Contents

Chapter 1

Integral Perfectoid Rings

1.1 The tilting functor

We say that a ring A of characteristic p is perfect if the Frobenius map $\Phi : A \to A$ is an isomorphism. If Φ is only surjective, we say that A is semi-perfect.

Definition 1.1.1. *(Tilting Functor)*
If A is a ring of characteristic p, set

$$A^{perf} := \varprojlim_{\Phi} A = \{(a_0, a_1, \dots) \in A^{\mathbb{N}} : a_i^p = a_{i-1} \textit{ for all } i \geq 1\}.$$

For any ring A, we set $A^\flat := (A/p)^{perf}$.

• As the notation suggests, for a ring A of characteristic p, A^{perf} is perfect. We say that A^{perf} is the perfection of A.
Indeed, given $(x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \dots) \in A^{\mathrm{perf}}$, we have $(x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \dots) = (x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, x^{\frac{1}{p^3}} \dots)^p$ and $(x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \dots)^p = 0$ if and only if $x^p = x = x^{\frac{1}{p}} = \cdots = 0$.
Moreover A^{perf} satisfies a universal property: if $f : S \to A$ is a ring homomorphism with S perfect of characteristic p, then it factors through $S \to A^{\mathrm{perf}} \to A$.
• $A^{\mathrm{perf}} \to A$ is surjective exactly when R is semi-perfect.

Lemma 1.1.2. *Let A be a ring and $\pi \in A$ be an element such that $p \in (\pi)$. Given $a, b \in A$ such that $a \equiv b\ [\pi]$, we have $a^{p^n} \equiv b^{p^n}\ [\pi^{n+1}]$ for all $n \geq 0$.*

Proof. We prove it by induction on n.
Assume that $a^{p^n} = b^{p^n} + \pi^{n+1}c$. Rising both sides to the p-th power, we get $a^{p^{n+1}} = b^{p^{n+1}} + p \cdot \pi^{n+1} \cdot d + \pi^{p(n+1)} \cdot c^p$ for some $d \in R$ since $p \mid \binom{p}{i}$. Hence the result. ∎

Example 1.1.3.

i) We have $\left(\mathbb{F}_p[T^{1/p^\infty}]/(T)\right)^{perf} \simeq \mathbb{F}_p\langle T^{1/p^\infty}\rangle$ (=T-adic completion).
Indeed, it is easy to see that the morphism

$$\mathbb{F}_p[T^{1/p^\infty}]/(T) \to \varprojlim_n \mathbb{F}_p[T^{1/p^\infty}]/(T^{p^n})$$
$$\left(x_n \mod [T]\right) \mapsto \left(x_n^{p^n} \mod [T^{p^n}]\right)$$

is well defined with inverse sending $(y_0 \mod [T], y_1 \mod [T^p], \dots)$ *to* $(y_0 \mod [T], y_1^{1/p} \mod [T], \dots)$.

ii) If $f : S \to R$ *is a surjective morphism of characteristic* p *rings with nilpotent kernel, then* $S^{perf} \simeq R^{perf}$.
Indeed, let $m \in \mathbb{N}$ *such that* $\mathrm{Ker}(f)^{p^m} = 0$. *We have a natural morphism*

$$\begin{aligned} f^{perf} : S^{perf} &\to R^{perf} \\ \big(s_n\big)_{n\in\mathbb{N}} &\mapsto \big(f(s_n)\big)_{n\in\mathbb{N}} \end{aligned}$$

Let $(r_n)_{n\in\mathbb{N}} \in R^{perf}$. *Let* $s'_{n+m} \in S$ *such that* $f(s'_{n+m}) = r_{n+m}$, *then* $s_n = {s'_{n+m}}^{p^m}$ *verifies* $f(s_n) = r_n$, $s_n^p = s_{n-1}$ *and is independent of the choice of* s'_{n+m}. *We see that* $f^{perf}\big((s_n)_{n\in\mathbb{N}}\big) = (r_n)_{n\in\mathbb{N}}$*; which shows that* f^{perf} *is an isomorphism.*

Lemma 1.1.4. *Assume that* A *is* π*-adically complete for some* π *such that* $p \in (\pi)$. *Then the projection maps* $A \to A/p \to A/\pi$ *induces bijections*

$$\lim_{x\mapsto x^p} A \xrightarrow{\sim} \varprojlim_{\Phi} A/p = A^\flat \xrightarrow{\sim} \varprojlim_{\Phi} A/\pi$$

of multiplicative monoids (since $x \to x^p$ *is not necessarily a morphism of rings).*
If one topologizes A *with the* π*-adic topology and* A/π *and* A/p *with the quotient topologies (corresponding with the discrete topology for* A/π*), then these bijections are homeomorphisms.*

Proof. We check that the map $\lim_{x\mapsto x^p} A \to \varprojlim_{\Phi} A/\pi$ is bijective. For the injectivity, fix $(a_n), (b_n) \in \lim_{x\mapsto x^p} A$ with $a_n \equiv b_n\ [\pi]\ \forall n$. Since $a_{n+k}^{p^k} = a_n$ for all $n, k \in \mathbb{N}$, we get by Lemma 1.1.2 that $a_n \equiv b_n\ [\pi^k]\ \forall k$. Since A is π-adically complete, we get that $a_n = b_n\ \forall n$.
Now for surjectivity, let $(\bar{a_n}) \in \lim_{\Phi} A/\pi$ with a set theoretical lift (a_n). Then $a_{n+k+1}^p \equiv a_{n+k}\ [\pi]\ \forall n, k$. By Lemma 1.1.2, for all n the sequence $k \mapsto a_{n+k}^{p^k}$ is Cauchy for the π-adic topology and has a limit b_n. We easily check that $b_n^p = b_{n_1}\ \forall n$ and b_n lift $\bar{a_n}$.
For the last part, the maps defined are clearly continuous, so we need to deal with the continuity in the other direction. An element of $\lim_{x\mapsto x^p} A$ say $(b_0, b_2, \dots)$ has a basis of neighborhoods of the form $U_{n,k} = \{(\alpha_1, \alpha_2, \dots) \in \lim_{x\mapsto x^p} A \mid \alpha_n \equiv b_n\ [\pi^k]\}$. But an element $(\bar{b_1}, \dots, \bar{b_{n+k}}, \bar{\alpha_{n+k+1}} \dots) \in \lim_{x\mapsto x^p} A/\pi$ verifies $\alpha_{n+k+i}^i \equiv b_{n+k}\ [\pi]$, thus by Lemma 1.1.2 we have $\alpha_{n+k+i}^{i+k} \equiv b_{n+k}^k \equiv b_n\ [\pi^k]$. Taking to the limit we see that its image by the inverse map lies in $U_{n,k}$ which gives us the desired continuity since the elements of the form $(\bar{b_1}, \dots, \bar{b_{n+k}}, \bar{\alpha_{n+k+1}} \dots)$ form a neighborhood of $(\bar{b_1}, \dots, \bar{b_n}, \dots)$. ■

Definition 1.1.5. *(sharp map)*
Given the bijection of multiplicative monoids $\lim_{x\mapsto x^p} A \xrightarrow{\sim} \lim_{\Phi} A/p = A^\flat$, *we project* $\lim_{x\mapsto x^p} A$ *on the last term to obtain a multiplicative map*

$$\begin{aligned} \sharp : A^\flat &\to A \\ f &\mapsto f^\sharp \end{aligned}$$

Its image is exactly those $f \in A$ *that admit a compatible system* $\{f^{\frac{1}{p^k}}\}$ *of p-powers.*

• By Lemma 1.1.4, we see that the sharp map (or untilting map) $\sharp : A^\flat \to A$ is continuous.
• Notice that the sharp map is generally not additive; in fact, if $b, c \in A^\flat$, it transforms under addition as follows

$$(b+c)^\sharp = \lim_{i\to\infty}((b^{\frac{1}{p^i}})^\sharp + (c^{\frac{1}{p^i}})^\sharp)^{p^i}$$

However, the composition $A^\flat \xrightarrow{\sharp \text{ mod p}} A/p$ is given by the projection on the last coordinate $A^\flat \simeq \lim\limits_{x\mapsto x^p} A/p \to A/p$, so it it a ring homomorphism.

1.2 Definition and first properties

Let A be a topological ring. We say that A is integral perfectoid if and only if there exists a non-zero divisor $\pi \in A$ such that

1. The topology on A is given by the π-adic topology and A is complete for this topology.
2. $p \in \pi^p A$
3. The morphism $\Phi : A/\pi A \to A/\pi^p A$ sending a to a^p is an isomorphism.

We call any such element π a perfectoid pseudo-uniformizer.

Example 1.2.1. *Let A be a perfectoid ring and $\pi \in A$ a pseudo-uniformizer. We define its algebra of perfectoid polynomials in several variables to be*

$$A\langle T_0^{1/p^\infty}, \dots T_n^{1/p^\infty}\rangle := \text{ the } \pi\text{-adic completion of } \bigcup_{i\geq 1} A[T_0^{1/p^i}, \dots, T_n^{1/p^i}]$$

then $A\langle T_0^{1/p^\infty}, \dots T_n^{1/p^\infty}\rangle$ is an integral perfectoid ring.

Lemma 1.2.2. *Let A be integral perfectoid, $\pi \in A$ a perfectoid pseudo-uniformizer. Then*

1. *Every element of $A/p\pi$ is a p-th power.*
2. *If an element $a \in A[\frac{1}{\pi}]$ satisfies $a^p \in A$ then $a \in A$.*
3. *After multiplying π by a unit, it has a compatible sequence of p-power roots $\pi^{\frac{1}{p}}, \pi^{\frac{1}{p^2}}, \dots \in A$.*

Proof.

1. Let $x \in A$. By surjectivity of Φ, there exist elements, $a_0, x_1 \in A$ such that $x = a_0^p + \pi^p x_1$; continuing the procedure for x_1, we get that $x = \sum_{i\geq 0} a_i^p \pi^{pi}$ for some $a_i \in A$. But $\sum_{i\geq 0} a_i^p \pi^{pi} \equiv (\sum_{i\geq 0} a_i \pi^i)^p$ $[p\pi]$ which proves the first property.
2. Let $l \geq 0$ be the smallest integer such that $p^l a \in A$. Assuming that $l > 0$, we get that $\pi^{pl} a^p \subset \pi^{pl} A \subset \pi^p A$. But by the third condition in the definition of an integral perfectoid ring, we have $\pi^l a \in \pi A$, and so $\pi^{l-1} a \in A$ which contradicts the minimality of l.

3. Since the Frobenius map is surjective on A/π^p, there exists an element of $\varprojlim_{\Phi} A/\pi^p$ of the form $(\pi \bmod \pi^p A, \dots)$. By the bijection in Lemma 1.1.4, there exists an element $a = (a_0, a_1, \dots) \in \lim\limits_{x\mapsto x^p} A$ such that $a_0 \equiv \pi\ [\pi^p]$, i.e., $a_0 = \pi(1+\pi^{p-1}k)$ for some $k \in A$; but since A is π-adically complete, $(1 + \pi^{p-1}k)$ is invertible.

■

Lemma 1.2.3. *Let A an integral perfectoid ring. Let $t \in A$ be an element satisfying the two first conditions in the definition of integral perfectoid rings. Then it satisfies the third one, i.e., it is a perfectoid pseudo-uniformizer.*

Proof. We need to show that $\Phi : A/t \to A/t^p$ is an isomorphism.
Let $\pi \in A$ be a perfectoid pseudo-uniformizer. By the first property in lemma 1.2.2, we have that the Frobenius map is surjective on A/p, hence it is also surjective on its quotient A/t. Since t and π define the same topology, $\{t^n A\}$ and $\{\pi^n A\}$ are confinal amongst each other. Thus $A[\frac{1}{\pi}] = A[\frac{1}{t}]$. If $a \in A$ satisfies $a^p \in t^p A$, then $\frac{a}{t} \in A[\frac{1}{\pi}]$ verifies $(\frac{a}{t})^p \in A$, so by the second property of lemma 1.2.2, we get that $\frac{a}{t} \in A$, i.e., $a \in tA$ which proves the injectivity. ■

Lemma 1.2.4. *A complete topological ring A of characteristic p is integral perfectoid if and only if it is perfect and the topology is π-adic for some non-zero divisor $\pi \in A$.*

Proof. Suppose that A is integral perfectoid. Let us show that it is perfect.
Take $a \in A$, by the isomorphism $\Phi : A/\pi \to A/\pi^p$, there exist $x_0, a_1 \in A$ such that $a = x_0^p + \pi^p a_1$, repeating this for a_1, we get that $a = x_0^p + (\pi x_1)^p + \dots + (\pi^k x_k) = (x_0 + \pi x_1 + \dots \pi^k x_k)^p$. By π-adic completeness of A, taking to the limit we get that $a = x^p$ for some $x \in A$.
Now let $a \in A$ such that $a^p = 0$, then by injectivity of Φ, we get that $a = \pi a_1$ for some $a_1 \in A$. Since π is a non-zero divisor, $a_1^p = 0$; continuing with that we get $\pi^n \mid a$ for all $n \geq 0$ which, by separability of A, implies that $a = 0$ and so A is perfect.
Conversely, suppose that A is perfect, and has the π-adic topology. Since the Frobenius sends bijectively πA to $\pi^p A$, the map $\Phi : A/\pi \to A/\pi^p$ is an isomorphism. ■

Lemma 1.2.5. *Let A be an integral perfectoid ring with a perfectoid pseudo-uniformizer π admitting p-power roots (which we can assume by lemma 3). Then $A^\flat$ is also an integral perfectoid ring having a perfectoid pseudo-uniformizer $\pi^\flat$ such that $\pi^{\flat\sharp} = \pi$. Moreover, the morphism of rings*

$$A^\flat/\pi^\flat \xrightarrow{\sharp\ mod\ \pi} A/\pi$$

is an isomorphism.

Proof. ■

Proof. The homeomorphism $A \lim\limits_{x\mapsto x^p} A/\pi A$ shows that A is an inverse limit of discrete rings, hence A is a complete topological ring. Let us show that this topology is the $\pi^\flat$-adic topology for some non-zero-divisor $\pi^\flat \in A^\flat$.

Let $\pi^\flat = (\pi, \pi^{\frac{1}{p}}, \dots) \in A^\flat$ be the corresponding element satisfying $(\pi^\flat)^\sharp = \pi$.
Let us show that $\pi^\flat$ is a non-zero divisor. Notice that for $n \geq 1$, we have a sequence

$$0 \to \pi^{1-1/p^n} A/\pi \to A/\pi \xrightarrow{\times \pi^{1/p^n}} A/\pi \xrightarrow{\varphi^n} A/\pi \to 0$$

which is exact since if $a \in A$ satisfies $a^{p^n} \in \pi A$, then $\frac{a}{\pi^{1/p^n}} \in A[\frac{1}{\pi}]$ satisfies $(\frac{a}{\pi^{1/p^n}})^{p^n} \in A$, and by the second property in lemma 1.2.2 we have $a \in \pi^{1/p^n} A$.
These sequences are also compatible in n and taking to the limit we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \pi^{1-1/p^n} A/\pi \arrow[r] & A/\pi \arrow[r, "\times \pi^{1/p^n}"] & A/\pi \arrow[r, "\varphi^n"] & A/\pi \arrow[r] & 0 \\
0 \arrow[r] & \pi^{1-1/p^{n+1}} A/\pi \arrow[r] \arrow[u, "0"] & A/\pi \arrow[r, "\times \pi^{1/p^n}"] \arrow[u, "\varphi"] & A/\pi \arrow[r, "\varphi^n"] \arrow[u, "\varphi"] & A/\pi \arrow[r] \arrow[u, "id"] & 0 \\
 & 0 \arrow[r] \arrow[u, dashed] & A^\flat \arrow[r, "\times \pi^\flat"] \arrow[u, dashed] & A^\flat \arrow[r, "\sharp \bmod \pi"] \arrow[u, dashed] & A/\pi \arrow[r] \arrow[u, dashed] & 0
\end{tikzcd}$$

since all the transition maps are either surjective (φ, id) or zero. Thus $\pi^\flat$ is a non-zero divisor of $A^\flat$ and the untilting map induces an isomorphism $A^\flat/\pi^\flat \xrightarrow{\sim} A/\pi$.
Now let us examine the topology on $A^\flat$. A basis of neighborhood of 0 in $A^\flat$ is given by $\mathrm{Ker}(p_n)$ with $p_n : A^\flat = \lim\limits_{x \mapsto x^p} A/\pi, (a_0, a_1, \dots) \mapsto a_n$. But since $p_n \circ \varphi^n = p_0$, with φ^n being an isomorphism, we get that $\mathrm{Ker}(p_n) = \varphi^n(\mathrm{Ker}(p_0)) = \phi^n(\pi^\flat A^\flat) = \pi^{\flat^{p^n}} A^\flat$ which shows that the topology on A is $\pi^\flat$-adic. Combining these two results, we get the first property in the definition of integral perfectoid rings. The second property is obvious and the third one comes from the fact that $A^\flat$ is perfect. Therfore, $A^\flat$ is integral perfectoid.

■

1.3 Tilting Correspondence for integral perfectoid rings

Lemma 1.3.1. *Suppose that R is a perfect ring of characteristic p. Let $t \in R$ be a non-zero-divisor such that R is t-adically complete, then $W(R)$ is $[t]$-adically complete.*
Moreover, if , $q \in W(R)$ an element such that $q \equiv p \mod [t]$, then q is a non-zero-divisor.

Proof. We have $[t] = (t, 0, 0, \dots)$ and for $a = (a_0, a_1, \dots) \in W(R)$ we have

$$[t] \cdot a = (ta_0, a_1 t^p, \dots) = 0 \text{ if and only if } a_i = 0 \ \forall i$$

therefore $[t]$ is a non-zero-divisor in $W(R)$.
If there exists $a \in W(R)$ such that $ap = [t]b$, then modding out by p gives $b = pb'$ for some $b' \in W(R)$ so that $a = [t]b'$ (since p is a non-zero-divisor in $W(R)$). Thus p is a non-zero-divisor in $W(R)/[t]$, which in turn, implies that q is a non-zero-divisor of $W(R)/[t]$ and that

$[t]$ is a non-zero-divisor in $W(R)/q$.
Now for each $k > 0$ consider the exact sequence

$$0 \to p^n W(R)/(p^{n+1}, [t]^k) \to W(R)/(p^{n+1}, [t]^k) \to W(R)/(p^n, [t]^k) \to 0$$

Since the transition functions are all surjective, taking the limit gives an exact sequence (see A&M prop.10.2)

$$0 \to p^n \widehat{W(R)/p^{n+1}} \to \widehat{W(R)/p^{n+1}} \to \widehat{W(R)/p^n} \to 0$$

where $\widehat{\cdot}$ denote the $[t]$-adic completion.
Since $R = W(R)/p$ is $[t]$-adically complete, we can suppose by induction that $\widehat{W(R)/p^n} = W(R)/p^n$. Since $p^n W(R)/p^{n+1} \simeq R$ as R-modules, we also get $p^n \widehat{W(R)/p^{n+1}} \simeq p^n W(R)/p^n$. Therefore, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & p^n W(R)/p^{n+1} \arrow[r] \arrow[d, "\wr"'] & W(R)/p^{n+1} \arrow[r] \arrow[d] & W(R)/p^n \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & p^n \widehat{W(R)/p^{n+1}} \arrow[r] & \widehat{W(R)/p^{n+1}} \arrow[r] & \widehat{W(R)/p^n} \arrow[r] & 0
\end{tikzcd}$$

A diagram chase shows that the middle arrow is an isomorphism. Hence $W(R)/p^{n+1}$ is $[t]$-adically complete.
The natural morphism $W(R) \to \widehat{W(R)} (= \varprojlim_k W(R)/[t]^k)$ is an isomorphism mod p^n for all $n > 0$. Since $\widehat{W(R)}$ is p-adically separated and $W(R)$ is p-adically complete, it follows that it is an isomorphism. So $W(R)$ is $[t]$-adically complete as desired.
Now let us show that q is a non-zero-divisor of $W(R)$. So write $q = p + [t]b$ and let $a \in W(R)$ such that $aq = 0$, i.e., $ap + [t]ba = 0$. But $ap \equiv 0 \mod [t]$ gives us $a \equiv 0 \mod [t]$ and so $a = [t]a'$ for some $a' \in W(R)$. Therfore, we have $[t]a'p + [t]^2 a'b = 0 \Rightarrow a'p + [t]a'b = 0$. Repeating the same thing for a' shows that $a \in \bigcap_{n>0} [t]^n W(R)$, and so $a = 0$ which gives us the desired result.

■

Theorem 1.3.2. *(Fontaine's θ-map)*
Let A be an integral perfectoid ring.

1 *There is a unique ring homomorphism*

$$\theta : W(A^\flat) \to A$$

which satisfies $\theta([b]) = b^\sharp$ for all $b \in A^\flat$.

2 *θ is surjective and its kernel is generated by a non-zero divisor*

3 *Given $\chi \in \mathrm{Ker}(\theta)$, χ generates its kernel if and only if $\chi = (\chi_0, \chi_1, \dots)$ with $\chi_1 \in A^{\flat\times}$.*

Proof. **1)** Since any element $b \in W(A^\flat)$ can be written uniquely as $b = \sum_{i\geq 0}[b_i]p^i$, we have a well defined map

$$\begin{aligned} \theta : W(A^\flat) &\to A \\ \sum_{i=0}^{\infty}[b_i]p^i &\mapsto \sum_{i=0}^{\infty} b_i^\sharp p^i \end{aligned}$$

So we only need to show that this map is a ring homomorphism. But since A is p-adically separated, we only need to show that it is a homomorphism $\mod p^n$ for all $n \in \mathbb{N}$. So fix $n \geq 0$.
Recall that we have a ring homomorphism

$$\begin{aligned} W_*^{(n)} : W(A) &\xrightarrow{W_*} A^{\mathbb{N}} \xrightarrow{p_n} A \\ (a_0, a_1, \dots) &\longrightarrow a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n \end{aligned}$$

Note that if $a_i \equiv a'_i \mod p$ then $a_i^{p^{n-i}} \equiv {a'_i}^{p^{n-i}} \mod p^{n-i+1}$ so that $p^i a_i^{p^{n-i}} \equiv p^i {a'_i}^{p^{n-i}} \mod p^{n+1}$. Therfore, $W_*^{(n)} \mod p^{n+1}$ depends only on the value of the Witt coordinates mod p, i.e., we get a commutative diagram

$$\begin{tikzcd} W(A) \arrow[r, "W_*^{(n)}"] \arrow[d, two heads] & A \arrow[d] \\ W(A/p) \arrow[r, "\overline{W_*^{(n)}}"] & A/p^{n+1} \end{tikzcd}$$

that shows that $\overline{W_*^{(n)}}$ is also a ring homomorphism.
We claim that the composition

$$W(A^\flat) \xrightarrow{W(\phi^{-n})} W(A^\flat) \xrightarrow{W(\sharp \bmod p)} W(A/p) \xrightarrow{\overline{W_*^{(n)}}} A/p^{n+1}$$

is exactly $\theta \mod p^{n+1}$. Indeed

$$\sum_{i=0}^{\infty}[b_i]p^i = (b_0, b_1^p, b_2^{p^2}, \dots) \mapsto (b_0^{1/p^n}, b_1^{1/p^{n-1}}, \dots) \mapsto (b_0^{1/p^n \sharp}, b_1^{1/p^{n-1}\sharp}, \dots) \mapsto \sum_{i=0}^{n}(b_i^{p^{i-n}\sharp})^{p^{n-i}}p^i = \sum_{i=0}^{n} b_i^\sharp p^i$$

The first two maps being ring homomorphisms induced by the universal property of the Witt vectors, we deduce that $\theta \mod p^{n+1}$ is a ring homomorphism as required.
2) Since A and $W(A^\flat)$ are p-adically complete, to prove the surjectivity of θ, it is enough to show that it is surjective $\mod p$. But this follows from the fact that $\sharp \mod p : A^\flat \to A/p$ is surjective.

Let $\pi \in A$ be a perfectoid pseudo-uniformizer admitting p-power roots and let $\pi^\flat = (\pi, \pi^{1/p}, \dots) \in A^\flat$. Since $p \in \pi^p A$ and θ is surjective, $p = \pi^\flat\theta(-z)$ for some $z \in W(A^\flat)$. Hence $\xi := p + [\pi^\flat]^p z \in \mathrm{Ker}(\theta)$. By lemma 1.3.1, ξ is a non-zero-divisor of $W(A^\flat)$ and $W(A^\flat)$ is $[\pi^\flat]$-adically complete. Since A is π-torsion free, one sees that $W(A^\flat)/\xi$ is an isomorphism if and only if it becomes an isomorphism when we mod out by $[\pi^\flat]$, i.e., we need

to check that $W(A^\flat)/(\xi, [\pi^\flat]) \to A/\pi$ is an isomorphism. But since $\xi \equiv p \mod [\pi^\flat]$, $W(A^\flat)/(\xi, [\pi^\flat]) \simeq A^\flat/\pi^\flat$ and the map identifies with $A^\flat/\pi^\flat \xrightarrow{\sharp \mod \pi} A/\pi$ which is an isomorphism by lemma 1.2.5.
3) First notice that the Witt vector expansion of ξ looks like

$$(\xi_0, \xi_1, \dots) = p + [\pi^\flat]x = (0, 1, 0, \dots) + (\pi^{\flat p}x_0, \pi^{\flat p^2}x_1, \dots) = (\pi^{\flat p}x_0, 1 + \pi^{\flat p^2}x_1, \dots)$$

In particular $\xi_1 \in A^{\flat\times}$ (since $A^\flat$ is $\pi^\flat$-adically complete).
Now let $\chi = (\chi_0, \chi_1, \dots) \in \mathrm{Ker}(\theta)$. $\chi = \beta\xi$ for some $\beta = (\beta_0, \beta_1, \dots) \in W(A^\flat)$, and we have

$$\chi = (\beta_0, \beta_1, \dots) \cdot (\xi_0, \xi_1, \dots) = (\beta_0\xi_0, \beta_1\xi_0^p + \beta_0^p\xi_1, \dots)$$

Therefore

$$\begin{aligned}
\mathrm{Ker}(\theta) = \chi W(A^\flat) &\Leftrightarrow \beta\xi W(A^\flat) = \xi W(A^\flat) \\
&\Leftrightarrow \beta \in W(A^\flat)^\times \text{ (since } \xi \text{ is a non-zero-divisor)} \\
&\Leftrightarrow \beta_0 \in A^{\flat\times} \text{ (} W(A^\flat) \text{ is } p\text{-adically complete and } W(A^\flat)/p = A^\flat \text{)} \\
&\Leftrightarrow \beta_0^p\xi_1 \in A^{\flat\times} \text{ (} \xi_1 \in A^{\flat\times} \text{)} \\
&\Leftrightarrow \chi_1 = \beta_1\xi_0^p + \beta_0^p\xi_1 \in A^{\flat\times} \text{ (since } A^\flat \text{ is } \pi^\flat\text{-adically complete and } \beta_0 \in \pi^\flat A^\flat \text{)}
\end{aligned}$$

■

Theorem 1.3.3. *(Tilting correspondence)*
Let A be an integral perfectoid ring A. The tilting induces an equivalence of categories

$$\textit{Perfectoid } A\textit{-algebras} \to \sim \textit{Perfectoid } A^\flat\textit{-algebras}$$
$$B \to B^\flat$$

whose inverse sends a perfectoid $A^\flat$-algebra to its untilt $C^\sharp := W(C) \otimes_{W(A^\flat),\theta} A$.

