Some technical stuff

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1 Preliminaries

Let A be a ring. We de fine quotient category $\mathbf{K}(A)$ is to be the category whose objects are cochain complexes of A-modules (the objects of $\mathbf{Comp}(\mathbf{Mod}_A)$) and whose morphisms are the chain homotopy equivalence classes of maps in $\mathbf{Comp}(\mathbf{Mod}_A)$. That is, $\mathrm{Hom}_{\mathbf{K}(A)}(A, B)$ is the set $\mathrm{Hom}_{\mathbf{Comp}(\mathbf{Mod}_A)}(A, B)/\sim$ of homotopy equivalence classes of maps in $\mathbf{Comp}(\mathbf{Mod}_A)$.
The derived category $\mathbf{D}(A)$ is defined to be the localisation $S^{-1}\mathbf{K}(A)$ of the category $\mathbf{K}(A)$ with respect to the collection S of quasi-isomorphisms. $\mathbf{K}(A)$ and $\mathbf{D}(A)$ are both triangulated categories.
In $\mathbf{D}(A)$, there exist a functor

$$- \otimes_A^{\mathbb{L}} - : \mathbf{D}(A) \to \mathbf{D}(A), \quad (L, K) \mapsto L \otimes_A^{\mathbb{L}} K$$

called the derived tensor product which is an exact functor of triangulated categories. We also have a functor

$$R\,\mathrm{Hom} : \mathbf{D}(A)^{\mathrm{op}} \times \mathbf{D}(A) \to \mathbf{D}(A), \quad (L, K) \mapsto R\,\mathrm{Hom}_A(K, L)$$

which is characterised by the formula

$$\mathrm{Hom}_{\mathbf{D}(A)}(K, R\,\mathrm{Hom}_A(L, M) = \mathrm{Hom}_{\mathbf{D}(A)}(K \otimes_A^{\mathbb{L}} L, M)$$

For $L, K \in \mathbf{D}(A)$, we define the i-th extension group of L by K to be

$$\mathrm{Ext}_A^i(L, K) := \mathrm{Hom}_{\mathbf{D}(A)}(L, K[i]) = \mathrm{Hom}_{\mathbf{D}(A)}(L[-i], K)$$

We have that $\mathrm{Ext}_A^i(L, K) = H^i(R\,\mathrm{Hom}_A(L, K))$

2 Derived completion

Let $K \in \mathbf{D}(A)$ and $f \in A$. We denote $T(K, f)$ the derived limit of the system

$$\cdots \to K \xrightarrow{f} K \xrightarrow{f} K$$

Lemma 2.1. *The following are equivalent*

1. $\mathrm{Ext}_A^n(A_f, K) = 0$ *for all* n,
2. $\mathrm{Hom}_{\mathbf{D}(A)}(E, K) = 0$ *for all* E *in* $\mathbf{D}(A_f)$,
3. $T(K, f) = 0$,
4. *for every* $p \in \mathbb{Z}$ *we have* $T(H^p(K), f) = 0$,

5. *for every* $p \in \mathbb{Z}$ *we have* $\operatorname{Hom}_A(A_f, H^p(K)) = 0$ *and* $\operatorname{Ext}^1_A(A_f, H^p(K)) = 0$,

6. $R\operatorname{Hom}_A(A_f, K) = 0$,

7. *the map* $\prod_{n\geq 0} K \to \prod_{n\geq 0} K$ *sending* $(x_0, x_1, \dots)$ *to* $(x_0 - fx_1, x_1 - fx_2, \dots)$ *is an isomorphism in* $\boldsymbol{D}(A)$.

