}^{\widetilde{\mathrm{SL}}_2(\mathbb{A}_F)}(\chi_V|\cdot|^{s+m/2}) = \{f : \widetilde{\mathrm{SL}}_2(\mathbb{A}) \to \mathbb{C} \mid f(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g) = |a|^{s+m/2}\chi_V(a) f(g)\}$$

Here the modulus function δ is defined as follows: first we let

$$\delta_v : B(F_v) \to \mathbb{R}^\times, \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto |\frac{a}{d}|_v^{1/2}$$

which we extend to a function $\delta_v : \mathrm{GL}_2(F_v) \to \mathbb{R}^\times$ by Iwasawa decomposition, and we led $\delta = \prod_v \delta_v$. Thus we can form the Eisenstein series:

$$E(s, g, \Phi) = \sum_{\gamma \in B(F) \backslash \mathrm{SL}_2(F)} \delta(\gamma g)^s \mathbf{r}(\gamma g)\Phi(0)$$

It has a meromorphic continuation to $s \in \mathbb{C}$ and a functional equation with center $s = 1 - m/2$.
Let r be the Witt index of V, i.e., the maximal dimension of F-subspaces of V consisting of elements of norm 0 (it is denoted by s in theorem 1.1). Then we always have $r \leq m/2$.

Theorem 2.3. *(Siegel-Weil) Assume that (V, Q) is anisotropic or $m - r > 2$, then:*

$$E(0, g, \Phi) = \kappa \frac{1}{\mathit{Vol}(SO_V(F) \backslash SO_V(\mathbb{A}_F))} \int_{SO_V(F) \backslash SO_V(\mathbb{A}_F)} \theta(g, h, \Phi) dh$$

with $\kappa = \begin{cases} 2 & \text{if } m = 1, 2 \\ 1 & \text{if } m > 2 \end{cases}$*, and the integration uses the Haar measure of total volume* 1.

Remark 2.4. The theorem implicitly states that the Eisenstein series is analytic at $s = 0$ and the integral on the RHS converges absolutely.

2.3 Explicit form of the metaplectic groups

We view the group $SL_2(\mathbb{A})$ (and similarly for other coefficient rings) as the subgroup of $Sp(V_\mathbb{A})$ given by $\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ with $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in \mathbb{A}$. We let $Mp_1(V_\mathbb{A})$ to be the inverse image of $\mathrm{SL}_2(\mathbb{A})$ inside $Mp(V_\mathbb{A})$.

Recall that for a group G, the set of isomorphism classes of central extensions

$$1 \to A \to E \to G \to 1$$

i.e., $A \subseteq Z(E)$ is classified by $H^2(G, A)$. Setting $E = A \times G$, this is given explicitly by defining the composition law on E for a cocycle $\alpha \in H^2(G, A)$ by:

$$(a, e) \cdot (a', e') = (\alpha(e, e')aa', ee')$$

Moreover, if G and A are locally compact topological groups, then a measurable cocycle α induces a unique locally compact topology on E compatible with the exact sequence.

Weil [Wei64] showed that there is a subgroup $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ of $Mp_1(\mathbb{A})$ satisfying the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, hook] & \widetilde{SL}_2(\mathbb{A}) \arrow[r] \arrow[d, hook] & \mathrm{SL}_2(\mathbb{A}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{T} \arrow[r] & Mp_1(\mathbb{A}) \arrow[r] & \mathrm{SL}_2(\mathbb{A}) \arrow[r] & 1
\end{tikzcd}$$

In other words, the 2-cocycle $\mathrm{SL}_2(\mathbb{A})^2 \to \mathbb{T}$ giving rise to the metaplectic extension is cohomologous to another one with values in μ_2.

Theorem 2.5. *[Kub67] Let $v = \infty, p$ be a place of $\mathbb{Q}$, and $(\cdot,\cdot)_v : \mathbb{Q}_v^\times \times \mathbb{Q}_v^\times \to \mu_2$ be the Hilbert symbol. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_v)$, set $x(\gamma) = c$ or d according to whether $c \neq 0$ or $= 0$. Then the map:*

$$\begin{aligned} a_p : \mathrm{SL}_2(\mathbb{Q}_v) \times \mathrm{SL}_2(\mathbb{Q}_v) &\to \mu_2 \\ (\gamma, \mu) &\mapsto (x(\gamma), x(\delta))_v (-x(\gamma)^{-1} x(\delta), x(\gamma\delta))_v \end{aligned}$$

defines a (measurable) cohomologically non-trivial 2-cocycle.

Proposition 2.6 ([Gel76]Prop 2.3)**.** *Let $v = \infty, p$ be a place of $\mathbb{Q}$. Then*

$$H^2(\mathrm{SL}_2(\mathbb{Q}_v), \mu_2) = \mu_2$$

In other words, there exists a unique (up to isomorphism) extension $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ of $\mathrm{SL}_2(\mathbb{Q}_v)$ by μ_2.

Remark 2.7. The topology on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ is not the product topology. If $\{U_n\}_n$ is a basis of neighborhoods of the identity in $\mathrm{SL}_2(\mathbb{Q}_v)$, then a basis of neighborhoods of the identity in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ is of the form $(U_n, 1)$ for U_n such that $\alpha(U_n, U_n)$ is identically one.

We modify Kubolta's cocycle by a coboundary as follows: let $s_p : \mathrm{SL}_2(\mathbb{Q}_p) \to \mu_2$ for a prime p be given by

$$s_p(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{cases} (c,d)_p & \text{if } c \neq d \text{ and } \mathrm{ord}_p(c) \equiv 1 \mod 2 \\ 1 & \text{otherwise} \end{cases}$$

and for $\mathbb{Q}_\infty = \mathbb{R}$, set $s_\infty = 1$. Then we define a new 2-cocycle:

$$\kappa_v(\gamma, \delta) = a_v(\gamma, \delta) s_v(\gamma) s_v(\delta) s_v(\gamma\delta)$$

By [Gel76, Prop 2.8] for a prime p, κ_p is trivial on $\Gamma_1(4)_p = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p) \mid c \equiv 0, a \equiv 1 \mod 4\}$. Therefore the product $\kappa^{(\infty)}(\gamma, \delta) = \prod_p \kappa_p(\gamma_p, \delta_p)$ is well defined for $\gamma, \delta \in \mathrm{SL}_2(\mathbb{A}^{(\infty)})$ and gives the metaplectic extension

$$1 \to \mu_2 \to \widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)}) \to \mathrm{SL}_2(\mathbb{A}^{(\infty)}) \to 1$$

For the infinite place, we will choose another cocycle defined by Shimura [Shi73]. We first define Shimura's symbol $\left(\frac{a}{b}\right)$ for an integer a and an odd integer $b \neq 0$ by:

(1) $\left(\frac{a}{b}\right) = 0$ if $(a; b) \neq 1$.

(2) If b is an odd prime, then $\left(\frac{a}{b}\right)$ is the Legendre symbol.

(3) If $b > 0$, $a \mapsto \left(\frac{a}{b}\right)$ is a character modulo b.

(4) If $a \neq 0$, $b \mapsto \left(\frac{a}{b}\right)$ is a character modulo $4a$ whose conductor is the conductor of $\mathbb{Q}[\sqrt{a}]/\mathbb{Q}$.

(5) $\left(\frac{a}{-1}\right) = 1$ or -1 accorting to whether $a > 0$ or $a < 0$.

(6) $\left(\frac{0}{\pm 1}\right) = 1$

Consider the theta function $\theta : \mathfrak{H} \to \mathbb{C}, \tau \mapsto \sum_{n \in \mathbb{Z}} \mathbf{e}_\infty(n^2\tau)$. We define for $\gamma \in \Gamma_0(4)$, $h(\gamma, \tau) := \theta(\gamma(\tau))/\theta(\tau)$. Then by [Shi73, 1.10],

$$h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) = \epsilon_d^{-1} \left(\frac{c}{d}\right) j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau)^{1/2} \quad \text{with } j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) = c\tau + d$$

where we choose a square root function $z^{1/2} = \sqrt{|z|}e^{i\pi\theta}$ for $z = |z|\mathbf{e}_\infty(\theta)$ with $-\pi < \theta \leq \pi$, and $\epsilon_d = i$ or 1 according to whether $d \equiv 3$ or $1 \mod 4$.
We can now realize

$$\widetilde{\mathrm{SL}}_2(\mathbb{R}) = \{(g, J(g,\tau)) \mid g \in \mathrm{SL}_2(\mathbb{R}),\ J(g,-) \text{ is holomorphic and } J^2(g,\tau) = j(g,\tau)\}$$

with multiplication given by $(g, J(g,\tau)(h, J(h,\tau)) = (gh, J(g,h(\tau))J(h,\tau))$ (because j is a cocycle). We thus have a central extension $\mu_2 \overset{\iota}{\hookrightarrow} \widetilde{\mathrm{SL}}_2(\mathbb{R}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{R})$ with $\iota(-1) = (\mathrm{id}, -1)$.
The above calcualtion shows that we have a section $\Gamma_0(4) \to \widetilde{\mathrm{SL}}_2(\mathbb{R}), \gamma \mapsto (\gamma, h(\gamma,\tau))$.

3 Waldspurger's formula

Let F be a number field and D be a quaternion algebra with ramification set Σ. Fix an embedding $K \hookrightarrow D$ for a quadratic extension K/F, then we have a decomposition:

$$D = K \oplus Kj \quad \text{ with } j^2 \in F^\times$$

We let $\eta : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ be the quadratic character associated to the extension K/F.

Consider the orthogonal space $(V, Q) = (D, N)$ and the orthogonal decomposition $V = V_1 \oplus V_2$ for $V_1 = K$ and $V_2 = Kj$.