Proof.
Let $\pi \in A$ be a perfectoid pseudo-uniformizer admitting p-power roots and $\pi^\flat = (\pi, \pi^{1/p}, \dots) \in A^\flat$ be the associated perfectoid pseudo-uniformizer of $A^\flat$. Let $\xi = p + [\pi^\flat]^p z \in W(A^\flat)$ be a generator of $\mathrm{Ker}(\theta)$.
First, we show that for a perfectoid A-algebra B, $(B^\flat)^\sharp = B$. For this, consider the commutative diagram

$$\begin{tikzcd}
W(B^\flat) \arrow[r, "\theta_B"] & B \\
W(A^\flat) \arrow[r, "\theta_A"] \arrow[u, "W(u)"] & A \arrow[u, "u"']
\end{tikzcd}$$

We see that the image of ξ in $W(B^\flat)$ lies in $\mathrm{Ker}(\theta_B)$ and since the second coordinate in the Witt vector expansion of ξ is in $A^\flat$ whose image under u is in $B^\flat$. By theorem 1.3.2, we get that $W(u)(\xi)$ generates $\mathrm{Ker}(\theta_B)$. Hence, the map

$$W(B^\flat) \otimes_{W(A^\flat)} A \longrightarrow B$$
$$b \otimes a \mapsto \theta_B(b)u(a)$$

On the other hand, let C be a perfectoid $A^\flat$-algebra. We show that $C^\sharp$ is a perfectoid A-algebra and that $(C^\sharp)^\flat = C$.
Since θ_A is surjective with kernel $\xi W(A^\flat)$, we get that $C^\sharp = W(C) \otimes_{W(A^\flat)} A \simeq W(C)/\xi$ where it is seen as an A-algebra via the identification $W(A^\flat)/\xi \simeq A$. is an isomorphism. Applying lemma 1.3.1 for $R = C$, $t = \pi^\flat$ and $q = \xi$, we get that π is a non-zero-divisor of $C^\sharp$ and that $C^\sharp$ is complete for the π-adic topology. It remains to show that $\Phi : C^\sharp/\pi \to C^\sharp/\pi^p$ is an isomorphism. But $\xi \equiv p \mod [\pi^\flat]^p$ so $C^\sharp/\pi \simeq W(C)/(\xi, \pi^\flat) \simeq C/\pi^\flat$ and $C^\sharp/\pi^p \simeq W(C)/(\xi, {\pi^\flat}^p) \simeq C/{\pi^\flat}^p$, and the map can be rewritten as $\Phi : C/\pi^\flat \to C/{\pi^\flat}^p$ which is indeed an isomorphism since $\pi^\flat$ is a perfectoid pseudo-uniformizer. Hence $C^\sharp$ is a perfectoid A-algebra. ■

Example 1.3.4. *Let A be an integral perfectoid ring and $A\langle T_0^{1/p^\infty}, \dots T_n^{1/p^\infty}\rangle$ its algebra of perfectoid polynomials in several variables. Its tilt $A\langle T_0^{1/p^\infty}, \dots T_n^{1/p^\infty}\rangle^\flat$ contains elements $T_i^\flat := (T_i, T_i^{1/p}, T_i^{1/p^2}, \dots)$ for $0 \leq i \leq n$. Therefore we get an $A^\flat$-algebra morphism*

$$A^\flat\langle U_0^{1/p^\infty}, \dots U_n^{1/p^\infty}\rangle \to A\langle T_0^{1/p^\infty}, \dots T_n^{1/p^\infty}\rangle^\flat$$

sending U_i to $T_i^\flat$ which becomes an isomorphism when we mod out by a pseudo-uniformizer $\pi^\flat \in A^\flat$. Since both rings are $\pi^\flat$-adically complete, we get that the morphism is actually an isomorphism and so $A\langle T_0^{1/p^\infty}, \dots T_n^{1/p^\infty}\rangle^\flat = A^\flat\langle {T_0^\flat}^{1/p^\infty}, \dots {T_n^\flat}^{1/p^\infty}\rangle$.

Chapter 2

Perfectoid Fields

We say that K is a non-Archimedean field if it is equipped with a valuation $|\cdot| : K^\times \to \mathbb{R}_{>0}$ for which it is **complete**.

2.1 Definition and first properties

Definition 2.1.1. *A perfectoid field K is a non-Archimedean field with residue characteristic p such that*

- *The value group $|K^\times| \subset \mathbb{R}_{\geq 0}$ is not discrete.*
- *K°/p is semi-perfect.*

Where the subring $K^\circ := \{x \in K \mid |x| \leq 1\}$ is called the valuation ring of K.
It is an open valuation ring with maximal ideal $K^{\circ\circ} := \{x \in K \mid |x| < 1\}$. A non-zero element $\pi \in K^{\circ\circ}$ is called a pseudo-uniformizer.

- Notice that the ring K° is an integral perfectoid ring with perfectoid pseudo-uniformizer any non-zero $\pi \in K^{\circ\circ}$ such that $|\pi^p| \geq |p|$.
- Perfectoid fields are Henselian since they are complete with respect to a rank 1 valuation.

Lemma 2.1.2. *Let K be a perfectoid field.*

1. *The value group is p-divisible.*
2. *$(K^{\circ\circ})^2 = K^{\circ\circ}$ and so the ring K° is not Noetherean.*

Proof. **1)** Let us call $x \in K^\circ$ small if $|p| < |x| \leq 1$. We first check the p-divisibility of $|x| \in |K^\times|$ for small x. The perfectoidness of K gives some $y, z \in K^\circ$ such that $y^p = x + pz$. Taking absolute value, we get by the non-Archimedean property that $|y|^p = |x|$, so $|x| \in |K^\times|$ is p-divisible.
For the general case, since $|K^\times|$ is not discrete, $|p|^{\mathbb{Z}} \subset |K^\times|$ is a strict inclusion, i.e., there exists $x \in K^\times$ with $|x| \notin |p|^{\mathbb{Z}}$. After rescaling by a suitable power of p, we can assume x to

be small. By total ordering of ideals in K° (it is a valuation ring), we must have $p = xy$ for small y. But then $|p| = |x||y|$, so $|p| \in |K^\times|$ is divisible by p. Since $|p|^{\mathbb{Z}}$ and $|x|$ for small x generate $|K^\times|$, we get the result.
2) Let $f \in K^{\circ\circ}$. By perfectoidness, we can write $f = g^p + ph$ for some $g \in K^{\circ\circ}$ and $h \in K^\circ$. The previous proof shows that $p \in (K^{\circ\circ})^2$ and so $f \in (K^{\circ\circ})$. Moreover, Nakayama's lemma shows that K° is not Noetherean. ■

Proposition 2.1.3. *Let $\pi \in K^\circ$ be a perfectoid pseudo-uniformizer with p-power roots and $\pi^\flat$ be the corresponding perfectoid pseudo-uniformizer in $K^{\circ\flat}$.*

1. *The ring $K^{\circ\flat}$ is a valuation ring of rank 1 and $K^\flat := K^{\circ\flat}[\frac{1}{\pi^\flat}]$ is a perfect field.*
2. *The ideal $\left({\pi^\flat}^{1/p^\infty}\right)$ is maximal.*
3. *The valuation topology on $K^\flat$ coming from 1. coincides with the one coming from the $\pi^\flat$-adic topology on $K^{\circ\flat}$. In this topology, $K^\flat$ is a perfectoid field and $K^{\circ\flat} = {K^\flat}^\circ$.*
4. *The value groups and the residue field of K and $K^\flat$ are canonically identified.*

Proof. **1)** First, we show that $K^{\circ\flat}$ is a domain. Since $K^{\circ\flat} \simeq \lim\limits_{x \mapsto x^p} K^\circ$ as multiplicative monoids, we can check the property on the latter. So let $a = (a_0, a_1, \dots), b = (b_0, b_1, \dots) \in \lim\limits_{x \mapsto x^p} K^\circ$ such that $a_n b_n = 0$ for all $n \geq 0$. But since K° is a domain, we can suppose that $a_0 = 0$, and given that $a_n = a_{n+1}^p$, we get by induction that $a = 0$ as required.
Now we show that $K^{\circ\flat}$ is a valuation ring. So let $a, b \in K^{\circ\flat}$ and $(a_n), (b_n) \in \lim\limits_{x \mapsto x^p} K^\circ$ be the corresponding elements. Since K° is a valuation ring, we have $a_0 \mid b_0$ or vice versa. Assume that $a_0 \mid b_0$. Then for valuation reasons, we must have $a_n \mid b_n$ for all $n \geq 0$. Hence $(a_n) \mid (b_n)$ in $\lim\limits_{x \mapsto x^p} K^\circ$, so $a \mid b$ in $K^{\circ\flat}$. This construction shows, that $K^{\circ\flat}$ is endowed with the valuation $|\cdot|^\flat := |\cdot| \circ \sharp : K^{\circ\flat} \to \mathbb{R}_{>0}$. In particular, $K^{\circ\flat}$ is of rank $\leq$ rank $K^\circ = 1$. But since $|\pi^\flat|^\flat = |\pi| \neq 1$, it is exactly of rank 1. In particular, inverting any non-zero unit in $K^{\circ\flat}$ such as $\pi^\flat$, produces the fraction field $K^\flat$.
2), By lemma 1.2.5 we know that $K^{\circ\flat}/\pi^\flat \simeq K^\circ/\pi$. And since the maximal ideal of K° is its nilradical (since K° is a rank 1 valuation ring), then the same must be true for $K^{\circ\flat}/\pi^\flat$. But the nilradical of $K^{\circ\flat}/\pi^\flat$ is the image of $\left({\pi^\flat}^{1/p^\infty}\right)$ (since $\left({\pi^\flat}^{1/p^\infty}\right) \subset$ nilradical and $K^{\circ\flat}/\left({\pi^\flat}^{1/p^\infty}\right)$ is perfect and thus reduced). So $\left({\pi^\flat}^{1/p^\infty}\right)$ is maximal.
3) Since $K^{\circ\flat}$ is a rank 1 valuation ring , the valuation topology coincides with the f-adic topology for any non-unit f. But $K^{\circ\flat}$ is complete for the $\pi^\flat$-adic topology, thus $K^\flat$ is a non-Archimedean field; and since it is perfect, we get that it is perfectoid.
4) The claim about residue fields follows from 2. using the identification $K^{\circ\flat}/\pi^\flat \simeq K^\circ/\pi$. For value groups, we trivially have $|{K^\flat}^\times|^\flat \subset |K^\times|$. To show equality, note that $|K^\times|$ is generated by $|x|$ for $x \in K^\circ$, $|p| < |x| < 1$ (proof of lemma 2.1.2). But by p-divisibility of the value group, we can always choose a perfectoid pseudo-uniformizer π such that $|\pi^p| = |x|$ and thus $|{\pi^\flat}^p| = |x|$. ■

Proposition 2.1.4. *For any continuous valuation $|\cdot| : K^\times \to \Gamma$ (of any rank), the function $|\cdot|^\flat := |\cdot| \circ \sharp : {K^\flat}^\times \to \Gamma$ is also a continuous valuation. This construction identifies the space of valuations on either fields.*

Proof. Fix a continuous valuation $|\cdot|$ on K. It is clear that $|\cdot|^\flat$ is multiplicative. Moreover, as the preimage of 0 by the sharp map is $\{0\}$ ($x^p = 0 \to x = 0$), we have $|f|^\flat = 0 \Leftrightarrow f = 0$. To check the non-Archimedean property, let $f = (f_n), g = (g_n) \in \varprojlim_{x \mapsto x^p} K \simeq K^\flat$, so $f^\sharp = f_0$, $g^\sharp = g_0$. We must check that

$$|f + g|^\flat \leq \max(|f|^\flat, |g|^\flat)$$

But this follows from

$$|f+g|^\flat = |(f+g)^\sharp| = |\lim_n (f_n+g_n)^{p^n}| = \lim_n |f_n+g_n|^{p^n} \leq \lim_n \max(|f_n|, |g_n|)^{p^n} = \lim_n \max(|f_0|, |g_0|)$$

where the second equality follows from the construction of the sharp map and the third equality from the continuity of $|\cdot|$ on K.
For the second part, write $|\cdot|_{\text{std}}$ for the non-Archimedean valuation on K. Observe that the valuation $|\cdot| : K^\times \to \Gamma$ is continuous if for one (or equivalently any) pseudo-uniformizer $f \in K^{\circ\circ}$, $|f|^n \to 0$ as $n \to \infty$. Using this remark, one checks the following about the valuation ring $R \subset K$ attached to $|\cdot|$:

- R contains $K^{\circ\circ}$ inside its maximal ideal as $|f| < 1$ for any $f \in K^{\circ\circ}$.
- We have $R \subset K^\circ$. Indeed, if not, then R has an element from $K^\circ[1/\pi] \setminus K^\circ$, i.e., an element of the form $\frac{a}{\pi^n}$ with $a \in K^\circ$ and $|a|_{\text{std}} > |\pi|_{\text{std}}$. But then $|\frac{\pi^n}{a}|_{\text{std}} < 1$, so $\frac{\pi^n}{a} \in K^{\circ\circ}$, and $\frac{\pi^n}{a}$ lies in the maximal ideal of R, which is absurd since its inverse is included in R.

Conversely, any valuation subring $R \subset K$ satisfying the above two conditions is continuous. Indeed by the correspondence between prime ideals and valuation rings above R (proposition $A.0.9$), the map $R \hookrightarrow K^\circ$ is a localization of R at its unique height 1 prime. So $K^{\circ\circ}$ must lie in all primes of R, and thus $|f|^n \to 0$ for $f \in K^{\circ\circ}$. Indeed, suppose the converse, then there exists $\gamma \in \Gamma$ such that $|f|^n > \gamma$ for all $n \geq 0$. But then considering the ideal $I = \{x \in R \mid |x| \leq \gamma\}$, $\sqrt{I}$ is a prime and $f \notin \sqrt{I}$: contradiction.
Now if we pass to the quotient, we get that the continuous valuations on K identify bijectively with valuation rings in $K^\circ/K^{\circ\circ}$. Repeating the same argument for $K^\flat$, we conclude using the identification $K^\circ/K^{\circ\circ} \simeq K^{\circ\flat}/K^{\flat\circ\circ}$. ∎

Example 2.1.5.
- *Let K be the completion of $\mathbb{Q}_p(p^{1/p^\infty})$, we claim that K is a perfectoid field. Clearly it is a non discrete Archimedean field, so it remains to show that K°/p is semi-perfect. But K°= p-adic completion of $\mathbb{Z}_p[p^{1/p^\infty}]$. Thus they have the same residue ring, i.e., $\mathbb{F}_p[T^{1/p^\infty}]/(T^p)$. From the two examples in Example* 1.1.3*, we get that $K^{\circ\flat} = \mathbb{F}_p\langle T^{1/p^\infty}\rangle$ and so $K^\flat = \mathbb{F}_p\widehat{\big((T)\big)(T^{1/p^\infty})}$.*
- *Let K be the completion of $\mathbb{Q}_p(\zeta_{p^\infty})$, we have K° = completion of $\mathbb{Z}_p[\zeta_{p^\infty}]$. Since modulo p we have $\frac{x^p-1}{x-1} = (x-1)^{p-1}$, then we get*

$$K^\circ/p \simeq \mathbb{F}_p[X^{1/p^\infty}]/(X-1)^{p-1} \simeq \mathbb{F}_p[T^{1/p^\infty}]/(T^{p-1})$$

where we use the substitution $T \to X - 1$. Hence K is perfectoid and by Example 1.*, we also get that $K^\flat = \mathbb{F}_p\widehat{\big((T)\big)(T^{1/p^\infty})}$.*

2.2 Tilting Correspondence for Perfectoid fields

In the previous examples, we saw that the perfectoid fields $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ and $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ have isomorphic tilts. In particular, the tilting functor $K \mapsto K^flat$ is not fully faithful on perfectoid fields over $\mathbb{Q}_p$. This is a consequence of working over a non-perfectoid base. We shall see in this subsection that if the base is perfectoid, then the tilting functor induces an equivalence.

Lemma 2.2.1. *Let K be a non-Archimedean field and $f(X) \in K[X]$ an irreducible monic polynomial such that $f(X) \in K^\circ$. Then $f(X) \in K^\circ[X]$.*

Proof. Write $f(X) = f_0 + f_1 X + \cdots + f^d$. Since f is irreducible, all of its roots have the same valuation, hence the Newton Polygon of f has only one slope which is the segment joining $\big(0, \mathrm{val}(f_0) \geq 0\big)$ and $\big(d, 0\big)$. All the points $\big(i, \mathrm{val}(f_i)\big)$ are above this line so we get $\mathrm{val}(f_i) \geq 0$ for all i. ∎

Proposition 2.2.2. *Let K be a perfectoid field. If $K^\flat$ is algebraically closed then K is algebraically closed.*

Proof. First, fix a perfectoid pseudo-uniformizer $\pi \in K$ with corresponding perfectoid pseudo-uniformizer $\pi^\flat$ in $K^\flat$.
Let $f(X) \in K[X]$ be an irreducible monic polynomial of degree d. We can suppose that $f(X) \in K^\circ[X]$ (otherwise we replace it by $\pi^{kd} f(\pi^{-k}X)$ for some $k \gg 1$).
Now we show that given $a \in K^\circ$ and $n \geq 0$ such that $|f(a)| \leq |\pi|^n$, there exists $\epsilon \in K^\circ$ such that $|\epsilon| \leq |\pi|^{n/d}$ and $|f(a+\epsilon)| \leq |\pi|^{n+1}$.
By the 4th property of proposition 2.1.3 and the fact that $K^\flat$ is algebraically closed, there exists $y \in K^\circ$ such that $|y|^d = |f(a)|$. Whence $g(X) := y^{-d} f(a + yX)$ is a monic irreducible polynomial in $K[X]$ whose constant coefficient is $g(0) = y^{-d} f(a) \in K^\circ$. By lemma 2.2.1, we have $g(X) \in K^\circ[X]$. Since $K^{\flat^\circ}/\pi^\flat \simeq K^\circ/\pi$ and $K^\flat$ is algebraically closed, there exists $b \in K^\circ$ such that $g(b) \equiv 0 \mod [\pi]$. We easily see that $\epsilon := yb$ checks the desired property.
Repeating the procedure and tending $n \to \infty$, we obtain a root of the polynomial f. ∎

Proposition 2.2.3.
Let M be a finite extension of $K^\flat$ of degree d. Then it's untilt $M^\sharp$ is a finite extension of K of the same degree.

Proof. Since $K^\flat$ is perfect, the extension $M/K^\flat$ is separable, i.e., the bilinear form $\mathrm{Tr}_{M/K^\flat}(\cdot,\cdot)$ is non degenerate.
Let $(e_1, \ldots, e_n)$ be a basis of M over $K^\flat$ and $(e_1^*, \ldots, e_n^*)$ be it's dual basis with respect to the Trace map.
Let $\pi^\flat \in K^{\flat^\circ}$ be a perfectoid pseudo-uniformizer, and fix $N \gg 0$ such that ${\pi^\flat}^N e_i \in M^\circ$, $\forall i$.
Define the morphisms:

$$f : (K^{\flat^\circ})^d \hookrightarrow M^\circ$$
$$(a_1, \ldots, a_d) \mapsto \sum_{i=0}^{d} \phi^{-n}({\pi^\flat}^N e_i) a_i$$

and

$$
\begin{aligned}
g : M^{\circ} &\longrightarrow (K^{\flat^{\circ}})^d \\
b &\mapsto \Big(\mathrm{Tr}_{M/K^\flat}(b\phi^{-n}({\pi^{\flat}}^{N} e_i^*)), \dots, \mathrm{Tr}_{M/K^\flat}(b\phi^{-n}({\pi^{\flat}}^{N} e_d^*))\Big)
\end{aligned}
$$

where ϕ is the Frobenius automorphism.
Then, we have:

$$
\begin{aligned}
f \circ g(b) &= \sum_{i=1}^{d} \phi^{-n}({\pi^{\flat}}^{N} e_i) \,\mathrm{Tr}_{M/K^\flat}\Big(b\phi^{-n}({\pi^{\flat}}^{N} e_i^*)\Big) \\
&= {\pi^{\flat}}^{\frac{2N}{p^n}} \sum_{i=1}^{d} \phi^{-n}(e_i)\,\mathrm{Tr}_{M/K^\flat}\big(b\phi^{-n}(e_i^*)\big) \\
&= {\pi^{\flat}}^{\frac{2N}{p^n}} b
\end{aligned}
$$

The last equality is given by expressing b in the basis $(\phi^{-n}(e_i))_{i=1,\dots,d}$ whose dual basis is $(\phi^{-n}(e_i^*))_{i=1,\dots,d}$.
Therfore, we get the inclusions ${\pi^{\flat}}^{\frac{2N}{p^n}} M^{\circ} \subset Im(f) \subset M^{\circ}$.
If we mod out by $\pi^\flat$, we get a morphism:

$$
\bar{f} : (K^{\flat^{\circ}}/\pi^\flat)^d \longrightarrow M^{\circ}/\pi^\flat
$$

Now since $K^{\flat^{\circ}}/\pi^\flat \simeq K^{\circ}/\pi$ and $M^{\circ}/\pi^\flat \simeq M^{\circ\sharp}/\pi$ via $\sharp$ mod π we can view $\bar{f}$ as

$$
\bar{f} : (K^{\circ}/\pi)^d \longrightarrow M^{\circ\sharp}/\pi
$$

with $\pi^{\frac{2N}{p^n}}(M^{\circ\sharp}/\pi) \subset Im(\bar{f})$ and $\mathrm{Ker}(\bar{f}) \subset \pi^{\alpha}(K^{\circ}/\pi)^d$ where $\alpha = 1 - \frac{2N}{p^n} > 0$ if we choose a large n.
Let $(\bar{m_i})_{i=1\dots d}$ be the image of the canonical basis of $(K^{\circ}/\pi)^d$ and $(m_i)_{i=1\dots d}$ be a lift to $M^{\circ\sharp}$ then the $(m_i)_{i=1\dots d}$ are free over K°.
Indeed, if $\beta_1 m_d + \dots + \beta_d m_d = 0$, then if we mod out by π we get that the $\beta_i \in \pi^{\alpha} K^{\circ}$.
Hence, if we divide the β_i by π^{α} and repeat the procedure, we get by π-adic separateness of K° and the fact that $\alpha > 0$ that $\beta_1 = \dots = \beta_d = 0$.
We thus have the inclusion $\pi^{\frac{2N}{p^n}} M^{\circ\sharp} \subset K^{\circ} m_1 \oplus \dots \oplus K^{\circ} m_d \subset M^{\circ\sharp}$.
Finally, inverting π gives $M^\sharp = K m_1 \oplus \dots \oplus K m_d$, so $M^\sharp$ is finite over K and $|M : K^\flat| = |M^\sharp : K|$. ■

Lemma 2.2.4.
Let F be a non-Archimedean field and $F_0 \subset F$ a dense subfield. The F is separably closed if and only if F_0 is separably closed.

Proof. The lemma follows directly from Krasner's lemma and the continuity of the roots of a polynomial with respect to its coefficients. ■

Theorem 2.2.5 (Tilting correspondence for perfectoid fields)**.**
Let K be a perfectoid field. Then:

1. *Any field extension L of K is also a perfectoid field.*
2. *$L \mapsto L^\flat$ is a degree preserving equivalence of categories.*

$$\{ \textit{ finite field extensions of } K\} \longrightarrow \{ \textit{ finite field extensions of } K^\flat\}$$

Proof. Since K is complete with respect to a valuation $|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$, the valuation $|\cdot|$ extends uniquely to any finite extension L/K, the resulting topology is the same as the topology viewed as a finite dimensional K-vector space and in particular L is complete with respect to this topology.
Thus the only thing that remains to prove in part 1) is that given a finite extension L/K, then every element L°/p is a p-th power.
In characteristic p this is true since every finite extension of a perfect field is still perfect. Hence we would like to deduce the result in characteristic 0 by tilting into characteristic p.
We have an equivalence of categories

$$\{\text{perfectoid fields over} K\} \longrightarrow \{\text{perfectoid fields over} K^\flat\}$$
$$L \mapsto L^\flat$$

Let M be a finite extension of $K^\flat$, M is perfectoid and by proposition 2.2.3, its untilt $M^\sharp$ is a finite extension of K of the same degree. Hence we have fully faithful functors:

$$\{\text{finite field extns of } K^\flat\} \longrightarrow \{\text{finite field extns of } K \text{ which are perfect }\} \subset \{ \text{ finite field extns of } K\}$$

We need show that the composition is surjective.
Let Q be a completion of an algebraic closure of $K^\flat$. Lemma 2.2.4 implies that Q is an algebraically closed (since it is perfect and separably closed) perfectoid field of characteristic p. Therfore, its untilt $Q^\sharp$ is an algebraically closed perfectoid field over K and the inclusion $K^\flat \subset M \subset Q$ gives the inclusion $K \subset M^\sharp \subset Q^\sharp$ for all M finite extensions over $K^\flat$.

