On the constituents of the mod p cohomology of Shimura curves

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Abstract

Let p be a prime number and K a finite unramified extension of $\mathbb{Q}_p$. When p is large enough with respect to $[K : \mathbb{Q}_p]$ and under mild genericity assumptions, we proved in our previous work that the admissible smooth representations π of $\mathrm{GL}_2(K)$ that occur in Hecke eigenspaces of the mod p cohomology of Shimura curves are of finite length. In this paper we obtain various refined results about the structure of subquotients of π, such as their Iwahori-socle filtrations and K_1-invariants, where K_1 is the principal congruence subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$. We also determine the Hilbert series of π as Iwahori-representation under these conditions.

Contents

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1 Introduction

1.1 The main results

Let p be a prime number, F a totally real number field and D a quaternion algebra of center F which is split at all p-adic places and at exactly one infinite place. In order to simplify this introduction we assume that p is inert in F (in the text we only need p unramified in F) and denote by v the unique p-adic place of F. Let $\mathbb{A}_F^{\infty,v}$ denote the ring of finite prime-to-v adèles of F and $\mathbb{F}$ a sufficiently large finite extension of $\mathbb{F}_p$. To any absolutely irreducible continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ and V^v a compact open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$, we associate the admissible smooth representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$:

$$\pi \stackrel{\text{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\overline{r}, H^1_{\text{ét}}(X_{V^vV_v} \times_F \overline{F}, \mathbb{F})), \tag{1}$$

where the inductive limit runs over compact open subgroups V_v of $(D \otimes_F F_v)^\times \cong \mathrm{GL}_2(F_v)$ and $X_{V^vV_v}$ is the smooth projective Shimura curve over F associated to D and V^vV_v. Throughout this introduction we fix π as in (1) such that $\pi \neq 0$. We assume moreover that $\overline{r}$ is sufficiently generic (*strongly generic* in the terminology of this paper, see below) and that a standard multiplicity one assumption, defined below, holds (this multiplicity one assumption is commonly referred to as "the minimal case").

In our previous work we established that π is of finite length. More precisely we showed that π is irreducible if $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is irreducible [BHH+25, Thm. 3.105(i)] and that π is of length at least 3 (if $F \neq \mathbb{Q}_p$) and at most $[F_v : \mathbb{Q}_p] + 1$ if π is reducible [BHH+, Thm. 1.1.1]. We moreover showed in *loc. cit.* that π is uniserial with distinct irreducible constituents if $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is nonsplit reducible. The goal of this paper, which is a continuation of our previous aforementioned paper, is to investigate the irreducible constituents of π when $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is reducible, especially in the more difficult case when $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is nonsplit. (We remark that all but two of the irreducible constituents are supersingular.) In particular, for any subquotient π' of π we determine its Iwahori-socle filtration, its invariants under the first principal congruence subgroup, and the dimension of its associated cyclotomic (φ, Γ)-module, showing in each case that the answer is *local*, i.e. only depends on $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ (as expected).

Let us describe our most important results in more detail.

We set $K \stackrel{\text{def}}{=} F_v$, $f \stackrel{\text{def}}{=} [K : \mathbb{Q}_p]$ and $q \stackrel{\text{def}}{=} p^f$. We denote by ω the mod p cyclotomic character of $\mathrm{Gal}(\overline{K}/K)$ (that we consider as a character of $K^\times$ via local class field theory, where uniformizers correspond to geometric Frobenius elements), and by ω_f, ω_{2f} Serre's fundamental characters of the inertia subgroup I_K of $\mathrm{Gal}(\overline{K}/K)$ of level f, $2f$ respectively. In this introduction, we say that $\overline{r}$ is *generic* if the following conditions are satisfied for $N \stackrel{\text{def}}{=} \max\{12, 2f + 7\}$:

(i) $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible;

(ii) for $w \nmid p$ such that either D or $\overline{r}$ ramifies at w, the framed deformation ring of $\overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ over the Witt vectors $W(\mathbb{F})$ is formally smooth;

(iii) $\overline{r}|_{I_K}$ is up to twist of form

$$\begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix} \text{ with } N \leq r_j \leq p - 3 - N$$

or

$$\begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & \\ & \omega_{2f}^{q(\text{same})} \end{pmatrix} \text{ with } \begin{cases} N \leq r_j \leq p - 3 - N, & j > 0 \\ N + 1 \leq r_0 \leq p - 2 - N. \end{cases}$$

Note that (iii) implies $p \geq \max\{27, 4f + 9\}$ and that (ii) can be made explicit ([Sho16], [BHH+23, Rk. 8.1.1]). We say that $\overline{r}$ is *strongly generic* if the above conditions are satisfied with $N \stackrel{\text{def}}{=}$

$\max\{12, 4f+1\}$. By [BHH$^+$23, Thm. 1.9] (for $\overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$ semisimple) and [Wan23, Thm. 6.3(ii)] (for $\overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$ non-semisimple) for $\overline{r}$ generic there is a unique integer $r \geq 1$ (the "multiplicity") such that, for any (absolutely) irreducible representation σ of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$, we have $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma, \pi) \in \{0, r\}$ (the notation $\overline{r}$ and r is somewhat unfortunate but is consistent with [BHH$^+$23, § 8]).

In the sequel we let $\overline{\rho} \stackrel{\text{def}}{=} \overline{r}^\vee|_{\mathrm{Gal}(\overline{K}/K)}$, where $\overline{r}^\vee$ is the dual of $\overline{r}$.

Let I (resp. I_1) be the subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$ of matrices which are upper triangular modulo p (resp. upper unipotent modulo p) and $K_1 \stackrel{\text{def}}{=} 1 + p\mathrm{M}_2(\mathcal{O}_K) \subseteq I_1$. Let $Z_1 \cong 1 + p\mathcal{O}_K$ be the center of I_1 (or K_1). For any admissible smooth representation π' of $\mathrm{GL}_2(K)$, we consider π'^{I_1} (resp. π'^{K_1}) as finite-dimensional representation of $I/I_1 \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ (resp. $\mathrm{GL}_2(\mathcal{O}_K)/K_1 \cong \mathrm{GL}_2(\mathbb{F}_q)$).

Suppose from now on that $\overline{r}$ is generic, that $r = 1$ and that $\overline{\rho}$ is nonsplit reducible. (Even though not explicitly written in the literature, representations $\overline{r}$ satisfying these conditions always exist, using suitable globalizations arguments (e.g. [GK14, Cor. A.3] and [EGS15, Appendix B]) and refining them to minimal level using [BD14, § 3.3].) Then, we recall that $\pi^{K_1} \cong D_0(\overline{\rho})$ [LMS22, HW18, Le19], where the $\mathrm{GL}_2(\mathcal{O}_K)$-representation $D_0(\overline{\rho})$ was defined in [BP12, Thm. 13.8]. As we recalled above, π is uniserial with distinct irreducible constituents, so any subquotient π' is uniquely a quotient π_1'/π_1 for some subrepresentations $\pi_1 \subseteq \pi_1' \subseteq \pi$. There is a strictly increasing filtration $D_0(\overline{\rho})_{\leq i}$ $(-1 \leq i \leq f)$ of $D_0(\overline{\rho}) = \pi^{K_1}$ defined in [Hu16, Prop. 5.2], and by [BHH$^+$, Thm. 4.3.15] there exist *unique* integers $-1 \leq i_0 \leq i_0' \leq f$ such that $\pi_1^{K_1} = D_0(\overline{\rho})_{\leq i_0}$ and $\pi_1'^{K_1} = D_0(\overline{\rho})_{\leq i_0'}$. Let $D_0(\overline{\rho})_i \stackrel{\text{def}}{=} D_0(\overline{\rho})_{\leq i}/D_0(\overline{\rho})_{\leq i-1}$ for $0 \leq i \leq f$. If $D_0(\overline{\rho}^{\mathrm{ss}})$ denotes the analog of $D_0(\overline{\rho})$ for $\overline{\rho}^{\mathrm{ss}}$, then there exists a decomposition $D_0(\overline{\rho}^{\mathrm{ss}}) = \bigoplus_{i=0}^f D_0(\overline{\rho}^{\mathrm{ss}})_i$ [BP12, Thm. 15.4] such that $D_0(\overline{\rho})_i \subseteq D_0(\overline{\rho}^{\mathrm{ss}})_i$ for all i. The following is one of our main results.

Theorem 1.1.1 (Corollary 6.3.9)**.** *Assume that $\overline{r}$ is generic, that $r = 1$ and that $\overline{\rho}$ is nonsplit reducible. Then for any nonzero subquotient π' of π we have*

$$\pi'^{K_1} \cong D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \oplus_{D_0(\overline{\rho})_{i_0+1}} (D_0(\overline{\rho})_{\leq i_0'}/D_0(\overline{\rho})_{\leq i_0})$$

as $\mathrm{GL}_2(\mathcal{O}_K)$-representations.

Note that if $\overline{\rho}$ is split reducible, we prove a stronger result in Proposition 5.1.

Using the theorem it is not hard to determine the I_1-invariants and the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of any subquotient π', as described in Theorem 1.1.2 below. In fact we do not know how to prove Theorem 1.1.1 directly but rather deduce it with the help of Theorem 1.1.2.

To state Theorem 1.1.2, we recall some more standard notation (for more details, see § 1.4). The set $\mathscr{P}$ parametrizes $\mathrm{JH}(D_0(\overline{\rho})^{I_1})$ and likewise $\mathscr{P}^{\mathrm{ss}} \supseteq \mathscr{P}$ parametrizes $\mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})^{I_1})$, where $\mathrm{JH}(\cdot)$ denotes the set of Jordan–Hölder factors (which are 1-dimensional here since I/I_1 is commutative). Given $\lambda \in \mathscr{P}^{\mathrm{ss}}$ let $\chi_\lambda : I/I_1 \to \mathbb{F}^\times$ denote the corresponding character and let $\ell(\lambda) \in \{0, 1, \ldots, f\}$ be the unique integer i such that $\chi_\lambda \in \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_i)$. Let $W(\overline{\rho})$ denote the set of Serre weights of $\overline{\rho}$ (cf. [BDJ10]), i.e. the irreducible subrepresentations of $\pi|_{\mathrm{GL}_2(\mathcal{O}_K)}$ or equivalently of $D_0(\overline{\rho})$ by its construction ([BP12, § 12]), and similarly define $W(\overline{\rho}^{\mathrm{ss}})$ using $D_0(\overline{\rho}^{\mathrm{ss}})$ (so that $W(\overline{\rho}^{\mathrm{ss}})$ contains $W(\overline{\rho})$). In other words, $W(\overline{\rho}) = \mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})))$, where

$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\cdot)$ denotes the $\mathrm{GL}_2(\mathcal{O}_K)$-socle. For $\sigma \in W(\overline{\rho}^{\mathrm{ss}})$ we let $\ell(\sigma) \stackrel{\mathrm{def}}{=} \ell(\lambda)$, where $\lambda \in \mathscr{P}^{\mathrm{ss}}$ parametrizes $\sigma^{I_1} \subseteq D_0(\overline{\rho}^{\mathrm{ss}})^{I_1}$.

Theorem 1.1.2 (Corollaries 6.2.2 and 6.2.4)**.** *Assume that $\overline{r}$ is generic, that $r = 1$ and that $\overline{\rho}$ is nonsplit reducible. Then for any nonzero subquotient π' of π we have:*

(i) $\mathrm{JH}(\pi'^{I_1}) = \{\chi_\lambda : \lambda \in \mathscr{P},\ i_0 < \ell(\lambda) \le i_0'$ *or* $\lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P},\ \ell(\lambda) = i_0 + 1\}$*;*

(ii) $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi') \cong \Big(\bigoplus_{\sigma \in W(\overline{\rho}), i_0 < \ell(\sigma) \le i_0'} \sigma\Big) \oplus \Big(\bigoplus_{\sigma \in W(\overline{\rho}^{\mathrm{ss}}) \setminus W(\overline{\rho}), \ell(\sigma) = i_0+1} \sigma\Big)$.

(Here part (i) is proved in Corollary 6.2.2 and part (ii) in Corollary 6.2.4.)

By Theorem 1.1.1 we can relate the rank of the (φ, Γ)-module of a subquotient π' to the K_1-invariants π'^{K_1} (in a way which is compatible with [BHH+25, § 2.4.3 Example 1, Conj. 1.4]), generalizing a result of Yitong Wang [Wan, Thm. 1.2] from subrepresentations to subquotients:

Corollary 1.1.3. *Assume that $\overline{r}$ is generic and that $r = 1$. Then for any subquotient π' of π we have*

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = |\,\mathrm{JH}(\pi'^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})|,$$

where $D_\xi^\vee(\pi')$ is the cyclotomic (φ, Γ)-module associated to π' in [BHH+25, § 2.1.1].

This is proved in Corollary 5.3 if $\overline{\rho}$ is semisimple and Corollary 6.3.10 otherwise.

The key in proving Theorem 1.1.2 (and hence Theorem 1.1.1) is the following result which determines the (dual of the) socle filtration of π' as an I-representation. To explain, let $\Lambda \stackrel{\mathrm{def}}{=} \mathbb{F}[\![I_1/Z_1]\!]$ denote the Iwasawa algebra of I_1/Z_1, which is a (noncommutative) noetherian local ring of Krull dimension $3f$. We denote by $\mathfrak{m}$ its maximal ideal. Since π has a central character, any subquotient π' of π is an admissible smooth representation of $\mathrm{GL}_2(K)/Z_1$ and hence its linear dual $\pi'^\vee \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{F}}(\pi', \mathbb{F})$ is a finitely generated Λ-module. For any $\lambda \in \mathscr{P}$ there are explicit graded ideals

$$\mathfrak{a}(\lambda) = \mathfrak{a}_1^f(\lambda) \subseteq \mathfrak{a}_1^{f-1}(\lambda) \subseteq \cdots \subseteq \mathfrak{a}_1^0(\lambda) \subseteq \mathfrak{a}_1^{-1}(\lambda) = \mathrm{gr}_{\mathfrak{m}}(\Lambda)$$

of $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$ (with commutative quotient rings of dimension f), cf. [BHH+, eq. (77)]. If M is a graded module and $k \in \mathbb{Z}$, we let $M(k)$ denote M with shifted grading $M(k)_n \stackrel{\mathrm{def}}{=} M_{n+k}$ for all $n \in \mathbb{Z}$. (With our conventions, note further that $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$ and $\mathrm{gr}_{\mathfrak{m}}(\pi'^\vee)$ are supported in *non-positive* degrees, i.e. the degree d part of $\mathrm{gr}_{\mathfrak{m}}(\pi'^\vee)$ equals $\mathfrak{m}^{-d}\pi'^\vee/\mathfrak{m}^{-d+1}\pi'^\vee$.)

Theorem 1.1.4 (Corollary 6.1.7)**.** *Assume that $\overline{r}$ is* strongly *generic, that $r = 1$ and that $\overline{\rho}$ is nonsplit reducible. Then for any subquotient π' of π we have an isomorphism of graded $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules with compatible I/I_1-actions,*

$$\mathrm{gr}_{\mathfrak{m}}(\pi'^\vee) \cong \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0'}(\lambda)}(-d_\lambda), \tag{2}$$

where $d_\lambda \stackrel{\mathrm{def}}{=} \max\{i_0 + 1 - \ell(\lambda), 0\}$.

For $\overline{\rho}$ semisimple the analogous result is [BHH$^+$, Cor. 3.2.7(ii)]. We remark that $\overline{r}$ generic, rather than strongly generic, is sufficient in case π' is a quotient of π.

Theorem 1.1.4 should be compared with [BHH$^+$, Cor. 4.4.6], which shows that

$$\mathrm{gr}_F(\pi'^\vee) \cong \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0'}(\lambda)}, \tag{3}$$

where F denotes the subquotient filtration induced by the $\mathfrak{m}$-adic filtration on $\pi^\vee$. It also generalizes [BHH$^+$, Thm. 2.1.2] (when $\pi' = \pi$) and [BHH$^+$, Cor. 4.4.5] (when $\pi' \subseteq \pi$), though under stronger genericity assumptions.

We point out the following interesting consequence of Theorem 1.1.4 when $f = 2$. Let π_s be defined analogously to π by using a global Galois representation $\overline{r}_s$ such that $\overline{\rho}^{\mathrm{ss}} \cong \overline{r}_s^\vee|_{\mathrm{Gal}(\overline{K}/K)}$. As $f = 2$ we know that π is uniserial of the form $\pi_0 - \pi_1 - \pi_2$ by [HW22, Thm. 10.37] and that $\pi_s \cong \pi_0 \oplus \pi_1' \oplus \pi_2$ by [BHH$^+$25, Thm. 3.105(ii)] for explicit principal series π_0, π_2 and some irreducible supersingular representations π_1, π_1'. Optimistically one may hope that $\pi_1 \cong \pi_1'$. By comparing Theorem 1.1.4 and [BHH$^+$, Cor. 3.2.7(ii)] we can provide the nontrivial evidence that $\mathrm{gr}_\mathfrak{m}(\pi_1^\vee) \cong \mathrm{gr}_\mathfrak{m}(\pi_1'^\vee)$, cf. Remark 6.1.11 (which also gives a weaker result for $f > 2$). Moreover, $\pi_1^{K_1} \cong \pi_1'^{K_1}$ as $\mathrm{GL}_2(\mathcal{O}_K)$-representations by comparing Theorem 1.1.1 with (the K_1-invariants in) Proposition 5.1.

In another direction, we determine the $\mathfrak{m}_{K_1}^2$-invariants of subquotients in case $\overline{\rho}$ is split reducible, where $\mathfrak{m}_{K_1}$ denotes the maximal ideal of the local ring $\mathbb{F}[\![K_1/Z_1]\!]$. We find, in particular, some weak evidence for the hope that π is semisimple in this case:

Proposition 1.1.5 (Proposition 5.1)**.** *Assume that $\overline{r}$ is generic, that $r = 1$ and that $\overline{\rho}$ is split reducible. For any subrepresentations $\pi_1 \subseteq \pi_2$ of π the induced sequence of $\mathrm{GL}_2(K)$-representations*

$$0 \to \pi_1[\mathfrak{m}_{K_1}^2] \to \pi_2[\mathfrak{m}_{K_1}^2] \to (\pi_2/\pi_1)[\mathfrak{m}_{K_1}^2] \to 0$$

is split exact.

Finally, we determine the Hilbert series of the associated graded module $\mathrm{gr}_\mathfrak{m}(\pi^\vee)$, namely the series $h_\pi(t) \stackrel{\mathrm{def}}{=} \sum_{n \geq 0} \dim_\mathbb{F}(\mathfrak{m}^n\pi^\vee/\mathfrak{m}^{n+1}\pi^\vee)t^n \in \mathbb{Z}[\![t]\!]$. If $\overline{\rho}$ is nonsplit reducible let $d_{\overline{\rho}} \in \{0, 1, \dots, f-1\}$, so that $2^{d_{\overline{\rho}}} = |W(\overline{\rho})|$.

Theorem 1.1.6 (Theorem 4.1)**.** *Assume that $\overline{r}$ is generic and that $r = 1$.*

(i) *If $\overline{\rho}$ is irreducible, then* $h_\pi(t) = \dfrac{(3+t)^f}{(1-t)^f} - 1.$

(ii) *If $\overline{\rho}$ is split reducible, then* $h_\pi(t) = \dfrac{(3+t)^f}{(1-t)^f} + 1.$

(iii) *If $\overline{\rho}$ is nonsplit reducible, then* $h_\pi(t) = 2^{f-d_{\overline{\rho}}} \cdot \dfrac{(1+t)^{f-d_{\overline{\rho}}}(3+t)^{d_{\overline{\rho}}}}{(1-t)^f}.$

This follows from the special case of Theorem 1.1.4 when $\pi' = \pi$, which we established earlier [BHH$^+$, Thm. 2.1.2]. We also determine the Hilbert series of $h_{\pi'}(t)$ for subquotients π' of π, in case $\overline{\rho}$ is split reducible. (It is possible to determine $h_{\pi'}(t)$ for nonsplit $\overline{\rho}$, but we did not find nice formulas in general.)

In fact, all of our results do not just apply to the global representation π defined in (1), but to an arbitrary smooth representation of $\mathrm{GL}_2(K)$ that satisfies axioms (i)–(v) in section 3.1. In section 7 we verify that a globally defined representation $\pi(\overline{\rho})$ satisfies all of these axioms (and actually relaxing condition (ii) at the beginning of the introduction, see section 7.1 for a detailed explanation).

1.2 Sketch proof of Theorem 1.1.4

In order to sketch the proof of the key Theorem 1.1.4 we assume for simplicity that $\pi' \overset{\text{def}}{=} \pi_2 = \pi/\pi_1$ is a quotient of π, which is where the main difficulty lies. Let N_2' denote the graded $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module on the right-hand side of the theorem, i.e. $N_2' \overset{\text{def}}{=} \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda)$ (as $i_0' = f$). Let $\overline{\mathfrak{m}}$ denote the unique maximal graded ideal of $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$. The proof of Theorem 1.1.4 breaks into three steps:

(a) Show that there exists a surjection $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \twoheadrightarrow N_2'/\overline{\mathfrak{m}}^3$.

(b) Show that $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3 \cong N_2'/\overline{\mathfrak{m}}^3$.

(c) Lift the isomorphism in (b) to an isomorphism $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \cong N_2'$.

For part (a), we let $\Theta_n \overset{\text{def}}{=} \pi[\mathfrak{m}^n]/\pi_1[\mathfrak{m}^n] \subseteq \pi_2[\mathfrak{m}^n] \subseteq \pi_2$ for some integer $n \geq 1$. Hence $\pi_2^\vee \twoheadrightarrow \Theta_n^\vee$ as Λ-modules and so $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\Theta_n^\vee)$. By using our previous work [BHH$^+$, Lemma 2.4.2] we can determine $\pi[\mathfrak{m}^n]$, and hence Θ_n, completely explicitly as an I-representation, provided $\overline{\rho}$ is sufficiently generic relative to n. For n sufficiently large (in fact, $n = i_0 + 4 \leq f + 4$ suffices) a computation shows that $\mathrm{gr}_{\mathfrak{m}}(\Theta_n^\vee)/\overline{\mathfrak{m}}^3 \cong N_2'/\overline{\mathfrak{m}}^3$ and (a) follows.

For part (b) we use some filtered and graded techniques. We first have an exact sequence of filtered Λ-modules,

$$0 \to C \to \pi_2^\vee/\mathfrak{m}^3 \to \pi^\vee/\mathfrak{m}^3 \to \pi_1^\vee/\mathfrak{m}^3 \to 0, \tag{4}$$

where $C = \mathrm{coker}(\mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi^\vee) \to \mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi_1^\vee))$. On the other hand, the exact sequence $0 \to \mathrm{gr}_F(\pi_2^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \to 0$ of graded $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules, where F denotes again the induced filtration on $\pi_2^\vee$, gives rise to the following exact sequence:

$$0 \to C' \to \mathrm{gr}_F(\pi_2^\vee)/\overline{\mathfrak{m}}^3 \to \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)/\overline{\mathfrak{m}}^3 \to \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)/\overline{\mathfrak{m}}^3 \to 0, \tag{5}$$

where C' is an analogous cokernel of graded modules, cf. (50).

We now compare dimensions of corresponding terms in the two exact sequences. By a subtle spectral sequence argument we see that $\mathrm{gr}(C)$ is a subquotient of C' (for a suitable filtration on C), so

$$\dim_{\mathbb{F}}(C) \leq \dim_{\mathbb{F}}(C'). \tag{6}$$

On the other hand,

$$\dim_{\mathbb{F}}(\pi_2^\vee/\mathfrak{m}^3) = \dim_{\mathbb{F}}(\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3) \geq \dim_{\mathbb{F}}(N_2'/\overline{\mathfrak{m}}^3) = \dim_{\mathbb{F}}(\mathrm{gr}_F(\pi_2^\vee)/\overline{\mathfrak{m}}^3), \tag{7}$$

where the inequality results from (a) and the last equality from (3). As the third (resp. fourth) nonzero terms in (4) and (5) evidently have the same dimensions, we deduce that $\dim_{\mathbb{F}}(C) - \dim_{\mathbb{F}}(\pi_2^\vee/\mathfrak{m}^3) = \dim_{\mathbb{F}}(C') - \dim_{\mathbb{F}}(\mathrm{gr}_F(\pi_2^\vee)/\overline{\mathfrak{m}}^3)$, hence equality holds in (6) and (7), so (b) follows (using (a)).

For part (c), we start with the map of graded modules $f : N_2' \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3$ from (b). By showing that N_2' admits a presentation of the form $\mathrm{gr}(\Lambda)(1)^{\oplus i} \oplus \mathrm{gr}(\Lambda)(2)^{\oplus j} \to \mathrm{gr}(\Lambda)^{\oplus k} \to N_2' \to 0$ for some integers $i, j, k \geq 0$ we can lift f to $\widetilde{f} : N_2' \to \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$. The map $\widetilde{f}$ is surjective by Nakayama's lemma and injective by a computation of cycles, using that N_2' is Cohen–Macaulay (and computing cycles using (3)).

1.3 Sketch proof of Theorem 1.1.1 and Theorem 1.1.2

Part (i) of Theorem 1.1.2 follows directly from Theorem 1.1.4, by evaluating both sides in degree 0. Part (ii) follows relatively easily by building on part (i), using that every irreducible $\mathrm{GL}_2(\mathcal{O}_K)$-representation has nonzero I_1-invariants.

The proof of Theorem 1.1.1 in subsection 6.3 is technically the most involved argument of this paper. Again the essential difficulty is when $\pi' \overset{\mathrm{def}}{=} \pi_2$ is a quotient of π, which we assume from now on.

It is relatively straightforward to understand the right-hand side of Theorem 1.1.1 as $\mathrm{GL}_2(\mathcal{O}_K)$-representation, namely $D_{i_0} \overset{\mathrm{def}}{=} D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \oplus_{D_0(\overline{\rho})_{i_0+1}} (D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0})$ (as $i_0' = f$), and to relate it to $\pi_2^{K_1}$:

(a) D_{i_0} is multiplicity free;

(b) $D_{i_0} \hookrightarrow \pi_2^{K_1}$ as $\mathrm{GL}_2(\mathcal{O}_K)$-representations;

(c) the embedding in (b) induces isomorphisms on I_1-invariants and $\mathrm{GL}_2(\mathcal{O}_K)$-socles.

(Here we use Theorem 1.1.2(i) and (ii) for (c).)

The main thrust for showing that the embedding in (b) is an isomorphism is the fact that $\pi_2[\mathfrak{m}^3]$ is multiplicity free (which follows from Theorem 1.1.4).

Suppose that the embedding in (b) is not an isomorphism. Take a minimal subrepresentation V' of $\pi_2^{K_1}$ not contained in D_{i_0}. Then $\tau' \overset{\mathrm{def}}{=} V'/(V' \cap D_{i_0})$ is the irreducible cosocle of V', and one can show that $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$. Note that $[V' : \tau'] \in \{1, 2\}$ by (a) and that $V' \ncong \tau'$ by (c).

Suppose first that $[V' : \tau'] = 1$, and to simplify notation we also assume that $f = 1$ (the argument easily generalizes to $f > 1$). In a first step we show that the radical $\mathrm{rad}(V')$ of V' is semisimple (Lemma 6.3.8(iii)). By known Ext^1 results there are 3 possibilities: (i) $\mathrm{rad}(V') \cong$

$\mu_0^-(\tau')$ or (ii) $\mathrm{rad}(V') \cong \mu_0^+(\tau')$ or (iii) $\mathrm{rad}(V') \cong \mu_0^-(\tau') \oplus \mu_0^+(\tau')$ for certain Serre weights $\mu_0^\pm(\tau')$ associated to the Serre weight τ' as defined in § 2.2. Case (i) is ruled out by the equality of I_1-invariants in (c) above. In cases (ii) and (iii) we enlarge V' slightly to $\widetilde{V'}$, where $V' \subseteq \widetilde{V'} \subseteq \pi_2^{K_1}$. In either case $\widetilde{V'}$ (but not V'!) is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \mathcal{W}$, where $\mathcal{W}$ is the I-representation

$$\mathcal{W} \cong \begin{pmatrix} \chi_{\mu_0^-(\tau')} \diagdown & \\ & \chi_{\tau'} \\ \chi_{\mu_0^+(\tau')} \diagup & \end{pmatrix},$$

where we write $\chi_\sigma \stackrel{\text{def}}{=} \sigma^{I_1}$ for any Serre weight σ. By Frobenius reciprocity we get a nonzero map $\mathcal{W} \to \pi_2[\mathfrak{m}^2]$ whose image is not contained in D_{i_0}. On the other hand, we show that $\tau' \in \mathrm{JH}(D_{i_0})$ and by following the same reasoning as above we obtain another nonzero map $\mathcal{W} \to \pi_2[\mathfrak{m}^2]$ whose image is contained in D_{i_0}. This shows that $[\pi_2[\mathfrak{m}^2] : \chi_{\tau'}] \geq 2$, contradicting the multiplicity freeness of $\pi_2[\mathfrak{m}^2]$.

The case where $[V' : \tau'] = 2$ is similar but more involved, using multiplicity freeness of $\pi_2[\mathfrak{m}^3]$.

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1.4 Notation and preliminaries

We normalize local class field theory so that uniformizers correspond to geometric Frobenius elements. We fix an embedding $\kappa_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and let $\kappa_j \stackrel{\text{def}}{=} \kappa_0 \circ \varphi^j$, where φ is the arithmetic Frobenius on $\mathbb{F}_q$. Given $J \subseteq \{0, \ldots, f-1\}$ we define $J^c \stackrel{\text{def}}{=} \{0, 1, \ldots, f-1\} \setminus J$. We let $I \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ denote the (upper) Iwahori subgroup of $\mathrm{GL}_2(K)$, I_1 the pro-p radical of I, Z_1 the center of I_1, and $K_1 \stackrel{\text{def}}{=} 1 + p\mathrm{M}_2(\mathcal{O}_K) \subseteq I_1$. We let $\Gamma \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbb{F}_q) \cong \mathrm{GL}_2(\mathcal{O}_K)/K_1$.

Let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. We will say that $\overline{\rho}$ is *n-generic* for some integer $n \geq 0$ if, up to twist, $\overline{\rho}|_{I_K}^{\mathrm{ss}} \not\cong \omega^{\pm 1} \oplus 1$ and either (using the notation of § 1.1)

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ & 1 \end{pmatrix} \qquad \text{with } n \leq r_j \leq p-3-n \text{ for all } 0 \leq j \leq f-1 \tag{8}$$

or

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & \\ & \omega_{2f}^{p^f(\text{same})} \end{pmatrix} \qquad \text{with } \begin{cases} n \le r_j \le p-3-n & \text{for } 0 < j \le f-1, \\ n+1 \le r_0 \le p-2-n & \text{for } j=0. \end{cases} \tag{9}$$

In particular, if $\overline{\rho}$ is n-generic then it is n-generic in the sense of [BHH+23, Def. 2.3.4], and $\overline{\rho}$ is 0-generic precisely when $\overline{\rho}$ is generic in the sense of [BP12, Def. 11.7] (note that the condition $\overline{\rho}|_{I_K}^{\mathrm{ss}} \not\cong \omega^{\pm 1} \oplus 1$ precisely rules out that $(r_0, \dots, r_{f-1}) \in \{(0, \dots, 0), (p-3, \dots, p-3)\}$ when $\overline{\rho}$ is reducible).

Attached to a 0-generic $\overline{\rho}$ we have a set $W(\overline{\rho})$ of Serre weights, i.e. irreducible representations of Γ over $\mathbb{F}$, defined in [BDJ10, § 3], and a multiplicity-free finite length Γ-representation $D_0(\overline{\rho})$ over $\mathbb{F}$, defined in [BP12, § 13], which is of the form $D_0(\overline{\rho}) = \bigoplus_{\tau \in W(\overline{\rho})} D_{0,\tau}(\overline{\rho})$, where each $D_{0,\tau}(\overline{\rho})$ is indecomposable (and multiplicity free) with socle the Serre weight τ ([BP12, § 15]).

Suppose that $\overline{\rho}$ is 0-generic. Recall the set $\mathscr{P}$ parametrizing $\mathrm{JH}(D_0(\overline{\rho})^{I_1})$, see [Bre14, § 4] (and denoted there by $\mathscr{P}\mathscr{D}$, resp. $\mathscr{P}\mathscr{I}\mathscr{D}$, if $\overline{\rho}$ is reducible, resp. irreducible). Recall also the subset $\mathscr{D} \subseteq \mathscr{P}$ parametrizing (the I_1-invariants of) the set of Serre weights in $W(\overline{\rho})$ (denoted in *loc. cit.* by $\mathscr{D}$ or $\mathscr{I}\mathscr{D}$ if $\overline{\rho}$ is reducible or irreducible respectively). We let $\mathscr{D}^{\mathrm{ss}} \subseteq \mathscr{P}^{\mathrm{ss}}$ denote the corresponding sets for the semisimplification $\overline{\rho}^{\mathrm{ss}}$ of $\overline{\rho}$, so $\mathscr{P} \subseteq \mathscr{P}^{\mathrm{ss}}$ and $\mathscr{D} \subseteq \mathscr{D}^{\mathrm{ss}}$.

Since we will be using this many times, we recall more precisely that if $\overline{\rho}$ is reducible, $\mathscr{P}^{\mathrm{ss}}$ denotes the set of f-tuples $(\lambda_0(x_0), \dots, \lambda_{f-1}(x_{f-1}))$ such that:

(i) $\lambda_j(x_j) \in \{x_j, x_j+1, x_j+2, p-3-x_j, p-2-x_j, p-1-x_j\}$;

(ii) if $\lambda_j(x_j) \in \{x_j, x_j+1, x_j+2\}$, then $\lambda_{j+1}(x_{j+1}) \in \{x_{j+1}, x_{j+1}+2, p-2-x_{j+1}\}$;

(iii) if $\lambda_j(x_j) \in \{p-3-x_j, p-2-x_j, p-1-x_j\}$, then $\lambda_{j+1}(x_{j+1}) \in \{x_{j+1}+1, p-3-x_{j+1}, p-1-x_{j+1}\}$

and $\mathscr{D}^{\mathrm{ss}}$ is the subset such that $\lambda_j(x_j) \in \{x_j, x_j+1, p-3-x_j, p-2-x_j\}$. Moreover, there exists a unique subset $J_{\overline{\rho}} \subseteq \{0, \dots, f-1\}$ such that

$$\begin{aligned} \mathscr{D} &= \Big\{\lambda \in \mathscr{D}^{\mathrm{ss}} : \lambda_j(x_j) \in \{x_j+1, p-3-x_j\} \Rightarrow j \in J_{\overline{\rho}}\Big\}, \\ \mathscr{P} &= \Big\{\lambda \in \mathscr{P}^{\mathrm{ss}} : \lambda_j(x_j) \in \{x_j+2, p-3-x_j\} \Rightarrow j \in J_{\overline{\rho}}\Big\}. \end{aligned} \tag{10}$$

In particular, $|W(\overline{\rho})| = 2^{|J_{\overline{\rho}}|}$.

For $\lambda \in \mathscr{P}$ we denote by χ_λ the character of H corresponding to λ. (More precisely, in [Bre14, § 4] a Serre weight σ_λ is associated to $\lambda \in \mathscr{P}$ and χ_λ is the action of $H = I/I_1$ on the 1-dimensional subspace $\sigma_\lambda^{I_1}$.) Set

$$J_\lambda \overset{\mathrm{def}}{=} \{j \in \{0, \dots, f-1\} : \lambda_j(x_j) \in \{x_j+1, x_j+2, p-3-x_j\}\} \tag{11}$$

and let $\ell(\lambda) \overset{\mathrm{def}}{=} |J_\lambda|$. By [BP12, § 11] the map $\lambda \mapsto J_\lambda$ induces a bijection between $\mathscr{D}^{\mathrm{ss}}$ and the set of subsets of $\{0, \dots, f-1\}$. Sometimes we will abuse notation and write $J_\tau \overset{\mathrm{def}}{=} J_\lambda$ and

$\ell(\tau) \stackrel{\text{def}}{=} \ell(\lambda)$ if $\tau \in W(\overline{\rho}^{\text{ss}})$ is parametrized by $\lambda \in \mathscr{D}^{\text{ss}}$. Given $\lambda \in \mathscr{D}^{\text{ss}}$ with corresponding subset $J = J_\lambda \subseteq \{0, \ldots, f-1\}$ we write $\delta(\lambda) \in \mathscr{D}^{\text{ss}}$ for the f-tuple defined by $\delta(\lambda)_j \stackrel{\text{def}}{=} \lambda_{j+1}$ for all $j \in \{0, \ldots, f-1\}$, and $\delta(J) \subseteq \{0, \ldots, f-1\}$ for the subset corresponding to $\delta(\lambda)$.

As in [BP12, § 1], given f integers $r_0, \ldots, r_{f-1} \in \{0, \ldots, p-1\}$ we denote by $(r_0, \ldots, r_{f-1})$ the Serre weight

$$\mathrm{Sym}^{r_0}\mathbb{F}^2 \otimes_{\mathbb{F}} (\mathrm{Sym}^{r_1}\mathbb{F}^2)^{\mathrm{Fr}} \otimes \cdots \otimes_{\mathbb{F}} (\mathrm{Sym}^{r_{f-1}}\mathbb{F}^2)^{\mathrm{Fr}^{f-1}},$$

where $\mathrm{GL}_2(\mathbb{F}_q)$ acts on $(\mathrm{Sym}^{r_j}\mathbb{F}^2)^{\mathrm{Fr}^j}$ via $\kappa_j : \mathbb{F}_q \hookrightarrow \mathbb{F}$. Following [HW22, § 2], we say that a Serre weight is *m-generic* for some integer $m \geq 0$ if, up to twist, $\sigma \cong (r_0, \ldots, r_{f-1})$, where $m \leq r_j \leq p-2-m$ for all $j \in \{0, \ldots, f-1\}$. We say that an $\mathbb{F}$-valued character χ of I is *m-generic* if $\chi = \sigma^{I_1}$ for some m-generic Serre weight σ. For any smooth character $\chi : I \to \mathbb{F}^\times$ we define $\chi^s \stackrel{\text{def}}{=} \chi(\Pi(\cdot)\Pi^{-1})$ with $\Pi \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. If σ is a Serre weight, we write χ_σ for the character of I/I_1 on σ^{I_1} and $\sigma^{[s]}$ for the unique Serre weight distinct from σ such that $\chi_{\sigma^{[s]}} = \chi_\sigma^s$. Finally, if $\chi, \chi' : I \to \mathbb{F}^\times$ are smooth characters such that $\mathrm{Ext}^1_{I/Z_1}(\chi', \chi) \neq 0$ we let $E_{\chi,\chi'}$ denote the unique nonsplit extension of χ' by χ, i.e. $0 \to \chi \to E_{\chi,\chi'} \to \chi' \to 0$. (The uniqueness follows from [Hu10, Lemme 2.4].)

Let $\mathfrak{m}_{K_1}$ denote the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![K_1/Z_1]\!]$ and let $\widetilde{\Gamma} \stackrel{\text{def}}{=} \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}^2$ (a finite-dimensional $\mathbb{F}$-algebra). We use the terminology "$\widetilde{\Gamma}$-representations" and "$\widetilde{\Gamma}$-modules" interchangeably.

We write $\widetilde{D}_0(\overline{\rho})$ for the finite-dimensional $\widetilde{\Gamma}$-representation over $\mathbb{F}$ constructed in [HW22, Prop. 4.3]. It is the unique (up to isomorphism) $\widetilde{\Gamma}$-representation which is maximal with respect to the following two properties:

(i) $\mathrm{soc}_{\widetilde{\Gamma}}\, \widetilde{D}_0(\overline{\rho}) \cong \bigoplus_{\sigma \in W(\overline{\rho})} \sigma$;

(ii) any Serre weight of $W(\overline{\rho})$ occurs in $\widetilde{D}_0(\overline{\rho})$ with multiplicity one.

Let $\Lambda \stackrel{\text{def}}{=} \mathbb{F}[\![I_1/Z_1]\!]$, a complete noetherian local ring with maximal ideal $\mathfrak{m} \stackrel{\text{def}}{=} \mathfrak{m}_{I_1/Z_1}$, and let $\mathrm{gr}(\Lambda) \stackrel{\text{def}}{=} \mathrm{gr}_{\mathfrak{m}}(\Lambda) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of Λ with respect to the $\mathfrak{m}$-adic filtration on Λ. The rings Λ and $\mathrm{gr}(\Lambda)$ are Auslander regular (see [BHH+23, Thm. 5.3.4] with [LvO96, Thm. III.2.2.5]). Recall ([BHH+25, § 3.1]) that we have an isomorphism of (noncommutative) algebras

$$\mathrm{gr}(\Lambda) \cong \bigotimes_{j \in \{0, \ldots, f-1\}} \mathbb{F}\langle y_j, z_j, h_j \rangle \tag{12}$$

with relations $[y_j, z_j] = h_j$, $[h_j, z_i] = [y_i, h_j] = 0$ for all $i, j \in \{0, \ldots, f-1\}$. We use increasing filtrations throughout, i.e. $F_n\Lambda = \mathfrak{m}^{-n}$ for $n \leq 0$, and the degrees of y_j and z_j (resp. h_j) are -1 (resp. -2). Define the graded ideal $J \stackrel{\text{def}}{=} (h_j, y_j z_j : 0 \leq j \leq f-1)$ of $\mathrm{gr}(\Lambda)$. As in [BHH+25, § 3] we define

$$R \stackrel{\text{def}}{=} \mathrm{gr}(\Lambda)/(h_j : 0 \leq j \leq f-1) \cong \mathbb{F}[y_j, z_j : 0 \leq j \leq f-1]$$

which is the largest commutative quotient of $\mathrm{gr}(\Lambda)$. We also define the following quotient of R:

$$\overline{R} \stackrel{\text{def}}{=} \mathrm{gr}(\Lambda)/J \cong R/(y_j z_j : 0 \leq j \leq f-1).$$

We recall from [BHH$^+$25, Def. 3.57] that given $\lambda \in \mathscr{P}$ we have an associated ideal $\mathfrak{a}(\lambda) = (t_0, \ldots, t_{f-1})$ of R, where the $t_j = t_j(\lambda)$ are defined as follows:

$$t_j \overset{\text{def}}{=} \begin{cases} z_j & \text{if} \quad \lambda_j(x_j) \in \{x_j, p-3-x_j\} \text{ and } j \in J_{\overline{\rho}} \\ y_j & \text{if} \quad \lambda_j(x_j) \in \{x_j+2, p-1-x_j\} \text{ and } j \in J_{\overline{\rho}} \\ y_j z_j & \text{if} \quad \lambda_j(x_j) \in \{x_j, p-1-x_j\} \text{ and } j \notin J_{\overline{\rho}} \\ y_j z_j & \text{if} \quad \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}. \end{cases} \tag{13}$$

Note that $(y_j z_j : 0 \le j \le f-1) \subseteq \mathfrak{a}(\lambda)$, so we often think of $\mathfrak{a}(\lambda)$ as ideal of $\overline{R}$.

Let $H \overset{\text{def}}{=} \begin{pmatrix} \mathbb{F}_q^\times & 0 \\ 0 & \mathbb{F}_q^\times \end{pmatrix} \cong I/I_1$. We write $\alpha_j : H \to \mathbb{F}^\times$ for the character defined by $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \kappa_j(ad^{-1})$. We recall that for any $j \in \{0, \ldots, f-1\}$ the element y_j (resp. z_j, resp. h_j) is an H-eigenvector with associated eigencharacter α_j (resp. α_j^{-1}, resp. the trivial character).

Suppose that H' is a compact p-adic analytic group and that π_1, π_2 are smooth representations of H' over $\mathbb{F}$. We write $\mathrm{Ext}^i_{H'}(\pi_1, \pi_2)$ for the i-th Ext group computed in the category of smooth representations of H' over $\mathbb{F}$. Dually, the functors $\mathrm{Tor}_i^{\mathbb{F}[\![H']\!]}(\pi_1^\vee, \pi_2^\vee)$ and $\mathrm{Ext}^i_{\mathbb{F}[\![H']\!]}(\pi_1^\vee, \pi_2^\vee)$ are computed in the abelian category of pseudocompact $\mathbb{F}[\![H']\!]$-modules. (See for example [Eme10, § 2].) If σ has finite length, we write $\mathrm{JH}(\sigma)$ for its set of irreducible constituents up to isomorphism.

Throughout this paper, if R is a filtered (resp. graded) ring, a morphism of filtered (resp. graded) R-modules $f : M \to N$ will always be a *filtered (resp. graded) morphism of degree zero*, i.e. satisfying $f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$. If R is any ring and M any left R-module, we recall that $\mathrm{Ext}^i_R(M, R)$ for $i \in \mathbb{Z}_{\ge 0}$ is a right R-module (for $i = 0$ the right R-action is given by $(fr)(m) \overset{\text{def}}{=} f(m)r$ for $r \in R$, $f \in \mathrm{Hom}_R(M, R)$ and $m \in M$) and we use the notation $\mathrm{E}^i_R(M) \overset{\text{def}}{=} \mathrm{Ext}^i_R(M, R)$.

2 Preliminaries

We establish structural results on finite-dimensional smooth mod p representations of $\mathrm{GL}_2(\mathcal{O}_K)$ which will extensively be used in section § 6.

If σ is a Serre weight, we write $\mathrm{Proj}_\Gamma\, \sigma$ (resp. $\mathrm{Inj}_\Gamma\, \sigma$) for a projective cover (resp. injective envelope) of σ in the category of $\mathbb{F}[\Gamma]$-modules. The objects $\mathrm{Proj}_{\widetilde{\Gamma}}\, \sigma$ and $\mathrm{Inj}_{\widetilde{\Gamma}}\, \sigma$ are defined similarly. We remark that $\mathrm{Proj}_\Gamma\, \sigma \cong \mathrm{Inj}_\Gamma\, \sigma$, as Γ is a finite group, but $\mathrm{Proj}_{\widetilde{\Gamma}}\, \sigma \not\cong \mathrm{Inj}_{\widetilde{\Gamma}}\, \sigma$ by [HW22, (2.6), Prop. 2.12].

2.1 Some Γ-representations

We collect a number of results on the combinatorics of Serre weights and injective envelopes.

Recall from [BP12, § 3] that given a Serre weight σ, there is an injective parametrization $\mathrm{JH}(\mathrm{Inj}_\Gamma\, \sigma) \hookrightarrow \mathcal{I}$ (where $\mathcal{I} \overset{\text{def}}{=} \mathcal{I}(x_0, \ldots, x_{f-1})$ is defined in [BP12, § 4]), which is bijective if σ is 1-generic. For $\mu \in \mathcal{I}$, we let $\mu(\sigma)$ be the Serre weight parametrized by μ as defined in [HW22, (2.2)] (recall that $\mu(\sigma)$ remains undefined if the formula in *loc. cit.* is not well defined). We say $\tau_1, \tau_2 \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$ are *compatible* (relative to σ) if the corresponding elements $\mu_1, \mu_2 \in \mathcal{I}$ are compatible in the sense of [BP12, Def. 4.10]. We write $\mu_2 \preceq \mu_1$ if $\mathcal{S}(\mu_2) \subseteq \mathcal{S}(\mu_1)$ and μ_2 is compatible with μ_1, where $\mathcal{S}(-) \subseteq \{0, \dots, f-1\}$ is the subset defined in [BP12, § 4]. It is easy to see that the set $\{\mu_2 \in \mathcal{I} : \mu_2 \preceq \mu_1\}$ is in bijection with all subsets of $\mathcal{S}(\mu_1)$, via $\mu_2 \mapsto \mathcal{S}(\mu_2)$, hence has cardinality $2^{|\mathcal{S}(\mu_1)|}$.

Recall from [BP12, Cor. 3.12] that given $\tau \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$ there exists a unique finite-dimensional Γ-representation $I(\sigma, \tau)$ such that $\mathrm{soc}_\Gamma I(\sigma, \tau) = \sigma$, $\mathrm{cosoc}_\Gamma I(\sigma, \tau) = \tau$ and $[I(\sigma, \tau) : \sigma] = 1$. By [BP12, Cor. 4.11], $\tau' \in \mathrm{JH}(I(\sigma, \tau))$ if and only if $\tau' = \mu'(\sigma)$ with $\mu' \preceq \mu$, where $\tau = \mu(\sigma)$.

Lemma 2.1.1. *Assume that $\overline{\rho}$ is 0-generic. Let $\sigma, \tau \in W(\overline{\rho}^{\mathrm{ss}})$. Then $\tau \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$. Moreover, $|\mathrm{JH}(I(\sigma, \tau))| = 2^{|J_\sigma \Delta J_\tau|}$ and*

$$\begin{aligned} \mathrm{JH}(I(\sigma, \tau)) &= \{\tau' \in W(\overline{\rho}^{\mathrm{ss}}) : J_\sigma \cap J_\tau \subseteq J_{\tau'} \subseteq J_\sigma \cup J_\tau\} && (14) \\ &= \{\tau' \in W(\overline{\rho}^{\mathrm{ss}}) : J_\sigma \Delta J_{\tau'} \subseteq J_\sigma \Delta J_\tau\}. && (15) \end{aligned}$$

In particular, $\mathrm{Ext}^1_\Gamma(\tau, \sigma) \neq 0$ if and only if $|J_\sigma \Delta J_\tau| = 1$.

Proof. First, from [HW18, Prop. 2.24] we deduce that $\tau \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$ and that any irreducible constituent of $I(\sigma, \tau)$ is also an element of $W(\overline{\rho}^{\mathrm{ss}})$.

Let $\lambda_\sigma, \lambda_\tau \in \mathscr{D}^{\mathrm{ss}}$ correspond to σ, τ, respectively. Then there exists a unique element $\mu \in \mathcal{I}$ such that $\lambda_\tau = \mu \circ \lambda_\sigma$. By above we know that

$$\mathrm{JH}(I(\sigma, \tau)) = \{\mu'(\sigma) : \mu' \preceq \mu \text{ in } \mathcal{I}\}. \tag{16}$$

We claim that $\mu'(\sigma)$ is well defined for any $\mu' \preceq \mu$; this will imply that $\mathrm{JH}(I(\sigma, \tau))$ has cardinality $2^{|\mathcal{S}(\mu)|}$. Given $\mu' \preceq \mu$ in $\mathcal{I}$, one checks using the table in the proof of [HW22, Lemma 2.6] that $\mu' \circ \lambda_\sigma \in \mathscr{D}^{\mathrm{ss}}$, so that $\mu'(\sigma) \in W(\overline{\rho}^{\mathrm{ss}})$ is the Serre weight corresponding to $\mu' \circ \lambda_\sigma$ (which is well defined, as $\overline{\rho}$ is 0-generic), proving the claim.

