Parity of ranks of abelian surfaces

Céline Maistret

joint with Vladimir Dokchitser and Holly Green

University of Bristol

November 30, 2021

Theorems (Dokchitser V., M.; Green H, M.)

Let K be a number field. Assuming finiteness of Ш, the Birch and Swinnerton-Dyer conjecture correctly predicts the parity of the rank of

- all semistable* principally polarized abelian surfaces over K,
- $E_1 \times E_2/K$, for elliptic curves E_1, E_2 with isomorphic 2-torsion groups.

*good ordinary reduction a places above 2.

Ranks of abelian varieties and conjectures

Mordell-Weil Theorem

Let A/K be an abelian variety over a number field

$$A(K) \simeq \mathbb{Z}^{rk(A)} \oplus T, \quad rk_A, |T| < \infty.$$

Birch and Swinnerton-Dyer conjecture

Granting analytic continuation of the L-function of A/K to $\mathbb{C}$,

$$rk(A) = ord_{s=1} L(A/K, s) =: rk_{an}(A).$$

Conjectural functional equation

The completed L-function $L^*(A/K, s)$ satisfies

$$L^*(A/K, s) = W(A)\, L^*(A/K, 2-s), \quad W(A) \in \{\pm 1\}.$$

Ranks of abelian varieties and conjectures

Mordell-Weil Theorem

Let A/K be an abelian variety over a number field

$$A(K) \simeq \mathbb{Z}^{rk(A)} \oplus T, \quad rk_A, |T| < \infty.$$

Birch and Swinnerton-Dyer conjecture

Granting analytic continuation of the L-function of A/K to $\mathbb{C}$,

$$rk(A) = ord_{s=1} L(A/K, s) =: rk_{an}(A).$$

Conjectural functional equation

The completed L-function $L^*(A/K, s)$ satisfies

$$L^*(A/K, s) = W(A)\, L^*(A/K, 2-s), \quad W(A) \in \{\pm 1\}.$$

Ranks of abelian varieties and conjectures

Mordell-Weil Theorem

Let A/K be an abelian variety over a number field

$$A(K) \simeq \mathbb{Z}^{rk(A)} \oplus T, \quad rk_A, |T| < \infty.$$

Birch and Swinnerton-Dyer conjecture

Granting analytic continuation of the L-function of A/K to $\mathbb{C}$,

$$rk(A) = ord_{s=1} L(A/K, s) =: rk_{an}(A).$$

Conjectural functional equation

The completed L-function $L^*(A/K, s)$ satisfies

$$L^*(A/K, s) = W(A)\ L^*(A/K, 2-s), \quad W(A) \in \{\pm 1\}.$$

Parity of analytic rank

Analytic rank

$$rk_{an}(A) := ord_{s=1} L(A/K, s).$$

Sign in functional equation

$$L^*(A/K, s) = W(A)\ L^*(A/K, 2-s), \quad W(A) \in \{\pm 1\}.$$

Consequence

$$(-1)^{rk_{an}(A)} = W(A).$$

Parity conjecture

B.S.D. modulo 2

$$(-1)^{rk(A)} \underset{BSD}{=} (-1)^{rk_{an}(A)} = W(A).$$

Global root number

The sign in the functional equation $W(A)$ is conjectured to be equal to the global root number of A:

$$W(A) = w(A).$$

Parity conjecture

$$(-1)^{rk(A)} = w(A).$$

Parity conjecture

B.S.D. modulo 2

$$(-1)^{rk(A)} \underset{BSD}{=} (-1)^{rk_{an}(A)} = W(A).$$

Global root number

The sign in the functional equation $W(A)$ is conjectured to be equal to the global root number of A:

$$W(A) = w(A).$$

Parity conjecture

$$(-1)^{rk(A)} = w(A).$$

Parity conjecture

B.S.D. modulo 2

$$(-1)^{rk(A)} \underset{BSD}{=} (-1)^{rk_{an}(A)} = W(A).$$

Global root number

The sign in the functional equation $W(A)$ is conjectured to be equal to the global root number of A:

$$W(A) = w(A).$$

Parity conjecture

$$(-1)^{rk(A)} = w(A).$$

Parity conjecture

B.S.D. modulo 2

$$(-1)^{rk(A)} \underset{BSD}{=} (-1)^{rk_{an}(A)} = W(A).$$

Global root number

The sign in the functional equation $W(A)$ is conjectured to be equal to the global root number of A:

$$W(A) = w(A) = \prod_v w_v(A)$$

Parity conjecture

$$(-1)^{rk(A)} = w(A)$$

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

Does it have a point of infinite order?