Let $N = \cup_M M^\sharp \subset Q^\sharp$. Let us show that N is dense in $Q^\sharp$.
Indeed, looking at the rings of integers we have:

$$N^\circ/\pi = \varinjlim_M M^{\circ\sharp}/\pi = \varinjlim_M M^\circ/\pi^\flat = K^{\flat^{\circ alg}}/\pi^\flat = Q^\circ/\pi^\flat = Q^{\circ\sharp}/\pi$$

where the fourth equality comes from the fact that $K^{\flat^{alg}}$ is dense in Q.
By lemma 2.2.4, we get that N is algebraically closed.
Therfore, if L is any finite extension of K, $L \subset N$ and hence there exists a finite extension M of $K^\flat$ such that $L \subset M^\sharp$. Up to taking the normalization of M, we can suppose it to be Galois. So we only need to prove that the injective map:

$$\{\text{subextensions of } M/K^\flat\} \longrightarrow \{ \text{ subextensions of } M^\sharp/K\}$$

is surjective. But $|M^\sharp : K| = |M : K^\flat|$ and Gal$(M^\sharp/K)$=Gal$(M/K^\flat)$ by the equivalence of categories between perfectoid fields over K and $K^\flat$. Therfore, $M^\sharp/K$ is Galois.

By the Galois correspondence, we get that the two sets are finite and have the same cardinality so the injection is a surjection.
Hence $L = E^\sharp$ for some E subextension of M and so L is perfectoid and we have the equivalence of categories.

■

Corollary 2.2.6.
Let K be a perfectoid field. Then, there is an isomorphism of topological groups $G_K \xrightarrow{\sim} G_{K^\flat}$.

2.3 Fontaine-Winterberger Theorem

Lemma 2.3.1. *Let F be field with a rank 1 valuation $|\cdot| : F^\times \to \mathbb{R}_{>0}$ such that F° is Henselian. Then the choice of embedding $F \to F^{sep}$ induces an isomorphism $G_{\widehat{F}} \xrightarrow{\sim} G_F$.*

Proof. By lemma 2.2.4, we have that $\widehat{F^{\mathrm{sep}}}$ is separably closed. Therfore, we have the inclusions $F^{\mathrm{sep}} \subset \widehat{F}^{\mathrm{sep}} \subset \widehat{F^{\mathrm{sep}}}$. So by restriction, we get a map $G_{\widehat{F}} \to G_F$.
The inclusions also show that F^{sep} is dense in $\widehat{F}^{\mathrm{sep}}$. Since F is Henselian, the extension to F^{sep} is unique and thus the automorphisms of F^{sep} are all isometries. We can thereby extend these automorphisms to $\widehat{F}^{\mathrm{sep}}$ which give the inverse of the above map. ■

Let k be a field of characteristic p. We define the perfect closure of k to be $k^{\mathrm{perf}} := \varinjlim_{\varphi} k$ which is the smallest perfect field containing k. For instance, $\mathbb{F}_p((T))^{\mathrm{perf}} = \mathbb{F}_p((T))(T^{1/p^\infty})$.

Lemma 2.3.2. *We have an isomorphism of topological groups $G_k \simeq G_{k^{perf}}$.*

Proof. Let L be a finite separable extension of k, to it we associate $L^{\mathrm{p}} = L \otimes_k k^{\mathrm{perf}}$ a finite separable extension of k^{perf}. Vice-versa, if M is a finite extension of k^{perf}, by the primitve element theorem, there exists an $\alpha \in M$ such that $M = k^{\mathrm{perf}}[\alpha]$. Let $P_\alpha \in k^{\mathrm{perf}}[X]$ be its minimal polynomial, then there exists a unique $n \geq 0$ such that $P_\alpha^{p^n} \in k[X]$ and is irreducible, we define $M^{\mathrm{imp}} = k[X]/\big(P_\alpha^{p^n}\big)$, which is independent of α.
We easily see that the two constructions are inverse of one another and are factorial. Therefore, we get the desired isomorphism of the Galois groups. ■

Combining all the previous results with corollary 2.2.6 and example 2.1.5, we get the Fontaine-Winterberger theorem:

Theorem 2.3.3. *There is an isomorphism of topological groups $G_{\mathbb{Q}_p(p^{1/p^\infty})} \simeq G_{\mathbb{F}_p((T))}$.*

2.4 Deeply ramified fields

A normed field F is called deeply ramified if for every finite extension L of F, the module of Kähler differentials Ω_{L°/F° is zero. In this section, we prove that perfectoid fields are deeply ramified.

Lemma 2.4.1. *Let L/K be a finite separable extension of normed fields. There exists a non-zero element $y \in L^\circ$ such that $y \cdot \Omega_{L^\circ/K^\circ} = 0$.*

Proof. Applying the first fundamental exact sequence for the extension of rings $K^\circ \to K \to L$, and since L is 0-smooth over K, we get an exact sequence

$$0 \to \Omega_{K/K^\circ} \otimes_K L \to \Omega_{L/K^\circ} \to \Omega_{L/K} \to 0$$

But by theorem C.0.11, $\Omega_{L/K} = 0$. Moreover $\Omega_{K/K^\circ} = 0$ since, by proposition C.0.10, it is a localization of $\Omega_{K^\circ/K^\circ} = 0$. Hence we get

$$S^{-1}\Omega_{L^\circ/K^\circ} = \Omega_{L/K^\circ} = 0$$

where $S = L^\circ \setminus \{0\}$. Therefore, for each $dx \in \Omega_{L^\circ/K^\circ}$, there exists $u \in S$ such that $u \cdot dx = 0$. In particular, if we choose a basis $x_1, \ldots, x_n \in L^\circ$ that form a basis of L over K, we can choose a $u \in L^\circ \setminus \{0\}$ such that $u \cdot dx_i = 0$. By choosing $t \in L^\circ \setminus \{0\}$ such that $t \cdot L^\circ \subset \bigoplus_i K^\circ x_i$, we see that $y = tu$ verifies $y \cdot \Omega_{L^\circ/K^\circ} = 0$. ■

We say that a normed field K is deeply ramified if for every finite extension L/K, the L°-module of Kähler differentials Ω_{L°/K° is zero.

Theorem 2.4.2. *A perfectoid field is deeply ramified.*

Proof. Let K be a perfectoid field and L a finite extension of K. Since perfectoid fields are either of characteristic 0 or perfect, the extension is separable. Therfore, by lemma 2.4.1, there exists a non-zero $y \in L^\circ$ such that $y \cdot \Omega_{L^\circ/K^\circ} = 0$.
Since any finite extension of a perfectoid field is also perfectoid, L is perfectoid. Hence for every $x \in L^\circ$, there exist $y, z \in L^\circ$ such that $x = y^p + pz$; thus $dx = py^{p-1}dy + pdz$. This shows by induction that $p^n\Omega_{L^\circ/K^\circ} = \Omega_{L^\circ/K^\circ}$ for all $n > 0$. Now let $n \in \mathbb{N}$ such that $|p^n| \leq |y|$ i.e., $y \mid p^n$. Since y annihilates Ω_{L°/K°, we have

$$0 = p^n\Omega_{\mathcal{O}_L/\mathcal{O}_K} = \Omega_{L^\circ/K^\circ}$$

■

Chapter 3

Perfectoid K-algebras

3.1 Banach K-algebras

Definition 3.1.1. *(K-Banach algebras)*
Let K be a non-Archimedean field. A Banach K-algebra R is a K-algebra equipped with a map $|\cdot| : R \longrightarrow \mathbb{R}_{\geq 0}$ extending the norm on K such that:

- $|f| = 0$ *if and only if f=0*

- $|fg| \leq |f||g|$ *with equality if $f \in K$*

- $|f + g| \leq \max(|f|, |g|)$

- *R is complete in the metric given by the norm*

The category of K-Banach algebras has as objects Banach K algebras and morphisms given by continuous maps.

Definition 3.1.2. *For a K-Banach algebra R, define the set $R^\circ \subset R$ of power bounded elements as*

$$R^\circ := \{f \in R| \quad \{|f^n|\} \text{ is bounded}\} = \{f \in R| \quad \{f^n\} \subset R \text{ is bounded}\}$$

R° is a subring which is open (as it contains the unit ball $R_{\leq 1}$) and the construction $R \mapsto R^\circ$ only depends on the topology of R.

Definition 3.1.3. *A K-Banach algebra R is uniform if R° is itself a bounded subset of R.*

Proposition 3.1.4. *Let $\pi \in K$ a pseudouniformizer. The following categories are equivalent:*

- *The category $\mathcal{C}$ of uniform Banach K-algebras R with continuous K-algebra maps.*

- *The category $\mathcal{D}_{tic}$ of π-adically complete and π-torsion free K°-algebras A with A totally integrally closed$^{(*)}$ in $A[\frac{1}{\pi}]$*

$(*)$ this means that given $f \in A[\frac{1}{\pi}]$ such that $f^{\mathbb{N}}$ lies in a finitely generated $A-$submodule of $A[\frac{1}{\pi}]$ then $f \in A$. This is a slightly more restrictive notion than integral closedness.

The functors between $\mathcal{C}$ and $\mathcal{D}_{tic}$ are $R \mapsto R^\circ$ and $A \mapsto A[\frac{1}{\pi}]$

Proof. We start by constructing the functor $\mathcal{C} \longrightarrow \mathcal{D}$. Let R be a uniform K-algebra. Then $R_{\leq 1} \subset R^\circ \subset R$, thus R° is an open subring. Besides, R° is bounded by assumption, so $R^\circ \subset R_{\leq r}$ for some $r > 0$. As R is a K-Banach algebra, we have $|\pi|^n \to 0$ as $n \to \infty$. In particular $\cap_n \pi^n R^n_{\leq r} = 0$. Therfore, R° is π-adically separated. Since R is complete for the metric topology, $R_{\leq r}$ is π-adically complete and so is its closed subgroup R° (since it is open).
Remains to show that R° is totally integrally closed in R. So let $f \in R$ such that $f^{\mathbb{N}}$ is contained in a finitely generated R°-submodule of R. Since $R = R^\circ[\frac{1}{\pi}]$, it follows that $f^{\mathbb{N}} \subset \frac{1}{\pi^k} R^\circ \subset \frac{1}{\pi^k} R_{\leq r}$ which shows that $f^{\mathbb{N}}$ is bounded so $f \in R^\circ$.
Since bounded elements are mapped to bounded elements via continuous K-linear maps, the extraction of R° from R is functorial so we obtain the desired functor F.

Conversely, let $A \in \mathcal{D}_{tic}$. We endow $R = A[\frac{1}{\pi}]$ with a Banach K-algebra structure by setting for $f \in R$

$$|f| = \inf\{\ |t| \mid f \in tA, \text{ for } t \in K\}$$

We check that this is indeed a K-algebra norm:
First, if $|f| = 0$, then $f \in \pi^n A$ for all $n \in \mathbb{N}$, by π-adic separatedness of A, we get $f = 0$.
Now, let $f \in tA$ and $g \in t'A$ for some $f, g \in R$ and $t, t' \in K$. Then $fg \in tt'A$, so $|fg| \leq |t||t'|$ passing to the inf we get $|fg| \leq |f||g|$. If we suppose that $|f| \leq |g|$, then we can take $|t| \leq |t'|$ and so $f + g \in t'A$ hence $|f + g| \leq |g| = \max(|g|, |f|)$.
For the completeness, a Cauchy sequence $(f_n)_{n\in\mathbb{N}} \in R$ is bounded, hence up to rescaling we can suppose $(f_n)_{n\in\mathbb{N}} \in A$ but by assumption A is π-adically complete hence complete for this metric, so we have the property.

Remark that we have $A \subset R_{\leq 1} \subset R^\circ$. We shall show that $A = R^\circ$; this will construct a functor $G : \mathcal{D}_{tic} \longrightarrow \mathcal{C}$ such that $F \circ G \simeq id$. So let $f \in R^\circ$. Then $f^{\mathbb{N}}$ is bounded; As $\{\pi^n A\}$ and $\{R_{\leq |t|}\}$ are confinal amongst each other, there must be some $c > 0$ such that $f^{\mathbb{N}} \subset \frac{1}{\pi^c} A$. Therfore, $f \in A$ as it is totally integrally closed in $A[\frac{1}{\pi}]$.

To finish the proof, we need to show that $G \circ F \simeq id$. This amounts to showing that given $R \in \mathcal{C}$; the given Banach norm $|\cdot|_{given}$ on R is equivalent to the one $|\cdot|_{R^\circ}$ coming from the construction above via $R = R^\circ[\frac{1}{\pi}]$. But the unit ball for $|\cdot|_{R^\circ}$ is exactly R° and we have $R_{|\cdot|_{given},\leq 1} \subset R^\circ$. On the other hand, since R is uniform, we have $R^\circ \subset R_{|\cdot|_{given},\leq r}$ for some $r > 0$. So we get the claim.

∎

3.2 Algebras over a Perfectoid Field

For the next two sections, we fix a perfectoid field K with tilt $K^\flat$. Let $\pi^\flat \in K^\flat$ be a pseudouniformizer such that $\pi := \pi^{\flat\sharp}$ satisfies $|p| \leq |\pi| < 1$. Then K°/π has characteristic p,and π has all p-power roots $\pi^{\frac{1}{p^k}} = (\pi^{\flat\frac{1}{p^k}})^\sharp$.

Definition 3.2.1. *Let A be a perfectoid K°-algebra. The set $A_* := \{f \in A[\frac{1}{\pi}] \mid K^{\circ\circ} f \in A\}$ is called the ring of almost elements of A.*

Lemma 3.2.2. *Let A be a K°-algebra such that π is a non-zero-divisor. Assume that $A \subset A[\frac{1}{\pi}]$ is integrally closed. Then*

1. *$\widehat{A} \subset \widehat{A}[\frac{1}{\pi}]$ is integrally closed.*
2. *$A_* \subset A[\frac{1}{\pi}]$ is integrally closed.*

Proof. First, we check that we can replace A by its separated quotient A/I where $I = \bigcap_{n\geq 0} \pi^n A$. If $g \in A$ such that $\pi g \in I$, then for all $n \geq 1$, we have $\pi g \in \pi^n A$ and since π is a non-zero-divisor, $g \in \pi^{n-1}A$, which shows that $g \in I$.
Now let $g \in A/I[\frac{1}{\pi}]$ be an element that satisfies a monic polynomial $h(X) \in A/I[X]$. Choose lifts $\tilde{g} \in A[\frac{1}{\pi}]$ and $\tilde{h}(X) \in A[X]$. Then $\tilde{h}(\tilde{g}) \in I[\frac{1}{\pi}] = I \subset A$. So $\tilde{g}$ is integral over A and thereby $\tilde{g} \in A$ and $g \in A/I$ as well. So we can reduce to the case where A i π-adically separated.
1) Let $g \in \widehat{A}[\frac{1}{\pi}]$ with g integral over $\widehat{A}$. Then $g = \pi^{-c}h$ for some $h \in \widehat{A}$ and $c \geq 0$. By integrality we can write

$$g^n = a_{n-1}g^{n-1} + a_{n-2}g^{n-2} + \cdots + a_0$$

for some $a_i \in \widehat{A}$. Multiplying by π^{cn} we get

$$h^n = \pi^c a_{n-1}h^{n-1} + \pi^{2c}a_{n-2}h^{n-2} + \cdots + \pi^{cn}a_0$$

By approximation, we can choose $h_0 \in A$ with $h_0 \equiv h \mod \pi^{cn}$ and $b_i \in A$ with $b_i \equiv a_i \mod \pi^{cn}$. So we can write

$$h_0^n = \pi^c b_{n-1}h_0^{n-1} + \pi^{2c}b_{n-2}h_0^{n-2} + \cdots + \pi_{cn}b_0 + \pi^{cn}d$$

for some $d \in A$. Dividing both sides by π^{cn}, we get that $g_0 = \pi^{-c}h_0 \in A[\frac{1}{\pi}]$ is integral over A and hence is in A. Thus $h_0 \in \pi^C A$. But since $h \equiv h_0 \mod \pi^{cn}$, we get that $h \in \pi^c\widehat{A}$ and so $g = \pi^{-c}h \in \widehat{A}$.
2) Let $g \in A[\frac{1}{\pi}]$ be an element integral over A_*. We get an equation

$$g^n = a_{n-1}g^{n-1} + a_{n-2}g^{n-2} + \cdots + a_0$$

with $a_i \in A_*$. Let $\epsilon \in K^{\circ\circ}$. Multiplying by ϵ^n we get

$$(\epsilon g)^n = a_{n-1}\epsilon(\epsilon g)^{n-1} + a_{n-2}\epsilon^2(\epsilon g)^{n-2} + \cdots + \epsilon^n a_0$$

Since $a_i \in A_*$, we have that $a_{n-i}\epsilon^i \in A$ for each $i \geq 1$. In particular, this shows that ϵg is integral over A and thereby an element of A. Since this is true for all $\epsilon \in K^{\circ\circ}$, we get that $g \in A_*$. ∎

Proposition 3.2.3. *In the context of proposition* 3.1.4*(but now we assume that* K *is perfectoid) the two categories mentioned in this are also equivalent to*

1. $\mathcal{D}_{ic}$ *of* t*-adically complete and* t*-torsion free* K°*-algebras* A *with* A *integrally closed in* $A[\frac{1}{\pi}]$ *and* $A \simeq A_*$.

2. $\mathcal{D}_{prc}$ *of* t*-adically complete and* t*-torsion free* K°*-algebras* A *with* A *p-root closed in* $A[\frac{1}{\pi}]$ *and* $A \simeq A_*$.

Proof. For $A \in \mathcal{D}_{tic}$ let us show that $A = A_*$. So let $f \in A[\frac{1}{\pi}]$ such that $\pi^{\frac{1}{p^n}} f \in A$ for all n. But this would imply that $f^{p^n} \in \frac{1}{\pi}A$ for all n; hence $f^{\mathbb{N}} \in \frac{1}{\pi}A$. By total integral closure, we get $f \in A$ as desired.
We have the inclusions $\mathcal{D}_{tic} \subset \mathcal{D}_{ic} \subset \mathcal{D}_{prc}$, so we only need to show that if $A \in \mathcal{D}_{prc}$ then it is totally integrally closed.
Let $f \in A[\frac{1}{\pi}]$ such that $f^{\mathbb{N}} \in \frac{1}{\pi^c}A$, in particular we have $(\pi^{\frac{c}{p^n}} f)^{p^n} \in A$; but A p-root closed in $A[\frac{1}{\pi}]$, so $\pi^{\frac{c}{p^n}} f \in A$. This is true for all n, hence $f \in A$. ∎

Corollary 3.2.4. *The category of uniform Banach* K*-algebras has all colimits.*

Proof. By proposition 3.2.3, it suffices to show the result in the category $\mathcal{D}_{ic}$.
Let $\langle (A_i)_{i\in I}, (f_{i,j})_{i\leq j}\rangle$ be a direct system where A_i are in $\mathcal{D}_{ic}$. Denote by A the π-adic completion of the underlying colimit of rings. Set A_{int} to be the integral closure of A in $A[\frac{1}{\pi}]$. We claim that $A_u := \big(\widehat{A_{int}}\big)_*$. Indeed by lemma 3.2.2 we have that $A_u \in \mathcal{D}_{ic}$. Plus suppose that we have a ring $B \in \mathcal{D}_{ic}$ and morphisms $\phi_i : A_i \to B$ compatible with the $f_{i,j}$. By the universal property of the colimit of rings, we get a map $\varinjlim_{i\in I} A_i \to B$. Since B is complete, it factors through a morphism $A \to B$ which induces a morphism $\phi : A[\frac{1}{\pi}] \to B[\frac{1}{\pi}]$. For an integral element $g \in A[\frac{1}{\pi}]$, let $P(X) \in A[X]$ be a monic polynomial such that $P(g) = 0$, then $\phi(P)\big(\phi(g)\big) = 0$ which by integral closedness of B shows that $\phi(\widehat{A_{int}}) \subset B$. Similarly, for an element $g \in (\widehat{A_{int}})_*$, we have $\phi(K^{\circ\circ}g) = K^{\circ\circ}\phi(g) \subset B$ and so $\phi(g) \in B$. This shows that ϕ restricts to a morphism $A_u \to B$ which is compatible with the $f_{i,j}$ and clearly unique. ∎

3.3 Tilting Correspondence for perfectoid K-algebras

Definition 3.3.1. *A Banach* K*-algebra* R *is perfectoid if* $R^\circ \subset R$ *is bounded, and the Frobenius map* $R^\circ/\pi \longrightarrow R^\circ/\pi$ *is surjective. With continuous morphisms as morphisms, this gives the category* $Perf_K$ *of perfectoid* K*-algebras.*

- Remark that for a perfectoid K-algebra R, the ring of bounded elements R° is an integral perfectoid ring.

Lemma 3.3.2. *Let* A *be a perfectoid* K°*-algebra. Then* $A = A_*$ *if and only if* $A^\flat = A^\flat_*$.

Proof.
($\Leftarrow$)Suppose that $A^\flat \simeq A^\flat_*$. By the tilting correspondence we get that $A = W(A^\flat)/\xi W(A^\flat)$ with $\xi = p + z[\pi^\flat]^p$ for some $z \in W(K^{\circ\flat})$.

We first show that if $\alpha \in W(A^\flat)$ verifies $[\pi^\flat]^{1-\frac{1}{p^n}} \mid \alpha \mod \xi$ for all n, then $[\pi^\flat] \mid \alpha \mod \xi$. Modding out by p, we get $\bar{\alpha} = {\pi^\flat}^{1-\frac{1}{p^n}}\bar{\alpha_n} + \bar{k_n}\bar{z}{\pi^\flat}^p$. Hence we get that ${\pi^\flat}^{1-\frac{1}{p^n}} | \bar{\alpha}$ for all n. Since $A^\flat \simeq A^\flat_*$, we get that $\pi^\flat | \bar{\alpha}$. Hence $[\pi^\flat] \mid \alpha \mod p$. Given that $\xi = p + z[\pi^\flat]^p$, we get that $[\pi^\flat] \mid \alpha \mod \xi$.
Now let $f \in A_*$. Write $f = \frac{\alpha}{\pi^c}$ with $c \geq 0$ and $\alpha \in A$. Since $\pi f \in A$ ie $\pi^{c-1} \mid \alpha$, we can suppose that $c = 1$. Using the isomorphism $A \simeq W(A^\flat)/\xi W(A^\flat)$ viewed as a K°-algebra via $K^\circ \simeq W(K^{\circ\flat})/\xi W(K^{\circ\flat})$, we stumble upon the previous situation. Hence we get that $\pi \mid \alpha$ and therefore $f \in A$, so $A = A_*$.

($\Rightarrow$) For the same reasons as before, we are reduced to show that if for some $\alpha \in A^\flat$, ${\pi^\flat}^{1-\frac{1}{p^n}} \mid \alpha$ for all n, then $\pi^\flat \mid \alpha$. Composing with the sharp map, this gives $\pi^{1-\frac{1}{p^n}} \mid \sharp\alpha$. Since $A = A_*$, we get that $\pi \mid \sharp\alpha$. Now given that the $\sharp$ map induces an isomorphism $\sharp : A^\flat/\pi^\flat A^\flat \xrightarrow{\simeq} A/\pi A$, we get that $\pi^\flat \mid \alpha$ as desired. ■

Theorem 3.3.3. *We have an equivalence of categories $Perf_K \simeq Perf_{K^\flat}$.*

Proof. By proposition 3.2.3, we have an equivalence of categories $\mathrm{Perf}_K \longleftrightarrow$ perfectoid K°-algebras in $\mathcal{D}_{prc}(K)$.
But by the second property in lemma 1.2.2, for a perfectoid K°-algebra, to be included in $\mathcal{D}_{prc}$ is equivalent to only have $A \simeq A_*$. But by lemma 3.3.2 we have that $A \simeq A_*$ if and only if $A^\flat \simeq A^\flat_*$.

$$\text{perfectoid } K^\circ\text{-algebras in } \mathcal{D}_{prc}(K) \longleftrightarrow \text{ perfectoid } K^{\flat\circ}\text{-algebras in } \mathcal{D}_{prc}(K^\flat)$$

To finish the proof, as in the first case we also have an equivalence of categories

$$\text{perfectoid } K^{\flat\circ}\text{-algebras in} \mathcal{D}_{prc}(K^\flat) \longleftrightarrow \mathrm{Perf}_{K^\flat}$$

■

Proposition 3.3.4. *The category of perfectoid K-algebras has all colimits.*

Proof. By theorem 3.3.3 and proposition 3.2.3, it suffices to show that the category of perfectoid ${K^\flat}^\circ$-algebras that are in $\mathcal{D}_{ic}$ admits all colimits. But the colimit (in the category of rings) of perfect rings is perfect and the operations of taking the integral closure, completeness and adjoining the almost elements preserve perfectness. By the proof of corollary 2, we get that in the category $\mathcal{D}_{ic}(K^\flat)$, the colimit of perfect rings is still perfect, and therefore the the colimit of perfectoid rings is still perfectoid. Hence the result. ■

Remark 3.3.5. *Notice by the construction of the colimit, we get that the colimit of perfectoid K-algebras is the π-adic completion of the colimit in the category of rings.*

Chapter 4

Adic spaces and perfectoid spaces

4.1 Adic spaces

4.1.1 Tate rings

Definition 4.1.1. *A topological ring A is called Tate if there exists an open subring A_0 such that the induced topology on A_0 is the t-adic topology for some $t \in A_0$ that becomes a unit in A. Any such A_0 is called a ring of definition and the element t is called a pseudo-uniformizer. The pair (A_0, t) is called couple of definition.*

- For any couple of definition (A_0, t), we have that $A = A_0[\frac{1}{t}]$. For if $f \in A$, the continuity of the multiplication shows that $t^n f \to 0$ and so we must have $t^n f \in A_0$ for some $n \gg 0$ (as A_0 is an open neighborhood of 0); therefore $f \in A_0[\frac{1}{t}]$.