Proof. Since $\operatorname{Ext}^n_A(A_f, K) = \operatorname{Hom}_{\mathbf{D}(A)}(A_f, K[n]) = \operatorname{Hom}_{\mathbf{D}(A)}(A_f[-n], K)$, we clearly have that 2) implies 1). And since $\operatorname{Ext}^n_A(A_f, K) = H^n(R\operatorname{Hom}_A(A_f, K))$, the equivalence between 1) and 6) is clear. Assume condition 1) and let $I^\bullet$ be a K-injective complex of A-modules representing K. By definition of $R\operatorname{Hom}$, condition 6) $\Leftrightarrow$ 1) signifies that the complex $\operatorname{Hom}_A(A_f, I^\bullet)$ is acyclic. Now for an element $E \in \mathbf{D}(A_f)$, let $M^\bullet$ be a complex of A_f-modules representing E. Then

$$\operatorname{Hom}_{\mathbf{D}(A)}(E, K) = \operatorname{Hom}_{\mathbf{K}(A)}(M^\bullet, I^\bullet) = \operatorname{Hom}_{\mathbf{K}(A_f)}(M^\bullet, \operatorname{Hom}_A(A_f, I^\bullet))$$

As $\operatorname{Hom}_A(A_f, I^\bullet))$ is a K-injective complex of A_f-modules (the above equation proves that), the fact that it is acyclic implies that it is homotopy equivalent to 0. Therefore $\operatorname{Hom}_{\mathbf{K}(A_f)}(M^\bullet, \operatorname{Hom}_A(A_f, I^\bullet)) = 0$ which proves 2).

Consider the following free resolution of A_f as an A-module

$$0 \to \bigoplus_{n\in\mathbb{N}} A \to \bigoplus_{n\in\mathbb{N}} A \to A_f \to 0$$

where the first map sends $(a_0, a_1, a_2 \dots)$ to $(a_0, a_1 - fa_0, a_2 - fa_1, \dots)$ and the second map sends $(a_0, a_1, a_2, \dots)$ to $a_0 + a_1/f + a_2/f^2 + \dots$. Applying $\operatorname{Hom}_A(-, I^\bullet)$, we get

$$0 \to \operatorname{Hom}_A(A_f, I^\bullet) \to \prod I^\bullet \to \prod I^\bullet \to 0$$

Since $\prod I^\bullet$ represents $\prod_{n\in\mathbb{N}} K$ this proves the equivalence of 1) and 7). Moreover, by definition of the derived limit, the above exact sequence shows that $T(K, f)$ is a representative of $R\operatorname{Hom}_A(A_f, K)$ in $\mathbf{D}(A)$. This gives the equivalence of 1) and 3).

We have a spectral sequence (I think this is just the spectral sequence that computes the cohomology of the Hom-bicomplex)

$$E_2^{p,q} = \operatorname{Ext}^q_A(A_f, H^p(K)) \Rightarrow \operatorname{Ext}^{p+q}_A(A_f, K)$$

It degenerates at E_2 since A_f has a projective resolution of length 1 (the above free resolution) and so there are only two non-zero rows ($q = 0, 1$), which gives us the exact sequence

$$0 \to \operatorname{Ext}^1_A(A_f, H^{p-1}(K)) \to \operatorname{Ext}^p_A(A_f, K) \to \operatorname{Hom}_A(A_f, H^p(K)) \to 0$$

This shows that 4) and 5) are equivalent to 1). □

Lemma 2.2. *Let* $I \subset A$ *be an ideal and* M *be an* A*-module.*

1. *If* M *is* I*-adically complete, then* $T(M, f) = 0$ *for every* $f \in I$

2. *If* $T(M, f) = 0$ *for every* $f \in I$*, and* I *is finitely generated, then the map* $M \to \varprojlim M/I^n M$ *is surjective*

Proof. 1) assume that M is p-adically complete. By 5. of lemma 2.1, it suffices to prove that $\mathrm{Ext}_A(A_f, M) = 0$ and $\mathrm{Hom}_A(A_f, M) = 0$. But

$$\mathrm{Hom}_A(A_f, M) = \mathrm{Hom}_A(A_f, \varprojlim M/I^n M) = \varprojlim \mathrm{Hom}_A(A_f, M/I^n M) = 0$$

since for every $n \geq 1$, $\mathrm{Hom}_A(A_f, M/I^n M) = 0$.
Now since Ext^1 characterises extensions, we need to show that every extension

$$0 \to M \to E \to A_f \to 0$$

is split. So for each $n \geq 1$, select a $e_n \in E$ mapping to $1/f^n$, and set $\delta_n = fe_{n+1} - e_n \in M$. So the element

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \dots$$

exists since M is f-adically complete and maps to $1/f^n$. Since $e'_n = fe^{n+1}$, we can define a splitting sending $1/f^n$ to e'_n.