From [HW22, Lemma 2.6(i)] we deduce that $\mathcal{S}(\lambda_\tau) = \mathcal{S}(\mu) \Delta \mathcal{S}(\lambda_\sigma)$, or equivalently $\mathcal{S}(\mu) = J_\sigma \Delta J_\tau$ (as $J_\lambda = \mathcal{S}(\lambda)$ for all $\lambda \in \mathscr{D}^{\mathrm{ss}}$), which implies $|\mathrm{JH}(I(\sigma, \tau))| = 2^{|J_\sigma \Delta J_\tau|}$. For the same reason, if $\tau' \in \mathrm{JH}(I(\sigma, \tau))$ corresponds to $\mu' \preceq \mu$ in $\mathcal{I}$ (i.e. $\tau' = \mu'(\sigma)$), then $\mathcal{S}(\mu') = J_\sigma \Delta J_{\tau'}$, so that (15) (equivalently (14)) follows from (16). The last assertion follows from $\mathrm{Ext}^1_\Gamma(\tau, \sigma) \neq 0$ if and only if $I(\sigma, \tau)$ has length 2. □

Recall from [BHH$^+$, § 3.1] and [BHH$^+$, Rk. 3.1.1] that given a character $\chi : I \to \mathbb{F}^\times$ with $\chi \neq \chi^s$, there is an injective parametrization $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi) \hookrightarrow \mathcal{P}$ (where $\mathcal{P} \stackrel{\mathrm{def}}{=} \mathcal{P}(x_0, \dots, x_{f-1})$ is defined in [BP12, § 2]), which is bijective if χ is 1-generic. Moreover, $\mathcal{P}$ is in bijection with the subsets of $\{0, \dots, f-1\}$ via the map $\xi \mapsto \mathcal{S}(\xi)$ of [BHH$^+$, eq. (36)]. With this parametrization, the socle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$ corresponds to the empty subset. For $J \subseteq \{0, 1, \dots, f-1\}$ let σ_J denote the constituent of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$ parametrized by $\xi \in \mathcal{P}$ such that $\mathcal{S}(\xi) = J$, if such a ξ exists. Finally, if $\sigma \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)$ is parametrized by $\xi \in \mathcal{P}$ we write $\mathcal{S}(\sigma) \stackrel{\mathrm{def}}{=} \mathcal{S}(\xi)$.

Lemma 2.1.2. *Assume that χ is 1-generic. Let $J, J' \subseteq \{0, 1, \dots, f-1\}$. Then $\sigma_{J'} \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma_J)$. Moreover, $|\mathrm{JH}(I(\sigma_J, \sigma_{J'}))| = 2^{|J \Delta J'|}$ and*

$$\begin{aligned}\mathrm{JH}(I(\sigma_J, \sigma_{J'})) &= \{\sigma_{J''} : J \cap J' \subseteq J'' \subseteq J \cup J'\} \\ &= \{\sigma_{J''} : J \,\Delta\, J'' \subseteq J \,\Delta\, J'\}.\end{aligned}$$

In particular, $\mathrm{Ext}^1_\Gamma(\sigma_{J'}, \sigma_J) \neq 0$ if and only if $|J \,\Delta\, J'| = 1$.

Proof. From [HW18, Cor. 2.22] applied to the principal series tame type $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}[\chi]$ we deduce that $I(\sigma_J, \sigma_{J'})$ exists and that $\mathrm{JH}(I(\sigma_J, \sigma_{J'})) \subseteq \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)$. As χ is 1-generic, $|\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)|$ has cardinality 2^f by above, namely σ_J is well defined for any subset $J \subseteq \{0, 1, \dots, f-1\}$. The proof is then essentially identical to the proof of Lemma 2.1.1, except that the bijection of $W(\overline{\rho}^{\mathrm{ss}})$ with $\{0, 1, \dots, f-1\}$, $\sigma \mapsto J_\sigma$ is replaced by the bijection of $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)$ with $\{0, 1, \dots, f-1\}$, $\sigma_J \mapsto J$. □

Lemma 2.1.3. *Let σ be a 1-generic Serre weight and $\tau_1, \tau_2 \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$. Assume that τ_1, τ_2 are compatible (relative to σ). Then $\tau_2 \in \mathrm{JH}(\mathrm{Inj}_\Gamma \tau_1)$ and $\mathrm{JH}(I(\tau_1, \tau_2)) \subseteq \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$.*

Proof. Let $\mu_1, \mu_2 \in \mathcal{I}$ correspond to τ_1, τ_2 respectively. Since μ_1, μ_2 are compatible, one checks that there exists a unique element in $\mathcal{I}$, denoted by $\mu_1 \cap \mu_2$ (resp. $\mu_1 \cup \mu_2$), which is compatible with μ_1 and μ_2 such that $\mathcal{S}(\mu_1 \cap \mu_2) = \mathcal{S}(\mu_1) \cap \mathcal{S}(\mu_2)$ (resp. $\mathcal{S}(\mu_1 \cup \mu_2) = \mathcal{S}(\mu_1) \cup \mathcal{S}(\mu_2)$). (See also [BP12, § 12] for the explicit construction of $\mu_1 \cap \mu_2$.)

Let $\tau_0, \tau_3 \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$ correspond to $\mu_0 \stackrel{\mathrm{def}}{=} \mu_1 \cap \mu_2$, $\mu_3 \stackrel{\mathrm{def}}{=} \mu_1 \cup \mu_2 \in \mathcal{I}$ respectively. We first assume $\tau_0 = \sigma$, equivalently $\mathcal{S}(\mu_1) \cap \mathcal{S}(\mu_2) = \emptyset$. We have $\tau_1, \tau_2 \in \mathrm{JH}(I(\sigma, \tau_3))$ by [BP12, Cor. 4.11], and the genericity assumption on σ implies that $I(\sigma, \tau_3)$ has length $2^{|\mathcal{S}(\mu_3)|}$. We deduce from [HW18, Lemma 2.20(iii)] that $\mathrm{JH}(I(\sigma, \tau_3)) = \mathrm{JH}(I(\tau_1, \tau_2))$ ($\tau_2 = \tau_1^c$ with the notation used there).

To treat the general case, we note that $I(\tau_0, \tau_3)$ exists and $\tau_1, \tau_2 \in \mathrm{JH}(I(\tau_0, \tau_3))$ by [BP12, Cor. 4.11], so we may view τ_1, τ_2 as Jordan–Hölder factors of $\mathrm{Inj}_\Gamma \tau_0$. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{I}(y_0, \dots, y_{f-1})$ be the element corresponding to $\tau_1, \tau_2, \tau_3 \in \mathrm{JH}(\mathrm{Inj}_\Gamma \tau_0)$. Using [HW22, Lemmas 2.1, 2.7] we get $\lambda_i \circ \mu_0 = \mu_i$ for $i = 1, 2, 3$. By [HW22, Lemma 2.6(i)] we have $\mathcal{S}(\mu_i) = \mathcal{S}(\lambda_i) \,\Delta\, \mathcal{S}(\mu_0)$ or equivalently $\mathcal{S}(\lambda_i) = \mathcal{S}(\mu_i) \,\Delta\, \mathcal{S}(\mu_0) = \mathcal{S}(\mu_i) \setminus \mathcal{S}(\mu_0)$, so $\mathcal{S}(\lambda_1) \cap \mathcal{S}(\lambda_2) = \emptyset$ and $\mathcal{S}(\lambda_1) \cup \mathcal{S}(\lambda_2) = \mathcal{S}(\lambda_3)$. Moreover, since μ_1, μ_2, μ_3 are compatible, $\lambda_1, \lambda_2, \lambda_3$ are also compatible by the table in the proof of [HW22, Lemma 2.6] (writing $\lambda_i = \mu_i \circ \mu_0^{-1}$, where $\mu_0^{-1} \in \mathcal{I}$ is the unique element defined by demanding $\mu_0^{-1} \circ \mu_0 = (x_0, \dots, x_{f-1})$), so that $\lambda_3 = \lambda_1 \cup \lambda_2$. Hence, by the previous paragraph we get $\mathrm{JH}(I(\tau_1, \tau_2)) = \mathrm{JH}(I(\tau_0, \tau_3))$, in particular $\mathrm{JH}(I(\tau_1, \tau_2)) \subseteq \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$ as $I(\tau_0, \tau_3)$ is a quotient of $I(\sigma, \tau_3)$. □

2.2 More Γ-representations

Recall from [HW22, Def. 2.9] that given $j \in \{0, \dots, f-1\}$ and $* \in \{+, -\}$ we define an f-tuple $\mu_j^* \in \bigoplus_{i=0}^{f-1}(\mathbb{Z} \pm x_i)$ as follows: if $f > 1$ then $(\mu_j^*)_{j-1}(x_{j-1}) \stackrel{\mathrm{def}}{=} p - 2 - x_{j-1}$, $(\mu_j^*)_j(x_j) \stackrel{\mathrm{def}}{=} x_j * 1$

and $(\mu_j^*)_i(x_i) \stackrel{\text{def}}{=} x_i$ for $i \notin \{j-1, j\}$, while if $f = 1$ then $\mu_0^*(x_0) \stackrel{\text{def}}{=} p - 2 - (*1) - x_0$. If σ is a 0-generic Serre weight corresponding to a tuple $(s_0, \dots, s_{f-1}) \in \{0, \dots, p-1\}^f$ we write $\mu_j^*(\sigma)$ for the Serre weight $\mu_j^*((s_0, \dots, s_{f-1})) \otimes \det^{e(\mu_j^*)(s_0,\dots,s_{f-1})}$, where $e(\mu_j^*) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$ is defined in [BP12, § 3]. (Note that $\mu_j^-(\sigma)$ is undefined if $f \geq 2$ and $s_j = 0$ and $\mu_j^+(\sigma)$ is undefined if $f = 1$ and $s_j = p - 2$.)

The following lemma is well known, but we state it for lack of convenient reference.

Lemma 2.2.1. *Suppose that $\sigma = (r_0, \dots, r_{f-1}) \otimes \eta$ is any Serre weight such that $\mu_i^-(\sigma)$ is defined. If $f = 1$ we moreover suppose that $0 < r_0 < p - 1$. Then the (unique up to isomorphism) nonsplit $\mathrm{GL}_2(\mathcal{O}_K)/Z_1$-extension $0 \to \mu_i^-(\sigma) \to V \to \sigma \to 0$ is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\sigma$ (hence is a Γ-representation), equivalently $\chi_\sigma \hookrightarrow V|_I$.*

Proof. This follows from [BP12, Thm. 2.4(iii) and Cor. 5.6(ii)]. ☐

Lemma 2.2.2. *Let σ be a 1-generic Serre weight. Let Q be a quotient of $\mathrm{Proj}_\Gamma \sigma$ such that*

(i) $\mathrm{soc}_\Gamma(Q) \cong \sigma^{\oplus r}$ *for some $r \geq 1$;*

(ii) $\mathrm{rad}_\Gamma(Q)/\mathrm{soc}_\Gamma(Q)$ *is nonzero and does not admit σ as a subquotient.*

Then $\mathrm{rad}_\Gamma(Q)/\mathrm{soc}_\Gamma(Q)$ is semisimple and there exists a subset $\mathcal{J} \subseteq \{0, 1, \dots, f-1\}$ such that

$$\mathrm{rad}_\Gamma(Q)/\mathrm{soc}_\Gamma(Q) \cong \bigoplus_{i \in \mathcal{J}} (\mu_i^+(\sigma) \oplus \mu_i^-(\sigma)).$$

Proof. By the same argument as in the proof of [HW22, Cor. 2.32] (using [HW22, Cor. 2.3] for Γ-representations instead of [HW22, Cor. 2.26] for $\widetilde{\Gamma}$-representations), we prove that $\mathrm{rad}_\Gamma(Q)/\mathrm{soc}_\Gamma(Q) \hookrightarrow \bigoplus_{i=0}^{f-1} (\mu_i^+(\sigma) \oplus \mu_i^-(\sigma))$ (in particular, it is semisimple). It thus suffices to show that $\mu_i^+(\sigma) \in \mathrm{JH}(Q)$ if and only if $\mu_i^-(\sigma) \in \mathrm{JH}(Q)$. Note that Q_σ surjects onto Q, where Q_σ is the largest quotient of $\mathrm{Proj}_\Gamma \sigma/\mathrm{rad}_\Gamma^3(\mathrm{Proj}_\Gamma \sigma)$ whose socle is σ-isotypic.

We now determine Q_σ more explicitly. Let $A'_{\sigma,i}$ $(0 \leq i \leq f-1)$ denote the Γ-representation of [HW18, Def. 2.5], which has $\mathrm{soc}_\Gamma(A'_{\sigma,i}) \cong \mathrm{cosoc}_\Gamma(A'_{\sigma,i}) \cong \sigma$ and $\mathrm{rad}_\Gamma(A'_{\sigma,i})/\mathrm{soc}_\Gamma(A'_{\sigma,i}) \cong \mu_i^+(\sigma) \oplus \mu_i^-(\sigma)$. Let A'_σ denote the fiber product of all $A'_{\sigma,i}$ over their common cosocle σ. (Up to twist this is dual to the notation A'_σ in [HW18].) Note that the natural injection $\mathrm{rad}_\Gamma(A'_\sigma) \hookrightarrow \bigoplus_i \mathrm{rad}_\Gamma(A'_{\sigma,i})$ is an isomorphism. (It surjects onto every factor, as $\mathrm{rad}_\Gamma(\cdot)$ preserves surjections, hence surjects onto the cosocle of the direct sum, which is multiplicity free.) Hence the cosocle of A'_σ is still σ. Also we obtain a surjection $\psi : Q_\sigma \twoheadrightarrow A'_\sigma$, and its kernel is σ-isotypic, because ψ induces an isomorphism after applying the functor $\mathrm{rad}_\Gamma(\cdot)/\mathrm{rad}_\Gamma^2(\cdot)$, e.g. by [BP12, Cor. 5.6(i)]. As $\mathrm{Ext}_\Gamma^1(\sigma, \sigma) = 0$ we have a surjection $\mathrm{soc}_\Gamma(\psi) : \mathrm{soc}_\Gamma(Q_\sigma) \twoheadrightarrow \mathrm{soc}_\Gamma(A'_\sigma) \cong \sigma^{\oplus f}$. On the other hand, $\mathrm{soc}_\Gamma(Q_\sigma) \cong \sigma^{\oplus f}$ by the dual version of [HW18, Prop. 2.11] (alternatively, see [AJL83, Thm. 4.3]), hence $Q_\sigma \cong A'_\sigma$.

Write $0 \to L \to A'_\sigma \to Q \to 0$, with L being the corresponding kernel. If $\mu_i^*(\sigma) \in \mathrm{JH}(L)$, then L has to contain the unique subrepresentation of $\mathrm{rad}_\Gamma(A'_{\sigma,i}) \subseteq A'_\sigma$ with cosocle $\mu_i^*(\sigma)$. In particular, the natural map $\mathrm{rad}_\Gamma(A'_{\sigma,i}) \hookrightarrow A'_\sigma \to Q$ has to vanish on the socle, and hence is zero (by condition (i)). This proves that $\mu_i^{-*}(\sigma) \in \mathrm{JH}(L)$, as desired. ☐

Recall again from [HW22, Def. 2.9] that given $j \in \{0, \ldots, f-1\}$ and $* \in \{+, -\}$ we define an f-tuple $\delta_j^* \in \bigoplus_{i=0}^{f-1}(\mathbb{Z} \pm x_i)$ by $(\delta_j^*)_j(x_j) \overset{\text{def}}{=} x_j * 2$ and $(\delta_j^*)_i(x_i) \overset{\text{def}}{=} x_i$ for $i \neq j$. If σ is a Serre weight corresponding to a tuple $(s_0, \ldots, s_{f-1}) \in \{0, \ldots, p-1\}^f$ we write $\delta_j^*(\sigma)$ for the Serre weight $\delta_j^*((s_0, \ldots, s_{f-1})) \otimes \det^{e(\delta_j^*)(s_0,\ldots,s_{f-1})}$ (which is defined only if $s_j * 2 \in \{0, \ldots, p-1\}$). It follows from the definition that $\chi_{\delta_j^*(\sigma)} = \chi_\sigma \alpha_j^{*1}$.

Lemma 2.2.3. *Assume that $\overline{\rho}$ is 1-generic and let $\sigma \in W(\overline{\rho}^{\mathrm{ss}})$. For any $0 \leq j \leq f-1$, there exists $* \in \{\pm\}$ such that*

$$\{\mu_j^+(\sigma), \mu_j^-(\sigma), \delta_j^+(\sigma), \delta_j^-(\sigma)\} \cap W(\overline{\rho}^{\mathrm{ss}}) = \{\mu_j^*(\sigma)\}. \tag{17}$$

Moreover, $J_{\mu_j^*(\sigma)} = J_\sigma \,\Delta\, \{j\}$.

Proof. Let $\sigma^c \in W(\overline{\rho}^{\mathrm{ss}})$ be determined by $J_{\sigma^c} = J_\sigma^c$. Then $\mathrm{JH}(I(\sigma, \sigma^c)) = W(\overline{\rho}^{\mathrm{ss}})$ by Lemma 2.1.1. Recall from § 2.1 that $\mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$ is parametrized by the set $\mathcal{I}$. Since $\delta_j^\pm \notin \mathcal{I}$ we deduce by [HW22, Lemmas 2.1, 2.7] that $\delta_j^\pm(\sigma)$ does not occur in $\mathrm{Inj}_\Gamma \sigma$ for any $0 \leq j \leq f-1$, hence $\delta_j^\pm(\sigma) \notin W(\overline{\rho}^{\mathrm{ss}})$.

Viewing σ^c as a constituent in $\mathrm{Inj}_\Gamma \sigma$, it is parametrized by an element $\lambda \in \mathcal{I}$. Since $|\mathrm{JH}(I(\sigma, \sigma^c))| = 2^f$, [BP12, Cor. 4.11] implies that $\mathcal{S}(\lambda)$ (defined above [BHH$^+$, Lemma 4.1.2]) equals $\{0, \ldots, f-1\}$. For each $0 \leq j \leq f-1$, there is a unique $* \in \{\pm\}$ such that μ_j^* is compatible (in the sense of [BP12, Def. 4.10]) with λ. By [BP12, Cor. 4.11] again and [HW22, Lemmas 2.1, 2.7], we deduce that exactly one of $\mu_j^\pm(\sigma)$ occurs in $\mathrm{JH}(I(\sigma, \sigma^c))$. The final claim is a direct check. □

2.3 Some $\widetilde{\Gamma}$-representations

We start by recalling some results from [HW22]. Let τ be a Serre weight and $\chi \overset{\text{def}}{=} \tau^{I_1}$. For $n \geq 1$ let $W_{\chi,n} \overset{\text{def}}{=} (\mathrm{Proj}_{I/Z_1} \chi)/\mathfrak{m}^n$, where $\mathrm{Proj}_{I/Z_1} \chi$ is the linear dual of the injective envelope $\mathrm{Inj}_{I/Z_1}(\chi^{-1})$ (a projective cover of χ in the dual category). It is a finite-dimensional representation of I/Z_1 over $\mathbb{F}$. We let $\overline{W}_{\chi,3}$ be the smallest quotient of $W_{\chi,3}$ such that $[W_{\chi,3} : \chi] = [\overline{W}_{\chi,3} : \chi]$. It is shown in [HW22, Lemma 3.2] that $\overline{W}_{\chi,3}$ fits into a short exact sequence

$$0 \to \bigoplus_{\chi'} E_{\chi,\chi'} \to \overline{W}_{\chi,3} \to \chi \to 0, \tag{18}$$

where the direct sum is taken over the characters χ' such that $\mathrm{Ext}^1_{I/Z_1}(\chi', \chi) \neq 0$. Then $\overline{W}_{\chi,3}$, and hence also $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \overline{W}_{\chi,3}$, is annihilated by $\mathfrak{m}_{K_1}^2$ [HW22, Cor. 3.3].

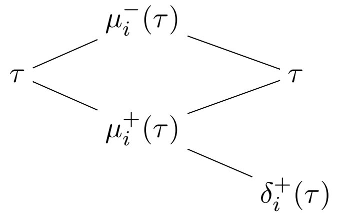
Recall from [HW22, Thm. 2.23] that given a 2-generic Serre weight σ and $\tau \in \mathrm{JH}(\mathrm{Inj}_{\widetilde{\Gamma}} \sigma)$, there exists a unique finite-dimensional $\widetilde{\Gamma}$-module $I(\sigma, \tau)$ such that $\mathrm{soc}_{\widetilde{\Gamma}} I(\sigma, \tau) = \sigma$, $\mathrm{cosoc}_{\widetilde{\Gamma}} I(\sigma, \tau) = \tau$ and $[I(\sigma, \tau) : \sigma] = 1$. (Note that this agrees with the definition of $I(\sigma, \tau)$ in § 2.1 if $\tau \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$.) We also recall that $I(\sigma, \tau)$ is multiplicity free by [HW22, Cor. 2.25].

Assume now that τ is 2-generic (so χ is 2-generic). For $0 \leq i \leq f-1$ and a sign $* \in \{\pm\}$ let $W_i^* = W_i^*(\chi)$ denote the unique uniserial I/Z_1-representation of the form $\chi - \chi\alpha_i^{*1} - \chi$. (It is a quotient of the I/Z_1-representation $\overline{W}_{\chi,3}$ in [HW22, § 3.1], see also § 3.2 below.) Let $Q_i^* = Q_i^*(\tau)$ denote the largest quotient of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^*$ with τ-isotypic socle. Then Q_i^* is a $\widetilde{\Gamma}$-representation by [HW22, Cor. 3.3].

Lemma 2.3.1. *Suppose that $i \in \{0, \dots, f-1\}$.*

(i) *The $\widetilde{\Gamma}$-representation Q_i^- is uniserial of the form $\tau - \mu_i^-(\tau) - \tau$.*

(ii) *The $\widetilde{\Gamma}$-representation Q_i^+ has the form*



In particular, $\operatorname{soc}_{\widetilde{\Gamma}}(Q_i^*) = \tau$ *for each* $* \in \{\pm\}$.

We remark that Lemma 2.3.1 does not determine $\ker(Q_i^+ \twoheadrightarrow \delta_i^+(\tau))$ uniquely up to isomorphism (this kernel is a suitable amalgam of the uniserial representations $\tau - \mu_i^-(\tau) - \tau$, $\tau - \mu_i^+(\tau) - \tau$, and this amalgam depends on a parameter in $\mathbb{F}^\times$), but this will not matter for us.

Proof. (i) Let Y_i^- denote a uniserial $\widetilde{\Gamma}$-representation of the form $\tau - \mu_i^-(\tau) - \tau$, which exists by taking a suitable quotient of the representation Θ_τ in [HW22, Prop. 3.12] (see also [HW22, Cor. 3.16]). By [HW22, Lemma 2.10] it is easy to see that Y_i^- is unique up to isomorphism (alternatively it follows once this lemma is proved).

We now show that part (i) holds, *assuming that* $W_i^- \hookrightarrow Y_i^-|_I$. Under this assumption, we show that the corresponding map $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^- \twoheadrightarrow Y_i^-$ is surjective. First note that $\mathrm{JH}(\operatorname{cosoc}_{\widetilde{\Gamma}}(\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^-)) \subseteq \{\tau, \delta_i^-(\tau)\}$, as $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi\alpha_i^{-1}$ (resp. $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$) has cosocle $\delta_i^-(\tau)$ (resp. τ). Hence if the map $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^- \twoheadrightarrow Y_i^-$ is not surjective, then it has image τ, which implies that it is trivial on $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi \subseteq \operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^-$, contradicting the injectivity of $W_i^- \hookrightarrow Y_i^-|_I$. The surjection $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^- \twoheadrightarrow Y_i^-$ shows that $Q_i^- = Y_i^-$ by definition of Q_i^-, as $[\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^- : \tau] = 2$, which completes part (i).

It remains to check that $W_i^- \hookrightarrow Y_i^-|_I$. Let $\chi' \stackrel{\mathrm{def}}{=} \chi\alpha_i^{-1}$ and let $E_{\chi,\chi'}$ be the I/Z_1-representation which is the unique nonsplit extension of χ' by χ (see § 1.4). By [BHH+25, Lemma 3.42(ii)] (applied with $\sigma = \tau$ and $\underline{Y}^{-\underline{i}}v = Y_i^{-1}v$) there is an injection $E_{\chi,\chi'} \hookrightarrow \tau|_I \hookrightarrow Y_i^-|_I$. As $\dim_{\mathbb{F}}(\tau) < q$, we know that $\tau|_I$ is multiplicity free by [BP12, Lemma 2.7]. Let $u \in \tau^{I_1}$ (resp. $v \in \tau$) be an H-eigenvector with eigencharacter χ (resp. $\chi' = \chi\alpha_i^{-1}$), so $E_{\chi,\chi'} = \mathbb{F}u \oplus \mathbb{F}v$. On the other hand, let $w \in Y_i^-$ be an H-eigenvector with eigencharacter χ, such that its image in $Y_i^-/\operatorname{soc}_{\widetilde{\Gamma}}(Y_i^-)$ is I_1-invariant. This is possible by Lemma 2.2.1.

We will prove that $\mathbb{F}u \oplus \mathbb{F}v \oplus \mathbb{F}w$ is I-stable (equivalently I_1-stable) and isomorphic to W_i^-. Note that $(g-1)w \in \operatorname{soc}_{\widetilde{\Gamma}}(Y_i^-) = \tau$ for all $g \in I_1$ (by the choice of w), and that w itself is not fixed by I_1, since otherwise there would be a surjection $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi \twoheadrightarrow Y_i^-$, which is impossible as τ has multiplicity 1 in $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi$. As $\mathbb{F}u\oplus\mathbb{F}v$ is I_1-stable it is enough to prove that $(g-1)w \in \mathbb{F}u\oplus\mathbb{F}v$ for $g \in I_1$. It then suffices to show that (note that Z_1 acts trivially on Y_i^-):

(a) $(g-1)w = 0$ for all $g \in \left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)$;

(b) $(g-1)w \in \tau^{H=\chi} = \mathbb{F}u$ for all $g \in \left(\begin{smallmatrix} 1+p\mathcal{O}_K & 0 \\ 0 & 1 \end{smallmatrix}\right)$;

(c) $(g-1)w \in \tau^{H=\chi'} = \mathbb{F}v$ for all $g \in \left(\begin{smallmatrix} 1 & 0 \\ p\mathcal{O}_K & 1 \end{smallmatrix}\right)$.

To prove (a), let $Y_j \overset{\mathrm{def}}{=} \sum_{a\in\mathbb{F}_q^\times} a^{-p^j}\left(\begin{smallmatrix} 1 & [a] \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{F}[\![\left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)]\!]$ for $0 \le j \le f-1$, so that $\mathbb{F}[\![\left(\begin{smallmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{smallmatrix}\right)]\!] = \mathbb{F}[\![Y_0,\dots,Y_{f-1}]\!]$. It is direct to check that Y_jw is an H-eigenvector with eigencharacter $\chi\alpha_j$. However, we see from [BP12, Lemma 2.7] that $\chi\alpha_j \notin \mathrm{JH}(\tau|_I)$ for all $0 \le j \le f-1$. Thus $Y_jw = 0$ for all j, so (a) holds.

Part (b) is obvious.

To prove (c), let $X_j \overset{\mathrm{def}}{=} \sum_{a\in\mathbb{F}_q^\times} a^{-p^j}\left(\begin{smallmatrix} 1 & 0 \\ [a] & 1 \end{smallmatrix}\right) \in \mathbb{F}[\![\left(\begin{smallmatrix} 1 & 0 \\ \mathcal{O}_K & 1 \end{smallmatrix}\right)]\!]$ for $0 \le j \le f-1$. Write $\tau = (r_0,\dots,r_{f-1})$ up to twist. By another application of [BP12, Lemma 2.7] we see that $\chi\alpha_j^{-(r_j+1)} \notin \mathrm{JH}(Y_i^-|_I)$ for all $j \ne i-1$, so using the $\mathrm{GL}_2(\mathcal{O}_K)$-action on Y_i^- we conclude that $X_j^{r_j+1}w = 0$ and hence $X_j^pw = 0$ for all $j \ne i-1$. On the other hand, $X_j^pX_{j'}^pw = 0$ for all j, j', as Y_i^- is a $\widetilde{\Gamma}$-representation. As $\mathbb{F}[\![\left(\begin{smallmatrix} 1 & 0 \\ p\mathcal{O}_K & 1 \end{smallmatrix}\right)]\!] = \mathbb{F}[\![X_0^p,\dots,X_{f-1}^p]\!]$, we deduce that $(g-1)w \in \mathbb{F}X_{i-1}^pw$, on which H acts by $\chi\alpha_{i-1}^{-p} = \chi'$.

(ii) Using (i) we determine the submodule structure of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W_i^+$ completely. This is done in Step 1 to Step 3 below. Write $\mathcal{S} \overset{\mathrm{def}}{=} \{0,1,\dots,f-1\}$ in what follows. For $J \subseteq \mathcal{S}$ let σ_J^0 (resp. σ_J^1, resp. σ_J^2) denote the constituent parametrized by J in the bottom $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi$ (resp. $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi\alpha_i$, resp. the top $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi$), (see § 2.1 for this parametrization). In particular, we write $\sigma_J \overset{\mathrm{def}}{=} \sigma_J^0 \cong \sigma_J^2$ and note that $\sigma_{\mathcal{S}} \cong \tau$. Note that the constituents σ_J occur with multiplicity 2, and that the σ_J^1 occur with multiplicity 1, cf. [BHH+, Lemma 4.3.3].

For $s \in \{0,1\}$ write V_J^s for the unique subrepresentation of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}E_{\chi,\chi\alpha_i} \subseteq \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W_i^+$ with cosocle σ_J^s (not to be confused with the Serre weight $\sigma_J^{[s]}$ of § 1.4!), or equivalently for the image of *any* nonzero map $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_J^s \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}E_{\chi,\chi\alpha_i}$. Write V_J^2 for the image of *some* map $\iota : \mathrm{Proj}_{\widetilde{\Gamma}}\sigma_J \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W_i^+$ such that the composite $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_J \xrightarrow{\iota} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W_i^+ \twoheadrightarrow \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi$ is nonzero. We claim that the V_J^2 are independent of the choice of ι (equivalently, $V_J^0 \subseteq V_J^2$). Indeed, as we recall at the beginning of § 2.3, W_i^+ is a quotient of $\overline{W}_{\chi,3}$. Using [HW22, Cor. 3.3] we see that $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\overline{W}_{\chi,3}$ is a $\widetilde{\Gamma}$-representation, so we can lift ι to $\phi : \mathrm{Proj}_{\widetilde{\Gamma}}\sigma_J \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\overline{W}_{\chi,3}$ such that the composite with $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\overline{W}_{\chi,3} \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi$ is nonzero. By [HW22, Prop. 3.10(i)] (and its proof) we get $[\mathrm{coker}(\phi) : \sigma_J] = 0$, hence by [HW22, Prop. 3.10(ii)] the image of ϕ is independent of any choices, and consequently the image V_J^2 of ι is well defined. Thus, to determine the submodule structure, it suffices to determine all minimal (proper) containments between the submodules of the form V_J^0, V_J^1, V_J^2. Here we say that a containment of two such modules is minimal if no other $V_{J'}^s$ lies strictly in between.

Step 1. By [BP12, Thm. 2.4] and [HW22, Lemma 3.7] the minimal containments among the V_J^0 and V_J^1 are given by

$$V_J^0 \subsetneq V^0_{J\sqcup\{k\}}, \quad V_J^1 \subsetneq V^1_{J\sqcup\{k\}} \qquad \text{for any } k \notin J; \tag{19}$$

$$V^0_{J\sqcup\{i\}} \subsetneq V_J^1 \qquad \text{if } i \notin J. \tag{20}$$

Likewise, [HW22, Lemma 3.8] shows that the minimal containments between submodules of the form V_J^1 and $V_{J'}^2$ are given by

$$V_J^1 \subsetneq V^2_{J\sqcup\{i\}} \qquad \text{if } i \notin J. \tag{21}$$

Likewise, by [BP12, Thm. 2.4], the minimal containments among the V_J^2 are given by

$$V_J^2 \subsetneq V^2_{J\sqcup\{k\}} \qquad \text{for any } k \notin J. \tag{22}$$

Step 2. We show that the minimal containments between submodules of the form V_J^0 and $V_{J'}^2$ are given by

$$V^0_{J\sqcup\{i\}} \subsetneq V_J^2 \qquad \text{if } i \notin J. \tag{23}$$

By dualizing (i) and replacing χ by χ^{-1} we deduce that the largest subrepresentation of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^+$ with cosocle $\sigma_\emptyset$ (and socle $\sigma_\emptyset$) is uniserial of the form

$$\sigma^0_\emptyset - \sigma^0_{\{i\}} - \sigma^2_\emptyset. \tag{24}$$

(Note that the middle constituent cannot be $\sigma^2_{\{i\}}$ by (22).)

Consider the statement

$$A(J_1,J_2): \qquad V^0_{J_1} \subsetneq V^2_{J_2} \text{ is a minimal containment.}$$

Note by (24) that

$$A(\{k\},\emptyset) \text{ holds if and only if } k=i. \tag{25}$$

Also note that

$$A(J_1,J_2) \quad\Rightarrow\quad |J_1 \,\Delta\, J_2| = 1 \quad\Rightarrow\quad (J_1 \subseteq J_2) \text{ or } (J_2 \subseteq J_1), \tag{26}$$

because if $A(J_1,J_2)$ holds, then $0 \neq \mathrm{Ext}^1_{\tilde\Gamma}(\sigma_{J_2},\sigma_{J_1}) = \mathrm{Ext}^1_\Gamma(\sigma_{J_2},\sigma_{J_1})$, where the equality follows from [BP12, Cor. 5.6(ii)], so that $|J_1 \,\Delta\, J_2| = 1$ by Lemma 2.1.2.

We now show that $A(J_1,J_2) \Rightarrow A(J_1 \sqcup \{k\}, J_2 \sqcup \{k\})$ if $J_1 \supseteq J_2$ and $k \notin J_1$. By $A(J_1,J_2)$ and (22) we deduce that $V^0_{J_1} \subseteq V^2_{J_2} \subseteq V^2_{J_2\sqcup\{k\}}$. In particular, $V^2_{J_2\sqcup\{k\}}$ admits a quotient Q whose socle is the cosocle of $V^0_{J_1}$ (which is isomorphic to σ_{J_1}).

We prove that Q is a Γ-representation isomorphic to $I(\sigma_{J_1},\sigma_{J_2\sqcup\{k\}})$, by first showing that (a) $\mathrm{JH}(Q) \subseteq \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi)$ and (b) Q is multiplicity free. For (a), suppose by contradiction that $V^0_{J_1} \subseteq V^1_J \subseteq V^2_{J_2\sqcup\{k\}}$ for some J. By Step 1 we deduce that $J_1 \subseteq J \sqcup \{i\} \subseteq J_2 \sqcup \{k\}$ (in particular, $i \notin J$). But $J_1 \subseteq J_2 \sqcup \{k\}$ implies equality by (26), contradicting $k \notin J_1$. For (b), suppose by contradiction that $V^0_{J_1} \subseteq V^0_J \subseteq V^2_J \subseteq V^2_{J_2 \sqcup \{k\}}$ for some J. By Step 1 this gives $J_1 \subseteq J \subseteq J_2 \sqcup \{k\}$, leading to the same contradiction as before. Using (a) and (b) we deduce that Q is a Γ-representation by [BP12, Cor. 5.7], so that $Q \cong I(\sigma_{J_1}, \sigma_{J_2 \sqcup \{k\}})$, as claimed.

As $Q \cong I(\sigma_{J_1}, \sigma_{J_2 \sqcup \{k\}})$ as Γ-representation we deduce by Lemma 2.1.2 and (26) that Q has length 4 and surjects onto the nonsplit extension $\sigma_{J_1 \sqcup \{k\}} - \sigma_{J_2 \sqcup \{k\}}$, i.e. there is a minimal containment $V^s_{J_1 \sqcup \{k\}} \subsetneq V^2_{J_2 \sqcup \{k\}}$ for some $s \in \{0, 2\}$. As $J_1 \supseteq J_2$ we deduce by (22) that $s = 0$. This establishes $A(J_1 \sqcup \{k\}, J_2 \sqcup \{k\})$.

Conversely, if $A(J_1 \sqcup \{k\}, J_2 \sqcup \{k\})$, then $V^0_{J_1} \subseteq V^0_{J_1 \sqcup \{k\}} \subseteq V^2_{J_2 \sqcup \{k\}}$ by (19). Using the same quotient Q as above, we again have $Q \cong I(\sigma_{J_1}, \sigma_{J_2 \sqcup \{k\}})$ as Γ-representation, which contains the nonsplit extension $\sigma_{J_1} - \sigma_{J_2}$, i.e. there is a minimal containment $V^0_{J_1} \subsetneq V^s_{J_2}$ for some $s \in \{0, 2\}$. By (19) we deduce that $s = 2$, so $A(J_1, J_2)$ holds. In particular, from (25) we deduce that $A(J \sqcup \{k\}, J)$ holds (for $k \notin J$) if and only if $k = i$.

In the preceding paragraph we dealt with all cases when $J_1 \supsetneq J_2$. If $J_1 \subsetneq J_2$, then we have $V^0_{J_1} \subsetneq V^0_{J_2} \subsetneq V^2_{J_2}$, so that $V^0_{J_1} \subsetneq V^2_{J_2}$ is not a minimal containment. We have thus confirmed the list of minimal containments between submodules of the form V^0_J and $V^2_{J'}$ in (23).

Step 3. Recall that Q_i^+ is the largest quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^+$ with socle $\tau \cong \sigma_{\mathcal{S}}$. As $V^0_{\mathcal{S}} \subseteq V^2_{\mathcal{S}}$, we have $\mathrm{soc}_{\widetilde{\Gamma}}(Q_i^+) = \tau$ and the submodules of Q_i^+ having irreducible cosocle are the images of the submodules V^s_J of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^+$ that contain $V^0_{\mathcal{S}}$. From Steps 1 and 2 we obtain precisely the following such submodules and containments:

$$\begin{array}{ccccc} & & V^2_{\mathcal{S}\setminus\{i\}} & & \\ & \nearrow & & \searrow & \\ V^0_{\mathcal{S}} & & & & V^2_{\mathcal{S}} \\ & \searrow & & \nearrow & \\ & & V^1_{\mathcal{S}\setminus\{i\}} & & \\ & & & \searrow & \\ & & & & V^1_{\mathcal{S}} \end{array}$$

This determines the submodule structure of Q_i^+ by Lemma 2.3.2 (taking $M = \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^+$, $\overline{M} = Q_i^+$, and all possible σ) below. It remains to observe that $\sigma^2_{\mathcal{S}\setminus\{i\}} \cong \mu_i^-(\tau)$, $\sigma^1_{\mathcal{S}\setminus\{i\}} \cong \mu_i^+(\tau)$, $\sigma^1_{\mathcal{S}} \cong \delta_i^+(\tau)$. □

Lemma 2.3.2. *Suppose that M is a finite length module over an artinian ring A, and that $\pi : M \twoheadrightarrow \overline{M}$ is a quotient morphism. Suppose that σ and τ are simple A-modules and that M_σ (resp. M_τ) is a submodule of M having cosocle σ (resp. τ). If the set of submodules of M having cosocle σ is totally ordered and $\pi(M_\sigma) \neq 0$, then*

$$M_\sigma \subseteq M_\tau \iff \pi(M_\sigma) \subseteq \pi(M_\tau).$$

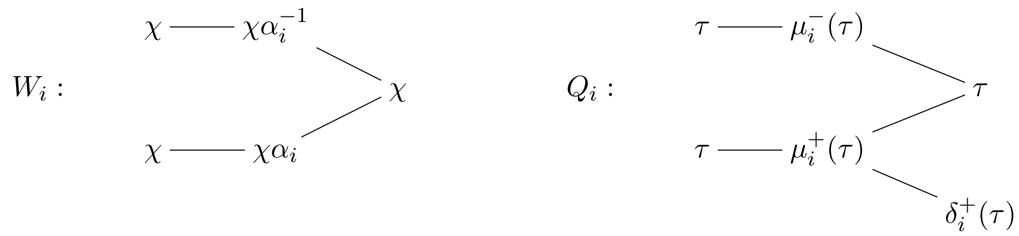
Proof. Let $N \overset{\text{def}}{=} \ker(\pi)$. For the nontrivial direction, we need to show that $M_\sigma \subseteq M_\tau + N$ implies $M_\sigma \subseteq M_\tau$. Let $\mathrm{Proj}_A\, \sigma$ be the projective cover of σ in the category of A-modules (which exists as

A is artinian). Pick $f : \mathrm{Proj}_A\,\sigma \to M$ that has image M_σ, and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(\mathrm{Proj}_A\,\sigma, M_\tau \oplus N) & \twoheadrightarrow & \mathrm{Hom}_A(\mathrm{Proj}_A\,\sigma, M_\tau + N) \\ \cong \uparrow & & \downarrow \\ \mathrm{Hom}_A(\mathrm{Proj}_A\,\sigma, M_\tau) \oplus \mathrm{Hom}_A(\mathrm{Proj}_A\,\sigma, N) & \xrightarrow{+} & \mathrm{Hom}_A(\mathrm{Proj}_A\,\sigma, M) \end{array}$$

As $M_\sigma \subseteq M_\tau + N$ and $\mathrm{Proj}_A\,\sigma$ is projective there exist $f_1 : \mathrm{Proj}_A\,\sigma \to M_\tau$ and $f_2 : \mathrm{Proj}_A\,\sigma \to N$ such that $f = f_1 + f_2$. By the condition on the submodules of M, we know that $\mathrm{im}(f_1) \subseteq \mathrm{im}(f_2)$ or $\mathrm{im}(f_2) \subseteq \mathrm{im}(f_1)$. In the first case, $\mathrm{im}(f) \subseteq \mathrm{im}(f_2) \subseteq N$, contradiction. Hence $M_\sigma = \mathrm{im}(f) \subseteq \mathrm{im}(f_1) \subseteq M_\tau$, as desired. □

Let $W_i = W_i(\chi)$ denote the fiber product of W_i^+ and W_i^- over their common cosocle χ. Let $Q_i = Q_i(\tau)$ denote the fiber product of Q_i^+ and Q_i^- over their common quotient $I(\mu_i^-(\tau), \tau)$ (cf. Lemma 2.3.1). We draw a diagram for W_i and Q_i, but keep in mind that the submodule structure is more complicated since the socle has multiplicities in each case:



Lemma 2.3.3. *The representation* $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i$ *has a unique quotient* Q *with socle* $\tau^{\oplus 2}$ *and such that* $[Q : \tau] = 3$*, and this quotient is isomorphic to* Q_i*. Moreover,* $\mathrm{JH}(Q_i) = \{\tau, \mu_i^-(\tau), \mu_i^+(\tau), \delta_i^+(\tau)\}$ *and* $Q_i/\operatorname{soc}_{\widetilde{\Gamma}}(Q_i)$ *is multiplicity free.*

Proof. By exactness of induction, $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i$ is the fiber product of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^\pm$ over $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i & \to & \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^+ \times \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^- & \to & \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \to & Q_i & \to & Q_i^+ \times Q_i^- & \xrightarrow{\alpha} & I(\mu_i^-(\tau), \tau) & \to & 0 \end{array}$$

(For the right square, note that the natural map $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i^* \twoheadrightarrow Q_i^* \twoheadrightarrow I(\mu_i^-(\tau), \tau)$ factors through the K_1-coinvariants $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}((W_i^*)_{K_1}) = \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(W_i^*/\chi)$, and hence through $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$ because $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(W_i^*/\chi)$ is multiplicity free and $\mu_i^-(\tau) \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)$.) By the snake lemma, since no constituent of $\ker(f)$ occurs in $Q_i^\pm$ (by Lemma 2.3.1), the left vertical map is surjective. Note that the map α sends $\operatorname{soc}_{\widetilde{\Gamma}}(Q_i^+ \times Q_i^-)$ to 0, so $\operatorname{soc}_{\widetilde{\Gamma}}(Q_i) = \operatorname{soc}_{\widetilde{\Gamma}}(Q_i^+) \times \operatorname{soc}_{\widetilde{\Gamma}}(Q_i^-) = \tau^{\oplus 2}$. As $[Q_i : \tau] = 3$, we deduce the existence of Q. Uniqueness of Q is clear, since $[Q_i : \tau] = [\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i : \tau] = 3$. The last statement follows from Lemma 2.3.1. □

Note that each Q_i surjects onto τ. For a nonempty subset $\mathcal{J} \subseteq \{0, 1, \dots, f-1\}$ let $Q_{\mathcal{J}} = Q_{\mathcal{J}}(\tau)$ denote the fiber product of all Q_i ($i \in \mathcal{J}$) over τ. Let $\chi \stackrel{\text{def}}{=} \chi_\tau$ and let $W_{\mathcal{J}} = W_{\mathcal{J}}(\chi)$ denote the fiber product of all W_i (equivalently of all $W_i^{\pm}$) for $i \in \mathcal{J}$ over their common cosocle χ.

Lemma 2.3.4.

(i) *The radical filtration of* $W_{\mathcal{J}}$ *is given by* $\chi^{\oplus 2|\mathcal{J}|} - \bigoplus_{i \in \mathcal{J}}(\chi\alpha_i \oplus \chi\alpha_i^{-1}) - \chi$. *Moreover,* $\operatorname{soc}_I(W_{\mathcal{J}}) \cong \operatorname{rad}_I^2(W_{\mathcal{J}}) \cong \chi^{\oplus 2|\mathcal{J}|}$.

(ii) *The* K_1*-coinvariants of* $W_{\mathcal{J}}$ *fit in a short exact sequence* $0 \to \bigoplus_{i \in \mathcal{J}} \chi\alpha_i \to (W_{\mathcal{J}})_{K_1} \to \chi \to 0$ *with cosocle* χ.

Proof. (i) By construction of $W_{\mathcal{J}}$ as a fiber product we have an inclusion $\iota : \bigoplus_{i \in \mathcal{J}, *} \operatorname{rad}_I(W_i^*) \hookrightarrow W_{\mathcal{J}}$. Its image is contained in $\operatorname{rad}_I(W_{\mathcal{J}})$ because $\operatorname{rad}_I(W_i^*) \subseteq W_{\mathcal{J}}$ is the unique subrepresentation with cosocle $\chi\alpha_i^{*1}$ and $\operatorname{rad}_I(W_{\mathcal{J}}) \twoheadrightarrow \operatorname{rad}_I(W_i^*)$. For length reasons, ι has to be an isomorphism onto $\operatorname{rad}_I(W_{\mathcal{J}})$, which shows that $\operatorname{cosoc}_I(W_{\mathcal{J}}) \cong \chi$. We also deduce the claims about $\operatorname{rad}_I(W_{\mathcal{J}}) / \operatorname{rad}_I^2(W_{\mathcal{J}})$ and $\operatorname{rad}_I^2(W_{\mathcal{J}})$, as $\operatorname{rad}_I(W_i^*) \cong (\chi - \chi\alpha_i^{*1})$. The last assertion easily follows as $\operatorname{rad}_I^2(W_{\mathcal{J}}) \subseteq \operatorname{soc}_I(W_{\mathcal{J}}) \subseteq \operatorname{soc}_I(\bigoplus_{i \in \mathcal{J}} W_i) \cong \chi^{\oplus 2|\mathcal{J}|}$ (note that $\operatorname{rad}_I^3(W_{\mathcal{J}}) = 0$).