- A subset $S \subset A$ is bounded if for a couple of definition (A_0, t), we have $S \subset t^{-n} A_0$ for some $n \geq 0$. If (A_1, s) is another couple of definition, since $\{s^n A_1\}_{n \in \mathbb{N}}$ is a basis of neighborhoods of 0, we have that $s^n A_1 \subset A_0$ for some $n \gg 0$, i.e., $A_1 \subset s^{-n} A_0$. But there exists a $k \geq 0$ such that $t^k s^{-n} \in A_0$ and thus $A_1 \subset t^{-k} A_0$ which shows that A_1 is bounded with respect to (A_0, t). Hence the definition of boundedness is independent of the choice of the couple of definition.

- An element $f \in A$ is powerbounded if the set $f^{\mathbb{N}}$ is bounded. The collection A° of all powerbounded elements is a subring of A containing all rings of definition of A. Conversely, for any $f \in A^\circ$ and (A_0, t) a couple of definition, the ring $A_0[f] \subset A$ is open and bounded and thus also a ring of definition. Therfore, A° is the union of all rings of definition of A.
  We have that A° is integrally closed in A. Indeed, let $f \in A$ be an integral element over A°, then it must be integral over some ring of definition A_0 of A and so the subring $A_0[f] \subset A$ is bounded and thus included in A°.
  We say that A is uniform if A° is bounded.

- An element $f \in A$ is topologically nilpotent if $f^n \to 0$. We denote by $A^{\circ\circ}$ the set of topologically nilpotent elements.
  Given a couple of definition (A_0, t), t is clearly topologically nilpotent. Conversely,

any $f \in A$ that is topologically nilpotent and a unit is a pseudo-uniformizer. Indeed if (A_0, t) is a couple of definition, up to replacing f by f^m for some $m \gg 0$, we can assume that $f \in tA_0$. But f becomes a unit in $A_0[\frac{1}{t}]$ so there exists a $g \in A_0$ such that $fg = t^m$ for some $m \gg 0$. Therfore, we have that $t^m A_0 \subset fA_0 \subset tA_0$ so that t and f induce the same topology and so f is a pseudo-uniformizer.

- We say that A is complete if A is complete in its topology; or equivalently, some (or any) ring of definition A_0 is complete for the t-adic topology for a pseudo-uniformizer $t \in A_0$.

4.1.2 Affinoid Tate rings

Definition 4.1.2. *An affinoid Tate ring is a pair (A, A^+) where A is a Tate ring and A^+ is an open integrally closed subring of A°. A morphism $(A, A^+) \to (B, B^+)$ of affinoid algebras is a continuous map $A \to B$ that carries A^+ into B^+.*

Definition 4.1.3. *(Adic Spectrum) Let (A, A^+) be an affinoid Tate ring. The adic spectrum Spa(A, A^+) is defined as the set of equivalence classes of valuations $x : A \to \Gamma \cup \{0\}$ (for varying Γ) such that*

1. $|f(x)| \leq 1$ *for* $f \in A^+$
2. x *is continuous, i.e.,* $x^{-1}\big(\Gamma_{\leq \gamma} \cup \{0\}\big)$ *is open for every* $\gamma \in \Gamma$.

We endow Spa(A, A^+) with the coarsest topology making the sets

$$\mathrm{Spa}(A, A^+)\langle \frac{f}{g} \rangle := \{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| \leq |g(x)| \neq 0\}$$

open for every $f, g \in A$.

Proposition 4.1.4. *Let (A, A^+) be an affinoid Tate ring and $U \subset$ Spa(A, A^+). We say that U is a rational subset of Spa(A, A^+) if and only if there exist $f_1, \ldots, f_n \in A$ that generate the unit ideal in A and $g \in A$ such that*

$$U = \mathrm{Spa}(A, A^+)\left\langle \frac{f_1, \ldots, f_n}{g} \right\rangle := \{\ x \in \mathrm{Spa}(A, A^+) \mid |f_i(x)| \leq |g(x)| \text{ for all } i\ \}$$

The rational subsets are closed under finite intersection and form a basis for the topology on Spa(A, A^+)

Remark 4.1.5.

- *for any $x \in$ Spa$(A, A^+)\left\langle \frac{f_1,\ldots,f_n}{g} \right\rangle$, we must have $|g(x)| \neq 0$ since otherwise, we would get $|f_i(x)| = 0$ for all i and thus $1 \in (f_i) \subset \mathfrak{p}_x$ which forces the equality $1 = |1| = 0$.*

- *Let $\pi \in A$ be a pseudo-uniformizer. Since π is a unit, we have $|\pi(x)| \neq 0$ for all $x \in Spa(A, A^+)$. So we are free to scale the parameters f_i and g by powers of π. In particular, we can choose the f_i and g to lie inside A^+. The condition that the f_i generate the unit ideal of A amount to the condition that the ideal of A^+ generated by the f_i's contains π^N for some $N \gg 0$ (i.e., the ideal is open). In this case we get*

$$Spa(A, A^+)\left\langle \frac{f_1, \dots, f_n}{g} \right\rangle = Spa(A, A^+)\left\langle \frac{f_1, \dots, f_n, \pi^N}{g} \right\rangle$$

Proposition 4.1.6. *Let (A, A^+) be an affinoid Tate ring such that A is a domain and π a pseudo-uniformizer. Let $x \in Spv(A, A^+)$ with valuation ring $A^+ \subset V \subset L := Frac\ A$. These facts are equivalent*

1. *$x \in Spa(A, A^+)$*
2. *$\forall g \in V \setminus \{0\}$, $\exists\ N \in \mathbb{N}$ such that $|\pi(x)^n| \leq |g(x)|\ \forall n \geq N$*
3. *V is π-adically separated*
4. *$L = V[\frac{1}{\pi}]$*
5. *$\exists\ \pi \in \mathfrak{p}_0 \subset V$ prime ideal of height 1*
6. *$\exists\ y \in Spa(A, A^+)$ such that $y \rightsquigarrow x$, $|\cdot|_y : A \longrightarrow \mathbb{R}_{\geq 0}$ and x, y define the same topology.*

Proof. 1) $\Rightarrow$ 2) Let $g \in V$, $\gamma = |g(x)|$. Since x is continuous $A_{\leq\gamma} := \{a \in A \mid |a(x)| \leq \gamma\}$ is open in A. Since $\pi^n \longrightarrow 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $\pi^n \in A_{\leq\gamma}$ ie $|\pi(x)^n| \leq |g(x)|$.
2) $\Rightarrow$ 3) taking $g \in \max V$ in 2) we get that $|\pi(x)| < 1$. Then, for all $g \in V$, take $N \in \mathbb{N}$ such that $|\pi(x)^N| \leq |g(x)| \Rightarrow |\pi(x)^{N+1}| < |g(x)| \Rightarrow \frac{g}{\pi^{N+1}} \notin V \Rightarrow g \notin \pi^{N+1}V$
2) $\Rightarrow$ 1) Let (A_0, π) be a ring of definition. $\exists\ m \in \mathbb{N}$ such that $\pi^m A_0 \subset A^+$ since A^+ is open. So for $\gamma \in \Gamma$ take $n \in \mathbb{N}$ such that $|\pi(x)^n| \leq \gamma$. Hence $\forall\ \alpha \in A_0$, $|\pi(x)^{n+m}\alpha(x)| \leq \gamma$ ie $\pi^{n+m}A_0 \subset A_{\leq\gamma}$ ie $A_{\leq\gamma}$ is open $\forall\gamma$ ie x is continuous.
3) $\Rightarrow$ 4) Let $g \in L \setminus V$ then $g^{-1} \in V$. By 3) there exists $N \in \mathbb{N}$ such that $g^{-1} \notin \pi^N V \Rightarrow \frac{g^{-1}}{\pi^N} \notin V \Rightarrow \pi^N g \in V \Rightarrow g \in V[\frac{1}{\pi}] \subset L$
4) $\Rightarrow$ 2) Let $g \in V \setminus \{0\}$, then $g^{-1} \in \frac{1}{\pi^n}V$ for some $n \gg 1 \Rightarrow \pi^n g^{-1} \in V \Rightarrow |\pi(x)^n| \leq |g(x)|$.
2) $\Leftrightarrow$ 5) First remark that A^+ being open and integrally closed, we have $\pi \in A^+ \subset V$. We also have that $\sqrt{(\pi)} \subset V$ is a non-zero prime ideal (proposition $A.0.9$).
5) $\Leftrightarrow \sqrt{(\pi)}$ is a minimal prime ideal of $V \Leftrightarrow \forall g \in V \setminus \{0\} \sqrt{(\pi)} \subset \sqrt{(g)} \Leftrightarrow \forall g \in V$, $\exists N \in \mathbb{N}$ such that $\pi^N \in (g) \Leftrightarrow \forall g \in V \setminus \{0\}$, $\exists N \in \mathbb{N}$ such that $|\pi(x)^n| \leq |g(x)|\ \forall n \geq N$.
5) $\Rightarrow$ 6) $V \subset W := V_{\mathfrak{p}_0}$ defines $y \in \mathrm{Spv}(L, A^+)$ such that $y \rightsquigarrow x$.
Fix $c \in (0, 1)$. We know that $\max W = \sqrt{(\pi)}$. We want to define a morphism of totally ordered groups

$$J : L^*/W^* \hookrightarrow \mathbb{R}_{>0}$$
$$\pi \mapsto c$$

Let $g \in \max W = \sqrt{(\pi)}$. W si π-adically separated, therefore $\forall n \in \mathbb{N}$, $\exists\ g(n) \in \mathbb{N}$ such that $g^n \in \pi^{g(n)}W$ but $g^n \notin \pi^{g(n)+1}W$. If we had such J we would get $c^{\frac{g(n)+1}{n}} \leq J(g) \leq c^{\frac{g(n)}{n}}$. So we define $J(g) = \inf c^{\frac{g(n)}{n}}$.
Now we show that x and y define the same topology on L. Since $V^* \subset W^*$ we have a surjective morphism of ordered groups $f : \Gamma_x = L^*/V^* \longrightarrow \Gamma_y = L^*/W^*$ verifying the commutative diagram

$$\begin{tikzcd} \Gamma_x \arrow[rr, "f"] & & \Gamma_y \subset \mathbb{R}_{>0} \\ & L^* \arrow[ul, "|\cdot|_x"] \arrow[ur, "|\cdot|_y"'] & \end{tikzcd}$$

with topologies defined by $\underline{x} : \{L_{\leq_x \gamma}\}_{\gamma \in \Gamma_x}$ and $\underline{y} : \{L_{\leq_y \tau}\}_{\tau \in \Gamma_y}$
We have $\forall \gamma \in \Gamma_x$, $L_{\leq_x \gamma} \subset L_{\leq_y f(\gamma)}$ and we know by 2) tha $\forall \gamma = |g(x)| \in \Gamma_x$, $\exists N \in \mathbb{N}$ such that $|\pi(x)^N| \leq \gamma$. Hence $L_{\leq_y c^{N+1}} = \pi^{N+1}W \subset \pi^N \max W \subset L_{\leq_x \gamma}$.
Therfore, x and y give the same topology.
6) $\Rightarrow$ 3) $y \in \text{Spa}(A, A^+) \Rightarrow W$ is π-adically separated $\Rightarrow V$ is π-adically separated.

■

Definition 4.1.7. *An affinoid field is a pair (L, L^+) such that*

1. *L is a non-Archimedean field.*
2. *$L^+ \subset L^\circ$ is an open valuation subring of L.*

Corollary 4.1.8.
To give $x \in Spa(A, A^+)$ is equivalent to giving $\gamma : (A, A^+) \longrightarrow (L, L^+)$ with (L, L^+) an affinoid field. γ being a continuous morphism that carries A^+ to L^+ such that Frac $\gamma(A)$ is dense in L and $\gamma(\pi)$ is a pseudo-uniformizer of L.

Proof.
($\Rightarrow$): Consider the prime ideal $\mathfrak{p} = \text{Supp } x \subset A$. x defines a continuous valuation on Frac $A/\mathfrak{p}$. Notice that $|\pi(x)| \neq 0$ since $\pi \in A^*$.
By the previous proposition $\exists\ y \rightsquigarrow x$ a continuous valuation $|\cdot|_y : \text{Frac } A/\mathfrak{p} \to \mathbb{R}_{>0}$. Set L to be the completion of Frac$A/\mathfrak{p}$ with respect to this valuation. Hence, L is a non-archimedain field.
We also have $A^+/(\mathfrak{p} \cap A^+) \subset R_x \subset R_y \subset \text{Frac } A/\mathfrak{p}$. So we set

$$L^+ := \text{ completion of } R_x \subset L^\circ = \text{ ring of integers of } L = \text{ completion of } R_y.$$

L^+ is open and is a valuation ring.
By construction we get the desired map $\gamma : A \longrightarrow L$ such that Frac $\gamma(A)$ is dense in L and $0 \neq \gamma(\pi) \in \max L^\circ$ so it is a pseudo-uniformizer.
The fact that Frac $\gamma(A)$ is dense in L ensures that L is unique.
($\Leftarrow$) Let $\gamma : A \longrightarrow L$ as in the statement of the corrolary. Since L^{circ} is $\gamma(\pi)$-adically separated, then so is L^+. By the previous proposition, we get that the valuation y on L defined by L^+ is continuous.

On the other hand, let (A_0, π) be a ring of definition. $\exists\ N \in \mathbb{N}$ such that $\pi^N A_0 \subset A^+$ since A^+ is open. So $\gamma(\pi^{N+h}A_0) \subset \gamma(\pi)^h L^+ \subset \gamma(\pi)^h L^\circ$ ie γ is continuous.
So composing both continuous maps $x : A \xrightarrow{\gamma} L \xrightarrow{y} \Gamma \cup \{0\}$ gives the desired element of $\mathrm{Spa}(A, A^+)$. ■

Proposition 4.1.9. *Let (A, A^+) be a complete affinoid Tate ring. Then*

1. $A^+ = \{f \in A \mid |f(x)| \leq 1,\ \forall\ x \in Spa(A, A^+)\}$
2. $A^* = \{g \in A \mid |g(x)| \neq 0,\ \forall\ x \in Spa(A, A^+)\}$
3. $Spa(A, A^+) = \emptyset \Leftrightarrow A = 0$

Proof. **1)** The inclusion of A^+ in $\{f \in A \mid |f(x)| \leq 1,\ \forall\ x \in \mathrm{Spa}(A, A^+)\}$ is clear by definition of the adic spectrum.
For the other inclusion, let $f \in A$ such $f \notin A^+$ and consider the ring $B = A^+[f^{-1}]$. Since A^+ is integrally closed, f^{-1} is not a unit in B (otherwise f would be integral over A^+) so there exist a maximal ideal $\mathfrak{m} \subset B$ such that $f{-1} \in \mathfrak{m}$. Let $\mathfrak{p} \subset \mathfrak{m}$ be a minimal prime ideal. By Proposition $A.0.3$, there exist a valuation ring $V \subset \mathrm{Frac}(B/\mathfrak{p})$ such that $\max(V) \cap B/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$. This gives a valuation $|\cdot|_V$ on B with support $\mathfrak{p}$.
Let π be a uniformizer of A, to extend the valuation on $A = A^+[\frac{1}{\pi}]$, we need to show that it is non-zero on π, i.e., $\pi \notin \mathfrak{p}$. So consider the multiplicative set $S = B \setminus \mathfrak{p}$, the injective map $A^+ \hookrightarrow A$ extends to an injective map $S^{-1}B \hookrightarrow S^{-1}A[f^{-1}]$ so that $S^{-1}A[f^{-1}] \neq 0$. Therefore there exist a non-zero prime ideal $\mathfrak{q}$ in $S^{-1}A[f^{-1}]$. Since $S \cap \mathfrak{q} = \emptyset$, we have that $\mathfrak{q} \cap B \subset \mathfrak{p}$ But by minimality of $\mathfrak{p}$, we have an equality. Since π is invertible in $S^{-1}A[f^{-1}]$, then $\pi \notin B\mathfrak{q} = \mathfrak{p}$ which is what we wanted.
Therefore, the valuation gives an element $x \in \mathrm{Spv}(A, A^+)$ such that $|f^{-1}| < 1$ and so $|f(x)| > 1$. So we need to modify x a bit so that it lies in $\mathrm{Spa}(A, A^+)$.
First, we show that π maps the maximal ideal of V. Indeed, for $m \gg 1$, we have $\pi^m f \in A^+$ and so $|\pi^m f| \leq 1$ but since $|f| > 1$ we must have $|\pi| < 1$, i.e., $\pi \in \max(V)$.
Next, we claim that $\bar{V} := V/I$, where $I := \bigcap_{n\in\mathbb{N}} \pi^n V$, is a valuation ring. For this, it suffice to show that I is a prime ideal of V and since V is a valuation ring, it amounts to check that it is a radical ideal. So let $g \in V$ such that $g^k \in I$ for some $k \in \mathbb{N}$. Then $g^k \in \pi^{nk}V$ for all n. Hence $\frac{g}{\pi^n} \in \mathrm{Frac}\ V$ has its k-th power in V and must thus lie in V. This gives $g \in \pi^n V$ for all n, so $g \in I$.
Therefore changing V with $\bar{V}$, we get a valuation in $\mathrm{Spa}(A, A^+)$ (by proposition 4.1.6) that satisfies $|f| > 1$ which shows the inclusion in the other direction.
2) One inclusion is clear. So assume $|f(x)| \neq 0\ \forall\ x \in \mathrm{Spa}(A, A^+)$. Consider π a pseudo-uniformizer of A, it is invertible in A, therefore up to a multiplication by a big power of π, we can assume that $f \in A^+$. We need to show that f is not contained in any prime ideal of A.
First notice that π lies in the Jacobson radical of A^+. Indeed, $\forall\ x \in A^+ \subset A^\circ$, $1 + x\pi$ is invertible in A^+ (its inverse is $\sum_{n=0}^{\infty}(-x\pi)^n$).
Now Let $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathfrak{p}^+ = \mathfrak{p} \cap A^+$. Since $\pi \notin \mathfrak{p}^+$ (π is invertible in A), and π lies in the Jacobson radical of A^+, there exists a prime ideal $\mathfrak{q}$ that strictly contains A^+. Then $(A^+/\mathfrak{p}^+)_\mathfrak{q}$ is a local ring strictly included in Frac $A^+/\mathfrak{p}^+$. So there is a valuation ring $V \subsetneq$ Frac $A^+/\mathfrak{p}^+$

that dominates $(A^+/\mathfrak{p}^+)_{\mathfrak{q}}$. Notice that the image of π in V is non-zero and is not invertible ($\pi \in \mathfrak{q}$).
By 1), we have that $\bar{V} = V/I$, where $I := \bigcap_{n\in\mathbb{N}} \pi^n V$, is a valuation ring. So consider the map $A^+ \longrightarrow \bar{V}$ where π is sent to a non-zero element. So we can extend it to a map $A = A^+[\frac{1}{\pi}] \longrightarrow \operatorname{Frac} \bar{V}$. The valuation given by $\bar{V}$ on $\operatorname{Frac} \bar{V}$ induces a valuation x on A such that $|a(x)| \leq 1$ for all $a \in A^+$ since A^+ is sent to $\bar{V}$ and such that $\bar{V}$ is π-adically separated by construction; so by the previous proposition, $x \in \operatorname{Spa}(A, A^+)$. Moreover, since $|f(x)| \neq 0$, $f \notin \operatorname{Supp}(x)$. But given that $\mathfrak{p}^+ \subset \operatorname{Supp}(x)$, and $f \in A^+$ (by assumption), we have that $f \notin \mathfrak{p}$ which we wanted to prove.

3) In the proof of 2) we saw that an element $\mathfrak{p} \in \operatorname{Spec}(A)$ gives rise to a valuation in $\operatorname{Spa}(A, A^+)$. Hence if $A \neq 0$, then $\operatorname{Spec}(A) \neq \emptyset$ so $\operatorname{Spa}(A, A^+) \neq \emptyset$. ■

4.1.3 Adic spaces

Lemma 4.1.10. *Let (A, A^+) be an affinoid Tate ring, $Y \subset$ Spa(A, A^+) a quasi-compact subset and $g \in A$ an element such that $|g(y)| \neq 0$ for all $y \in Y$. Then there exists an open neighborhood $0 \in V \subset A$ such that $|y(f)| \leq |y(g)|$ for all $f \in V$, $y \in Y$.*

Proof. Let (A_0, π) be a pair of definition. Given $y \in Y$, continuity of y implies that $\{f \in A \mid |y(f)| < |y(g)|\}$ is open in A, whence it contains $\pi^N A_0$ for large N. This shows that $Y \subset \bigcup_{N\geq 1} \operatorname{Spa}(A, A^+)\left\langle \frac{\pi^N}{g} \right\rangle$. Thus the quasi-compactness of Y implies that $Y \subset \operatorname{Spa}(A, A^+)\left\langle \frac{\pi^N}{g} \right\rangle$. For some fixed N. But then, for all $y \in Y$ and $a \in A_0$, we have

$$|y(\pi^{N+1}a)| = |y(\pi a)||y(\pi^N)| < |y(\pi^N)| \leq |y(g)|$$

So $V := \pi^{N+1} A_0$ works. ■

Lemma 4.1.11. *Let (A, A^+) be a complete affinoid Tate ring, let $f_1, \ldots, f_n \in A$ generating the unit ideal and let $g \in A$. Then there exists an open neighborhood $U \subset A$ of 0 such that for all $g' \in g + U$ and all $f_i' \in f_i + U$, the elements $f_1', \ldots, f_n'$ also generate an open ideal of A^+ and*

$$\mathit{Spa}(A, A^+)\left\langle \frac{f_1', \ldots, f_n'}{g'} \right\rangle = \mathit{Spa}(A, A^+)\left\langle \frac{f_1, \ldots, f_n}{g} \right\rangle$$

Proof. Let (A_0, π) be a pair of definition. For convenience of notation, write $f_0 = g$ and set

$$X_i = \operatorname{Spa}(A, A^+)\left\langle \frac{f_0, f_1, \ldots, f_n}{f_i} \right\rangle$$

for $i = 0, \ldots, n$. X_i is quasi-compact and $|x(f_i)| \neq 0$ for all $x \in X_i$; therefore the previous lemma shows that there exists $N \gg 0$ such that $|x(f_i)| > |x(a)|$ for all $x \in X_i$ and all $a \in \pi^N A_0$.
Let us show that $U := \pi^N A_0$ works. So let $f_i' \in f_i + \pi^N A_0$, $g' \in g + \pi^N A_0$. We must prove that

$$X_0 = \operatorname{Spa}(A, A^+)\left\langle \frac{f_1', \ldots, f_n'}{g'} \right\rangle$$

$\subseteq$: Let $x \in X_0$. Since $f_i' - f_i \in \pi^N A_0$, we have $|x(f_i)| > |x(f_i' - f_i)|$ (by the choice of N); also $x \in X_0$ so $|x(f_0)| \geq |x(f_i)|$. Combining these inequalities we get

$$|x(f_i')| = |x(f_i + (f_i' - f_i))| \leq |x(f_0)| = |x(f_0 + (f_0' - f_0))| = |x(f_0')|$$

so that $x \in \operatorname{Spa}(A, A^+)\left\langle \frac{f_1', \ldots, f_n'}{g'} \right\rangle$.

$\supseteq$: Let $x \in \operatorname{Spa}(A, A^+) \setminus X_0$. Write $|x(f_j)| = \max(|x(f_0)|, \ldots, |x(f_n)|)$, then $x \in X_j$; so we have $|x(f_j)| > |x(f_j - f_j')| \Rightarrow |x(f_j)| = |x(f_j')|$ and we get

$$|x(g')| \leq \max(|x(f_0)|, |x(f_0 - g')|) \leq |x(f_j)| = |x(f_j')|$$

So $x \notin \operatorname{Spa}(A, A^+)\left\langle \frac{f_1', \ldots, f_n'}{g'} \right\rangle$.