Let π be an irreducible cuspidal automorphic representation of $D_{\mathbb{A}}^\times$ with central character $\omega_\pi : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$, and let $\chi : K^\times \setminus \mathbb{A}_K^\times \to \mathbb{C}^\times$ be a character with $\omega_\pi \cdot \chi_{|\mathbb{A}_F^\times} = 1$. We define the toric period integral:

$$P_\chi(f) = \int_{K^\times\backslash\mathbb{A}_K^\times/\mathbb{A}_F^\times} f(t)\chi(t)\ \mathrm{d}t \quad \text{ for } f \in \pi$$

then $P_\chi \in \mathrm{Hom}_{K^\times}(\pi \otimes \chi, \mathbb{C})$.

For any $\Phi \in \mathcal{S}(V_{\mathbb{A}_F} \times \mathbb{A}_F^\times)$, we have a theta series:

$$\theta(g, h, \Phi) = \sum_{x\in V, u\in F^\times} \mathbf{r}(g,h)\Phi(x,u) \quad \text{ for } g \in \mathrm{GL}_2(\mathbb{A}_F), h \in D_{\mathbb{A}_F}^\times \times D_{\mathbb{A}_F}^\times$$

Let σ be the Jacquet-Langlands transfer of π to GL_2/F. For any $\varphi \in \sigma$, we define the normalized global Shimizu lifting:

$$\Theta(\Phi,\varphi)(h) : \frac{\zeta(2)}{L(1,\pi,\mathrm{ad})} \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)} \varphi(g)\theta(g,h,\Phi)\ \mathrm{d}g \quad \text{ for } h \in D_{\mathbb{A}_F}^\times \times D_{\mathbb{A}_F}^\times$$

This defines an automorphic form $\Theta(\Phi,\varphi) \in \pi \otimes \widetilde{\pi}$ (the reason for this normalisation will be appearent in the next lemma). Let

$$\mathcal{F} : \pi \otimes \widetilde{\pi} \to \mathbb{C}$$

be the canonical bilinear map defined by the Petersson pairing.

For an additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}$, we consider the Whittaker model $\mathcal{W}(\psi,\sigma)$ of σ. For $\varphi \in \sigma$ and $x \in F^\times$, let:

$$W_{x,\varphi}(g) = \int_{F\backslash\mathbb{A}_F} \varphi(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} g)\mathbf{e}_{\mathbb{A}_F}(-x\alpha)\ \mathrm{d}\alpha$$

then the map $\varphi \mapsto W_{x,\varphi}$ induces a Hecke equivariant isomorphism $\sigma \xrightarrow{\sim} \mathcal{W}(\sigma, \mathbf{e}_{\mathbb{A}_F}(x-))$.

Lemma 3.1. *For any $\varphi \in \sigma$ and decomposable $\Phi = \otimes_v \Phi_v \in \mathcal{S}(V_{\mathbb{A}_F} \times \mathbb{A}_F^\times)$, we have:*

$$\mathcal{F}\big(\Theta(\Phi,\varphi)\big) = \prod_v \frac{\zeta_v(2)}{L(1,\pi_v,\mathrm{ad})} \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} W_{\varphi,-1,v}(g)\mathbf{r}(g)\Phi_v(1,1)\ dg$$

For $\Phi \in \mathcal{S}(V_{\mathbb{A}_F} \times \mathbb{A}_F^\times)$, we can form the mixed theta-Eisenstein series:

$$I(s,g,\Phi) = \sum_{\gamma \in P(F)\backslash GL_2(F)} \delta(\gamma g)^s \sum_{(x_1,u)\in V_1\times F^\times} \mathbf{r}(\gamma g)\Phi(x_1,u)$$

Define its χ-component:

$$I(s,g,\chi,\Phi) = \int_{T(F)\backslash T(\mathbb{A}_F)} \chi(t) I(s,g,\mathbf{r}(t,1)\Phi)\ \mathrm{d}t$$

(Here $(t,1)$ is seen as an element of O_V). For any $\varphi \in \sigma$, we introduce the Petersson pairing:

$$P(s,\chi,\Phi,\varphi) = \int_{Z(\mathbb{A}_F)\,\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \varphi(g) I(s,g,\chi,\Phi)\ \mathrm{d}g$$

Proposition 3.2. *[Wal85, Prop. 4] If we have decomposable $\Phi = \otimes_v \Phi_v$ and $\varphi = \otimes_v \varphi_v$, then:*

$$P(s,\chi,\Phi,\varphi) = \prod_v P_v(s,\chi_v,\Phi_v,\varphi_v)$$

where:

$$P_v(s,\chi_v,\Phi_v,\varphi_v) = \int_{Z(F_v)\backslash T(F_v)} \chi(t) \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} \delta_v(g)^s W_{-1,\varphi_v}(g)\mathbf{r}(g)\Phi_v(t^{-1},Q(t))\ dg\,dt$$

Proof. Writing the explicit formula for $I(s,g,\chi,\Phi)$, we get that $P(s,\chi,\Phi,\varphi)$ is equal to:

$$\int_{Z(\mathbb{A}_F)P(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \varphi(g)\delta(g)^s \int_{T(F)\backslash T(\mathbb{A}_F)} \chi(t) \sum_{(x_1,u)\in V_1\times F^\times} \mathbf{r}(g,(t,1))\Phi(x_1,u)\ \mathrm{d}g\mathrm{d}t$$

We decompose the first integral as a double integral:

$$\int_{Z(\mathbb{A}_F)P(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \mathrm{d}g = \int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)P(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{N(F)\backslash N(\mathbb{A}_F)} \mathrm{d}n\mathrm{d}g$$

and using the expression of the Whittaker model and of $\mathbf{r}$ on elements of $N(\mathbb{A}_F)$, we get:

$$\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)P(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \delta(g)^s \int_{T(F)\backslash T(\mathbb{A}_F)} \chi(t) \sum_{(x_1,u)\in V_1\times F^\times} W_{-Q(x_1)u,\varphi}(g)\mathbf{r}(g,(t,1))\Phi(x_1,u)\mathrm{d}t\mathrm{d}g$$

Since φ is cuspidal, $W_{0,\varphi} = 0$. This way we can change variables $(x_1,u) \mapsto (x, Q(x_1^{-1})u)$ to obtain the following expression:

$$\sum_{(x_1,u)\in K^\times\times F^\times} W_{-u,\varphi}(g)\mathbf{r}(g,(t,1))\Phi(x_1,Q(x_1)^{-1}u) = \sum_{(x_1,u)\in K^\times\times F^\times} W_{-u,\varphi}(g)\mathbf{r}(g,(tx_1,1))\Phi(1,u)$$

Since $T(F) = K^\times$ and $\int_{T(F)\backslash T(\mathbb{A}_F)} \sum_{x_1\in T(F)} = \int_{T(\mathbb{A}_F)}$, the integral becomes:

$$\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)P(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \delta(g)^s \int_{T(\mathbb{A}_F} \chi(t) \sum_{u\in F^\times} W_{-u,\varphi}(g)\mathbf{r}(g,(t,1))\Phi(1,u)\ \mathrm{d}t\mathrm{d}g$$

By a straightforward calculation, we have $W_{-u,\varphi}(\begin{pmatrix}1 & 0\\ 0 & u^{-1}\end{pmatrix} g) = W_{-1,\varphi}(g)$ and $|u|_{\mathbb{A}_F} = 1$, the integral is equal to:

$$\int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)P(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \delta(g)^s \int_{T(\mathbb{A}_F} \chi(t) \sum_{u\in F^\times} W_{-1,\varphi}(\begin{pmatrix}1 & 0\\ 0 & u^{-1}\end{pmatrix} g)\mathbf{r}(\begin{pmatrix}1 & 0\\ 0 & u^{-1}\end{pmatrix} g, (t,1))\Phi(1,1)\ \mathrm{d}t\mathrm{d}g$$

The sum over $u \in F^\times$ collapses with the quotient over $P(F)$, thus we get the following expression:

$$P(s,\chi,\Phi,\varphi) = \int_{Z(\mathbb{A}_F)N(\mathbb{A}_F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \delta(g)^s \int_{T(\mathbb{A}_F)} \sum_{u\in F^\times} \chi(t) W_{-1,\varphi}(g)\mathbf{r}(g)\Phi(t^{-1}, Q(t))\ \mathrm{d}t\mathrm{d}g$$

We may decompose the inside integral as $\int_{Z(\mathbb{A}_F\backslash T(\mathbb{A}_F)}\int_{Z(\mathbb{A}_F)}$ and move the first integral outside. Then using the fact that $\omega_\sigma \cdot \chi_{\mathbb{A}_F^\times} = 1$, we obtain:

$$P(s,\chi,\Phi,\varphi) = \int_{Z(\mathbb{A}_F)\backslash T(\mathbb{A}_F)} \chi(t) \int_{N(\mathbb{A}_F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \delta(g)^s W_{-1,\varphi}(g)\mathbf{r}(g)\Phi(t^{-1}, Q(t))\ \mathrm{d}t\mathrm{d}g$$

□

When everything is unramified, Waldspurger computed these integrals (cf. lemma 2 in [Wal85]) and got:

$$P_v(s,\chi_v,\Phi_v,\varphi_v) = \frac{L((s+1)/2, \pi_v, \chi_v)}{L(s+1, \eta_v)}$$

So we may define a normalised integral P_v° by:

$$P_v^\circ(s,\chi_v,\Phi_v,\varphi_v) = \frac{L(s+1, \eta_v)}{L((s+1)/2, \pi_v, \chi_v)} P_v(s,\chi_v,\Phi_v,\varphi_v)$$