(ii) By (i) it is clear that $W_{\mathcal{J}}$ has a unique quotient, say $\mathcal{E}$, which fits in a short exact sequence as in the statement. By [Hu10, Lemma 2.4(ii)] $\mathcal{E}$ is annihilated by $\mathfrak{m}_{K_1}$, so that $(W_{\mathcal{J}})_{K_1} \twoheadrightarrow \mathcal{E}$. We prove that this is an isomorphism. Since $(W_{\mathcal{J}})_{K_1}$ has cosocle χ by (i), we have a surjection $\operatorname{Proj}_{I/K_1} \chi \twoheadrightarrow (W_{\mathcal{J}})_{K_1}$, which kills $\operatorname{soc}_I(\operatorname{Proj}_{I/K_1} \chi) = \chi$ because $\dim_{\mathbb{F}}(W_{\mathcal{J}})_{K_1} \leq 4|\mathcal{J}| + 1 < p^f = \dim_{\mathbb{F}}(\operatorname{Proj}_{I/K_1} \chi)$. Since $\operatorname{Proj}_{I/K_1} \chi / \operatorname{soc}_I(\operatorname{Proj}_{I/K_1} \chi)$ is multiplicity free [BHH+23, Lemma 6.1.3], it follows that $(W_{\mathcal{J}})_{K_1}$ is multiplicity free, hence $(W_{\mathcal{J}})_{K_1}$ is a quotient of $W_{\mathcal{J}} / \operatorname{rad}_I^2(W_{\mathcal{J}})$ by (i). To conclude it suffices to prove that $\chi\alpha_i^{-1}$ does not occur in $(W_{\mathcal{J}})_{K_1}$. Otherwise, $(W_{\mathcal{J}})_{K_1}$ would surject onto $E_{\chi\alpha_i^{-1},\chi}$ which is not annihilated by $\mathfrak{m}_{K_1}$ by [Hu10, Lemma 2.4(ii)] again, contradiction. $\square$

Lemma 2.3.5. *The representation* $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{J}}$ *has a unique quotient* Q *with socle* $\tau^{\oplus 2|\mathcal{J}|}$ *and such that* $[Q : \tau] = 2|\mathcal{J}| + 1$*, and this quotient is isomorphic to* $Q_{\mathcal{J}}$*. Moreover,* $\operatorname{JH}(Q_{\mathcal{J}}) = \{\tau, \mu_i^{\pm}(\tau), \delta_i^{+}(\tau) : i \in \mathcal{J}\}$ *and* $Q_{\mathcal{J}} / \operatorname{soc}_{\widetilde{\Gamma}}(Q_{\mathcal{J}})$ *is multiplicity free.*

Proof. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{J}} & \longrightarrow & \prod_{i \in \mathcal{J}} \operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_i & \longrightarrow & (\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)^{\oplus(|J|-1)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Q_{\mathcal{J}} & \longrightarrow & \prod_{i \in \mathcal{J}} Q_i & \xrightarrow{\alpha} & \tau^{\oplus(|J|-1)} & \longrightarrow & 0
\end{array}$$

We claim that the left vertical map g is surjective. As $\operatorname{coker}(g)$ is a quotient of $\ker(f)$ and $\operatorname{JH}(\ker(f)) \cap \operatorname{JH}(\prod_{i \in \mathcal{J}} Q_i) = \{\mu_i^-(\tau) : i \in \mathcal{J}\}$ it follows that all the constituents of $\operatorname{coker}(g)$ are of the form $\mu_i^-(\tau)$ for some $i \in \mathcal{J}$, hence it suffices to show that $Q_{\mathcal{J}}$ cannot surject onto any $\mu_i^-(\tau)$, $i \in \mathcal{J}$. This is true, as $\mu_i^-(\tau)$ occurs with multiplicity one in $\prod_{i \in \mathcal{J}} Q_i$ (by Lemma 2.3.3) and

$Q_{\mathcal{J}} \twoheadrightarrow Q_i \twoheadrightarrow Q_i^- \twoheadrightarrow I(\mu_i^-(\tau), \tau)$. By construction, the components of the map α are obtained as composition $Q_i \twoheadrightarrow Q_i^- \twoheadrightarrow \tau$ (or are zero), so the map sends $\prod_{i \in \mathcal{J}} \mathrm{soc}_{\widetilde{\Gamma}}(Q_i)$ to 0 (by Lemma 2.3.3 for Q_i^-). Hence $\mathrm{soc}_{\widetilde{\Gamma}}(Q_{\mathcal{J}}) = \prod_{i \in \mathcal{J}} \mathrm{soc}_{\widetilde{\Gamma}}(Q_i) = \tau^{\oplus 2|\mathcal{J}|}$ and together with $[\prod_{i \in \mathcal{J}} Q_i : \tau] = 3|\mathcal{J}|$ we deduce $[Q_{\mathcal{J}} : \tau] = 3|\mathcal{J}| - (|\mathcal{J}| - 1) = 2|\mathcal{J}| + 1$, hence we obtain the existence of Q (by taking $Q = Q_{\mathcal{J}}$). Uniqueness and the last statement again follow easily. □

Recall that $\mathcal{J}$ is a fixed subset of $\mathcal{S} = \{0, \dots, f-1\}$. Let $\Theta_{\mathcal{J}} = \Theta_{\mathcal{J}}(\tau) \subseteq Q_{\mathcal{J}}$ denote the largest subrepresentation such that $\mathrm{cosoc}_{\widetilde{\Gamma}}(\Theta_{\mathcal{J}}) \cong \tau$. (This exists and $[\Theta_{\mathcal{J}} : \tau] = [Q_{\mathcal{J}} : \tau] = 2|\mathcal{J}| + 1$: by [HW22, Prop. 3.10], noting that $W_{\mathcal{S}} = \overline{W}_{\chi,3}$, the representation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{S}}$ has a largest subrepresentation with cosocle τ; the same is then true for any quotient representation, in particular for $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{J}} \twoheadrightarrow Q_{\mathcal{J}}$.) We note that $\mathrm{JH}(\Theta_{\mathcal{J}}) = \{\tau, \mu_i^{\pm}(\tau) : i \in \mathcal{J}\}$, with $\Theta_{\mathcal{J}} / \mathrm{soc}_{\widetilde{\Gamma}}(\Theta_{\mathcal{J}})$ multiplicity free. For $i \in \mathcal{J}$ let $\Psi_i = \Psi_i(\tau) \subseteq Q_{\mathcal{J}}$ be the unique subrepresentation such that $\mathrm{cosoc}_{\widetilde{\Gamma}}(\Psi_i) \cong \delta_i^+(\tau)$. Then $\Psi_i \cong I(\tau, \delta_i^+(\tau))$, which is uniserial of shape $\tau - \mu_i^+(\tau)$-$- \delta_i^+(\tau)$, as

$$I(\tau, \delta_i^+(\tau)) \hookrightarrow \ker(Q_i^+ \twoheadrightarrow I(\mu_i^-(\tau), \tau)) \hookrightarrow \ker(Q_i \twoheadrightarrow \tau) \hookrightarrow Q_{\mathcal{J}}$$

(the first inclusion coming from Lemma 2.3.1(ii)). In particular, $\mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i) \subseteq \Theta_{\mathcal{J}}$ for all $i \in \mathcal{J}$.

Lemma 2.3.6. *The representation $Q_{\mathcal{J}}$ is the colimit of the diagram $(\Theta_{\mathcal{J}} \hookleftarrow \mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i) \hookrightarrow \Psi_i)_{i \in \mathcal{J}}$ (with $2|\mathcal{J}| + 1$ objects and $2|\mathcal{J}|$ morphisms).*

Proof. We claim that $\mathrm{cosoc}_{\widetilde{\Gamma}}(Q_{\mathcal{J}}) \cong \tau \oplus \bigoplus_{i \in \mathcal{J}} \delta_i^+(\tau)$. Suppose first that $Q_{\mathcal{J}} \twoheadrightarrow \sigma$ for some irreducible σ. Then

$$0 \neq \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{J}}, \sigma) = \mathrm{Hom}_I(W_{\mathcal{J}}, \sigma) = \mathrm{Hom}_I((W_{\mathcal{J}})_{K_1}, \sigma).$$

But $(W_{\mathcal{J}})_{K_1}$ is an extension of χ by $\bigoplus_{i \in \mathcal{J}} \chi \alpha_i^{-1}$ by the last assertion in [Hu10, Lemma 2.4(ii)]. Hence $\sigma \in \{\tau, \delta_i^+(\tau) : i \in \mathcal{J}\}$. Conversely, it is enough to note that $Q_{\mathcal{J}} \twoheadrightarrow Q_i \twoheadrightarrow Q_i^+ \twoheadrightarrow \delta_i^+(\tau) \oplus \tau$ for all $i \in \mathcal{J}$ by Lemma 2.3.1(ii).

Hence $Q_{\mathcal{J}} = \Theta_{\mathcal{J}} + \sum_{i \in \mathcal{J}} \Psi_i$. Write $\mathcal{J} = \{0 \leq i_1 < \cdots < i_n \leq f-1\}$. Let $R_k \stackrel{\text{def}}{=} \Theta_{\mathcal{J}} + \sum_{j=1}^k \Psi_{i_j}$, with the convention $R_0 = \Theta_{\mathcal{J}}$. We will prove by induction that $R_k \cong R_{k-1} \oplus_{\mathrm{rad}_{\widetilde{\Gamma}}(\Psi_{i_k})} \Psi_{i_k}$ for $1 \leq k \leq n$, which will complete the proof. It suffices to show that $R_{k-1} \cap \Psi_{i_k} = \mathrm{rad}_{\widetilde{\Gamma}}(\Psi_{i_k})$. This is clear, as $\mu_{i_k}^+(\tau) \in \mathrm{JH}(\Theta_{\mathcal{J}})$, which gives the inclusion $\supseteq$, and as $\delta_{i_k}^+(\tau) \notin \mathrm{JH}(R_{k-1})$. (Recall that these constituents occur with multiplicity one in $Q_{\mathcal{J}}$.) □

3 Abstract setting

Let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous 0-generic representation as in § 1.4 and let π denote a smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$. In this section we introduce and study certain assumptions on π (relative to $\overline{\rho}$) that play a key role in our work.

3.1 Assumptions

From now until the end of this paper, we assume that π satisfies assumptions (i) (with $r = 1$) and (ii) in [BHH$^+$25, § 3.3.2] and assumption (iv) (with $r = 1$) in [BHH$^+$, § 2.1], i.e.

(i) we have $\pi^{K_1} \cong D_0(\overline{\rho})$ as $\mathrm{GL}_2(\mathcal{O}_K)$-representations (in particular, π is admissible) and π has central character $\det(\overline{\rho})\omega^{-1}$;

(ii) for any $\lambda \in \mathscr{P}$ we have $[\pi[\mathfrak{m}^3] : \chi_\lambda] = [\pi[\mathfrak{m}] : \chi_\lambda]$;

(iv) for any smooth character $\chi : I \to \mathbb{F}^\times$ and any $i \geq 0$, $\mathrm{Ext}^i_{I/Z_1}(\chi, \pi) \neq 0$ only if $[\pi[\mathfrak{m}] : \chi] \neq 0$, in which case

$$\dim_{\mathbb{F}} \mathrm{Ext}^i_{I/Z_1}(\chi, \pi) = \binom{2f}{i}.$$

For later reference we also recall assumption (iii) of [BHH$^+$25, § 3.3.5], *though we will not assume it until § 5*:

(iii) there is a $\mathrm{GL}_2(K)$-equivariant isomorphism of Λ-modules

$$\mathrm{E}^{2f}_\Lambda(\pi^\vee) \cong \pi^\vee \otimes (\det(\overline{\rho})\omega^{-1}),$$

where $\mathrm{E}^{2f}_\Lambda(\pi^\vee)$ is endowed with the $\mathrm{GL}_2(K)$-action defined in [Koh17, Prop. 3.2].

Finally, we introduce a further assumption which will be used only in § 6 (namely to verify equation (6) in the introduction).

(v) We have

$$\dim_{\mathbb{F}} \mathrm{Tor}^{\mathrm{gr}(\Lambda)}_1(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) = \dim_{\mathbb{F}} \mathrm{Tor}^\Lambda_1(\Lambda/\mathfrak{m}^3, \pi^\vee),$$

where $\overline{\mathfrak{m}} = (y_j, z_j : 0 \leq j \leq f-1)$ denotes the unique maximal graded ideal of $\mathrm{gr}(\Lambda)$ (see (12)).

We first note the following consequence:

Lemma 3.1.1. *Suppose assumptions* (i) *and* (iv) *hold. Let $\chi : I \to \mathbb{F}^\times$ be a character such that $\chi \notin \mathrm{JH}(\pi^{I_1})$ and Q be a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$. Then $\mathrm{Ext}^i_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(Q, \pi) = 0$ for $i \in \{0, 1\}$. In particular, this result holds when $Q = \tau$ is a Serre weight such that $\chi_\tau \notin \mathrm{JH}(\pi^{I_1})$ by taking $\chi = \chi_\tau$.*

Proof. Using [Bre14, Prop. 4.2], the assumption on χ implies that $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi) \cap W(\overline{\rho}) = \emptyset$. Thus for any *subquotient* Q of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$ we have $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(Q, \pi) = 0$, as $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma, \pi) \neq 0$ if and only if $\sigma \in W(\overline{\rho})$ by assumption (i).

Consider the short exact sequence $0 \to V \to \operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi \to Q \to 0$, where V is the corresponding kernel. It induces a short exact sequence

$$\operatorname{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(V,\pi) \to \operatorname{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(Q,\pi) \to \operatorname{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi,\pi) = 0,$$

where the first term vanishes by the last paragraph and the last term vanishes by assumption (iv) using Shapiro's lemma. The result follows. ☐

Remark 3.1.2. Even if it is not needed for this paper, it is natural to ask if $\operatorname{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau,\pi) = 0$ for any Serre weight $\tau \notin W(\overline{\rho})$. It is possible to prove this for a globally defined representation $\pi(\overline{\rho})$ as in [BHH$^+$, § 2.6], in a similar way to [BHH$^+$, Prop. 2.6.3], but we don't know how to deduce this property using only assumptions (i)–(iv).

3.2 Consequences of the assumptions

Recall from § 1.4 that $\mathfrak{m}_{K_1}$ denotes the maximal ideal of the Iwasawa algebra $\mathbb{F}[\![K_1/Z_1]\!]$ and $\widetilde{\Gamma} = \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}^2$ (which is a finite-dimensional $\mathbb{F}$-algebra). Let π be an admissible smooth $\mathrm{GL}_2(K)$-representation satisfying assumptions (i), (ii) and (iv) above with 2-generic underlying $\overline{\rho}$. In this subsection we explicitly determine the finite-dimensional $\widetilde{\Gamma}$-module $\pi[\mathfrak{m}_{K_1}^2]$.

Consider the $\widetilde{\Gamma}$-representation $\widetilde{D}_0(\overline{\rho})$ from § 1.4. As $\overline{\rho}$ is 2-generic (which is precisely strongly generic in the sense of [HW22, Def. 4.4]) one can deduce from [HW22, Prop. 4.1, Thm. 4.6] that $\widetilde{D}_0(\overline{\rho})$ is multiplicity free and has a direct sum decomposition

$$\widetilde{D}_0(\overline{\rho}) = \bigoplus_{\sigma\in W(\overline{\rho})} \widetilde{D}_{0,\sigma}(\overline{\rho}),$$

where $\widetilde{D}_{0,\sigma}(\overline{\rho})$ is the largest subrepresentation of $\operatorname{Inj}_{\widetilde{\Gamma}} \sigma$ containing σ with multiplicity one and no other Serre weights of $W(\overline{\rho})$ (see also [BHH$^+$23, Thm. 8.4.2]). In particular, $\operatorname{soc}_{\widetilde{\Gamma}}(\widetilde{D}_{0,\sigma}(\overline{\rho})) = \sigma$.

Lemma 3.2.1. *Let τ' be a Serre weight.*

(i) *If $\tau' \notin W(\overline{\rho})$, then* $\operatorname{Ext}^1_{\widetilde{\Gamma}}(\tau', \widetilde{D}_0(\overline{\rho})) = 0$.

(ii) *If $\tau' \in \mathrm{JH}(D_0(\overline{\rho})) \setminus W(\overline{\rho})$, then* $\operatorname{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau', D_0(\overline{\rho})) \cong \operatorname{Ext}^1_{\widetilde{\Gamma}}(\tau', D_0(\overline{\rho})) = 0$.

Proof. (i) This follows from the maximality of $\widetilde{D}_0(\overline{\rho})$ recalled above.

(ii) The first isomorphism is a general fact, because both τ' and $D_0(\overline{\rho})$ are annihilated by $\mathfrak{m}_{K_1}$ (so any extension between them is automatically annihilated by $\mathfrak{m}_{K_1}^2$). The second one follows from (i) and the fact that $\operatorname{Hom}_{\widetilde{\Gamma}}(\tau', \widetilde{D}_0(\overline{\rho})/D_0(\overline{\rho})) = 0$ (as $\widetilde{D}_0(\overline{\rho})$ is multiplicity free). ☐

Let τ be a Serre weight and $\chi \overset{\text{def}}{=} \tau^{I_1}$ and recall the representation $\overline{W}_{\chi,3}$ from § 2.3. If τ (hence χ) is 2-generic then by [HW22, Prop. 3.10(i)] for any Jordan–Hölder factor τ' of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \overline{W}_{\chi,3}$ there exists a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant morphism

$$\phi_{\tau'} : \operatorname{Proj}_{\widetilde{\Gamma}} \tau' \to \operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \overline{W}_{\chi,3} \tag{27}$$

such that $[\mathrm{coker}(\phi_{\tau'}) : \tau'] = 0$. Note that by [HW22, Prop. 3.10(ii)], the image $\mathrm{im}(\phi_{\tau'})$ is unique (even though $\phi_{\tau'}$ need not be unique up to scalar).

For the following, we emphasize that $\phi_{\tau'}$ depends on τ (we always take $\chi \stackrel{\mathrm{def}}{=} \tau^{I_1}$).

Lemma 3.2.2. *Assume that $\tau \in W(\overline{\rho})$, so τ is 2-generic. Then $\mathrm{JH}(\mathrm{coker}(\phi_\tau)) \cap W(\overline{\rho}) = \emptyset$, and*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau', \tau) = 0$$

for any $\tau' \in \mathrm{JH}(\mathrm{coker}(\phi_\tau))$.

Proof. The first assertion is proved in [HW22, Cor. 4.14] (taking $\chi = \tau^{I_1}$ there). The second is essentially a consequence of [HW22, Cor. 3.11]. To see this, let τ' be a Serre weight such that $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau', \tau) \neq 0$ (this is equivalent to $\mathrm{Ext}^1_{\Gamma}(\tau', \tau) \neq 0$ by [BHH+, Lemma 4.3.4], noting that τ' is automatically 0-generic by [BP12, Cor. 5.6(ii)]). We need to prove that $\tau' \notin \mathrm{JH}(\mathrm{coker}(\phi_\tau))$. By [HW22, Prop. 3.12(ii)] (where $\mathscr{E}(\tau)$ in *loc. cit.* is the set of Serre weights τ'' such that $\mathrm{Ext}^1_{\Gamma}(\tau'', \tau) \neq 0$), we know that τ' is a Jordan–Hölder factor of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \overline{W}_{\chi,3}$, so we have a morphism $\phi_{\tau'}$ as in (27). Since $[\mathrm{coker}(\phi_{\tau'}) : \tau'] = 0$, it suffices to prove that $\mathrm{im}(\phi_{\tau'}) \subseteq \mathrm{im}(\phi_\tau)$. But this follows from [HW22, Cor. 3.11(a), (c)]. Indeed, if $\tau' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)$, then we conclude by [HW22, Cor. 3.11(a)], as $J(\tau) = \{0, \dots, f-1\}$ in the notation of *loc. cit.*; if $\tau' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi')$ for some $\chi' \in \mathrm{JH}(\overline{W}_{\chi,3})$ with $\chi' \neq \chi$, then we conclude by [HW22, Cor. 3.11(c)]. ☐

Corollary 3.2.3. *If $\tau \in W(\overline{\rho})$, then*

$$\mathrm{Ext}^1_{\widetilde{\Gamma}}(\mathrm{coker}(\phi_\tau), \widetilde{D}_0(\overline{\rho})) = \mathrm{Ext}^1_{\widetilde{\Gamma}}(\mathrm{coker}(\phi_\tau), \tau) = 0.$$

Proof. Using $\mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau', \tau) = \mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau', \tau)$ for any Serre weight τ', the first term is 0 by dévissage from Lemma 3.2.1(i) and the first assertion in Lemma 3.2.2, and the second term is 0 by dévissage from the second assertion in Lemma 3.2.2. ☐

Lemma 3.2.4. *Assume that $\tau \in W(\overline{\rho})$. Then $\mathrm{coker}(\phi_\tau)$ has a direct sum decomposition*

$$\mathrm{coker}(\phi_\tau) \cong \bigoplus_{j=0}^{f-1} \mathrm{coker}(\phi_\tau)_j, \tag{28}$$

where $\mathrm{coker}(\phi_\tau)_j$ is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi\alpha_j$ for $0 \leq j \leq f-1$. Moreover,

(i) *if $\chi\alpha_j \in \mathrm{JH}(\pi^{I_1})$, then $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}\big(\mathrm{coker}(\phi_\tau)_j, D_0(\overline{\rho})\big) = 0$;*

(ii) *if $\chi\alpha_j \notin \mathrm{JH}(\pi^{I_1})$, then $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}\big(\mathrm{coker}(\phi_\tau)_j, \pi\big) = 0$.*

Remark 3.2.5. Although it will not be used in this paper, we have the following explicit description of $\mathrm{coker}(\phi_\tau)_j$: it is the unique quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi\alpha_j$ consisting of the Jordan–Hölder factors parametrized by the subsets of $\{0, \dots, f-1\}$ that contain j.

Proof. By construction, $\mathrm{im}(\phi_\tau)$ contains the image of any morphism $\mathrm{Proj}_{\widetilde{\Gamma}}\,\tau \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\overline{W}_{\chi,3}$, and in particular contains the subrepresentation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi^{\oplus 2f} \subseteq \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\overline{W}_{\chi,3}$ (recall that $\chi^{\oplus 2f} \subseteq \overline{W}_{\chi,3}$ by (18) and that $\mathrm{cosoc}_\Gamma\big(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi\big) = \tau$). Thus, the quotient map $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\overline{W}_{\chi,3} \twoheadrightarrow \mathrm{coker}(\phi_\tau)$ factors through $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,W_{\chi,2} \twoheadrightarrow \mathrm{coker}(\phi_\tau)$. Recall that $W_{\chi,2}$ fits into a short exact sequence

$$0 \to \bigoplus_{j=0}^{f-1}(\chi\alpha_j \oplus \chi\alpha_j^{-1}) \to W_{\chi,2} \to \chi \to 0.$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(q) & \longrightarrow & \mathrm{Proj}_{\widetilde{\Gamma}}\,\tau & \xrightarrow{\;q\;} & \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi & \longrightarrow & 0\\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \phi_\tau} & & \big\| & & \\
0 & \longrightarrow & \bigoplus_{j=0}^{f-1}\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi\alpha_j^{\pm 1} & \longrightarrow & \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,W_{\chi,2} & \longrightarrow & \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi & \longrightarrow & 0
\end{array}$$

so that we have a surjection $\gamma : \bigoplus_{j=0}^{f-1}\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi\alpha_j^{\pm 1} \twoheadrightarrow \mathrm{coker}(\phi_\tau)$ by the snake lemma. As $\bigoplus_{j=0}^{f-1}\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi\alpha_j^{\pm 1}$ is multiplicity free (for instance by [BHH$^+$, Lemma 4.3.3]) we deduce an isomorphism $\mathrm{coker}(\phi_\tau) \cong \bigoplus_{j=0}^{f-1}\mathrm{coker}(\phi_\tau)_j^{\pm}$, where $\mathrm{coker}(\phi_\tau)_j^{\pm} \overset{\mathrm{def}}{=} \gamma(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi\alpha_j^{\pm 1})$ (in particular it is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi\alpha_j^{\pm 1}$). If $\mathrm{coker}(\phi_\tau)_j^- \neq 0$ then $\mathrm{coker}(\phi_\tau)$ and a fortiori $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,W_{\chi,2}$ would surject onto $\delta_j^-(\tau)$ (the cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi\alpha_j^{-1}$). But this is not true by Frobenius reciprocity, as one checks that $\mathrm{Hom}_I(W_{\chi,2}, \delta_j^-(\tau)) = 0$ by [BHH$^+$25, Lemma 3.42(ii)]. We thus get the decomposition (28) by taking $\mathrm{coker}(\phi_\tau)_j \overset{\mathrm{def}}{=} \mathrm{coker}(\phi_\tau)_j^+$.

(i) By Lemma 3.2.1(ii) and the first statement in Lemma 3.2.2, it suffices to show that $\mathrm{JH}(\mathrm{coker}(\phi_\tau)_j) \subseteq \mathrm{JH}(D_0(\overline{\rho}))$ when $\chi\alpha_j \in \mathrm{JH}(\pi^{I_1})$. In fact, we prove the following stronger statement: if $\chi' \in \mathrm{JH}(\pi^{I_1})$ then $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi') \subseteq \mathrm{JH}(D_0(\overline{\rho}))$. By [BHH$^+$, eq. (54)] we have $\mathrm{JH}(\mathrm{Proj}_\Gamma\,\sigma) \subseteq \mathrm{JH}(D_0(\overline{\rho}))$ for any $\sigma \in W(\overline{\rho})$. Now, since $\chi' \in \mathrm{JH}(\pi^{I_1})$, we have $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi') \cap W(\overline{\rho}) \neq \emptyset$ by [Bre14, Prop. 4.2]. Thus it suffices to prove that

$$\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi') \subseteq \mathrm{JH}(\mathrm{Proj}_\Gamma\,\sigma') \tag{29}$$

for any $\sigma' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi')$.

We prove (29) for any character $\chi' : I \to \mathbb{F}^\times$ satisfying $\chi' \neq \chi'^s$. Let $\mathrm{Proj}_{W(\mathbb{F})[\Gamma]}\,\sigma'$ be the projective cover of σ' in the category of $W(\mathbb{F})[\Gamma]$-modules. Let $[\chi'] : I \to W(\mathbb{F})^\times$ be the Teichmüller lift of χ'. Since $\sigma' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\,\chi')$, there is a non-zero morphism $\gamma : \mathrm{Proj}_{W(\mathbb{F})[\Gamma]}\,\sigma' \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}[\chi']$. Inverting p, the latter representation is irreducible over $W(\mathbb{F})[1/p]$ as $[\chi'] \neq [\chi'^s]$, so γ is surjective after inverting p. We conclude by the Brauer–Nesbitt theorem (cf. [Ser77, Chap. 15, Thm. 32]).

(ii) It is a direct consequence of Lemma 3.1.1. $\square$

Lemma 3.2.6. *Let $\tau \in W(\overline{\rho})$ and Q be a quotient of* $\mathrm{Proj}_{\widetilde{\Gamma}}\tau$ *such that* $\mathrm{rad}_{\widetilde{\Gamma}}(Q) \subseteq \widetilde{D}_0(\overline{\rho})$ *(hence* $\mathrm{rad}_{\widetilde{\Gamma}}(Q)$ *is multiplicity free). Then Q is a quotient of* $\mathrm{im}(\phi_\tau)$.

Proof. In this proof, if M is a finite-dimensional $\widetilde{\Gamma}$-module, we write $\mathrm{rad}(M)$, $\mathrm{soc}(M)$ and $\mathrm{cosoc}(M)$ for $\mathrm{rad}_{\widetilde{\Gamma}}(M)$, $\mathrm{soc}_{\widetilde{\Gamma}}(M)$ and $\mathrm{cosoc}_{\widetilde{\Gamma}}(M)$ respectively. We may assume that $Q \neq 0$. Since $\mathrm{rad}(Q)$ is multiplicity free by assumption, we have $[\mathrm{rad}(Q):\tau] \leq 1$ and $[Q:\tau] \leq 2$. Since $\mathrm{rad}(Q) \subseteq \widetilde{D}_0(\overline{\rho}) = \bigoplus_{\sigma\in W(\overline{\rho})} \widetilde{D}_{0,\sigma}(\overline{\rho})$ (which is multiplicity free), we have a decomposition

$$\mathrm{rad}(Q) = \bigoplus_{\sigma\in W(\overline{\rho})} V_\sigma$$

for some subrepresentations $V_\sigma \subseteq \widetilde{D}_{0,\sigma}(\overline{\rho})$. If $V_\sigma \neq 0$, let Q_σ be the quotient of Q by its largest subrepresentation in which σ does not occur, so $\mathrm{soc}(Q_\sigma) \cong \sigma$ (even if $\sigma = \tau$, as $\mathrm{cosoc}(Q_\sigma) = \tau$) and $0 \to V_\sigma \to Q_\sigma \to \tau \to 0$. Assume first $\sigma \neq \tau$ and $V_\sigma \neq 0$. By [HW22, Lemma 4.10], the natural morphism

$$\mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,\sigma) \to \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,V_\sigma) \tag{30}$$

is an isomorphism. Since Q_σ has cosocle τ, we deduce that $V_\sigma = \sigma$. Assume next that $\sigma = \tau$ and also $V_\tau \neq 0$. Then [HW22, Cor. 4.9] implies that the natural inclusion $D_{0,\tau}(\overline{\rho}) \hookrightarrow \widetilde{D}_{0,\tau}(\overline{\rho})$ induces an isomorphism

$$\mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau, D_{0,\tau}(\overline{\rho})) \xrightarrow{\sim} \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau, \widetilde{D}_{0,\tau}(\overline{\rho})). \tag{31}$$

Letting $A \overset{\mathrm{def}}{=} V_\tau \cap D_{0,\tau}(\overline{\rho}) \neq 0$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,A) & \longrightarrow & \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,V_\tau) & \longrightarrow & \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,V_\tau/A) \\
& \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 \longrightarrow & \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,D_{0,\tau}(\overline{\rho})) & \xrightarrow{\sim} & \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,\widetilde{D}_{0,\tau}(\overline{\rho})) & \longrightarrow & \mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,\widetilde{D}_{0,\tau}(\overline{\rho})/D_{0,\tau}(\overline{\rho}))
\end{array}$$

where all the vertical arrows are easily seen to be injective (as $\widetilde{D}_{0,\tau}(\overline{\rho})$ is multiplicity free). A diagram chase together with (31) shows that the class of Q_τ in $\mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,V_\tau)$ lies in the image of $\mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau,A)$. Since Q_τ has cosocle τ, we have $A = V_\tau$, namely $V_\tau \subseteq D_{0,\tau}(\overline{\rho})$. Altogether we get $\mathrm{rad}(Q) \subseteq D_0(\overline{\rho})$.

Now we prove the lemma. If τ does not occur in $\mathrm{rad}(Q)$, then $[Q:\tau] = 1$ and $V_\tau = 0$. Moreover, the discussion in the last paragraph implies that $\mathrm{rad}(Q) = \bigoplus_{\sigma\in\mathrm{JH}(\mathrm{soc}(Q))}\sigma$ (provided $\mathrm{rad}(Q) \neq 0$). Thus Q is a Γ-representation by [BHH$^+$, Lemma 4.3.4] (and the first sentence in its proof), and [HW22, Cor. 3.14] (applied with $m = 0$) implies that Q is a certain quotient of Θ_τ, where Θ_τ in *loc. cit.* is a quotient of $\mathrm{im}(\phi_\tau)$ constructed in [HW22, Prop. 3.12]. As a consequence, Q is a quotient of $\mathrm{im}(\phi_\tau)$.

If τ occurs in $\mathrm{rad}(Q)$, then τ must occur in $\mathrm{soc}(Q)$ as $\mathrm{rad}(Q) \subseteq D_0(\overline{\rho})$ and $\tau \in \mathrm{JH}(\mathrm{soc}(D_0(\overline{\rho})))$. In this case Q satisfies the following conditions:

(1) $[Q:\tau] = 2$, $\tau \hookrightarrow \mathrm{soc}(Q)$, and $\mathrm{cosoc}(Q) \cong \tau$;

(2) $\mathrm{rad}(Q)$ is a subrepresentation of $D_0(\overline{\rho})$.

It is proved in the last paragraph of the proof of [HW22, Prop. 4.18] that such a representation is a quotient of Θ_τ, hence of $\mathrm{im}(\phi_\tau)$. The argument goes as follows. Firstly, by [HW22, Lemma 4.10] condition (2) implies that $\mathrm{JH}(\mathrm{soc}(Q))$ is contained in $\{\tau\} \cup \mathscr{E}(\tau)$, where $\mathscr{E}(\tau)$ denotes the set of Serre weights τ' such that $\mathrm{Ext}^1_\Gamma(\tau', \tau) \neq 0$ (equivalently, $\mathrm{Ext}^1_\Gamma(\tau, \tau') \neq 0$). Secondly, using (1) and the fact $\mathrm{Ext}^1_{\widetilde{\Gamma}}(\tau, \tau) = 0$, one shows that the socle of $C \overset{\mathrm{def}}{=} Q/\tau$ is contained in $\mathscr{E}(\tau)$ and, using again [HW22, Lemma 4.10] and $\mathrm{cosoc}_{\widetilde{\Gamma}}(Q) = \tau$, that C fits in a short exact sequence

$$0 \to S \to C \to \tau \to 0$$

for some subrepresentation S of $\bigoplus_{\tau' \in \mathscr{E}(\tau)} \tau'$. Then we conclude by [HW22, Cor. 3.14]. □

Lemma 3.2.7. *Assume that $\tau \in W(\overline{\rho})$. Then $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{Proj}_{\widetilde{\Gamma}} \tau, \pi)$ has dimension 1 over $\mathbb{F}$.*

Proof. **Step 1.** We prove that $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\tau, \pi/\pi[\mathfrak{m}_{K_1}]) = 0$. Suppose by contradiction that $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\tau, \pi/\pi[\mathfrak{m}_{K_1}]) \neq 0$. The pullback of τ gives a subrepresentation $V \subseteq \pi|_{\mathrm{GL}_2(\mathcal{O}_K)}$ which (using assumption (i)) fits into a nonsplit extension

$$0 \to D_0(\overline{\rho}) \to V \to \tau \to 0. \tag{32}$$

Note that V is a $\widetilde{\Gamma}$-representation but not a Γ-representation. By the projectivity of $\mathrm{Proj}_{\widetilde{\Gamma}} \tau$, there exists a $\widetilde{\Gamma}$-equivariant morphism $q : \mathrm{Proj}_{\widetilde{\Gamma}} \tau \to V$ whose composition with $V \twoheadrightarrow \tau$ is the natural surjection $\mathrm{Proj}_{\widetilde{\Gamma}} \tau \twoheadrightarrow \tau$. Let V_τ denote the image of q, which has cosocle τ. Clearly V_τ satisfies the conditions (on Q) in Lemma 3.2.6, so there exists a surjection $\mathrm{im}(\phi_\tau) \twoheadrightarrow V_\tau$ and we denote by β the composition $\mathrm{im}(\phi_\tau) \twoheadrightarrow V_\tau \hookrightarrow V$.

We introduce a 3-step filtration on $M \overset{\mathrm{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \overline{W}_{\chi,3}$ as follows. Let $S \subseteq \{0, \ldots, f-1\}$ be the set of indices j such that $\chi\alpha_j \in \mathrm{JH}(\pi^{I_1})$. Put $M_2 \overset{\mathrm{def}}{=} \mathrm{im}(\phi_\tau) \subseteq M$ and

$$M_1 \overset{\mathrm{def}}{=} \ker \big(M \twoheadrightarrow \bigoplus_{j \notin S} \mathrm{coker}(\phi_\tau)_j\big),$$

where we used (28). Then $0 \subseteq M_2 \subseteq M_1 \subseteq M$ with

$$M_1/M_2 \cong \bigoplus_{j \in S} \mathrm{coker}(\phi_\tau)_j, \quad M/M_1 \cong \bigoplus_{j \notin S} \mathrm{coker}(\phi_\tau)_j. \tag{33}$$

By Lemma 3.2.2, Lemma 3.2.4(i) and (32), (33) we have $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(M_1/M_2, V) = 0$, so the natural morphism $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(M_1, V) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(M_2, V)$ is surjective. Thus we can lift β to $\beta' : M_1 \to V$, which we view as a morphism $\beta' : M_1 \to \pi$ (as $V \subseteq \pi$). Next, since $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(M/M_1, \pi) = 0$ by Lemma 3.2.4(ii) and (33), we can further lift β' to a morphism $\beta'' : M \to \pi$. By Frobenius reciprocity, we obtain an I-equivariant morphism $\overline{W}_{\chi,3} \to \pi|_I$ whose image is then contained in $\pi[\mathfrak{m}^3]$ and has cosocle χ. As $\tau \in W(\overline{\rho})$ we have $\chi \hookrightarrow \pi^{I_1} = \pi[\mathfrak{m}]$. By assumption (ii) we conclude that χ does not appear in $\pi[\mathfrak{m}^3]/\pi[\mathfrak{m}]$, hence $\overline{W}_{\chi,3} \to \pi|_I$ factors through $\overline{W}_{\chi,3} \twoheadrightarrow \chi \hookrightarrow \pi|_I$. Correspondingly, β'' itself factors through $M \twoheadrightarrow \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi \to \pi$, so $\mathrm{im}(\beta'')$ is contained in $\pi[\mathfrak{m}_{K_1}]$. But this is not true by construction of β'', contradiction.

Step 2. Suppose by contradiction that $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{Proj}_{\widetilde{\Gamma}} \tau, \pi) \geq 2$. By [HW22, Prop. 4.18], which requires $\overline{\rho}$ to be 2-generic and condition (a) at the beginning of [HW22, § 4.3] to hold, we also have $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_\tau, \pi) \geq 2$. (Recall from [HW22, § 3.3] that Θ_τ is the smallest quotient of $\mathrm{Proj}_{\widetilde{\Gamma}} \tau / \mathrm{rad}^3_{\widetilde{\Gamma}} (\mathrm{Proj}_{\widetilde{\Gamma}} \tau)$ such that $[\mathrm{Proj}_{\widetilde{\Gamma}} \tau / \mathrm{rad}^3_{\widetilde{\Gamma}} (\mathrm{Proj}_{\widetilde{\Gamma}} \tau) : \tau] = [\Theta_\tau : \tau]$ and that Θ_τ fits into a short exact sequence

$$0 \to \bigoplus_{\tau'} E_{\tau,\tau'} \to \Theta_\tau \to \tau \to 0,$$

where the direct sum is taken over the Serre weights τ' such that $\mathrm{Ext}^1_\Gamma(\tau', \tau) \neq 0$; see [HW22, Cor. 3.16].) Thus, there exists a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant morphism $\gamma : \Theta_\tau \to \pi$ which does not factor through the cosocle of Θ_τ. Since $\pi[\mathfrak{m}_{K_1}] \cong D_0(\overline{\rho})$ is multiplicity free and $\tau \cong \mathrm{cosoc}_{\widetilde{\Gamma}}(\Theta_\tau)$ occurs in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$, we deduce that $\mathrm{im}(\gamma)$ is not contained in $\pi[\mathfrak{m}_{K_1}]$. However, $\mathrm{rad}_{\widetilde{\Gamma}}(\Theta_\tau)$ is annihilated by $\mathfrak{m}_{K_1}$ (by [HW22, Cor. 3.16]), so the image U of $\mathrm{rad}_{\widetilde{\Gamma}}(\Theta_\tau)$ is contained in $\pi[\mathfrak{m}_{K_1}]$. The inclusions $U \subseteq \pi[\mathfrak{m}_{K_1}] \subseteq \pi$ induce natural maps

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, U) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, \pi[\mathfrak{m}_{K_1}]) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, \pi). \tag{34}$$

The first map is injective, because by assumption (i) either $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\tau, \pi[\mathfrak{m}_{K_1}]/U) = 0$ (if $\tau \in \mathrm{JH}(U)$) or the map $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\tau, \pi[\mathfrak{m}_{K_1}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\tau, \pi[\mathfrak{m}_{K_1}]/U)$ is surjective (if $\tau \notin \mathrm{JH}(U)$). The second map is also injective by Step 1. However, viewing $\mathrm{im}(\gamma)$ as a (non-zero) element in $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, U)$, which is sent to 0 via the natural map $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, U) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, \mathrm{im}(\gamma))$, we conclude that $\mathrm{im}(\gamma)$ is sent to zero via (34) since the latter factors through $\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, U) \to \mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\tau, \mathrm{im}(\gamma))$. This gives the desired contradiction. □

Proposition 3.2.8. *Suppose that π satisfies assumptions* (i), (ii) *and* (iv) *with a 2-generic underlying $\overline{\rho}$. Then*

$$\pi[\mathfrak{m}^2_{K_1}] \cong \widetilde{D}_0(\overline{\rho}). \tag{35}$$

Proof. It follows from Lemma 3.2.7 that $[\pi[\mathfrak{m}^2_{K_1}] : \sigma] = 1$ for any $\sigma \in W(\overline{\rho})$ (recall that $[\pi[\mathfrak{m}^2_{K_1}] : \sigma]$ is precisely the dimension of $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{Proj}_{\widetilde{\Gamma}} \sigma, \pi)$). From the construction of $\widetilde{D}_0(\overline{\rho})$ we deduce an inclusion $\pi[\mathfrak{m}^2_{K_1}] \subseteq \widetilde{D}_0(\overline{\rho})$. Suppose the inclusion is strict, and choose a Serre weight $\tau \hookrightarrow \widetilde{D}_0(\overline{\rho})/\pi[\mathfrak{m}^2_{K_1}]$. Let $V_\tau \subseteq \widetilde{D}_0(\overline{\rho})$ be a subrepresentation with cosocle τ and such that the composition $V_\tau \hookrightarrow \widetilde{D}_0(\overline{\rho}) \twoheadrightarrow \widetilde{D}_0(\overline{\rho})/\pi[\mathfrak{m}^2_{K_1}]$ coincides with the chosen inclusion $\tau \hookrightarrow \widetilde{D}_0(\overline{\rho})/\pi[\mathfrak{m}^2_{K_1}]$. As $D_0(\overline{\rho}) \subseteq \pi[\mathfrak{m}^2_{K_1}]$ by assumption (i), we have $\tau \in \mathrm{JH}(\widetilde{D}_0(\overline{\rho})) \setminus \mathrm{JH}(D_0(\overline{\rho}))$, so in particular $\chi_\tau \notin \mathrm{JH}(\pi^{I_1})$. Applying $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(-, \pi)$ to $0 \to \mathrm{rad}_{\widetilde{\Gamma}}(V_\tau) \to V_\tau \to \tau \to 0$ and using Lemma 3.1.1, we obtain an isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(V_\tau, \pi) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{rad}_{\widetilde{\Gamma}}(V_\tau), \pi).$$

Thus, the natural inclusion $\mathrm{rad}_{\widetilde{\Gamma}}(V_\tau) \subseteq \pi$ lifts to an embedding $V_\tau \hookrightarrow \pi$, whose image is contained in $\pi[\mathfrak{m}^2_{K_1}]$ as V_τ is annihilated by $\mathfrak{m}^2_{K_1}$. This gives a contradiction as $\tau \notin \mathrm{JH}(\pi[\mathfrak{m}^2_{K_1}])$ ($\widetilde{D}_0(\overline{\rho})$ being multiplicity free). □

4 On the Hilbert series of π

Let π be a smooth mod p representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying assumptions (i), (ii) and (iv) of § 3. In this section we compute the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$.

If $M = \bigoplus_{n\leq 0} M_n$ is a graded $\mathbb{F}$-vector space with $\dim_{\mathbb{F}} M_n < +\infty$ for all n, we define the *Hilbert series*

$$h_M(t) \stackrel{\text{def}}{=} \sum_{n\geq 0} \dim_{\mathbb{F}}(M_{-n})t^n \in \mathbb{Z}[\![t]\!].$$

In particular, if Π is any admissible smooth representation of $\mathrm{GL}_2(K)$ the Hilbert series $h_\Pi(t) \stackrel{\text{def}}{=} h_{\mathrm{gr}_{\mathfrak{m}}(\Pi^\vee)}(t) \in \mathbb{Z}[\![t]\!]$ is defined.

Theorem 4.1. *Assume that $\overline{\rho}$ is 9-generic and that π satisfies assumptions* (i), (ii) *and* (iv) *of § 3.*

(i) *If $\overline{\rho}$ is irreducible, then* $h_\pi(t) = \dfrac{(3+t)^f}{(1-t)^f} - 1.$

(ii) *If $\overline{\rho}$ is split reducible, then* $h_\pi(t) = \dfrac{(3+t)^f}{(1-t)^f} + 1.$

(iii) *If $\overline{\rho}$ is nonsplit reducible and $d_{\overline{\rho}} \stackrel{\text{def}}{=} |J_{\overline{\rho}}|$ (so $d_{\overline{\rho}} < f$, see* (10) *for $J_{\overline{\rho}}$), then* $h_\pi(t) = 2^{f-d_{\overline{\rho}}} \cdot \dfrac{(1+t)^{f-d_{\overline{\rho}}}(3+t)^{d_{\overline{\rho}}}}{(1-t)^f}.$

Remark 4.2. Note that the denominator of $h_\pi(t)$ equals $(1-t)^f$ expresses the fact that the Gelfand–Kirillov dimension of π equals f. (By [BHH$^+$23, Lemma 5.1.3], the Gelfand–Kirillov dimension of π equals the dimension of $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ as an $\overline{R}$-module, hence equals the dimension of $(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))_{\overline{\mathfrak{m}}}$ as an $\overline{R}_{\overline{\mathfrak{m}}}$-module by [BH93, Ex. 1.5.25], hence equals the exponent of $(1-t)$ in the denominator of $h_\pi(t)$ by [Mat89, Thms. 13.2, 13.4], cf. the discussion on [Mat89, p. 97].)

Remark 4.3. Note that if we put $t = 0$ we recover the dimension formula for $D_0(\overline{\rho})^{I_1} = \pi^{I_1}$ in [BP12, Thm. 1.1].

Proof. We first recall that, under our assumptions, by [BHH$^+$, Thm. 2.1.2] we have

$$\mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \cong N \stackrel{\text{def}}{=} \bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda),$$

where $\mathfrak{a}(\lambda)$ is the ideal of R associated to $\lambda \in \mathscr{P}$ in (13). It remains to determine $h_N(t)$.

We note the following elementary but useful formulas. First, if M, M' are two graded $\mathbb{F}$-vector spaces, then

$$h_{M\otimes M'}(t) = h_M(t)h_{M'}(t). \tag{36}$$

Second, we have for any integer $n \geq 0$:

$$\frac{1}{2}\Big[(2+x)^n - (2-x)^n\Big] = \sum_{0\leq i\leq n,\ i \text{ odd}} \binom{n}{i} 2^{n-i}x^i, \tag{37}$$

$$\frac{1}{2}\Big[(2+x)^n + (2-x)^n\Big] = \sum_{0\leq i\leq n,\ i \text{ even}} \binom{n}{i} 2^{n-i}x^i. \tag{38}$$

By definition of N, we have $h_N(t) = \sum_{\lambda\in\mathscr{P}} h_{R/\mathfrak{a}(\lambda)}(t)$. Recalling $\mathfrak{a}(\lambda) = (t_j; 0 \leq j \leq f-1)$ with $t_j \in \{y_j, z_j, y_jz_j\}$ and noting that

$$h_{\mathbb{F}[y_i]}(t) = h_{\mathbb{F}[z_i]}(t) = 1/(1-t), \quad h_{\mathbb{F}[y_i,z_i]/(y_iz_i)}(t) = (1+t)/(1-t),$$

we obtain by (36) that

$$h_{R/\mathfrak{a}(\lambda)}(t) = \frac{(1+t)^{|\mathcal{A}(\lambda)|}}{(1-t)^f},$$

where $\mathcal{A}(\lambda) \stackrel{\text{def}}{=} \{j : t_j = y_jz_j\}$. Hence, we are reduced to counting the cardinality of $\lambda \in \mathscr{P}$ such that $|\mathcal{A}(\lambda)| = s$ for a given $0 \leq s \leq f$.

(i) Given $\lambda \in \mathscr{P}$, we define an element $\overline{\lambda} \in \mathscr{D}$ as follows:

$$\overline{\lambda}_0(x_0) \stackrel{\text{def}}{=} \begin{cases} x_0 - 1 & \text{if } \lambda_0(x_0) \in \{x_0 - 1, x_0 + 1\}, \\ p-2-x_0 & \text{if } \lambda_0(x_0) \in \{p-2-x_0, p-x_0\}, \\ \lambda_0(x_0) & \text{otherwise} \end{cases}$$

and if $j \neq 0$,

$$\overline{\lambda}_j(x_j) \stackrel{\text{def}}{=} \begin{cases} x_j & \text{if } \lambda_j(x_j) \in \{x_j, x_j + 2\}, \\ p-3-x_j & \text{if } \lambda_j(x_j) \in \{p-1-x_j, p-3-x_j\}, \\ \lambda_j(x_j) & \text{otherwise.} \end{cases}$$

It is easy to see that $\overline{\lambda} \in \mathscr{D}$. By [BHH+25, Def. 3.57], we have $t_j = y_jz_j$ if and only if $\lambda_j(x_j) \in \{x_j + 1, p-2-x_j\}$ if $j \neq 0$ (resp. $\lambda_0(x_0) \in \{x_0, p-1-x_0\}$), thus $\mathcal{A}(\lambda) = \mathcal{A}(\overline{\lambda})$. On the other hand, given $\overline{\lambda} \in \mathscr{D}$, there exist exactly $2^{|\{0,\dots,f-1\}\setminus\mathcal{A}(\overline{\lambda})|}$ elements $\lambda \in \mathscr{P}$ giving rise to $\overline{\lambda}$. As a consequence we have $h_N(t) = Q_N(t)/(1-t)^f$ with

$$Q_N(t) \stackrel{\text{def}}{=} \sum_{0\leq s\leq f,\ s \text{ odd}} 2^{f-s} \cdot 2\binom{f}{s}(1+t)^s = (3+t)^f - (1-t)^f,$$

where the first equality follows from Lemma 4.5 below and the second from (37) (with $x = 1+t$). The result follows.

(ii) The proof is similar to (i) using Lemma 4.5(ii) below and (38).

(iii) Let $\overline{\mathscr{P}} \subseteq \mathscr{P}$ be the subset introduced in the proof of [BHH+25, Prop. 3.61], namely $\lambda \in \overline{\mathscr{P}}$ if and only if

$$\lambda_j(x_j) \in \{x_j, x_j + 1, p-1-x_j, p-2-x_j, p-3-x_j\} \tag{39}$$

and $\lambda_j(x_j) = p-1-x_j$ implies $j \notin J_{\overline{\rho}}$ (recall from (10) that $\lambda_j(x_j) = p-3-x_j$ implies $j \in J_{\overline{\rho}}$). In the proof of [BHH+25, Prop. 3.61] a map $\mathscr{P} \to \overline{\mathscr{P}}$, $\lambda \mapsto \overline{\lambda}$ is defined, which satisfies $\mathcal{A}(\lambda) = \mathcal{A}(\overline{\lambda})$ and for any $\overline{\lambda} \in \overline{\mathscr{P}}$, there exist exactly $2^{|\{0,\ldots,f-1\}\setminus\mathcal{A}(\overline{\lambda})|}$ elements λ in $\mathscr{P}$ giving rise to $\overline{\lambda}$. Using Lemma 4.5(iii) below (with $|\mathcal{A}(\lambda)| = f - d_{\overline{\rho}} + s$), we then obtain $h_N(t) = Q'_N(t)/(1-t)^f$, where

$$Q'_N(t) \overset{\text{def}}{=} \sum_{0 \le s \le d_{\overline{\rho}}} 2^{d_{\overline{\rho}}-s} \cdot 2^{f-d_{\overline{\rho}}} \binom{d_{\overline{\rho}}}{s} (1+t)^{(f-d_{\overline{\rho}})+s} = 2^{f-d_{\overline{\rho}}}(1+t)^{f-d_{\overline{\rho}}}(3+t)^{d_{\overline{\rho}}}$$

(recall $d_{\overline{\rho}} = |J_{\overline{\rho}}|$), proving the result. ☐

Remark 4.4. We note that our proof determines $h_N(t)$ in each case without any genericity conditions on $\overline{\rho}$.

Lemma 4.5.