Now let us prove that $f_1', \ldots, f_n'$ also generate an open ideal of A^+. We can take N big enough so that there exist $\alpha_1, \ldots, \alpha_n \in A^+$ such that $\pi^{N-1} = \alpha_1 f_1 + \cdots + \alpha_n f_n$. Substituting f_i by $f_i' + (f_i - f_i')$ we get $\pi^{N-1} = \alpha_1 f_1' + \cdots + \alpha_n f_n' + \pi^N k$. Since A is complete $1 - \pi k$ is invertible (this is the only point where we use that A is complete). So we get that $\pi^{N-1} \in (f_i', \ldots, f_n')$. ∎

Proposition 4.1.12. *Let (A, A^+) be an affinoid Tate ring. The canonical map $g : Spa(\widehat{A}, \widehat{A^+}) \longrightarrow Spa(A, A^+)$ (given by extending a valuation by uniform continuity) is a homeomorphism identifying rational subsets.*

Proof. Clearly the map g is bijective. So let $U \in \operatorname{Spa}(\widehat{A}, \widehat{A^+})$ a rational subset. We need to show that $g(U)$ is rational. Write $U = \operatorname{Spa}(\widehat{A}, \widehat{A^+})\left\langle \frac{f_1, \ldots, f_n}{g} \right\rangle$. By the previous lemma, we can choose $f_1, \ldots, f_n$ to be in A. Now since the $f_1, \ldots, f_n$ generate an open ideal of $\widehat{A^+}$ there exists $N \geq 0$ such that $|x(\pi^N)| \leq |x(g)|$ for all $x \in U$. Therfore, the rational subset $V := \operatorname{Spa}(A, A^+)\left\langle \frac{f_1, \ldots, f_n, \pi^N}{g} \right\rangle$ verifies $V = g(U)$ as desired. ∎

Theorem 4.1.13. *Let (A, A^+) be an affinoid Tate ring. Let $U \subset X := Spa(A, A^+)$ be a rational subset. Then there exists a unique complete affinoid Tate (A, A^+)-algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ satisfying:*

1. *The map $Spa((\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \longrightarrow Spa(A, A^+)$ has image contained in U.*
2. *if (S, S^+) is a complete affinoid Tate (A, A^+)-algebra, then the map $Spa(S, S^+) \longrightarrow Spa(A, A^+)$ factors through $Spa(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$.*

Moreover the map $Spa(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \longrightarrow U$ is a homeomorphism identifying rational subsets of the source with rational subsets of X contained in U.

Proof. Choose a couple of definition (A_0, π). We can write $U = \operatorname{Spa}(\widehat{A}, \widehat{A^+})\left\langle \frac{f_1, \ldots, f_n}{g} \right\rangle$ for $f_i, g \in A_0$ with $\pi^N \in (f_i) \subset A_0$ for some $N \geq 0$. Let $B_0 := A_0[\frac{f_i}{g}] \subset A[\frac{1}{g}]$ and set $B := A[\frac{1}{g}]$. Since the ideal $(f_1, \ldots, f_n)$ contain a power of π, we have $B = B_0[\frac{1}{\pi}]$. We view B as a Tate A-algebra with couple of definition (B_0, π). Let B^+ be the integral closure of the subring of B generated by $A^+[\frac{f_i}{g}]$. Then (B, B^+) is an affinoid (A, A^+)-algebra. We define

$(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to be its completion.
The image of $\mathrm{Spa}((\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \longrightarrow \mathrm{Spa}(A, A^+)$ has image contained in U: for any $x \in \mathrm{Spa}((\mathcal{O}_X(U), \mathcal{O}_X^+(U))$, $|\frac{f_i(x)}{g(x)}| \leq 1$ as $\frac{f_i}{g} \in \mathcal{O}_X^+(U)$.
Now let (S, S^+) be any complete affinoid (A, A^+)-algebra such that the induced map $\mathrm{Spa}(S, S^+) \longrightarrow \mathrm{Spa}(A, A^+)$ has image contained in U. For any $x \in Spa(S, S^+)$, we have $|f_i(x)| \leq |g_i(x)| \neq 0$ by hypothesis. Proposition 4.1.9 shows that $g \in S^*$ and $\frac{f_i}{g} \in S^+$. Since $S^+ \subset S^\circ$ and S° being the direct limit of all rings of definition of C, we can choose a ring of definition $S_0 \subset C$ that contains the image of A_0 as well as the elements $\frac{f_i}{g} \in S$. This gives a map $B_0 \to S_0$ that induces a map $B \to S$ of tate A-algebras by inverting π. Passing to the π-adic completion, we get a map $\mathcal{O}_X(U) \to S$ of Tate A-algebras. As $\frac{f_i}{g} \in S^+$ and S^+ is integrally closed, the map $B \to S$ carries B^+ into C^+ and hence the map $\mathcal{O}_X(U) \to S$ carries $\mathcal{O}_X^+(U)$ into S^+ giving the desired map $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (S, S^+)$ of complete affinoid (A, A^+)-algebras.
For the last assertion, by proposition 4.1.12, it suffices to show that $\Psi : Y := \mathrm{Spa}(B, B^+) \to U$ is a homeomorphism preserving rational subsets. The injectivity is clear as $A \to B$ is a localization. For the surjectivity, recall that giving $x \in \mathrm{Spa}(A, A^+)$ is equivalent to giving a map $\gamma : (A, A^+) \to (L, L^+)$ where (L, L^+) is an affinoid field such that Frac $\gamma(A)$ is dense in L. Now for $x \in U$, we have $\gamma(g) \in L^*$ so the map factors through B and we have $\gamma(\frac{f_i}{g}) \in L^+$ so B^+ is sent into L^+, this gives the preimage of x in Y.
We need to show that Ψ carries rational subsets of Y into rational subsets of X contained in U. For this, consider V a rational subset of Y. We can write $V = Y\langle\frac{b_1,\dots,b_n}{c}\rangle$ for some $b_i \in B_0$ generating the unit ideal of B. As $B_0 \subset A_0[\frac{1}{g}]$, and as g is a unit of B, we can rescale the b_i, c by a power of g so that they lift to some $\tilde{b_i}, \tilde{c}$ in A_0. Thus we have $\Psi(U) = X\langle\frac{\tilde{b_1},\dots,\tilde{b_n}}{\tilde{c}}\rangle$. Since U is quasi-compact, then so is its image. Using lemma 4.1.10 as for all $x \in \Psi(U)$, $|x(\tilde{c})| \neq 0$, there exists $N \geq 1$ such that $|x(\pi^N)| \leq |x(\tilde{c})|$ for all $x \in \Psi(U)$. Therefore, $\Psi(U) = X\langle\frac{\tilde{b_1},\dots,\tilde{b_n},\pi^N}{\tilde{c}}\rangle$ is a rational subset. ∎

Definition 4.1.14. *(Stalks) Let (A, A^+) be an affinoid Tate ring, and let $x \in X =$Spa(A, A^+). We define the stalks*

$$\mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U) \ \text{and}\ \mathcal{O}_{X,x}^+ := \varinjlim_{U \ni x} \mathcal{O}_X(U)^+$$

where both direct limits run through the rational open sets U containing x and are computed in the category of rings.

Proposition 4.1.15. *Let (A, A^+) be an affinoid Tate ring with a pseudo-uniformizer $t \in A^+$, and let $x \in X =$Spa(A, A^+).*

1. *The valuation $f \mapsto |f(x)|$ extends to the stalk $\mathcal{O}_{X,x}$ and we have $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x} \mid |f(x)| \leq 1\}$.*
2. *The stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal given by $\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \mid |f(x)| = 0\}$.*
3. *The stalk $\mathcal{O}_{X,x}^+$ is local with maximal ideal $\{f \in \mathcal{O}_{X,x} \mid |f(x)| < 1\}$ (in particular $\mathfrak{m}_x$ is an ideal both in $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^+$).*

4. *Write $k(x)$ for the residue field of $\mathcal{O}_{X,x}$ and let $k^+(x) \subset k(x)$ be the image of $\mathcal{O}^+_{X,x}$. The valuation ring corresponding to the induced valuation on $k(x)$ is $k^+(x)$ and we have isomorphisms $\left(\widehat{\mathcal{O}_{X,x}}, \widehat{\mathcal{O}^+_{X,x}}\right) \xrightarrow{\sim} \left(\widehat{k(x)}, \widehat{k^+(x)}\right) \xrightarrow{\sim} \left(L_x, L_x^+\right)$ where (L_x, L_x^+) is the affinoid field corresponding to x in corollary 4.1.8 and where the completion is taken with respect to the t-adic topology.*

Proof. **1)** Let U be an open subset of X containing x. Using the universal property of theorem 4.1.13, there exists a unique map $\left(\mathcal{O}_X(U), \mathcal{O}^+_X(U)\right) \to \left(L_x, L_x^+\right)$ of affinoid Tate $(A, A^+)-$algebras. By passage to the limit, we get a map $\mathcal{O}_{X,x} \to L_x$ sending $\mathcal{O}^+_{X,x}$ to L^+_x. This map induces the desired valuation on $\mathcal{O}_{X,x}$ and we see that $\mathcal{O}^+_{X,x} \subset \{f \in \mathcal{O}_{X,x} \mid |f(x)| \leq 1\}$. For the other inclusion, let $\overline{f} \in \mathcal{O}_{X,x}$ be such that $|\overline{f}(x)| \leq 1$, then we can represent it by some $f \in \mathcal{O}_X(U)$ with $|f(x)| \leq 1$. But then $x \in V = U\langle \frac{1,f}{1} \rangle \subset U$ and so $f \in \mathcal{O}^+_X(V)$; thus $\overline{f} \in \mathcal{O}^+_{X,x}$.
2) Let $g \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x$. We need to check that g is invertible.
We have that $|g(x)| \neq 0$ so there exists an $n \gg 0$ such that $|g(x)| \geq t^n$. Let U be a rational open set such that $g \in \mathcal{O}_X(U)$, then $V = U\langle \frac{t^n}{g} \rangle$ is a rational open subsets and g is invertible in $\mathcal{O}_X(V)$, and thus also in $\mathcal{O}_{X,x}$.
3) We need to check that any $g \in \mathcal{O}^+_{X,x}$ outside of the ideal $\{f \in \mathcal{O}_{X,x} \mid |f(x)| < 1\}$ is invertible. Let U be a rational open set such that $g \in \mathcal{O}_X(U)$. By the choice of g we have $|g(x)| = 1$, but then $x \in V = U\langle \frac{1}{g} \rangle$ and g is invertible in $\mathcal{O}^+_X(V)$. Thus g is invertible in $\mathcal{O}^+_{X,x}$.
4) It is clear that $\mathcal{O}_{X,x}$ induces a valuation on $k(x)$ and that the corresponding valuation ring is $k^+(x)$. Moreover, by 2., the map $A \to \mathcal{O}_{X,x} \to L_x$ factors through $\mathrm{Frac}(A/\mathfrak{p}_x)$ (where $\mathfrak{p}_x$ is the support of x). But by construction of L_x, the image of $\mathrm{Frac}(A/\mathfrak{p}_x)$ is dense, then so is the image of $\mathfrak{O}_{X,x}$. And thus by t-adic completion, we get $\widehat{\mathcal{O}_{X,x}} \to L_x$ is surjective. Now since $\mathrm{Ker}\left(\mathcal{O}_{X,x} \to L_x\right) = \mathfrak{m}_x \subset \bigcap_{n \geq 0} t^n \mathcal{O}_{X,x}$ (t is invertible in $\mathcal{O}_{X,x}$) and so after taking t-adic completion, we get that $\widehat{\mathcal{O}_{X,x}} \to L_x$ is an isomorphism. Now since the maps discussed above are of affinoid Tate rings, we get the desired result. ■

Definition 4.1.16. *(Structure presheaf)*
Let (A, A^+) be an affinoid Tate ring. We define the structure presheaf $\mathcal{O}_X$ on X =Spa(A, A^+) by setting for an arbitrary open set $W \subset X$

$$\mathcal{O}_X(W) := \varprojlim_{U \subset W} \mathcal{O}_X(U) \text{ and } \mathcal{O}^+_X(W) := \varprojlim_{U \subset W} \mathcal{O}^+_X(U)$$

where the directs limits are taken over all rational open subsets U contained in W; and are computed in the category of rings.

Since rational open subsets form a basis for the topology, the stalks defined in definition 13 agree with the stalks of the presheafs $\mathcal{O}_X$ and $\mathcal{O}^+_X$. In particular, for $x \in W$, we get a valuation $f \mapsto |f(x)|$ on $\mathcal{O}_X(W)$ and we have

$$\mathcal{O}^+_X(W) = \{f \in \mathcal{O}_X(W) \mid |f(x)| \leq 1 \ \text{ for all } x \in W\}$$

This shows that the presheaf $\mathcal{O}^+_X$ is completely determined by the data of $\mathcal{O}_X$ and of a valuation on the stalk.

It is not true in general that $\mathcal{O}_X$ is a sheaf. One could be tempted to take the sheafification but in that case we lose the universal property in theorem 4.1.13. This motivates the next definition.

Definition 4.1.17. *An affinoid Tate ring is sheafy if the structure presheaf $\mathcal{O}_X$ is a sheaf.*

Definition 4.1.18. *(Adic Spaces)*
An adic space is a pair $\big(X, \mathcal{O}_X, \{v_x\}_{x\in X}\big)$ where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings and such that $\big(X, \mathcal{O}_X\big)$ is a locally ringed space, and v_x a valuation on the stalk $\mathcal{O}_{X,x}$ for every $x \in X$. Moreover, we want that it is locally of the form $Spa(A, A^+)$ for some sheafy affinoid pair (A, A^+).

4.2 Perfectoid Spaces

4.2.1 Perfectoid affinoid algebras

Definition 4.2.1. *A perfectoid affinoid K-algebra is a pair (A, A^+) such that*

1. *(A, A^+) is an affinoid K-algebra*
2. *A is a perfectoid K-algebra.*

Lemma 4.2.2. *Given a perfectoid K-algebra A, it is equivalent to give*

1. *$A^+ \subset A^\circ$ open integrally closed subring*
2. *an integrally closed subring $\overline{A^+} \subset A^\circ/\mathfrak{m}A^\circ$*

Proof. Given A^+ open integrally closed, there exists $N \in \mathbb{N}$ such that $\pi^N A^\circ \in A^+$; therefore for all $f \in A^\circ$, $(\pi^{\frac{1}{p^n}} f)^{Np^n} \in A^+ \Rightarrow \pi^{\frac{1}{p^n}} f \in A^+$. So we have the inclusion $\mathfrak{m}A^\circ \subset A^+$. We thus define $\overline{A^+} := A^+/\mathfrak{m}A^\circ$. We prove that $\overline{A^+}$ is integrally closed. For this, take $\overline{P}(X) \in \overline{A^+}[X]$ monic such that $\overline{P}(\overline{g}) = 0$ for some $\overline{g} \in A^\circ/\mathfrak{m}A^\circ$. Lifting we get $P(X) \in A^+[X]$ monic such that $P(g) \in \mathfrak{m}A^\circ \subset A^+$, so $Q(X) = P(X) - P(g)$ is monic and verifies $Q(g) = 0$; hence $g \in A^+ \Rightarrow \overline{g} \in \overline{A^+}$.

Vice versa, given $\overline{A^+} \subset A^\circ/\mathfrak{m}A^\circ$, let $A^+ := \{g \in A^\circ \mid \overline{g} \in \bar{A}^+\} \subset A^\circ$. A^+ is open as A° is open and we have for every $g \in A^\circ$ such that $P(g) = 0$ for $P(X) \in A^+[X]$ monic, $\overline{P}(\overline{g}) = 0 \Rightarrow \overline{g} \in \overline{A^+} \Rightarrow g \in A^+$ so A^+ is integrally closed. ■

Proposition 4.2.3. *We have an equivalence of categories*

$$\text{Perfectoid affinoid } K\text{-algebras} \longleftrightarrow \text{ Perfectoid affinoid } K^\flat\text{-algebras}$$
$$(R, R^+) \longleftrightarrow (R^\flat, R^{\flat,+})$$

such that $R \to R^\flat$ is given by the equivalence of $K/K^\flat$-algebras. Moreover, R^+ is an integral perfectoid ring and we have $R^{\flat,+} = R^{+,\flat}$.

Proof. We have an isomorphism $R^\circ/\pi R^\circ \simeq R^{\flat\circ}/\pi^\flat R^{\flat\circ}$. Since the image of $(\pi^{\frac{1}{p^\infty}})$ in $R^\circ/\pi R^\circ$ is its nilradical, and the same is true for $(\pi^{\flat\frac{1}{p^\infty}})$, we get that $R^\circ/\mathfrak{m}R^\circ \simeq R^{\flat\circ}/\mathfrak{m}^\flat R^{\flat\circ}$. We define $R^{\flat,+}$ via the pullback square

$$\begin{array}{ccc} R^{\flat,+} & \longrightarrow & R^{\flat\circ} \\ \downarrow & & \downarrow \\ R^+/\mathfrak{m}R^\circ & \longrightarrow & R^\circ/\mathfrak{m}R^\circ \end{array}$$

By lemma 4.2.2 applied for the perfectoid field $K^\flat$, we have that $R^{\flat,+}$ is an open integrally closed subring of $R^{\flat^\circ}$. This lemma also shows that we cover all open integrally closed subrings of $R^{\flat^\circ}$ which gives the equivalence
Now we prove that R^+ is an integral perfectoid ring. Since open and thus closed in R°, we get that R^+ is π-adically complete. So it remains to show that $R^+/\pi^{\frac{1}{p}} \xrightarrow{\phi} R^+/\pi$ is an isomorphism. But since $R^\circ/\mathfrak{m}R^\circ$ is perfect, the map is surjective. Injectivity follows by integral closedness of A^+.
We can thus take the tilt of R^+ giving us the commutative diagram

$$\begin{array}{ccc} R^{+,\flat} & \longrightarrow & R^{\flat\circ} \\ \downarrow & & \downarrow \\ R^+/\mathfrak{m}R^\circ & \longrightarrow & R^\circ/\mathfrak{m}R^\circ \end{array}$$

which is a pullback square since $R^+/\mathfrak{m}R^\circ$ is perfect. This gives us the desired equality $R^{+,\flat} = R^{\flat,+}$.

■

4.2.2 Tilting rational subsets

Fix a perfectoid field K, write $\mathfrak{m} \subset K^\circ$ and $\mathfrak{m}^\flat \subset K^{\circ\flat}$ for the corresponding maximal ideals. Choose a unifomizer $\pi^\flat \in K^\flat$, and let $\pi = \pi^{\flat\sharp}$. Fix a perfectoid affinoid K-algebra (R, R^+) with tilt $(R^\flat, R^{\flat+})$. Write $X := \mathrm{Spa}(R, R^+)$ and $X^\flat := \mathrm{Spa}(R^\flat, R^{\flat+})$ for the attached adic spectra.
Recall that we have a continuous multiplicative map $\sharp : R^\flat \to R$. For any $x \in X$, $x : R \to \Gamma \cup \{0\}$, the composition $R^\flat \xrightarrow{\sharp} R \xrightarrow{x} \Gamma \cup \{0\}$ gives a point $x^\flat$ in $X^\flat$. Denote this map by $\Phi : X \to X^\flat$.

Lemma 4.2.4. *Let A,B be perfectoid K°-algebra and let $f : B \to A$ be a morphism of K algebras. Then f is almost an isomorphism if and only if the induced map of perfectoid Tate rings $\tilde{f} : A[\frac{1}{\pi}] \to B[\frac{1}{\pi}]$ is an isomorphism.*

Proof. $(\Rightarrow)$ Suppose that f is an almost isomorphism. Then the kernel and cokernel are in particular killed by π, whence $A[\frac{1}{\pi}] \to B[\frac{1}{\pi}]$ is an isomorphism.
$(\Leftarrow)$ Suppose that we have an isomorphism $\tilde{f} : A[\frac{1}{\pi}] \xrightarrow{\simeq} B[\frac{1}{\pi}] := R$. We may view $A \subset B$ as integral perfectoid subrings of definition of R. That and the fact that A is p-closed, gives $R^{\circ\circ} \subset A$; so $\pi^{\frac{1}{p^n}} B \subset A$ for all $n \in \mathbb{N}^*$. ■

Proposition 4.2.5. *Let $f_1, \dots, f_n, g \in R^{+\flat}$ where f_n is a power of $\pi^\flat$. Let C be the $R^{+\flat}$-subalgebra of $R^{+\flat}[\frac{1}{g}]$ generated by $f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}}$ for $i = 1, \dots, n$ and $k \geq 1$; similarly, let B be the R^+-subalgebra of $R^+[\frac{1}{g^\sharp}]$ generated by the $f_i^{\sharp\frac{1}{p^k}}/g^{\sharp\frac{1}{p^k}}$ for $i = 1, \dots, n$ and $k \geq 1$. Then*

1. *The kernel of the surjection*

$$\psi^\flat : R^{+\flat}[X_i^{\frac{1}{p^\infty}}] \longrightarrow C, \quad X_i^{\frac{1}{p^k}} \mapsto f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}}$$

is almost the ideal generated by $g^{\frac{1}{p^k}}X_i^{\frac{1}{p^k}} - f_i^{\frac{1}{p^k}}$ for $i = 1, \dots, n$ and $k \geq 1$.

2. *Similarly, the kernel of the surjection*

$$\psi : R^{+}[X_i^{\frac{1}{p^\infty}}] \longrightarrow B, \quad X_i^{\frac{1}{p^k}} \mapsto f_i^{\sharp\frac{1}{p^k}}/g^{\sharp\frac{1}{p^k}}$$

is almost the ideal generated by $g^{\sharp\frac{1}{p^k}}X_i^{\frac{1}{p^k}} - f_i^{\sharp\frac{1}{p^k}}$ for $i = 1, \dots, n$ and $k \geq 1$.

3. *The $\pi^\flat$-adic completion $\widehat{C}$ is an integral perfectoid $R^{+\flat}$ algebra.*
Let $\widehat{C}^\sharp$ be its intilt with corresponding generic fiber $\widehat{C}^\sharp[\frac{1}{\pi}]$ (a perfectoid Tate R-algebra).

4. *There exists a unique continuous map of R-algebras $\widehat{B}[\frac{1}{\pi}] \to \widehat{C}^\sharp[\frac{1}{\pi}]$; it is an isomorphism (therefore $\widehat{B}[\frac{1}{\pi}]$ is a pefrectoid K-algebra); it restricts to an injective almost surjection $\widehat{B} \hookrightarrow \widehat{C}^\sharp$ and it sends $f_i^{\sharp\frac{1}{p^k}}/g^{\sharp\frac{1}{p^k}}$ to $f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}}$ for all $i = 1, \dots, n$ and $k \geq 1$.*

5. *$\widehat{C}^\sharp$ is integral over $\widehat{B}$.*

6. *$\widehat{C}$ is integral over its subring $\widehat{R^{+\flat}[\frac{f_i}{g}]}$ and the difference between the rings is killed by a power of $\pi^\flat$; similarly, $\widehat{B}$ is integral over its subring $\widehat{R^{+}[\frac{f_i^\sharp}{g^\sharp}]}$ and the difference between the two rings is killed by a power of π.*

Proof. **3)** C is obviously perfect and $\pi^\flat$ is a non-zero divisor of it; therefore its completion $\widehat{C}$, equipped with the $\pi^\flat$-adic topology, is an integral perfectoid $R^{+\flat}$-algebra.

2) Let $J \subset R^+[X_i^{\frac{1}{p^\infty}}]$ be the ideal generated by the given elements; we obviously have the inclusion $J \subset \operatorname{Ker}\psi$ and we want to show that is an almost equality. Since the Frobenius map acts isomorphically on both J and $\operatorname{Ker}\psi$, it is enough to show that $\operatorname{Ker}\psi/J$ vanishes after inverting $\pi^\flat$, i.e., that

$$R^{+\flat}[\frac{1}{\pi^\flat}][X_i^{\frac{1}{p^\infty}}]/\langle g^{\frac{1}{p^k}}X^{\frac{1}{p^k}} - f_i^{\frac{1}{p^k}} : i, k\rangle \to C[\frac{1}{\pi^\flat}]$$

is injective. Since f_n is a power of $\pi^\flat$, f_n is invertible on both sides and so g is also invertible on both sides thanks to the relation $gX_n - f_n$. Hence we can rewrite both terms

$$R^{+\flat}[\frac{1}{\pi^\flat}, \frac{1}{g}][X_i^{\frac{1}{p^\infty}}]/\langle X^{\frac{1}{p^k}} - f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}} : i, k\rangle \longrightarrow C[\frac{1}{\pi^\flat}] = R^{+\flat}[\frac{1}{\pi^\flat}, \frac{1}{g}]$$

which is clearly an isomorphism.
2) and **4)**: The untilts $(f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}})^\sharp \in \widehat{C}^\sharp$ satisfy

$$g^{\sharp\frac{1}{p^k}}(f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}})^\sharp = (g^{\frac{1}{p^k}}((f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}}))^\sharp = f_i^{\frac{1}{p^k}\sharp} = f_i^{\sharp\frac{1}{p^k}}$$

so g is invertible in $\widehat{C}^\sharp[\frac{1}{\pi}]$ and thus there is a unique map of R^+ algebras $e : B \to \widehat{C}^\sharp[\frac{1}{\pi}]$; it sends $f_i^{\sharp\frac{1}{p^k}}/g^{\sharp\frac{1}{p^k}}$ to $(f_i^{\frac{1}{p^k}}/g^{\frac{1}{p^k}})^\sharp$ and has image in $\widehat{C}^\sharp$. Taking π-adic completion extends this map to $\widehat{e} : \widehat{B} \to \widehat{C}^\sharp$ to obtain the desired map.
Now consider the composition

$$e \circ \psi : R^+[X_i^{\frac{1}{p^\infty}}]/J \xrightarrow{\psi} B \xrightarrow{e} \widehat{C}^\sharp$$

whose reduction modulo π identifies with the reduction modulo $\pi^\flat$ of $\psi^\flat$. The latter map is an almost isomorphism by 1), whence $e \circ \psi$ is an almost isomorphism modulo π. Since $R^+[X_i^{\frac{1}{p^\infty}}]/J$ is π-adically separated and B is π-torsion free, we get that ψ is almost injective; this proves 2).
Now since $e \circ \psi$ is an almost isomorphism modulo π and ψ is surjective, we get that e is an almost isomorphism modulo π. Given that $\widehat{B}/(\pi) \simeq B/(\pi)$, the induced map $\widehat{e} : \widehat{B} \to \widehat{C}^\sharp$ is an almost isomorphism modulo π. Since $\widehat{B}$ is π-adically separated and $\widehat{C}^\sharp$ is π-adically complete, we get that $\widehat{e}$ is an almost isomorphism. Therfore, $\widehat{B}[\frac{1}{\pi}] \to \widehat{C}^\sharp[\frac{1}{\pi}]$ is an isomorphism and since $\widehat{B}$ is π-torsion free, it restricts to an injection almost surjection $\widehat{B} \hookrightarrow \widehat{C}^\sharp$.