2) Assume that $I = (f_1, \dots, f_r)$ and that $T(M, f_i) = 0$ for $i = 1, \dots, r$. One easily shows that if $M \to \varprojlim M/f_i^n M$ is surjective for every f_i, then $M \to \varprojlim M/I^n M$ is surjective. So we can assume that $I = (f)$ and that $T(M, f) = 0$.
Consider some $x_n \in M$ for $n \geq 0$ and the extension

$$0 \to M \to E \to A_f \to 0$$

where $E = (M \oplus \bigoplus Ae_n)/\langle x_n - fe_{n+1} + e_n \rangle$. Again by 5. of lemma 2.1, this extension is split, so we obtain an element that we can write $x + e_0$ ($x \in M$) that generates a copy of A_f in E $x + e_0 = x - x_0 + fe_1 = x - x_0 - fx_1 + f^2 e_2 = \dots$ By the snake lemma, we have $M/f^n M = E/f^n E$ and since $x + e_0 \in f^n E$, we get that $x = x_0 + fx_1 + \dots + f^{n-1}x_{n-1} \mod f^n M$. Which shows the surjectivity of the desired map. □

Definition 2.3. Let I be an ideal of A and $K \in \mathbf{D}(A)$. We say that K is derived complete with respect to I if for every $f \in I$ we have $T(K, f) = 0$. We denote by $\mathbf{D}_{comp}(A) = \mathbf{D}_{comp}(A, I)$ the full subcategory of $\mathbf{D}(A)$ consisting of derived complete objects with respect to I.
If M is an A-module, we say that M is derived complete with respect to I if $M[0] \in \mathbf{D}(A)$ is derived complete with respect to I.

Corollary 2.4. *If the ideal $I \subset A$ is finitely generated, and M is an A-module, then the following are equivalent*

1. *M is I-adically complete,*
2. *M is derived complete with respect to I and I-adically separated*

Proof. Direct consequence of 2.2. □

Proposition 2.5. *Let I be a finitely generated ideal of a ring A. The inclusion functor $\boldsymbol{D}_{comp}(A, I) \to \boldsymbol{D}(A)$ has a left adjoint, i.e, there exist a map sending any object K of $\boldsymbol{D}(A)$ to a derived complete object $K^\wedge$ of $\boldsymbol{D}(A)$ such that the map*

$$\mathrm{Hom}_{\boldsymbol{D}(A)}(K^\wedge, E) \to \mathrm{Hom}_{\boldsymbol{D}(A)}(K, E)$$

is a bijection whenever E is derived complete. In fact, if A is generated by $f_1, \dots, f_r \in A$, we have

$$K^\wedge = R\,\mathrm{Hom}\left((A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0}, f_{i_1}} \to \dots \to A_{f_1, \dots, f_r}), K\right)$$

Proof. Let $K^\wedge$ be defined as above. Then the map of complexes

$$(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0},f_{i_1}} \to \cdots \to A_{f_1,\dots,f_r}) \to A$$

induces a map $K \to K^\wedge$. It suffices to show that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.
Let $f \in A$ We have

$$\begin{aligned} R\operatorname{Hom}_A(A_f, K^\wedge) &= R\operatorname{Hom}\big(A_f, R\operatorname{Hom}\big((A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0},f_{i_1}} \to \cdots \to A_{f_1,\dots,f_r}), K\big)\big) \\ &= R\operatorname{Hom}\big(A_f \otimes^{\mathbb{L}}_A (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0},f_{i_1}} \to \cdots \to A_{f_1,\dots,f_r}), K\big) \\ &= R\operatorname{Hom}\big((A_f \to \prod_{i_0} A_{ff_{i_0}} \to \prod_{i_0<i_1} A_{ff_{i_0},f_{i_1}} \to \cdots \to A_{ff_1,\dots,f_r}), K\big) \end{aligned}$$