(i) *If $\overline{\rho}$ is irreducible, then $|\mathcal{A}(\lambda)|$ is odd for all $\lambda \in \mathscr{D}$. For any subset $J \subseteq \{0,\ldots,f-1\}$ with $|J|$ odd, there exist exactly 2 elements $\lambda \in \mathscr{D}$ such that $\mathcal{A}(\lambda) = J$. As a consequence, for any $0 \le s \le f$ which is odd, the set of $\lambda \in \mathscr{D}$ with $|\mathcal{A}(\lambda)| = s$ has cardinality $2\binom{f}{s}$.*

(ii) *If $\overline{\rho}$ is split reducible, then $|\mathcal{A}(\lambda)|$ is even for all $\lambda \in \mathscr{D}$. For any subset $J \subseteq \{0,\ldots,f-1\}$ with $|J|$ even, there exist exactly 2 elements $\lambda \in \mathscr{D}$ such that $\mathcal{A}(\lambda) = J$. As a consequence, for any $0 \le s \le f$ which is even, the set of $\lambda \in \mathscr{D}$ with $|\mathcal{A}(\lambda)| = s$ has cardinality $2\binom{f}{s}$.*

(iii) *If $\overline{\rho}$ is nonsplit reducible, then $J_{\overline{\rho}}^c \subseteq \mathcal{A}(\lambda)$ for any $\lambda \in \overline{\mathscr{P}}$ (where $\overline{\mathscr{P}}$ is defined in the proof of Theorem 4.1(iii)), and for any $J \subseteq J_{\overline{\rho}}$ the set of $\lambda \in \overline{\mathscr{P}}$ with $\mathcal{A}(\lambda) = J \sqcup J_{\overline{\rho}}^c$ has cardinality $2^{f-d_{\overline{\rho}}}$. In particular, we always have $f - d_{\overline{\rho}} \le |\mathcal{A}(\lambda)| \le f$, and for any $0 \le s \le d_{\overline{\rho}}$ the set of $\lambda \in \overline{\mathscr{P}}$ with $|\mathcal{A}(\lambda)| = f - d_{\overline{\rho}} + s$ has cardinality $2^{f-d_{\overline{\rho}}}\binom{d_{\overline{\rho}}}{s}$.*

Proof. (i) By the definition of $\mathcal{A}(\lambda)$, we have

$$\mathcal{A}(\lambda) = \{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\} \text{ if } j \neq 0, \text{ or } \lambda_0(x_0) \in \{x_0, p-1-x_0\} \text{ if } j = 0\}. \quad (40)$$

By definition of $\mathscr{D}$ (see [BP12, § 11]) and of $\mathcal{A}(\lambda)$, we check that $\lambda_j(x_j)$ is determined by $\lambda_{j-1}(x_{j-1})$ and the value of $\mathbf{1}_{\mathcal{A}(\lambda)}(j)$ for any j. For example, if $\lambda_0(x_0) = x_0$ and $f \ge 2$, then $\lambda_1(x_1) = x_1$ (resp. $\lambda_1(x_1) = p-2-x_1$) if $1 \notin \mathcal{A}(\lambda)$ (resp. $1 \in \mathcal{A}(\lambda)$). This implies that $\lambda \in \mathscr{D}$ is determined by $\lambda_0(x_0)$ and $\mathcal{A}(\lambda)$. Moreover, one checks that:

- $|\mathcal{A}(\lambda) \cap \{1,\ldots,f-1\}|$ is even if $\lambda_0(x_0) \in \{x_0, p-1-x_0\}$ (by showing that $|\{j \neq 0 : \lambda_j(x_j) = p-2-x_j\}| = |\{j \neq 0 : \lambda_j(x_j) = x_j+1\}|$), and is odd if $\lambda_0(x_0) \in \{x_0-1, p-2-x_0\}$ (by showing that $|\{j \neq 0 : \lambda_j(x_j) = p-2-x_j\}| = |\{j \neq 0 : \lambda_j(x_j) = x_j+1\}| \pm 1$). As a consequence, $|\mathcal{A}(\lambda)|$ is always odd by (40).
- Conversely, if $\lambda_0(x_0) \in \{x_0, p-1-x_0\}$ (resp. $\lambda_0(x_0) \in \{x_0-1, p-2-x_0\}$) and $J \subseteq \{1,\ldots,f-1\}$ is even (resp. odd), then there exists a unique $\lambda \in \mathscr{D}$ with given value at $j=0$ and such that $J = \mathcal{A}(\lambda) \cap \{1,\ldots,f-1\}$.

Thus, for any $J \subseteq \{0, \dots, f-1\}$ with $|J|$ odd, there exist exactly two $\lambda \in \mathscr{D}$ with $\mathcal{A}(\lambda) = J$. The result follows from this.

(ii) The proof is similar to (and simpler than) (i). In this case, one has $\mathcal{A}(\lambda) = \{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}\}$ and it follows directly from the definition of $\mathscr{D}$ that the subsets $|\{j : \lambda_j(x_j) = x_j + 1\}|$ and $|\{j : \lambda_j(x_j) = p - 2 - x_j\}|$ of $\mathbb{Z}/f\mathbb{Z}$ are interlaced, i.e. between any two distinct elements of one subset there exists an element of the other, and hence of the same cardinality.

(iii) By the proof of [BHH+25, Lemma 3.59], there is a bijection between $\overline{\mathscr{P}}$ and $\mathscr{D}^{\mathrm{ss}}$ as follows: $\lambda \in \overline{\mathscr{P}}$ corresponds to $\mu \in \mathscr{D}^{\mathrm{ss}}$ defined by

$$\mu_j(x_j) \stackrel{\text{def}}{=} \begin{cases} p-3-x_j & \text{if } \lambda_j(x_j) = p-1-x_j, \\ \lambda_j(x_j) & \text{otherwise.} \end{cases}$$

One checks that $\mathcal{A}(\lambda) = (\mathcal{A}(\mu) \cap J_{\overline{\rho}}) \sqcup J_{\overline{\rho}}^c$, so in particular $J_{\overline{\rho}}^c \subseteq \mathcal{A}(\lambda)$. Thus for a given $J \subseteq J_{\overline{\rho}}$,

$$|\{\lambda \in \overline{\mathscr{P}} : \mathcal{A}(\lambda) = J \sqcup J_{\overline{\rho}}^c\}| = |\{\mu \in \mathscr{D}^{\mathrm{ss}} : \mathcal{A}(\mu) \cap J_{\overline{\rho}} = J\}|,$$

where $\mathcal{A}(\mu)$ is formed with respect to $\overline{\rho}^{\mathrm{ss}}$. If $|J|$ is even, then for any $J' \subseteq J_{\overline{\rho}}^c$ with $|J'|$ being even, there exist exactly 2 elements $\mu \in \mathscr{D}^{\mathrm{ss}}$ such that $\mathcal{A}(\mu) = J \sqcup J'$ by (ii), so the cardinality of $\mu \in \mathscr{D}^{\mathrm{ss}}$ satisfying $\mathcal{A}(\mu) \cap J_{\overline{\rho}} = J$ is $\sum_{0 \le i \le f - d_{\overline{\rho}},\ i \text{ even}} 2\binom{f-d_{\overline{\rho}}}{i} = 2^{f-d_{\overline{\rho}}}$. Similarly, if $|J|$ is odd, then the cardinality of $\mu \in \mathscr{D}^{\mathrm{ss}}$ satisfying $\mathcal{A}(\mu) \cap J_{\overline{\rho}} = J$ is $\sum_{0 \le i \le f - d_{\overline{\rho}},\ i \text{ odd}} 2\binom{f-d_{\overline{\rho}}}{i} = 2^{f-d_{\overline{\rho}}}$. This proves the second statement and the last one easily follows. □

In the rest of this section, we assume that $\overline{\rho}$ is split reducible. For $\lambda \in \mathscr{P}$, recall the set $J_\lambda \subseteq \{0, \dots, f-1\}$ defined in (11). For $i \in \{0, \dots, f\}$ put

$$N_{(i)} \stackrel{\text{def}}{=} \bigoplus_{\lambda \in \mathscr{P}, |J_\lambda| = i} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda).$$

The following result computes the Hilbert series of $N_{(i)}$.

Proposition 4.6. *Assume $\overline{\rho}$ is split reducible. Then for any $0 \le i \le f$,*

$$h_{N_{(i)}}(t) = \frac{2 \sum\limits_{0 \le s \le i} \binom{f}{2s}\binom{f-2s}{i-s}(1+t)^{2s}}{(1-t)^f}.$$

Remark 4.7. Together with [BHH+, Cor. 3.2.7](ii), Proposition 4.6 gives the Hilbert series $h_{\pi'}(t)$ for any subquotient π' of π if $\overline{\rho}$ is split reducible and $\max\{9, 2f+1\}$-generic.

Proof. Since $\overline{\rho}$ is split reducible, $|\mathcal{A}(\lambda)| = 2|\{j : \lambda_j(x_j) = x_j + 1\}|$ by the proof of Lemma 4.5(ii). Since $\{j : \lambda_j(x_j) = x_j + 1\} \subseteq J_\lambda$, we deduce $|\mathcal{A}(\lambda)|/2 \le |J_\lambda|$. Fix $0 \le i \le f$. As in the proof of Theorem 4.1, we have

$$h_{N_{(i)}}(t) = \frac{\sum_{0 \le s \le i} |\mathscr{P}_{i,s}|(1+t)^{2s}}{(1-t)^f}$$

where $\mathscr{P}_{i,s} \stackrel{\text{def}}{=} \{\lambda \in \mathscr{P} : |J_\lambda| = i, |\mathcal{A}(\lambda)| = 2s\}$. Thus, it suffices to show $|\mathscr{P}_{i,s}| = 2\sum_{0 \le s \le i} \binom{f}{2s}\binom{f-2s}{i-s}$.

Let $\lambda \in \mathscr{P}_{i,s}$ (with $0 \le s \le i \le f$) and write $\mathcal{A}(\lambda) = \{0 \le j_1 < j_1' < \cdots < j_s < j_s' < f\}$. Assume first $\lambda_{j_1}(x_{j_1}) = x_{j_1} + 1$; we call it case $+$. Then one checks that λ is uniquely determined by $(\mathcal{A}(\lambda), J_\lambda \setminus \mathcal{A}(\lambda))$ as follows:

- $\lambda_{j_k} = x_{j_k} + 1$ and $\lambda_{j_k'}(x_{j_k'}) = p - 2 - x_{j_k'}$ for $1 \le k \le s$ by the proof of Lemma 4.5(ii);
- if $j_k < j < j_k'$ for some k, then $\lambda_j(x_j) \in \{x_j, x_j + 2\}$, and $\lambda_j(x_j) = x_j + 2$ if and only if $j \in J_\lambda \setminus \mathcal{A}(\lambda)$;
- if $j_k' < j < j_{k+1}$ for some k (in $\mathbb{Z}/f\mathbb{Z}$), then $\lambda_j(x_j) \in \{p-1-x_j, p-3-x_j\}$, and $\lambda_j(x_j) = p - 3 - x_j$ if and only if $j \in J_\lambda \setminus \mathcal{A}(\lambda)$.

Conversely, an element $\lambda \in \mathscr{P}$ satisfying the above conditions belongs to $\mathscr{P}_{i,s}$ with $i = |\{j : \lambda_j(x_j) = \{x_j + 1, x_j + 2, p - 3 - x_j\}\}|$. Similar statements hold if $\lambda_{j_1}(x_{j_1}) = p - 2 - x_{j_1}$; we call it case $-$.

The above discussion implies that sending λ to $\big(\mathcal{A}(\lambda), \text{case } \pm, J_\lambda \setminus \mathcal{A}(\lambda)\big)$ gives a bijection between $\mathscr{P}_{i,s}$ and the set of triples $(J, \pm, J')$ satisfying

$$J \subseteq \{0, \ldots, f-1\}, \quad |J| = 2s, \quad J' \subseteq J^c, \quad |J'| = i - s.$$

Thus $|\mathscr{P}_{i,s}| = 2\sum_{0 \le s \le i} \binom{f}{2s}\binom{f-2s}{i-s}$ as desired. $\square$

5 On the structure of subquotients of π in the semisimple case

We determine the $\mathfrak{m}_{K_1}^2$-torsion of any subquotient of π, where π is any smooth mod p representation of $\mathrm{GL}_2(K)$ satisfying assumptions (i)–(iv) of § 3 and the underlying Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ is semisimple and sufficiently generic. By [BHH+25, Cor. 3.90] and Proposition 3.2.8 we may and will assume that $\overline{\rho}$ is split reducible.

Proposition 5.1. *Assume that $\overline{\rho}$ is split reducible and* $\max\{9, 2f+1\}$*-generic.*

(i) *Let π' be a subquotient of π. Then there exists a (unique) subset $\Sigma' \subseteq \{0, \ldots, f\}$ such that*

$$\pi'[\mathfrak{m}_{K_1}^2] \cong \bigoplus_{i \in \Sigma'} \widetilde{D}_0(\overline{\rho})_i,$$

where $\widetilde{D}_0(\overline{\rho})_i \overset{\text{def}}{=} \bigoplus_{\sigma \in W(\overline{\rho}),\ |J_\sigma| = i} \widetilde{D}_{0,\sigma}(\overline{\rho})$ for $0 \le i \le f$.

(ii) *Let $\pi_1 \subseteq \pi_2$ be subrepresentations of π. Then the induced sequence of $\widetilde{\Gamma}$-modules*

$$0 \to \pi_1[\mathfrak{m}_{K_1}^2] \to \pi_2[\mathfrak{m}_{K_1}^2] \to (\pi_2/\pi_1)[\mathfrak{m}_{K_1}^2] \to 0$$

is split exact.

Remark 5.2. As a consequence of Proposition 5.1(ii), if $\pi_1 \subseteq \pi_2$ are subrepresentations of π, then the induced sequence of Γ-representations

$$0 \to \pi_1^{K_1} \to \pi_2^{K_1} \to (\pi_2/\pi_1)^{K_1} \to 0$$

is split exact. This strengthens [BHH$^+$, Lemma 3.2.6].

Proof. We first prove (i) for any subrepresentation $\pi' = \pi_1$. Let $\Sigma' = \Sigma_1$ be the unique subset such that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1) = \bigoplus_{\ell(\sigma)\in\Sigma_1}\sigma$. (See [BHH$^+$, Prop. 3.2.2] for the existence of Σ_1.) First, since $\pi_1[\mathfrak{m}_{K_1}^2] \subseteq \pi[\mathfrak{m}_{K_1}^2]$, we deduce from Proposition 3.2.8 that

$$\pi_1[\mathfrak{m}_{K_1}^2] \subseteq \bigoplus_{i\in\Sigma_1}\widetilde{D}_0(\overline{\rho})_i.$$

Denote by Q the quotient $\big(\bigoplus_{i\in\Sigma_1}\widetilde{D}_0(\overline{\rho})_i\big)/\pi_1[\mathfrak{m}_{K_1}^2]$; we want to prove $Q = 0$. By [HW22, Thm. 4.6], $\widetilde{D}_0(\overline{\rho})$ is multiplicity free, so $\mathrm{JH}(Q) \cap W(\overline{\rho}) = \emptyset$. Consider the natural morphisms

$$Q \hookrightarrow \pi[\mathfrak{m}_{K_1}^2]/\pi_1[\mathfrak{m}_{K_1}^2] \hookrightarrow (\pi/\pi_1)[\mathfrak{m}_{K_1}^2] \hookrightarrow \pi/\pi_1$$

which induce an embedding $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(Q) \hookrightarrow \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi/\pi_1)$. But $\mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi/\pi_1)) \subseteq W(\overline{\rho})$ by [BHH$^+$, Lemma 3.2.6], so we must have $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(Q) = 0$, equivalently $Q = 0$.

(ii) As in the proof of [BHH$^+$, Cor. 3.2.5], it suffices to treat the special case $\pi_2 = \pi$. We again define Σ_1 by the equality $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1) = \bigoplus_{\ell(\sigma)\in\Sigma_1}\sigma$, so $\pi_1[\mathfrak{m}_{K_1}^2] = \bigoplus_{i\in\Sigma_1}\widetilde{D}_0(\overline{\rho})_i$ by the preceding paragraph. Thus there is an inclusion $\bigoplus_{i\notin\Sigma_1}\widetilde{D}_0(\overline{\rho})_i \cong \pi[\mathfrak{m}_{K_1}^2]/\pi_1[\mathfrak{m}_{K_1}^2] \subseteq (\pi/\pi_1)[\mathfrak{m}_{K_1}^2]$. Suppose that this is not an equality. Then $(\pi/\pi_1)[\mathfrak{m}_{K_1}^2]$ contains a subrepresentation V which fits into a *nonsplit* $\widetilde{\Gamma}$-extension

$$0 \to \bigoplus_{i\notin\Sigma_1}\widetilde{D}_0(\overline{\rho})_i \to V \to \tau \to 0 \tag{41}$$

for some Serre weight τ. (The extension is nonsplit by [BHH$^+$, Lemma 3.2.6].) We have $\tau \in W(\overline{\rho})$ by Lemma 3.2.1(i) and we let again $\chi \overset{\mathrm{def}}{=} \tau^{I_1}$.

By the projectivity of $\mathrm{Proj}_{\widetilde{\Gamma}}\tau$, there exists a $\widetilde{\Gamma}$-equivariant morphism $\beta : \mathrm{Proj}_{\widetilde{\Gamma}}\tau \to V$ whose composition with $V \twoheadrightarrow \tau$ is the natural projection $\mathrm{Proj}_{\widetilde{\Gamma}}\tau \twoheadrightarrow \tau$. Let V_β denote the image of β, which has cosocle τ. By (41), V_β satisfies the conditions in Lemma 3.2.6, so it is a quotient of $\mathrm{im}(\phi_\tau)$, namely β factors through $\mathrm{im}(\phi_\tau) \to V$.

By Corollary 3.2.3 we have $\mathrm{Ext}^1_{\widetilde{\Gamma}}(\mathrm{coker}(\phi_\tau), V) = 0$ by dévissage using (41). Hence, using the short exact sequence $0 \to \mathrm{im}(\phi_\tau) \to \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\overline{W}_{\chi,3} \to \mathrm{coker}(\phi_\tau) \to 0$, we can lift the map $\mathrm{im}(\phi_\tau) \to V$ of the previous paragraph to

$$\beta' : \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\overline{W}_{\chi,3} \to V \ (\hookrightarrow \pi/\pi_1).$$

The splitting statement in [BHH$^+$, Cor. 3.2.5] with $n = 3$ implies that the natural sequence

$$0 \to \mathrm{Hom}_I(\overline{W}_{\chi,3}, \pi_1) \to \mathrm{Hom}_I(\overline{W}_{\chi,3}, \pi) \to \mathrm{Hom}_I(\overline{W}_{\chi,3}, \pi/\pi_1) \to 0$$

is exact, so combined with Frobenius reciprocity we obtain a morphism

$$\beta'' : \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\overline{W}_{\chi,3} \to \pi$$

whose composition with $\pi \twoheadrightarrow \pi/\pi_1$ gives β'. By [BHH$^+$23, Prop. 6.4.6], any I-equivariant morphism $\overline{W}_{\chi,3} \to \pi$ factors through $\overline{W}_{\chi,3} \twoheadrightarrow \chi$, hence β'' factors as $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \overline{W}_{\chi,3} \twoheadrightarrow \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi \to \pi$. In particular, the image of β'' is contained in π^{K_1} and has cosocle τ. Since τ occurs in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$ and not elsewhere in π^{K_1} (as π^{K_1} is multiplicity free), the image of β'' (hence also the image of β) is just τ. This gives a contradiction, proving (ii), as V is a nonsplit extension by assumption and V_β has cosocle τ.

Finally, (i) is a direct consequence of the first paragraph of the proof and of (ii). □

We can now prove Theorem 1.1.3 in the semisimple case.

Corollary 5.3. *Assume that $\overline{\rho}$ is semisimple and* $\max\{9, 2f+1\}$*-generic. Then for any subquotient π' of π we have*

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = |\,\mathrm{JH}(\pi'^{K_1}) \cap W(\overline{\rho})|.$$

Proof. We have $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = |\,\mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\,\pi')|$ by [BHH$^+$, Cor. 3.2.7(i)] (if $\overline{\rho}$ is split reducible) and [BHH$^+$25, Prop. 3.87(ii)] (if $\overline{\rho}$ is irreducible, noting that $\pi' = \pi$ by [BHH$^+$25, Cor. 3.90] in that case). It suffices to show that $\mathrm{JH}(\pi'^{K_1}) \cap W(\overline{\rho}) = \mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\,\pi')$. If $\overline{\rho}$ is irreducible this is clear, as $\pi'^{K_1} = \pi^{K_1} = D_0(\overline{\rho})$ by assumption (i). If $\overline{\rho}$ is split reducible, then $\pi'^{K_1} = \bigoplus_{i\in\Sigma'} D_0(\overline{\rho})_i$ by Proposition 5.1, keeping the notation there, and the result follows. □

6 On the structure of subquotients of π in the non-semisimple case

We prove many results on the structure of subquotients of π as I- and $\mathrm{GL}_2(\mathcal{O}_K)$-representations.

From now on π denotes an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying assumptions (i)–(v) of § 3, with underlying Galois representation $\overline{\rho}$ which is nonsplit reducible and 0-generic.

The main results of this section include the description of the I_1- and K_1-invariants as well as of the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of any subquotient of π. These results all depend on determining the I_1-socle filtration of any subquotient π' of π (equivalently, the associated graded module of $\pi'^\vee$ for the $\mathfrak{m}$-adic filtration), which is the subject of subsection 6.1.

We again suppose that $\pi_1 \subseteq \pi$ is a subrepresentation of π and let $\pi_2 \overset{\mathrm{def}}{=} \pi/\pi_1$. Let $i_0 \overset{\mathrm{def}}{=} i_0(\pi_1) \in \{-1, \dots, f\}$, cf. [BHH$^+$, Thm. 4.3.15]. To simplify notation, for $\lambda \in \mathscr{P}$ we let $d_\lambda \overset{\mathrm{def}}{=} \max\{i_0 + 1 - |J_\lambda|, 0\}$.

6.1 The graded module of subquotient representations of π

We describe $\mathrm{gr}_{\mathfrak{m}}(\pi'^\vee)$, where π' is any subquotient of π (Corollary 6.1.7). We start with quotients $\pi_2 = \pi/\pi_1$ of π:

Theorem 6.1.1. *Assume that $\overline{\rho}$ is* $\max\{9, 2f+3\}$*-generic. We have an isomorphism of graded* $\mathrm{gr}(\Lambda)$*-modules with compatible H-actions,*

$$\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \cong \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda). \tag{42}$$

We recall the ideal $\mathfrak{a}_1^{i_0}(\lambda)$ of $\overline{R}$ from [BHH+, (77)]. Let $J_1 \stackrel{\text{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p-1-x_j\}$ and $J_2 \stackrel{\text{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = x_j\}$. If $i_0 \geq |J_\lambda|$, then $\mathfrak{a}_1^{i_0}(\lambda)$ is the ideal generated by $\mathfrak{a}(\lambda)$ (cf. (13)) and all $\prod_{j \in J_1'} y_j \prod_{j \in J_2'} z_j$, where $J_1' \subseteq J_1$, $J_2' \subseteq J_2$, $|J_1'| + |J_2'| = i_0 + 1 - |J_\lambda|$. Otherwise, $\mathfrak{a}_1^{i_0}(\lambda) = \overline{R}$. In particular, $\mathfrak{a}_1^{i_0}(\lambda) = \mathfrak{a}(\lambda)$ if $|J_1| + |J_2| < i_0 + 1 - |J_\lambda|$.

The grading shifts in (42) are such that all nonzero direct summands contribute in degree 0, but vanish in degree 1. Note also from the definitions that $\mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}(\lambda) = 0$ if $|\{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) \in \{p-1-x_j, x_j\}\}| < d_\lambda$. (The converse is also true, by comparing equations [BHH+, (81) and (82)], or alternatively see the proof of [BHH+, Cor. 4.4.7].)

Remark 6.1.2. Theorem 6.1.1 implies that $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$ is Cohen–Macaulay or zero as $\mathrm{gr}(\Lambda)$-module. (By [BHH+, Prop. 4.4.3] and [BHH+, Cor. 4.4.5], each nonzero $\mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}(\lambda)$ is Cohen–Macaulay, as the Cohen–Macaulay property is closed under direct summands, shifts in grading, and direct sums.)

Remark 6.1.3. Theorem 6.1.1 shows that $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$ is killed by the ideal J, i.e. is an $\overline{R}$-module. This is a priori not obvious. A similar comment applies to Corollary 6.1.7.

Remark 6.1.4. When $i_0 = f$, Theorem 6.1.1 is trivially true, because $\pi_2 = 0$ and $\mathfrak{a}_1^f(\lambda) = \mathfrak{a}(\lambda)$ for all $\lambda \in \mathscr{P}$. (By [BHH+, Lemma 4.1.4] we have $f+1-|J_\lambda| = |J_1|+|J_2|+|J_{\lambda^*}|+1 > |J_1|+|J_2|$, where $\lambda \mapsto \lambda^*$ is the involution of $\mathscr{P}$ defined in [BHH+25, Def. 3.62].) When $i_0 = f-1$, Theorem 6.1.1 can be proved as follows, assuming that $\overline{\rho}$ is only $\max\{9, 2f+1\}$-generic. By [BHH+, Thm. 4.4.8(ii)], π_2 is irreducible in this case, so π_2 is the principal series $\mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_1 \otimes \chi_2\omega^{-1})$ by [HW22, Prop. 10.8], where $\overline{\rho} \cong \left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ and hence $\chi_1|_{I_K} = \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j}$ and $\chi_2|_{I_K} = 1$. (Here $B(K)$ denotes the Borel subgroup of upper-triangular matrices of $\mathrm{GL}_2(K)$.) We apply the combinatorial Proposition 6.1.10 below (or argue directly) to deduce

$$\bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda) \cong \chi_{\lambda'}^{-1} \otimes \frac{\overline{R}}{(z_j : 0 \leq j \leq f-1)} \oplus \chi_{\lambda''}^{-1} \otimes \frac{\overline{R}}{(y_j : 0 \leq j \leq f-1)}, \tag{43}$$

where $\lambda', \lambda'' \in \mathscr{P}^{\mathrm{ss}}$ are given by $\lambda_j'(x_j) = p-3-x_j$, $\lambda_j''(x_j) = x_j + 2$ for all j. We calculate $\chi_{\lambda'} = (\chi_2|_{I_K})\omega^{-1} \otimes \chi_1|_{I_K}$ and $\chi_{\lambda''} = \chi_1|_{I_K} \otimes (\chi_2|_{I_K})\omega^{-1}$. We conclude by [BHH+25, Prop. 3.76(ii)].

Lemma 6.1.5. *Suppose that M is a graded* $\mathrm{gr}(\Lambda)$*-module. Let $N \stackrel{\text{def}}{=} (\mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}(\lambda))(-d_\lambda)$ for some $\lambda \in \mathscr{P}$. Then the natural map*

$$\mathrm{HOM}_{\mathrm{gr}(\Lambda)}(N, M)_0 \to \mathrm{HOM}_{\mathrm{gr}(\Lambda)}(N, M/\mathrm{gr}_{\leq -3} M)_0$$

is an isomorphism.

Recall that $\mathrm{HOM}(N, M)_0$ denotes the graded morphisms $N \to M$ (of degree 0).

Proof. **Step 1.** Suppose for the moment that S is a graded ring and N any finitely presented graded S-module. For any subset $D \subseteq \mathbb{Z}$ we say that N *has relations in degrees* D as S-module if there exists a graded exact sequence of the form $\bigoplus_{i=1}^n S(-d_i) \to S^{\oplus m} \to N \to 0$ with $d_i \in D$ for all i (in particular, N is generated by its degree 0 part).

We claim that if N has relations in degrees D as S/I-module, where I is an ideal of S that is generated by finitely many homogeneous elements s_i whose degrees are contained in D, then the same is true as S-module.

To see this, by assumption we can find a surjective graded homomorphism $(S/I)^{\oplus m} \to N \to 0$, whose kernel is generated by finitely many homogeneous elements x_j whose degrees are contained in D for all j. Lift each x_j to a homogeneous element $\widetilde{x}_j$ of $S^{\oplus m}$ of the same degree. By composition we have a surjective morphism $S^{\oplus m} \to N \to 0$ of graded S-modules. Its kernel is generated by all $s_i e_k$ (where e_k denotes the standard $\mathbb{F}$-basis of $S^{\oplus m}$) and all $\widetilde{x}_j$, as desired.

Step 2. We show that $N = (\mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}(\lambda))(-d_\lambda)$ has relations in degrees $\{-1, -2\}$ as $\mathrm{gr}(\Lambda)$-module. Since N is a graded $\overline{R}/\mathfrak{a}(\lambda)$-module and $\overline{R}/\mathfrak{a}(\lambda)$ is obtained from $\mathrm{gr}(\Lambda)$ by quotienting by an ideal generated by homogeneous elements of degrees -1 and -2, by Step 1 it suffices to show that N has relations in degrees $\{-1, -2\}$ as $\overline{R}/\mathfrak{a}(\lambda)$-module.

Note that $j \in J_1 \sqcup J_2$ implies that $t_j = y_j z_j$ and let $d \stackrel{\text{def}}{=} d_\lambda$ for short. By interchanging y_j and z_j for some j and permuting $\{0, 1, \dots, f-1\}$, we may assume that

$$N = (y_{i_1} \cdots y_{i_d} : 0 \le i_1 < \cdots < i_d < e)(-d)$$

as graded $\overline{R}/\mathfrak{a}(\lambda)$-module, for some $0 \le e \le f$. This module is generated by the elements $X_I \stackrel{\text{def}}{=} \prod_{i \in I} y_i$ (of degree 0) for subsets $I \subseteq E \stackrel{\text{def}}{=} \{0, 1, \dots, e-1\}$ with $|I| = d$. We claim that the relations are generated by

$$\begin{aligned} z_i X_I &= 0 && \text{for all } i \in I; \\ y_i X_{I' \setminus \{i\}} &= y_j X_{I' \setminus \{j\}} && \text{for all } i \neq j \text{ in } I' \subseteq E,\ |I'| = d+1. \end{aligned} \tag{44}$$

Note that $\overline{R}/\mathfrak{a}(\lambda)$ has as $\mathbb{F}$-basis all monomials of the form $\prod_j w_j^{\ge 0}$ with $w_j \in \{y_j, z_j\} \setminus \{t_j\}$. If $\sum_I f_I X_I = 0$ with $f_I \in \overline{R}/\mathfrak{a}(\lambda)$, then using relations (44), without loss of generality, f_I does not contain any z_i ($i \in I$) and y_i ($i \in E \setminus I$, $i < \min I$). The map $f_I \mapsto f_I X_I$ is injective for such f_I, and moreover for every monomial term in $f_I X_I$, I is the set of d largest elements i of E such that y_i divides it. This shows that $f_I = 0$ for all I, proving that we have found all relations, and indeed the relations are in degree -1.

Step 3. By Step 2 we have an exact sequence $\bigoplus_{i=1}^n \mathrm{gr}(\Lambda)(-d_i) \to \mathrm{gr}(\Lambda)^{\oplus m} \to N \to 0$ with

$d_i \in \{-1,-2\}$ for all i. We get a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{HOM}_{\mathrm{gr}(\Lambda)}(N,M)_0 & \longrightarrow & \mathrm{HOM}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda),M)_0^{\oplus m} & \longrightarrow & \bigoplus_{i=1}^{n} \mathrm{HOM}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)(-d_i),M)_0 \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{HOM}_{\mathrm{gr}(\Lambda)}(N,\overline{M})_0 & \longrightarrow & \mathrm{HOM}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda),\overline{M})_0^{\oplus m} & \longrightarrow & \bigoplus_{i=1}^{n} \mathrm{HOM}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)(-d_i),\overline{M})_0 \end{array}$$

where $\overline{M} \stackrel{\text{def}}{=} M/\operatorname{gr}_{\le -3} M$. As $\mathrm{HOM}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)(-i),M)_0 = M_i$ for all $i \in \mathbb{Z}$, the middle and right vertical arrows are isomorphisms, hence so is the left one. □

Fix $n \ge 1$, which we will specify later, and assume that $\overline{\rho}$ is $(2n-1)$-generic. Recall $\tau \stackrel{\text{def}}{=} \tau^{(n)} \subseteq \pi$ from [BHH+, § 2.4], so $\tau = \bigoplus_{\lambda\in\mathscr{P}} \tau_\lambda$ with $\tau_\lambda \stackrel{\text{def}}{=} \tau_\lambda^{(n)}$ and $\mathrm{soc}_I(\tau_\lambda) \cong \chi_\lambda$. Let $\overline{\tau} \stackrel{\text{def}}{=} \tau[\mathfrak{m}^n] = \pi[\mathfrak{m}^n]$ (last statement of [BHH+, Lemma 2.4.2)] and $\overline{\tau}_\lambda \stackrel{\text{def}}{=} \tau_\lambda[\mathfrak{m}^n]$ for $\lambda \in \mathscr{P}$, so $\overline{\tau} = \bigoplus_{\lambda\in\mathscr{P}} \overline{\tau}_\lambda$. Let Θ denote the image of $\overline{\tau}$ in π_2. As τ is multiplicity free by [BHH+, Cor. 2.4.3(i)] (for $r=1$), we have $\overline{\tau} \cap \pi_1 = \bigoplus_{\lambda\in\mathscr{P}}(\overline{\tau}_\lambda \cap \pi_1)$ and $\Theta = \bigoplus_{\lambda\in\mathscr{P}} \Theta_\lambda$, where Θ_λ is the image of $\overline{\tau}_\lambda$ in π_2. For the same reason,

$$F_{-i}\Theta^\vee = \mathfrak{m}^i\overline{\tau}^\vee \cap \Theta^\vee = \Big(\bigoplus_{\lambda\in\mathscr{P}} \mathfrak{m}^i\overline{\tau}_\lambda^\vee\Big) \cap \Theta^\vee = \bigoplus_{\lambda\in\mathscr{P}} (\mathfrak{m}^i\overline{\tau}_\lambda^\vee \cap \Theta^\vee) = \bigoplus_{\lambda\in\mathscr{P}} F_{\lambda,-i}\Theta_\lambda^\vee \tag{45}$$

for all $i \in \mathbb{Z}_{\ge 0}$, where F (resp. F_λ) denotes the filtration on $\Theta^\vee$ (resp. $\Theta_\lambda^\vee$) induced from the $\mathfrak{m}$-adic filtration on $\overline{\tau}^\vee$ (resp. $\overline{\tau}_\lambda^\vee$). In particular, $\mathrm{gr}_F(\Theta^\vee) = \bigoplus_{\lambda\in\mathscr{P}} \mathrm{gr}_{F_\lambda}(\Theta_\lambda^\vee)$ and $F_{-n}\Theta^\vee = 0$.

Suppose that $n \ge 1$ and that $\overline{\rho}$ is $(2n-1)$-generic. The following lemma determines the submodule structure of $\tau_\lambda^{(n)}[\mathfrak{m}^n]$ (and hence of $\pi[\mathfrak{m}^n] = \tau^{(n)}[\mathfrak{m}^n]$ if $r=1$ by [BHH+, Cor. 2.4.3(ii)]), since $\bigoplus_{\lambda\in\mathscr{P}} \tau_\lambda^{(n)}$ is multiplicity free by [BHH+, Cor. 2.4.3(i)] (as $\overline{\rho}$ is $(2n-1)$-generic).

Lemma 6.1.6. *Suppose that $\overline{\rho}$ is $(n-1)$-generic and keep the above notation. Suppose that $\lambda \in \mathscr{P}$. For any $\chi, \chi' \in \mathrm{JH}(\tau_\lambda^{(n)}[\mathfrak{m}^n])$ with $\mathrm{Ext}^1_{I/Z_1}(\chi,\chi') \neq 0$, upon perhaps interchanging χ and χ', there exist $\ell_j \in \mathbb{Z}$ for $0 \le j \le f-1$, $\varepsilon \in \{\pm 1\}$, and $0 \le j_0 \le f-1$ such that*

(i) $\chi = \chi_\lambda \prod_j \alpha_j^{\ell_j}$ *and* $\chi' = \chi\alpha_{j_0}^{\varepsilon}$;

(ii) $\ell_j \ge 0$ *if* $t_j = y_j$, $\ell_j \le 0$ *if* $t_j = z_j$, *and* $\sum_j |\ell_j| < n$;

(iii) $|\ell_{j_0} + \varepsilon| < |\ell_{j_0}|$;

(iv) $E_{\chi',\chi}$ *(the unique nonsplit extension of χ by χ', see § 1.4) is a subquotient of* $\tau_\lambda^{(n)}[\mathfrak{m}^n]$.

In particular, either $E_{\chi,\chi'}$ or $E_{\chi',\chi}$ occurs as subquotient of $\tau_\lambda^{(n)}[\mathfrak{m}^n]$.

Proof. By construction of $\tau_\lambda^{(n)}$ (cf. the proofs of [BHH+, Lemma 2.4.1] and [HW22, Prop. 9.19]), as $\chi \in \tau_\lambda^{(n)}[\mathfrak{m}^n]$ we can write $\chi = \chi_\lambda \prod_j \alpha_j^{\ell_j}$ for some $\ell_j \in \mathbb{Z}$ satisfying condition (ii). As

$\mathrm{Ext}^1_{I/Z_1}(\chi, \chi') \neq 0$ we have $\chi' = \chi\alpha_{j_0}^{\varepsilon}$ for some $\varepsilon \in \{\pm 1\}$ and some $0 \leq j_0 \leq f-1$. By the genericity condition we deduce that condition (ii) holds for $(\ell_0, \dots, \ell_{j_0} + \varepsilon, \dots, \ell_{f-1})$. (As $\overline{\rho}$ is $(n-1)$-generic, $|\ell_j| < n \leq \frac{p-1}{2}$, and $\varepsilon p^{j_0} + \sum_{j=0}^{f-1} \ell_j p^j \equiv \sum_{j=0}^{f-1} \ell'_j p^j \pmod{p^f - 1}$ for integers $|\ell'_j| < \frac{p-1}{2}$ implies $(\ell_0, \dots, \ell_{j_0} + \varepsilon, \dots, \ell_{f-1}) = (\ell'_0, \dots, \ell'_{f-1})$.) Interchanging χ and χ', if necessary, we may assume that (iii) holds. Then (iv) holds, as the nonsplit extension of $\alpha_{j_0}^{\ell_{j_0}}$ by $\alpha_{j_0}^{\ell_{j_0}+\varepsilon}$ occurs in the j_0-th tensor factor defining $\tau_\lambda^{(n)}$. □

Proof of Theorem 6.1.1. By Remark 6.1.4 we can assume throughout the proof that $i_0 \leq f-2$. Let N'_2 denote the right-hand side of the theorem, i.e. $N'_2 \stackrel{\text{def}}{=} \bigoplus_{\lambda \in \mathscr{P}} N'_{2,\lambda}$ with

$$N'_{2,\lambda} \stackrel{\text{def}}{=} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda).$$

Step 1. We show that $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3 \twoheadrightarrow N'_2/\overline{\mathfrak{m}}^3$ as graded $\mathrm{gr}(\Lambda)$-modules with compatible H-actions.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)} & \longrightarrow & \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\overline{R}}{\mathfrak{a}(\lambda)} & \longrightarrow & \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\overline{R}}{\mathfrak{a}_1^{i_0}(\lambda)} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{gr}_{F'}(\pi_2^\vee) & \longrightarrow & \mathrm{gr}(\pi^\vee) & \longrightarrow & \mathrm{gr}(\pi_1^\vee) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}_F(\Theta^\vee) & \longrightarrow & \mathrm{gr}(\overline{\tau}^\vee) & \longrightarrow & \mathrm{gr}((\overline{\tau} \cap \pi_1)^\vee) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \bigoplus_{\lambda \in \mathscr{P}} \mathrm{gr}_{F_\lambda}(\Theta_\lambda^\vee) & \longrightarrow & \bigoplus_{\lambda \in \mathscr{P}} \mathrm{gr}(\overline{\tau}_\lambda^\vee) & \longrightarrow & \bigoplus_{\lambda \in \mathscr{P}} \mathrm{gr}((\overline{\tau}_\lambda \cap \pi_1)^\vee) & \longrightarrow & 0
\end{array}$$

of graded $\mathrm{gr}(\Lambda)$-modules with compatible H-actions, where F' denotes the filtration on $\pi_2^\vee$ induced by the $\mathfrak{m}$-adic filtration on $\pi^\vee$ and we recall that F denotes the filtration on $\Theta^\vee$ induced by the $\mathfrak{m}$-adic filtration on $\overline{\tau}^\vee$. The top vertical maps are isomorphisms by [BHH+, Cor. 4.4.5] (see also the proof of [BHH+, Prop. 4.4.3]). From $\overline{\tau} = \pi[\mathfrak{m}^n]$ we get $(\overline{\tau} \cap \pi_1)[\mathfrak{m}^n] = \pi_1[\mathfrak{m}^n]$. Hence the middle and right vertical maps are isomorphisms in degrees $> -n$, and so the same is true of the left vertical map. The middle vertical composition $\bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes (\mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}(\lambda)) \to \bigoplus_{\lambda \in \mathscr{P}} \mathrm{gr}(\overline{\tau}_\lambda^\vee)$ is an isomorphism in degree 0, respecting the direct sum decomposition (by H-equivariance). As its domain is generated by its degree 0 part as $\mathrm{gr}(\Lambda)$-module, it follows that the middle vertical composition respects the direct sum decomposition, and hence the same is true for the left and right vertical maps. We deduce that for each $\lambda \in \mathscr{P}$ the morphism

$$N'_{2,\lambda}(d_\lambda) = \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)} \to \mathrm{gr}_{F_\lambda}(\Theta_\lambda^\vee) \tag{46}$$

of graded $\mathrm{gr}(\Lambda)$-modules is an isomorphism in degrees $> -n$.

We now show that

$$F_{\lambda,-d_\lambda-i}(\Theta_\lambda^\vee) = \mathfrak{m}^i\Theta_\lambda^\vee \quad \text{for any } \lambda \in \mathscr{P},\ i \geq 0. \tag{47}$$

To see this, note that if $d_\lambda = 0$, then $\mathfrak{a}_1^{i_0}(\lambda) = \overline{R}$, hence the λ-part of the above commutative diagram shows that the natural map $\mathrm{gr}_{F_\lambda}(\Theta_\lambda^\vee) \hookrightarrow \mathrm{gr}(\overline{\tau}_\lambda^\vee)$ is an isomorphism, which implies that the natural map $\Theta_\lambda^\vee \hookrightarrow \overline{\tau}_\lambda^\vee$ of filtered Λ-modules is an isomorphism, as desired. Suppose now that $d_\lambda > 0$. We first obtain from the previous diagram the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \chi_\lambda^{-1} \otimes \dfrac{\mathfrak{a}_1^{i_0}(\lambda) + \mathfrak{m}^n}{\mathfrak{a}(\lambda) + \mathfrak{m}^n} & \longrightarrow & \chi_\lambda^{-1} \otimes \dfrac{\overline{R}}{\mathfrak{a}(\lambda) + \mathfrak{m}^n} & \longrightarrow & \chi_\lambda^{-1} \otimes \dfrac{\overline{R}}{\mathfrak{a}_1^{i_0}(\lambda) + \mathfrak{m}^n} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{gr}_{F_\lambda}(\Theta_\lambda^\vee) & \longrightarrow & \mathrm{gr}(\overline{\tau}_\lambda^\vee) & \longrightarrow & \mathrm{gr}((\overline{\tau}_\lambda \cap \pi_1)^\vee) & \longrightarrow & 0
\end{array}$$

By the definition of $\mathfrak{a}(\lambda)$ the middle isomorphism shows that

$$\mathrm{JH}(\overline{\tau}_\lambda^\vee) = \Big\{\chi_\lambda^{-1} \prod_j \alpha_j^{\ell_j} : \ell \in L\Big\},$$

where

$$L \overset{\text{def}}{=} \Big\{\ell = (\ell_j)_{j=0}^{f-1} : \ell_j \leq 0 \text{ if } t_j = y_j,\ \ell_j \geq 0 \text{ if } t_j = z_j,\ \sum_j |\ell_j| < n\Big\},$$

as well as

$$\mathrm{JH}(\mathfrak{m}^i\overline{\tau}_\lambda^\vee) = \Big\{\chi_\lambda^{-1} \prod_j \alpha_j^{\ell_j} : \ell \in L,\ i \leq \sum_j |\ell_j|\Big\}.$$

By the definition of $\mathfrak{a}_1^{i_0}(\lambda)$ the left isomorphism shows that

$$\mathrm{JH}(\Theta_\lambda^\vee) = \Big\{\chi_\lambda^{-1} \prod_j \alpha_j^{\ell_j} : \ell \in L,\ |\{j \in J_1 : \ell_j > 0\}| + |\{j \in J_2 : \ell_j < 0\}| \geq d_\lambda\Big\},$$

where we recall that $J_1 = \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p-1-x_j\}$, $J_2 = \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = x_j\}$. In particular, $\Theta_\lambda^\vee \subseteq \mathfrak{m}^{d_\lambda}\overline{\tau}_\lambda^\vee$ and hence $\mathfrak{m}^i\Theta_\lambda^\vee \subseteq \Theta_\lambda^\vee \cap \mathfrak{m}^{d_\lambda+i}\overline{\tau}_\lambda^\vee$ for all $i \geq 0$. Conversely, to show $\Theta_\lambda^\vee \cap \mathfrak{m}^{d_\lambda+i}\overline{\tau}_\lambda^\vee \subseteq \mathfrak{m}^i\Theta_\lambda^\vee$, by multiplicity freeness it suffices to show that $\mathrm{JH}(\Theta_\lambda^\vee) \cap \mathrm{JH}(\mathfrak{m}^{d_\lambda+i}\overline{\tau}_\lambda^\vee) \subseteq \mathrm{JH}(\mathfrak{m}^i\Theta_\lambda^\vee)$. Take $\chi \overset{\text{def}}{=} \chi_\lambda^{-1}\prod_j \alpha_j^{\ell_j}$ with $\ell \in L$, $|\{j \in J_1 : \ell_j > 0\}| + |\{j \in J_2 : \ell_j < 0\}| \geq d_\lambda$, and $d_\lambda + i \leq \sum_j |\ell_j|$. With the help of Lemma 6.1.6 it is easy to show that there exist characters $\chi_{i'} \in \mathrm{JH}(\Theta_\lambda^\vee)$ $(0 \leq i' \leq i)$ with $\chi_0 \overset{\text{def}}{=} \chi$ and such that the unique nonsplit extension $E_{\chi_{i'-1},\chi_{i'}}$ (of $\chi_{i'}$ by $\chi_{i'-1}$) occurs as subquotient of $\Theta_\lambda^\vee$ for all $0 < i' \leq i$. (If $i > 0$ then we find χ_1 as follows: if there is $j \notin J_1 \sqcup J_2$ such that $\ell_j \neq 0$, choose such a j; otherwise, choose j such that $|\ell_j| > 0$ and is as small as possible. Then $\chi_1 \overset{\text{def}}{=} \chi_0\alpha_j^{-\mathrm{sgn}(\ell_j)}$ is still an element of $\mathrm{JH}(\Theta_\lambda^\vee)$, and we have decreased $\sum_j |\ell_j|$ by 1. Proceed inductively to find all $\chi_{i'}$.) We deduce that χ occurs in $\mathrm{rad}^i\,\Theta_\lambda^\vee = \mathfrak{m}^i\Theta_\lambda^\vee$, proving (47).

We now let $n \overset{\text{def}}{=} i_0 + 4$. As $d_\lambda \le i_0 + 1 < n - 2$ we obtain from (46) and (47) an isomorphism of graded $\mathrm{gr}(\Lambda)$-modules

$$N'_{2,\lambda}/\overline{\mathfrak{m}}^3 = N'_{2,\lambda}/\mathrm{gr}_{\le -3} N'_{2,\lambda} \cong \mathrm{gr}_{F_\lambda}(\Theta_\lambda^\vee(-d_\lambda))/\mathrm{gr}_{F_\lambda,\le -3}(\Theta_\lambda^\vee(-d_\lambda)) = \mathrm{gr}(\Theta_\lambda^\vee)/\overline{\mathfrak{m}}^3.$$

Hence

$$\mathrm{gr}(\pi_2^\vee)/\overline{\mathfrak{m}}^3 \twoheadrightarrow \mathrm{gr}(\Theta^\vee)/\overline{\mathfrak{m}}^3 \cong \bigoplus_{\lambda \in \mathscr{P}} \mathrm{gr}(\Theta_\lambda^\vee)/\overline{\mathfrak{m}}^3 \cong N'_2/\overline{\mathfrak{m}}^3, \tag{48}$$

as desired.

Step 2. We show that $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3 \cong N'_2/\overline{\mathfrak{m}}^3$ as graded $\mathrm{gr}(\Lambda)$-modules with compatible H-actions.

From the cohomology long exact sequence we get

$$0 \to \mathrm{coker}\big(\mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi^\vee) \to \mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi_1^\vee)\big) \to \pi_2^\vee/\mathfrak{m}^3 \to \pi^\vee/\mathfrak{m}^3 \to \pi_1^\vee/\mathfrak{m}^3 \to 0. \tag{49}$$

We let

$$C \overset{\text{def}}{=} \mathrm{coker}\big(\mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi^\vee) \to \mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi_1^\vee)\big)$$

and give it the induced filtration as quotient of $\mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi_1^\vee)$.

First we show that $\mathrm{gr}(C)$ is a subquotient of

$$C' \overset{\text{def}}{=} \mathrm{coker}\big(\mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) \to \mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee))\big). \tag{50}$$

Notice that $\mathrm{gr}(C)$ is a quotient of

$$\mathrm{coker}\big(\mathrm{gr}(\mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi^\vee)) \to \mathrm{gr}(\mathrm{Tor}_1^\Lambda(\Lambda/\mathfrak{m}^3, \pi_1^\vee))\big),$$

because if we have a filtered exact sequence $X \to Y \to C \to 0$ with C carrying the induced filtration, then $\mathrm{coker}(\mathrm{gr}(X) \to \mathrm{gr}(Y))$ surjects onto $\mathrm{gr}(C)$ by [LvO96, Thm. I.4.2.4(1)]. Then, as in the proof of [BHH$^+$, Prop. 2.4.9], we consider the morphism of spectral sequences that converges to this morphism:

$$\begin{array}{ccc} E_i^r & \Longrightarrow & \mathrm{Tor}_i^\Lambda(\Lambda/\mathfrak{m}^3, \pi^\vee) \\ \downarrow & & \downarrow \\ E_i'^r & \Longrightarrow & \mathrm{Tor}_i^\Lambda(\Lambda/\mathfrak{m}^3, \pi_1^\vee). \end{array}$$

(Referring to that proof, we have $E_i^0 = \mathrm{gr}(\Lambda/\mathfrak{m}^3 \otimes_\Lambda M_i) \cong \mathrm{gr}(\Lambda/\mathfrak{m}^3) \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}(M_i)$ by [LvO96, Lemma I.6.14], so $E_i^1 = \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$.) Assumption (v) says that $\dim_{\mathbb{F}} E_1^\infty = \dim_{\mathbb{F}} E_1^1$. It easily follows that $\mathrm{coker}(E_1^{r+1} \to E_1'^{r+1})$ is a subquotient of $\mathrm{coker}(E_1^r \to E_1'^r)$ for any $r \ge 1$ (recall that $E_1'^{r+1}$ is a subquotient of $E_1'^r$, while $E_1^{r+1} = E_1^r$ by the preceding sentence). This implies the claim by taking r sufficiently large.