5) Let B' be the integral closure of $\widehat{B}$ in $\widehat{C}^\sharp$. Since $\widehat{C}^\sharp$ is pth-root closed in the perfectoid tate ring $\widehat{B}[\frac{1}{\pi}]$, B' is also pth-root closed in $\widehat{B}[\frac{1}{\pi}]$ so it is integral perfectoid. But B' contains $f_i^{\sharp\frac{1}{p^k}}/g^{\sharp\frac{1}{p^k}}$ for all $k \geq 0$ so $B'^\flat$ contains C, whence its completeness forces $B'^\flat = \widehat{C}$; the tilting correspondence gives $B' = \widehat{C}^\sharp$.

6) By assumption we have $f_n = \pi^N$ for some $N \in \mathbb{N}$. Thus we have

$$\pi^{\flat Nn}\prod_{i=1}^{n}(\frac{f_i}{g})^{\frac{1}{p^{a_i}}} = \prod_{i=1}^{n}\pi^{\flat N}(\frac{f_i}{g})^{\frac{1}{p^{a_i}}} = \prod_{i=1}^{n} f_i^{\frac{1}{p^{a_i}}} g^{1-\frac{1}{p^{a_i}}}\frac{f_n}{g} \in R^{+\flat}[\frac{f_i}{g}]$$

Therfore, $\pi^{\flat Nn}C \subset R^{+\flat}[\frac{f_i}{g}] \subset C$, taking the $\pi^\flat$-adic completion we get the inclusion $\pi^{\flat Nn}\widehat{C} \subset \widehat{R^{+\flat}[\frac{f_i}{g}]} \subset \widehat{C}$. Since C is integral over $R^{+\flat}[\frac{f_i}{g}]$, $\widehat{C}$ is integral over $R^{+\flat}[\frac{f_i}{g}]$. Indeed, let $f \in \widehat{C}$ and take $\tilde{f} \in C$ such that $f - \tilde{f} \in \pi^{\flat Nn}\widehat{C}$. There exists a monic polynomial $P(X) \in R^{+\flat}[\frac{f_i}{g}][X]$ such that $P(\tilde{f}) = 0$; thus $P(f) \in \pi^{\flat Nn}\widehat{C} \subset \widehat{R^{+\flat}[\frac{f_i}{g}]}$ so that f is the zero of the monic polynomial $Q(X) := P(X) - P(f) \in \widehat{R^{+\flat}[\frac{f_i}{g}]}$.
The exact same argument works on the untilted side. ■

Corollary 4.2.6. *Let $U \subset X^\flat$ be a rational subset. $\mathcal{O}_X(U)$ is a perfectoid K-algebra and $\mathcal{O}_{X^\flat}(U^\flat)$ is a perfectoid $K^\flat$-algebra. Moreover, The perfectoid affinoid K-algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ tilts to the perfectoid affinoid $K^\flat$-algebra $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.*

Proof. Let B,$\widehat{B}, C, \widehat{C}$ be as in the statement of the previous proposition. We have $\mathcal{O}_X(U) = \widehat{R[\frac{1}{g^\sharp}]} = \widehat{R^+[\frac{f_i^\sharp}{g^\sharp}]}[\frac{1}{\pi}] = \widehat{B}[\frac{1}{\pi}]$ the last equality is given by 6) of the last proposition. Similarly, we also have $\mathcal{O}_{X^\flat}(U^\flat) = \widehat{C}[\frac{1}{\pi^\flat}]$. Therfore, 3) and 4) of the last proposition show that $\mathcal{O}_X(U)$ and $\mathcal{O}_{X^\flat}(U^\flat)$ are perfectoid K and $K^\flat$ algebras respectively and that there is an isomorphism of K-algebras $\mathcal{O}_X(U) \to \mathcal{O}_{X^\flat}(U^\flat)^\sharp$. Identifying these perfectoid K-algebras, we then have

$$\begin{aligned}
\mathcal{O}_X^+(U) &= \text{ integral closure of } \widehat{R^+[\frac{f_i^\sharp}{g^\sharp}]} \text{ in } \mathcal{O}_X(U) \quad \text{(by definition)} \\
&= \text{ integral closure of } \widehat{B} \text{ in } \widehat{B}[\frac{1}{\pi}] \quad \text{(by 6. of previous Prop.)} \\
&= \text{integral closure of } \widehat{C}^\sharp \text{ in } \widehat{C}^\sharp[\frac{1}{\pi}] \quad \text{(by 5. of previous Prop.)} \\
&= \text{ (integral closure of } \widehat{C} \text{ in } \widehat{C}^\sharp[\frac{1}{\pi^\flat}])^\sharp \quad \text{(since tilting is compatible with integral closures)} \\
&= \text{ (integral closure of } R^{+\flat}[\frac{f_i}{g}] \text{ in } \mathcal{O}_{X^\flat}(U^\flat))^\sharp \quad \text{(by 6. of previous Prop.)} \\
&= \mathcal{O}_{X^\flat}^+(U^\flat)^\sharp \quad \text{(by definition)}
\end{aligned}$$

■

Lemma 4.2.7. *(Scholze approximation lemma)*
Let $R = K\langle T_0^{\frac{1}{p^\infty}}, \ldots, T_n^{\frac{1}{p^\infty}} \rangle$. Let $f \in R^\circ$ be a homogeneous element of degree $d \in \mathbb{N}[\frac{1}{p}]$ (f lies in the π-adic completion of $\bigoplus_{k_1,\ldots,k_n \in \mathbb{N}[\frac{1}{p}],\ \sum_i k_i = d} K^\circ T^{k_1} \cdots T^{k_n}$). Then for any $c \geq 0$ and any $\epsilon > 0$ there exists an element

$$g_{c,\epsilon} \in R^{\flat\circ} = K^{\flat\circ}\langle T_0^{\frac{1}{p^\infty}}, \ldots, T_n^{\frac{1}{p^\infty}} \rangle$$

homogeneous of degree d such that for all $x \in Spa(R, R^\circ)$, we have

$$|f(x) - g_{c,\epsilon}^\sharp(x)| \leq |\pi(x)|^{1-\epsilon} \max(|f(x)|, |\pi(x)|^c)$$

Remark 4.2.8. *Note that for $\epsilon < 1$, the given estimate says in particular that for all $x \in X$, we have*

$$\max(|f(x)|, |\pi(x)|^c) = \max(|g_{c,\epsilon}^\sharp(x)|, |\pi^c|)$$

Proof. We fix $\epsilon > 0$, we can suppose that $\epsilon < 1$ and $\epsilon \in \mathbb{N}[\frac{1}{p}]$. We prove inductively that for any c we can find some $\epsilon(c) \in \mathbb{N}^*[\frac{1}{p}]$ and some

$$g_c \in R^{\flat\circ} = K^{\flat\circ}\langle T_0^{\frac{1}{p^\infty}}, \ldots, T_n^{\frac{1}{p^\infty}} \rangle$$

homogeneous of degree d such that for all $x \in X = \mathrm{Spa}(R, R^\circ)$, we have

$$|f(x) - g_c^\sharp(x)| \leq |\pi(x)|^{1-\epsilon+\epsilon(c)} \max(|f(x), |\pi(x)|^c)$$

Now fix $0 < a < \epsilon$ in $\mathbb{N}[\frac{1}{p}]$. We will argue by induction increasing from c to $c' = c + a$. Since $|\pi(x)| \leq 1$ for all x, we can replace $\epsilon(c)$ by something smaller, so we assume $\epsilon(c) \leq \epsilon - a$. For the case $c = 0$, we can take any element $g \in R^\flat$ such that $f \equiv g^\sharp \mod [\pi]$ and we can take $\epsilon(0) = 0$.
Let $U_c = X^\flat\langle \frac{g_c, {\pi^\flat}^c}{{\pi^\flat}^c} \rangle$ where $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat\circ})$. By the previous remark, its preimage is $U_c^\sharp = X\langle \frac{f, \pi^c}{\pi^c} \rangle$. This implies that

$$h := f - g_c^\sharp \in \pi^{c+1-\epsilon+\epsilon(c)} \mathcal{O}_X^+(U_c^\sharp)$$

Lemma 4.2.4 shows that $R^\circ\langle (\frac{g_c^\sharp}{\pi^c})^{\frac{1}{p^\infty}} \rangle \hookrightarrow \mathcal{O}_X(U_c^\sharp)^\circ$ is almost an equality. and so if we choose some $0 < \epsilon(c') < \epsilon(c)$ in $\mathbb{N}[\frac{1}{p}]$, we get that

$$h \in \pi^{c+1-\epsilon+\epsilon(c')} R^\circ \langle (\frac{g_c^\sharp}{\pi^c})^{\frac{1}{p^\infty}} \rangle$$

But h is homogeneous of degree $\leq d$, so that h lies in the π-adic completion of

$$\bigoplus_{i \in \mathbb{N}[\frac{1}{p}], 0 \leq i \leq 1} \pi^{c+1-\epsilon+\epsilon(c')} (\frac{g_c^\sharp}{\pi^c})^i R^\circ_{\deg = d - di}$$

Hence we can find elements $r_i \in R^\circ$ homogeneous of degree $d - di$ such that $r_i \to 0$, with

$$h = \sum_{i \in \mathbb{N}[\frac{1}{p}], 0 \leq i \leq 1} \pi^{c+1-\epsilon+\epsilon(c')} (\frac{g_c^\sharp}{\pi^c})^i r_i$$

this means that if we fix a $\varphi : \mathbb{N}[\frac{1}{p}] \cap [0, 1] \to \mathbb{N}$ then $r_{\varphi(i)} \to 0$.
Since $R^\circ/(\pi) \simeq R^{\flat\circ}/(\pi^\flat)$ we can choose $s_i \in R^{\flat\circ}$ homogeneous of degree $d - di$, $s_i \to 0$ such that π divides $r_i - s_i^\sharp$. Now set

$$g_{c'} = g_c + \sum_{i \in \mathbb{N}[\frac{1}{p}], 0 \leq i \leq 1} (\pi^\flat)^{c+1-\epsilon+\epsilon(c')} (\frac{g_c}{(\pi^\flat)^c})^i s_i$$

Let us show the induction step, i.e., that for all $x \in X$, we have

$$|f(x) - g_{c'}^\sharp(x)| \leq |\pi(x)|^{1-\epsilon+\epsilon(c')} \max(|f(x)|, |\pi(x)|^{c'})$$

Assume first that $|f(x)| > |\pi(x)|^c$. Then we have $|g_c^\sharp(x)| = |f(x)| > |\pi(x)|^c$ by the remark. Plus we have

$$|(\pi^\flat)^{c+1-\epsilon+\epsilon(c')} (\frac{g_c}{(\pi^\flat)^c})^i s_i^\sharp(x)| \leq |\pi(x)|^{1-\epsilon+\epsilon(c')} |f(x)|$$

Indeed, since $s_i^\sharp \in R^\circ$, $|s_i^\sharp(x)| \leq 1$; so neglecting them, the left-hand side is maximal when $i = 1$ ($|\frac{g_c}{(\pi^\flat)^c}(x)| \geq 1$) in which case the two sides are equal. Therfore, we have

$$\begin{aligned}
|f(x) - g_{c'}^\sharp(x)| &= |h(x) - (g_{c'}^\sharp - g_c^\sharp)(x)| \\
&\leq \max(|h(x) - (g_{c'} - g_c)^\sharp(x)|, |(g_{c'} - g_c)^\sharp(x) - (g_{c'}^\sharp - g_c^\sharp)(x)|) \\
&\leq \max(|h(x)|, |(g_{c'} - g_c)^\sharp(x)|, |\pi(x)| \cdot |g_{c'}^\sharp(x)|, |\pi(x)| \cdot |g_c^\sharp(x)|) \\
&\leq |\pi(x)|^{1-\epsilon+\epsilon(c')} |f(x)|
\end{aligned}$$

where we used the fact that $|(g_{c'}-g_c)^\sharp(x)| \leq |\pi(x)|^{1-\epsilon+\epsilon(c')}|f(x)|$ (by the previous estimate) and the induction hypothesis $|h(x)| \leq |\pi(x)|^{1-\epsilon+\epsilon(c)}|f(x)| \leq |\pi(x)|^{1-\epsilon+\epsilon(c')}|f(x)|$; and we also used the fact that $|(a+b)^\sharp - a^\sharp - b^\sharp| \leq |\pi(x)| \cdot \max(|a^\sharp(x)|, |b^\sharp(x)|)$ (since $\pi \mid p$).
Now we treat the case $|f(x)| \leq |\pi(x)|^c$. We claim that in fact

$$|f(x) - g_{c'}^\sharp(x)| \leq |\pi(x)|^{c'+1-\epsilon+\epsilon(c')}$$

in this case, which clearly implies the inequality. For this, it is enough to see that $f - g_{c'}^\sharp$ is an element of $\pi^{1+c}\mathcal{O}_X(U_c^\sharp)$, because $c+1 > c'+1-\epsilon+\epsilon(c')$ (remember $\epsilon(c') < \epsilon - a$). But we have

$$\frac{g_{c'}}{(\pi^\flat)^c} = \frac{g_c}{(\pi^\flat)^c} + \sum_i (\pi^\flat)^{1-\epsilon+\epsilon(c')}(\frac{g_c}{(\pi^\flat)^c})^i s_i$$

with all terms being in $O_{X^\flat}(U_c)^\circ$. Hence since $O_{X^\flat}(U_c)^\circ$ tilts to $O_X(U_c^\sharp)^\circ$ by corollary 4.2.6 , we get

$$\frac{g_{c'}^\sharp}{\pi^c} = \frac{g_c^\sharp}{\pi^c} + \sum_i \pi^{1-\epsilon+\epsilon(c')}(\frac{g_c^\sharp}{\pi^c})^i r_i \mod \pi O_X(U_c^\sharp)^\circ$$

Multiplying by π^c, this rewrites as

$$f - g_{c'}^\sharp = f - g_c^\sharp - h = 0 \mod \pi^{c+1}\mathcal{O}_X(U_c^\sharp)^\circ$$

which gives the desired estimate. ■

Corollary 4.2.9. *Let (R, R^+) be a perfectoid affinoid K-algebra, with tilt $(R^\flat, R^{\flat+})$. Then*

1. *For any $f \in R$ and any $c \in \mathbb{N}[\frac{1}{p}]$, $\epsilon \in \mathbb{N}^*[\frac{1}{p}]$, there exists some $g_{c,\epsilon} \in R^\flat$ such that*

$$|(f - g_{c,\epsilon}^\sharp)(x)| \leq |\pi(x)|^{1-\epsilon}\max(|f(x)|, |\pi(x)|^c)$$

for all $x \in X$.

2. *Given $f, g \in R$ and an integer $c \geq 0$, there exist $a, b \in R^\flat$ such that*

$$X\langle\frac{f, \pi^c}{g}\rangle = X\langle\frac{a^\sharp, \pi^c}{b^\sharp}\rangle$$

3. *Every rational subset of X is the preimage of a rational subset of $X^\flat$.*

Proof. **1)** It suffices to prove the corollary for $f \in R^+$. Indeed, if we had this case, then given any $f \in R$, $c \in \mathbb{N}[\frac{1}{p}]$, we can set $g_c = {\pi^\flat}^{-n} h_{c+n}$ where n is picked so that $\pi^n f \in R^+$ and $h_{c+n} \in R^{\flat+}$ is chosen so that $|(\pi^n f - h_{c+n}^\sharp)(x)| \leq |\pi(x)|^{1-\epsilon}\max(|\pi^n f(x)|, |\pi(x)|^{c+n})$. Thus we reduce to the case of $f \in R^+$.
We also may assume that c is an integer. Indeed if $d \leq c$, then

$$\max(|f(x)|, |\pi(x)|^c) \leq \max(|f(x)|, |\pi(x)|^c)$$

Recall the partial order $x \succ y \Leftrightarrow x \rightsquigarrow y \Leftrightarrow y \in \bar{\{x\}}$ for $x, y \in \mathrm{Spa}(R, R^+)$(proposition $A.0.9$). For a fixed $y \in \mathrm{Spa}(R, R^+)$, the set $\{x \mid x \succ y\}$ is totally ordered and its maximal element y_0 lies in $\mathrm{Spa}(R, R^\circ)$. Proving the corollary for y_0 and for a slightly bigger ϵ implies the corollary for y. Indeed, we have

$$|(f - g^\sharp_{c,\epsilon})(y_0)| < |\pi(y_0)|^{1-\epsilon} \max(|f(y_0), |\pi(y_0)|^c) \Rightarrow |(f - g^\sharp_{c,\epsilon})(y)| < |\pi(y)|^{1-\epsilon} \max(|f(y), |\pi(y)|^c)$$

Therefore, it is enough to check the inequality at maximal points and so we can assume that $R^+ = R^\circ$.
Now let $f \in R^+$ and $c \geq 0$ an integer. We can choose $g_0, \dots, g_c \in R^{\flat +}$ and $f_{c+1} \in R^+$ such that

$$f = g_0^\sharp + g_1^\sharp \pi + \cdots + g_c^\sharp \pi^c + f_{c+1}\pi^{c+1}$$

Write $f_0 = \sum_{i=0}^c g_i^\sharp \pi^i$, so that $f = f_0 + \pi^{c+1} f_{c+1}$. Using the strict non-Archimedean inequality, we see that the right side of the desired inequality is left unchanged if we replace f by f_0, we can thus easily check that any choice of g_c that solves the inequality for f_0 also solves it for f. Thus we can assume $f = f_0$. Now consider the map

$$\mu : P := K\langle T_0^{\frac{1}{p^\infty}}, \dots, T_c^{\frac{1}{p^\infty}} \rangle \to R$$

of perfectoid K-algebras carrying T_i to $g_i^\sharp$; this map satisfies $\mu(P^\circ) \subset R^\circ = R^+$ which induces the map $\mathrm{Spa}(R, R^\circ) \to \mathrm{Spa}(P, P^\circ)$ (given by the composition by μ. By lemma 4.2.7 applied to $\sum_{i=0}^c T_i \pi^i \in P^\circ$, there exists some $h_0 \in P^{\flat\circ}$ such that $h = \mu^\flat(h_0) \in R^{\flat +}$ satisfies

$$|(f - h^\sharp)(x)| \leq \pi^{1-\epsilon} \max(|f(x), |\pi(x)|^c)$$

for all $x \in \mathrm{Spa}(R, R^+)$ as wanted.

2) Using 1) we can choose $a, b \in R^\flat$ such that

$$|(g - b^\sharp)(x)| < \max(|g(x)|, |\pi(x)|^c)$$

and

$$\max(|f(x), |\pi(x)|^c) = \max(|a^\sharp(x)|, |\pi(x)|^c)$$

Now say $x \in X\langle \frac{f, \pi^c}{g} \rangle$. As $|\pi(x)^c| \leq |g(x)|$, we get by the first inequality $|g(x) - b^\sharp(x)| < |g(x)|$. By the strict non-Archimedean inequality, this can only happen if $|b^\sharp(x)| = |g(x)|$, so we get $|\pi(x)|^c \leq |b^\sharp(x)|$. Moreover, the second equality shows that we either have $|a^\sharp(x)| \leq |\pi(x)|^c$ or $a^\sharp(x)| = |f(x)|$; both these cases give $|a^\sharp(x)| \leq |g(x)| = |b^\sharp(x)|$ proving that $x \in X\langle \frac{a^\sharp, \pi^c}{b^\sharp} \rangle$. Conversely, say $x \in X\langle \frac{a^\sharp, \pi^c}{b^\sharp} \rangle$. If $|\pi^c(x)| > |g(x)|$ then by the first inequality and the strict non-Archimedean inequality, this would imply that $|g(x)| = |b^\sharp(x)|$ but this means that $|b^\sharp(x)| = |g(x)| < |\pi(x)^c|$: contradiction. Therfore, we must have $|\pi(x)^c| \leq |g(x)|$. This implies, as seen before, that $|g(x)| = b^\sharp(x)|$. It remains to check that that $|f(x)| \leq |g(x)|$. If not, we must have $|f(x)| > |g(x)| \geq \pi^c$. By the second equality, this means that

$|a^\sharp(x)| = |f(x)| > |g(x)| = |b^\sharp(x)|$ which contradicts the assumption on x. Therfore, $x \in X\langle \frac{f,\pi^c}{g} \rangle$.

3) the general case for a rational subset $U = X\langle \frac{f_1,\ldots,f_n}{g} \rangle$ with $f_n = \pi^c$ for some integer $c \geq 1$, observe that $U = \bigcap_{i=1}^{n-1} X\langle \frac{f_i,\pi^c}{g} \rangle$ so the result follows from 2. ■

Theorem 4.2.10. *The map $\psi : X \to X^\flat$ is a homeomorphism sending rational in X subsets to rational subsets in $X^\flat$.*

Proof.
Injectivity follows from the fact that $\psi : X \to X^\flat$ is continuous, X is T_0 and that a rational open in X is a preimage of a rational open in $X^\flat$ (corollary 4.2.9).
Now let us show surjectivity. Consider $x \in X^\flat$. The open valuation ring $\widehat{k(x)^+}$ is the π-adic completion of the direct limit of the perfect rings $\mathcal{O}_{X^\flat}(U^\flat)$ as $U^\flat$ ranges through rational open subsets of $X^\flat$ containing x. So $\widehat{k(x)^+}$ is perfect and thus perfectoid. Inverting π we have that $\widehat{k}(x)$ is a perfectoid K-algebra and so the couple $(\widehat{k(x)}, \widehat{k(x)^+})$ is a perfectoid affinoid field. Moreover, the point x defines a map $(R^\flat, R^{+\flat}) \to (\widehat{k(x)}, \widehat{k(x)^+})$. Since tilting preserve perfectoid affinoid fields, this map untilts to a map $(R, R^+) \to (L, L^+)$ whee (L, L^+) is a perfectoid affinoid field. By corrolary 1), this corresponds to a point $y \in \mathrm{Spa}(R, R^+)$. The valuation $\psi(y)$ is defined by the map $R^\flat \xrightarrow{\sharp} R \to L$ which coincides with the map $R^\flat \to \widehat{k(x)} \xrightarrow{\sharp} L$ defining x. So $\psi(y) = x$ which proves surjectivity. ■

Corollary 4.2.11. *For $x \in X$, the affinoid field $(\widehat{k(x)}, \widehat{k(x)^+})$ is perfectoid with tilt $(\widehat{k(x^\flat)}, \widehat{k(x^\flat)^+})$.*

Proof. We have that $\widehat{k(x)^+}$ is the π-adic completion of the direct limit of the perfectoid K°-algebras $\mathcal{O}_X^+(U)$ as U ranges through the rational subsets of X. By remark 1, this gives that $\widehat{k(x)} = \widehat{k(x)^+}[\frac{1}{\pi}$ is the colimit of the $\mathcal{O}_X(U)$ in the category of perfectoid K-algebras and thus $\widehat{k(x)}$ is perfectoid. Its tilt is the colimit of the tilts of $\mathcal{O}_X(U)$, i.e., the colimit of the $\mathcal{O}_X(U^\flat)$ which is $\widehat{k(x^\flat)}$ (since the rational neighborhoods of x are sent bijectively to the rational neighborhoods of $x^\flat$) . ■

Theorem 4.2.12. *(Criterion of K. Buzzard and A. Verberkmoes)*
We say say that an affinoid K-algebra (A, A^+) is stably uniform if for all $U \subset$Spa(A, A^+) rational open, the Banach K-algebra $\mathcal{O}_X(U)$ is uniform, i.e., $\mathcal{O}_X(U)^\circ$ is bounded.
If (A, A^+) is stably uniform then it is sheafy.

We deduce form this criterion that we can associate to any perfectoid affinoid K-algebra (R, R^+) an affinoid adic space X =Spa(R, R^+). We call these spaces affinoid perfectoid spaces.

Definition 4.2.13. *(Perfectoid Spaces) A perfectoid space is an adic space over K that is locally isomorphic to an affinoid perfectoid space.*

Theorem 4.2.14. *Any perfectoid space X admits a tilt $X^\flat$. This operation identifies the category of perfectoid spaces over K and the category of perfectoid spaces over $K^\flat$.*

The topological spaces of X and $X^\flat$ are identified, and X is affinoid perfectoid if and only if $X^\flat$ is affinoid perfectoid. Moreover, for any affinoid perfectoid subspace $U \subset X$, the pair $\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big)$ is a perfectoid affinoid K-algebra with tilt $\big(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}(U^\flat)^+\big)$.

Appendix A

Valuation rings

Definition A.0.1. *Let V be an integral domain and let K be its fraction field. We say that V is a valuation ring if for every $x \in K^{\times}$, either $x \in V$ or $x^{-1} \in V$.*

Consider two local rings A and B with respective maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$. We say that B dominates A and we write $A \preceq B$ if $A \subset B$ and $\mathfrak{m} = A \cap \mathfrak{n}$.
We see that if B dominates A then we have an inclusion of residue fields $A/\mathfrak{m} \subset B/\mathfrak{n}$.
For a field K, the relation B dominates A defines a partial order on the set of local rings contained in K.
This relation gives another characterization of valuation rings.

Theorem A.0.2. *Let V be an integral domain and let K be a field containing V. The following conditions are equivalent*

1. *V is a valuation ring*
2. *K is the fraction field of V and the set of principal ideals of V is totally ordered by inclusion.*
3. *K is the fraction field of V and the set of ideals of V is totally ordered by inclusion.*
4. *K is the fraction field of V and V is a maximal element in the set of local rings contained in K and ordered by the relation of domination.*

Proof.