This normalized integral P_v° will be regular at $s = 0$ and equal to

$$\frac{L(1/2, \pi_v, \chi_v)L(1, \pi_v, \mathrm{ad})}{\zeta_v(2)L(1,\eta_v)} \int_{Z(F_v)\backslash T(F_v)} \chi_v(t)\mathcal{F}(\pi(t)\Theta(\Phi_v,\varphi_v))\mathrm{d}t$$

by lemma 3.1. This can be written as $\alpha_v(\Theta(\Phi_v,\varphi_v))$ with $\alpha_v \in \mathrm{Hom}(\pi_v \otimes \widetilde{\pi}_v, \mathbb{C})$ given by integration of matrix coefficients:

$$\alpha_v(f_1 \otimes f_2) = \frac{L(1/2, \pi_v, \chi_v)L(1, \pi_v, \mathrm{ad})}{\zeta_v(2)L(1,\eta_v)} \int_{Z(F_v)\backslash T(F_v)} \chi(t)\mathcal{F}\chi_v(t)\langle \pi(t) f_1, f_2\rangle\ \mathrm{d}t$$

and we define the global element $\alpha := \otimes_v \alpha_v \in \mathrm{Hom}(\pi \otimes \widetilde{\pi}, \mathbb{C})$. We thus get:

Proposition 3.3. *We have that:*

$$P(0,\chi,\Phi,\varphi) = \frac{L(1/2,\pi,\chi)}{L(1,\eta)} \prod_v \alpha_v(\Theta(\Phi_v,\varphi_v))$$

We thus get to the main theorem of [Wal85]:

Theorem 3.4. *For $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$, we have:*

$$P_\chi(f_1) \cdot P_{\chi^{-1}}(f_2) = \frac{\zeta_F(2)L(1/2,\pi,\chi)}{8L(1,\eta)^2 L(1,\pi,\mathrm{ad})} \alpha(f_1 \otimes f_2)$$

4 Doi-Naganuma Lift

Let $E/\mathbb{Q}$ be a real quadratic field extension so that $E = \mathbb{Q}(\sqrt{\Delta})$ with $\Delta > 0$ *squarefree*. We write $\mathrm{Gal}(E/\mathbb{Q}) = \{1, \sigma\}$ and note that we fixed an embedding $E \hookrightarrow \mathbb{R}$ so that the set of embeddings of E in $\mathbb{R}$ is identified with $\mathrm{Gal}(E/\mathbb{Q})$. Let D be a quaternion algebra over $\mathbb{Q}$ and $D_E = D \otimes_{\mathbb{Q}} E$. We will consider the following quadratic spaces (V, Q):

($D_\sigma^\pm$) Let $(D_\sigma^\pm) = \{x \in D_E \mid x^\sigma = \pm x^\iota\}$, and $Q^\pm(x) = xx^\sigma = \pm xx^\iota = \pm N(x) \in \mathbb{Q}$. Then,

$$S(x,y) = S^\pm(x,y) = \pm \operatorname{Tr}_{D_E/E}(xy^\iota) = \operatorname{Tr}_{D_E/E}(xy^\sigma) \in \mathbb{Q}$$

We have $m = 4$. Indeed, we have a decomposition over $\mathbb{C}$:

$$D_E \otimes_{\mathbb{Q}} \mathbb{C} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$$

with σ interchanging the components $M_2(\mathbb{C})$, and we have:

$$D_\sigma^\pm \otimes_{\mathbb{Q}} \mathbb{C} = \{(X, \pm X^\iota) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \mid X \in M_2(\mathbb{C})\}$$

which has dimension 4 over $\mathbb{C}$.

($D_0^\pm$) Let $D_0^\pm = \{x \in D_\sigma^\pm \mid \operatorname{Tr}(x) = x + x^\iota = 0\}$ and $Q^\pm(x) = xx^\sigma = \pm N(x)$. Note that $D_0^- = \{x \in D \mid \operatorname{Tr}(x) = 0\}$ and that $D_0^+ = \sqrt{\Delta} D_0^- \subset D_E$ (so that as quadratic spaces D_0^+ is isomorphic to D_0^- with the norm multiplied by $-\Delta$) . Then the lemma 1.11 gives the Clifford algebras and groups associated to these spaces. In particular, the even Clifford group is isomorphic to $D^\times$ and the morphism of algebraic groups $\tau_{D_0} : D^\times \twoheadrightarrow SO_{D_0}$ is given by $a \mapsto (v \mapsto ava^{-1})$.

($Z^\pm$) Let $Z^\pm = \{x \in D_\sigma^\pm \mid x^\iota = x\} = \delta_\pm \mathbb{Q}$ with $\delta_+ = 1$ and $\delta_- = \sqrt{\Delta}$ with $S^\pm(\delta_\pm x, \delta_\pm y) = \operatorname{Tr}(\delta_\pm x(\delta_\pm y)^\sigma) = \pm 2\delta_\pm^2 xy$. So $Q^\pm(\delta_\pm x) = \pm \delta_\pm^2 x^2$, the space (Z^+, Q^+) is positive definite, and (Z^-, Q^-) is negative definite.

Note that if we don't need to refer to the sign of $D_\sigma^\pm$ we just write D_σ instead.

We may let $a \in D_E$ act on $D_\sigma^\pm$ by $v \mapsto a^\iota v a^\sigma$ as:

$$(a^\iota v a^\sigma)^\sigma = a^{\iota\sigma} v^\sigma a = \pm a^{\iota\sigma} v^\iota a = \pm (a^\iota v a^\sigma)^\iota$$

This preserves Q up to a scalar $N(a)N(a)^\sigma \in \mathbb{Q}$, and so we get a morphism of linear algebraic groups:

$$\widetilde{\tau} : D_E^\times \to GO_{D_\sigma} \tag{4.1}$$

Given the inclusion of quadratic spaces $(D_\sigma^\pm, \pm N) \subseteq (D_E, \pm N)$ and following (1.7), we define the $\mathbb{Q}$-linear map:

$$p : D_\sigma^\pm \to R \subset M_2(D_E) = A(D_E)$$
$$x \mapsto \begin{pmatrix} 0 & \pm x \\ x^\iota & 0 \end{pmatrix}$$

where $R = \{\begin{pmatrix} a & b \\ b^\sigma & a^\sigma \end{pmatrix} \mid a, b \in D_E\}$. Since $\dim_{\mathbb{Q}} R = 2^4$, we get that $R = A(D_\sigma)$. We also have that $A^+(D_\sigma) = \{\begin{pmatrix} a & 0 \\ 0 & a^\sigma \end{pmatrix} \mid a \in D_E\} \cong D_E$ where we make the identification by the projection $\begin{pmatrix} a & 0 \\ 0 & a^\sigma \end{pmatrix} \mapsto a^\sigma$. By (1.8) and (1.5), the even Clifford group is equal to:

$$G_{D_\sigma}^+ = \{a \in D_E \mid N(a) \in \mathbb{Q}^\times\}$$

and the morphism to the special orthogonal group is given explicitly by:

$$\tau_{D_\sigma} : G_{D_\sigma}^+ \to SO_{D_\sigma}$$
$$a \mapsto (x \mapsto a^\sigma x a^{-1})$$

By lemma 1.9, this map is surjective with kernel $\mathbb{Q}^\times$.

4.1 Choices of D for a fixed D_E

Pick $\alpha \in D_\sigma^\pm \cap D_E^\times$, and consider $x^{\sigma_\alpha} = \alpha x^\sigma \alpha^{-1}$. Then,

$$(x^{\sigma_\alpha})^{\sigma_\alpha} = \alpha(\alpha^\sigma (x^\sigma)^\sigma \alpha^{\sigma,-1})\alpha^{-1} = \alpha(\pm\alpha^\iota x \pm \alpha^{\iota,-1})\alpha^{-1} = x$$

Thus we get a new action of $\mathrm{Gal}(E/\mathbb{Q})$ on D_E, and the fixed points $D_\alpha = H^0(E/\mathbb{Q}, D_E) = \{x \in D_E \mid x\alpha = \alpha x^\sigma\}$ under this new action is a quaternion algebra over $\mathbb{Q}$ with $D_E = D_\alpha \otimes_\mathbb{Q} E$.

Lemma 4.1. *With the above notation we have:*

(1) If B is a central simple $\mathbb{Q}$-subalgebra of D_E of dimension 4, then there exists $\alpha \in D_\sigma \cap D_E^\times$ such that $B = D_\alpha$.

(2) We have that $\alpha = x\beta x^{\iota\sigma}$ for $\beta \in D_\sigma \cap D^\times E$ and $x \in D_E^\times$ if and only if $D_\alpha \cong D_\beta$ as quaternion algebras over $\mathbb{Q}$, and in this case, we have $D_\alpha = xD_\beta x^{-1}$ inside D_E.

(3) We have that $D_\alpha = D$ if and only if $\alpha \in D_\sigma \cap E^\times$.

Proof. Let D be a quaternion $\mathbb{Q}$-subalgebra $B \subseteq D_E$. Then we have an action of $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ on D_E such that $H^0(E/\mathbb{Q}, D_E) = B$. This is given by the action on the second factor in $D_E = B \otimes_\mathbb{Q} E$. Write this action by α_σ, then $x \mapsto (x^\sigma)^{\sigma_\alpha}$ is an E-linear automorphism of D_E. By the Noether-Skolem theorem, it is an inner automorphism, and so there exists $\alpha \in D_E^\times$ such that:

$$x^\sigma = \alpha x^{\sigma_\alpha}\alpha^{-1} \quad \forall x \in E$$

Since $(x^{\sigma_\alpha})^{\sigma_\alpha} = x$, we see that $\alpha^\sigma\alpha \in Z(D_E) = E$. In particular $(\alpha^\sigma\alpha)\alpha = \alpha(\alpha^\sigma\alpha)$, and so dividing on both sides by α, we get that α and α^σ commute. Then $(\alpha\alpha^\sigma)^\sigma = \alpha^\sigma\alpha = \alpha\alpha^\sigma$ which shows that $\alpha\alpha^\sigma \in \mathbb{Q}$. Thus $\alpha^\sigma = z\alpha^\iota$ for some $z \in \mathbb{Q}^\times$, and $\alpha^{\sigma\iota} = z\alpha$. Therefore α is an eigenvalue of $\sigma\iota$ which is of order 2 (σ and ι commute), and so $z = \pm 1$ which gives that $\alpha^\sigma = \pm\alpha^\iota$. If z does not match with the sign of $D_\sigma^\pm$, we replace α with $\sqrt{\Delta}\alpha$. We have $B = D_\alpha$ which shows (1). □