From the sequence (49) we see that

$$\begin{aligned} \dim_{\mathbb{F}}(C) &= \dim_{\mathbb{F}}(\pi_2^\vee/\mathfrak{m}^3) - \dim_{\mathbb{F}}(\pi^\vee/\mathfrak{m}^3) + \dim_{\mathbb{F}}(\pi_1^\vee/\mathfrak{m}^3) \\ &= \dim_{\mathbb{F}}(\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3) - \dim_{\mathbb{F}}(\mathrm{gr}(\pi^\vee)/\overline{\mathfrak{m}}^3) + \dim_{\mathbb{F}}(\mathrm{gr}(\pi_1^\vee)/\overline{\mathfrak{m}}^3). \end{aligned} \tag{51}$$

By Step 1 we know that

$$\begin{aligned}\dim_{\mathbb{F}}(\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3) &\geq \dim_{\mathbb{F}}(N_2'/\overline{\mathfrak{m}}^3) = \sum_{\lambda\in\mathscr{P}} \dim_{\mathbb{F}}(N_{2,\lambda}'/\overline{\mathfrak{m}}^3) \\ &= \sum_{\lambda\in\mathscr{P}} \dim_{\mathbb{F}}(N_{2,\lambda}/\overline{\mathfrak{m}}^3) = \dim_{\mathbb{F}}(\mathrm{gr}_{F'}(\pi_2^\vee)/\overline{\mathfrak{m}}^3),\end{aligned} \tag{52}$$

where $N_{2,\lambda} \stackrel{\text{def}}{=} \chi_\lambda^{-1}\otimes(\mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}(\lambda)) = N_{2,\lambda}'(d_\lambda)$ and we used [BHH$^+$, Cor. 4.4.5] for the last equality. Combining equations (51), (52) together with the fact that $\mathrm{gr}(C)$ is a subquotient of C' (cf. (50)) we obtain

$$\dim_{\mathbb{F}}(C') \geq \dim_{\mathbb{F}}(C) \geq \dim_{\mathbb{F}}(\mathrm{gr}_{F'}(\pi_2^\vee)/\overline{\mathfrak{m}}^3) - \dim_{\mathbb{F}}(\mathrm{gr}(\pi^\vee)/\overline{\mathfrak{m}}^3) + \dim_{\mathbb{F}}(\mathrm{gr}(\pi_1^\vee)/\overline{\mathfrak{m}}^3). \tag{53}$$

The exact sequence

$$0 \to C' \to \mathrm{gr}_{F'}(\pi_2^\vee)/\overline{\mathfrak{m}}^3 \to \mathrm{gr}(\pi^\vee)/\overline{\mathfrak{m}}^3 \to \mathrm{gr}(\pi_1^\vee)/\overline{\mathfrak{m}}^3 \to 0$$

shows that equality holds in (53), and hence in (52). As equality holds in (52), the surjection $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)/\overline{\mathfrak{m}}^3 \twoheadrightarrow N_2'/\overline{\mathfrak{m}}^3$ in (48) has to be an isomorphism.

Step 3. By Lemma 6.1.5 we get a graded morphism (of degree 0) $N_2' \to \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$, which has to be surjective by the graded Nakayama lemma. Recall that N_2' is Cohen–Macaulay by Remark 6.1.2. By Step 4 of the proof of [BHH$^+$, Prop. 4.4.3] we have $\mathcal{Z}(N_2') = \mathcal{Z}(N_2^{i_0}) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee))$. Using the same argument as in the last paragraph of the proof of [BHH$^+$, Prop. 4.4.3] (i.e. N_2' is Cohen–Macaulay and the two modules have the same cycle), we deduce that the morphism $N_2' \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$ is an isomorphism. □

Corollary 6.1.7. *Assume that $\overline{\rho}$ is $(4f+1)$-generic. Suppose $\pi' = \pi_1'/\pi_1$ is any nonzero subquotient, where $\pi_1 \subsetneq \pi_1' \subseteq \pi$. Then we have an isomorphism of graded $\mathrm{gr}(\Lambda)$-modules with compatible H-actions,*

$$\mathrm{gr}_{\mathfrak{m}}(\pi'^\vee) \cong \bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0'}(\lambda)}(-d_\lambda), \tag{54}$$

where $-1 \leq i_0 \stackrel{\text{def}}{=} i_0(\pi_1) < i_0' \stackrel{\text{def}}{=} i_0(\pi_1') \leq f$ and $d_\lambda \stackrel{\text{def}}{=} \max\{i_0+1-|J_\lambda|, 0\}$.

Proof. We first assume that $f \geq 2$. Then $4f+1 \geq \max\{9, 2f+3\}$, so we may assume that $i_0' \leq f-1$ by Theorem 6.1.1. Let N' denote the right-hand side of (54). Let $\pi_2 \stackrel{\text{def}}{=} \pi/\pi_1$ and $\pi_2' \stackrel{\text{def}}{=} \pi/\pi_1'$, so

$$0 \to \pi' \to \pi_2 \to \pi_2' \to 0.$$

Let $d_\lambda' \stackrel{\text{def}}{=} \max\{i_0'+1-|J_\lambda|, 0\}$.

Step 1. We show that $N'/\overline{\mathfrak{m}}^n \cong \mathrm{gr}(\pi'^\vee)/\overline{\mathfrak{m}}^n$, where $n \stackrel{\text{def}}{=} \max\{i_0'-i_0, 2\}+1 (\leq f+1)$.

Let $n' \stackrel{\text{def}}{=} n + i_0' + 1 (\leq 2f+1)$ and let $\overline{\tau} \stackrel{\text{def}}{=} \tau^{(n')}[\mathfrak{m}^{n'}]$. Note that by assumption $\overline{\rho}$ is $(2n'-1)$-generic. Define $\Theta = \bigoplus_{\lambda\in\mathscr{P}} \Theta_\lambda$ (resp. $\Theta' = \bigoplus_{\lambda\in\mathscr{P}} \Theta_\lambda'$), as the image of $\overline{\tau}$ in π_2 (resp. π_2'). Then $\Theta_\lambda'^\vee \subseteq \Theta_\lambda^\vee$ for all $\lambda \in \mathscr{P}$. By (47) applied to $\Theta_\lambda^\vee$ and $\Theta_\lambda'^\vee$ we have

$$\mathfrak{m}^i\Theta_\lambda^\vee \cap \Theta_\lambda'^\vee = \mathfrak{m}^{i+d_\lambda}\overline{\tau}_\lambda^\vee \cap \Theta_\lambda'^\vee = \mathfrak{m}^{i+d_\lambda-d_\lambda'}\Theta_\lambda'^\vee \tag{55}$$

for all $i \in \mathbb{Z}$.

From (47) and (46) we have

$$\mathrm{gr}(\Theta_\lambda^\vee)(d_\lambda) \cong \mathrm{gr}_{F_\lambda}(\Theta_\lambda^\vee) \cong \left(\chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}\right)_{\geq -n'+1},$$

using the notation $(\cdot)_{\geq -n'+1}$ as in [BHH$^+$, Lemma 2.2.7], hence

$$\mathrm{gr}(\Theta_\lambda^\vee) \cong \left(\chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda)\right)_{\geq -n'+d_\lambda+1} \tag{56}$$

and likewise for $\mathrm{gr}(\Theta_\lambda'^\vee)$. As $i_0+1 \geq d_\lambda$, by Theorem 6.1.1 the natural surjection $\mathrm{gr}(\pi_2^\vee) \twoheadrightarrow \mathrm{gr}(\Theta^\vee)$ is an isomorphism in degrees $\geq -n' + i_0 + 2$, and likewise for $\mathrm{gr}(\pi_2'^\vee)$ in degrees $\geq -n' + i_0' + 2$. As $-n+1 = -n' + i_0' + 2 > -n' + i_0 + 2$, we obtain that the natural surjections $\pi_2^\vee \twoheadrightarrow \Theta^\vee$ and $\pi_2'^\vee \twoheadrightarrow \Theta'^\vee$ induce isomorphisms

$$\pi_2^\vee/\mathfrak{m}^i\pi_2^\vee \xrightarrow{\sim} \Theta^\vee/\mathfrak{m}^i\Theta^\vee, \qquad \pi_2'^\vee/\mathfrak{m}^i\pi_2'^\vee \xrightarrow{\sim} \Theta'^\vee/\mathfrak{m}^i\Theta'^\vee \tag{57}$$

for all $0 \leq i \leq n$.

Suppose $0 \leq i \leq n$. From the exact sequence $\pi_2'^\vee/\mathfrak{m}^i\pi_2'^\vee \to \pi_2^\vee/\mathfrak{m}^i\pi_2^\vee \to \pi'^\vee/\mathfrak{m}^i\pi'^\vee \to 0$ and (57), we obtain that

$$\pi'^\vee/\mathfrak{m}^i\pi'^\vee \cong \bigoplus_{\lambda\in\mathscr{P}} \Theta_\lambda^\vee/(\Theta_\lambda'^\vee + \mathfrak{m}^i\Theta_\lambda^\vee).$$

By the line above together with (57) we see that the kernel of $\pi_2^\vee/\mathfrak{m}^i\pi_2^\vee \twoheadrightarrow \pi'^\vee/\mathfrak{m}^i\pi'^\vee$ is identified with

$$\bigoplus_{\lambda\in\mathscr{P}} (\Theta_\lambda'^\vee + \mathfrak{m}^i\Theta_\lambda^\vee)/\mathfrak{m}^i\Theta_\lambda^\vee \cong \bigoplus_{\lambda\in\mathscr{P}} \Theta_\lambda'^\vee/\mathfrak{m}^{i+d_\lambda-d'_\lambda}\Theta_\lambda'^\vee,$$

where the isomorphism follows from (55), and hence we have an exact sequence

$$0 \to \bigoplus_{\lambda\in\mathscr{P}} \Theta_\lambda'^\vee/\mathfrak{m}^{i+d_\lambda-d'_\lambda}\Theta_\lambda'^\vee \to \bigoplus_{\lambda\in\mathscr{P}} \Theta_\lambda^\vee/\mathfrak{m}^i\Theta_\lambda^\vee \to \pi'^\vee/\mathfrak{m}^i\pi'^\vee \to 0.$$

Therefore the filtration on the left term induced by the $\mathfrak{m}$-adic filtration on the middle term is the $\mathfrak{m}$-adic filtration up to a shift by $d'_\lambda - d_\lambda$. Taking graded pieces for $i = n$, we obtain

$$0 \to \bigoplus_{\lambda\in\mathscr{P}} (\mathrm{gr}(\Theta_\lambda'^\vee)/\overline{\mathfrak{m}}^{n+d_\lambda-d'_\lambda})(d'_\lambda - d_\lambda) \to \bigoplus_{\lambda\in\mathscr{P}} \mathrm{gr}(\Theta_\lambda^\vee)/\overline{\mathfrak{m}}^n \to \mathrm{gr}(\pi'^\vee)/\overline{\mathfrak{m}}^n \to 0$$

By (56) and its analogue for $\mathrm{gr}(\Theta_\lambda'^\vee)$ we obtain

$$0 \to \left(\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0'}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda)\right)_{\geq -n+1} \to \left(\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda)\right)_{\geq -n+1} \to \mathrm{gr}(\pi'^\vee)/\overline{\mathfrak{m}}^n \to 0.$$

Therefore, $\mathrm{gr}(\pi'^\vee)/\overline{\mathfrak{m}}^n \cong (N')_{\geq -n+1} \cong N'/\overline{\mathfrak{m}}^n$ (the second isomorphism holds since N' is generated by its elements of degree 0, which follows from the definition of the ideal $\mathfrak{a}_1^{i_0}(\lambda)$), as we wanted to show.

Step 2. We lift the isomorphism $N'/\overline{\mathfrak{m}}^n \xrightarrow{\sim} \mathrm{gr}(\pi'^\vee)/\overline{\mathfrak{m}}^n = (\mathrm{gr}(\pi'^\vee))_{\geq -n+1}$ to a homomorphism $N' \to \mathrm{gr}(\pi'^\vee)$. Consider the short exact sequence of graded $\mathrm{gr}(\Lambda)$-modules,

$$0 \to \frac{\mathfrak{a}_1^{i_0'}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda) \to \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda) \to \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0'}(\lambda)}(-d_\lambda) \to 0.$$

Going back to the proof of Lemma 6.1.5 (noting that $n \geq 3$), we know that the middle term has relations generated in degrees -1, -2, and the left term has generators in degree $d_\lambda - d'_\lambda$. As $0 \leq d'_\lambda - d_\lambda \leq i'_0 - i_0$, the right term has relations in degree $-1, -2, \dots, -\max\{i'_0 - i_0, 2\} = -n+1$. (Note that when $d'_\lambda - d_\lambda = 0$ then $d'_\lambda = d_\lambda = 0$ by definition since $i_0 < i'_0$, and hence $\mathfrak{a}_1^{i_0}(\lambda) = \mathfrak{a}_1^{i'_0}(\lambda) = \overline{R}$.) By Step 3 of the proof of Lemma 6.1.5, we deduce the desired lifting $N' \to \mathrm{gr}(\pi'^\vee)$.

Step 3. Using Steps 1 and 2, we conclude that the homomorphism $N' \to \mathrm{gr}(\pi'^\vee)$ is an isomorphism exactly as in Step 3 of the proof of Theorem 6.1.1. (Note that N' is Cohen–Macaulay by [BHH$^+$, Cor. 4.4.6]. Also note that $\mathcal{Z}(\mathrm{gr}(\pi'^\vee)) = \mathcal{Z}(\mathrm{gr}(\pi_2^\vee)) - \mathcal{Z}(\mathrm{gr}(\pi_2'^\vee)) = \mathcal{Z}(N_2^{i_0}) - \mathcal{Z}(N_2^{i'_0}) = \mathcal{Z}(N')$.)

Finally we assume that $f = 1$, and we just assume that $\overline{\rho}$ is 0-generic. We prove that π has length 2 and fits into a short exact sequence $0 \to \pi_0 \to \pi \to \pi_1 \to 0$, where π_0, π_1 are irreducible principal series as in [BHH$^+$25, Cor. 3.92], namely they are dual to each other in the sense that

$$\mathrm{E}^2_\Lambda(\pi_i^\vee) \cong \pi_{1-i}^\vee \otimes (\det(\overline{\rho})\omega^{-1}). \tag{58}$$

Indeed, by assumption (i) we know that $W(\overline{\rho}) = \{\sigma_0\}$ is a singleton and it is easy to see that $\pi_0 \overset{\mathrm{def}}{=} \langle \mathrm{GL}_2(K) \cdot \sigma_0 \rangle$ is an irreducible principal series (as in the proof of [BHH$^+$25, Cor. 3.92]) and that $\pi_0 = \mathrm{soc}_{\mathrm{GL}_2(K)}(\pi)$. Using assumption (iii) and (58), we deduce that π has a quotient isomorphic to π_1. We need to prove that $V = 0$, where $V \overset{\mathrm{def}}{=} \ker(\pi \twoheadrightarrow \pi_1)/\pi_0$. By [BHH$^+$25, Thm. 3.67] (with $r = 1$) there is a surjection of $\mathrm{gr}(\Lambda)$-modules with compatible H-action

$$N \overset{\mathrm{def}}{=} \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \tag{59}$$

and $m(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) \leq 4$ by [BHH$^+$25, Cor. 3.71], where $m(-)$ denotes the total multiplicity of $\overline{R}$-modules. On the other hand, [BHH$^+$25, Prop. 3.76(ii)] implies that $\mathrm{gr}_{\mathfrak{m}}(\pi_i^\vee)$ is an $\overline{R}$-module and $m(\mathrm{gr}_{\mathfrak{m}}(\pi_i^\vee)) = 2$ for $i = 0, 1$, so we deduce $m(V) = 0$ by the additivity of $m(-)$, equivalently $\dim_{\mathbb{F}} V < +\infty$. However, this forces $\mathrm{Ext}^1_{\mathrm{GL}_2(K)}(V, \pi_0) = 0$ by [Eme10, Lemma 4.3.9, Prop. 4.3.32(1)], hence $V = 0$ (as $\pi_0 = \mathrm{soc}_{\mathrm{GL}_2(K)}(\pi)$). In all, we deduce that π has length 2 and that (59) is an isomorphism (as the graded module N in (59) is Cohen–Macaulay) which determines $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$. Moreover, using [BHH$^+$25, Prop. 3.76(ii)] again we check that $\mathrm{gr}_{\mathfrak{m}}(\pi'^\vee)$ is as in (54) for any proper subquotient π' of π. □

Corollary 6.1.8. *Assume $\overline{\rho}$ is $\max\{9, 2f+3, 2n+2i_0+1\}$-generic for some $n \geq 1$. If $\pi' = \pi'_1/\pi_1$ is any subquotient, where $\pi_1 \subsetneq \pi'_1 \subseteq \pi$, then $\pi'[\mathfrak{m}^n]$ is multiplicity free as I-representation.*

Proof. Since π' injects into $\pi_2 = \pi/\pi_1$ and hence $\pi'[\mathfrak{m}^n] \subseteq \pi_2[\mathfrak{m}^n]$ we are reduced to the case where $\pi' = \pi_2$. We need to show that $\mathrm{gr}(\pi_2^\vee)/\overline{\mathfrak{m}}^n$ is multiplicity free. By Theorem 6.1.1, and as

$\mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}(\lambda)$ is generated by elements of degree $-d_\lambda$ for each $\lambda \in \mathscr{P}$, it is equivalent to showing that

$$\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes \left(\frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)}(-d_\lambda)\right)_{>-n}$$

is multiplicity free. Hence it is sufficient that

$$\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes \left(\frac{\overline{R}}{\mathfrak{a}(\lambda)}\right)_{>-(n+d_\lambda)}$$

is multiplicity free. This follows from [BHH$^+$, Lemma 2.3.7], as $n + d_\lambda \le n + i_0 + 1$ and as $N/\mathcal{I}^{(n+i_0+1)}N$ surjects onto $N/\overline{\mathfrak{m}}^{n+i_0+1}N = N_{>-(n+i_0+1)}$ (as h_j kills N). □

Remark 6.1.9. Just like in Remark 6.1.4 there is an easier proof when $i_0(\pi_1) \in \{f-1, f\}$ with $\overline{\rho}$ being $\max\{9, 2n-1, 2f+1\}$-generic. In the first case, the multiplicity freeness follows from (43), by applying [BHH$^+$, Lemma 2.3.6(ii)] with $m = 2n-2$ and $\lambda = \lambda'' \in \mathscr{P}^{\mathrm{ss}}$ (so $t_j = y_j$ for all j); in the second case it is trivial.

We conclude this section with a further result regarding the structure of π. Let π_s be an admissible smooth representation satisfying assumptions (i)–(v) with respect to $\overline{\rho}^{\mathrm{ss}}$. The most optimistic expectation, at least for globally defined $\pi = \pi(\overline{\rho})$ and $\pi_s = \pi(\overline{\rho}^{\mathrm{ss}})$ as in § 7.1 (cf. the comments after [BP12, Thm. 19.10]), is that $\pi^{\mathrm{ss}} \cong \pi_s$ and moreover that π and π_s both have length $f+1$. (In fact, the first expectation implies the second by Remark 6.2.3 below.) The following proposition provides new evidence for this expectation. For any $\lambda' \in \mathscr{P}^{\mathrm{ss}}$ we let $\mathfrak{a}^{\mathrm{ss}}(\lambda')$ denote the ideal of R generated by all $t_j^{\mathrm{ss}} = t_j^{\mathrm{ss}}(\lambda') \in \{y_j, z_j, y_j z_j\}$, which are defined as in (13) but for the Galois representation $\overline{\rho}^{\mathrm{ss}}$. Recall $\mathfrak{a}^{\mathrm{ss}}(\lambda') \supseteq \ker(R \to \overline{R})$, so we often think of it as ideal of $\overline{R}$.

Proposition 6.1.10. *For any $-1 \le i_0 \le f-1$ we have an isomorphism*

$$\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0+1}(\lambda)}(-d_\lambda) \cong \bigoplus_{\lambda'\in\mathscr{P}^{\mathrm{ss}},\, \ell(\lambda')=i_0+1} \chi_{\lambda'}^{-1} \otimes \frac{\overline{R}}{\mathfrak{a}^{\mathrm{ss}}(\lambda')}$$

of graded $\mathrm{gr}(\Lambda)$*-modules with compatible H-actions.*

Remark 6.1.11. For any $-1 \le i_0 \le f-1$ let $\pi' \overset{\mathrm{def}}{=} \pi_1'/\pi_1$, where $i_0(\pi_1) = i_0$ and $i_0(\pi_1') = i_0+1$ (if such π_1, π_1' exist) and let π_s' denote the subquotient of π_s corresponding to the subset $\mathscr{P}' \overset{\mathrm{def}}{=} \{\lambda' \in \mathscr{P}^{\mathrm{ss}},\ \ell(\lambda') = i_0+1\}$ in [BHH$^+$, Cor. 3.2.7(ii)] (if it exists). If π' and π_s' exist (for example, if π and π_s have length $f+1$), then by Corollary 6.1.7 and [BHH$^+$, Cor. 3.2.7(iii)] (provided that $\overline{\rho}$ is $\max\{9, 4f+1\}$-generic), Proposition 6.1.10 asserts that

$$\mathrm{gr}_{\mathfrak{m}}(\pi'^{\vee}) \cong \mathrm{gr}_{\mathfrak{m}}(\pi_s'^{\vee}) \tag{60}$$

as graded $\mathrm{gr}(\Lambda)$-modules with compatible H-actions. If $i_0+1 \in \{0, f\}$, we even know that $\pi' \cong \pi_s'$ are isomorphic principal series (compare [HW22, Prop. 10.8] with [BHH$^+$25, Cor. 3.92]). More interestingly, if $f = 2$ and $i_0 = 0$ we know that π' and π_s' exist (and are supersingular) by [HW22, Thm. 1.7], [BHH$^+$25, Cor. 3.92], and hence (60) holds (provided $\overline{\rho}$ is $\max\{9, 4f+1\}$-generic).

Proof. Fix $\lambda \in \mathscr{P}$. As usual, let $J_1 \overset{\text{def}}{=} \{j \in J_\rho^c : \lambda_j(x_j) = p-1-x_j\}$ and $J_2 \overset{\text{def}}{=} \{j \in J_\rho^c : \lambda_j(x_j) = x_j\}$, and let $J \overset{\text{def}}{=} J_1 \sqcup J_2$. We show that

$$\chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0+1}(\lambda)}(-d_\lambda) \cong \bigoplus_{J' \subseteq J,\ |J_\lambda|+|J'|=i_0+1} \chi_{\lambda'(J')}^{-1} \otimes \frac{\overline{R}}{\mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))}, \tag{61}$$

where $\lambda'(J') \in \mathscr{P}^{\mathrm{ss}}$ is defined by

$$\lambda'(J')_j(x_j) \overset{\text{def}}{=} \begin{cases} p-3-x_j & \text{if } j \in J_1' \overset{\text{def}}{=} J' \cap J_1, \\ x_j+2 & \text{if } j \in J_2' \overset{\text{def}}{=} J' \cap J_2, \\ \lambda_j(x_j) & \text{otherwise.} \end{cases}$$

It is easy to check that this implies the proposition, by taking a direct sum over all $\lambda \in \mathscr{P}$.

If $i_0+1-|J_\lambda| < 0$, then (61) trivially holds: by definition, $\mathfrak{a}_1^{i_0}(\lambda) = \mathfrak{a}_1^{i_0+1}(\lambda) = \overline{R}$, so both sides are zero.

Suppose that $i_0+1-|J_\lambda| \geq 0$, so $d_\lambda = i_0+1-|J_\lambda|$. Note that $\chi_{\lambda'(J')}^{-1} = \chi_\lambda^{-1} \prod_{j \in J_1'} \alpha_j \prod_{j \in J_2'} \alpha_j^{-1}$, and $\prod_{j \in J_1'} \alpha_j \prod_{j \in J_2'} \alpha_j^{-1}$ gives the action of H on the degree d_λ polynomial

$$p_{J'} \overset{\text{def}}{=} \prod_{j \in J_1'} y_j \prod_{j \in J_2'} z_j \in R \cong \mathbb{F}[y_j, z_j : 0 \leq j \leq f-1].$$

Therefore, by twisting both sides by $\chi_\lambda(d_\lambda)$, it suffices to show that

$$\frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0+1}(\lambda)} \cong \bigoplus_{J' \subseteq J,\ |J'|=d_\lambda} p_{J'} \otimes \frac{\overline{R}}{\mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))}.$$

We have

$$\frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0+1}(\lambda)} \cong \frac{I(J_1,J_2,d_\lambda)+\mathfrak{a}(\lambda)}{I(J_1,J_2,d_\lambda+1)+\mathfrak{a}(\lambda)} \cong \frac{I(J_1,J_2,d_\lambda)}{I(J_1,J_2,d_\lambda+1)+I(J_1,J_2,d_\lambda)\cap\mathfrak{a}(\lambda)}$$

(recall the ideals $I(J_1,J_2,d_\lambda)$ from [BHH+, Def. 4.2.4]). Note that $I(J_1,J_2,d_\lambda) = (p_{J'} : J' \subseteq J,\ |J'| = d_\lambda)$ and $\mathfrak{a}(\lambda) = (t_j : 0 \leq j \leq f-1)$ with $t_j = y_j z_j$ for all $j \in J$ (see equation (13)). By [HH11, Prop. 1.2.1] the ideal $I(J_1,J_2,d_\lambda)\cap\mathfrak{a}(\lambda)$ is generated by the monomials $p_{J'}z_j$ ($j \in J_1'$), $p_{J'}y_j$ ($j \in J_2'$), and $p_{J'}t_j$ ($j \notin J'$), where $J' \subseteq J$ runs through all subsets with $|J'| = d_\lambda$. Hence the ideal $I(J_1,J_2,d_\lambda+1)+I(J_1,J_2,d_\lambda)\cap\mathfrak{a}(\lambda)$ is generated by the monomials $p_{J'}z_j$ ($j \in J_1' \sqcup (J_2 \setminus J_2')$), $p_{J'}y_j$ ($j \in J_2' \sqcup (J_1 \setminus J_1')$), and $p_{J'}t_j$ ($j \notin J$), where again $J' \subseteq J$ runs through all subsets with $|J'| = d_\lambda$. Since, from equation (13) and the definition of $\lambda'(J')$, $\mathfrak{a}^{\mathrm{ss}}(\lambda'(J')) = \mathfrak{a}(\lambda) + (z_j : j \in J_1' \sqcup (J_2 \setminus J_2')) + (y_j : j \in J_2' \sqcup (J_1 \setminus J_1'))$ we deduce that

$$I(J_1,J_2,d_\lambda+1)+I(J_1,J_2,d_\lambda)\cap\mathfrak{a}(\lambda) = \sum_{J' \subseteq J,\ |J'|=d_\lambda} p_{J'} \cdot \mathfrak{a}^{\mathrm{ss}}(\lambda'(J')).$$

In particular,

$$\frac{I(J_1,J_2,d_\lambda)}{I(J_1,J_2,d_\lambda+1)+I(J_1,J_2,d_\lambda)\cap\mathfrak{a}(\lambda)} \cong \frac{\sum_{J' \subseteq J,\ |J'|=d_\lambda} p_{J'}R}{\sum_{J' \subseteq J,\ |J'|=d_\lambda} p_{J'}\mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))},$$

so for each index J', multiplication induces a homomorphism

$$p_{J'} \otimes \frac{R}{\mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))} \to \frac{I(J_1,J_2,d_\lambda)}{I(J_1,J_2,d_\lambda+1)+I(J_1,J_2,d_\lambda)\cap \mathfrak{a}(\lambda)}$$

and passing to the direct sum induces a surjective homomorphism

$$\bigoplus_{J'\subseteq J,\, |J'|=d_\lambda} p_{J'} \otimes \frac{R}{\mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))} \twoheadrightarrow \frac{I(J_1,J_2,d_\lambda)}{I(J_1,J_2,d_\lambda+1)+I(J_1,J_2,d_\lambda)\cap \mathfrak{a}(\lambda)},$$

which we need to show is an isomorphism. Suppose that we have an element $f = (p_{J'} \otimes [f_{J'}])_{J'}$ in the kernel, or equivalently that $\sum_{J'} p_{J'} f_{J'} = \sum_{J'} p_{J'} g_{J'}$ in R for some $g_{J'} \in \mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))$. By replacing $f_{J'}$ by $f_{J'} - g_{J'}$ we may assume that $g_{J'} = 0$ for all J', i.e. that $\sum_{J'} p_{J'} f_{J'} = 0$. Fix J' and let $\mathfrak{b}(J') \stackrel{\mathrm{def}}{=} (z_j : j \in J_2 \setminus J_2') + (y_j : j \in J_1 \setminus J_1') \subseteq \mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))$. From $p_{J'} f_{J'} = -\sum_{J''\neq J'} p_{J''} f_{J''}$ we deduce that $p_{J'} f_{J'} \in \mathfrak{b}(J')$. As multiplication by $p_{J'}$ is injective on $R/\mathfrak{b}(J')$ (a polynomial ring), it follows that $f_{J'} \in \mathfrak{b}(J') \subseteq \mathfrak{a}^{\mathrm{ss}}(\lambda'(J'))$, so $f = 0$, as desired. □

6.2 I_1-invariants and $\mathrm{GL}_2(\mathcal{O}_K)$-socle of subquotient representations of π

We describe the I_1-invariants of subquotients of π. We deduce that π is multiplicity free and give a description of the $\mathrm{GL}_2(\mathcal{O}_K)$-socles of subquotients of π.

We start with the I_1-invariants of quotients $\pi_2 = \pi/\pi_1$ of π:

Proposition 6.2.1. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. Let $i_0 = i_0(\pi_1)$ with $-1 \le i_0 \le f$ be as in [BHH$^+$, Thm. 4.3.15]. Then $\pi_2^{I_1}$ is multiplicity free and*

$$\mathrm{JH}(\pi_2^{I_1}) = \{\chi_\lambda : \lambda \in \mathscr{P}, |J_\lambda| > i_0 \text{ or } \lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}, |J_\lambda| = i_0+1\}.$$

Proof. As H-representations $\pi_2^{I_1}$ is dual to the degree 0 part of $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$, namely, $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$. Comparing Theorem 6.1.1 and [BHH$^+$, Cor. 4.4.5], we see that there is an isomorphism $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \cong \mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}_F(\pi_2^\vee)$ compatible with H-actions (but not gradings), where F denotes the filtration on $\pi_2^\vee$ induced by the $\mathfrak{m}$-adic filtration on $\pi^\vee$. The result then follows from [BHH$^+$, Cor. 4.4.7]. □

We can generalize to subquotients:

Corollary 6.2.2. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. Suppose $\pi' = \pi_1'/\pi_1$ is any nonzero subquotient, where $\pi_1 \subsetneq \pi_1' \subseteq \pi$. Let $i_0 \stackrel{\mathrm{def}}{=} i_0(\pi_1)$, $i_0' \stackrel{\mathrm{def}}{=} i_0(\pi_1')$, so $-1 \le i_0 < i_0' \le f$. Then π'^{I_1} is multiplicity free and*

$$\mathrm{JH}(\pi'^{I_1}) = \{\chi_\lambda : \lambda \in \mathscr{P},\ i_0 < |J_\lambda| \le i_0' \text{ or } \lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P},\ |J_\lambda| = i_0+1\}.$$

Proof. Since we have injections $(\pi_1')^{I_1}/\pi_1^{I_1} \hookrightarrow \pi'^{I_1} \hookrightarrow (\pi/\pi_1)^{I_1}$, we deduce from [BHH$^+$, Cor. 4.3.16] and Proposition 6.2.1 that

$$\begin{aligned}\{\chi_\lambda : \lambda \in \mathscr{P}, i_0 < |J_\lambda| \le i_0'\} &\subseteq \mathrm{JH}(\pi'^{I_1}) \\ &\subseteq \{\chi_\lambda : \lambda \in \mathscr{P}, i_0 < |J_\lambda| \text{ or } \lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}, |J_\lambda| = i_0+1\}.\end{aligned}$$

In particular, π'^{I_1} is multiplicity free. On the other hand, note that π'^{I_1} is the kernel of the natural map $(\pi/\pi_1)^{I_1} \to (\pi/\pi_1')^{I_1}$. For any $\lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$ with $|J_\lambda| = i_0 + 1$, $\chi_\lambda \in \mathrm{JH}((\pi/\pi_1)^{I_1})$ and maps to zero in $(\pi/\pi_1')^{I_1}$ by Proposition 6.2.1, so $\chi_\lambda \in \mathrm{JH}(\pi'^{I_1})$ for such λ.

To finish the proof it remains to show that χ_λ for $\lambda \in \mathscr{P}, |J_\lambda| > i_0'$ does not occur in π'^{I_1}. By [BHH$^+$, Cor. 4.3.16], χ_λ does not occur in $\pi_1'^{I_1}$ and $\pi_1^{I_1}$, and by Proposition 6.2.1 it occurs in $(\pi/\pi_1)^{I_1}$ with multiplicity 1. Hence the χ_λ-eigenspace of $(\pi/\pi_1)^{I_1}$ is the image of the χ_λ-eigenspace of π^{I_1} and so maps to zero in $H^1(I_1/Z_1, \pi_1)$. The following diagram then shows that χ_λ cannot occur in $\pi'^{I_1} = (\pi_1'/\pi_1)^{I_1}$, since it does not occur in $\pi_1'^{I_1}$:

$$\begin{array}{ccccccc}
0 \longrightarrow & \pi_1^{I_1} & \longrightarrow & \pi_1'^{I_1} & \longrightarrow & (\pi_1'/\pi_1)^{I_1} & \longrightarrow H^1(I_1/Z_1, \pi_1) \\
& \| & & \downarrow & & \downarrow & \| \\
0 \longrightarrow & \pi_1^{I_1} & \longrightarrow & \pi^{I_1} & \longrightarrow & (\pi/\pi_1)^{I_1} & \longrightarrow H^1(I_1/Z_1, \pi_1).
\end{array}$$
□

Remark 6.2.3. Let π_s be an admissible smooth representation satisfying assumptions (i)–(v) with respect to $\overline{\rho}^{\mathrm{ss}}$. We remark that if $\pi^{\mathrm{ss}} \cong \pi_s$ (as one might hope is true in analogy with $\mathrm{GL}_2(\mathbb{Q}_p)$) or even just $\pi^{\mathrm{ss}} \cong \pi_s^{\mathrm{ss}}$, then π and π_s have length exactly $f+1$ provided $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. (Sketch proof: we calculate $(\pi^{\mathrm{ss}})^{I_1}$ using Corollary 6.2.2 and $(\pi_s^{\mathrm{ss}})^{I_1}$ using [BHH$^+$, Cor. 3.2.5]. If $\Sigma \subseteq \{0, 1, \dots, f\}$ denotes the subset of elements of the form $i_0(\pi_1)+1$ for some subrepresentation $\pi_1 \subsetneq \pi$, then we deduce that $\mathscr{P}^{\mathrm{ss}} = \mathscr{P} \cup \{\lambda \in \mathscr{P}^{\mathrm{ss}} : |J_\lambda| \in \Sigma\}$. As $\overline{\rho}$ is non-semisimple, $J_{\overline{\rho}} \neq \{0, 1, \dots, f-1\}$ and one easily shows that for any $1 \le k \le f$ there exists $\lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$ with $|J_\lambda| = k$. We deduce that $\Sigma = \{0, 1, \dots, f\}$, i.e. π has length $f+1$.)

Corollary 6.2.4. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. Suppose $\pi' = \pi_1'/\pi_1$ is any nonzero subquotient, where $\pi_1 \subsetneq \pi_1' \subseteq \pi$. Let $i_0 \overset{\mathrm{def}}{=} i_0(\pi_1)$, $i_0' \overset{\mathrm{def}}{=} i_0(\pi_1')$, so $-1 \le i_0 < i_0' \le f$. Then*

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi') \cong \bigoplus_{\sigma \in W(\overline{\rho}), i_0 < \ell(\sigma) \le i_0'} \sigma \oplus \bigoplus_{\sigma \in W(\overline{\rho}^{\mathrm{ss}}) \setminus W(\overline{\rho}), \ell(\sigma) = i_0+1} \sigma. \tag{62}$$

In particular, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')$ is multiplicity free.

For the proof, recall from [BHH$^+$, § 4.3.4] that $D_0(\overline{\rho})$ admits an increasing filtration $0 = D_0(\overline{\rho})_{\le -1} \subsetneq D_0(\overline{\rho})_{\le 0} \subsetneq \cdots \subsetneq D_0(\overline{\rho})_{\le i} \subsetneq \cdots \subsetneq D_0(\overline{\rho})_{\le f} = D_0(\overline{\rho})$, where $D_0(\overline{\rho})_{\le i}$ is the largest Γ-subrepresentation of $D_0(\overline{\rho})$ not containing any $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\tau) > i$ as subquotient. We set $D_0(\overline{\rho})_i \overset{\mathrm{def}}{=} D_0(\overline{\rho})_{\le i}/D_0(\overline{\rho})_{\le i-1}$, $D_{0,\sigma}(\overline{\rho})_{\le i} \overset{\mathrm{def}}{=} D_{0,\sigma}(\overline{\rho}) \cap D_0(\overline{\rho})_{\le i}$ and $D_{0,\tau}(\overline{\rho})_i \overset{\mathrm{def}}{=} D_0(\overline{\rho})_i \cap D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$ for $\sigma \in W(\overline{\rho})$ and $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ (see *loc. cit.* and note that $D_0(\overline{\rho})_i \hookrightarrow D_0(\overline{\rho}^{\mathrm{ss}})_i$ using [BP12, Prop. 13.1] so that the latter intersection is taken in $D_0(\overline{\rho}^{\mathrm{ss}})_i$, see also [Hu16, Prop. 5.2]).

Proof. By Corollary 6.2.2, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')$ is multiplicity free. We prove the inclusion "$\supseteq$". If $\sigma \in W(\overline{\rho}), i_0 < \ell(\sigma) \le i_0'$, then $\sigma \subseteq \pi_1'|_{\mathrm{GL}_2(\mathcal{O}_K)}$ but $\sigma \not\subseteq \pi_1|_{\mathrm{GL}_2(\mathcal{O}_K)}$ by [BHH$^+$, Cor. 4.3.17], so $\sigma \subseteq \pi'|_{\mathrm{GL}_2(\mathcal{O}_K)}$. If $\sigma \in W(\overline{\rho}^{\mathrm{ss}}), \ell(\sigma) = i_0 + 1$, then $\sigma \subseteq D_0(\overline{\rho})_{i_0+1}$ by [BHH$^+$, eq. (66)], which by [BHH$^+$, Thm. 4.3.15] injects into

$$\pi_1'^{K_1}/\pi_1^{K_1} = D_0(\overline{\rho})_{\le i_0'}/D_0(\overline{\rho})_{\le i_0} \hookrightarrow \pi'^{K_1}.$$

(In particular, the right-hand side of (62) injects into $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1'^{K_1}/\pi_1^{K_1})$.)

Now we prove the inclusion "$\subseteq$". Suppose that τ is a Serre weight such that $\tau \subseteq \pi'|_{\mathrm{GL}_2(\mathcal{O}_K)}$. By Corollary 6.2.2, we know $\tau^{I_1} = \chi_\lambda$, where either $\lambda \in \mathscr{P}$, $i_0 < |J_\lambda| \le i_0'$ or $\lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$, $|J_\lambda| = i_0 + 1$.

If $\lambda \in \mathscr{P}$, $i_0 < |J_\lambda| \le i_0'$, then τ^{I_1} lifts to $\pi_1'^{I_1}$ by [BHH+, Cor. 4.3.16] and hence τ is the image of a morphism $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda \to \pi_1' \twoheadrightarrow \pi'$. In particular, $\tau \hookrightarrow \pi_1'^{K_1}/\pi_1^{K_1} \cong D_0(\overline{\rho})_{\le i_0'}/D_0(\overline{\rho})_{\le i_0}$, so by dévissage and [BHH+, eq. (66)], $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ with $i_0 < \ell(\tau) \le i_0'$. Suppose that $\tau \notin W(\overline{\rho})$ (or we are done). By [BHH+, Lemma 4.1.1] we deduce that the image of the above morphism $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda \to \pi_1' \subseteq \pi$ is isomorphic to $I(\sigma, \tau)$, where $\sigma \in W(\overline{\rho})$ is determined (via equation (11)) by $J_\sigma = J_{\overline{\rho}} \cap J_\lambda$. As the image of the composition is τ, it follows that $\mathrm{rad}_\Gamma(I(\sigma,\tau)) \subseteq \pi_1^{K_1} = D_0(\overline{\rho})_{\le i_0}$, so from [BHH+, Lemma 4.1.3] (taking τ' such that $|J_{\tau'}| = |J_\tau| - 1$) and [BHH+, eq. (66)] we deduce that $\ell(\tau) = i_0 + 1$, as desired.

Suppose that $\lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$, $|J_\lambda| = i_0 + 1$. From the proof of Corollary 6.2.2 we know that the 1-dimensional subspace $(\pi')^{I=\chi_\lambda}$ is the image of the morphism $W(\chi_\mu, \chi_\lambda) \hookrightarrow \pi_1' \twoheadrightarrow \pi'$, where J_1, J_2 and $\mu \in \mathscr{P}$ are defined as in equations [BHH+, eq. (60), eq. (61)]. By Frobenius reciprocity we have a corresponding morphism $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu, \chi_\lambda) \to \pi_1'$ and we denote by V its image. Note that $V/(V \cap \pi_1) \cong \tau$. By [BHH+, Lemma 4.3.9(i)] we have $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V) \cong \sigma$, where $\sigma \in W(\overline{\rho})$ is determined by $J_\sigma = J_{\overline{\rho}} \cap J_\mu$, and by [BHH+, Lemma 4.1.1] and [BHH+, eq. (61)] it has parameter

$$\begin{aligned} \mathcal{S}(\sigma) &= \{j : \mu_j(x_j) \in \{x_j, \underline{x_j + 1}, p-2-x_j, p-3-x_j\}\} \\ &= \{j : \lambda_j(x_j) \in \{x_j, \underline{x_j+1}, \underset{\cdots\cdots}{x_j+2}, p-2-x_j, \underline{p-3-x_j}\}\} \end{aligned}$$

inside $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu$ (see § 2.1 for $\mathcal{S}(\sigma)$). Here we use the convention (from the proof of [BHH+, Lemma 4.1.1]) that an underlined entry is only allowed when $j \in J_{\overline{\rho}}$, and similarly that an entry with a dotted underline is only allowed when $j \notin J_{\overline{\rho}}$. Let $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ be determined by $J_{\tau'} = J_\lambda$, and by [BHH+, Lemma 3.1.3] it has parameter

$$\mathcal{S}(\tau') = \{j : \lambda_j(x_j) \in \{x_j, x_j+1, p-2-x_j, p-3-x_j\}\}$$

inside $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda$. As

$$(\mathcal{S}(\sigma) \sqcup J_1) \setminus J_2 = \{j : \lambda_j(x_j) \in \{x_j, \underline{x_j+1}, p-2-x_j, p-3-x_j\}\} \subseteq \mathcal{S}(\tau'),$$

and $J_2 \subseteq \mathcal{S}(\sigma) \setminus \mathcal{S}(\tau')$ (and since $\mathcal{S}(\sigma) \cap J_1 = \mathcal{S}(\tau') \cap J_2 = \emptyset$) we deduce from [BHH+, Prop. 4.3.6(ii), Prop. 4.3.8] that $\tau' \in \mathrm{JH}(V^{K_1})$. Since $\ell(\tau') = i_0 + 1$, it follows from [BHH+, Thm. 4.3.15] that $\tau' \notin \mathrm{JH}(\pi_1^{K_1})$, so $\tau' \cong \tau$. Thus $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ and $\ell(\tau) = i_0 + 1$, i.e. τ appears in the right-hand side of (62), as desired. $\square$

Remark 6.2.5. The proof shows that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi') = \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1'^{K_1}/\pi_1^{K_1})$, but by (62) it is bigger than $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1')/\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1)$, in general.

Recall from subsection 1.4 the $\widetilde{\Gamma}$-representation $\widetilde{D}_0(\overline{\rho})$ and from Proposition 3.2.8 that $\pi[\mathfrak{m}_{K_1}^2] \cong \widetilde{D}_0(\overline{\rho})$.

We now define the increasing filtration $(\widetilde{D}_0(\overline{\rho})_{\le i})_{-1 \le i \le f}$ on $\widetilde{D}_0(\overline{\rho})$ by letting $\widetilde{D}_0(\overline{\rho})_{\le i}$ be the largest $\widetilde{\Gamma}$-subrepresentation of $\widetilde{D}_0(\overline{\rho})$ that does not contain any $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\tau) > i$ as subquotient. Equivalently, it is the largest subrepresentation V of $\mathrm{Inj}_{\widetilde{\Gamma}}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi)$ such that

(i) $[V : \sigma] = 1$ for all $\sigma \in W(\overline{\rho})$, $\ell(\sigma) \le i$;

(ii) $[V : \tau] = 0$ for all $\tau \in W(\overline{\rho}^{\rm ss})$, $\ell(\tau) > i$.

Then $\widetilde{D}_0(\overline{\rho})_{\le i} = \bigoplus_{\sigma \in W(\overline{\rho})} \widetilde{D}_{0,\sigma}(\overline{\rho})_{\le i}$ for a unique subrepresentation $\widetilde{D}_{0,\sigma}(\overline{\rho})_{\le i} \subseteq \widetilde{D}_0(\overline{\rho})_{\le i}$.

We remark (though will not need) that $\widetilde{D}_0(\overline{\rho})_{\le i} \cap D_0(\overline{\rho}) = D_0(\overline{\rho})_{\le i}$ and that all properties of the filtration $D_0(\overline{\rho})_{\le i}$ before [BHH+, Lemma 4.3.14] generalize to the filtration $\widetilde{D}_0(\overline{\rho})_{\le i}$.

Corollary 6.2.6. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. Let $i_0 = i_0(\pi_1)$ with $-1 \le i_0 \le f$ be as in [BHH+, Thm. 4.3.15]. Then*

$$\pi_1[\mathfrak{m}_{K_1}^2] \cong \widetilde{D}_0(\overline{\rho})_{\le i_0}.$$

Proof. (Cf. the proof of Proposition 5.1(i).) By [BHH+, Thm. 4.3.15] we have $\pi_1[\mathfrak{m}_{K_1}] = D_0(\overline{\rho})_{\le i_0}$, and as $\pi[\mathfrak{m}_{K_1}^2]$ is multiplicity free we deduce from the injection $\pi_1[\mathfrak{m}_{K_1}^2]/\pi_1[\mathfrak{m}_{K_1}] \hookrightarrow \pi[\mathfrak{m}_{K_1}^2]/\pi[\mathfrak{m}_{K_1}]$ that no element of $W(\overline{\rho}^{\rm ss})$ occurs in $\pi_1[\mathfrak{m}_{K_1}^2]/\pi_1[\mathfrak{m}_{K_1}]$. As a consequence,

$$\pi_1[\mathfrak{m}_{K_1}^2] \subseteq \widetilde{D}_0(\overline{\rho})_{\le i_0}.$$

Let $Q \stackrel{\rm def}{=} \widetilde{D}_0(\overline{\rho})_{\le i_0}/\pi_1[\mathfrak{m}_{K_1}^2]$, which injects into $\pi[\mathfrak{m}_{K_1}^2]/\pi_1[\mathfrak{m}_{K_1}^2]$ and hence into $\pi_2[\mathfrak{m}_{K_1}^2]$. If $Q \ne 0$, pick an irreducible subrepresentation $\sigma \subseteq Q \subseteq \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$. Then $\sigma \in W(\overline{\rho}^{\rm ss})$ and $\ell(\sigma) > i_0$ by Corollary 6.2.4 (with $i_0' = f$), contradicting that σ contributes to $\widetilde{D}_0(\overline{\rho})_{\le i_0}$. Hence $Q = 0$, as we wanted to show. □

6.3 K_1–invariants of subquotient representations of π

We describe the K_1-invariants of subquotients of π (Corollary 6.3.9). The proofs in this section are subtle (and sometimes technical), in particular use the results of the preceding two sections and certain $\widetilde{\Gamma}$-representations that are not multiplicity free (Lemma 2.3.1).

We start with the K_1-invariants of quotients $\pi_2 = \pi/\pi_1$ of π:

Theorem 6.3.1. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. Let $i_0 = i_0(\pi_1)$ with $-1 \le i_0 \le f$ be as in [BHH+, Thm. 4.3.15]. We have*

$$\pi_2^{K_1} \cong D_0(\overline{\rho}^{\rm ss})_{i_0+1} \oplus_{D_0(\overline{\rho})_{i_0+1}} (D_0(\overline{\rho})/D_0(\overline{\rho})_{\le i_0}).$$

To prepare for the proof, we first need some lemmas.

Recall from § 1.4 that $D_0(\overline{\rho}) = \bigoplus_{\sigma \in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})$, and from [BP12, § 13] that $D_{0,\sigma}(\overline{\rho})$ is maximal (for the inclusion) with respect to the two properties $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_{0,\sigma}(\overline{\rho})) = \sigma$ and $\mathrm{JH}(D_{0,\sigma}(\overline{\rho})/\sigma) \cap W(\overline{\rho}) = \emptyset$. In particular, $D_{0,\sigma}(\overline{\rho}^{\rm ss}) \subseteq D_{0,\sigma}(\overline{\rho})$.

We now define and study the important subrepresentation $\mathcal{W} = \bigoplus_{\sigma \in W(\overline{\rho})} \mathcal{W}_\sigma \subseteq D_0(\overline{\rho})$ as well as its image $\mathcal{W}_2 = \bigoplus_{\sigma \in W(\overline{\rho})} \mathcal{W}_{2,\widetilde{\sigma}} \subseteq D_0(\overline{\rho})/D_0(\overline{\rho})_{\le i_0}$.