1) $\Rightarrow$ 2) Let x and y be non-zero elements of V. Then we either have $\frac{x}{y} \in V$ or $\frac{y}{x} \in V$, i.e., $x \mid y$ or $y \mid x$.
2) $\Rightarrow$ 3) Let I and J be two ideals of J and suppose that $I \not\subset J$. So let $a \in I \setminus J$. By total order on principal ideals, for each $b \in J$ we either have $a \mid b$ or $b \mid a$. But $b \mid a$ means that $a \in J$ which contradicts the choice of a and so $b \in I$. This is true for all $b \in J$ and so $J \subset I$.
3) $\Rightarrow$ 4) Since V contains a maximal ideal, it must be unique by the total ordering of ideals and so V is local. Suppose that there exists a local ring $V' \subset K$ that dominates V and let $x \in V'$ be a non-zero element. Then $x = \frac{a}{b}$ with $a, b \in V$. But we either have $b \mid a$ or $a \mid b$. In the first case we have $x \in V$. And in the second, we get that $x^{-1} \in V$; more precisely

$x^{-1} \in \mathfrak{m}$ (since x^{-1} is not invertible in V). But given that V' dominates V, we have that x^{-1} is contained in the maximal ideal of V' and is invertible in V': contradiction. So $x \in V$ and we have $V = V'$ so that V is maximal with respect to the relation of domination.
4) ⇒ 1) Denote by $\mathfrak{m}$ the maximal ideal of V. Let $x \in K$ such that $x \notin V$ and set $V' = V[x]$. There is no prime $\mathfrak{p}$ of V' that lies over $\mathfrak{m}$ (meaning that $\mathfrak{p} \cap V = \mathfrak{m}$), otherwise $V'_{\mathfrak{p}}$ would be a local ring that dominates V which contradicts the maximality of V. Thus we have $\mathfrak{m}V' = V'$, i.e., we can write $1 = v_0 + v_1 x + \cdots + v_n x^n$ for some $v_i \in V$. Dividing this equation by x^n we get that x^{-1} is integral over V. Therfore, the ring $V[x^{-1}]$ is integral over V and therefore (by the lying over property) has a prime that lies over $\mathfrak{m}$. By the same reasoning as before, we get that $V[x^{-1}] = V$ and so $x^{-1} \in V$.

■

Proposition A.0.3. *Let A be an integral domain contained in a field K and $h : A \to \Omega$ be a morphism of A to an algebraically closed field Ω, then there exists a valuation ring $V \subset K$ and a morphism $f : V \to L$ which extends h and such that $\max(V) = f^{-1}(0)$.*
In particular, any local subring A of a field K is dominated by at least one valuation ring V of K.

Proof. We consider the set $\sum = \{(B, f) \mid B \subset K$ and $f : B \to \Omega$ extending $h\}$. We partially order this set by $(B, f) \leq (C, g)$ if $B \subset C$ and $g_{|B} = f$. Any totally ordered subset $\big((B_\alpha, f_\alpha)\big)$ of $\sum$ has an upper bound (B, f) with B consisting of the union of all the B_α and f being the obvious extending map. Hence, by Zorn's lemma the set $\sum$ contains a maximal elements (V, f). We claim that (V, f) is the desired valuation ring and morphism.
If we denote by $\mathfrak{p}$ the kernel of $f : V \to L$ (which is prime), then we can extend f to the local ring $V_{\mathfrak{p}}$, and thus by maximality $V_{\mathfrak{p}} = V$ hence V is a local ring and $\mathfrak{p}$ is its maximal ideal.
Now we show that for an $x \in K$, either $x \in V$ or $x^{-1} \in V$. If x is integral over V, then by algebraic closedness of Ω, f extends to a map $V[x] \to \Omega$. But (V, f) is a maximal element and so $V[x] = V$, i.e., $x \in V$.
If x is not integral over V, then we have $x \notin V[x^{-1}]$. So x^{-1} is not invertible in $V[x^{-1}]$ and therefore lies in a maximal ideal $\mathfrak{q}$. Consider the morphisms $\phi : V \to V[x^{-1}]$ and $p : V[x^{-1}] \to V[x^{-1}]/\mathfrak{q}$. Then since $p(x^{-1}) = 0$, we get that $p(V[x^{-1}]) = p \circ \phi(V)$. Given that $V[x^{-1}]/\mathfrak{q}$ is a field and V is local then $V[x^{-1}]/\mathfrak{q} = V/\mathfrak{p}$ so that the morphism $f \circ p : V[x^{-1}] \to \Omega$ extends f which by the maximality of (V, f) gives that $V[x^{-1}] = V$, i.e., $x^{-1} \in V$. ■

Definition A.0.4. *Given a ring A, a valuation on A is a map $|\cdot| : A \to \Gamma \cup \{0\}$ such that*

- $\big(\Gamma, \cdot, \leq\big)$ *is a totally ordered Abelian group* $(\forall \alpha, \beta, \gamma \in \Gamma, \quad \alpha \leq \beta \Leftrightarrow \alpha \cdot \gamma \leq \beta \cdot \gamma)$. *We extend $\leq$ to $\Gamma \cup \{0\}$ so that $0 \leq \alpha$ and $\alpha \cdot 0 = 0$ $\forall \alpha \in Gamma$.*
- $|0| = 0$ *and* $|1| = 1$
- $|\cdot|$ *is multiplicative, i.e.,* $|x \cdot y| = |x| \cdot |y|$
- $|x + y| \leq \max(|x|, |y|)$ *for all* $x, y \in A$

We define the support of $|\cdot|$ to be supp$(|\cdot|) := \{x \in A \mid |x| = 0\}$. It is a prime ideal of A.

Remark A.0.5. *Given a valuation $|\cdot|$, this defines a valuation $|\cdot|' : Frac(A/supp(|\cdot|) \to \Gamma \cup \{0\}$ such that $|x|' = 0$ if and only if $x = 0$. Conversely, given a prime $\mathfrak{p}$ of A and a valuation $|\cdot|' : Frac(A/\mathfrak{p}) \to \Gamma \cup \{0\}$, we get a valuation $|\gamma| : A \to \Gamma \cup \{0\}$ by composing $|\cdot|'$ with the map $A \to Frac(A/\mathfrak{p})$.*

Proposition A.0.6. *Let $|\cdot| : K \to \Gamma \cup \{0\}$ be a valuation on a field K. Then the set $A = \{x \in K \mid |x| \leq 1\}$ is a valuation ring with maximal ideal $\mathfrak{m} = \{x \in K \mid |x| < 1\}$.*
Conversely, to any valuation ring V of K we can associate a valuation $|\cdot|_V : K \to \Gamma \cup \{0\}$ on K such that V is the inverse image of $\{\alpha \in Gamma \mid \alpha \leq 1\}$.

Proof. By the properties of the valuation we get that A is a ring. Moreover for $x \in K^\times$, since Γ is totally ordered, we either have $|x| \geq 1$ or $|x^{-1}| \geq 1$, i.e., $x \in A$ or $x^{-1} \in A$.
For the converse, we set Γ to be the quotient of $K^\times$ by the group $V^\times$ of invertible elements of V. The divisibility relation on V defines a partial order on Γ ($xV^\times \leq yV^\times \Leftrightarrow \exists\, z \in V, x = zy$) which is compatible with the group structure and since for every $xV^\times, yV^\times \in \Gamma$ we either have $xy^{-1} \in V$ or $yx^{-1} \in V$, then this order is total. The canonical mapping $K^\times \to \Gamma$ gives the desired valuation when extended at 0. ■

Given two valuations $|\cdot| : A \to \Gamma$ and $\widetilde{|\cdot|} : A \to \widetilde{\Gamma}$ on A.
We say that $|\cdot|$ and $\widetilde{|\cdot|}$ are equivalent and we write $|\cdot| \sim \widehat{|\cdot|}$ if one of the following equivalent conditions hold

- There exists an isomorphism of totally ordered groups

$$\begin{tikzcd} \Gamma_{|\cdot|} \arrow[rr, "\sim"] & & \Gamma_{\widetilde{|\cdot|}} \\ & A \arrow[ul, "|\cdot|"'] \arrow[ur, "\widetilde{|\cdot|}"'] & \end{tikzcd}$$

  where $\Gamma_{|\cdot|}$ denotes the image of $|\cdot|$ (same for $\widetilde{|\cdot|}$.

- $|a| \leq |b| \Leftrightarrow \widetilde{|a|} \leq \widetilde{|b|}$

- supp$(|\cdot|)$ = supp $(\widetilde{|\cdot|}) = \mathfrak{p}$ and the induced valuation on Frac$(A/\mathfrak{p})$ has the same valuation ring.

Definition A.0.7. *Let A be a domain and K its fraction field. We define the valuation spectrum of A to be*

$$X = Spv(K, A) := \{|\cdot| : K \to \Gamma \cup \{0\} \text{ valuation} \mid |a| \leq 1\ \forall a \in A\}/\sim$$

For every $x \in X$, we write $|f(x)|$ for the associated valuation of $f \in K$, and we denote $A_x \subset K$ to be the associated valuation ring.

From what have been said above on valuations, we have a bijection

$$\begin{aligned} X &\xrightarrow{\sim} \{A \subset V \subset K \text{ valuation ring }\} \\ x &\mapsto A_x \end{aligned}$$

For $f_1, \dots, f_n \in A$ and $g \neq 0 \in A$ we set

$$X_{f_1,\dots,f_n;g} := \{x \in X \mid f_i(x)| \leq |g(x)|\}$$

So that we define a topology on X by allowing the sets $X_{f_1,\dots,f_n;g}$ to be a basis for the topology.

Lemma A.0.8. *Let A be an integral domain with fraction field K. Let $\widetilde{A}$ be the normalization of A in K. Then $\widetilde{A}$ is the intersection of valuation rings containing A.*

Proof. First, since notice that valuation rings are normal. Indeed suppose that $x \notin V$ is integral over V, then $x \in V[x^{-1}]$. But given that V is a valuation ring, we have $x^{-1} \in V$, thus $x \in V$: contradiction. Therfore, we get that $\widetilde{A}$ is in the intersection of all valuation rings containing A.
On the other hand, let $x \in K$ such that $x \notin \widetilde{A}$. We have $x \notin A[x^{-1}] \subset K$. Hence x^{-1} is not a unit in $A[x^{-1}]$ and is thus contained in a maximal ideal $\mathfrak{m}$. By proposition 15, $A[x^{-1}]_{\mathfrak{m}}$ is dominated by a valuation ring V. Hence we have $A \subset A[x^{-1}]_{\mathfrak{m}} \subset V$ such that $x^{-1} \in \max(A[x^{-1}]_{\mathfrak{m}}) \subset \max(V)$ and so x^{-1} is not invertible in V, i.e., $x \notin V$. ■

We define a sheaf $\mathcal{O}_X$ on X =Spv(K, A) by setting for every $U \subset X$ open subset

$$\mathcal{O}_X(U) := \bigcap_{x \in U} A_x$$

We have that for every, $f_1, \dots, f_n \in A$, $g \neq 0 \in A$

$$\mathcal{O}_X(X_{f_1,\dots f_n;g}) = \bigcap_{x \in X,\ A[\frac{f_1}{g},\dots,\frac{f_n}{g}] \subset A_x} A_x = A[\widetilde{\frac{f_1}{g}, \dots \frac{f_n}{g}}]$$

In fact we have Spv$(K, A[\frac{f_1}{g}, \dots, \frac{f_n}{g}] \simeq X_{f_1,\dots,f_n;g}$.

Proposition A.0.9. *Let V be a valuation ring with fraction field K.*

1. *We have a $1:1$ correspondence between $V \subset W \subset K$ valuation rings and $\mathfrak{p} \subset V$ prime ideals given by*

$$W \longrightarrow \max W$$
$$V_{\mathfrak{p}} \longleftarrow \mathfrak{p}$$

   In particular Spv(K, V) is a totally ordered where $v \leq w \Leftrightarrow v \in \bar{\{w\}}$

2. *$\forall I \subset V$, $\sqrt{I} \subset V$ is a prime ideal.*

Proof. If $\mathfrak{p} \subset V$ is a prime ideal, then $V \subset V_{\mathfrak{p}} \subset K$. Since V is a valuation ring then so is $V_{\mathfrak{p}}$ and by the localization properties we have $\max V_{\mathfrak{p}} = \max V_{\mathfrak{p}} \cap V = \mathfrak{p}$.
Now let $V \subset W$ be a valuation ring. Let $\alpha \in \max W$.
Suppose $\alpha \notin V$, then $\alpha^{-1} \in V \subset W \Rightarrow \alpha\alpha^{-1} = 1 \in \max W$: contradiction. Hence $\max W \subset V$ is a prime ideal of V. We also get the inclusion $V_{\max W} \subset W$; but by the maximality property of valuation rings, we get an equality.
3) Let $I \subset V$ be an ideal. We have $\sqrt{I} = \cap_{I \subset \mathfrak{p}} \mathfrak{p}$. Let $x, y \in V$ such that $xy \in \sqrt{I}$ but $x \notin \sqrt{I}$ and $y \notin \sqrt{I}$ hence there are prime ideals $\mathfrak{p}, \mathfrak{q}$ such that $x \notin \mathfrak{p}$ and $y \notin \mathfrak{q}$. Since ideals are totally ordered, we can suppose that $\mathfrak{p} \subset \mathfrak{q}$, then $x, y \notin \mathfrak{p}$ but $xy \in \mathfrak{p}$: contradiction.

■

Appendix B

Witt Vectors

In this section we mainly follow the book of J.P.Serre "Corps Locaux" and the course notes of Pierre Colmez on local fields.
We fix a prime number p.

Let R be a perfect ring of characteristic p. We say that A is a perfect p-ring of residual ring R, if there exists a non-zero divisor $\pi \in A$ such that A is separated and complete for the π-adic topology and such that $A/\pi = R$ (thus $p \in \pi A$).
If $\pi = p$ then A is said to be strict.
Since R is perfect, we have $A^\flat \simeq R$, and the sharp map induces a map $[\cdot] : R \to A$ called the Teichmüller lift which is multiplicative and verifies $\overline{[r]} = r$ for all $r \in R$. By π-adic separatedness of A, and the fact that π is a non-zero-divisor, every element a of A can be written uniquely in the form

$$a = \sum_{i=0}^{\infty} [a_i]\pi^i$$

For example, consider Λ to be a set of indices, we define $S_\Lambda := \mathbb{Z}[X_\alpha^{1/p^\infty}]_{\alpha\in\Lambda} = \bigcup_{m\in\mathbb{N}} \mathbb{Z}[X_\alpha^{1/p^m}]_{\alpha\in\Lambda}$. We write $\widehat{S_\Lambda}$ for the completion of S_Λ for the p-adic topology. We have $\overline{S_\Lambda} := \widehat{S_\Lambda}/p = S_\Lambda/p = \mathbb{F}_p[X_\alpha^{1/p^\infty}]_{\alpha\in\Lambda}$ which is perfect. Therfore, $\widehat{S_\Lambda}$ is a strict perfect p-ring.

Now we proceed to the construction of the ring of Witt vectors for a general commutative ring R.
Consider the sequence of polynomials in $\mathbb{Z}[X_i]_{i\in\mathbb{N}}$

$$W_n = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$$

If we write $Z' = \mathbb{Z}[\frac{1}{p}]$, then we can easily show that the X_i can be expressed as polynomials in $Z'[W_i]_{i\in\mathbb{N}}$, i.e., $X_0 = W_0$, $X_1 = \frac{1}{p}W_1 - \frac{1}{p}W_0^p$, etc...

Theorem B.0.1. *Let $\phi \in \mathbb{Z}[X, Y]$. There exists a sequence of polynomials $(\varphi_0, \dots, \varphi_n, \dots)$*

in $\mathbb{Z}[X_i, Y_j]_{i,j\in\mathbb{N}}$ *that uniquely verify*

$$W_n(\varphi_0, \dots, \varphi_n, \dots) = \phi(W_n(X_0, \dots), W_n(Y_0, \dots)) \quad \forall n \in \mathbb{N}$$

Proof. Consider the polynomials $P_n \in Z'[X_i]_{i\in\mathbb{N}}$ such that $X_n = P_n(W_0, \dots, W_i, \dots) = W_n(P_1, \dots, P_i, \dots)$. Then letting

$$\varphi_n(X_i, Y_j) = P_n\Big(\phi\big(W_0(X_0, \dots), W_0(Y_0, \dots)\big), \dots, \phi\big(W_i(X_0, \dots), \dots, W_i(Y_0, \dots)\big), \dots\Big)$$

we see that the $(\varphi_n)_{n\in\mathbb{N}}$ exist in $Z'[X_i, Y_j]_{i,j\in\mathbb{N}}$ and are unique. So we need to show that they are actually in $\mathbb{Z}[X_i, Y_j]_{i,j\in\mathbb{N}}$.

Consider the ring $\widehat{S_{\mathbb{N}\amalg\mathbb{N}}} = \widehat{\mathbb{Z}[X_i^{1/p^\infty}, Y_j^{1/p^\infty}}]_{i,j\in\mathbb{N}}$ and let $x' = \sum_{i=0}^{\infty} X_i^{1/p^i} p^i$ and $y' = \sum_{j=0}^{\infty} Y_j^{1/p^j} p^j$. Since $\phi(x', y') \in \widehat{S}$, it can be uniquely written as

$$\phi(x', y') = \sum_{n=0}^{\infty} [\overline{\psi_n}]^{1/p^n} p^n \qquad \overline{\psi_n} \in \overline{S_{\mathbb{N}\amalg\mathbb{N}}}$$

Write ψ_n for a representative of $\overline{\psi_n}$. We will prove that the φ_n are in $\mathbb{Z}[X_i, Y_j]_{i,j\in\mathbb{N}}$ and that they are congruent mod p to the ψ_n.

First modding out everything by p^n we get

$$\phi(\sum_{k\le n} X_k^{1/p^k} p^k, \sum_{k\le n} Y_k^{1/p^k} p^k) \equiv \sum_{k\le n} [\overline{\psi_k}(X_i, Y_j)]^{1/p^k} p^k \mod p^{n+1}$$

Replacing the X_is and Y_js by $X_i^{p^n}$ and $Y_j^{p^n}$, we get

$$\psi(W_n(X_i), W_n(Y_j)) \equiv \sum_{k\le n} [\overline{\psi_k}(X_i^{p^n}, Y_j^{p^n})]^{1/p^k} p^k \mod p^{n+1}$$

But $\overline{\psi_k}(X_i^{p^n}, Y_j^{p^n}) = \overline{\psi_k}(X_i, Y_j)^{p^n}$ and since $[\,\cdot\,]$ is multiplicative, we see that $W_n(\varphi_0, \dots, \varphi_n) \equiv \sum_{k\le n} [\overline{\psi_k}]^{p^{n-k}} p^k \mod p^{n+1}$. Given that $[\overline{\psi_k}] \equiv \psi_k \mod p$, we have by lemma 1, $[\overline{\psi_i}]^{p^{n-i}} \equiv \psi_i^{p^{n-i}} \mod p^{n-i+1}$. Therfore,

$$W_n(\varphi_0, \dots, \varphi_n) \equiv W_n(\psi_0, \dots, \psi_n) \mod p^{n+1} \quad (*)$$

.

Since φ_0 and ψ_0 can both be taken $\phi(X_0, Y_0)$, we can reason by induction on n and suppose that all the $\varphi_k \in \mathbb{Z}[X_i, Y_j]_{i,j\in\mathbb{N}}$ for $k < n$ and that $\varphi_k \equiv \psi_k \mod p$.

We deduce by $(*)$ and by the expression of W_n that $p^n\varphi_n \equiv p^n\psi_n \mod p^{n+1}$. Hence φ_n has integer coefficients and $\varphi_n \equiv \psi_n \mod p$. ■

Now let $(S_0, \dots, S_n, \dots)$ (resp. $(P_0, \dots, P_n, \dots)$ be the polynomials $(\varphi_n)_{n\mathbb{N}}$ associated to the polynomial $\phi(X, Y) = X + Y$ (resp.$\phi(X, Y) = XY$).

Let R be a commutative ring. We would like to give the set $R^{\mathbb{N}}$ a structure of a commutative ring by setting for $\mathfrak{a} = (a_0, a_1, \dots)$ and $\mathfrak{b} = (b_0, b_1, \dots)$ in $R^{\mathbb{N}}$

$$\mathfrak{a} + \mathfrak{b} := (S_0(a_i, b_j), S_1(a_i, b_j), \dots)$$
$$\mathfrak{a} \cdot \mathfrak{b} := (P_0(a_i, b_j), P_1(a_i, b_j), \dots)$$

Theorem B.0.2. *These laws of composition turn $R^{\mathbb{N}}$ into a commutative ring called the ring of Witt vectors with coefficients in R and denoted by $W(R)$.*

Proof. By the above theorem, we get a homomorphism

$$\begin{aligned} W_* : W(R) &\to R^{\mathbb{N}} \\ (r_i)_{i\in\mathbb{N}} &\mapsto (W_0(r_i), W_1(r_i), \dots) \end{aligned}$$

Indeed for example $\big(W_n(S_0(a_i,b_j), S_1(a_i,b_j), \dots)\big)_{n\in\mathbb{N}} = \big(W_n(a_i) + W_n(b_j)\big)_{n\in\mathbb{N}}$.
The morphism W_* is an isomorphism if p is invertible in R (in this case, the inverse map is given by the polynomials P_i expressing the X_i in terms of the W_i), and we see that $W(R)$ is a commutative element with unit $1 = (1, 0, \dots, 0, \dots)$.
But if a theorem is true for a ring R, it is true for its subrings and its quotients. And since it is true for $Z'[T_\alpha]_{\alpha\in\Lambda}$, it is also true for $\mathbb{Z}[T_\alpha]_{\alpha\in\Lambda}$ and hence for every ring R. ∎

We see that in particular the maps $W_*^{(n)} : W(R) \to R$ defined to be the composition of W_* with the projection on the n-th coordinate, are ring homomorphisms.

Let $x \in R$, we define $T(x) = (x, 0, \dots)$. Hence we get a map $T : R \to W(R)$. When p is invertible in R, $W_* \circ r : x \mapsto (x, x^p, x^{p^2}, \dots)$. We deduce that for a general R

$$\begin{aligned} T(xy) &= T(x)T(y) \quad \forall\, x, y \in R \\ T(x)\cdot(y_0, y_1, \dots) &= (xy_0, x^p y_1, x^{p^2} y_2, \dots) \quad \forall\, x, y_i \in A \end{aligned}$$

Now we go back to the case of a perfect ring R of characteristic p. Recall that there exist polynomials $(\varphi_n)_{n\in\mathbb{N}}$ that verifying

$$W_n(\varphi_0(X_i, Y_j), \varphi_1(X_i, Y_j), \dots) = \phi\big(W_n(X_0, X_1, \dots), W_n(Y_0, Y_1, \dots)\big)$$

. We showed that $\varphi_n \equiv \psi_n \mod p$ for all $n \in \mathbb{N}$ where $(\psi_n)_{n\in\mathbb{N}}$ are the polynomials verify

$$\phi(x', y') = \sum_{n=0}^{\infty} [\psi_n(X_i, Y_j)]^{1/p^n} p^n$$

for $x' = \sum_{i=0}^{\infty} X_i^{1/p^i} p^i$ and $y' = \sum_{i=0}^{\infty} Y_i^{1/p^i} p^i$ in $\widehat{S_{\mathbb{N}\amalg\mathbb{N}}}$. Therefore, since R is of characteristic p, we can assume that $\varphi_n = \psi_n$ for all $n \in \mathbb{N}$.
Assume that $\phi(X, Y) = pX$ and let $(\varphi_n)_{n\in\mathbb{N}}$ be the corresponding polynomials. Since they are uniquely defined, the map $(x_0, x_1, \dots) \mapsto (\varphi_0(x_0, x_1, \dots), \varphi_1(x_0, x_1, \dots), \dots)$ correspond to the multiplication by p in the ring of Witt vectors of R.
But in the ring $\widehat{S}_{\mathbb{N}\amalg\mathbb{N}}$ we have

$$\phi(x', y') = p \cdot x' = \sum_{n=0}^{\infty} [\psi_n(X_i, Y_j)]^{1/p^n} p^n = \sum_{n=1}^{\infty} [X_{n-1}]^{1/p^{n-1}} p^n$$

we get that $\psi_0 = 0$ and $\psi_n(X_i, Y_j) = X_{n-1}^p$ for $n > 0$.
Therfore, for an element $r = (r_0, r_1, \dots) \in W(R)$

$$p \cdot r = (0, r_0^p, r_1^p, \dots)$$

On the other hand, if $r \in W(R)$ is of the form $r = (0, r_0, r_1, \dots)$ then $r = pr'$ for $r' = (r_0^{1/p}, r_1^{1/p}, \dots)$ (by perfectness of R). Therfore, we get an exact sequence :

$$0 \to pW(R) \to W(R) \xrightarrow{W_*^{(0)}} R \to 0$$

More generally we get that $p^n W(R) = \{(r_0, r_1, \dots) \in W(R) \mid r_0 = \dots = r_{n-1} = 0\}$, therefore $W(R)$ is p-adically separated.
Now since $S_n(X_i, Y_j)$ is a polynomial in only $X_1, Y_1, \dots, X_n, Y_n$, we get that $x + p^n W(R) = \{(r_0, r_1, \dots) \in W(R) \mid r_0 = x_0, \dots, r_n = x_n\}$ (we also use the fact that $(-1) \cdot (r_0, r_1, \dots) = (-r_0, -r_1, \dots)$).
Therfore, we easily see that $W(R)$ is p-adically complete. By perfectness of R, p is a non-zero divisor; so $W(R)$ is a strict perfect p-ring with residue ring R.
Since the map T verifies $T(x) \mod p = x$ for all $x \in R$ and is multiplicative, then it is identified with the Teichmüller lift. Moreover, we have

$$\sum_{i=0}^{\infty} [r_i] p^i = (r_0, r_1^p, r_2^{p^2}, \dots)$$

Indeed, looking at the sum in $\widehat{S_{\mathbb{N} \amalg \mathbb{N}}}$

$$x' + y' = \sum_{n=0}^{\infty} [\overline{S_n}(X_i, Y_j)]^{1/p^n} p^n$$

if set $Y_1 = Y_2 = \dots = 0$, we see that $\overline{S_0}(X_i, Y_0, 0, \dots) = X_0 + Y_0$ and $\overline{S_n}(X_i, Y_0, 0, \dots) = X_n$. This shows that in $W(R)$,

$$(r_0, r_1, r_2, \dots) + (x, 0, 0, \dots) = (r_0 + x, r_1, r_2, \dots)$$

which, together with the formula for multiplication by p, shows the above equality.