5 Rankin convolution

5.1 Adelic fourier expansion of cuspforms of integral weight

Let $F \in S_\kappa(\Gamma_0(C), \varphi)$ be a cusp form of weight $\kappa \in \mathbb{N}$, $\varphi : (\mathbb{Z}/C\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character where we let $\varphi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \varphi(d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(C)$. Then we have:

$$F(\gamma(\tau)) = \varphi(\gamma)F(\tau)j(\gamma,\tau)^\kappa \quad \text{for all} \gamma \in \Gamma_0(C)$$

Since $\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_+^\times \cong \widehat{\mathbb{Z}}^\times$, by composing with the projection map $\widehat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/C\mathbb{Z})^\times$ we extend φ to a character $\varphi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$. Using strong approximation, $\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\widehat{\Gamma}_0(C)\,\mathrm{SL}_2(\mathbb{R})$ and we lift F to $\mathbf{F} : \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C}$ by putting:

$$\mathbf{F}(\alpha u) = \varphi^*(u)F(u_\infty \cdot i)j(u_\infty, i)^{-\kappa}$$

for $\alpha \in \mathrm{SL}_2(\mathbb{Q})$, $u \in \widehat{\Gamma}_0(C)\,\mathrm{SL}_2(\mathbb{R})$, and $\varphi^* = \varphi^{-1}$.
Define an idele character $\boldsymbol{\varphi} : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ by $\boldsymbol{\varphi}(x) = \varphi^*(x)|x|_\mathbb{A}^{-\kappa}$. Write the Fourier expansion of F as $F(\tau) = \sum_{n=1}^\infty a_n(F)\mathbf{e}_\infty(n\tau)$. For $g \in B(\widehat{\mathbb{Z}})B(\mathbb{R})$ with $g = \begin{pmatrix} x & yx^{-1} \\ 0 & x^{-1} \end{pmatrix}$ with $x \in \widehat{\mathbb{Z}}^\times\mathbb{R}_+^\times$, we find for $\tau = g_\infty \cdot i = x_\infty^2 i + y_\infty$,

$$\mathbf{F}(g) = \varphi^*(x^\infty)^{-1}x_\infty^\kappa \sum_{n=1}^\infty a_n(F)\mathbf{e}_\infty(n\tau) = \boldsymbol{\varphi}(x)^{-1}\sum_{n=1}^\infty a_n(F)\exp(-2\pi n x_\infty^2)\mathbf{e}_\infty(ny_\infty)$$

Let $v(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in N(\mathbb{A})$, then for $g = v(y)\mathrm{diag}[x, x^{-1}]$, write $\mathbf{F}(x, y) := \mathbf{F}(g)$. Since $\mathbf{F}(x, y + u) = \mathbf{F}(v(u)g) = \mathbf{F}(g) = \mathbf{F}(x, y)$ for $u \in \mathbb{Q}$, we have the adelic Fourier expansion of $\mathbf{F}(x, y)$ with respect to $y \in \mathbb{A}$:

$$\mathbf{F}(x, y) = \sum_{u \in \mathbb{Q}} a_{\mathbf{F}}(u; x)\mathbf{e}_{\mathbb{A}}(uy)$$

For $t \in \mathbb{Q}^\times$, we have $\mathrm{diag}[t, t^{-1}]v(y)\mathrm{diag}[x, x^{-1}] = v(t^2x)\mathrm{diag}[tx, (tx)^{-1}]$ we have:

$$\sum_{u \in \mathbb{Q}} a_{\mathbf{F}}(u, x)\mathbf{e}_{\mathbb{A}}(uy) = \mathbf{F}(x, y) = \mathbf{F}(tx, t^2y) = \sum_{u \in \mathbb{Q}} a_{\mathbf{F}}(u, tx)\mathbf{e}_{\mathbb{A}}(ut^2y)$$

By the uniqueness of the Fourier expansion, we get:

$$\begin{aligned} &a_{\mathbf{F}}(u, x) = a_{\mathbf{F}}(t^{-2}u, tx) \quad \text{for } t \in \mathbb{Q}^\times, u \in \mathbb{Q} \\ &a_{\mathbf{F}}(u, x) = \begin{cases} \boldsymbol{\varphi}(x)^{-1}a_u(F)\exp(-2\pi u x_\infty^2) & \text{if } u \in \mathbb{N}^\times \\ 0 & \text{if } u \notin \mathbb{N}^\times \end{cases} \quad \text{for } x \in \widehat{\mathbb{Z}}^\times\mathbb{R}_+^\times \end{aligned} \tag{5.1}$$

Suppose that $t \in \widehat{\mathbb{Z}}^\times\mathbb{R}_+^\times$, then given that $\mathrm{diag}[t, t^{-1}] \in \widehat{\Gamma}_0(C)\,\mathrm{SL}_2(\mathbb{R})$, we get by definition of $\mathbf{F}$:

$$\mathbf{F}(xt, y) = \mathbf{F}(v(y)\begin{pmatrix} xt & 0 \\ 0 & (xt)^{-1} \end{pmatrix}) = \mathbf{F}(g\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})) = \varphi^*(t)^{-1}\mathbf{F}(g\begin{pmatrix} t_\infty & 0 \\ 0 & t_\infty^{-1} \end{pmatrix}) = \varphi^*(t)^{-1}\mathbf{F}(xt_\infty, y)$$

so that

$$\sum_{u \in \mathbb{Q}} a_{\mathbf{F}}(u, xt)\mathbf{e}(uy) = \varphi^*(t)^{-1}\sum_{u \in \mathbb{Q}} a_{\mathbf{F}}(u, xt_\infty)\mathbf{e}(uy)$$

By unicity of the Fourier expansion, we get that

$$a_{\mathbf{F}}(u, xt) = \varphi^*(t)^{-1}a_{\mathbf{F}}(u, xt_\infty) \tag{5.2}$$

For $x \in \mathbb{Q}^\times(\mathbb{A}^\times)^2 = \mathbb{Q}^\times(\widehat{\mathbb{Z}}^\times)^2\mathbb{R}_+^\times$, write $x = ua^2$ for $u \in \mathbb{Q}^\times$ with $a \in \mathbb{A}^\times$, and define:

$$a_{\mathbf{F}}(ua^2) := \boldsymbol{\varphi}(a)a_{\mathbf{F}}(u, a)\exp(2\pi a_\infty^2 u_\infty)$$

Note that by comparing the two Fourier expansions, we have that if $a \in \widehat{\mathbb{Z}}^\times\mathbb{R}_+^\times$,

$$a_{\mathbf{F}}(ua^2) = \begin{cases} a_u(F) & \text{if } u \in \mathbb{N}^\times \\ 0 & \text{if } u \notin \mathbb{N}^\times \end{cases}$$

If $ua^2 = tb^2$ for some $t \in \mathbb{Q}^\times$ and $b \in \mathbb{A}^\times$, then there exists $q \in \mathbb{Q}^\times$ and $s \in \mathbb{A}^\times$ with $s^2 = 1$, $s_\infty = 1$, $q^2 = u/t$, and $b = qsa$. Then we get that:

$$\begin{aligned} a_{\mathbf{F}}(ua^2) &= \boldsymbol{\varphi}(a)a_{\mathbf{F}}(u, a)\exp(2\pi a_\infty^2 u_\infty) \\ &= \boldsymbol{\varphi}(aqs)\varphi^*(s)^{-1}a_{\mathbf{F}}(q^{-2}u, qs_\infty a)\exp(2\pi(a_\infty s_\infty q_\infty)^2 q_\infty^{-2}u_\infty) \\ &= \boldsymbol{\varphi}(b)a_{\mathbf{F}}(t, b)\exp(2\pi b_\infty^2 t_\infty) = a_{\mathbf{F}}(tb^2) \end{aligned}$$

by (5.4) and (5.2). This shows that $a_{\mathbf{F}}(x)$ is well defined, and we get that for $x \in \widehat{\mathbb{Z}}^\times\mathbb{R}_+^\times$ and $y \in \mathbb{Q}^\times$,

$$\mathbf{F}(x, y) = \mathbf{F}(v(y)\mathrm{diag}(x, x^{-1})) = \boldsymbol{\varphi}^{-1}(x)\sum_{u \in \mathbb{Q}} a_{\mathbf{F}}(ux^2)\exp(-2\pi n x_\infty^2)\mathbf{e}_{\mathbb{A}}(uy)$$

5.2 Adelic Fourier expansion of cuspforms of half integral weight

Let $f \in S_{k/2}(\Gamma_0(M), \psi)$ for k odd and ψ an even Dirichet character modulo M. Then $f(\gamma \cdot \tau) = \psi(\gamma) f(\tau) h(\gamma, \tau)^k$ for $\gamma \in \Gamma_0(M)$, $h(\gamma, \tau) = \frac{\theta(\gamma \cdot \tau)}{\theta(\tau)}$ and $\psi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \psi(d)$. We extend ψ to a character $\psi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$, and we lift f to $\mathbf{f} : \mathrm{SL}_2(\mathbb{Q}) \backslash Mp(\mathbb{A}) \to \mathbb{C}$ by putting:

$$\mathbf{f}(\alpha(u, \zeta J(u_\infty, \tau))) = \psi^*(u) f(u_\infty \cdot i) \zeta^k J(u_\infty, i)^{-k} \tag{5.3}$$

for $\alpha \in \mathrm{SL}_2(\mathbb{Q}) \subset Mp(\mathbb{A})$, $(u, J(u_\infty, \tau)) \in \widehat{\Gamma}_0(M) Mp(\mathbb{R})$, and $\zeta \in \mathbb{T}$; regarding $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A}) \subset Mp(\mathbb{A})$.