Lemma 6.3.2. *Assume that $\overline{\rho}$ is 0-generic. There exists a unique subrepresentation $\mathcal{W} \subseteq D_0(\overline{\rho})$ such that $\mathrm{JH}(\mathcal{W}) = W(\overline{\rho}^{\mathrm{ss}})$. Moreover, $\mathcal{W}$ has a direct sum decomposition $\mathcal{W} = \bigoplus_{\sigma\in W(\overline{\rho})} \mathcal{W}_\sigma$, where $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathcal{W}_\sigma) = \sigma$ and $\mathrm{JH}(\mathcal{W}_\sigma) = \{\tau \in W(\overline{\rho}^{\mathrm{ss}}) : J_{\overline{\rho}} \cap J_\tau = J_\sigma\}$ (in particular, $\mathcal{W}_\sigma \subseteq D_{0,\sigma}(\overline{\rho})$ for all $\sigma \in W(\overline{\rho})$). The cosocle $\widetilde{\sigma}$ of $\mathcal{W}_\sigma$ is irreducible and we have $J_{\widetilde{\sigma}} = J_\sigma \sqcup J_{\overline{\rho}}^c$.*

Proof. This is a direct consequence of [BHH+, Lemma 4.1.3]. □

For $\sigma \in W(\overline{\rho})$ let $\widetilde{\sigma} \in W(\overline{\rho}^{\mathrm{ss}})$ be the element such that $J_{\widetilde{\sigma}} = J_\sigma \sqcup J_{\overline{\rho}}^c$, thus $\mathcal{W}_\sigma \cong I(\sigma, \widetilde{\sigma})$. Clearly, $\sigma \mapsto \widetilde{\sigma}$ gives a bijection between $W(\overline{\rho})$ and $\{\tau \in W(\overline{\rho}^{\mathrm{ss}}) : J_\tau \supseteq J_{\overline{\rho}}^c\}$. For convenience, we write $\mathcal{W}_{\widetilde{\sigma}} \overset{\mathrm{def}}{=} \mathcal{W}_\sigma$.

Let $\mathcal{W}_2 \subseteq \pi_2$ (resp. $\mathcal{W}_{2,\widetilde{\sigma}}$) be the image of $\mathcal{W}$ (resp. $\mathcal{W}_{\widetilde{\sigma}}$) in $D_0(\overline{\rho})/D_0(\overline{\rho})_{\le i_0} \cong \pi^{K_1}/\pi_1^{K_1} \subseteq \pi_2^{K_1}$. In particular, $\mathrm{JH}(\mathcal{W}_2) = \{\tau \in W(\overline{\rho}^{\mathrm{ss}}) : \ell(\tau) \ge i_0 + 1\}$. As $\mathcal{W}$ is multiplicity free, we have $\mathcal{W}_2 = \bigoplus_{\sigma\in W(\overline{\rho})} \mathcal{W}_{2,\widetilde{\sigma}}$ (note that, contrary to $\mathcal{W}_{\widetilde{\sigma}}$, $\sigma \notin \mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}})$ in general).

Lemma 6.3.3. *Assume that $\overline{\rho}$ is 0-generic and that $\sigma \in W(\overline{\rho})$.*

(i) *We have*
$$\mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}}) = \{\tau \in W(\overline{\rho}^{\mathrm{ss}}) : J_{\overline{\rho}} \cap J_\tau = J_\sigma,\ \ell(\tau) \ge i_0 + 1\}.$$
Moreover, if $\ell(\sigma) \ge i_0 + 1$, then $\mathcal{W}_{2,\widetilde{\sigma}} = \mathcal{W}_{\widetilde{\sigma}} \cong I(\sigma, \widetilde{\sigma})$.

(ii) *Suppose that $\tau, \tau' \in \mathrm{JH}(\mathcal{W}_2)$ are such that $\mathrm{Ext}^1_\Gamma(\tau', \tau) \ne 0$. Then the nonsplit extension $\tau - \tau'$ occurs in $\mathcal{W}_2$ as a subquotient if and only if $J_{\tau'} = J_\tau \sqcup \{j\}$ for some $j \notin J_{\overline{\rho}}$.*

(iii) *We have*
$$\mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}}) = \bigoplus \{\tau \in W(\overline{\rho}^{\mathrm{ss}}) : J_{\overline{\rho}} \cap J_\tau = J_\sigma,\ \ell(\tau) = \max\{i_0 + 1, \ell(\sigma)\}\}.$$

Proof. (i) Note that $\mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}}) \subseteq \mathrm{JH}(\mathcal{W}_2) \cap \mathrm{JH}(\mathcal{W}_{\widetilde{\sigma}})$, and equality has to hold since both sides form a partition of $\mathrm{JH}(\mathcal{W}_2)$ as σ varies. The first formula follows, and it implies the second part, as $\mathcal{W}_{2,\widetilde{\sigma}}$ is a quotient of $\mathcal{W}_{\widetilde{\sigma}} \cong I(\sigma, \widetilde{\sigma})$.

(ii) "$\Leftarrow$": The nonsplit extension $\tau - \tau'$ is isomorphic to $I(\tau, \tau')$. Assuming $J_{\tau'} = J_\tau \sqcup \{j\}$ for some $j \notin J_{\overline{\rho}}$, let $\sigma \in W(\overline{\rho})$ be determined by $J_\sigma = J_{\overline{\rho}} \cap J_\tau = J_{\overline{\rho}} \cap J_{\tau'}$. By [BHH+, Lemma 4.1.3] we know that τ occurs in $I(\sigma, \tau')$ and that τ' occurs in $I(\sigma, \widetilde{\sigma})$. From $I(\tau, \tau') \twoheadleftarrow I(\sigma, \tau') \hookrightarrow I(\sigma, \widetilde{\sigma}) \twoheadrightarrow \mathcal{W}_{2,\widetilde{\sigma}}$ and the multiplicity freeness of $\mathcal{W}_{\widetilde{\sigma}} \cong I(\sigma, \widetilde{\sigma})$, we deduce that $\tau - \tau'$ occurs as subquotient of $\mathcal{W}_{2,\widetilde{\sigma}}$.

"$\Rightarrow$": if $\tau - \tau'$ occurs as subquotient, then $\tau, \tau' \in \mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}})$ for some $\sigma \in W(\overline{\rho})$. Thus $J_\tau \,\Delta\, J_{\tau'} \subseteq J_{\overline{\rho}}^c$ by (i) and moreover $|J_\tau \,\Delta\, J_{\tau'}| = 1$ by Lemma 2.1.1, i.e. $J_{\tau'} = J_\tau \sqcup \{j\}$ or $J_\tau = J_{\tau'} \sqcup \{j\}$ for some $j \notin J_{\overline{\rho}}$. If we had $J_\tau = J_{\tau'} \sqcup \{j\}$, then $\tau' - \tau$ would occur as subquotient by "$\Leftarrow$". This contradicts the fact that, by multiplicity freeness, at most one of $\tau - \tau'$, $\tau' - \tau$ can occur. Hence $J_{\tau'} = J_\tau \sqcup \{j\}$.

(iii) This follows from (i) and (ii). □

Lemma 6.3.4. *Assume that $\overline{\rho}$ is 1-generic. Let $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$.*

(i) *If $\sigma \in W(\overline{\rho})$, then the natural morphism*

$$\mathrm{Ext}^1_\Gamma(\tau', \mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}})) \to \mathrm{Ext}^1_\Gamma(\tau', \mathcal{W}_{2,\widetilde{\sigma}}) \tag{63}$$

is surjective.

(ii) *If $0 \to \mathcal{W}_{2,\widetilde{\sigma}} \to V \to \tau' \to 0$ is a nonsplit extension of Γ-representations and $V' \subseteq V$, $V' \nsubseteq \mathcal{W}_{2,\widetilde{\sigma}}$ is any subrepresentation with cosocle τ', then $\mathrm{rad}_\Gamma(V')$ is semisimple and $[V' : \tau'] = 1$.*

Proof. Consider a nonsplit Γ-extension $0 \to \mathcal{W}_{2,\widetilde{\sigma}} \to V \to \tau' \to 0$, and let $V' \subseteq V$, $V' \nsubseteq \mathcal{W}_{2,\widetilde{\sigma}}$ be such that $\mathrm{cosoc}_\Gamma(V') \cong \tau'$. Write $\mathrm{soc}_\Gamma(V') = \bigoplus_{i=1}^n \tau_i$ and let $m \overset{\text{def}}{=} \max\{i_0 + 1, \ell(\sigma)\}$. (Note that, if $\ell(\sigma) \geq i_0 + 1$, then $n = 1$ and $\tau_1 = \sigma$ by the last statement of Lemma 6.3.3(i).) As V is nonsplit, $\mathrm{soc}_\Gamma(V) = \mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}})$, so by Lemma 6.3.3(iii) we deduce that $\tau_i \in W(\overline{\rho}^{\mathrm{ss}})$, $\ell(\tau_i) = m$, and the τ_i are pairwise distinct.

We claim that V' is multiplicity free, or equivalently that $[V' : \tau'] = 1$. If not, then V' has a quotient $\overline{V}'$ with $\mathrm{soc}_\Gamma(\overline{V}') = \tau'$ and $[\overline{V}' : \tau'] = 2$, and we get a contradiction by Lemmas 2.2.2 and 2.2.3 applied with $Q = \overline{V}'$ and $\sigma = \tau'$, as $\mathrm{JH}(V) \subseteq W(\overline{\rho}^{\mathrm{ss}})$.

By Lemma 6.3.3(i) and (iii) we know that

$$\mathrm{JH}(\mathrm{rad}_\Gamma(V')) \subseteq \mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}}) \subseteq \{\tau \in W(\overline{\rho}^{\mathrm{ss}}) : \ell(\tau) \geq m\} \tag{64}$$

and also that $\tau \in \mathrm{JH}(\mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}}))$ implies $\ell(\tau) = m$. As $\mathrm{Ext}^1_\Gamma(\tau', \mathrm{rad}_\Gamma(V')) \neq 0$ we obtain by dévissage that $\mathrm{Ext}^1_\Gamma(\tau', \tau) \neq 0$ for some constituent τ of $\mathrm{rad}_\Gamma(V')$ and hence, by the last assertion of Lemma 2.1.1 and (64), we deduce that $\ell(\tau') \geq \ell(\tau) - 1 \geq m - 1$.

We claim that $\ell(\tau') \neq m$. As V' is multiplicity free by above, V' admits a unique quotient $\overline{V}'$ such that $\mathrm{soc}_\Gamma(\overline{V}') = \tau_1$ (recall that $\tau_1 \subseteq \mathrm{soc}_\Gamma(V')$), so $\overline{V}' \cong I(\tau_1, \tau')$ by [BP12, Cor. 3.12]. Assume by contradiction that $\ell(\tau') = \ell(\tau_1) = m$, and note that $\tau' \ncong \tau_1$ by multiplicity freeness of V'. By Lemma 2.1.1 applied to $\overline{V}' \cong I(\tau_1, \tau')$, we deduce that $\overline{V}'$ has a Jordan–Hölder constituent $\tau'' \neq \tau'$ (e.g. that corresponding to $J_{\tau_1} \cap J_{\tau'} \subsetneq J_{\tau_1}$) satisfying $|J_{\tau''}| < |J_{\tau_1}| = m$. This contradicts (64), proving the claim.

Arguing as in the previous paragraph (replacing τ_1 by $\tau_i \subseteq \mathrm{soc}_\Gamma(V')$), we have a surjection $V' \twoheadrightarrow I(\tau_i, \tau')$ and hence $\mathrm{rad}_\Gamma(V') \twoheadrightarrow \mathrm{rad}_\Gamma(I(\tau_i, \tau'))$ for each $1 \leq i \leq n$. As $\mathrm{rad}_\Gamma(V') \subseteq \mathcal{W}_{2,\widetilde{\sigma}}$ we conclude that $\mathrm{JH}(\mathrm{rad}_\Gamma(I(\tau_i, \tau'))) \subseteq \mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}})$. By Lemma 2.1.1 applied to $I(\tau_i, \tau')$ and Lemma 6.3.3(i) we deduce that

$$\{J : J \,\Delta\, J_{\tau_i} \subseteq J_{\tau'} \,\Delta\, J_{\tau_i} \text{ for some } i \text{ and } J \neq J_{\tau'}\} \subseteq \{J : J_\sigma \subseteq J \subseteq J_{\widetilde{\sigma}},\ |J| \geq m\}. \tag{65}$$

Fix $1 \leq i \leq n$. If $J_{\tau_i} \cap J_{\tau'} \neq J_{\tau'}$, then by (65) applied to $J_{\tau_i} \cap J_{\tau'}$ we deduce that $|J_{\tau_i} \cap J_{\tau'}| \geq m = |J_{\tau_i}|$. Hence $J_{\tau_i} \cap J_{\tau'}$ equals $J_{\tau'}$ or J_{τ_i}, i.e. $J_{\tau'} \subseteq J_{\tau_i}$ or $J_{\tau_i} \subseteq J_{\tau'}$ for any i.

If $\ell(\tau') \in \{m-1, m+1\}$, then by above $|J_{\tau'} \,\Delta\, J_{\tau_i}| = 1$ for all i, and it follows from Lemma 2.1.1 that $I(\tau_i, \tau')$ has length 2. The natural map $V' \to \bigoplus_i I(\tau_i, \tau')$ is injective, as it is injective on socles, so $\mathrm{rad}_\Gamma(V') \hookrightarrow \bigoplus_i \mathrm{rad}_\Gamma(I(\tau_i, \tau')) = \bigoplus_i \tau_i$. We conclude that $\mathrm{rad}_\Gamma(V')$ is semisimple. Hence the class $[V]$ of V, which is by construction the image of $[V']$ under $\mathrm{Ext}^1_\Gamma(\tau', \mathrm{rad}_\Gamma(V')) \to$

$\mathrm{Ext}^1_\Gamma(\tau', \mathcal{W}_{2,\widetilde{\sigma}})$, is in fact the image of $[V']$ under the composition

$$\mathrm{Ext}^1_\Gamma(\tau', \mathrm{rad}_\Gamma(V')) \to \mathrm{Ext}^1_\Gamma(\tau', \mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}})) \to \mathrm{Ext}^1_\Gamma(\tau', \mathcal{W}_{2,\widetilde{\sigma}}),$$

i.e. is in the image of (63).

We suppose finally till the end of that proof that $\ell(\tau') > m+1$, and we will derive a contradiction (so that this case does not happen). Note that the assumption implies $(J_\sigma \subsetneq) J_{\tau_i} \subsetneq J_{\tau'}$ for all i. If there exists $j \in J_{\tau'} \setminus J_{\widetilde{\sigma}}$, then $J = J_{\tau_1} \sqcup \{j\}$ belongs to the left-hand side of (65) (using $\ell(\tau') \neq m+1$), but (obviously) not to its right-hand side, a contradiction. Hence $J_{\tau'} \subseteq J_{\widetilde{\sigma}}$, and thus $\tau' \in \mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}})$ (using Lemma 6.3.3(i)). Let V'' denote the unique subrepresentation of $\mathcal{W}_{2,\widetilde{\sigma}}$ with cosocle τ'.

We now show that $V' \cong V''$. As both V' and V'' are multiplicity free, it is enough to show that $\mathrm{soc}_\Gamma(V') = \mathrm{soc}_\Gamma(V'')$ by (the dual of) [BP12, Prop. 3.6, Cor. 3.11]. (The references imply that $\mathrm{Proj}_\Gamma \, \tau'$ admits a maximal multiplicity-free quotient R. As R is multiplicity free, the quotients of R are determined by their socles.) If $\ell(\sigma) \geq i_0+1$, this is obvious, as $\mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}}) \cong \sigma$ is irreducible. If $\ell(\sigma) \leq i_0$, then by Lemma 6.3.3(ii) and (iii) we have $\mathrm{soc}_\Gamma(V'') \cong \bigoplus_J \tau_J$, where the direct sum runs over all the $J \subseteq \{0, \ldots, f-1\}$ such that $J_\sigma \subseteq J \subseteq J_{\tau'}$ and $|J| = i_0+1$. (Here, τ_J is the element of $W(\overline{\rho}^{\mathrm{ss}})$ such that $J_{\tau_J} = J$, see § 1.4.) Thus $\mathrm{soc}_\Gamma(V') \subseteq \mathrm{soc}_\Gamma(V'')$, since $J_{\tau_i} \subseteq J_{\tau'}$ for all $1 \leq i \leq n$. We claim that if $\tau_J \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$ (for some $J \subseteq \{0, \ldots, f-1\}$ such that $J_\sigma \subseteq J \subseteq J_{\tau'}$ and $|J| = i_0+1$), then $\tau_{(J \sqcup \{j\}) \setminus \{j'\}} \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$ for any $j \in J_{\tau'} \setminus J$ and any $j' \in J \setminus J_\sigma$. To see this: from $\mathrm{JH}(I(\tau_J, \tau')) \subseteq \mathrm{JH}(V')$ we get $\tau_{J \sqcup \{j\}} \in \mathrm{JH}(\mathrm{rad}_\Gamma(V'))$ and from Lemma 6.3.3(ii) we deduce that $\tau_{(J \sqcup \{j\}) \setminus \{j'\}} \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$, as desired. As $|J_\sigma| < |J| = i_0+1 < |J_{\tau'}| - 1$, by iteration of the claim above we conclude that any $J \subseteq \{0, \ldots, f-1\}$ such that $J_\sigma \subseteq J \subseteq J_{\tau'}$ and $|J| = i_0+1$ satisfies $\tau_J \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$. Hence $\mathrm{soc}_\Gamma(V'') \subseteq \mathrm{soc}_\Gamma(V')$, so indeed $V' \cong V''$.

We next claim that $\mathrm{rad}_\Gamma(V') = \mathrm{rad}_\Gamma(V'')$ is indecomposable. We already know that $\mathrm{rad}_\Gamma(V')$, $\mathrm{rad}_\Gamma(V'')$ are isomorphic subrepresentations of $\mathcal{W}_{2,\widetilde{\sigma}}$, and $\mathcal{W}_{2,\widetilde{\sigma}}$ is multiplicity free, so $\mathrm{rad}_\Gamma(V') = \mathrm{rad}_\Gamma(V'')$. The indecomposability is obvious if $\ell(\sigma) \geq i_0+1$, as $\mathrm{soc}_\Gamma(V') = \mathrm{soc}_\Gamma(V'')$ is then irreducible, so suppose $\ell(\sigma) \leq i_0$. Following the argument of the previous paragraph, we know that the uniserial representations of the form $\tau_J - \tau_{J \sqcup \{j\}}$ and $\tau_{(J \sqcup \{j\}) \setminus \{j'\}} - \tau_{J \sqcup \{j\}}$ occur as subquotients of $\mathcal{W}_{2,\widetilde{\sigma}}$ by Lemma 6.3.3(ii) and hence of $\mathrm{rad}_\Gamma(V')$. This shows by the same iteration as in the preceding paragraph that all constituents of $\mathrm{soc}_\Gamma(V')$ lie in the same indecomposable component of $\mathrm{rad}_\Gamma(V')$. Therefore $\mathrm{rad}_\Gamma(V')$ is indecomposable.

By the preceding two paragraphs, we can pick an isomorphism $f : V' \xrightarrow{\sim} V''$. By indecomposability of $\mathrm{rad}_\Gamma(V')$, we may rescale f so that $f|_{\mathrm{rad}_\Gamma(V')}$ is the identity on $\mathrm{rad}_\Gamma(V') = \mathrm{rad}_\Gamma(V'')$. This means that V' and V'' define the same class in $\mathrm{Ext}^1_\Gamma(\tau', \mathrm{rad}_\Gamma(V'))$, up to scalar, so some linear combination splits, implying $\tau' \in \mathrm{JH}(\mathrm{soc}_\Gamma(V))$. Since $\mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}}) = \mathrm{soc}_\Gamma(V)$ by definition of V, this contradicts that $\ell(\tau') > m+1$ (if $\tau \in \mathrm{JH}(\mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}}))$, then $\ell(\tau) = m$). □

Proposition 6.3.5. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. Let $i_0 = i_0(\pi_1)$ with $-1 \leq i_0 \leq f$ be as in [BHH+, Thm. 4.3.15] and $\pi_2 \stackrel{\mathrm{def}}{=} \pi/\pi_1$. Then $D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}$ injects into $\pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$.*

Proof. Since $D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} = \bigoplus_{\tau \in W(\overline{\rho}^{\mathrm{ss}}), \ell(\tau) = i_0+1} D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$ and is multiplicity free, it suffices to prove that $D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$ injects into $\pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$ for any $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\tau) = i_0+1$. Let $\lambda \in \mathscr{D}^{\mathrm{ss}}$ be the element corresponding to τ. As in Step 2 of the proof of [BHH$^+$, Thm. 4.3.15] we define

$$J_1 \stackrel{\text{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p-3-x_j\}, \quad \widetilde{J}_1 \stackrel{\text{def}}{=} \{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}\},$$

the element $\mu \in \mathscr{P}$ by $\mu_j(x_j) = p-1-x_j$ if $j \in J_1$ and $\mu_j(x_j) = \lambda_j(x_j)$ otherwise, and the character χ'' by

$$\chi'' \stackrel{\text{def}}{=} \chi_\mu \prod_{j \in J_1 \sqcup \widetilde{J}_1} \alpha_j^{-1}.$$

We then have $W(\chi_\mu, \chi'') \hookrightarrow \pi|_I$, hence a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant morphism as in Step 4 of the proof of [BHH$^+$, Thm. 4.3.15]:

$$\widetilde{\kappa} : \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu^s, \chi''^s) \to \pi|_{\mathrm{GL}_2(\mathcal{O}_K)}.$$

Let $\sigma_1 \in W(\overline{\rho})$ and $\tau_1 = \delta(\tau) \in W(\overline{\rho}^{\mathrm{ss}})$ be as in Step 4 of the proof of [BHH$^+$, Thm. 4.3.15], so that in particular $\mathrm{im}(\widetilde{\kappa})$ has socle σ_1 and $I(\sigma_1, \tau_1)$ embeds into $\mathrm{im}(\widetilde{\kappa})^{K_1}$. By [BHH$^+$, Lemma 4.1.3], for any $\tau' \in \mathrm{JH}(I(\sigma_1, \tau_1))$ with $\tau' \neq \tau_1$, we have $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\tau') < \ell(\tau_1) = i_0 + 1$, hence $\mathrm{rad}_\Gamma(I(\sigma_1, \tau_1)) \subseteq \pi_1$ and $I(\sigma_1, \tau_1) \nsubseteq \pi_1$ by [BHH$^+$, Thm. 4.3.15] and [BHH$^+$, eq. (68)].

Consider the composite morphism

$$\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu^s, \chi''^s) \to \pi|_{\mathrm{GL}_2(\mathcal{O}_K)} \twoheadrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)};$$

we claim that it factors through

$$\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\lambda^s, \chi''^s) \to \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}. \tag{66}$$

It suffices to prove that the image of $W(\chi_\mu^s, \chi''^s) \hookrightarrow \pi|_I \twoheadrightarrow \pi_2|_I$ has socle χ_λ^s or equivalently that the image of $W(\chi_\mu, \chi'') \hookrightarrow \pi|_I \twoheadrightarrow \pi_2|_I$ has socle χ_λ. This follows from Proposition 6.2.1 and the following two facts:

(a) under the morphism $W(\chi_\mu, \chi'') \hookrightarrow \pi|_I$, $\mathrm{rad}_I(W(\chi_\mu, \chi_\lambda))$ is sent into π_1, as any constituent is of the form χ_ν with $\nu \in \mathscr{P}^{\mathrm{ss}}$, $\ell(\nu) < i_0 + 1$ (this follows from [BHH$^+$, Lemma 4.3.1], the recipe [BHH$^+$, eq. (61)], and since $\ell(\lambda) = i_0 + 1$);

(b) by the discussion after [BHH$^+$, eq. (73)] we have in particular that

$$\big(\mathrm{JH}(W(\chi_\lambda, \chi'')) \setminus \{\chi_\lambda\}\big) \cap \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})^{I_1}) = \emptyset.$$

We note by [Bre14, Prop. 4.2] that fact (b) is equivalent to

$$\mathrm{JH}\left(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\lambda^s, \chi''^s) / \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda^s\right) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset. \tag{67}$$

Let V be the image of (66) and $V_\lambda \subseteq V$ be the image of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda^s$ in π_2, so that V/V_λ is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\lambda^s, \chi''^s)/\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda^s$ and hence $\mathrm{JH}(V/V_\lambda) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset$ by (67). From the exact sequence $0 \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V_\lambda) \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V) \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V/V_\lambda)$, as

$$\mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V)) \subseteq \mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2)) \subseteq W(\overline{\rho}^{\mathrm{ss}})$$

(by Corollary 6.2.4) and $\mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V/V_\lambda)) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset$, we deduce that the natural map $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V) \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V/V_\lambda)$ is zero, i.e. $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V_\lambda) = \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V)$.

We claim that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V_\lambda) = \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V) = \tau_1$. By the first paragraph, the representation τ_1 injects into $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V)$, hence into $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V_\lambda)$. Conversely, if $\tau_2 \subseteq \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V_\lambda)$, then we obtain surjections $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda^s \twoheadrightarrow V_\lambda \twoheadrightarrow I(\tau_2, \tau^{[s]})$ and the final representation surjects onto $I(\delta(\tau), \tau^{[s]})$ by [BP12, Lemma 12.8(ii)] and [BP12, Lemma 15.2] (with $\mathcal{S}^- = \mathcal{S}^+ = \emptyset$ here). As $\delta(\tau) = \tau_1$ occurs in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V_\lambda)$, by multiplicity freeness of V_λ we deduce that $\tau_2 = \tau_1$.

As in Step 4 of the proof of [BHH+, Thm. 4.3.15], [BHH+, Lemma 4.3.13] and the preceding paragraph imply that V contains $D_{0,\delta(\tau)}(\overline{\rho}^{\mathrm{ss}})$, hence $D_{0,\delta(\tau)}(\overline{\rho}^{\mathrm{ss}})$ injects into $\pi_2^{K_1}$. As $\ell(\delta(\tau)) = \ell(\tau)$ and $\delta(\cdot)$ is periodic, we deduce that $D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$ also injects into $\pi_2^{K_1}$, as desired. $\square$

We define the Γ-representation $D_{i_0} \overset{\mathrm{def}}{=} D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \oplus_{D_0(\overline{\rho})_{i_0+1}} (D_0(\overline{\rho})/D_0(\overline{\rho})_{\le i_0})$.

Lemma 6.3.6. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic.*

(i) *D_{i_0} is multiplicity free.*

(ii) *D_{i_0} injects into $\pi_2^{K_1}$.*

(iii) *We have $D_{i_0}^{I_1} \cong \pi_2^{I_1}$ and $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_{i_0}) \cong \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2)$. Both representations are multiplicity free. In particular, $\mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_{i_0})) \subseteq W(\overline{\rho}^{\mathrm{ss}})$.*

(iv) *We have $\mathcal{W}_2 \subseteq D_{i_0}$ and $\mathrm{JH}(D_{i_0}/\mathcal{W}_2) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset$.*

(v) *Let τ' be a Serre weight. If $\mathrm{Ext}^1_\Gamma(\tau', D_{i_0}) \neq 0$, then either $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ or*

$$\tau' \in \bigcup_{j \ge i_0+2} \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_j) \setminus \mathrm{JH}(D_0(\overline{\rho})_j).$$

Proof. (i) By construction, D_{i_0} is a successive extension of the form

$$D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} - D_0(\overline{\rho})_{i_0+2} - \cdots - D_0(\overline{\rho})_f. \tag{68}$$

More precisely, it inherits from [BHH+, eq. (65)] a filtration with graded pieces $D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}$, $D_0(\overline{\rho})_{i_0+2}, \ldots, D_0(\overline{\rho})_f$. Since $D_0(\overline{\rho})_j$ injects into $D_0(\overline{\rho}^{\mathrm{ss}})_j$ for all j and $D_0(\overline{\rho}^{\mathrm{ss}})$ is multiplicity free, D_{i_0} is also multiplicity free.

(ii) Clearly, $\pi^{K_1}/\pi_1^{K_1} \cong D_0(\overline{\rho})/D_0(\overline{\rho})_{\le i_0}$ injects into $\pi_2^{K_1}$. Recall that

$$\Sigma \overset{\mathrm{def}}{=} \mathrm{soc}_\Gamma(D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}) = \mathrm{soc}_\Gamma(D_0(\overline{\rho})_{i_0+1})$$

and that $\mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}/\Sigma) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset$. Hence the restriction maps

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}, \pi_2) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})_{i_0+1}, \pi_2) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Sigma, \pi_2).$$

are injective using Corollary 6.2.4 (applied to π_2), and therefore bijective by Proposition 6.3.5 (for each $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\tau) = i_0 + 1$ we have an injection $D_{0,\tau}(\overline{\rho}^{\mathrm{ss}}) \hookrightarrow \pi_2$). Thus any injection

$f : D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0} \hookrightarrow \pi_2^{K_1}$ can be extended to a map $\widetilde{f} : D_{i_0} \to \pi_2$. From the short exact sequence

$$0 \to D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0} \to D_{i_0} \to D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}/D_0(\overline{\rho})_{i_0+1} \to 0,$$

together with $\mathrm{soc}_\Gamma(D_{i_0}) \subseteq W(\overline{\rho}^{\mathrm{ss}})$ (which follows from (68)) and $W(\overline{\rho}^{\mathrm{ss}}) \cap \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}/D_0(\overline{\rho})_{i_0+1}) = \emptyset$ (which follows from [BHH$^+$, eq. (66)]), we deduce that $\mathrm{soc}_\Gamma(D_{i_0}) = \mathrm{soc}_\Gamma(D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0})$. We conclude that $\widetilde{f}$ is also injective and (ii) follows.

(iii) We claim that the inclusion $D_{i_0}^{I_1} \subseteq \pi_2^{I_1}$, deduced from (ii), is in fact an equality. Indeed, [BHH$^+$, Cor. 4.3.16] and [BHH$^+$, Lemma 3.1.3] show respectively that

$$\{\chi_\lambda : \lambda \in \mathscr{P},\, |J_\lambda| \geq i_0 + 1\} \subseteq \mathrm{JH}\left(\left(D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0}\right)^{I_1}\right)$$

and

$$\{\chi_\lambda : \lambda \in \mathscr{P}^{\mathrm{ss}},\, |J_\lambda| = i_0 + 1\} \subseteq \mathrm{JH}\left(D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}^{I_1}\right).$$

By the definition of D_{i_0} we obtain inclusions

$$\{\chi_\lambda : \lambda \in \mathscr{P}, |J_\lambda| > i_0 \text{ or } \lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}, |J_\lambda| = i_0 + 1\} \subseteq \mathrm{JH}(D_{i_0}^{I_1}) \subseteq \mathrm{JH}(\pi_2^{I_1}),$$

so that equality holds by Proposition 6.2.1.

The equality of $\mathrm{soc}_\Gamma(D_{i_0})$ and $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2)$ follows from the chain of inclusions

$$\mathrm{soc}_\Gamma(D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0}) \subseteq \mathrm{soc}_\Gamma(D_{i_0}) \subseteq \mathrm{soc}_\Gamma(\pi_2^{K_1}) = \mathrm{soc}_\Gamma(D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0}),$$

where the first inclusion follows from the construction of D_{i_0}, the second from part (ii) and the last equality follows from Remark 6.2.5.

(iv) By definition, $\mathcal{W}_2 \subseteq D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0} \subseteq D_{i_0}$. We have an exact sequence

$$0 \to (D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0})/\mathcal{W}_2 \to D_{i_0}/\mathcal{W}_2 \to D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}/D_0(\overline{\rho})_{i_0+1} \to 0$$

and a surjection $D_0(\overline{\rho})/\mathcal{W} \twoheadrightarrow (D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0})/\mathcal{W}_2$. As no constituents of $D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}/D_0(\overline{\rho})_{i_0+1}$ and $D_0(\overline{\rho})/\mathcal{W}$ lie in $W(\overline{\rho}^{\mathrm{ss}})$ (see the proof of (ii) in the former case), we deduce the final claim.

(v) If $\mathrm{Ext}^1_\Gamma(\tau', D_{i_0}) \neq 0$, then $\mathrm{Ext}^1_\Gamma(\tau', D) \neq 0$ for one of the graded pieces appearing in (68). If $\mathrm{Ext}^1_\Gamma(\tau', D_0(\overline{\rho})_j) \neq 0$ for some $j \geq i_0+2$, then using the exact sequence $0 \to D_0(\overline{\rho})_j \to D_0(\overline{\rho}^{\mathrm{ss}})_j \to R_j \to 0$, where R_j is the corresponding quotient, we see that either $\mathrm{Ext}^1_\Gamma(\tau', D_0(\overline{\rho}^{\mathrm{ss}})_j) \neq 0$ or $\mathrm{Hom}_\Gamma(\tau', R_j) \neq 0$. In the latter case, $\tau' \in \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_j) \setminus \mathrm{JH}(D_0(\overline{\rho})_j)$ by multiplicity freeness.

Therefore, $\mathrm{Ext}^1_\Gamma(\tau', D_{i_0}) \neq 0$ implies $\tau' \in \bigcup_{j \geq i_0+2} \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_j) \setminus \mathrm{JH}(D_0(\overline{\rho})_j)$ or for some $j \geq i_0 + 1$ we have $\mathrm{Ext}^1_\Gamma(\tau', D_0(\overline{\rho}^{\mathrm{ss}})_j) \neq 0$. The latter implies $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ by [HW18, Lemmas 2.25, 2.26] (and recalling $D_0(\overline{\rho}^{\mathrm{ss}})_j = \bigoplus_{\tau \in W(\overline{\rho}^{\mathrm{ss}}), \ell(\tau)=j} D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$). □

The following result strengthens Lemma 6.3.6(v).

Corollary 6.3.7. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. If $\mathrm{Ext}^1_\Gamma(\tau', D_{i_0}) \neq 0$ for some Serre weight τ', then $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$.*

Proof. By Lemma 6.3.6(v), it suffices to show $\mathrm{Ext}^1_\Gamma(\tau', D_{i_0}) = 0$ if $\tau' \in \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_j) \setminus \mathrm{JH}(D_0(\overline{\rho})_j)$ for some $j \geq i_0 + 2$. Fix such a Serre weight τ', so $\tau' \notin \mathrm{JH}(D_{i_0})$ (and $\tau' \notin \mathrm{JH}(D_0(\overline{\rho}))$). By contradiction let $0 \to D_{i_0} \to V \to \tau' \to 0$ be a nonsplit extension of Γ-representations and $V' \subseteq V$, $V' \nsubseteq D_{i_0}$ any subrepresentation with cosocle τ'. Say $\tau' \in \mathrm{JH}(D_{0,\tau''}(\overline{\rho}^{\mathrm{ss}}))$, for $\tau'' \in W(\overline{\rho}^{\mathrm{ss}})$, $\ell(\tau'') = j > i_0 + 1$. Note that $\tau' \not\cong \tau''$, as $\tau' \notin W(\overline{\rho}^{\mathrm{ss}})$ by [BHH+, eq. (66)]. Pick any $\kappa \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$, so $\kappa \in W(\overline{\rho}^{\mathrm{ss}})$ by Lemma 6.3.6(iii) and $V' \twoheadrightarrow I(\kappa, \tau')$, so $\tau'' \in \mathrm{JH}(I(\kappa, \tau')) \subseteq \mathrm{JH}(V')$ by [BP12, Lemma 12.8(ii)]. As $\tau' \not\cong \tau''$, τ'' even occurs in $\mathrm{rad}_\Gamma(V') \subseteq D_{i_0}$, so τ'' occurs in $\mathcal{W}_2 \subseteq D_{i_0}$ by Lemma 6.3.6(iv). Define $\sigma \in W(\overline{\rho})$ by $J_\sigma = J_{\overline{\rho}} \cap J_{\tau''}$, so $\tau'' \in \mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}})$ (where $\mathcal{W}_2$ and $\mathcal{W}_{2,\widetilde{\sigma}}$ are defined just before Lemma 6.3.3). We claim that $\tau' \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$, which gives a contradiction by [BHH+, eq. (67)] since $\mathrm{JH}(\mathrm{Inj}_\Gamma \sigma) \subseteq \mathrm{JH}(D_0(\overline{\rho}))$ (cf. [BHH+, eq. (54)]).

If $\ell(\sigma) \geq i_0 + 1$, then by the last assertion of Lemma 6.3.3(i) the socle of the unique subrepresentation of $\mathcal{W}_2$ with cosocle τ'' is σ, so $\sigma \hookrightarrow V'$ and hence $V' \twoheadrightarrow I(\sigma, \tau')$, which implies $\tau' \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$, as claimed.

If $\ell(\sigma) \leq i_0$, the socle of the unique subrepresentation of $\mathcal{W}_2$ with cosocle τ'' is the direct sum of all $\tau_J \in W(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\tau_J) = i_0 + 1$, $J_\sigma \subseteq J \subseteq J_{\tau''} (\subseteq J_\sigma \sqcup J^c_{\overline{\rho}})$, by Lemma 6.3.3, and each such τ_J must inject into V'. (Again, τ_J is the element of $W(\overline{\rho}^{\mathrm{ss}})$ such that $J_{\tau_J} = J$.) We deduce as before that $\tau' \in \mathrm{JH}(\mathrm{Inj}_\Gamma \tau_J)$ for each such τ_J, equivalently $\tau_J \in \mathrm{JH}(\mathrm{Inj}_\Gamma \tau')$ by [BP12, Lemma 3.2]. To prove the claim it suffices to prove the following two statements:

(a) for any two subsets $J, J' \subseteq \{0, \ldots, f-1\}$, if $\tau_J, \tau_{J'} \in \mathrm{JH}(\mathrm{Inj}_\Gamma \tau')$, then $\tau_{J \cap J'} \in \mathrm{JH}(\mathrm{Inj}_\Gamma \tau')$;

(b) $\bigcap_J J = J_\sigma$, where J runs over all elements in $X \overset{\text{def}}{=} \{J : J_\sigma \subseteq J \subseteq J_{\tau''}, |J| = i_0 + 1\}$.

For (a), we first observe that by [BP12, Lemma 12.6] $\tau_J, \tau_{J'}$ are automatically compatible in the sense that their corresponding elements in $\mathcal{I}$ are compatible (relative to τ'). Thus Lemma 2.1.3 implies that $\mathrm{JH}(I(\tau_J, \tau_{J'})) \subseteq \mathrm{JH}(\mathrm{Inj}_\Gamma \tau')$, in particular $\tau_{J\cap J'} \in \mathrm{JH}(\mathrm{Inj}_\Gamma \tau')$ using Lemma 2.1.2. For (b), we note that $\ell(\sigma) \leq i_0$ and $\ell(\tau'') \geq i_0 + 2$, so $J_\sigma \subsetneq J \subsetneq J_{\tau''}$ for any $J \in X$. Fix $J \in X$ and $j' \in J_{\tau''} \setminus J$. Then $J_j \overset{\text{def}}{=} (J \sqcup \{j'\}) \setminus \{j\} \in X$ for any $j \in J \setminus J_\sigma$. It is direct to check that $J_\sigma = J \cap (\bigcap_j J_j)$, from which (b) follows. $\square$

For $\sigma \in W(\overline{\rho})$ and $0 \leq i \leq f$ let us define for convenience the Γ-representations

$$D_{0,\sigma}(\overline{\rho})_{(i)} \overset{\text{def}}{=} \frac{D_{0,\sigma}(\overline{\rho})_{\leq i}}{D_{0,\sigma}(\overline{\rho})_{\leq i-1}} \cong \bigoplus_{\tau \in W(\overline{\rho}^{\mathrm{ss}}), \ell(\tau) = i, J_\sigma = J_{\overline{\rho}} \cap J_\tau} D_{0,\tau}(\overline{\rho})_i$$

(using [BHH+, eq. (70)]) and

$$D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}})_{(i)} \overset{\text{def}}{=} \bigoplus_{\tau \in W(\overline{\rho}^{\mathrm{ss}}), \ell(\tau) = i, J_\sigma = J_{\overline{\rho}} \cap J_\tau} D_{0,\tau}(\overline{\rho}^{\mathrm{ss}}).$$

(Note that $D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}})_{(i)}$ depends on $\overline{\rho}$, not just on $\overline{\rho}^{\mathrm{ss}}$!) Hence

$$D_0(\overline{\rho})_i = \bigoplus_{\sigma \in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})_{(i)} \qquad \text{and} \qquad D_0(\overline{\rho}^{\mathrm{ss}})_i = \bigoplus_{\sigma \in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}})_{(i)}. \tag{69}$$

(In the second case, note that $W(\overline{\rho}^{\mathrm{ss}}) = \coprod_{\sigma\in W(\overline{\rho})}\{\tau \in W(\overline{\rho}^{\mathrm{ss}}) : J_{\overline{\rho}} \cap J_\tau = J_\sigma\}$.) Note that the injection $D_0(\overline{\rho})_i \hookrightarrow D_0(\overline{\rho}^{\mathrm{ss}})_i$ (cf. above [BHH$^+$, eq. (66)]) respects the direct sum decompositions (69), as $D_{0,\tau}(\overline{\rho})_i = D_0(\overline{\rho})_i \cap D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$ (cf. above [BHH$^+$, eq. (70)]).

Lemma 6.3.8. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic.*

(i) *There is a direct sum decomposition $D_{i_0} = \bigoplus_{\sigma\in W(\overline{\rho})} D_{i_0,\sigma}$, where*

$$D_{i_0,\sigma} \overset{\mathrm{def}}{=} D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}})_{(i_0+1)} \oplus_{D_{0,\sigma}(\overline{\rho})_{(i_0+1)}} (D_{0,\sigma}(\overline{\rho})/D_{0,\sigma}(\overline{\rho})_{\le i_0}) \qquad \text{for } \sigma \in W(\overline{\rho}).$$

Moreover, $D_{i_0,\sigma} = D_{0,\sigma}(\overline{\rho})$ if $\ell(\sigma) \ge i_0 + 1$.

(ii) *Fix $\sigma \in W(\overline{\rho})$. We have a natural injection $\mathcal{W}_{2,\widetilde{\sigma}} \hookrightarrow D_{i_0,\sigma}$ and $\mathrm{JH}(D_{i_0,\sigma}/\mathcal{W}_{2,\widetilde{\sigma}}) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset$. Moreover,*

$$\mathrm{soc}_\Gamma(D_{i_0,\sigma}) \cong \bigoplus_{\tau\in W(\overline{\rho}^{\mathrm{ss}}),\ell(\tau)=\max\{i_0+1,\ell(\sigma)\},J_\sigma=J_{\overline{\rho}}\cap J_\tau} \tau. \qquad (70)$$

(iii) *Fix $\sigma \in W(\overline{\rho})$ and let $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$. Suppose that $0 \to D_{i_0,\sigma} \to V \to \tau' \to 0$ is a nonsplit extension of Γ-representations and $V' \subseteq V$, $V' \not\subseteq D_{i_0,\sigma}$ is any subrepresentation with cosocle τ'. If $[V' : \tau'] = 1$, then $\mathrm{rad}_\Gamma(V')$ is semisimple and contained in $\mathcal{W}_{2,\widetilde{\sigma}} \subseteq D_{i_0,\sigma}$. If $[V' : \tau'] = 2$, then $\tau' \cong \sigma$ and $\ell(\sigma) \ge i_0 + 1$.*

Proof. (i) Recall that $D_0(\overline{\rho}) = \bigoplus_{\sigma\in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})$, compatibly with filtrations. As $D_0(\overline{\rho})_{i_0+1} \hookrightarrow D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}$, and the injection respects the direct sum decompositions (69), the direct sum decomposition claimed in (i) follows.

We now prove the last claim of (i). If $\ell(\sigma) \ge i_0 + 1$, then $D_{0,\sigma}(\overline{\rho})_{\le i_0} = 0$, and $D_{0,\sigma}(\overline{\rho})_{(i_0+1)} = D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}})_{(i_0+1)} = 0$ if $\ell(\sigma) > i_0 + 1$. If $\ell(\sigma) = i_0 + 1$, then it follows from the definitions of $D_{0,\sigma}(\overline{\rho})_{\le i_0+1}$ and of $D_0(\overline{\rho})_{\le i_0+1}$ (cf. [BHH$^+$, § 4.3.4]), and from the inclusion $D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}}) \subseteq D_{0,\sigma}(\overline{\rho})$ (cf. the introduction to [BHH$^+$, § 4]) that $D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}}) \subseteq D_{0,\sigma}(\overline{\rho})_{\le i_0+1}$, so $D_{0,\sigma}(\overline{\rho})_{(i_0+1)} = D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}})_{(i_0+1)}(= D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}}))$.

(ii) Consider the diagram of Γ-representations

$$\begin{array}{ccccc} \mathcal{W} & \hookrightarrow & D_0(\overline{\rho}) & & \\ \downarrow & & \downarrow & & \\ \mathcal{W}_2 & \hookrightarrow & D_0(\overline{\rho})/D_0(\overline{\rho})_{\le i_0} & \hookrightarrow & D_{i_0} \end{array}$$

Each term in this diagram has a natural direct sum decomposition indexed by $\sigma \in W(\overline{\rho})$. The top horizontal arrow preserves the decompositions because $D_0(\overline{\rho})$ is multiplicity free and as $\mathrm{JH}(\mathcal{W}_\sigma) \subseteq \mathrm{JH}(D_{0,\sigma}(\overline{\rho}))$ for all $\sigma \in W(\overline{\rho})$ (Lemma 6.3.2), which implies that the bottom horizontal map preserves the decompositions. The property $\mathrm{JH}(D_{i_0,\sigma}/\mathcal{W}_{2,\widetilde{\sigma}}) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset$ then follows from Lemma 6.3.6(iv). By Lemma 6.3.6(iii) and (iv) we know that $\mathrm{soc}_\Gamma(\mathcal{W}_2) = \mathrm{soc}_\Gamma(D_{i_0})$, and hence $\mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}}) = \mathrm{soc}_\Gamma(D_{i_0,\sigma})$ for each σ. Formula (70) follows from Lemma 6.3.3(iii).

(iii) As V is a nonsplit extension, we have $\mathrm{soc}_\Gamma(V) = \mathrm{soc}_\Gamma(D_{i_0,\sigma})$ hence $\mathrm{JH}(\mathrm{soc}_\Gamma(V)) \subseteq \mathrm{JH}(\mathrm{soc}_\Gamma(D_{i_0})) \subseteq W(\overline{\rho}^{\mathrm{ss}})$ by Lemma 6.3.6(iii). Since $D_{i_0,\sigma}$ is multiplicity free by Lemma 6.3.6(i), for each $\tau \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$ there is a unique largest quotient V'_τ of V' with socle τ (and cosocle τ').

We first show that

$$[\mathrm{soc}_\Gamma(V') : \tau'] = 0 \ \Leftrightarrow\ [V' : \tau'] = 1 \ \Rightarrow\ \mathrm{rad}_\Gamma(V') \subseteq \mathcal{W}_{2,\widetilde{\sigma}} \qquad (71)$$
$$\Rightarrow\ \mathrm{rad}_\Gamma(V') \text{ is semisimple.}$$

If $[\mathrm{soc}_\Gamma(V') : \tau'] = 0$ then $[V' : \tau] = 1$ for each $\tau \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$ as $D_{i_0,\sigma}$ is multiplicity free, hence $V'_\tau \cong I(\tau, \tau')$ by [BP12, Cor. 3.12]. As V' injects into $\bigoplus_{\tau\in\mathrm{JH}(\mathrm{soc}_\Gamma(V'))} V'_\tau$, we deduce that $\mathrm{rad}_\Gamma(V')$ injects into $\bigoplus_{\tau\in\mathrm{JH}(\mathrm{soc}_\Gamma(V'))} \mathrm{rad}_\Gamma(V'_\tau)$, so $[\mathrm{rad}_\Gamma(V') : \tau'] = 0$ as $[\mathrm{rad}_\Gamma(V'_\tau) : \tau'] = 0$ for all τ, i.e. $[V' : \tau'] = 1$. The converse of the first implication in (71) is obvious. Still assuming $[\mathrm{soc}_\Gamma(V') : \tau'] = 0$, by Lemma 2.1.1 we obtain $\mathrm{JH}(V'_\tau) \subseteq W(\overline{\rho}^{\mathrm{ss}})$ for all $\tau \in \mathrm{JH}(\mathrm{soc}_\Gamma(V'))$, thus $\mathrm{JH}(V') \subseteq W(\overline{\rho}^{\mathrm{ss}})$. This implies that $\mathrm{rad}_\Gamma(V') \subseteq \mathcal{W}_{2,\widetilde{\sigma}}$ by Lemma 6.3.6(iv), so $\mathrm{rad}_\Gamma(V')$ is semisimple by Lemma 6.3.4(ii) (applied to the pushout of $0 \to \mathrm{rad}_\Gamma(V') \to V' \to \tau' \to 0$ along the injection $\mathrm{rad}_\Gamma(V') \subseteq \mathcal{W}_{2,\widetilde{\sigma}}$, which is still nonsplit, as it is contained in V which is nonsplit by assumption).