Using Parity conjecture

$$(-1)^{rk(E)} = \prod_v w_v = w_\infty \cdot w_5 \cdot w_{13}$$

$$w_5 = w_{13} = 1, \qquad w_\infty = -1$$

E has odd rank

$$(-1)^{rk(E)} = -1 \cdot 1 \cdot 1 = -1.$$

E has a point of infinite order over $\mathbb{Q}$.

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

Does it have a point of infinite order?

Using Parity conjecture

$$(-1)^{rk(E)} = \prod_v w_v = w_\infty \cdot w_5 \cdot w_{13}$$

$$w_5 = w_{13} = 1, \qquad w_\infty = -1$$

E has odd rank

$$(-1)^{rk(E)} = -1 \cdot 1 \cdot 1 = -1.$$

E has a point of infinite order over $\mathbb{Q}$.

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

Does it have a point of infinite order?

Using Parity conjecture

$$(-1)^{rk(E)} = \prod_v w_v = w_\infty \cdot w_5 \cdot w_{13}$$

$$w_5 = w_{13} = 1, \qquad w_\infty = -1$$

E has odd rank

$$(-1)^{rk(E)} = -1 \cdot 1 \cdot 1 = -1.$$

E has a point of infinite order over $\mathbb{Q}$.

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

Does it have a point of infinite order?

Using Parity conjecture

$$(-1)^{rk(E)} = \prod_v w_v = w_\infty \cdot w_5 \cdot w_{13}$$

$$w_5 = w_{13} = 1, \qquad w_\infty = -1$$

E has odd rank

$$(-1)^{rk(E)} = -1 \cdot 1 \cdot 1 = -1.$$

E has a point of infinite order over $\mathbb{Q}$.

Known results

Česnavičius; Coates-Fukaya-Kato-Sujatha; Dokchitser-Dokchitser; Kramer-Tunnell; Monsky; Morgan; Nekovář.

(p)-parity conjecture is known for

- elliptic curves over $\mathbb{Q}$,
- elliptic curves over K admitting a p-isogeny,
- elliptic curves over totally real number field when $p \neq 2$ (all non CM cases and some CM cases for $p = 2$),

$\Rightarrow$ open for elliptic curves over number fields in general,

- Jacobians of hyperelliptic curves base-changed from a subfield of index 2,
- abelian varieties admitting a suitable isogeny.

Computing the parity of rank of abelian varieties

$$(-1)^{rk(A)} = w(A).$$

Computing the parity of the rank of elliptic curves

BSD 1

$$rk_E = ord_{s=1} L(E, s)$$

BSD 2 (BSD quotient)

$$lim_{s=1} \frac{L(E,s)}{(s-1)^{rk_E}} = \frac{\Omega_{\mathbb{R}} \prod_p c_p Reg_E |Ш|}{|E_{tors}|^2}$$

Theorem (Cassels) Isogeny invariance of B.S.D. quotient

Assuming $Ш(E)$ is finite, if $\phi : E \to E'$ is an isogeny defined over $\mathbb{Q}$ then

$$\frac{\Omega_{\mathbb{R}} \prod_p c_p Reg_E |Ш(E)|}{|E_{tors}|^2} = \frac{\Omega'_{\mathbb{R}} \prod_p c'_p Reg'_E |Ш(E')|}{|E'_{tors}|^2}$$