The last equality is true by looking at the definition of the derived tensor product and noticing that the complex $(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0},f_{i_1}} \to \cdots \to A_{f_1,\dots,f_r})$ is K-flat (every element of the complex is a flat A-module).
Now for $f \in I$, the complex

$$(A_f \to \prod_{i_0} A_{ff_{i_0}} \to \prod_{i_0<i_1} A_{ff_{i_0},f_{i_1}} \to \cdots \to A_{ff_1,\dots,f_r})$$

is 0 in $\mathbf{D}(A)$ by corollary 4.4. Hence $R\operatorname{Hom}_A(A_f, K^\wedge) = 0$, so $K^\wedge$ is derived complete by lemma 2.1.
Conversely, by the same lemma 2.1, we have $R\operatorname{Hom}_A(A_f, K) = 0$ for each $f = f_{i_0} \cdots f_{i_p}$, hence $K \to K^\wedge$ is an isomorphism in $D(A)$. ☐

Lemma 2.6. *Let $I \subset A$ be an ideal and let (K_n) be an inverse system of objects of $\boldsymbol{D}(A)$ such that for all $f \in I$, there exist $e = e(n, f)$ such that f^e is zero on K_n. Then for $K \in \boldsymbol{D}(A)$, the object $K' = K \otimes^{\mathbb{L}}_A K_n$ is derived complete with respect to I.*

Proof. The category of derived complete objects being preserved under $R\lim$, it suffices to show that each $K \otimes^{\mathbb{L}}_A K_n$ is derived complete. But by assumption, for all $f \in I$, there exist e such that f^e is zero in $K \otimes^{\mathbb{L}}_A K_n$. Hence $T(K \otimes^{\mathbb{L}}_A K_n, f)$=0. ☐

2.1 Some useful facts in the principal case

In this subsection, we assume that $I = (f)$ for some $f \in A$. One can prove -I am definitely not doing that here but it is just technical- that in this case, we have

$$K^\wedge = R\lim\big(K \otimes^{\mathbb{L}}_A (A \xrightarrow{f^n} A)\big)$$

At least one can see directly from lemma 2.6 that this object is derived complete.

Lemma 2.7. *Let $f \in A$. If there exist an integer $c \geq 1$ such that $A[f^c] = A[f^{c+1}] = \dots$, then for all $n \geq 1$, there exist maps*

$$(A \xrightarrow{f^n} A) \to A/f^n, \quad \text{and} \quad A/(f^{n+c}) \to (A \xrightarrow{f^n} A)$$

in $\boldsymbol{D}(A)$ inducing an isomorphism of the pro-objects $\{A/f^nA\}$ and $\{(A \xrightarrow{f^n} A)\}$ in $\boldsymbol{D}(A)$.

Proof. The first map is given by the following commutative diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{f^n} & A & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

For the second arrow, first we define a map

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & A/A[f^c] & \xrightarrow{f^{n+c}} & A & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow{\scriptstyle f^c} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{f^n} & A & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

But since the arrow $A/A[f^c] \xrightarrow{f^{n+c}} A$ is injective, the first row is quasi-isomorphic to $A/f^{n+c}A$ which gives the second map. □

Lemma 2.8. *Let A be a ring and $f \in \mathbb{A}$. We have the naive derived completion $K \mapsto K' = R\lim(K \otimes_A^{\mathbb{L}} A/f^nA)$ and $K \mapsto K^\wedge = R\lim\big(K \otimes_A^{\mathbb{L}} (A \xrightarrow{f^n} A)\big)$. The natural transformation $K^\wedge \mapsto K'$ is an isomorphism if and only if the f-power torsion of A is bounded.*

Proof. We won't need the only if part, so we will only prove the if part. But by lemma 2.7, the pro-objects $\{A/f^nA\}$ and $\{(A \xrightarrow{f^n} A)\}$ are isomorphic. The result follows from lemma 091B (Stack project). □

3 p-complete flatness

Definition 3.1. Given $a, b \in \mathbb{Z} \cup \{\infty\}$, we say that $M \in \mathbf{D}(A)$ has Tor amplitude $[a, b]$ if for any A-module N, we have $M \otimes_A^{\mathbb{L}} N \in \mathbf{D}^{[a,b]}(A)$. If $a = b$, we say that M has Tor amplitude concentrated in degree a.