To conclude, it suffices to show that $\tau' \not\cong \sigma$ or $\ell(\sigma) \le i_0$ imply $[\mathrm{soc}_\Gamma(V') : \tau'] = 0$, hence $[V' : \tau'] = 1$. If $\tau' \not\cong \sigma$ and $\ell(\sigma) \ge i_0+1$, then $D_{i_0,\sigma} = D_{0,\sigma}(\overline{\rho})$ by part (i), so $\mathrm{soc}_\Gamma(V') = \sigma$ and we are done. If $\ell(\sigma) \le i_0$, assume by contradiction that $\tau' \subseteq \mathrm{soc}_\Gamma(V')$. Then $\tau' \subseteq \mathrm{soc}_\Gamma(V') \subseteq D_{i_0,\sigma}$, so $\tau' \subseteq \mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}})$ by part (ii), so $\tau' \notin W(\overline{\rho})$ and $\ell(\tau') = i_0 + 1$ by Lemma 6.3.3(iii). By [HW22, Cor. 2.32], $V'_{\tau'}$ has 3 socle layers, of the shape

$$\tau' - \mathrm{soc}_1(V'_{\tau'}) - \tau',$$

and $\mathrm{soc}_1(V'_{\tau'})$ contains at least one element of $W(\overline{\rho}^{\mathrm{ss}})$ by Lemmas 2.2.2 and 2.2.3, say τ''. Since $\tau', \tau'' \in W(\overline{\rho}^{\mathrm{ss}})$ and $\tau' - \tau''$ is a subquotient of $D_{i_0,\sigma}$ (hence of $\mathcal{W}_{2,\widetilde{\sigma}}$ by part (ii) and Lemma 6.3.6(iv)), we have $J_{\tau''} = J_{\tau'} \sqcup \{j''\}$ for some $j'' \notin J_{\overline{\rho}}$ by Lemma 6.3.3(ii). On the other hand, as $\tau' \notin W(\overline{\rho})$ there exists $j' \in J_{\tau'} \setminus J_{\overline{\rho}}$ (so $j' \ne j''$). Let $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ be the element corresponding to $J_{\tau''} \setminus \{j'\}$. Then $\ell(\tau) = i_0 + 1$ and the extension $\tau - \tau''$ occurs in V' as a subquotient by Lemma 6.3.3(ii). By part (ii) and Lemma 6.3.3(iii) we have $\tau \subseteq \mathrm{soc}_\Gamma(\mathcal{W}_{2,\widetilde{\sigma}}) \subseteq \mathrm{soc}_\Gamma(D_{i_0,\sigma})$. Thus τ occurs in $\mathrm{soc}_\Gamma(V')$ (using that $D_{i_0,\sigma}$ containing $\mathrm{rad}_\Gamma(V')$ is multiplicity free). As $\tau \ne \tau'$ we have $[V' : \tau] = 1$ so τ occurs in $\mathrm{soc}_\Gamma(V')$. As $\tau \ne \tau'$, V' has quotient V'_τ which is isomorphic to $I(\tau, \tau')$ (arguing as in the case $[\mathrm{soc}_\Gamma(V') : \tau'] = 0$). By Lemma 2.1.1 and as $J_\tau = (J_{\tau'} \sqcup \{j''\}) \setminus \{j'\}$, it has length 4 and a constituent $\tau''' \in W(\overline{\rho}^{\mathrm{ss}})$, with $J_{\tau'''} = J_\tau \setminus \{j''\}$ in $\mathrm{soc}_1(I(\tau, \tau'))$. But this implies that the nonsplit extension $\tau - \tau'''$ occurs in $D_{i_0,\sigma}$ (hence in $\mathcal{W}_{2,\widetilde{\sigma}}$), which is impossible by Lemma 6.3.3(ii). Hence $[\mathrm{soc}_\Gamma(V') : \tau'] = 0$, as desired. □

Proof of Theorem 6.3.1. If $i_0 = f$ this is trivial (both sides are zero). If $i_0 = f - 1$, then as in Remark 6.1.4 we have $\pi_2 \cong \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_1 \otimes \chi_2\omega^{-1})$, where $\overline{\rho} \cong \left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$. Theorem 6.3.1 boils down to showing $\mathrm{Ind}_I^K(\chi_1 \otimes \chi_2\omega^{-1}) \cong D_0(\overline{\rho}^{\mathrm{ss}})_f$. This is true by [BP12, Rk. 14.9(i)]. From now on we will assume that $i_0 \le f - 2$. Furthermore, if $i_0 = -1$ the result is just assumption (i) from Section 3.1, so we will assume $i_0 \ge 0$ and so $f \ge 2$.

Recall that $D_{i_0} \subseteq \pi_2^{K_1}$ and that both representations have the same I_1-invariants and the same $\mathrm{GL}_2(\mathcal{O}_K)$-socle by Lemma 6.3.6(ii), (iii).

For a contradiction, assume $D_{i_0} \subsetneq \pi_2^{K_1}$ with Q being the quotient. Pick a Serre weight τ' which injects into Q. Then we obtain an extension of Γ-representations

$$0 \to D_{i_0} \to V \to \tau' \to 0,$$

which is nonsplit by the above discussion, and moreover $V \subseteq \pi_2^{K_1}$. In other words, V defines a nonzero element in $\mathrm{Ext}^1_\Gamma(\tau', D_{i_0})$. Hence, by Corollary 6.3.7, we have $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$.

For $\sigma \in W(\overline{\rho})$ let V_σ denote the quotient of V defined as the pushout of $V \hookleftarrow D_{i_0} \twoheadrightarrow D_{i_0,\sigma}$, so that $V \hookrightarrow \bigoplus_{\sigma \in W(\overline{\rho})} V_\sigma$ (recall $D_{i_0} = \bigoplus_{\sigma \in W(\overline{\rho})} D_{i_0,\sigma}$ from Lemma 6.3.8(i)). (We caution the reader that, despite the notation, $\sigma \notin \mathrm{JH}(V_\sigma)$ in general.) If there exists $\sigma' \in W(\overline{\rho})$ such that $\tau' \in \mathrm{JH}(D_{i_0,\sigma'})$ (there can be at most one such σ' by Lemma 6.3.6(i)) and $[V_{\sigma'}] = 0$ in $\mathrm{Ext}^1_\Gamma(\tau', D_{i_0,\sigma'})$, then choose any splitting $s : \tau' \hookrightarrow V_{\sigma'}$ and let V' be the image of any morphism $\mathrm{Proj}_\Gamma\, \tau' \to V$ whose composition with $V \twoheadrightarrow V_{\sigma'}$ is the map $\mathrm{Proj}_\Gamma\, \tau' \twoheadrightarrow \tau' \xrightarrow{s} V_{\sigma'}$. Otherwise, let $V' \subseteq V$, $V' \not\subseteq D_{i_0}$ denote any subrepresentation with cosocle τ'. In either case, $V' \subseteq V$, $V' \not\subseteq D_{i_0}$ and $\mathrm{cosoc}_\Gamma(V') \cong \tau'$. Moreover, $\mathrm{rad}_\Gamma(V') \subseteq \mathrm{rad}_\Gamma(V) \subseteq D_{i_0}$ is multiplicity free and

$$\mathrm{JH}(\mathrm{soc}_\Gamma(V')) \subseteq \mathrm{JH}(\mathrm{soc}_\Gamma(D_{i_0})) \subseteq W(\overline{\rho}^{\mathrm{ss}}) \tag{72}$$

by Lemma 6.3.6(i) and (iii).

For any $\sigma \in W(\overline{\rho})$ let V'_σ denote the image of $V' \hookrightarrow V \twoheadrightarrow V_\sigma$, so $\mathrm{cosoc}_\Gamma(V'_\sigma) \cong \tau'$ and $V'_\sigma \not\subseteq D_{i_0,\sigma}$. We show that

$$[V_\sigma] = 0 \text{ in } \mathrm{Ext}^1_\Gamma(\tau', D_{i_0,\sigma}) \text{ if and only if } V'_\sigma \cong \tau'. \tag{73}$$

If $[V_\sigma] = 0$ and $\tau' \in \mathrm{JH}(D_{i_0,\sigma})$, then $V'_\sigma \cong \tau'$ by construction in the preceding paragraph. If $[V_\sigma] = 0$ and $\tau' \notin \mathrm{JH}(D_{i_0,\sigma})$, then $V'_\sigma \cong \tau'$ (the unique subrepresentation of $D_{i_0,\sigma} \oplus \tau'$ with cosocle τ'). Conversely, the "if" direction of (73) is true, as $V'_\sigma \not\subseteq D_{i_0,\sigma}$ provides the splitting.

For later reference we show that

$$\mathrm{rad}_\Gamma(V') \cong \bigoplus_{\sigma \in W(\overline{\rho})} \mathrm{rad}_\Gamma(V'_\sigma). \tag{74}$$

By construction, $V' \hookrightarrow \bigoplus_{\sigma \in W(\overline{\rho})} V'_\sigma$, so $\mathrm{rad}_\Gamma(V') \hookrightarrow \bigoplus_{\sigma \in W(\overline{\rho})} \mathrm{rad}_\Gamma(V'_\sigma)$. As V' surjects onto V'_σ, we deduce $\mathrm{rad}_\Gamma(V')$ surjects onto $\mathrm{rad}_\Gamma(V'_\sigma)$ for all σ. Since the $\mathrm{rad}_\Gamma(V'_\sigma) \subseteq D_{i_0,\sigma}$ for $\sigma \in W(\overline{\rho})$ have disjoint constituents, it follows that the injection $\mathrm{rad}_\Gamma(V') \hookrightarrow \bigoplus_{\sigma \in W(\overline{\rho})} \mathrm{rad}_\Gamma(V'_\sigma)$ is an isomorphism.

We now distinguish cases.

Step 1. Assume $[V' : \tau'] = 1$. We will show that $\chi_{\tau'}$ contributes *twice* to $\pi_2[\mathfrak{m}^2]$, but this contradicts Corollary 6.1.8.

We claim that $\mathrm{rad}_\Gamma(V')$ is semisimple. By (74) it suffices to show that $\mathrm{rad}_\Gamma(V'_\sigma)$ is semisimple for any $\sigma \in W(\overline{\rho})$. If $[V_\sigma] = 0$, then $V'_\sigma \cong \tau'$ by (73) and we are done. If $[V_\sigma] \neq 0$ we deduce that $\mathrm{rad}_\Gamma(V'_\sigma)$ is semisimple by Lemma 6.3.8(iii) (using the assumption $[V' : \tau'] = 1$). This establishes the claim.

In the following three paragraphs we show that for any Γ-subrepresentation $V'' \subseteq \pi_2^{K_1}$ such that $\mathrm{cosoc}_\Gamma(V'') \cong \tau'$ and $\mathrm{rad}_\Gamma(V'')$ is semisimple there exist $\mathcal{J} \subseteq \{0, \dots, f-1\}$ and a map $\widetilde{f} : Q_{\mathcal{J}} = Q_{\mathcal{J}}(\tau') \to \pi_2^{K_1}$ such that $\Theta_{\mathcal{J}} = \Theta_{\mathcal{J}}(\tau')$ surjects onto V'', where $Q_{\mathcal{J}}(\tau')$ (resp. $\Theta_{\mathcal{J}}(\tau')$) was defined just before Lemma 2.3.5 (resp. Lemma 2.3.6).

Note that $\tau \in \mathrm{JH}(\mathrm{rad}_\Gamma(V''))$ implies $\mathrm{Ext}^1_\Gamma(\tau', \tau) \neq 0$ and $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ (by (72), as $\mathrm{rad}_\Gamma(V'') \subseteq \mathrm{soc}_\Gamma(V'')$). For $* \in \{\pm\}$ define $\mathcal{J}^* \stackrel{\mathrm{def}}{=} \{0 \le i \le f-1 : \mu_i^*(\tau') \hookrightarrow V''\}$, so that $\mathrm{rad}_\Gamma(V'') \cong \bigoplus_{i \in \mathcal{J}^+} \mu_i^+(\tau') \oplus \bigoplus_{i \in \mathcal{J}^-} \mu_i^-(\tau')$ by [BHH+, Lemma 4.3.4]. We note that $\mathcal{J}^+ \cap \mathcal{J}^- = \emptyset$ by Lemma 2.2.3.

Suppose that $i \in \mathcal{J}^+$ (or more generally that $\mu_i^+(\tau') \hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$). We claim that the nonsplit extension $\mu_i^+(\tau') - \delta_i^+(\tau')$ embeds into D_{i_0}. By Corollary 6.2.4 (applied to π_2) we have two cases. If $\mu_i^+(\tau') \in W(\overline{\rho})$ and $\ell(\mu_i^+(\tau')) \ge i_0 + 1$, then $\mu_i^+(\tau') - \delta_i^+(\tau')$ embeds into $D_{0,\mu_i^+(\tau')}(\overline{\rho})$ (by Lemma 2.2.3 and the definition of $D_{0,\mu_i^+(\tau')}(\overline{\rho})$), so into D_{i_0}. If $\mu_i^+(\tau') \in W(\overline{\rho}^{\mathrm{ss}}) \setminus W(\overline{\rho})$ and $\ell(\mu_i^+(\tau')) = i_0 + 1$, then $\mu_i^+(\tau') - \delta_i^+(\tau')$ similarly embeds into $D_{0,\mu_i^+(\tau')}(\overline{\rho}^{\mathrm{ss}}) \subseteq D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \subseteq D_{i_0}$. In either case we deduce from Lemma 2.2.1 (note $\mu_i^+(\tau') = \mu_i^-(\delta_i^+(\tau'))$) that

$$\chi_{\delta_i^+(\tau')} \in \mathrm{JH}(\pi_2^{I_1}) \quad \forall\, i \in \mathcal{J}^+, \tag{75}$$

and that

$$\chi_{\delta_i^+(\tau')} \in \mathrm{JH}(\pi^{I_1}) \quad \text{if } i \in \mathcal{J}^+ \text{ and } \mu_i^+(\tau') \in W(\overline{\rho}). \tag{76}$$

Let $\mathcal{J} \stackrel{\mathrm{def}}{=} \mathcal{J}^+ \sqcup \mathcal{J}^-$. Now we show that there exists a map $\widetilde{f} : Q_{\mathcal{J}} = Q_{\mathcal{J}}(\tau') \to \pi_2^{K_1}$ such that $\Theta_{\mathcal{J}} = \Theta_{\mathcal{J}}(\tau')$ surjects onto V''. Clearly there exists a surjection $f : \Theta_{\mathcal{J}} / \mathrm{soc}_{\widetilde{\Gamma}}(\Theta_{\mathcal{J}}) \twoheadrightarrow V''$, which is unique up to scalar (both sides are multiplicity free, with cosocle τ', and we know all their constituents). We show that $f : \Theta_{\mathcal{J}} \twoheadrightarrow \Theta_{\mathcal{J}} / \mathrm{soc}_{\widetilde{\Gamma}}(\Theta_{\mathcal{J}}) \twoheadrightarrow V'' \subseteq \pi_2^{K_1}$ can be extended to a map $Q_{\mathcal{J}} \to \pi_2^{K_1}$. By Lemma 2.3.6 (with $\Psi_i = \Psi_i(\tau')$) it suffices to show that the map $f|_{\mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i)}$ extends to Ψ_i for all $i \in \mathcal{J}$. (The extension is automatically unique, as $\delta_i^+(\tau') \not\hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$ by Lemma 2.2.3.) Note that $f|_{\mathrm{soc}_{\widetilde{\Gamma}}(\Psi_i)} = 0$, as $\mathrm{soc}_{\widetilde{\Gamma}}(\Psi_i) \subseteq \mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i) \subseteq \Theta_{\mathcal{J}}$ and $f(\mathrm{soc}_{\widetilde{\Gamma}}(\Theta_{\mathcal{J}})) = 0$. If $i \in \mathcal{J}^-$, $f|_{\mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i)} = 0$, as $\mu_i^+(\tau') \not\hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$ by Corollary 6.2.4 and Lemma 2.2.3, so the extension to Ψ_i is trivial. If $i \in \mathcal{J}^+$, $f|_{\mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i)}$ factors through an injection of $\mu_i^+(\tau')$ into $\pi_2^{K_1}$ and by above this extends to an injection of $\mu_i^+(\tau') - \delta_i^+(\tau')$ into $\pi_2^{K_1}$.

By Lemma 2.3.5 the map $\widetilde{f}$ gives rise to a homomorphism $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{J}} \twoheadrightarrow Q_{\mathcal{J}} \to \pi_2^{K_1}$, and by Frobenius reciprocity we get a map $\overline{f} : W_{\mathcal{J}} \to \pi_2^{K_1}|_I$. Since $\overline{f}$ factors through $W_{\mathcal{J}} \twoheadrightarrow (W_{\mathcal{J}})_{K_1}$ which is killed by $\mathfrak{m}^2$ by Lemma 2.3.4(ii), we get $\overline{f}(W_{\mathcal{J}}) \subseteq \pi_2[\mathfrak{m}^2]$. In particular, as $\mathrm{cosoc}_I(W_{\mathcal{J}}) \cong \chi_{\tau'}$, we see that $\chi_{\tau'}$ contributes to $\pi_2[\mathfrak{m}^2]$.

We now specialize to $V'' \stackrel{\mathrm{def}}{=} V'$, in which case $\overline{f}(W_{\mathcal{J}}) \not\subseteq D_{i_0}$. If $\overline{f}(W_{\mathcal{J}}) \subseteq \pi_2[\mathfrak{m}] = \pi_2^{I_1}$, then $\overline{f}(W_{\mathcal{J}}) \subseteq D_{i_0}^{I_1} \subseteq D_{i_0}$ by the first statement of Lemma 6.3.6(iii), contradiction. Since $\overline{f}(W_{\mathcal{J}})$ is K_1-invariant, by Lemma 2.3.4(ii) we deduce that $\mathrm{rad}_I(\overline{f}(W_{\mathcal{J}})) \subseteq \bigoplus_{i \in \mathcal{J}} \chi_{\tau'} \alpha_i$. If $\chi_{\tau'} \alpha_i \hookrightarrow \overline{f}(W_{\mathcal{J}}) \subseteq \pi_2$, then Frobenius reciprocity gives us a nonzero map $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\tau'} \alpha_i \to \mathrm{im}(\widetilde{f}) \subseteq \pi_2$, and hence $[\mathrm{im}(\widetilde{f}) : \delta_i^+(\tau')] \neq 0$. By construction, $[\mathrm{im}(\widetilde{f}) : \delta_i^+(\tau')] = 0$ for all $i \in \mathcal{J}^-$, so $\mathrm{rad}_I(\overline{f}(W_{\mathcal{J}})) \subseteq \bigoplus_{i \in \mathcal{J}^+} \chi_{\tau'} \alpha_i$. (In fact, we have $\mathrm{rad}_I(\overline{f}(W_{\mathcal{J}})) = \bigoplus_{i \in \mathcal{J}^+} \chi_{\tau'} \alpha_i$, but we will not need this.)

Choose now $i \in \mathcal{J}^+$ such that $\chi_{\tau'}\alpha_i \subseteq \overline{f}(W_{\mathcal{J}})$. Let $\lambda' \in \mathscr{D}^{\mathrm{ss}}$ correspond to $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$, and for $i \in \mathcal{J}^+$ we let $\lambda^{(i)} \in \mathscr{D}^{\mathrm{ss}}$ correspond to $\mu_i^+(\tau') \in W(\overline{\rho}^{\mathrm{ss}})$ and $\lambda'^{(i)} \in \mathscr{P}^{\mathrm{ss}}$ correspond to $\delta_i^+(\tau')$. (See § 1.4 for $\mathscr{D}^{\mathrm{ss}}$ and $\mathscr{P}^{\mathrm{ss}}$.) More precisely, note that

$$(\lambda'_i, \lambda_i^{(i)}, \lambda_i'^{(i)}) \in \Big\{(x_i, x_i+1, x_i+2), (p-3-x_i, p-2-x_i, p-1-x_i)\Big\},$$
$$\lambda'_{i-1} = p-2-\lambda_{i-1}^{(i)} = \lambda_{i-1}'^{(i)}, \qquad \text{and}$$
$$\lambda'_j = \lambda_j^{(i)} = \lambda_j'^{(i)} \qquad \forall\, j \notin \{i-1, i\}.$$

In particular, it follows that

$$J_{\lambda^{(i)}} = J_{\lambda'^{(i)}} = J_{\lambda'} \,\Delta\, \{i\}, \qquad \ell(\lambda^{(i)}) = \ell(\lambda'^{(i)}) = \ell(\lambda') \pm 1. \tag{77}$$

Recall that the I-representations $\Theta = \bigoplus_{\mu\in\mathscr{P}} \Theta_\mu$ and $\overline{\tau} = \bigoplus_{\mu\in\mathscr{P}} \overline{\tau}_\mu$ were defined just before Lemma 6.1.6 (taking $n = i_0 + 4 (\le f+2)$ and noting that by assumption $\overline{\rho}$ is $(2n-1)$-generic). Recall that in Step 2 of the proof of Theorem 6.1.1 we showed that the natural map (48) is an isomorphism. Equivalently, $\Theta[\mathfrak{m}^3] = \pi_2[\mathfrak{m}^3]$, and hence $\pi_2[\mathfrak{m}^2] = \bigoplus_{\mu\in\mathscr{P}} \Theta_\mu[\mathfrak{m}^2]$. Moreover, using the exact sequence $0 \to \overline{\tau}_\mu \cap \pi_1 \to \overline{\tau}_\mu \to \Theta_\mu \to 0$ and the dual of (47) (recalling that $F_{\mu,-i}\Theta_\mu^\vee = \mathfrak{m}^i\overline{\tau}_\mu^\vee \cap \Theta_\mu^\vee$, cf. (45)), we see that

$$\Theta_\mu[\mathfrak{m}^i] = \frac{\overline{\tau}_\mu[\mathfrak{m}^{i+d_\mu}]}{(\overline{\tau}_\mu \cap \pi_1)[\mathfrak{m}^{i+d_\mu}]}$$

for all $i \ge 0$ (recall that $d_\mu = \max\{i_0+1-\ell(\mu), 0\}$). By (75) we have $\chi_{\tau'}\alpha_i = \chi_{\lambda'^{(i)}} \in \mathrm{JH}(\pi_2^{I_1}) = \mathrm{JH}(\pi_2[\mathfrak{m}]) = \mathrm{JH}(\Theta[\mathfrak{m}])$, and moreover it occurs in $\overline{\tau}_{\mu^{(i)}}$, where $\mu^{(i)} \in \mathscr{P}$ is obtained from $\lambda'^{(i)}$ as μ is obtained from λ in [BHH$^+$, eq. (61)]. As the I-representation $\overline{\tau}$ is multiplicity free ([BHH$^+$, Cor. 2.4.3(ii)]) we deduce that $\chi_{\tau'}\alpha_i$ occurs in $\Theta_{\mu^{(i)}}[\mathfrak{m}]$. Since the nonsplit extension $\chi_{\tau'}\alpha_i - \chi_{\tau'}$ is a quotient of $\overline{f}(W_{\mathcal{J}}) \subseteq \pi_2[\mathfrak{m}^2] = \Theta[\mathfrak{m}^2]$, it follows again by multiplicity freeness of $\overline{\tau}$ that $\chi_{\tau'}$ occurs in the direct summand $\Theta_{\mu^{(i)}}[\mathfrak{m}^2]$ of $\Theta[\mathfrak{m}^2]$ as well. Dually, $\chi_{\tau'}^{-1}$ occurs in $\mathrm{gr}_{\mathfrak{m},-1}(\Theta_{\mu^{(i)}}^\vee)$.

Define $J_1^{(i)}$, $J_2^{(i)}$ exactly as in [BHH$^+$, eq. (60)] (with $\lambda'^{(i)}$ instead of λ) and let

$$\widetilde{J}_1^{(i)} \stackrel{\text{def}}{=} \{j \notin J_{\overline{\rho}} : \mu_j^{(i)} = p-1-x_j\} = \{j \notin J_{\overline{\rho}} : \lambda_j'^{(i)} \in \{p-3-x_j, p-1-x_j\}\},$$
$$\widetilde{J}_2^{(i)} \stackrel{\text{def}}{=} \{j \notin J_{\overline{\rho}} : \mu_j^{(i)} = x_j\} = \{j \notin J_{\overline{\rho}} : \lambda_j'^{(i)} \in \{x_j, x_j+2\}\},$$

so $J_1^{(i)} \subseteq \widetilde{J}_1^{(i)}$, $J_2^{(i)} \subseteq \widetilde{J}_2^{(i)}$, $\widetilde{J}_1^{(i)} \cap \widetilde{J}_2^{(i)} = \emptyset$. Note that $\ell(\mu^{(i)}) = \ell(\lambda'^{(i)}) - |J_1^{(i)}| - |J_2^{(i)}| = \ell(\lambda^{(i)}) - |J_1^{(i)}| - |J_2^{(i)}|$, where the first equality was noted just after [BHH$^+$, eq. (61)]. We claim that

$$d_{\mu^{(i)}} = |J_1^{(i)}| + |J_2^{(i)}|. \tag{78}$$

If $\mu_i^+(\tau') \notin W(\overline{\rho})$, then $\ell(\lambda^{(i)}) = \ell(\mu_i^+(\tau')) = i_0 + 1$. If $\mu_i^+(\tau') \in W(\overline{\rho})$ and $\ell(\lambda^{(i)}) \ge i_0 + 1$, then by above $\lambda'^{(i)} \in \mathscr{P}$ (as $\chi_{\delta_i^+(\tau')} \in \mathrm{JH}(\pi^{I_1})$ by (76)), so $\mu^{(i)} = \lambda'^{(i)}$ and $J_1^{(i)} = J_2^{(i)} = \emptyset$. The claim follows.

As noted just after [BHH$^+$, eq. (61)] we have $\chi_{\lambda'^{(i)}} = \chi_{\mu^{(i)}} \prod_{j\in J_1^{(i)}} \alpha_j^{-1} \prod_{j \in J_2^{(i)}} \alpha_j$, or equivalently (using $\chi_{\lambda'}\alpha_i = \chi_{\lambda'^{(i)}}$):

$$\chi_{\lambda'}^{-1} = \chi_{\mu^{(i)}}^{-1}\alpha_i \prod_{j\in J_1^{(i)}} \alpha_j \prod_{j\in J_2^{(i)}} \alpha_j^{-1}. \tag{79}$$

Recall from (46) and (47) that $\mathrm{gr}_{\mathfrak{m},-1}(\Theta_{\mu^{(i)}}^\vee) \cong \chi_{\mu^{(i)}}^{-1} \otimes \mathrm{gr}_{-1-d_{\mu^{(i)}}}(\mathfrak{a}_1^{i_0}(\mu^{(i)})/\mathfrak{a}(\mu^{(i)}))$ (using $-1-d_{\mu^{(i)}} > -n$), and that

$$\frac{\mathfrak{a}_1^{i_0}(\mu^{(i)})}{\mathfrak{a}(\mu^{(i)})} \cong \frac{I(\widetilde{J}_1^{(i)}, \widetilde{J}_2^{(i)}, d_{\mu^{(i)}})}{I(\widetilde{J}_1^{(i)}, \widetilde{J}_2^{(i)}, d_{\mu^{(i)}}) \cap \mathfrak{a}(\mu^{(i)})} \tag{80}$$

by [BHH$^+$, eq. (77)]. As $\chi_{\lambda'}^{-1}$ occurs in $\mathrm{gr}_{\mathfrak{m},-1}(\Theta_{\mu^{(i)}}^\vee)$, we deduce that there exists a monomial $m \in I(\widetilde{J}_1^{(i)}, \widetilde{J}_2^{(i)}, d_{\mu^{(i)}})$, $m \notin \mathfrak{a}(\mu^{(i)})$ of degree $d_{\mu^{(i)}}+1$ that has H-eigencharacter $\chi_{\lambda'}^{-1}\chi_{\mu^{(i)}}$. By (79) the monomial

$$m' \overset{\text{def}}{=} \begin{cases} y_i \prod_{j\in J_1^{(i)}} y_j \prod_{j\in J_2^{(i)}} z_j & \text{if } i \notin J_2^{(i)}, \\ \prod_{j\in J_1^{(i)}} y_j \prod_{j\in J_2^{(i)}\setminus\{i\}} z_j & \text{if } i \in J_2^{(i)} \end{cases}$$

has the same H-eigencharacter. Since in addition m and m' have degree at most 2 in each variable (by [BHH$^+$, Def. 4.2.4] in the first case) and are not multiples of $y_j z_j$ for any j (as $m \notin \mathfrak{a}(\mu^{(i)})$ in the first case), we deduce using $p-1>4$ that $m = m'$. Hence m' has degree $d_{\mu^{(i)}}+1$. Using (78) it follows that $i \notin J_2^{(i)}$ and $m = y_i \prod_{j\in J_1^{(i)}} y_j \prod_{j \in J_2^{(i)}} z_j$. If $i \in J_{\overline{\rho}}$, then $\mu_i^{(i)} = \lambda_i'^{(i)} \in \{x_i+2, p-1-x_i\}$ (by [BHH$^+$, eq. (60), eq. (61)]) and hence $t_i = y_i$ (where $t_i = t_i(\mu^{(i)})$ is defined in (13)), so m is a multiple of $t_i \in \mathfrak{a}(\mu^{(i)})$, contradiction. Hence $i \in J_{\overline{\rho}}^c$, and we deduce moreover that $\lambda_i'^{(i)} \neq x_i+2$ from $i \notin J_2^{(i)}$ and [BHH$^+$, eq. (60)].

We are left with the case where $\lambda_i'^{(i)} = p-1-x_i$ and $i \in J_{\overline{\rho}}^c$. In this case $J_{\tau'} = J_{\mu_i^+(\tau')} \sqcup \{i\}$ by (77) since $\lambda'_i = p-3-x_i$, hence $\tau' \notin W(\overline{\rho})$ (as $i \in J_{\overline{\rho}}^c$). Moreover, as $\mu_i^+(\tau') \hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$, we have $\ell(\tau') = \ell(\mu_i^+(\tau'))+1 > i_0+1$, so $\tau' \in \mathrm{JH}(\mathcal{W}_2)$, so $\tau' \in \mathrm{JH}(\mathcal{W}_{2,\widetilde{\sigma}})$, where $\sigma \in W(\overline{\rho})$ is determined by $J_\sigma = J_{\overline{\rho}} \cap J_{\tau'}$ (Lemma 6.3.3(i)). We claim that the unique subrepresentation W' of $\mathcal{W}_{2,\widetilde{\sigma}} \subseteq \mathcal{W}_2 \subseteq D_{i_0}$ having cosocle τ' has semisimple radical. We have two cases by Corollary 6.2.4. If $\mu_i^+(\tau') \in W(\overline{\rho})$ and $\ell(\mu_i^+(\tau')) \geq i_0+1$, then $\sigma \cong \mu_i^+(\tau')$ and $W' \cong (\mu_i^+(\tau') - \tau')$ by Lemma 6.3.3(i), (iii). If $\mu_i^+(\tau') \in W(\overline{\rho}^{\mathrm{ss}})$ and $\ell(\mu_i^+(\tau')) = i_0+1$, then $\ell(\sigma) = |J_{\overline{\rho}} \cap J_{\tau'}| \leq |J_{\mu_i^+(\tau')}| = i_0+1$ (using $i \in J_{\overline{\rho}}^c$) and $\ell(\tau') = i_0+2$, so $\mathrm{rad}_\Gamma(W')$ is semisimple by Lemma 6.3.3(ii), (iii), proving the claim. By the third paragraph of Step 1 (applied with $V'' = V'$, resp. W'), we obtain two morphisms $Q_{\mathcal{S}} \to \pi_2^{K_1}$ that are linearly independent, as $V' \nsubseteq D_{i_0}$ and $W' \subseteq D_{i_0}$. (Note that the subset $\mathcal{J} \subseteq \mathcal{S}$ may differ in the two cases, but $Q_{\mathcal{S}}$ surjects onto any $Q_{\mathcal{J}}$, and likewise $W_{\mathcal{S}}$ surjects onto any $W_{\mathcal{J}}$.) Since we showed that the maps $\overline{f} : W_{\mathcal{S}} \to \pi_2$ corresponding to $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{S}} \twoheadrightarrow Q_{\mathcal{S}} \to \pi_2^{K_1}$ have images contained in $\pi_2[\mathfrak{m}^2]$, we deduce that $\chi_{\tau'}$ contributes *twice* to $\pi_2[\mathfrak{m}^2]$, as we wanted to show.

Step 2. Assume $[V' : \tau'] = 2$. We will show that $\chi_{\tau'}$ contributes *twice* to $\pi_2[\mathfrak{m}^3]$, but this contradicts Corollary 6.1.8. (Note that $2n+2i_0+1 \leq 6+2(f-2)+1 = 2f+3$ when $n=3$ and as we assumed $i_0 \leq f-2$.)

From (74) and since $\mathrm{rad}_\Gamma(V') \subseteq \mathrm{rad}_\Gamma(V) \subseteq D_{i_0}$ is multiplicity free we deduce that $\tau' \in \mathrm{JH}(\mathrm{rad}_\Gamma(V'_\sigma))$ for a unique $\sigma \in W(\overline{\rho})$. By (73) and the last statement of Lemma 6.3.8(iii) (applied to $V'_\sigma \subseteq V_\sigma$) it follows that $\tau' \cong \sigma \in W(\overline{\rho})$ and $\ell(\tau') \geq i_0 + 1$, hence $[V'_{\tau'} : \tau'] = 2$. Using $\mathrm{rad}_\Gamma(V'_{\tau'}) \subseteq D_{i_0,\tau'}$ and $\ell(\tau') \geq i_0+1$ it follows moreover from Lemma 6.3.8(i) that $\mathrm{soc}_\Gamma(V'_{\tau'}) = \tau'$, and hence $\tau' \hookrightarrow \mathrm{rad}_\Gamma(V')$ by (74). The natural surjection $\mathrm{rad}_\Gamma(V') \twoheadrightarrow \mathrm{rad}_\Gamma(V'/\tau')$ induces an isomorphism $\mathrm{rad}_\Gamma(V')/\tau' \xrightarrow{\sim} \mathrm{rad}_\Gamma(V'/\tau')$. We apply again (74) to deduce that

$$\mathrm{rad}_\Gamma(V')/\tau' \cong \bigoplus_{\sigma \neq \tau'} \mathrm{rad}_\Gamma(V'_\sigma) \oplus (\mathrm{rad}_\Gamma(V'_{\tau'})/\tau'). \tag{81}$$

As in the second paragraph of Step 1, $\mathrm{rad}_\Gamma(V'_\sigma)$ is semisimple for all $\sigma \neq \tau'$, and $\mathrm{rad}_\Gamma(V'_{\tau'})/\tau'$ is semisimple by Lemma 2.2.2. In conclusion, $\mathrm{rad}_\Gamma(V'/\tau')$ is semisimple. As $\mathrm{cosoc}_\Gamma(V'/\tau') \cong \tau'$ we can write, as in the fourth paragraph of Step 1,

$$\bigoplus_{\sigma \neq \tau'} \mathrm{rad}_\Gamma(V'_\sigma) \cong \bigoplus_{i \in \mathcal{J}^+} \mu_i^+(\tau') \oplus \bigoplus_{i \in \mathcal{J}^-} \mu_i^-(\tau') \tag{82}$$

and (using moreover Lemma 2.2.2),

$$\mathrm{rad}_\Gamma(V'_{\tau'})/\tau' \cong \bigoplus_{i \in \mathcal{J}'} (\mu_i^+(\tau') \oplus \mu_i^-(\tau')) \tag{83}$$

for some subsets $\mathcal{J}^+$, $\mathcal{J}^-$, $\mathcal{J}'$ of $\{0, 1, \dots, f-1\}$. For $i \in \mathcal{J}^*$, $* \in \{\pm\}$ we have $\mu_i^*(\tau') \hookrightarrow V' \hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$ by (74). Therefore $\mathcal{J}^+ \cap \mathcal{J}^- = \emptyset$ by Corollary 6.2.4 and Lemma 2.2.3, and note that $(\mathcal{J}^+ \sqcup \mathcal{J}^-) \cap \mathcal{J}' = \emptyset$ as $\mathrm{rad}_\Gamma(V') \subseteq D_{i_0}$ is multiplicity free. Let $\mathcal{J} \overset{\mathrm{def}}{=} \mathcal{J}' \sqcup \mathcal{J}^+ \sqcup \mathcal{J}^-$, and write $Q_\mathcal{J} = Q_\mathcal{J}(\tau')$, $\Theta_\mathcal{J} = \Theta_\mathcal{J}(\tau')$, $\Psi_i = \Psi_i(\tau')$ as in Step 1.

We show that $\dim_\mathbb{F} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_\mathcal{J}, \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \geq 2$. It suffices to show that $\dim_\mathbb{F} \mathrm{Hom}_{\widetilde{\Gamma}}(\Theta_\mathcal{J}, V') \geq 2$. As we have a trivial map $\Theta_\mathcal{J} \twoheadrightarrow \tau' \hookrightarrow V'$, it suffices to find a map whose image is not irreducible. We follow the argument in [HW22, Cor. 3.14]. By (81), (82), and (83) there exists a surjection $f : \Theta_\mathcal{J}/\mathrm{soc}_{\widetilde{\Gamma}}(\Theta_\mathcal{J}) \twoheadrightarrow V'/\tau'$ (just as in the sixth paragraph of Step 1). The same proof as in [HW22, Cor. 3.13] shows that $\mathrm{Ext}^1_{\widetilde{\Gamma}}(\Theta_\mathcal{J}, \tau') = 0$, so we can lift f to $\widetilde{f} : \Theta_\mathcal{J} \to V'$ whose image is not contained in $\tau' \subseteq V'$, as desired.

We show that the restriction map

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(Q_\mathcal{J}, \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_\mathcal{J}, \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \tag{84}$$

is surjective (even an isomorphism). By Lemma 2.3.6 it suffices to show that the restriction map

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Psi_i, \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i), \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \tag{85}$$

is an isomorphism for any $i \in \mathcal{J}$. The map (85) is injective, as $\Psi_i \cong (\tau' - \mu_i^+(\tau') - \delta_i^+(\tau'))$ and $\delta_i^+(\tau') \not\hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$ by Lemma 2.2.3, so it suffices to show that the map (85) is surjective. By Lemma 2.2.1, $\mathrm{rad}_{\widetilde{\Gamma}}(\Psi_i) \cong (\tau' - \mu_i^+(\tau'))$ is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\mu_i^+(\tau')}$ (note that $\mu_i^-(\mu_i^+(\tau')) = \tau'$ as $f \geq 2$), so by Lemma 6.3.6(iii) (and Frobenius reciprocity) we see that the right-hand side of (85) is at most 1-dimensional. It thus suffices to show that $\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Psi_i, \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \neq 0$ for all $i \in \mathcal{J}$.

Suppose $i \in \mathcal{J}$. If $\mu_i^+(\tau') \hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$, then the nonsplit extension $\mu_i^+(\tau') - \delta_i^+(\tau')$ embeds into $D_{i_0} \subseteq \pi_2^{K_1}$ exactly as in the fifth paragraph of Step 1, so we are done. Otherwise, $i \in \mathcal{J}^-$ and hence $\mu_i^-(\tau') \in W(\overline{\rho}^{\mathrm{ss}})$. As $\tau' \in W(\overline{\rho})$ and $\ell(\tau') \geq i_0 + 1$ (see the second paragraph of Step 2), the uniserial representation $\Psi_i = (\tau' - \mu_i^+(\tau') - \delta_i^+(\tau'))$ injects into $\widetilde{D}_0(\overline{\rho})$ (by the definition of $\widetilde{D}_0(\overline{\rho})$ in § 1.4, noting that $\mu_i^+(\tau'), \delta_i^+(\tau') \notin W(\overline{\rho})$ by Lemma 2.2.3) and even into $\widetilde{D}_0(\overline{\rho})/\widetilde{D}_0(\overline{\rho})_{\leq i_0} \hookrightarrow \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}$, where this last injection comes from Corollary 6.2.6. We have proved that (84) is an isomorphism.

As $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{J}}$ surjects onto $Q_{\mathcal{J}}$ by Lemma 2.3.5, we deduce from the surjectivity of (84) and from $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_{\mathcal{J}}, \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \geq 2$ that

$$\dim_{\mathbb{F}} \mathrm{Hom}_I(W_{\mathcal{J}}, \pi_2|_I) = \dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\mathcal{J}}, \pi_2|_{\mathrm{GL}_2(\mathcal{O}_K)}) \geq 2.$$

As $\mathrm{cosoc}_I(W_{\mathcal{J}}) = \chi_{\tau'}$ and $W_{\mathcal{J}}$ is killed by $\mathfrak{m}^3$, it follows that $\chi_{\tau'}$ occurs at least twice in $\pi_2[\mathfrak{m}^3]$, as we wanted to show. □

Corollary 6.3.9. *Assume that $\overline{\rho}$ is $\max\{9, 2f+3\}$-generic. Suppose $\pi' = \pi_1'/\pi_1$ is any nonzero subquotient, where $\pi_1 \subsetneq \pi_1' \subseteq \pi$. Let $i_0 \overset{\mathrm{def}}{=} i_0(\pi_1)$, $i_0' \overset{\mathrm{def}}{=} i_0(\pi_1')$, so $-1 \leq i_0 < i_0' \leq f$. Then*

$$\pi'^{K_1} \cong D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \oplus_{D_0(\overline{\rho})_{i_0+1}} (D_0(\overline{\rho})_{\leq i_0'}/D_0(\overline{\rho})_{\leq i_0}).$$

Proof. Note that π'^{K_1} is the kernel of the natural map $(\pi/\pi_1)^{K_1} \to (\pi/\pi_1')^{K_1}$. Let us write again $D_{i_0} \overset{\mathrm{def}}{=} D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \oplus_{D_0(\overline{\rho})_{i_0+1}} (D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0})$. By Theorem 6.3.1 the above natural map is identified with a map

$$\theta : D_{i_0} \to D_{i_0'}.$$

Let $\theta_0 \overset{\mathrm{def}}{=} \theta|_{D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0}}$. We have $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi/\pi_1') \subseteq \{\sigma \in W(\overline{\rho}^{\mathrm{ss}}) : \ell(\sigma) \geq i_0' + 1\}$ by Corollary 6.2.4, and $\mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1}) \cap W(\overline{\rho}^{\mathrm{ss}})$ is disjoint from that set since $i_0 < i_0'$, so $D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \subseteq \ker(\theta)$ and hence $\ker(\theta) = D_0(\overline{\rho}^{\mathrm{ss}})_{i_0+1} \oplus_{D_0(\overline{\rho})_{i_0+1}} \ker(\theta_0)$. On the other hand, by comparison with $\pi^{K_1} = D_0(\overline{\rho})$ we see that θ_0 is the natural surjection $D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0} \twoheadrightarrow D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0'}$. The result follows. □

We can now extend [Wan, Thm. 1.2] to subquotients.

Corollary 6.3.10. *Keep the notation and assumptions of Corollary 6.3.9. Then we have*

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = |\,\mathrm{JH}(\pi'^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})| = \sum_{i_0 < i \leq i_0'} \binom{f}{i}.$$

Proof. By exactness of $D_\xi^\vee$ and [BHH+, Cor. 4.4.2] we know that the two outside terms are equal. By Corollary 6.3.9 (using (68), [BHH+, eq. (67)] and that $W(\overline{\rho}^{\mathrm{ss}}) \subseteq \mathrm{JH}(D_0(\overline{\rho}))$), we deduce that $\mathrm{JH}(\pi'^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}}) = \{\sigma \in W(\overline{\rho}^{\mathrm{ss}}) : i_0 < \ell(\sigma) \leq i_0'\}$, and the result follows. □

7 Global arguments

In this section we prove that certain globally defined smooth mod p representations of $\mathrm{GL}_2(K)$ satisfy assumption (v) of § 3 (besides assumptions (i)–(iv) of § 3).

7.1 Global setting

We define smooth mod p representations of $\mathrm{GL}_2(K)$ that arise from the mod p étale cohomology of suitable Shimura curves, and recall why they satisfy assumptions (i)–(iv) of § 3. We then show in § 7.2 below that they furthermore satisfy assumption (v).

Let F be a totally real number field in which p is unramified, and let S_p denote the set of places of F above p. For each finite place w of F we denote by F_w the completion of F at w. We fix a quaternion algebra D over F, with center F such that D splits at all places in S_p and at exactly one infinite place. We let S_D denote the set of places of F at which D ramifies.

We fix a continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ and define $S_{\overline{r}}$ to be the set of places where $\overline{r}$ ramifies. We write $\overline{r}_w$ for $\overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$. We assume that:

- $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible;
- if $p = 5$, then the image of $\overline{r}(\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1})))$ in $\mathrm{PGL}_2(\mathbb{F})$ is not isomorphic to A_5;
- for all $w \in S_p$, $\overline{r}_w$ is 0-generic;
- for all $w \in S_D$, $\overline{r}_w$ is non-scalar.

We now fix $v \in S_p$ and let $\psi : G_F \to W(\mathbb{F})^\times$ be the Teichmüller lift of $\omega \det(\overline{r})$. Following [EGS15, § 6.5] with the corrections of [BHH+23, Rk. 8.1.3] (see also [BD14, § 3.3, § 3.4]) we have a compact open subgroup U^v of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$ and a smooth representation of $U^v(\mathbb{A}_F^{\infty,v})^\times$ on a finite-dimensional $\mathbb{F}$-vector space which we denote by $\overline{M}^v$, and on which $(\mathbb{A}_F^{\infty,v})^\times$ acts by ψ^{-1}. (Following the notation of [EGS15, § 6.5] this is the mod p-reduction of the inflation to $\prod_{w \in S \setminus \{v\}} K_w \prod_{w \notin S \cup \{v\}} (\mathcal{O}_D)_w^\times$ of the $\prod_{w \in S \setminus \{v\}} K_w$-representation L over $W(\mathbb{F})$ of *loc. cit.*, where furthermore $(\mathbb{A}_F^{\infty,v})^\times$ acts via ψ^{-1}. Again, the representation L should be corrected following [BHH+23, Rk. 8.1.3(i)], in particular ψ in [EGS15, § 6.5] should be replaced everywhere by its inverse.) We set $\overline{\rho} \overset{\mathrm{def}}{=} \overline{r}_v^\vee$ and following [BD14, eq. (3.3)] (which treats the case where $\overline{r}$ is split at all $w \in S_p$, but generalizes to the remaining cases by [EGS15, § 6.5]) we define the "local factor"

$$\pi(\overline{\rho}) \overset{\mathrm{def}}{=} \mathrm{Hom}_{U^v}\Big(\overline{M}^v, \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\overline{r}, \varinjlim_{V} H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})\big)\Big)[\mathfrak{m}'_{\overline{r}}], \tag{86}$$

where X_V denotes the smooth projective Shimura curve over F associated to V constructed with the convention "$\varepsilon = -1$" (see [BD14, § 3.1] and [BDJ10, § 2]), the colimit runs over all compact open subgroups of $(D \otimes_F \mathbb{A}_F^\infty)^\times$, and $\mathfrak{m}'_{\overline{r}}$ is the maximal ideal denoted $\mathfrak{m}'$ in [BD14, § 3.3] and in [EGS15, p. 50] (though the context of *loc. cit.* is slightly different since they use patching functors). We assume from now on that

- $\mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\overline{r}, \varinjlim_V H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})\big) \neq 0$.

In particular $\pi(\overline{\rho}) \neq 0$ by [BD14, Thm. 3.7.1] under the condition that $\overline{r}$ is reducible at all $w \in S_p$, but the proof extends to the general case using the material of [EGS15, § 6.5].

We define the ring R_∞ as in [BHH+23, § 8.1], with the set S in *loc. cit.* taken to be $S_D \cup S_{\overline{r}}$ and the rings $R^{\psi_w}_{\overline{r}_w}$ ($w \in (S_D \cup S_{\overline{r}}) \setminus S_p$) and $R^{(0,-1),\tau_w,\psi_w}_{\overline{r}_w}$ ($w \in S_p \setminus \{v\}$) of *loc. cit.* replaced by the rings $R^{\min}_w$ of [EGS15, § 6.5]. By [EGS15, Thm. 7.2.1] and [BD14, Lemma 3.4.1] the rings $R^{\min}_w$ are formally smooth over $W(\mathbb{F})$ (of dimension 3 or $3+3[F_w:\mathbb{Q}_p]$ according to whether $w \in S_p$ or not), so that R_∞ is formally smooth over $W(\mathbb{F})$ of relative dimension $4|S_D \cup S_{\overline{r}}| + 2[F_v:\mathbb{Q}_p] + q - 1$ for some integer $q \geq [F:\mathbb{Q}]$.

We can now follow the construction of [EGS15, § 6.4] (where the definition of $S(\sigma)_{\mathfrak{m}}$ of *loc. cit.* should be corrected as explained in [BHH+23, Rk. 8.1.3(iii)]). We obtain a patching functor (in the sense of [EGS15, § 6.1]) M_∞ defined on the category of continuous representations of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ on finite type $W(\mathbb{F})$-modules with central character ψ^{-1}, and taking values in the category of R_∞-modules of finite type, such that

$$M_\infty(\sigma_v)/\mathfrak{m}_\infty \cong \big(\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_{F_v})}(\sigma_v, \pi(\overline{\rho}))\big)^\vee$$

and which moreover satisfies $\dim_{\mathbb{F}}(M_\infty(\sigma_v)/\mathfrak{m}_\infty) \leq 1$ for any Serre weight σ_v, by [EGS15, § 6.5]. Furthermore the construction of [DL21, Thm. 6.2] gives a finitely generated module $\mathbb{M}_\infty$ over $R_\infty[\![\mathrm{GL}_2(\mathcal{O}_{F_v})]\!]$ such that $\mathbb{M}_\infty/\mathfrak{m}_\infty \cong \pi(\overline{\rho})^\vee$ and

$$M_\infty(\sigma_v) = \mathrm{Hom}^{\mathrm{cont}}_{W(\mathbb{F})[\![\mathrm{GL}_2(\mathcal{O}_{F_v})]\!]}(\mathbb{M}_\infty, \sigma_v^\vee)^\vee$$

(where $(-)^\vee \stackrel{\mathrm{def}}{=} \mathrm{Hom}^{\mathrm{cont}}_{W(\mathbb{F})}(-, E/W(\mathbb{F}))$).

Proposition 7.1.1. *If $\overline{\rho}$ is 12-generic then $\pi(\overline{\rho})$ satisfies assumptions* (i)–(iv) *with "$r=1$".*

Proof. The proofs of the results of [BHH+23, § 8.2, § 8.3, Thms. 8.4.1, 8.4.2, 8.4.3], [Wan23, § 6] go through verbatim for our $\pi(\overline{\rho})$, replacing all occurrences of r in *loc. cit.* by 1. (Note that the hypothesis [BHH+23, § 8.1, item (iii)(b)] and [Wan23, § 1, item (ii)] are satisfied as all $R^{\min}_w$ above are formally smooth over $W(\mathbb{F})$.) In particular [BHH+23, Thm. 1.9], [Wan23, Thm. 6.3(ii)] hold, with $r=1$, for $\pi(\overline{\rho})$ so that $\pi(\overline{\rho})$ satisfies assumption (i) and (ii) (for the latter, using [BHH+23, Prop. 6.4.6] which holds for a not necessarily semisimple $\overline{\rho}$). Similarly [BHH+23, Thm. 1.10], [Wan23, Thm. 6.3(i)] hold for $\pi(\overline{\rho})$ so that $\pi(\overline{\rho})$ satisfies assumption (iii) (via [HW22, Thm. 8.2]). Finally the proofs of [BHH+, Lemma 2.6.2, Prop. 2.6.3] go through verbatim replacing r and π in *loc. cit.* with 1 and $\pi(\overline{\rho})$ respectively, so $\pi(\overline{\rho})$ satisfies assumption (iv). (Note that assumption (i) is also satisfied by the main result of [LMS22, HW18, Le19].) □

Remark 7.1.2. Using [EGS15, § 6.5] it should be possible to modify the definition of $\pi(\overline{\rho})$ and generalize the above result to the definite case (i.e. when D is ramified at all infinite places).