Computing the parity of the rank of elliptic curves

BSD 1

$$rk_E = ord_{s=1} L(E, s)$$

BSD 2 (BSD quotient)

$$lim_{s=1} \frac{L(E, s)}{(s-1)^{rk_E}} = \frac{\Omega_{\mathbb{R}} \prod_p c_p Reg_E |Ш|}{|E_{tors}|^2}$$

Theorem (Cassels) Isogeny invariance of B.S.D. quotient

Assuming $Ш(E)$ is finite, if $\phi : E \to E'$ is an isogeny defined over $\mathbb{Q}$ then

$$\frac{\Omega_{\mathbb{R}} \prod_p c_p Reg_E |Ш(E)|}{|E_{tors}|^2} = \frac{\Omega'_{\mathbb{R}} \prod_p c'_p Reg'_E |Ш(E')|}{|E'_{tors}|^2}$$

Computing the parity of the rank of elliptic curves

BSD 1

$$rk_E = ord_{s=1} L(E,s)$$

BSD 2 (BSD quotient)

$$lim_{s=1} \frac{L(E,s)}{(s-1)^{rk_E}} = \frac{\Omega_{\mathbb{R}} \prod_p c_p Reg_E |Ш|}{|E_{tors}|^2}$$

Theorem (Cassels) Isogeny invariance of B.S.D. quotient

Assuming $Ш(E)$ is finite, if $\phi : E \to E'$ is an isogeny defined over $\mathbb{Q}$ then

$$\frac{\Omega_{\mathbb{R}} \prod_p c_p Reg_E |Ш(E)|}{|E_{tors}|^2} = \frac{\Omega'_{\mathbb{R}} \prod_p c'_p Reg'_E |Ш(E')|}{|E'_{tors}|^2}$$

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

$E/\mathbb{Q}$ admits a 2-isogeny

Using Cassel's theorem

$$c_5 = c_{13} = 1, \quad c'_5 = c'_{13} = 2, \quad \Omega_{\mathbb{R}} = 2\Omega'_{\mathbb{R}}$$

$$\Rightarrow \frac{Reg_E}{Reg_{E'}} = \frac{|Ш(E)||E'(\mathbb{Q})_{tors}|^2 \Omega_{\mathbb{R}} \prod_p c_p}{|Ш(E')||E(\mathbb{Q})_{tors}|^2 \Omega'_{\mathbb{R}} \prod_p c'_p} = \frac{\Omega_{\mathbb{R}} \prod_p c_p}{\Omega'_{\mathbb{R}} \prod_p c'_p} \cdot \square = \frac{2}{4} \cdot \square \neq 1$$

$\Rightarrow E$ has a point of infinite order.

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x,\ \Delta_E = 5 \cdot 13$

$E/\mathbb{Q}$ admits a 2-isogeny

Using Cassel's theorem

$$c_5 = c_{13} = 1, \quad c'_5 = c'_{13} = 2, \quad \Omega_{\mathbb{R}} = 2\Omega'_{\mathbb{R}}$$

$$\Rightarrow \frac{Reg_E}{Reg_{E'}} = \frac{|\text{Ш}(E)||E'(\mathbb{Q})_{tors}|^2 \Omega_{\mathbb{R}} \prod_p c_p}{|\text{Ш}(E')||E(\mathbb{Q})_{tors}|^2 \Omega'_{\mathbb{R}} \prod_p c'_p} = \frac{\Omega_{\mathbb{R}} \prod_p c_p}{\Omega'_{\mathbb{R}} \prod_p c'_p} \cdot \square = \frac{2}{4} \cdot \square \neq 1$$

$\Rightarrow E$ has a point of infinite order.