Definition 3.2. Fix $M \in \mathbf{D}(A)$ and $a, b \in \mathbb{Z} \cup \{\infty\}$.

- We say that M has p-complete Tor amplitude $\in [a, b]$ if $M \otimes_A^{\mathbb{L}} A/pA \in \mathbf{D}(A/pA)$ has Tor amplitude concentrated in $[a, b]$. If $a = b$, we say that $M \in \mathbf{D}(A)$ has p-complete Tor amplitude concentrated in degree a.
- We say that M is p-completely (faithfully) flat if $M \otimes_A^{\mathbb{L}} A/pA \in \mathbf{D}(A/pA)$ is concentrated in degree 0 and is a (faithfully) flat A/pA-module.

Note that $M \in \mathbf{D}(A)$ having Tor amplitude concentrated in degree 0 just means that M is concentrated in degree 0 and is a flat A-module.
Therefore $M \in \mathbf{D}(A)$ is p-completely flat if and only if it has p-complete Tor amplitude concentrated in degree 0.

Remark 3.3. One can replace in the definition A/pA by A/p^nA for every $n \geq 1$ without changing its meaning.
Indeed, suppose that we have an extension of rings $R \to S$ with $S = R/I$ for an ideal I such that $I^2 = 0$ (I is canonically an S-module). Then $M \in \mathbf{D}(R)$ has tor amplitude in $[a, b]$ if and only if $M \otimes_R^{\mathbb{L}} S \in \mathbf{D}(S)$ has tor amplitude in $[a, b]$.
The only if part, is just a consequence of the stability of the tor amplitude under base change. And for the if part, consider the exact triangle $I \to R \to S$. Applying $M \otimes_R^{\mathbb{L}} -$ gives an exact triangle

$$(M \otimes_R^{\mathbb{L}} S) \otimes_S^{\mathbb{L}} I \to M \to M \otimes_R^{\mathbb{L}} S$$

The leftmost term is in $\mathbf{D}^{[a,b]}(R)$ so tensoring with an R-module N we get an object of $\mathbf{D}^{[a,b]}(R)$. Also by hypothesis we have $(M \otimes^{\mathbb{L}}_R S) \otimes^{\mathbb{L}}_R N = (M \otimes^{\mathbb{L}}_R S) \otimes^{\mathbb{L}}_S (N \otimes^{\mathbb{L}}_R S) \in \mathbf{D}^{[a,b]}(R)$. Therefore $M \otimes^{\mathbb{L}}_R N \in \mathbf{D}^{[a,b]}(R)$.

Lemma 3.4. *Fix $M \in \boldsymbol{D}(A)$ and $a, b \in \mathbb{Z} \cup \{\infty\}$. Let $\widehat{M} \in \boldsymbol{D}(A)$ be the derived p-completion of M. The following are equivalent*

1. *M has p-complete* Tor *amplitude in $[a, b]$ (resp. is p-completely (faithfully) flat)*
2. *$\widehat{M}$ has p-complete* Tor *amplitude in $[a, b]$ (resp. is p-completely (faithfully) flat)*

Proof. The map $M \mapsto \widehat{M}$ induces an isomorphism $M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \widehat{M} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Indeed, we have for every $N \in \mathbf{D}(A)$,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}, N) &\cong \mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M} \otimes^{\mathbb{L}}_{A} (\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A), N) \\ &\cong \mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M}, R\,\mathrm{Hom}_A(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A, N))\end{aligned}$$

$R\,\mathrm{Hom}_A(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A, N)$ can easily be seen to be derived p-complete (use 2. of lemma 2.1). Hence by proposition 2.5, we get that

$$\begin{aligned}Hom_{\mathbf{D}(A)}(M \otimes^{\mathbb{L}}_{A} \mathbb{Z}/p\mathbb{Z}, N) &\cong \mathrm{Hom}_{\mathbf{D}(A)}(\widehat{M}, R\,\mathrm{Hom}_A(\mathbb{Z}/p\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} A, N)) \\ &\cong \mathrm{Hom}_{\mathbf{D}(A)}(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}, N)\end{aligned}$$