Note that $B(\mathbb{A})$ is canonically lifted to $Mp(\mathbb{A})$ by the Weil representation, and this lifting coincides with the splitting $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow Mp(\mathbb{A})$. Define the idele character $\boldsymbol{\psi} : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ by $\boldsymbol{\psi}(a) = \psi^*(a)|a|_{\mathbb{A}}^{-k/2}$. Then letting $f(\tau) = \sum_{n=1}^{\infty} a_n(f) \mathbf{e}_\infty(n\tau)$, we put for $v(y)\mathrm{diag}[x, x^{-1}] = \begin{pmatrix} x & yx^{-1} \\ 0 & x^{-1} \end{pmatrix} \in B(\widehat{\mathbb{Z}})B(\mathbb{R}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A})$:

$$\mathbf{f}(x, y) := \mathbf{f}(v(y)\mathrm{diag}[x, x^{-1}]) = \psi^*(x^{-1}) f(x_\infty^2 i + y_\infty) x_\infty^{k/2} = \boldsymbol{\psi}^{-1}(x) \sum_{n=1}^{\infty} a_n(f) \exp(-2\pi n x_\infty^2) \mathbf{e}_\infty(n y_\infty)$$

Noting that $\mathbf{f}(x, y+u) = \mathbf{f}(x, y)$ for $u \in \mathbb{Q}$, $\mathbf{f}(a, u)$ has a Fourier expansion over $y \in \mathbb{A}$ of the form:

$$\mathbf{f}(x, y) = \sum_{u \in \mathbb{Q}} a_{\mathbf{f}}(u; x) \mathbf{e}_{\mathbb{A}}(uy)$$

As before, we get by uniqueness of the Fourier expansion that:

$$\begin{aligned} a_{\mathbf{f}}(u, x) &= a_{\mathbf{f}}(t^{-2}u, tx) \quad \text{for } t \in \mathbb{Q}^\times, u \in \mathbb{Q} \\ a_{\mathbf{f}}(u, x) &= \begin{cases} \boldsymbol{\psi}(x)^{-1} a_u(F) \exp(-2\pi u x_\infty^2) & \text{if } u \in \mathbb{N}^\times \\ 0 & \text{if } u \notin \mathbb{N}^\times \end{cases} \quad \text{for } x \in \widehat{\mathbb{Z}}^\times \mathbb{R}_+^\times \end{aligned} \tag{5.4}$$

Define

5.3 Adelic Rankin product

Lemma 5.1. *The natural map $\pi : B(\mathbb{Q}) \backslash B(\mathbb{A}) C_\infty \twoheadrightarrow \mathrm{SL}_2(\mathbb{Q}) \backslash Mp(\mathbb{A})$ is an isomorphism.*

Proof. By strong approximation, $\mathrm{SL}_2(\mathbb{A}^{(\infty)}) = \mathrm{SL}_2(\mathbb{Q})K$ for an open subgroup K of $\widehat{\Gamma}_0(4)$. By Iwasawa decomposition, we have that $B(\mathbb{R})C_\infty = Mp(\mathbb{R})$ so that $Mp(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})KC_\infty$. Thus we have a natural continuous surjection:

$$\pi_K : B_K := B(\mathbb{Q}) \backslash B(\mathbb{A}) K C_\infty \twoheadrightarrow \mathrm{SL}_2(\mathbb{Q}) \backslash Mp(\mathbb{A})$$

For $x \in Mp(\mathbb{A})$ and an open neighborhood U of x, there exists a compact open $K \subset \widehat{\Gamma}_0(4)$ such that $xK \subseteq U$. But knowing that π_K is surjective, we have $xK \cap \pi_1(B(\mathbb{Q}) \backslash B(\mathbb{A}) C_\infty) \neq \emptyset$, which shows that π_1 has dense image. Since $\mathrm{SL}_2(\mathbb{Q}) \backslash Mp(\mathbb{A})$ is locally compact, we can consider a system of open neighborhoods $\{X_n\}_{n \geq 0}$ of a point $x \in \mathrm{SL}_2(\mathbb{Q}) \backslash Mp(\mathbb{A})$. Let $\{Y_n\}_{n \geq 0}$ be a system of open compact subsets of $B(\mathbb{Q}) \backslash B(\mathbb{A}) C_\infty$ such that $Y_{n+1} \subset Y_n$ and $Y_n \subset \pi^{-1}(X_n)$. Then given that $\mathrm{SL}_2(\mathbb{Q}) \backslash Mp(\mathbb{A})$ is Hausdorff, we get that $\cap_{n \geq 0} \pi_1(Y_n) = \{x\}$ and so π_1 is surjective.

Now if $\pi_1(bu) = \pi_1(b'u')$ for $b, b' \in B(\mathbb{A})$ and $u, u' \in C_\infty$, then there is a $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ such that $\gamma bu = b'u'$. By projecting down to $\mathrm{SL}_2(\mathbb{A})$ and comparing the finite part, we find that $\gamma \in B(\mathbb{Q})$ which shows the injectivity of π_1. $\square$

6 Computing the Period

6.1 Symmetric domain for $O(n,2)$

Suppose that $m = \dim V = n+2$ and that $V_{\mathbb{R}}$ has signature $(n,2)$. We would like to make explicit the symmetric domain $GO_V^+(\mathbb{R})/\mathbb{R}^\times C$ for a maximal compact subgroup $C \subset GO_V^+(\mathbb{R})$.
We start with the following complex submanifold of $V_{\mathbb{C}}$:

$$\mathcal{Y}(Q) = \{v \in V_{\mathbb{C}} \mid Q(v) = 0 \text{ and } S(v,\overline{v}) < 0\}$$

Since S is indefinite over $\mathbb{C}$, the space $\mathcal{Y}(Q)$ is always non-empty, and $g \in GO_V^+(\mathbb{R})$ acts on $\mathcal{Y}(Q)$ by $v \mapsto gv$.

Take $v \in \mathcal{Y}(Q)$ and write W for the subspace of $V_{\mathbb{R}}$ spanned over $\mathbb{R}$ by $2\mathrm{Re}(v) = v + \overline{v}$ and $2\mathrm{Im}(v) = iv - i\overline{v}$. Then we have:

$$\begin{aligned} Q(v+\overline{v}) &= 2S(v,\overline{v}) < 0 \\ Q(iv - i\overline{v}) &= 2S(v,\overline{v}) < 0 \\ S(v+\overline{v}, iv - i\overline{v}) &= -iS(v,\overline{v}) + iS(\overline{v},v) = 0 \end{aligned}$$

This shows that $S_{|W}$ is negative definite, and so $S_{|W^\perp}$ is positive definite. Now define the positive linear bilinear form:

$$P_v(x,y) = S(x_{W^\perp}, y_{W^\perp}) - S(x_W, y_W)$$

for the orthogonal projections x_W to W and $x_{W^\perp}$ to $W^\perp$ of x. If $g \in GO_V^+(\mathbb{R})$ fixes $v \in \mathcal{Y}(Q)$, then g fixes by definition the positive definite form P_v. Thus g has to be in the compact subgroup O_{P_v} made up of orthogonal matrices preserving P_v. On the other hand, if we have to $v, w \in \mathcal{Y}(Q)$, then by Sylvester's theorem, we can find $g \in GO_V^+(\mathbb{R})$ such that $gv = w$ and hence $GO_V^+(\mathbb{R})/O_{P_v} \cong \mathcal{Y}(Q)$.

Now we make explicit the domain $\mathcal{Y}(Q)$ as a hermitian bounded matrix domain:

Proposition 6.1. *[Hid06, Prop. 2.1] There is a $\mathbb{C}$-linear isomorphism $A : V_{\mathbb{C}} \xrightarrow{\sim} \mathbb{C}^{n+2}$ such that:*

$$S(x,y) = {}^t(Ax)R(Ay) \quad \text{and} \quad S(\overline{x},y) = {}^t(\overline{Ax})T(Ay)$$

where R and T are real symmetric matrices given by:

$$R = \begin{pmatrix} \mathrm{id}_n & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & -\mathrm{id}_2 \end{pmatrix}$$

With A as in the proposition, the map $g \mapsto AgA^{-1}$ gives an isomorphism of Lie groups:

$$\iota : GO_V^+(\mathbb{R}) \xrightarrow{\sim} G(Q,T) = \{g \in \mathrm{GL}_{n+2}(\mathbb{C}) \mid {}^t gRg = \nu(g)R,\ {}^t\overline{g}Tg = \nu(g)Q \text{ for some } \nu(g) \in \mathbb{R}^\times\}$$

and the map $v \mapsto Av$ gives an isomorphism of complex manifolds:

$$j : \mathcal{Y}^+(Q) \xrightarrow{\sim} \mathcal{Y}(R,T) = \{u \in \mathbb{C}^{n+2} \mid {}^t uRu = 0 \text{ and } {}^t uTu < 0\}$$