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

Lemma (Dokchitser-Dokchitser)

If ϕ is an isogeny of degree d such that $\phi^\vee \phi = \phi\phi^\vee = [d]$ then

$$\frac{Reg_E}{Reg_{E'}} = d^{rk(E)} \cdot \square$$

$c_5 = c_{13} = 1, \quad c_5' = c_{13}' = 2, \quad \Omega_{\mathbb{R}} = 2\Omega_{\mathbb{R}}'$

$$\Rightarrow \frac{Reg_E}{Reg_{E'}} = \frac{|Ш(E)||E'(\mathbb{Q})_{tors}|^2 \Omega_{\mathbb{R}} \prod_p c_p}{|Ш(E')||E(\mathbb{Q})_{tors}|^2 \Omega_{\mathbb{R}}' \prod_p c_p'} = \frac{\Omega_{\mathbb{R}} \prod_p c_p}{\Omega_{\mathbb{R}}' \prod_p c_p'} \cdot \square = \frac{2}{4} \cdot \square = 2 \cdot \square$$

$\Rightarrow E$ has odd rank

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

Lemma (Dokchitser-Dokchitser)

If ϕ is an isogeny of degree d such that $\phi^\vee\phi = \phi\phi^\vee = [d]$ then

$$\frac{Reg_E}{Reg_{E'}} = d^{rk(E)} \cdot \square$$

$$c_5 = c_{13} = 1, \quad c'_5 = c'_{13} = 2, \quad \Omega_{\mathbb{R}} = 2\Omega'_{\mathbb{R}}$$

$$\Rightarrow \frac{Reg_E}{Reg_{E'}} = \frac{|Ш(E)||E'(\mathbb{Q})_{tors}|^2\Omega_{\mathbb{R}}\prod_p c_p}{|Ш(E')||E(\mathbb{Q})_{tors}|^2\Omega'_{\mathbb{R}}\prod_p c'_p} = \frac{\Omega_{\mathbb{R}}\prod_p c_p}{\Omega'_{\mathbb{R}}\prod_p c'_p}\cdot\square = \frac{2}{4}\cdot\square = 2\cdot\square$$

$\Rightarrow E$ has odd rank

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x$, $\Delta_E = 5 \cdot 13$

Lemma (Dokchitser-Dokchitser)

If ϕ is an isogeny of degree d such that $\phi^\vee \phi = \phi\phi^\vee = [d]$ then

$$\frac{Reg_E}{Reg_{E'}} = d^{rk(E)} \cdot \square$$

$$c_5 = c_{13} = 1, \quad c'_5 = c'_{13} = 2, \quad \Omega_\mathbb{R} = 2\Omega'_\mathbb{R}$$

$$\Rightarrow \frac{Reg_E}{Reg_{E'}} = \frac{|Ш(E)||E'(\mathbb{Q})_{tors}|^2 \Omega_\mathbb{R} \prod_p c_p}{|Ш(E')||E(\mathbb{Q})_{tors}|^2 \Omega'_\mathbb{R} \prod_p c'_p} = \frac{\Omega_\mathbb{R} \prod_p c_p}{\Omega'_\mathbb{R} \prod_p c'_p} \cdot \square = \frac{2}{4} \cdot \square = 2 \cdot \square$$

$\Rightarrow E$ has odd rank

Computing the parity of the rank

For an elliptic curve E with a p isogeny ϕ to E'

$$p^{rk(E)} = \frac{\Omega_E}{\Omega_{E'}} \prod_\ell \frac{c_\ell}{c'_\ell} \cdot \square$$

For an elliptic curve E with a p isogeny ϕ to E'

$$(-1)^{rk(E)} = (-1)^{ord_p\left(\frac{\Omega_E}{\Omega_{E'}} \prod_\ell \frac{c_\ell}{c'_\ell}\right)}$$

In general

For an abelian variety A with an isogeny ϕ satisfying $\phi\phi^\vee = [p]$

$$(-1)^{rk(A)} = (-1)^{ord_p\left(\frac{\Omega_A}{\Omega_{A'}} \prod_\ell \frac{c_\ell(A)}{c_\ell(A')} \frac{|Ш(A)|}{|Ш(A')|}\right)}$$

Computing the parity of the rank

For an elliptic curve E with a p isogeny ϕ to E'

$$p^{rk(E)} = \frac{\Omega_E}{\Omega_{E'}} \prod_{\ell} \frac{c_\ell}{c'_\ell} \cdot \square$$