which shows the claim.
Now notice that we have $A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = (\cdots \to 0 \to A \xrightarrow{p} A \to 0 \to \dots)$ is quasi-isomorphic to $(\cdots \to 0 \to A[p] \xrightarrow{0} A/pA \to 0 \to \dots)$. This induces an isomorphism

$$\begin{aligned}M \otimes^{\mathbb{L}}_A A/pA \oplus M[1] \otimes^{\mathbb{L}}_A A[p] &\cong M \otimes^{\mathbb{L}}_A (A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}) \\ &\cong M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \\ &\cong \widehat{M} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \\ &\cong \widehat{M} \otimes^{\mathbb{L}}_A A/pA \oplus \widehat{M}[1] \otimes^{\mathbb{L}}_A A[p]\end{aligned}$$

Since the morphism induced from $M \to \widehat{M}$ by $- \otimes^{\mathbb{L}}_A (A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ respects the summands, we get that $M \otimes^{\mathbb{L}}_A A/pA \cong \widehat{M} \otimes^{\mathbb{L}}_A A/pA$, which gives the result. □

Lemma 3.5. *Let $A \to B$ be a map of rings, $M \in \boldsymbol{D}(A)$ and $a, b \in \mathbb{Z} \cap \{\infty\}$.*

1. *If $M \in \boldsymbol{D}(A)$ has p-complete* Tor *amplitude in $[a, b]$ (resp. p-completely (faithfully) flat), then the same holds true for $M \otimes^{\mathbb{L}}_A B \in \boldsymbol{D}(B)$.*
2. *If $A \to B$ is p-completely faithfully flat, then the converse of* 1. *holds true.*

Proof. This is immediate from the discrete case. □

Lemma 3.6. *Suppose that A has p^∞-torsion and let $M \in \boldsymbol{D}(A)$ be derived p-complete with p-complete tor amplitude in $[a, b]$, $a, b \in \mathbb{Z} \cup \{\infty\}$. Then $M \in \boldsymbol{D}^{[a,b]}(A)$.*

Proof. By lemma 2.8, M is the derived limit of $M \otimes^{\mathbb{L}}_A A/p^n A$. But by remark 3.3, all $M \otimes^{\mathbb{L}}_A A/p^n A \in \mathbf{D}^{[a,b]}(A/p^n A)$. Looking at the long exact sequence of cohomology from the exact triangle

$$M \to \prod_n M \otimes^{\mathbb{L}}_A A/p^n A \to \prod_n M \otimes^{\mathbb{L}}_A A/p^n A$$

and noticing that the maps on the highest degree $H^b(M \otimes^{\mathbb{L}}_A A/p^n A)$ are surjective, we get that $M \in \mathbf{D}^{[a,b]}(A)$. □

Lemma 3.7. *Suppose that A has bounded p^∞-torsion.*

1. *If $M \in \boldsymbol{D}(A)$ is derived p-complete and p-completely flat then it is a classically p-complete A-module concentrated in degree 0, with bounded p^∞-torsion, such that M/p^nM is flat over A/p^nA for every $n \geq 1$. Moreover, for every $n \geq 1$, the map*
$$M \otimes_A A[p^n] \to M[p^n]$$
is an isomorphism.

2. *Conversely, if N is a classically p-adically complete A-module with bounded p^∞-torsion such that N/p^nN is flat over A/p^nA for all $n \geq 1$, then $N[0] \in \boldsymbol{D}(A)$ is p-completely flat.*

Proof. **1)** Lemma 3.6 implies that M is concentrated in degree 0. The condition that M is p-completely flat implies that $M \otimes_A^{\mathbb{L}} A/p^nA$ is a flat A/p^nA-module for all $n \geq 1$. But

$$\begin{aligned} M \otimes_A^{\mathbb{L}} A/p^nA = M \otimes_A^{\mathbb{L}} (\cdots \to A \xrightarrow{p^n} A \to 0 \to \dots) = (\cdots \to M \xrightarrow{p^n} M \to 0 \to \dots) \\ \cong (\cdots \to M/p^nM \to \dots) \in \mathbf{D}^{[0,0]}(A/p^nA) \end{aligned}$$