These two maps are equivariant, i.e, $\iota(g)j(v) = j(gv)$.
Let us show that $\mathcal{Y}(Q,T)$ has two connected components. So writing $u = {}^t(u_1, \cdots, u_{n+2}) \in \mathcal{Y}(R,T)$, we get:

$$(\sum_{i=1}^n u_i^2) - 2u_{n+1}u_{n+2} = {}^t uRu = 0$$

$$\sum_{i=1}^n |u_i|^2 < |u_{n+1}|^2 + |u_{n+2}|^2 = {}^t\overline{u}Qu < 0$$

If we suppose that $|u_{n+1}| = |u_{n+2}|$, then

$$\sum_{i=1}^{n} |u_i|^2 > |\sum_{i=1}^{n} u_i^2| = 2|u_{n+1}u_{n+2}| = |u_{n+1}|^2 + |u_{n+2}|^2$$

a contradiction. Thus we either have $|u_{n+1}| > |u_{n+2}|$ or $|u_{n+2}| > |u_{n+1}|$. These two cases split the domain $\mathcal{Y}(Q,T)$ into two pieces of connected components.
To see that each component is connected, we may assume that $|u_{n+2}| > |u_{n+1}|$ by interchanging the coordinates if necessary, and so $u_{n+2} \neq 0$. Put $z_i = \frac{u_i}{u_{n+2}}$ for $i \leq n$ and define $z = {}^t(z_1, \dots, z_n)$. Then $\frac{u_{n+1}}{u_{n+2}} = \frac{{}^t zz}{2}$, and defining:

$$\mathfrak{Z} = \mathfrak{Z}_n = \{z \in \mathbb{C}^n \mid {}^t z\overline{z} < 1 + \frac{1}{4}|{}^t zz|^2 < 2\}$$

we see that $\mathbb{C}^\times \times \mathfrak{Z}$ is isomorphic to the component of $\mathcal{Y}(R,T)$ given by $|u_{n+2}| > |u_{n+1}|$ via

$$(\lambda, z) \mapsto \lambda \mathcal{P}(z)$$

where $\mathcal{P}(z) = {}^t(z, \frac{{}^t zz}{2}, 1)$. We define an action of $g \in GO_V^+(\mathbb{R})$ on $\mathfrak{Z}$ and a factor of automorphic $\mu(g,z)$ for $z \in \mathfrak{Z}$ by:

$$\iota(g)\mathcal{P}(z) = \mathcal{P}(g(z))\mu(g,z)$$

We now look into spherical functions on $V_{\mathbb{C}}$. Choose a basis $v_1, \dots, v_m$ of V so that we have an identification of $V_{\mathbb{R}}$ with $\mathbb{R}^m$ by $v \mapsto (x_1, \dots, x_m)$ for $v = \sum_i x_i v_i$. We take the dual basis v_j^* so that $S(v_j^*, v_i) = \delta_{ij}$, and define a second degree homogenious differential operator Δ by:

$$\Delta = \sum_{ij} S(v_i^*, v_j^*) \frac{\partial^2}{\partial x_i \partial x_j}$$

A polynomial function $\eta : V_{\mathbb{R}} \to \mathbb{C}$ is called a spherical function if $\Delta\eta = 0$. Writing $S = (S(v_i, v_j))$, we have that this definition does not depend on the choice of the basis v_i because $\Delta = {}^t\partial S^{-1}\partial$ for $\partial = {}^t(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m})$. And since $\partial({}^t wSx) = {}^t Sw = Sw$ for a constant vector w, we find that for $k \geq 2$,

$$\begin{aligned}
\Delta({}^t wSx)^k &= {}^t\partial S^{-1}\partial({}^t wSx)^k \\
&= k^t\partial(S^{-1}Sw)({}^t wSx)^{k-1} \\
&= k^t({}^t w\partial({}^t wSx)^{k-1}) \\
&= k(k-1)({}^t wSw)({}^t wSx)^{k-2}
\end{aligned}$$

Thus the polynomial function $x \mapsto S(w,x)^k$ for $k \geq 2$ is spherical if and only if $Q(w) = 0$. In fact, all homogenious spherical polynomials of degree $k \geq 2$ are a linear combination of $S(w,x)^k$ for a finite set of spherical vectors w with $Q(w) = 0$. In particular, for $v \in \mathcal{Y}^+(Q)$, then function $x \mapsto S(v,x)^k$ is a spherical function. We define a Schwartz function Ψ on $V_{\mathbb{R}}$ for each $\tau = x + iy \in \mathfrak{H}$ and $v \in \mathcal{Y}(Q)$ by:

$$\Psi(\tau; v)(w) = \mathbf{e}(\frac{1}{2}(S[w]x + iP_v[w]y)) = \exp(i\pi(S[w]x + iP_v[w]y)) \tag{6.1}$$

where $S[w] = S(w,w)$.

Now we go back to our case and suppose that D is indefinite, so that we can fix an isomorphism $D_E \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$ and,

$$V_{\mathbb{R}} = D_\sigma^{\pm} \otimes_{\mathbb{Q}} \mathbb{R} \cong \{(X, \pm X^\iota) \in M_2(\mathbb{R}) \oplus M_2(\mathbb{R}) \mid X \in M_2(\mathbb{R})\} \cong M_2(\mathbb{R})$$

which has signature $(2,2)$. With these identifications, the morphism (4.1) becomes:

$$\begin{aligned}(\mathrm{GL}_2(\mathbb{R})\times\mathrm{GL}_2(\mathbb{R}))/\{\pm(1,1)\} &\hookrightarrow GO_V(\mathbb{R})\\ (X_1,X_2) &\mapsto (M\mapsto X_2MX_1^{\iota})\end{aligned}$$

with $\nu((X_1,X_2))=\det(X_1X_2)$. Since the symmetric space of $GO_V(\mathbb{R})$ has dimension 2 over $\mathbb{C}$, the above morphism has to be onto on the identity connected component. Also the symmetric space of $GO_V(\mathbb{R})$ has four connected components ($\mathcal{Y}(Q)$ has two), the above morphism has to be surjective and so it is an isomorphism. Given that the symmetric space of $\mathrm{GL}_2^+(\mathbb{R})\times\mathrm{GL}_2^+(\mathbb{R})$ is $\mathfrak{H}\times\mathfrak{H}$, we find that $\mathfrak{Z}=\mathfrak{H}\times\mathfrak{H}$. But let us make this more explicit.

Since $V_{\mathbb{C}}\cong M_2(\mathbb{C})$ with $S^{\pm}=\pm\,\mathrm{Tr}$, we have from the definition that:

$$\begin{aligned}\mathcal{Y}^{+}&=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb{C})\mid ad=bc,\ a\overline{d}-b\overline{c}+d\overline{a}-c\overline{b}<0\}\\ \mathcal{Y}^{-}&=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb{C})\mid ad=bc,\ a\overline{d}-b\overline{c}+d\overline{a}-c\overline{b}>0\}\end{aligned}$$

Pick $v=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathcal{Y}^-\sqcup\mathcal{Y}^+$ and suppose that $c=0$. Then by the defining equation of $\mathcal{Y}^{\pm}$, $ad=0$ and so $0=a\overline{d}+d\overline{a}>0$ (or <0) which is a contradiction. Thus $c\neq0$, and define $z=\frac{a}{c}$ and $w=\frac{-d}{c}$. Then $-zw=\frac{b}{c}$ and,

$$v=cp(z,w)\quad\text{ with } p(z,w)=\begin{pmatrix}z&-wz\\1&-w\end{pmatrix}=-{}^{t}(z,1)(w,1)J$$

where $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. By the equation defining $\mathcal{Y}^{\pm}$, we have:

$$S^{\pm}(p(z,w),\overline{p(z,w)})=\pm(z-\overline{z})(w-\overline{w})=\pm4\mathrm{Im}(z)\mathrm{Im}(w)\neq0 \tag{6.2}$$

Then we clearly have that $\mathcal{Y}^-\sqcup\mathcal{Y}^+\cong\mathbb{C}^{\times}\times(\mathbb{C}-\mathbb{R})^2$ via $cp(z,w)\leftarrow\!\shortmid (c,(z,w))$. The action of $(X_1,X_2)\in GO_{D_\sigma}(\mathbb{R})$ is given explicitly by:

$$X_2p(z,w)X_1^{\iota}=p(X_2\cdot z,X_1\cdot w)j(X_2,z)j(X_1,w) \tag{6.3}$$

Thus $(X_1,X_2)\cdot(z,w)=(X_2\cdot z,X_1\cdot w)$ and the factor of automorphy $\mu((X_1,X_2),(z,w))=j(X_2,z)j(X_1,w)$. Let us also define a spherical function:

$$v\mapsto[v;z,w]^k=S^{+}[v,p(z,w)]^k$$

for a positive integer $k>0$.

As before, let W be the subspace of $V_{\mathbb{R}}$ generated by $\mathrm{Re}(p(z,w))$ and $\mathrm{Im}(p(z,w))$ and decompose $V_{\mathbb{R}}=W\oplus W^{\perp}$. A direct calculation shows that $W^{\perp}$ is generated by the real and imaginary part of $p(z,\overline{w})$. If $\mathrm{Im}(z)\mathrm{Im}(w)>0$, then by (6.2), we have that S^+ is >0 on W and S^+ is <0 on $W^{\perp}$ (the opposite for S^-). Let $P^{\pm}$ be the positive majorant of $S^{\pm}$ given by the above decomposition (cf [Hid20]), then we have that:

$$P^{\pm}(x,y)=\pm S^{\pm}(x_W,y_W)\mp S^{\pm}(x_{W^{\perp}},y_{W^{\perp}})$$

To compute $P^{\pm}[v]$, note that $P^{\pm}[v]\pm S^{\pm}[v]=\pm2S^{\pm}(v_W,v_W)$. So writing $v=cp(z,w)+\overline{c}p(\overline{z},\overline{w})+x$ with $x\in W^{\perp}$ and $c\in\mathbb{C}$, we have:

$$\begin{aligned}P^{\pm}[v]\pm S^{\pm}[v]&=\pm2S^{\pm}(cp(z+w)+\overline{c}p(\overline{z},\overline{w}),cp(z+w)+\overline{c}p(\overline{z},\overline{w}))\\ &=\pm4|c|^2S^{\pm}(p(z,w),p(\overline{z},\overline{w}))\\ &=4|c|^2(w-\overline{w})(z-\overline{z})\geq0\end{aligned}$$

Now if $\mathrm{Im}(z)\mathrm{Im}(w) < 0$, then replacing W by $W^{\perp}$, w by $\overline{w}$, and repeating the calculations, we get that $P^{\pm}[v] \pm S^{\pm}[v] = 4|c|^2(\overline{w} - w)(z - \overline{z}) \geq 0$.