7.2 Verifying assumption (v)

We keep the setup of § 7.1. The goal of this section is to prove the following result:

Proposition 7.2.1. *Assume that $\overline{\rho}$ is 9-generic. If $\pi(\overline{\rho})$ satisfies assumptions* (i), (ii) *and* (iv) *of § 3, then it also satisfies assumption* (v).

To simplify notation we let $\pi \overset{\text{def}}{=} \pi(\overline{\rho})$ and assume that it satisfies assumptions (i), (ii) and (iv) in the remainder of this section.

Remark 7.2.2. In fact, we will even establish a canonical isomorphism

$$\mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^n, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) \cong \mathrm{gr}(\mathrm{Tor}_1^{\Lambda}(\Lambda/\mathfrak{m}^n, \pi^\vee))$$

for $n = 3$. We remark that the $n = 2$ case can be proved by a similar, but significantly shorter, argument. The $n = 1$ case was established in [BHH$^+$, Cor. 2.5.1(i)] (taking $i = 1$ there).

The proof of Proposition 7.2.1 requires a number of preliminary results.

Lemma 7.2.3. *Assume that $\overline{\rho}$ is 0-generic. Then for any $\lambda \in \mathscr{P}$ we have*

$$\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda) \cap W(\overline{\rho}) = \{\tau \in W(\overline{\rho}) : J'^{\min} \subseteq J_\tau \subseteq J'^{\max}\},$$

where

$$J'^{\min} \overset{\text{def}}{=} \{j \in J_{\overline{\rho}} : \lambda_j \in \{x_j + 2, p - 3 - x_j\}\},$$
$$J'^{\max} \overset{\text{def}}{=} \{j \in J_{\overline{\rho}} : \lambda_j \notin \{x_j, p - 1 - x_j\}\}.$$

Proof. Note that we can replace λ with $\lambda^{[s]}$ (see [BHH$^+$, eq. (52)]) without changing the validity of the lemma. By [Bre14, Prop. 4.3] we have

$$\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda^s) \cap W(\overline{\rho}) = \{\sigma_J : J^{\min} \subseteq J \subseteq J^{\max}\},$$

where $\sigma_J \in W(\overline{\rho})$ denotes the Serre weight defined by $\nu_J \overset{\text{def}}{=} \mu_J \circ \lambda$, with $\mu_J \in \mathcal{P}$ determined by $\mu_{J,j} \in \{p - 2 - x_j, p - 1 - x_j\}$ if and only if $j \in J$. As $\sigma_J \in W(\overline{\rho})$, we deduce that $\mu_J \circ \lambda \in \mathscr{D}$ by [HW22, Lemmas 2.1, 2.7]. Also recall from [Bre14, Prop. 4.3] that

$$J^{\min} = \delta(\{j : \lambda_j \in \{p - 1 - x_j, x_j + 2\} \text{ or } (\lambda_j = x_j + 1, j \notin J_{\overline{\rho}}\}),$$
$$J^{\max} = \delta(\{j : \lambda_j \notin \{p - 3 - x_j, x_j\} \text{ and } (\lambda_j = p - 2 - x_j \Rightarrow j \in J_{\overline{\rho}})\}).$$

Let $J(\lambda) \overset{\text{def}}{=} \{j : \lambda_j \in \{p - 3 - x_j, p - 2 - x_j, p - 1 - x_j\}\}$. As $\nu_J = \mu_J \circ \lambda$, we have

$$j \in \delta(J_{\nu_J}) \iff \nu_{J,j} \in \{p - 3 - x_j, p - 2 - x_j\} \iff j \in J \,\Delta\, J(\lambda).$$

Equivalently,

$$J_{\nu_J} = \delta^{-1}(J) \,\Delta\, K, \text{ where } K \overset{\text{def}}{=} \delta^{-1}(J(\lambda)) = \{j : \lambda_j \in \{p - 3 - x_j, p - 1 - x_j, x_j + 1\}\}. \quad (87)$$

From basic set theory, $J^{\min} \subseteq J \subseteq J^{\max}$ if and only if $J'^{\min} \subseteq \delta^{-1}(J) \,\Delta\, K \subseteq J'^{\max}$, where

$$J'^{\min} \overset{\text{def}}{=} (\delta^{-1}(J^{\min}) \setminus K) \sqcup (K \setminus \delta^{-1}(J^{\max})),$$
$$J'^{\max} \overset{\text{def}}{=} (\delta^{-1}(J^{\max}) \setminus K) \sqcup (K \setminus \delta^{-1}(J^{\min})).$$

Finally, it follows from the definitions that

$$J'^{\min} = \{j : \lambda_j = x_j + 2\} \sqcup \{j : \lambda_j = p - 3 - x_j\},$$
$$J'^{\max} = \{j \in J_{\overline{\rho}} : \lambda_j \in \{x_j + 2, p - 2 - x_j\}\} \sqcup \{j \in J_{\overline{\rho}} : \lambda_j \in \{p - 3 - x_j, x_j + 1\}\}. \qquad \square$$

Lemma 7.2.4. *Suppose that $A_0 = \mathbb{F}[x_1, \dots, x_n]$ and $A = \mathbb{F}[\![x_1, \dots, x_n]\!]$. If I_0, J_0 are ideals of A_0, then $I_0 A \cap J_0 A = (I_0 \cap J_0)A$ as ideals of A.*

Proof. This is a special case of [Mat89, Thm. 7.4(ii)], as A is flat over A_0. □

The following lemma follows exactly as in [LLHLM20, Lemma 3.6.2] and [EGS15, Prop. 8.1.1].

Lemma 7.2.5. *Suppose V is a finite length smooth representation of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$ that is multiplicity free. Suppose that the scheme-theoretic supports of the R_∞-modules $M_\infty(\sigma)$ (cf. [BHH+, § 2.6]) are reduced and do not share any irreducible components for σ running through $\mathrm{JH}(V)$. Then*

$$\mathrm{Ann}_{R_\infty} M_\infty(V) = \bigcap_{\sigma \in \mathrm{JH}(V)} \mathrm{Ann}_{R_\infty} M_\infty(\sigma).$$

Lemma 7.2.6. *Suppose $R = \mathbb{F}[X_j, Y_j\ (1 \le j \le k)]$, $I = (X_j Y_j\ (1 \le j \le k),\ Y_j Y_{j'}\ (1 \le j < j' \le k))$. We have*

$$\dim_{\mathbb{F}} \mathrm{Tor}_i^R(\mathbb{F}, R/I) = \begin{cases} 1 & \text{if } i = 0, \\ i\binom{k+1}{i+1} & \text{if } i > 0. \end{cases}$$

Proof. Note that R/I is the Stanley–Reisner ring $\mathbb{F}[\Delta]$ associated to the simplicial complex Δ whose minimal non-faces are $\{X_j, Y_j\}$ $(1 \le j \le k)$ and $\{Y_j, Y_{j'}\}$ $(1 \le j < j' \le k)$ [BH93, § 5]. Also, $\dim_{\mathbb{F}} \mathrm{Tor}_i^R(\mathbb{F}, R/I)$ is the rank of the degree i term in any minimal graded free resolution of R/I as R-module. Let $\mathscr{V} \stackrel{\text{def}}{=} \{X_1, \dots, Y_k\}$ denote the set of vertices. By [BH93, Thm. 5.5.1, Thm. 5.3.2] we have

$$\dim_{\mathbb{F}} \mathrm{Tor}_i^R(\mathbb{F}, R/I) = \sum_{\mathscr{W} \subseteq \mathscr{V}} \dim_{\mathbb{F}} \widetilde{H}_{|\mathscr{W}|-i-1}(|\Delta_{\mathscr{W}}|; \mathbb{F}), \tag{88}$$

where $\Delta_{\mathscr{W}}$ denotes the subcomplex obtained by all faces of Δ whose vertices are contained in $\mathscr{W}$ with geometric realization $|\Delta_{\mathscr{W}}|$, and where $\widetilde{H}_j$ denotes the j-th reduced homology group (by convention, $H_{-1}(\emptyset; \mathbb{F}) = \mathbb{F}$). By definition of Δ, if $\mathscr{W}$ contains at least two X_j, then $|\Delta_{\mathscr{W}}|$ is contractible (so the term indexed by $\mathscr{W}$ in (88) vanishes). Similarly, if $\mathscr{W}$ contains X_j for precisely one j, but it does not contain Y_j, then $|\Delta_{\mathscr{W}}|$ is contractible. If $\mathscr{W}$ contains X_j for precisely one j and it also contains Y_j, then $|\Delta_{\mathscr{W}}|$ is homotopic to a disjoint union of 2 points. If $\mathscr{W}$ contains no X_j, then $|\Delta_{\mathscr{W}}|$ is a disjoint union of $|\mathscr{W}|$ points. If $|\Delta_{\mathscr{W}}|$ is homotopic to a disjoint union of $s \ge 2$ points, the term $\dim_{\mathbb{F}} \widetilde{H}_{|\mathscr{W}|-i-1}(|\Delta_{\mathscr{W}}|; \mathbb{F})$ equals $s-1$ in degree $i = |\mathscr{W}| - 1$ and 0 otherwise.

Now let us compute $\dim_{\mathbb{F}} \mathrm{Tor}_i^R(\mathbb{F}, R/I)$ via (88). If $i > 0$, the only contribution then comes from the $\binom{k}{i+1}$ subsets $\mathscr{W}$ of $\{Y_1, \dots, Y_k\}$ of cardinality $i+1$ (each contributing i) and the $k \cdot \binom{k-1}{i-1}$ subsets $\mathscr{W}$ that contain precisely one X_j and also Y_j (each contributing 1). The lemma easily follows. □

Recall from § 2.3 that for $n \ge 1$ we denote $W_{\chi,n} = (\mathrm{Proj}_{I/Z_1} \chi)/\mathfrak{m}^n \cong \chi \otimes_{\mathbb{F}} \Lambda/\mathfrak{m}^n$. (For $n = 2, 3$ the structure of $W_{\chi,n}$ is completely explicit, see [HW22, § 3.1].) For $\lambda \in \mathscr{P}$ we let

$$k_\lambda \stackrel{\text{def}}{=} |\{0 \le j \le f-1 : t_j \ne y_j z_j\}|.$$

(Recall that the t_j, depending on λ, are defined in (13).) We recall that $\overline{R}_\infty = R_\infty \otimes_{\mathcal{O}} \mathbb{F}$.

Proposition 7.2.7. *Assume that $\overline{\rho}$ is 2-generic. Then for any $\lambda \in \mathscr{P}$ we can find compatible isomorphisms*

$$\overline{R}_\infty \cong \mathbb{F}[\![X_j, Y_j \ (1 \le j \le \ell), Z_m \ (\ell < m \le N)]\!]$$

for some integer $N \ge 2f$ and

$$M_\infty(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\chi_\lambda,2}) \cong \overline{R}_\infty/(X_jY_j \ (1 \le j \le \ell), \ Y_iY_j \ (1 \le i < j \le k_\lambda), \ Z_m(\ell < m \le 2f)),$$

where $\ell \overset{\mathrm{def}}{=} |J_{\overline{\rho}}|$.

Proof. Let $\chi \overset{\mathrm{def}}{=} \chi_\lambda$ and $V_\chi \overset{\mathrm{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\chi,2}$. We have

$$M_\infty(V_\chi)/\mathfrak{m}_\infty \cong \mathrm{Hom}_I((\mathrm{Proj}_{I/Z_1}\chi)/\mathfrak{m}^2, \pi)^\vee \cong \mathrm{Hom}_I(\mathrm{Proj}_{I/Z_1}\chi, \pi[\mathfrak{m}^2])^\vee, \tag{89}$$

and this is one-dimensional by [HW22, Thm. 1.3(ii)], so $M_\infty(V_\chi)$ is a cyclic $\overline{R}_\infty$-module.

We first show that

$$\mathrm{JH}(V_\chi) \cap W(\overline{\rho}) = \Big\{\sigma \in W(\overline{\rho}) : |(J_\sigma \setminus J'') \,\Delta\, J'| \le 1\Big\}, \tag{90}$$

where $J' \overset{\mathrm{def}}{=} \{j \in J_{\overline{\rho}} : \lambda_j \in \{x_j + 2, p-3-x_j\}\}$ and $J'' \overset{\mathrm{def}}{=} \{j \in J_{\overline{\rho}} : \lambda_j \in \{x_j+1, p-2-x_j\}\}$. Note that Lemma 7.2.3 applied to χ gives $J'^{\min} = J'$ and $J'^{\max} = J' \sqcup J''$. On the other hand, by [BHH$^+$, Lemma 2.3.6(ii)] (with $m = 1$), $\chi' \overset{\mathrm{def}}{=} \chi\alpha_j^{\pm 1}$ occurs in π^{I_1} if and only if $j \in J_{\overline{\rho}}$ and $\lambda_j \in \{x_j, p-3-x_j\}$ (resp. $\lambda_j \in \{x_j+2, p-1-x_j\}$) if the sign is positive (resp. negative), and in each such case Lemma 7.2.3 applied to χ' gives $J'^{\min} = J' \,\Delta\, \{j\}$ and $J'^{\max} = J' \sqcup J''$. In other words,

$$\begin{aligned}\mathrm{JH}(V_\chi) \cap W(\overline{\rho}) &= \Big\{\sigma \in W(\overline{\rho}) : \exists\, K \subseteq J_{\overline{\rho}} \setminus J'', |K| \le 1, J' \,\Delta\, K \subseteq J_\sigma \subseteq (J' \,\Delta\, K) \sqcup J''\Big\}\\ &= \Big\{\sigma \in W(\overline{\rho}) : \exists\, K \subseteq J_{\overline{\rho}} \setminus J'', |K| \le 1, J_\sigma \setminus J'' = J' \,\Delta\, K\Big\}\\ &= \Big\{\sigma \in W(\overline{\rho}) : \exists\, K \subseteq J_{\overline{\rho}} \setminus J'', |K| \le 1, (J_\sigma \setminus J'') \,\Delta\, J' = K\Big\},\end{aligned}$$

which is equivalent to (90).

As $\overline{\rho}$ is nonsplit, we may assume without loss of generality that $0 \notin J_{\overline{\rho}}$ if f is odd. Let

$$\mu \overset{\mathrm{def}}{=} \begin{cases}(x_0+1, p-2-x_1, x_2+1, p-2-x_3, \dots, p-2-x_{f-1}) & \text{if } f \text{ is even},\\ (x_0, p-2-x_1, x_2+1, p-2-x_3, \dots, p-2-x_{f-2}, x_{f-1}+1) & \text{if } f \text{ is odd},\end{cases}$$

so $\mu \in \mathscr{P}$ and for all $j \in J_{\overline{\rho}}$ we have $\mu_j \in \{x_j+1, p-2-x_j\}$. Then [Bre14, Prop. 4.3] or Lemma 7.2.3 imply that $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu) \supseteq W(\overline{\rho})$. Observe that if $\sigma \in W(\overline{\rho})$, then σ is parametrized, in the notation of [Bre14], by the set

$$J_{\sigma,\mu} \overset{\mathrm{def}}{=} \{j \text{ even} : j \in \delta(J_\sigma)\} \sqcup \{j \text{ odd} : j \notin \delta(J_\sigma)\}.$$

(If f is odd we take $0 \le j \le f-1$, not just $j \in \mathbb{Z}/f\mathbb{Z}$!) On the other hand, in the same situation, the minimal/maximal subsets in [Bre14, Prop. 4.3] equal

$$\begin{aligned}J^{\min} &= \delta(\{j \text{ even} : j \notin J_{\overline{\rho}}, j \ne 0 \text{ if } f \text{ odd}\})\\ &= \{j \text{ odd} : j \notin \delta(J_{\overline{\rho}})\}\end{aligned}$$

and

$$\begin{aligned} J^{\max} &= \delta(\{j \text{ even} : j \neq 0 \text{ if } f \text{ odd}\} \sqcup \{j \text{ odd} : j \in J_{\overline{\rho}}\}) \\ &= \{j \text{ odd}\} \sqcup \{j \text{ even} : j \in \delta(J_{\overline{\rho}})\}. \end{aligned}$$

In particular, we deduce that

$$\begin{aligned} J_{\sigma,\mu} \setminus J^{\min} &= \{j \text{ even} : j \in \delta(J_\sigma)\} \sqcup \{j \text{ odd} : j \in \delta(J_{\overline{\rho}} \setminus J_\sigma)\}, \\ J^{\max} \setminus J^{\min} &= \delta(J_{\overline{\rho}}). \end{aligned} \tag{91}$$

Let τ^0 denote the lattice in a tame principal series type obtained by inducing the Teichmüller lift of χ_μ from I to $\mathrm{GL}_2(\mathcal{O}_K)$, and let $\tau \stackrel{\text{def}}{=} \tau^0[1/p]$. By [EGS15, Thm. 7.2.1] the corresponding fixed-determinant framed local deformation ring $R_{\overline{\rho}}^{\tau,\psi,\square}$ is isomorphic to $\mathcal{O}[\![x_j, y_j, z_m : j \in J^{\max} \setminus J^{\min}, 1 \leq m \leq f+3-\ell]\!]/(x_j y_j : \text{all } j)$ (of relative dimension $f+3$), where $\ell \stackrel{\text{def}}{=} |J^{\max} \setminus J^{\min}| = |J_{\overline{\rho}}|$. The full fixed-determinant framed local deformation ring $R_{\overline{\rho}}^{\psi,\square}$ is a power series ring in $3f+3$ variables. It is not hard to see that we can choose an isomorphism $R_{\overline{\rho}}^{\psi,\square} \cong \mathcal{O}[\![X_j, Y_j, Z_m : j \in J^{\max} \setminus J^{\min}, \ell < m \leq 3f+3-\ell]\!]$ such that $R_{\overline{\rho}}^{\tau,\psi,\square} = R_{\overline{\rho}}^{\psi,\square}/((X_j Y_j : \text{all } j) + I_Z)$, where $I_Z \stackrel{\text{def}}{=} (Z_m : \ell < m \leq 2f)$.

Hence by [EGS15, Thm. 10.1.1] we deduce that

$$\begin{aligned} R_\infty &\cong \mathcal{O}[\![X_j, Y_j, Z_m : j \in J^{\max} \setminus J^{\min}, \ell < m \leq N]\!], \\ M_\infty(\tau^0) &\cong R_\infty/((X_j Y_j : \text{all } j) + I_Z), \end{aligned}$$

for some integer $N \geq 3f+3-\ell$. From [EGS15, Thm. 7.2.1(4), Lemma 10.1.12] it follows that

$$\mathrm{Ann}_{\overline{R}_\infty} M_\infty(\sigma) = (X_j : j \in J_{\sigma,\mu} \setminus J^{\min},\ Y_j : j \in J^{\max} \setminus J_{\sigma,\mu}) + I_Z. \tag{92}$$

(In fact, to compare with the conventions of [EGS15] we have to replace $(J^{\min}, J^{\max}, J_{\sigma,\mu})$ by $((J^{\max})^c, (J^{\min})^c, J_{\sigma,\mu}^c)$, cf. the proof of [EGS15, Lemma 7.4.1], but this amounts to interchanging X_j and Y_j for each j.)

By Lemma 7.2.5 we have

$$\begin{aligned} \mathrm{Ann}_{\overline{R}_\infty} M_\infty(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu) &= \bigcap_{\sigma \in W(\overline{\rho})} \mathrm{Ann}_{\overline{R}_\infty} M_\infty(\sigma), \\ I_\chi \stackrel{\text{def}}{=} \mathrm{Ann}_{\overline{R}_\infty} M_\infty(V_\chi) &= \bigcap_{\sigma \in \mathrm{JH}(V_\chi) \cap W(\overline{\rho})} \mathrm{Ann}_{\overline{R}_\infty} M_\infty(\sigma). \end{aligned}$$

We will make several changes of variables, which will not affect the final result. Up to interchanging X_j's and Y_j's, we deduce from equations (91) and (92) that

$$I_\chi \cong \bigcap_{\sigma \in \mathrm{JH}(V_\chi) \cap W(\overline{\rho})} \Big((X_j : j \in \delta(J_\sigma),\ Y_j : j \in \delta(J_{\overline{\rho}} \setminus J_\sigma)) + I_Z \Big).$$

Shifting the indices in X_j and Y_j by one (to get rid of the $\delta(\cdot)$), and applying (90) we get

$$I_\chi \cong \bigcap_{J \subseteq J_{\overline{\rho}},\ |(J \setminus J'') \Delta J'| \leq 1} \Big((X_j : j \in J,\ Y_j : j \in J_{\overline{\rho}} \setminus J) + I_Z \Big). \tag{93}$$

As $J' \cap J'' = \emptyset$ we have $(J \setminus J'') \,\Delta\, J' = (J \,\Delta\, J') \setminus J''$. Interchanging variables X_j and Y_j for all $j \in J'$, which has the effect of replacing J by $J \,\Delta\, J'$ in (93) we get

$$I_\chi \cong \bigcap_{J \subseteq J_{\overline{\rho}},\ |J \setminus J''| \le 1} \Big((X_j : j \in J,\ Y_j : j \in J_{\overline{\rho}} \setminus J) + I_Z \Big).$$

By Lemma 7.2.4 and [BH93, Thm. 5.1.4], this intersection equals

$$(X_j Y_j,\ Y_{j_1} Y_{j_2} : \text{all } j \in J_{\overline{\rho}};\ j_1 < j_2 \text{ both contained in } J_{\overline{\rho}} \setminus J'') + I_Z. \tag{94}$$

(The corresponding simplicial complex has facets $\{X_j : j \notin J, Y_j : j \in J\}$, hence minimal non-faces $\{Z_m\}$ for all m, $\{X_j, Y_j\}$ for all j, and $\{Y_{j_1}, Y_{j_2}\}$ for all $j_1 < j_2$ both contained in $J_{\overline{\rho}} \setminus J''$.) As $k_\lambda = |\{j \in J_{\overline{\rho}} : \lambda_j \notin \{x_j + 1, p - 2 - x_j\}\}| = |J_{\overline{\rho}} \setminus J''|$, we are done. ☐

Proposition 7.2.8. *Assume that $\overline{\rho}$ is 2-generic. Then for any $\lambda \in \mathscr{P}$ we have*

$$\dim_{\mathbb{F}} \operatorname{Ext}^i_{I/Z_1}(W_{\chi_\lambda,2}, \pi) = \begin{cases} 1 & \text{if } i = 0, \\ 2f + \binom{k_\lambda}{2} & \text{if } i = 1, \\ 2f^2 + (k_\lambda^2 - k_\lambda - 1)f - \binom{k_\lambda + 1}{3} & \text{if } i = 2. \end{cases}$$

Proof. Let $\chi \overset{\text{def}}{=} \chi_\lambda$ and $k \overset{\text{def}}{=} k_\lambda$ for short.

By [BHH+, Lemma 2.6.2], $\operatorname{Ext}^i_{I/Z_1}(W_{\chi,2}, \pi)$ is dual to $\operatorname{Tor}_i^{\overline{R}_\infty}(\mathbb{F}, M_\infty(\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W_{\chi,2}))$, hence by Proposition 7.2.7 to $\operatorname{Tor}_i^{\overline{R}_\infty}(\mathbb{F}, \overline{R}_\infty / I_\infty)$, where

$$\begin{aligned} \overline{R}_\infty &= \mathbb{F}[\![X_j, Y_j\ (1 \le j \le \ell), Z_m\ (\ell < m \le N)]\!], \\ I_\infty &= (X_j Y_j\ (1 \le j \le \ell),\ Y_i Y_j\ (1 \le i < j \le k),\ Z_m (\ell < m \le 2f)) \subseteq \overline{R}_\infty, \end{aligned}$$

where $\ell = |J_{\overline{\rho}}|$. Let

$$\begin{aligned} R &\overset{\text{def}}{=} \mathbb{F}[X_j, Y_j\ (1 \le j \le \ell), Z_m\ (\ell < m \le N)], \\ I &\overset{\text{def}}{=} (X_j Y_j\ (1 \le j \le \ell),\ Y_i Y_j\ (1 \le i < j \le k),\ Z_m (\ell < m \le 2f)) \subseteq R. \end{aligned}$$

As $\overline{R}_\infty$ is flat over R and $\overline{R}_\infty / I_\infty = \overline{R}_\infty \otimes_R R/I$, by considering minimal graded free resolutions we deduce an isomorphism

$$\operatorname{Tor}_i^R(\mathbb{F}, R/I) \cong \operatorname{Tor}_i^{\overline{R}_\infty}(\mathbb{F}, \overline{R}_\infty / I_\infty).$$

It remains to compute $\operatorname{Tor}_i^R(\mathbb{F}, R/I)$ for $i \le 2$. We let

$$\begin{aligned} R^{(1)} &\overset{\text{def}}{=} \mathbb{F}[X_j, Y_j\ (1 \le j \le k)], & I^{(1)} &\overset{\text{def}}{=} (X_j Y_j\ (1 \le j \le k),\ Y_i Y_j\ (1 \le i < j \le k)), \\ R^{(2,j)} &\overset{\text{def}}{=} \mathbb{F}[X_j, Y_j], & I^{(2,j)} &\overset{\text{def}}{=} (X_j Y_j), \\ R^{(3,m)} &\overset{\text{def}}{=} \mathbb{F}[Z_m], & I^{(3,m)} &\overset{\text{def}}{=} (Z_m), \\ R^{(4,n)} &\overset{\text{def}}{=} \mathbb{F}[Z_n], & I^{(4,n)} &\overset{\text{def}}{=} (0), \end{aligned}$$

so that

$$R \cong R^{(1)} \otimes_{\mathbb{F}} \bigotimes_{k<j\le \ell} R^{(2,j)} \otimes_{\mathbb{F}} \bigotimes_{\ell<m\le 2f} R^{(3,m)} \otimes_{\mathbb{F}} \bigotimes_{n>2f} R^{(4,n)},$$
$$R/I \cong R^{(1)}/I^{(1)} \otimes_{\mathbb{F}} \bigotimes_{k<j\le \ell} R^{(2,j)}/I^{(2,j)} \otimes_{\mathbb{F}} \bigotimes_{\ell<m\le 2f} R^{(3,m)}/I^{(3,m)} \otimes_{\mathbb{F}} \bigotimes_{n>2f} R^{(4,n)}/I^{(4,n)}.$$

By using the tensor product of a minimal graded free resolution of $R^{(1)}/I^{(1)}$ and of the minimal graded free resolutions

$$0 \to R^{(2,j)} \xrightarrow{X_jY_j} R^{(2,j)} \to R^{(2,j)}/I^{(2,j)} \to 0$$
$$0 \to R^{(3,m)} \xrightarrow{Z_m} R^{(3,m)} \to R^{(3,m)}/I^{(3,m)} \to 0$$
$$0 \to R^{(4,n)} \to R^{(4,n)}/I^{(4,n)} \to 0$$

we obtain that

$$\begin{aligned}\dim_{\mathbb{F}} \operatorname{Tor}_i^R(\mathbb{F}, R/I) &= \sum_{j=0}^{i} \binom{2f-k}{i-j} \dim_{\mathbb{F}} \operatorname{Tor}_j^{R^{(1)}}(\mathbb{F}, R^{(1)}/I^{(1)}),\\ &= \binom{2f-k}{i} + \sum_{j=1}^{i} j \cdot \binom{2f-k}{i-j}\binom{k+1}{j+1}\end{aligned}$$

by Lemma 7.2.6. We conclude by a short calculation. □

Corollary 7.2.9. *Assume that $\overline{\rho}$ is 3-generic. Then for any $\lambda \in \mathscr{P}$ we have*

$$\dim_{\mathbb{F}} \operatorname{Ext}^1_{I/Z_1}(W_{\chi_\lambda,3}, \pi) \ge 2f^2 + f + \binom{k_\lambda+1}{3}.$$

Remark 7.2.10. We will see below (in the proof of Proposition 7.2.1) that equality holds, at least under a stronger genericity condition. By the proof of this corollary, this implies in fact that the natural map $\operatorname{Ext}^2_{I/Z_1}(W_{\chi_\lambda,3}, \pi) \to \operatorname{Ext}^2_{I/Z_1}(\chi_\lambda \otimes \mathfrak{m}^2/\mathfrak{m}^3, \pi)$ is injective.

Proof. Again let $\chi \overset{\text{def}}{=} \chi_\lambda$. By [HW22, Thm. 1.3], we have $\operatorname{Hom}_{I/Z_1}(W_{\chi,3}, \pi) = \operatorname{Hom}_{I/Z_1}(\chi, \pi)$. The exact sequence $0 \to \chi \otimes \mathfrak{m}^2/\mathfrak{m}^3 \to W_{\chi,3} \to W_{\chi,2} \to 0$ thus gives rise to a long exact sequence

$$\begin{aligned}0 \to \operatorname{Hom}_{I/Z_1}(\chi \otimes \mathfrak{m}^2/\mathfrak{m}^3, \pi) \to \operatorname{Ext}^1_{I/Z_1}(W_{\chi,2}, \pi) \to \operatorname{Ext}^1_{I/Z_1}(W_{\chi,3}, \pi)\\ \to \operatorname{Ext}^1_{I/Z_1}(\chi \otimes \mathfrak{m}^2/\mathfrak{m}^3, \pi) \to \operatorname{Ext}^2_{I/Z_1}(W_{\chi,2}, \pi).\end{aligned} \tag{95}$$

Let $J \overset{\text{def}}{=} \{0 \le j \le f-1 : t_j \ne y_jz_j\}$ (where again the t_j are defined in (13)). Let $\varepsilon_j \overset{\text{def}}{=} -1$ if $t_j = y_j$, $\varepsilon_j \overset{\text{def}}{=} +1$ if $t_j = z_j$. Note that, by [BHH+, Lemma 2.3.6(ii)] (with $m = 2$), $\operatorname{JH}(\chi \otimes \mathfrak{m}^2/\mathfrak{m}^3) \cap \operatorname{JH}(\pi^{I_1})$ consists of χ (occurring $2f$ times in $\chi \otimes \mathfrak{m}^2/\mathfrak{m}^3$) and all $\chi\alpha_i^{\varepsilon_i}\alpha_j^{\varepsilon_j}$ for $\{i < j\} \subseteq J$ (each occurring once in $\chi \otimes \mathfrak{m}^2/\mathfrak{m}^3$). Hence by assumption (iv) we deduce that

$$\dim_{\mathbb{F}} \operatorname{Ext}^i_{I/Z_1}(\chi \otimes \mathfrak{m}^2/\mathfrak{m}^3, \pi) = \begin{cases} 2f + \binom{k_\lambda}{2} & \text{if } i = 0, \\ 2f(2f + \binom{k_\lambda}{2}) & \text{if } i = 1. \end{cases}$$

By Proposition 7.2.8 we deduce that the first map in (95) is an isomorphism, so

$$\begin{aligned}\dim_{\mathbb{F}} \operatorname{Ext}^1_{I/Z_1}(W_{\chi,3},\pi) &\geq \dim_{\mathbb{F}} \operatorname{Ext}^1_{I/Z_1}(\chi\otimes \mathfrak{m}^2/\mathfrak{m}^3,\pi) - \dim_{\mathbb{F}} \operatorname{Ext}^2_{I/Z_1}(W_{\chi,2},\pi)\\ &= 2f\left(2f+\binom{k_\lambda}{2}\right) - \left(2f^2+(k_\lambda^2-k_\lambda-1)f-\binom{k_\lambda+1}{3}\right)\\ &= 2f^2+f+\binom{k_\lambda+1}{3},\end{aligned}$$

where we used Proposition 7.2.8 again. □

Lemma 7.2.11. *Assume that $\overline{\rho}$ is $(2n+1)$-generic. Suppose that $\chi : I \to \mathbb{F}^\times$, $J, J' \subseteq \{0,1,\dots,f-1\}$, $i_j, i'_{j'} \in \mathbb{Z}\setminus\{0\}$ for all $j\in J$, $j'\in J'$ such that $\sum_{j\in J}|i_j| \leq n$ and $\sum_{j\in J'}|i'_j|\leq n$. If $\chi\prod_{j\in J}\alpha_j^{i_j}\in \mathrm{JH}(\pi^{I_1})$ and $\chi\prod_{j\in J'}\alpha_j^{i'_j}\in\mathrm{JH}(\pi^{I_1})$, then*

(i) *$|i_j - i'_j|\leq 1$ for all $j\in J\cap J'$;*

(ii) *$\chi\prod_{j\in J''}\alpha_j^{i''_j}\in \mathrm{JH}(\pi^{I_1})$ for any $J\cap J'\subseteq J''\subseteq J\cup J'$, where $i''_j = i_j$ if $j\in J$ and $i''_j = i'_j$ if $j\in J''\setminus J$.*

Proof. Let $\chi' \stackrel{\text{def}}{=} \chi\prod_{j\in J}\alpha_j^{i_j}\in\mathrm{JH}(\pi^{I_1})$ and $\chi'' \stackrel{\text{def}}{=} \chi\prod_{j\in J'}\alpha_j^{i'_j}\in\mathrm{JH}(\pi^{I_1})$. Write $\chi' = \chi_\lambda$ for some $\lambda\in\mathscr{P}$. Since

$$\chi'' = \chi' \prod_{j\in J\cap J'}\alpha_j^{i'_j - i_j}\prod_{j\in J\setminus J'}\alpha_j^{-i_j}\prod_{j\in J'\setminus J}\alpha_j^{i'_j},$$

part (i) immediately follows from [BHH+, Lemma 2.3.6(ii)] (with $m=2n$). The same lemma implies part (ii) as well, by noting that

$$\chi\prod_{j\in J''}\alpha_j^{i''_j} = \chi'\prod_{j\in J\setminus J''}\alpha_j^{-i_j}\prod_{j\in J''\setminus J}\alpha_j^{i'_j}$$

and since the assumptions imply that $J''\setminus J\subseteq J'\setminus J$ and $J\setminus J''\subseteq J\setminus J'$. □

Proof of Proposition 7.2.1. It suffices to establish a canonical isomorphism

$$\operatorname{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) \cong \mathrm{gr}(\operatorname{Tor}_1^{\Lambda}(\Lambda/\mathfrak{m}^3,\pi^\vee)).$$

Just as in the proof of [BHH+, Cor. 2.4.8, Cor. 2.5.1] it suffices to show that

$$\dim_{\mathbb{F}}\operatorname{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) \leq \dim_{\mathbb{F}}\operatorname{Tor}_1^{\Lambda}(\Lambda/\mathfrak{m}^3,\pi^\vee),$$

and then equality has to hold.

Step 1. We first show that

$$\begin{aligned}&\dim_{\mathbb{F}}\operatorname{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee))\\ &\qquad = \sum_{\lambda\in\mathscr{P}}\left(4f^3+(6-4k_\lambda)f^2+(2k_\lambda^2-2k_\lambda+1)f-\frac{1}{6}k_\lambda(k_\lambda-1)(2k_\lambda-1)\right).\end{aligned}$$

From [BHH+, Thm. 2.1.2] we have $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \cong \bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)$. Fix $\lambda \in \mathscr{P}$ and let $J \overset{\text{def}}{=} \{0 \le j \le f-1 : t_j \ne y_j z_j\}$, $k \overset{\text{def}}{=} |J| = k_\lambda$. It will suffice to show that

$$\dim_{\mathbb{F}} \mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, R/\mathfrak{a}(\lambda)) = 4f^3 + (6-4k)f^2 + (2k^2-2k+1)f - \frac{1}{6}k(k-1)(2k-1). \quad (96)$$

We will compute this Tor_1 using an explicit free resolution of $R/\mathfrak{a}(\lambda)$.

Recall from [BHH+, eq. (19)] that $\mathrm{gr}(\Lambda)$ (resp. $\mathrm{gr}(\Lambda)_j$) is the universal enveloping algebra of the Lie algebra $\bigoplus_{j=0}^{f-1} \mathfrak{g}_j$ (resp. $\mathfrak{g}_j$) over $\mathbb{F}$, where $\mathfrak{g}_j$ has $\mathbb{F}$-basis y_j, z_j, h_j, subject to $[y_j, z_j] = h_j$, h_j is central, and $[\mathfrak{g}_j, \mathfrak{g}_{j'}] = 0$ for all $j \ne j'$. In the following we use the Poincaré–Birkhoff–Witt bases for these Lie algebras, for the ordering $y_0, \dots, y_{f-1}, z_0, \dots, z_{f-1}, h_0, \dots, h_{f-1}$. In particular, $\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3$ has $\mathbb{F}$-basis given by all ordered monomials whose degree is at least -2, where y_j, z_j have degree -1 and h_j has degree -2, and its dimension equals $2f^2 + 4f + 1$.

Note that $R/\mathfrak{a}(\lambda)$ is the tensor product of $\mathrm{gr}(\Lambda)_j/(t_j, h_j)$ over $\mathbb{F}$ for all $0 \le j \le f-1$. Recall from [HW22, Lemma 9.6] the minimal gr-free resolution of $\mathrm{gr}(\Lambda)_j/(t_j, h_j)$ as $\mathrm{gr}(\Lambda)_j$-module:

$$G_\bullet^{(j)}: \quad 0 \to \mathrm{gr}(\Lambda)_j \xrightarrow{(-h_j, t_j)} \mathrm{gr}(\Lambda)_j \oplus \mathrm{gr}(\Lambda)_j \xrightarrow{\binom{t_j}{h_j}} \mathrm{gr}(\Lambda)_j \to 0,$$

where we ignore the grading and H-actions. (Compared to [HW22, Lemma 9.6] we applied the involution $(\alpha, \beta) \mapsto (-\alpha-\beta, \beta)$ to the middle term in case $t_j = y_j z_j$.) Then $\mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3, R/\mathfrak{a}(\lambda))$ is obtained as the first homology of the complex $\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3 \otimes_{\mathrm{gr}(\Lambda)} G_\bullet$, where $G_\bullet$ is the tensor product complex of all $G_\bullet^{(j)}$, $0 \le j \le f-1$. Note that $G_0 \cong \mathrm{gr}(\Lambda)$, $G_1 \cong \bigoplus_{j=0}^{f-1} \mathrm{gr}(\Lambda)^{\oplus 2}$, $G_2 \cong \bigoplus_{j=0}^{f-1} \mathrm{gr}(\Lambda) \oplus \bigoplus_{0\le i<j\le f-1} (\mathrm{gr}(\Lambda)^{\oplus 2}) \otimes_{\mathrm{gr}(\Lambda)} (\mathrm{gr}(\Lambda)^{\oplus 2})$.

For the purpose of the calculation we may and it will be convenient to assume that $t_j = z_j$ for all $j \in J$ (by interchanging y_j and z_j, if necessary).

The morphism $\partial_1 : G_1/\overline{\mathfrak{m}}^3 \to G_0/\overline{\mathfrak{m}}^3$ is given by $\binom{t_j}{h_j}$ in the j-th component, so its image in $\mathrm{gr}(\Lambda)/\overline{\mathfrak{m}}^3$ has $\mathbb{F}$-spanning vectors given by

$$\begin{cases} y_j z_j, h_j & \text{if } j \notin J, \\ t_j, y_i t_j, z_i t_j, h_j & \text{if } j \in J, \end{cases}$$

where $0 \le i \le f-1$ is arbitrary. As the term $t_i t_j = t_j t_i$ gets counted twice for any $\{i < j\} \subseteq J$, we see that

$$\dim_{\mathbb{F}} \mathrm{im}(\partial_1) = 2(f-k) + (2f+2)k - \binom{k}{2} = 2f(k+1) - \binom{k}{2}.$$

Since $\dim_{\mathbb{F}} G_1/\overline{\mathfrak{m}}^3 = 4f^3 + 8f^2 + 2f$, we deduce that

$$\dim_{\mathbb{F}} \ker(\partial_1) = 4f^3 + 8f^2 - 2kf + \binom{k}{2}.$$

The image of the morphism $\partial_2 : G_2/\overline{\mathfrak{m}}^3 \to G_1/\overline{\mathfrak{m}}^3$ is generated by $(-h_j, t_j)_j$ for all j and $(t_j, 0)_i - (t_i, 0)_j$, $(h_j, 0)_i - (0, t_i)_j$, $(0, h_j)_i - (0, h_i)_j$ for all $i \ne j$ as a $\mathrm{gr}(\Lambda)$-module. (Here the subscript j denotes the j-th component of $G_1/\overline{\mathfrak{m}}^3 \cong \bigoplus_{j=0}^{f-1}(\operatorname{gr}(\Lambda)/\overline{\mathfrak{m}}^3)^{\oplus 2}$.) As an $\mathbb{F}$-vector space we get spanning vectors

$$\begin{aligned} (h_j,0)_i-(0,t_i)_j &\quad 0\le i,j\le f-1,\\ (t_j,0)_i-(t_i,0)_j &\quad 0\le i<j\le f-1,\\ (0,h_j)_i-(0,h_i)_j &\quad 0\le i<j\le f-1, \end{aligned}$$

and

$$-(0,wt_i)_j \quad \text{if } i\in J, \text{ any } j, \tag{97}$$

$$(wt_j,0)_i-(wt_i,0)_j \quad \text{if } \{i<j\}\subseteq J, \tag{98}$$

$$(wt_j,0)_i \quad \text{if } i\notin J,\ j\in J, \tag{99}$$

where $w\in\{y_0,\dots,y_{f-1},z_0,\dots,z_{f-1}\}$ is arbitrary. By the Poincaré–Birkhoff–Witt Theorem, the only linear relations occur in (97), where $(0,t_jt_\ell)_i$ is listed twice for any $\{j<\ell\}\subseteq J$ and any i; in (98), where for any $\{i<j<\ell\}\subseteq J$ the elements

$$(t_\ell t_j,0)_i-(t_\ell t_i,0)_j,\ (t_it_\ell,0)_j-(t_it_j,0)_\ell,\ (t_jt_i,0)_\ell-(t_jt_\ell,0)_i,$$

add to zero, and in (99), where $(t_jt_\ell,0)_i$ is listed twice for any $\{j<\ell\}\subseteq J$, $i\notin J$. Therefore,

$$\begin{aligned}\dim_{\mathbb{F}}\operatorname{im}(\partial_2) &= f^2+\binom{f}{2}+\binom{f}{2}+2f^2k+2f\binom{k}{2}+2fk(f-k)-f\binom{k}{2}-\binom{k}{3}-(f-k)\binom{k}{2}\\ &= 2f^2(2k+1)-f(2k^2+1)+2\binom{k+1}{3}.\end{aligned}$$

We finally check that $\dim_{\mathbb{F}}\ker(\partial_1)-\dim_{\mathbb{F}}\operatorname{im}(\partial_2)$ equals the right-hand side of (96), as desired.

Step 2. We show that

$$\dim_{\mathbb{F}}\operatorname{Tor}_1^{\Lambda}(\Lambda/\mathfrak{m}^3,\pi^\vee)\ge\sum_{\lambda\in\mathscr{P}}\left(4f^3+(6-4k_\lambda)f^2+(2k_\lambda^2-2k_\lambda+1)f-\frac{1}{6}k_\lambda(k_\lambda-1)(2k_\lambda-1)\right).$$

Note that

$$\operatorname{Tor}_1^{\Lambda}(\Lambda/\mathfrak{m}^3,\pi^\vee)\cong\operatorname{Tor}_1^{\mathbb{F}[\![I/Z_1]\!]}(\mathbb{F}[\![I/Z_1]\!]\otimes_\Lambda\Lambda/\mathfrak{m}^3,\pi^\vee)\cong\bigoplus_{\chi:I\to\mathbb{F}^\times}\operatorname{Tor}_1^{\mathbb{F}[\![I/Z_1]\!]}(W_{\chi,3},\pi^\vee), \tag{100}$$

and this is dual to $\bigoplus_{\chi:I\to\mathbb{F}^\times}\operatorname{Ext}^1_{I/Z_1}(W_{\chi,3},\pi)$ by [BHH$^+$, Lemma 2.6.2].

By assumption (iv), $\operatorname{Ext}^1_{I/Z_1}(W_{\chi,3},\pi)=0$ if $\operatorname{JH}(W_{\chi,3})\cap\operatorname{JH}(\pi^{I_1})=\emptyset$. Assume that $\operatorname{JH}(W_{\chi,3})\cap\operatorname{JH}(\pi^{I_1})\neq\emptyset$ and let $0\le i<3$ be minimal such that $\operatorname{JH}(\chi\otimes\mathfrak{m}^i/\mathfrak{m}^{i+1})\cap\operatorname{JH}(\pi^{I_1})\neq\emptyset$. From Lemma 7.2.11 (with $n=2$) we deduce for this i that $\operatorname{JH}(\chi\otimes\mathfrak{m}^i/\mathfrak{m}^{i+1})\cap\operatorname{JH}(\pi^{I_1})$ is a singleton. For any $\lambda\in\mathscr{P}$ and $0\le i<3$ let $X_{\lambda,i}$ be the set of all χ such that $\operatorname{JH}(\chi\otimes\mathfrak{m}^i/\mathfrak{m}^{i+1})\cap\operatorname{JH}(\pi^{I_1})=\{\chi_\lambda\}$ and $\operatorname{JH}(\chi\otimes\mathfrak{m}^j/\mathfrak{m}^{j+1})\cap\operatorname{JH}(\pi^{I_1})=\emptyset$ for all $0\le j<i$. Let $X_\lambda\overset{\text{def}}{=}\bigsqcup_{0\le i<3}X_{\lambda,i}$. It will be sufficient to show that

$$\sum_{\chi\in X_\lambda}\dim_{\mathbb{F}}\operatorname{Ext}^1_{I/Z_1}(W_{\chi,3},\pi)\ge 4f^3+(6-4k)f^2+(2k^2-2k+1)f-\frac{1}{6}k(k-1)(2k-1),$$

where $k \overset{\text{def}}{=} k_\lambda$ for short.

If $\chi \in X_{\lambda,0}$, then $\chi = \chi_\lambda$ and

$$\dim_{\mathbb{F}} \operatorname{Ext}^1_{I/Z_1}(W_{\chi,3}, \pi) \geq 2f^2 + f + \binom{k+1}{3}$$

by Corollary 7.2.9. Moreover, $|X_{\lambda,0}| = 1$.

Suppose that $\chi \in X_{\lambda,1}$. Then the unique (up to scalar) nonzero morphism $\operatorname{Proj}_{I/Z_1} \chi_\lambda \to W_{\chi,3}$ factors through a morphism $i : W_{\chi_\lambda,2} \to W_{\chi,3}$, as the image is contained in $\operatorname{rad} W_{\chi,3} = \mathfrak{m} W_{\chi,3}$. Moreover, i is injective by [BHH+23, Lemma 6.1.2] and any Jordan–Hölder factor of $\operatorname{coker}(i)$ is not contained in $\operatorname{JH}(\pi^{I_1})$ by Lemma 7.2.11 (with $n = 2$). (We know the constituents of $W_{\chi,3}$ and their multiplicities by [BHH+23, (44)].) Hence i induces an isomorphism $\operatorname{Ext}^1_{I/Z_1}(W_{\chi,3}, \pi) \xrightarrow{\sim} \operatorname{Ext}^1_{I/Z_1}(W_{\chi_\lambda,2}, \pi)$, which has dimension $2f + \binom{k}{2}$ by Proposition 7.2.8. From [BHH+, Lemma 2.3.6(ii)] applied with $\sum_j |i_j| \leq 1$ it follows that $|X_{\lambda,1}| = 2f - k$.

If $\chi \in X_{\lambda,2}$, then $\operatorname{JH}(W_{\chi,3}) \cap \operatorname{JH}(\pi^{I_1}) = \{\chi_\lambda\}$ (with multiplicity one), so $\operatorname{Ext}^1_{I/Z_1}(W_{\chi,3}, \pi) \xleftarrow{\sim} \operatorname{Ext}^1_{I/Z_1}(\chi_\lambda, \pi)$, which has dimension $2f$ by assumption (iv). We claim that $|X_{\lambda,2}| = 2f^2 - 2kf + \binom{k+1}{2}$. Let again $J \overset{\text{def}}{=} \{0 \leq j \leq f-1 : t_j \neq y_j z_j\}$, which depends on λ. Let $\varepsilon_j \overset{\text{def}}{=} -1$ if $t_j = y_j$, $\varepsilon_j \overset{\text{def}}{=} +1$ if $t_j = z_j$, and $\varepsilon_j \in \{\pm 1\}$ arbitrary for $j \notin J$. Note that for integers $i_j \in \mathbb{Z}$ $(0 \leq j \leq f-1)$ such that $\sum_j |i_j| \leq 2$ we have $\chi_\lambda \prod_j \alpha_j^{\varepsilon_j i_j} \in \operatorname{JH}(\pi^{I_1})$ if and only if $i_j \in \{0,1\}$ if $j \in J$ and $i_j = 0$ if $j \notin J$, cf. [BHH+, Lemma 2.3.6(ii)] (with $m = 2$). Using Lemma 7.2.11 (with $n = 2$) we deduce

$$\begin{aligned} X_{\lambda,2} = \{&\chi_\lambda \alpha_j^{-\varepsilon_j} \alpha_{j'}^{-\varepsilon_{j'}} (\{j \leq j'\} \subseteq J),\ \chi_\lambda \alpha_j^{-\varepsilon_j} \alpha_{j'}^{\pm 1} (j \in J, j' \notin J), \\ &\chi_\lambda \alpha_j^{\pm 2} (j \notin J),\ \chi_\lambda \alpha_j^{\pm 1} \alpha_{j'}^{\pm 1} (\{j < j'\} \subseteq J^c)\}, \end{aligned}$$

which has cardinality

$$\binom{k+1}{2} + 2k(f-k) + 2(f-k) + 4\binom{f-k}{2} = 2f^2 - 2kf + \binom{k+1}{2}.$$

We conclude by

$$\begin{aligned} \sum_{\chi \in X_\lambda} \dim_{\mathbb{F}} \operatorname{Ext}^1_{I/Z_1}(W_{\chi,3}, \pi) &\geq \left(2f^2 + f + \binom{k+1}{3}\right) + (2f-k)\left(2f + \binom{k}{2}\right) \\ &\quad + \left((2f^2 - 2kf + \binom{k+1}{2}\right)(2f) \\ &= 4f^3 + (6-4k)f^2 + (2k^2 - 2k + 1)f - \frac{1}{6}k(k-1)(2k-1). \qquad \square \end{aligned}$$

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