For an elliptic curve E with a p isogeny ϕ to E'

$$(-1)^{rk(E)} = (-1)^{ord_p\left(\frac{\Omega_E}{\Omega_{E'}} \prod_\ell \frac{c_\ell}{c'_\ell}\right)}$$

In general

For an abelian variety A with an isogeny ϕ satisfying $\phi\phi^\vee = [p]$

$$(-1)^{rk(A)} = (-1)^{ord_p\left(\frac{\Omega_A}{\Omega_{A'}} \prod_\ell \frac{c_\ell(A)}{c_\ell(A')} \frac{|Ш(A)|}{|Ш(A')|}\right)}$$

Computing the parity of the rank

For an elliptic curve E with a p isogeny ϕ to E'

$$p^{rk(E)} = \frac{\Omega_E}{\Omega_{E'}} \prod_{\ell} \frac{c_\ell}{c'_\ell} \cdot \square$$

For an elliptic curve E with a p isogeny ϕ to E'

$$(-1)^{rk(E)} = (-1)^{ord_p\left(\frac{\Omega_E}{\Omega_{E'}} \prod_\ell \frac{c_\ell}{c'_\ell}\right)}$$

In general

For an abelian variety A with an isogeny ϕ satisfying $\phi\phi^\vee = [p]$

$$(-1)^{rk(A)} = (-1)^{ord_p\left(\frac{\Omega_A}{\Omega_{A'}} \prod_\ell \frac{c_\ell(A)}{c_\ell(A')} \frac{|Ш(A)|}{|Ш(A')|}\right)}$$

Parity conjecture

$$(-1)^{rk(A)} = w(A).$$

Proving the parity conjecture

For an abelian variety A with an isogeny ϕ satisfying $\phi\phi^\vee = [p]$

$$(-1)^{rk(A)} = (-1)^{ord_p\left(\frac{\Omega_A}{\Omega_{A'}} \prod_\ell \frac{c_\ell(A)}{c_\ell(A')} \frac{|Ш(A)|}{|Ш(A')|}\right)} \stackrel{?}{=} w(A) = \prod_v w_v(A)$$

Parity conjecture

$$(-1)^{rk(A)} = w(A).$$

Proving the parity conjecture

For an abelian variety A with an isogeny ϕ satisfying $\phi\phi^\vee = [p]$

$$(-1)^{rk(A)} = (-1)^{ord_p\left(\frac{\Omega_A}{\Omega_{A'}} \prod_\ell \frac{c_\ell(A)}{c_\ell(A')} \frac{|Ш(A)|}{|Ш(A')|}\right)} \stackrel{?}{=} w(A) = \prod_v w_v(A)$$

Parity conjecture for principally polarized abelian surfaces

▼ Types of p.p. abelian surfaces

Theorem (see Gonzales-Guàrdia-Rotger)

Let A/K be a principally polarized abelian surface defined over a number field K. Then A is one of the following three types:

- $A \simeq_K J(C)$, where C/K is a smooth curve of genus 2,
- $A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K,
- $A \simeq_K Res_{F/K}E$, where $Res_{F/K}E$ is the Weil restriction of an elliptic curve defined over a quadratic extension F/K.

Parity conjecture for principally polarized abelian surfaces

▼ Types of p.p. abelian surfaces

Theorem (see Gonzales-Guàrdia-Rotger)

Let A/K be a principally polarized abelian surface defined over a number field K. Then A is one of the following three types:

- $A \simeq_K J(C)$, where C/K is a smooth curve of genus 2,
- $A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K,
- $A \simeq_K Res_{F/K}E$, where $Res_{F/K}E$ is the Weil restriction of an elliptic curve defined over a quadratic extension F/K.

Parity conjecture for principally polarized abelian surfaces

▼ Types of p.p. abelian surfaces

Theorem (see Gonzales-Guàrdia-Rotger)

Let A/K be a principally polarized abelian surface defined over a number field K. Then A is one of the following three types:

- $A \simeq_K J(C)$, where C/K is a smooth curve of genus 2,
- $A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K,
- $A \simeq_K Res_{F/K}E$, where $Res_{F/K}E$ is the Weil restriction of an elliptic curve defined over a quadratic extension F/K.