Since $S^{\pm}(v, p(z,w)) = \overline{c}S^{\pm}(p(z,w), \overline{p(z,w)}) = \pm\overline{c}(w - \overline{w})(z - \overline{z})$, we get that:

$$P^{\pm}[v] = \mp S^{\pm}[v] + \frac{|[v,z,w]|^2}{|\mathrm{Im}(z)\mathrm{Im}(w)|}$$

Write $\tau^{\pm} = \begin{cases} \overline{\tau} & \text{in case } + \\ -\tau & \text{in case } - \end{cases}$ and define a Schwartz function Ψ_k on $V_{\mathbb{R}}$ for $(\tau, z, w) \in \mathfrak{H} \times (\mathbb{C} - \mathbb{R})^2$ and $0 \leq k \in \mathbb{Z}$:

$$\Psi_k(\tau; z, w)(v) = \mathrm{Im}(\tau)\frac{[v, \overline{z}, w]^k}{(z - \overline{z})^k(w - \overline{w})^k}\mathbf{e}_{\infty}\big(N(v)\tau^{\pm} + i\frac{\mathrm{Im}(\tau)}{2|\mathrm{Im}(z)\mathrm{Im}(w)|}|[v, z, \overline{w}]|^2\big)$$

We choose a Bruhat function $\phi^{(\infty)} : D_{\sigma, \mathbb{A}^{(\infty)}} \to \mathbb{C}$, and put:

$$\phi = \phi_k = \phi^{(\infty)} \otimes \Psi_k$$

and consider Siegel's theta series $\theta(\phi_k) = \theta(\phi_k)(\tau, z, w) = \sum_{v \in D_{\sigma}} \phi(v)$.
From (6.3), we have by direct computation that for $g \in D_E^{\times}$:

$$[g^{\sigma,-1} v g^{\iota,-1}; z, w] = N(gg^{\sigma})^{-1} j(g^{\sigma} z) j(g, w)[v; g^{\sigma} z, gw] \tag{6.4}$$

and since $\mathrm{Im}(gz) = N(g)\frac{\mathrm{Im}(z)}{|j(g,z)|^2}$, we get that:

$$\frac{[v; g^{\sigma}\overline{z}, g\overline{w}]}{\mathrm{Im}(g^{\sigma}z)\mathrm{Im}(gw)} = j(g^{\sigma}, z) j(g, w)\frac{[g^{\sigma,-1} v g^{\iota,-1}; \overline{z}, \overline{w}]}{\mathrm{Im}(z)\mathrm{Im}(w)}$$

multiplying on both sides with $[v; g^{\sigma} z, gw]$, we get:

$$\frac{|[v; g^{\sigma} z, gw]|^2}{\mathrm{Im}(g^{\sigma}z)\mathrm{Im}(gw)} = N(gg^{\sigma})\frac{|[g^{\sigma,-1} v g^{\iota,-1}; z, w]|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}$$

Thus for $\gamma \in D_E^{\times}$ with $N(\gamma) = 1$ and $\phi^{(\infty)}(\gamma^{\sigma,-1} v \gamma^{\iota,-1}) = \phi^{(\infty)}(v)$, we have:

$$\theta(\phi_k)(\gamma^{\sigma} z, \gamma w) = \theta(\phi_k)(z, w) j(\gamma^{\sigma}, z)^k j(\gamma, \overline{w})^k \tag{6.5}$$

6.2 Differential form coming from theta series

Let $L_E(n; A)$ be the space of homogenious polynomials for each pari (X, Y) and (X', Y') of variables of degree n with coefficients in A for an E-algebra A. Suppose that $D_E \otimes_{\mathbb{Q}} A \cong M_2(A) \times M_2(A)$ for two projections inducing the identity and σ. We let $\gamma \in D_E$ act on $P(X, Y; X', Y') \in L_E(n; A)$ via $(\gamma \cdot P)(X, Y; X', Y') = P((X, Y)^t \gamma^{\iota}; (X', Y')^t \gamma^{\sigma\iota})$. Then,

$$\Theta(z, w) = \Theta(\tau; z, w) := \theta(\phi_k)(\tau; z, w)(X - \overline{w}Y)^{k-2}(X' - zY')^{k-2}\, \mathrm{d}z \wedge \mathrm{d}\overline{w}$$

is a $\mathcal{C}^{\infty}$ differential form with values in $L_E(k-2, \mathbb{C})$. Since $\gamma \cdot (X - zY)^{k-2} = j(\gamma, z)^{k-2}(X - \gamma(z)Y)^{k-2}$ and $\mathrm{d}\gamma(z) = \det(\gamma) j(\gamma, z)^{-2}\mathrm{d}z$, we have that:

$$\begin{aligned} \gamma^*\Theta(z, w) &= \Theta(\gamma^{\sigma}(z), \gamma(w)) \\ &= \theta(\tau; \gamma^{\sigma}(z), \gamma(w))(X - \gamma(w)Y)^{k-2}(X' - \gamma^{\sigma}(z)Y')^{k-2}\, \mathrm{d}\gamma^{\sigma}(z)\mathrm{d}\gamma(\overline{w}) \\ &= \gamma \cdot \Theta(z, w) \end{aligned}$$

where we write $\gamma \cdot \Theta$ for the action of γ on the value in $L_E(k-2, \mathbb{C})$. We write $\Theta(\tau; z); = \Theta(\tau; z, z)$.
We let $L(n; A) = L_{\mathbb{Q}}(n; A)$ be the space of homogenious polynomials of degree n in the variables

(X,Y) with coefficients in A. If $D\otimes_{\mathbb{Q}} A \cong M_2(A)$, we let $\gamma \in D$ act on $P(X,Y) \in L(n,A)$ via $(\gamma\cdot P)(X,Y) = P((X,Y)^t\gamma^\iota)$. Then by the Clebsch-Gordan decomposition, we have:

$$L_E(n,A)_{|D^\times} \cong L_{\mathbb{Q}}(n;A)\otimes L_{\mathbb{Q}}(n;A) \cong \bigoplus_{j=0}^{n} L_{\mathbb{Q}}(2n-2j;A)$$

We write $\pi : L_E(n,A) \to L_{\mathbb{Q}}(0,A) = A$ for the $\mathrm{SL}_2(\mathbb{R})$-equivariant projection given by:

$$\pi(P) = \frac{1}{n!^2}\nabla^n P \quad \text{where } \nabla = \frac{\partial^2}{\partial X\partial Y'} - \frac{\partial^2}{\partial Y\partial X'}$$

Then we have that:

$$\begin{aligned}\frac{1}{n!^2}\nabla^n(X^{n-i}Y^iX'^{n-j}Y'^j) &= \frac{1}{n!^2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\partial^{2k}}{(\partial X)^k(\partial Y')^k}\frac{\partial^{2(n-k)}}{(\partial Y)^{n-k}(\partial X')^{n-k}}(X^{n-i}Y^iX'^{n-j}Y'^j)\\ &= \begin{cases} (-1)^j\binom{n}{i}^{-1} & \text{if } n=i+j\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus we get from $(X-zY)^n(X'-\overline{z}Y')^n = \sum_{i,j=0}^{n}(-1)^{i+j}\binom{n}{i}\binom{n}{j}z^i\overline{z}^jX^{n-i}Y^iX'^{n-j}Y'^j$ that:

$$\pi((X-zY)^n(X'-\overline{z}Y')^n) = \sum_{i=0}^{n}\binom{n}{i}(-1)^iz^i\overline{z}^{n-i} = (\overline{z}-z)^n$$

6.3 Factoring the Theta series

We split the quadratic space as:

$$(D^{\pm}_{\sigma}, \pm N) = (Z^{\pm}, \pm N_{|Z^{\pm}}) \oplus (D^{\pm}_0, \pm N_{|D^{\pm}_0})$$

Then D^+_0 (resp. D^-_0) is 3 dimensional of signature $(1,2)$ (resp. $(2,1)$) and Z^+ (resp. Z^-) has signature $(1,0)$ (resp. $(0,1)$). We assume that there are Schwartz-Bruhat functions $\phi_Z \in \mathcal{S}(Z^{(\infty)}_{\mathbb{A}})$ and $\phi_0 \in \mathcal{S}(D^{(\infty)}_{0,\mathbb{A}})$ such that for $\mathfrak{z} \in Z^{(\infty)}_{\mathbb{A}}$ and $\mathfrak{n} \in D^{(\infty)}_{0,\mathbb{A}}$, we have a tensor product decomposition:

$$\phi^{(\infty)}(\mathfrak{z}+\mathfrak{n}) = (\phi_Z\otimes\phi_0)(\mathfrak{z}+\mathfrak{n}) := \phi_Z(\mathfrak{z})\phi_0(\mathfrak{n})$$

of the Schwartz-Bruhat function in order to factor the theta series.