So $M \otimes_A^{\mathbb{L}} A/p^nA = M/p^nM$ is a flat A/p^nA-module for all $n \geq 1$. Moreover, by lemma 2.8, M is the limit of $M \otimes_A^{\mathbb{L}} A/p^nA = M/p^nM$ so it is classically p-complete. $\square$

Corollary 3.8. *Let $A \to B$ be a map of derived p-complete rings.*

1. *If A has bounded p^∞-torsion and $A \to B$ is p-completely flat, then B has bounded p^∞-torsion.*

2. *Conversely, if B has bounded p^∞-torsion and $A \to B$ is p-completely faithfully flat, then A has bounded p^∞-torsion.*

3. *Assume that A and B both have bounded p^∞-torsion. Then the map $A \to B$ is p-completely flat (resp. p-completely faithfully flat) if and only if $A/p^n \to B/p^nB$ is flat (resp. faithfully flat) for all $n \geq 1$.*

4 Appendix

4.1 Derived Limit

Let $\mathcal{D}$ be a triangulated category and (K_n, f_n) be an inverse system of objects of $\mathcal{D}$. We say that an object K of $\mathcal{D}$ is a derived limit of the system (K_n) if the product $\prod K_n$ exists and there is a distinguished triangle

$$K \to \prod K_n \to \prod K_n \to K[1]$$

where the map $\prod K_n \to \prod K_n$ is given by $(k_n) \mapsto (k_n - f_{n+1}(k_{n+1}))$. In this case, we denote $K = R\lim K_n$.

4.2 The Koszul Complex

Definition 4.1. Let R be a ring, E an R-module and $\varphi : E \to R$ an R-module map. We define the Koszul complex $\mathbf{K}_\bullet(\varphi)$ to be the commutative differential graded algebra verifying

1. the underlying graded algebra is the exterior algebra $\wedge(E)$

2. the derivation $d : \mathbf{K}_\bullet(\varphi) \to \mathbf{K}_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = \mathbf{K}_1(\varphi)$

If $e_1 \wedge \cdots \wedge e_n$ is one of the generators of degree n in $\mathbf{K}_\bullet(\varphi)$, then

$$d(e_1 \wedge \cdots \wedge e_n) = \sum_i (-1)^{i+1} \varphi(e_i) e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n$$

If $f_1, \ldots, f_n \in R$, the Koszul complex on $f_1, \ldots, f_n$, denoted by $\mathbf{K}_\bullet(f_\bullet)$ is the Koszul complex associated to the map $(f_1, \ldots, f_r) : R^r \to R$.

Lemma 4.2. *Let $e \in E$ and $f = \varphi(e) \in R$. Then we have*

$$f = de + ed$$

as endomorphisms of $\boldsymbol{K}_\bullet(\varphi)$.
In particular, multiplication by f_i is homotopic to zero on $\boldsymbol{K}_\bullet(f_\bullet)$. So the homology module $H_i(\boldsymbol{K}_\bullet(f_\bullet))$ are annihilated by $(f_1, \ldots, f_r)$

Proof. We have $d(ea) = d(e)a - ed(a) = fa - ed(a)$. □

Lemma 4.3. *The alternating Cech complex*

$$R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0}, f_{i_1}} \to \cdots \to R_{f_1 \ldots f_r}$$

is the colimit of the Koszul complexes $\boldsymbol{K}_\bullet(f_\bullet^n)$.

Proof. The transition maps $\mathbf{K}_\bullet(f_\bullet^n) \to \mathbf{K}_\bullet(f_\bullet^{n+1})$ send $e_{i_0} \wedge \cdots \wedge e_{i_p}$ to $f_{i_0} \ldots f_{i_p} e_{i_0} \wedge \cdots \wedge e_{i_p}$. Hence by sending each Koszul complex to the complex $R \to \prod_{i_0} R \to \prod_{i_0 < i_1} R \to \cdots \to R$ (the obvious map), we get the result by noticing that $R_g = \operatorname{colim}(\cdots \to R \xrightarrow{g} R \xrightarrow{g} R)$. □

Corollary 4.4. *If $(f_1, \ldots, f_r) = R$ then the alternating Cech complex is acyclic.*

Proof. This combines lemma 4.2 and 4.3. □

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