Parity conjecture for principally polarized abelian surfaces

Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Jac(C) with $C : y^2 = f(x)$ and $deg(f) = 6$
- Reduce to Jacobians with specific 2-torsions
  - Regulator constant
- Use BSD invariance under isogeny to compute parity of rank
  - Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(J)} = \prod_v \lambda_v$
- Compute λ_v for all v
  - Ω_J, c_ℓ, μ_v
- Compare λ_v and $w_v(J)$
  - $(-1)^{rk(J)} = \prod_v \lambda_v$ $\qquad \prod_v \lambda_v = \prod_v w_v(J) = w(J)$

Parity conjecture for principally polarized abelian surfaces

Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Jac(C) with $C : y^2 = f(x)$ and $deg(f) = 6$
- Reduce to Jacobians with specific 2-torsions
  - Regulator constant

Theorem: Regulator constants (T. Dokchitser V. Dokchitser)

Suppose
• $C : y^2 = f(x)$ is semistable,
• K_f = splitting field of f,
• Parity conjecture holds for J/L for all $K \subseteq L \subseteq K_f$ with $Gal(K_f/L) \subseteq C_2 \times D_4$.
Then the parity conjecture holds for J/K.

Parity conjecture for principally polarized abelian surfaces

Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Types of p.p. abelian surfaces
- Reduce to Jacobians with specific 2-torsions
  - $C : y^2 = f(x)$ with $Gal(f) \subseteq C_2 \times D_4$
- Use BSD invariance under isogeny to compute parity of rank
  - Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(J)} = \prod_v \lambda_v$
- Compute λ_v for all v
  - Ω_J, c_ℓ, μ_v
- Compare λ_v and $w_v(J)$
  - $(-1)^{rk(J)} = \prod_v \lambda_v$ $\qquad$ $(-1)^{rk(J)} = \prod_v w_v(J) = w(J)$

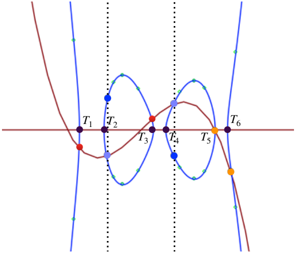
Parity conjecture for principally polarized abelian surfaces

Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Types of p.p. abelian surfaces
- Reduce to Jacobians with specific 2-torsions
  - $C : y^2 = f(x)$ with $Gal(f) \subseteq C_2 \times D_4$
- Use BSD invariance under isogeny to compute parity of rank

▼

2 torsions: $J(\overline{K})[2] = \{[T_i, T_k], i \neq k\} \cup \{0\}$, where $T_i = (x_i, 0) \in C(\overline{K})$.



Proposition

If $Gal(f) \subseteq C_2 \times D_4$ then J admits a **Richelot isogeny** Φ s.t. $\Phi\Phi^\vee = [2]$.

Parity conjecture for principally polarized abelian surfaces

Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Types of p.p. abelian surfaces
- Reduce to Jacobians with specific 2-torsions
  - $C : y^2 = f(x)$ with $Gal(f) \subseteq C_2 \times D_4$
- Use BSD invariance under isogeny to compute parity of rank
  - $Gal(f) \subseteq C_2 \times D_4 \Rightarrow$ Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(J)} \prod_v \lambda_v$
- Compute λ_v for all v
  - Ω_J, c_ℓ, μ_v
- Compare λ_v and $w_v(J)$
  - $(-1)^{rk(J)} = \prod_v \lambda_v$ $\qquad$ $(-1)^{rk(J)} = \prod_v w_v(J) = w(J)$

Parity of the rank as a product of local terms

Using BSD invariance under isogeny

For a Jacobian J with a Richelot isogeny ϕ to J' (i.e. $\phi\phi^\vee = [2]$)

$$(-1)^{rk(J)} = (-1)^{ord_2\left(\frac{\Omega_J}{\Omega_{J'}} \prod_\ell \frac{c_\ell(J)}{c_\ell(J')} \frac{|\text{Ш}(J)|}{|\text{Ш}(J')|}\right)}$$