Next we study the decomposition of the infinite part. First decompose the spherical polynomial $[v;z,\overline{z}]$. For $\mathfrak{z} \in Z^{\pm} = \mathbb{Q}\delta_{\pm}$ and $\mathfrak{n} \in D_0$, we have:

$$[\mathfrak{z}+\mathfrak{n};\overline{z},z]^k = ([\mathfrak{z};\overline{z},z]+[\mathfrak{n};\overline{z},z])^k = \sum_{j=0}^{k}\binom{k}{j}\mathfrak{z}^j(\overline{z}-z)^j[\mathfrak{n};\overline{z},z]^{k-j}$$

Note that since $p(\overline{z},z) = \begin{pmatrix}\overline{z} & -\overline{z}z\\ 1 & -z\end{pmatrix}$ so that $\mathrm{Re}(p(\overline{z},z)) \in D_{0,\mathbb{R}}$ and $\mathrm{Im}(p(\overline{z},z)) \in Z_{\mathbb{R}}$. Thus $[\mathfrak{n},\overline{z},z] = S^+(\mathfrak{n},\mathrm{Re}(p(\overline{z},z))) \in \mathbb{R}$. Hence,

$$|[\mathfrak{z}+\mathfrak{n};z,\overline{z}]|^2 = ([\mathfrak{n};\overline{z},z]+\mathfrak{z}(z-\overline{z}))([\mathfrak{n};\overline{z},z]-\mathfrak{z}(z-\overline{z})) = |[\mathfrak{n};z,\overline{z}]|^2 - \mathfrak{z}^2(z-\overline{z})^2$$

Now set

$$\Psi^Z_j(\tau)(\mathfrak{z}) = \mathfrak{z}^j\mathbf{e}_\infty\left(\mathfrak{z}^2\tau^{\pm} - i\frac{\mathrm{Im}(\tau)(z-\overline{z})^2\mathfrak{z}^2}{2\mathrm{Im}(z)^2}\right) = \begin{cases}\mathfrak{z}^j\mathbf{e}_\infty(\mathfrak{z}^2\tau) & \text{in case } +\\ \mathfrak{z}^j\mathbf{e}_\infty(-\mathfrak{z}^2\overline{\tau}) & \text{in case } -\end{cases}$$

and,

$$\Psi_j^{D_0}(\tau,z)(\mathfrak{n}) = (z-\overline{z})^{-j}[\mathfrak{n};\overline{z},z]^j\mathbf{e}_\infty\left(N(\mathfrak{n})\tau^{\pm}+i\frac{\mathrm{Im}(\tau)|[\mathfrak{n};z,\overline{z}]|^2}{2\mathrm{Im}(z)^2}\right).$$
$$= (z-\overline{z})^{-j}S^{+}(\mathfrak{n},\mathrm{Re}(p(\overline{z},z)))^j\mathbf{e}_\infty\left(N(\mathfrak{n})\tau^{\pm}+i\frac{\mathrm{Im}(\tau)S^{+}(\mathfrak{n},\mathrm{Re}(p(\overline{z},z)))^2}{2\mathrm{Im}(z)^2}\right)$$

By the calculations above, we get that:

$$\mathrm{Im}(\tau)^{-1}(z-\overline{z})^k\phi_k(\tau;z,z) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\phi_Z\Psi_j^Z(\tau)\otimes\phi_0\Psi_{k-j}^{D_0}(\tau;z) \tag{6.6}$$

6.4 The period integral

We assume that the level of $\theta(\phi_k)$ with respect to τ is of the form $\Gamma_\tau = \Gamma_0(M)$ for some integer $M>0$. For $\alpha \in D_\sigma^{\pm}\cap D_E^{\times}$, let:

$$\widehat{\Gamma}_\alpha = \{x\in O_\alpha(\mathbb{A}^{(\infty)}) \mid \phi^{(\infty)}(xv) = \phi^{(\infty)}(v)\ \forall v\in D_{0,\mathbb{A}^{(\infty)}}\}$$

and $\mathrm{Sh}_\alpha = \mathrm{Sh}_{\alpha,\phi} = O_\alpha(\mathbb{Q})\backslash O_\alpha(\mathbb{A})/\widehat{\Gamma}_\alpha C_\alpha$ where C_α is a maximal compact subgroup of $O_\alpha(\mathbb{R})$. We write $z = x+iy$, $\tau = \xi+i\eta$, $\mathrm{Sh}=\mathrm{Sh}_\delta$, and consider for $F\in S_k^{\mp}(\Gamma_\tau,\varphi\chi_{D_\sigma}^{\pm})$ and $n=k-2$:

$$P'_\delta(F) := \int_{\mathrm{Sh}}\int_{\Gamma_\tau\backslash\mathfrak{H}} n!^{-2}\nabla^n\Theta(\tau;z,z)F(\tau)\eta^{k-2}\ \mathrm{d}\xi\mathrm{d}\eta$$
$$= \int_{\mathrm{Sh}}\int_{\Gamma_\tau\backslash\mathfrak{H}} (\overline{z}-z)^n\theta(\phi_k)F(\tau)\eta^{k-2}\ \mathrm{d}z\wedge\mathrm{d}\overline{z}\ \mathrm{d}\xi\mathrm{d}\eta$$

since $\mathrm{d}z\wedge\mathrm{d}\overline{z} = -2i\mathrm{d}x\wedge\mathrm{d}y$ and $-\frac{1}{4}(\overline{z}-z)^2y^{-2}=1$, the above integral is equal to:

$$\frac{i}{2}\int_{\Gamma_\tau\backslash\mathfrak{H}}\left(\int_{\mathrm{Sh}}(\overline{z}-z)^k\theta(\phi_k)(\tau;z,z)y^{-2}\ \mathrm{d}x\mathrm{d}y\right)F(\tau)\eta^{k-2}\ \mathrm{d}\xi\mathrm{d}\eta$$

Choose a lattice L of D_σ and assume $L = L_Z\oplus L_0$ for lattices $L_Z\subset Z$ and $L_0\subset D_0^{\pm}$. We take ϕ_0 to be the characteristic function of $\widehat{L}_0\subset D_0\otimes_{\mathbb{Q}}\mathbb{A}^{(\infty)}$ and we choose in subsection 6.5 the finite part ϕ_Z of ϕ^Z which has open support in $\widehat{L}_Z^*$.

6.5 Choice of ϕ_Z

6.6 Siegel Weil formula

Since $O_Z(\mathbb{R}) = \{\pm 1\}$, the action of $g\in O_V(\mathbb{R})$ on $\Psi_j^Z(i)(\mathfrak{z}) = \mathfrak{z}^j\mathbf{e}_\infty(\mathfrak{z}^2 i)$ is trivial. For $g_\tau = \begin{pmatrix}1 & \mathrm{Re}(\tau)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathrm{Im}(\tau)^{1/2} & 0\\ 0 & \mathrm{Im}(\tau)^{1/2}\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$, we have that $g_\tau(i)=\tau$, and so:

$$\mathbf{r}_Z(g_\tau)\Psi_j^Z(i)(\mathfrak{z}) = \mathbf{r}_Z(\begin{pmatrix}1 & \mathrm{Re}(\tau)\\ 0 & 1\end{pmatrix})\mathrm{Im}(\tau)^{\frac{1+2j}{4}}\mathfrak{z}^j\mathbf{e}_\infty(\mathrm{Im}(\tau)\mathfrak{z}^2 i)$$
$$= \mathrm{Im}(\tau)^{\frac{1+2j}{4}}\mathfrak{z}^j\mathbf{e}_\infty(\mathfrak{z}^2(\pm\mathrm{Re}(\tau)+i\mathrm{Im}(\tau)))$$
$$= \mathrm{Im}(\tau)^{\frac{1+2j}{4}}\Psi_j^Z(\tau)(\mathfrak{z})$$

Since the even Clifford algebra of D_0 is D, we have by theorem 1.9 that $SO_{D_0}(\mathbb{R})\cong PGL_2(\mathbb{R})$ by τ_{D_0} with the action on the matrices by conjugation, and for $g_z = \begin{pmatrix}\mathrm{Im}(z)^{\frac{1}{2}} & \mathrm{Re}(z)\mathrm{Im}(z)^{-\frac{1}{2}}\\ 0 & \mathrm{Im}(z)^{\frac{-1}{2}}\end{pmatrix}\in PGL_2(\mathbb{R})$ we have by (6.4):

$$\mathbf{r}_{D_0}(g_\tau,g_z)\Psi_j^{D_0}(i;i)(\mathfrak{n}) = \mathbf{r}_{D_0}(g_\tau)\Psi_j^{D_0}(i;z)(\mathfrak{n}) = \mathrm{Im}(\tau)^{(3+2j)/4}\Psi_j^{D_0}(\tau;z)(\mathfrak{n})$$

As the Siegel-Weil formula is stated with respect to the theta series of variable $g \in O_{D_0}(\mathbb{A})$ and not with respect to z, we lift $\theta(\phi_k)(\tau; z, z)$ to a function $\boldsymbol{\theta}(\phi_k)(\tau; g)$ on $O_{D_0}(\mathbb{A})$ in the standard way by:

$$\boldsymbol{\theta}(\phi_k)(\tau; g) = \theta(\phi_k)(\tau; g \cdot i, g \cdot i)|j(g, i)|^{-2k}$$

then we have by (6.6) that:

$$\begin{aligned}\boldsymbol{\theta}(\phi_k)(\tau; g_z) = (z - \overline{z})^k \theta(\phi_k)(\tau; z, z) &= \eta \sum_{j=0}^{k} (-1)^j \binom{k}{j} \theta(\phi_j^Z)(\tau) \theta(\phi_{k-j}^{D_0})(\tau; z) \\ &= \eta \sum_{j=0}^{k} (-1)^j \binom{k}{j} \theta(\phi_j^Z)(\tau) \mathbf{r}_{D_0}(g_z) \theta(\phi_{k-j}^{D_0})(\tau; i)\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathrm{Sh}} \eta^{-1}(z - \overline{z})^k \theta(\phi_k)(\tau; z, z) \frac{\mathrm{d}x \mathrm{d}y}{y^2} &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \theta(\phi_j^Z)(\tau) \int_{O_{D_0}(\mathbb{Q}) \backslash O_{D_0}(\mathbb{A}) / \widehat{\Gamma}_\delta C_\delta} \mathbf{r}_{D_0}(g) \theta(\phi_{k-j}^{D_0})(\tau; i) \, \mathrm{d}\mu_g \\ &= \mathfrak{m} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \theta(\phi_j^Z)(\tau) E(\phi_{k-j}^{D_0})\end{aligned}$$

by the Siegel-Weil formula (theorem 2.3). Here we normalize the Haar measure $\mathrm{d}\mu_g$ on $O_\delta(\mathbb{A})$ so that it has volume 1 on $\widehat{\Gamma}_\delta C_\delta$, and $\mathfrak{m}$ satisfies $\mathrm{d}\mu_g = \frac{\mathfrak{m}}{2} \mathrm{d}\omega_{O_\delta}$ for the Tamagawa measure $\mathrm{d}\omega_\delta$ of O_δ (the factor of $\frac{1}{2}$ is because $\int \mathrm{d}\omega_{O_\delta} = 2$).

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