Lemma B.0.3. *Let Λ to be a set of indices and A to be a perfect p-ring of residual ring R. If $\bar{f} : \overline{S_\Lambda} \to R$ is a morphism of rings and if $\tilde{f} : \overline{S_\Lambda} \to A$ is a multiplicative map lifting $\bar{f}$, then there exists a unique morphism of rings $\widehat{f} : \widehat{S_\Lambda} \to A$ such that $\widehat{f} \circ [\cdot] = \tilde{f}$.*

Proof. We should clearly have $f(\sum_{i=0}^{\infty} [x_i] p^i) = \sum_{i=0}^{\infty} \tilde{f}(x_i) p^i$, which proves the uniqueness. Now let $f : S_\Lambda \to A$ sending X_α^{1/p^n} to $\tilde{f}(X_\alpha^{1/p^n})$ for $\alpha \in \Lambda$ and $n \in \mathbb{N}$. We extend by continuity to a morphism $\widehat{f} : \widehat{S_\Lambda} \to A$ whose reduction mod p agrees with $\bar{f}$ on the X_α^{1/p^n}. Since they generate $\overline{S_\Lambda}$ the two maps are equal; which gives the desired property. ■

If $(r_\alpha)_{\alpha \in \Lambda}$ is a set of generators of R as an $\mathbb{F}_p$-algebra, get a map $\bar{f} : \overline{S_\Lambda} \to R$ sending X_α^{1/p^n} to r_α^{1/p^n} for all $\alpha \in \Lambda$ and $n \in \mathbb{N}$. We also get a multiplicative map $\tilde{f} = [\cdot] \circ \bar{f} : \overline{S_\Lambda} \to W(R)$. By the previous lemma applied to $A = W(R)$, there exists a morphism of rings $\widehat{f} : \widehat{S_\Lambda} \to W(R)$ making the following diagram commute

$$\begin{tikzcd}
\widehat{S_\Lambda} \arrow[r, "\widehat{f}"] \arrow[d, shift left] & W(R) \arrow[d, shift left] \\
\overline{S_\Lambda} \arrow[r, "\bar{f}"] \arrow[u, shift left, "{[\ ]}"] \arrow[ur, "\tilde{f}"] & R! \arrow[u, shift left, "{[\ ]}"]
\end{tikzcd}$$

Since $\bar{f}$ is surjective, we get that $\widehat{f}$ is also surjective, and inspecting the commutative diagram shows that

$$\mathrm{Ker}(\widehat{f}) = \{\sum_{i=0}^{\infty}[s_i]p^i \in \widehat{S_\Lambda} \mid s_i \in \mathrm{Ker}(\bar{f}) \ \forall\ i \in \mathbb{N}\}$$

Theorem B.0.4. *Let R be a perfect ring of characteristic p. Then the ring of Witt vectors satisfy the following universal property: if A is a p-ring of residue ring R' and if $th\bar{e}ta : R \to R'$ is a ring homomorphism with a multiplicative map $\tilde{\theta} : R \to A$ lifting $\bar{\theta}$, then there exists a unique ring morphism $\theta : W(R) \to A$ such that $\theta([x]) = \tilde{\theta}(x)$ for all $x \in R$.*

Proof. Applying the previous lemma to the composition $\overline{S_\Lambda} \xrightarrow{\bar{f}} R \xrightarrow{\bar{\theta}} R'$ and $\overline{S_\Lambda} \xrightarrow{\bar{f}} R \xrightarrow{\tilde{\theta}} A$, we obtain a commutative diagram

$$\begin{tikzcd}
\widehat{S_\Lambda} \arrow[rr, "\Theta"] \arrow[d, shift left] & & A \arrow[d, shift left] \\
\overline{S_\Lambda} \arrow[r, "\bar{f}"] \arrow[u, shift left, "{[\ ]}"] & R \arrow[r, "\theta"] \arrow[ur, "\tilde{\theta}"] & R' \arrow[u, shift left, "{[\ ]}"]
\end{tikzcd}$$

we see that $\mathrm{Ker}(\widehat{f}) \subset \mathrm{Ker}(\Theta)$, therefore Θ factors through $W(R)$ which shows the universal property. ■

These result could be established for any strict p-ring of residue ring R. Therfore, by the universal property, we conclude that $W(R)$ is the unique strict p-ring of residue ring R up to isomorphism.

Appendix C

Kähler Differentials

Let k be a ring, A a k-algebra and M an A-module.

Definition C.0.1. *A derivation $D : A \to M$ is a map such that that*

- $D(a+b) = D(a) + D(b)$
- $D(ab) = aD(b) + bD(a)$

The set of all derivations from A to M is denoted $Der(A, M)$. We equip it with a structure of A-module in the obvious way.
If moreover $D(w) = 0$ for all $w \in k$, we say that D is a k-derivation and we denote, $Der_k(A, M)$ the subset of $Der(A, M)$ of k-derivations.

Remark C.0.2. *Given a commutative diagram in the category of k-algebras*

$$\begin{tikzcd} B \arrow[rr, "f"] & & A \\ & C \arrow[ul, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

we say that h is a lift of g to B.
Let $N = \operatorname{Ker} f \subset B$. If h' is another lift of g, then $h - h'$ is a morphism from C to N. If $N^2 = 0$, then N is an $f(B)$-module and by $g : C \to f(B) \subset A$, a C-module. $h - h' : C \to N$ is a k-derivation. Indeed

$$\begin{aligned} (h-h')(ab) &= h(a)h(b) - h'(a)h'(b) \\ &= h(a)(h(b) - h'(b)) + h'(b)(h(a) - h'(b)) \\ &= a.(h-h')(b) + b.(h-h')(a) \end{aligned}$$

since $h(a)$ is a lift of $g(a)$ and $h'(b)$ is a lift of $g(b)$.
Conversely, if $D \in Der_k(C, N)$, then clearly $h + D$ is a lift of g.
Now for an A-module M, we give the A-module $A \oplus M$ the structure of a k-algebra by defining the product as

$$(a, m)(a', m') = (aa', am' + a'm) \quad \text{for } a, a' \in A \text{ and } m, m' \in M$$

we write $A * M$ for this algebra.

Note that M is an ideal of $A * M$ with square zero. And the projection $\pi : A * M \to A$ is a morphism of k-algebras as well. Considering the commutative diagram

$$\begin{tikzcd}
A * M \arrow[rr, "\pi"] & & A \\
& A \arrow[ul, "h"] \arrow[ur, "id"'] &
\end{tikzcd}$$

we see by the previous remark that, writing $h(a) = (a, da)$, we get that h is a morphism of k-algebras precisely when d is a k-derivation from A to M. Thus $Der_k(A, M)$ identifies with the set of sections of π.

Proposition C.0.3. *The functor*

$$\begin{aligned} A - Mod &\longrightarrow A - Mod \\ M &\mapsto \quad Der_k(A, M) \end{aligned}$$

is representable, i.e., there exists an A-module denoted $\Omega_{A/k}$ and a k-derivation $d : A \to \Omega_{A/k}$ such that the map

$$\begin{aligned} Hom_A(\Omega_{A/k}, M) &\longrightarrow \; Der_k(A, M) \\ f &\mapsto \; f \circ d \end{aligned}$$

Proof. Define the morphism of k-algebras $\mu : A \otimes_k A \to A$ given by $\mu(x \otimes y) = xy$. Let $I = \operatorname{Ker} \mu$, $\Omega_{A/k} := I/I^2$ and $B = (A \otimes_k A)/I^2$, then we have an exact sequence

$$0 \to \Omega_{A/k} \to B \to A \to 0$$

Consider the commutative diagram

$$\begin{tikzcd}
B \arrow[rr, "\mu"] & & A \\
& A \arrow[ul, "\lambda_1"] \arrow[ul, bend left, "\lambda_2"] \arrow[ur, "1_A"'] &
\end{tikzcd}$$

where $\lambda_1 : 1 \mapsto a \otimes a$ and $\lambda_2 : a \mapsto a \otimes 1$.
By Remark 1, we get that $d := \lambda_1 - \lambda_2$ is a k-derivation from A to Ω_{A_k}.
If $D \in \mathrm{Der}_k(A, M)$, the map

$$\begin{aligned} \varphi : A \otimes_k A &\longrightarrow A * M \\ x \otimes y &\mapsto (xy, xDy) \end{aligned}$$

is a morphism of k-algebras verifying $\varphi(I) \subset M$. Since $M^2 = 0$, we get a morphism of A-modules $f : I/I^2 = \Omega_{A/k} \to M$. Moreover, for $a \in A$ we have

$$f(da) = f(1 \otimes a - a \otimes 1 \mod I^2) = \varphi(1 \otimes a) - \varphi(a \otimes 1) = Da - aD(1) = Da$$

so that $D = f \circ d$. Which gives surjectivity of the desired map.
Moreover, $\Omega_{A/k}$ has the A-module structure induced by multiplication by $a \otimes 1$ in $A \otimes A$ (or the multiplication by $1 \otimes a$ since the difference lies in I. From the formula

$$a \otimes a' = (a \otimes 1)(1 \otimes a' - a' \otimes 1) + aa' \otimes 1$$

we see that $\Omega_{A/k}$ is generated as an A-module by $\{da \mid a \in A\}$ which give sthe uniqueness of the linear map satisfying $D = f \circ d$. ■

Remark C.0.4. *If A is generated as a k-algebra by a subset $U \subset A$, then $\Omega_{A/k}$ is generated as an A-module by $\{da \mid a \in U\}$. Indeed, given $a \in A$, there exists a polynomial $f(X) \in k[X_1, \dots, X_n]$ and $a_i \in U$ such that $a = f(a_1, \dots, a_n)$. Differentiating, we get*

$$da = \sum_{i=1}^{n} f_i(a_1, \dots, a_n) da_i \text{ where } f_i = \partial f / \partial X_i$$

In particular, if $A = k[X_1, \dots, X_n]$, then $\Omega_{A/k} = AdX_1 \oplus \dots \oplus AdX_n$, since the derivation $D_i = \partial / \partial X_i$ verifies $D_i(X_j) = \delta_{i,j}$.

Definition C.0.5. *We say that a k-algebra A is 0-smooth over k if it has the following property: for any k-algebra C with an ideal N such that $N^2 = 0$, then given the commutative diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & C/N \\ k \arrow[r] & C \arrow[u] \end{tikzcd}$$

there exists a morphism of k-algebras v such that the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] \arrow[dr, "v"] & C/N \\ k \arrow[r] & C \arrow[u] \end{tikzcd}$$

is commutative. Moreover, we say that A is 0-unramified over k if there is at most one such v. When A is both 0-smooth and 0-unramified, we say that A is 0-etale.

By remark 1, the condition of being 0-unramified is that $\Omega_{A/k} = 0$ (it would imply that $d = \lambda_1 - \lambda_2 = 0$).

If A is a ring and $S \subset A$ is a multiplicative subset, then the localization A_S is 0-etale over A.
Indeed, given a commutative diagram

$$\begin{tikzcd} S^{-1}A \arrow[r, "u"] \arrow[d] & C/N \\ A \arrow[r, "t"] & C \arrow[u] \end{tikzcd}$$

then for $s \in S$, $t(s)$ is invertible in C/N; say there is $c \in C, n \in N$ such that $t(s)y = 1+n$, then $t(s)y(1-n) = (1+n)(1-n) = 1$. Thus $t(S) \subset C^*$, by the universal property of the localization, there is a unique map $v : S^{-1}1 \to C$ making the lower part of the diagram

$$\begin{tikzcd} S^{-1}A \arrow[r, "u"] \arrow[d] \arrow[dr, "v"] & C/N \\ A \arrow[r, "t"] & C \arrow[u] \end{tikzcd}$$

commutes, but this map would also make the whole diagram commutes.

Lemma C.0.6. *A sequence of A-modules*

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is exact if for every A-module T

$$\hom_A(M'', T) \xrightarrow{g^*} \hom_A(M, T) \xrightarrow{f^*} \hom_A(M', T)$$

the sequence is exact.

Proof. Suppose that $\hom_A(M'', T) \xrightarrow{g^*} \hom_A(M, T) \xrightarrow{f^*} \hom_A(M', T)$ is exact for every T. For $T = M''$, we have $f^* \circ g^*(id_{M''}) = g \circ f = 0$.
Now take $T = M/\mathrm{Im} f$ and let $r : M \to M/\mathrm{Im} f$ be the canonical reduction. Clearly we have $f^*(r) = 0$, hence by exactness of the above sequence, there exists a morphism $s : M'' \to M/\mathrm{Im} f$ such that $s \circ g = r$; therefore we get that $\mathrm{Ker}\, g \subset \mathrm{Im} f$. ∎

Theorem C.0.7. *(First fundamental exact sequence)*
A composite $k \xrightarrow{f} A \xrightarrow{g} B$ of ring homomorphisms lead to an exact sequence of B-modules

$$\Omega_{A/k} \otimes B \xrightarrow{\alpha} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \tag{C.1}$$

where the maps are given by $\alpha(d_{A/k}a \otimes b) = b \cdot d_{B/k}g(a)$ and $\beta(d_{B/k}b) = d_{B/A}b$.
If moreover B is 0-smooth over A, then the sequence

$$0 \to \Omega_{A/k} \otimes B \xrightarrow{\alpha} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \to 0 \tag{C.2}$$

is a split exact sequence.

Proof. For every B-module T we have an exact sequence

$$0 \to \ \mathrm{Der}_A(B, T) \to \ \mathrm{Der}_k(B, T) \to \ \mathrm{Der}_k(A, T) \tag{C.3}$$

Using the identification given in proposition 1, we obtain the exact sequence

$$0 \to \hom_B(\Omega_{B/A}, T) \xrightarrow{\beta^*} \hom_B(\Omega_{B/k}, T) \xrightarrow{\tilde{\alpha}^*} \hom_A(\Omega_{A/k}, T)$$

where $d_{B/A} = \beta \circ d_{B/k}$ and $d_{B/k} \circ g = \tilde{\alpha} \circ d_{A/k}$.
Since we have an isomorphism of B-modules $\hom_A(\Omega_{A/k}, T) \xrightarrow{\simeq} \hom_B(\Omega_{A/k} \otimes_A B, T)$ we get the exact sequence

$$0 \to \hom_B(\Omega_{B/A}, T) \xrightarrow{\beta^*} \hom_B(\Omega_{B/k}, T) \xrightarrow{\alpha^*} \hom_B(\Omega_{A/k} \otimes_A B, T)$$

where $\alpha : \Omega_{A/k} \otimes_A B \to \Omega_{B/k}$ is given by $\alpha(x \otimes b) = b \cdot \tilde{\alpha}(x)$.
This sequence is exact for every B-module T, therefore by lemma 1 we get the desired exact sequence

$$\Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \to 0$$

Now suppose that B is 0-smooth over A. Choose $D \in \mathrm{Der}_k(A, T)$ and consider the diagram

$$\begin{tikzcd} B \arrow[r, "id_B"] & B \\ A \arrow[u, "g"] \arrow[r, "\varphi"] & B * T \arrow[u, "\pi"'] \end{tikzcd}$$

with $\pi(a) = (g(a), Da)$.
By assumption, there exists a section of π $h : B \to B * T$ which can be added to the diagram leaving it commutative. By remark 1, $h = (\mathrm{id}_B, D')$ where D' is an k-derivation from B to T and it verifies $D = D' \circ g$. Therfore, the map $\mathrm{Der}_k(B, T) \to \mathrm{Der}_k(A, T)$ in the sequence (3) is surjective for all B-module T. By lemma 1, this gives the exactness of the sequence (2).
Now take $T = \Omega_{A/k} \otimes B$ and define D by $D(a) = d_{A/k}(a) \otimes 1$, so that $D = D' \circ g$. Let $\alpha' : \Omega_{B/k} \to \Omega A/k \otimes B$ be the map such that $D' = \alpha' \circ d_{B/k}$. Using the expression of α, we see that $\alpha' \circ \alpha = \mathrm{id}_T$, therefore (2) is split. ■

Now consider the case $k \xrightarrow{f} A \xrightarrow{g} B$ where g is surjective; set $\mathfrak{m} := \mathrm{Ker}\, g$ so that $B = A/\mathfrak{m}$, then we would have $\Omega_{B/A} = 0$ since every A-derivation kills A.

Theorem C.0.8. *(Second fundamental exact sequence)*
In the above notation, we have an exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \to 0 \tag{C.4}$$

where δ is the B-linear map defined by $\delta(x \mod \mathfrak{m}^2) = d_{A/k}x \otimes 1$.
Moreover, if B is 0-smooth over k then

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega_{B/k} \to 0 \tag{C.5}$$

is a split exact sequence.

Proof. First notice that δ is well defined. Indeed $\forall\ x, y \in \mathfrak{m}$, $d_{A/k}xy \otimes 1 = xd_{A/k}y \otimes 1 + yd_{A/k}x \otimes 1 = d_{A/k}y \otimes \bar{x} + d_{A/k}x \otimes \bar{y} = 0$.
Consider the sequence(4). Since $\Omega_{B/A} = 0$, by the first fundamental sequence we get that

α is surjective. Hence it remains to prove that $\operatorname{Im}\delta = \operatorname{Ker}\alpha$. So as before, considering an arbitrary B-module T and the sequence

$$0 \to \hom_B(\Omega_{B/k}, T) \xrightarrow{\alpha^*} \hom_B(\Omega_{A/k} \otimes_A B, T) \xrightarrow{\delta^*} \hom_B(\mathfrak{m}/\mathfrak{m}^2, T)$$

we need to show that it is exact.
Via the identification of proposition 1, we can rewrite it as

$$\begin{array}{ccccc} 0 \to \operatorname{Der}_k(B,T) & \longrightarrow & \operatorname{Der}_k(A,T) & \longrightarrow & \hom_B(\mathfrak{m}/\mathfrak{m}^2, T) \\ s \circ d_{B/k} & \mapsto & s \circ \alpha \circ d_{A/k} & & \\ & & t \circ d_{A/k} & \mapsto & [t] \circ \delta \end{array} \tag{C.6}$$

where $[t] : \Omega_{A/k} \otimes_A B \to T$, $x \otimes b \mapsto b \cdot t(x)$. Concretely, the second map takes a derivation $D \in \operatorname{Der}_k(A,T)$ to it's restriction $D_{|\mathfrak{m}} \in \hom_B(\mathfrak{m}/\mathfrak{m}^2, T)$.
First, we have $[\tilde{\alpha}] \circ \delta = \alpha \circ \delta : (x \mod \mathfrak{m}^2) \mapsto d_{A/k}x \otimes 1 \mapsto d_{A/k}g(x) = 0$. Therfore, $\delta^* \circ \alpha^* = 0$.
Now let $D = t \circ d_{A/k}$ such that $[t] \circ \delta = 0$. Then $\forall\ x \in \mathfrak{m}$, $t(d_{A/k})(x) = D(x) = 0$. So D factors through g and thus $D \in \operatorname{Im}\alpha^*$.

Suppose now that B is 0-smooth over k, then looking at the commutative diagram

$$\begin{tikzcd} B \arrow[r, "id_B"] & B \\ k \arrow[u] \arrow[r] & A/\mathfrak{m}^2 \arrow[u, "g"'] \end{tikzcd}$$

we see by definition of 0-smoothness that there is a k-morphism $s : B \to A/\mathfrak{m}^2$ such that $g \circ s = \mathrm{id}_B$. Now considering the diagram

$$\begin{tikzcd} A/\mathfrak{m}^2 \arrow[rr, "g"] & & B \\ & A/\mathfrak{m}^2 \arrow[ul, "s\circ g"] \arrow[ul, bend left, "\mathrm{id}_A"] \arrow[ur, "g"'] & \end{tikzcd}$$

we get by remark 1. that $D = \mathrm{id}_A - s \circ g \in \operatorname{Der}_k(A, \mathfrak{m}/\mathfrak{m}^2)$.
If $\psi \in \hom_B(\mathfrak{m}/\mathfrak{m}^2, T)$, then the composite D' of

$$A \to A/\mathfrak{m}^2 \xrightarrow{D} \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\psi} T$$

is an element of $Der_k(A,T)$ satisfying $D_{|\mathfrak{m}} = \psi$. Indeed, for $x \in \mathfrak{m}$, letting $\bar{x} = x \mod \mathfrak{m}^2$, we have

$$D'(x) = \psi(D(\bar{x})) = \psi(\bar{x} - s \circ g(\bar{x})) = \psi(\bar{x}).$$

Therfore, the last map in (6) is surjective. If we set $T = \mathfrak{m}/\mathfrak{m}^2$, then we see that (5) is a split exact sequence. ■

Proposition C.0.9. *(Base change)*
Let $k \to k'$ be a morphism of rings. For any k-algebra A, let $A' = A \otimes_k k'$ be its base change. There is a canonical isomorphism of A'-modules

$$\Omega_{A'/k'} \simeq \Omega_{A/k} \otimes_k k'$$

Proof. By Yoneda's Lemma, it suffices to prove that there is a natural bijection between $\hom_{A'}(\Omega_{A'/k'}, -)$ and $\hom_{A'}(\Omega_{A/k} \otimes_k k', -) \simeq \hom_A(\Omega_{A/k}, -)$. But by proposition 1., this is equivalent to giving a natural bijection between $\mathrm{Der}_{k'}(A', -)$ and $\mathrm{Der}_k(A, -)$.
But for an A'-module T, we have a natural bijection

$$\begin{aligned} \mathrm{Der}_{k'}(A', T) &\longrightarrow \mathrm{Der}_k(A, T) \\ D' &\mapsto (\ a \mapsto D'(a \otimes 1)\ ) \end{aligned}$$

who's inverse is given by

$$\begin{aligned} \mathrm{Der}_k(A, T) &\longrightarrow \mathrm{Der}_{k'}(A', T) \\ D &\mapsto (\ x \otimes y \mapsto y \cdot D(x)\ ) \end{aligned}$$

indeed, we have $D'(xx' \otimes yy') = yy'D(xx') = yy'xD(x') + yy'x'D(x) = x \otimes y \cdot D'(x' \otimes y') + x' \otimes y' \cdot D'(x \otimes y)$.
Thus we the desired natural bijection. ∎

Proposition C.0.10. *Let k be a ring, A a k-algebra and $S \subset A$ a multiplicatively closed subset. Then we have a natural isomorphism*

$$S^{-1}\Omega_{A/k} \simeq \Omega_{S^{-1}A/k}$$

Proof. Since S^{-1} is 0-smooth over A, by the first fundamental exact sequence we have that

$$0 \to \Omega_{A/k} \otimes_A S^{-1}A \to \Omega_{S^{-1}A/k} \to \Omega_{S^{-1}A/A} \to 0$$

is exact. Thus we only need to show that $\Omega_{S^{-1}A/A} = 0$. But since it is an S^1A-module, localizing at S does nothing to it, so we have

$$\Omega_{S^{-1}A/A} = \Omega_{S^{-1}A/A} \otimes_A S^{-1}A$$

By the base change formula we have $\Omega_{S^{-1}A/A} \otimes_A S^{-1}A \simeq \Omega_{S^{-1}A \otimes_A S^{-1}A/A} \simeq \Omega_{S^{-1}A/S^{-1}A} = 0$ (since $S^{-1}A \otimes_A S^{-1}A \simeq S^{-1}A$). ∎

Theorem C.0.11. *Let L/K be a separable algebraic extension. Then L is 0-etale over K (in particular $\Omega_{L/K} = 0$). Moreover, for any subfield $k \subset K$ we have*

$$\Omega_{L/k} = \Omega_{K/k} \otimes_K L$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{u} & C/N \\ \uparrow & & \uparrow \\ K & \longrightarrow & C \end{array}$$

such that $N^2 = 0$. If L' is an intermediate field that is finite over K, by the primitive element theorem $L = K(\alpha)$. Let $f(X) \in K[X]$ be the minimum polynomial of α over K so that $L = K[X]/(f)$ and $f'(\alpha) \neq 0$.
To lift $u_{|L'} : L' \to C/N$ to C, we only need to find an element $y \in C$ such that $f(y) = 0$ and $y \equiv u(\alpha) \mod N$. For that, let y be any lift of $u(\alpha)$. We have $f(y) \equiv f(u(\alpha)) \equiv u(f(\alpha)) \equiv 0 \mod N$. Moreover, we have $N^2 = 0$, so we get

$$f(y + \eta) = f(y) + f'(y) \cdot \eta$$

but $f'(\alpha)$ is a unit of L, so that $u(f'(\alpha)) \equiv f'(y) \mod N$ is a unit of C/N but since $N^2 = 0$ $f'(y)$ is a unit of C. Thus if we take $\eta = -\frac{f(y)}{f'(y)} \in N$, we have $f(y + \eta) = 0$. The K-algebra morphism $v : L' \to C$ obtained by taking α to $y + \eta$ is a lifting of $u_{|L'}$.
Now notice that for every separable algebraic element α over K with minimal polynomial $f(X)$ we have

$$0 = d(f(\alpha)) = f'(\alpha)d\alpha$$

But since $f'(\alpha) \neq 0$, we have $d\alpha = 0$; so that $\Omega_{L/K} = 0$ (since it is generated by $d\alpha$ for $\alpha \in L$. We also have $\Omega_{L'/K} = 0$, which by definition 2 (0-ramified), implies that the lift v is unique. Thus gluing the unique lifts of the finite extensions $K \subset L' \subset L$, we obtain a lift $v : L \to C$ which is unique. Hence L is 0-etale.
Using the fact that L is 0-smooth over K and that $\Omega_{L/K} = 0$, the first fundamental exact sequence gives us the desired equality.

■

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