Theorem

Assume that $Gal(f) \subseteq C_2 \times D_4$. Then

$$(-1)^{rk(J)} = \prod_v (-1)^{ord_2\left(\frac{c_v \mu_v}{c'_v \mu'_v}\right)},$$

where c_v, c'_v denote the Tamagawa numbers of J and J' respectively and $\mu_v = 2$ if C is deficient at v, $\mu_v = 1$ otherwise (cf Poonen-Stoll).

Parity of the rank as a product of local terms

Using BSD invariance under isogeny

For a Jacobian J with a Richelot isogeny ϕ to J' (i.e. $\phi\phi^\vee = [2]$)

$$(-1)^{rk(J)} = (-1)^{ord_2\left(\frac{\Omega_J}{\Omega_{J'}} \prod_\ell \frac{c_\ell(J)}{c_\ell(J')} \frac{|\text{Ш}(J)|}{|\text{Ш}(J')|}\right)}$$

Theorem

Assume that $Gal(f) \subseteq C_2 \times D_4$. Then

$$(-1)^{rk(J)} = \prod_v (-1)^{ord_2\left(\frac{c_v \mu_v}{c'_v \mu'_v}\right)},$$

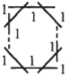
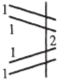
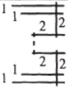
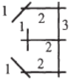
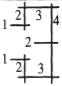
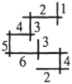
where c_v, c'_v denote the Tamagawa numbers of J and J' respectively and $\mu_v = 2$ if C is deficient at v, $\mu_v = 1$ otherwise (cf Poonen-Stoll).

Parity conjecture for principally polarized abelian surfaces

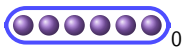
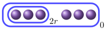
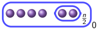
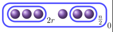
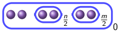
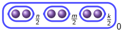
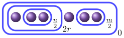
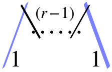
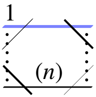
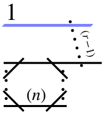
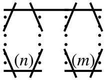
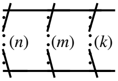
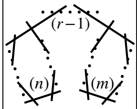
Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Types of p.p. abelian surfaces
- Reduce to Jacobians with specific 2-torsions
  - $C : y^2 = f(x)$ with $Gal(f) \subseteq C_2 \times D_4$
- Use BSD invariance under isogeny to compute parity of rank
  - $Gal(f) \subseteq C_2 \times D_4 \Rightarrow$ Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(J)} = \prod_v (-1)^{ord_2(\frac{c_v \mu_v}{c'_v \mu'_v})}$,
- Compute λ_v for all v
  - Ω_J, c_ℓ, μ_v
- Compare λ_v and $w_v(J)$
  - $$(-1)^{rk(J)} = \prod_v \lambda_v \qquad (-1)^{rk(J)} = \prod_v w_v(J) = w(J)$$

Local arithmetic of elliptic curves

Kodaira symbol	I_0	I_n $(n \geq 1)$	II	III	IV	I_0^*	I_n^* $(n \geq 1)$	IV^*	III^*	II^*
Special fiber $\tilde{\mathcal{C}}$ (The numbers indicate multiplicities)										
$m =$ number of irred. components	1	n	1	2	3	5	$5+n$	7	8	9
$E(K)/E_0(K) \cong \tilde{\mathcal{E}}(k)/\tilde{\mathcal{E}}^0(k)$	(0)	$\frac{\mathbb{Z}}{n\mathbb{Z}}$	(0)	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$ n even; $\frac{\mathbb{Z}}{4\mathbb{Z}}$ n odd	$\frac{\mathbb{Z}}{3\mathbb{Z}}$	$\frac{\mathbb{Z}}{2\mathbb{Z}}$	(0)
$\tilde{\mathcal{E}}^0(k)$	$\tilde{E}(k)$	k^*	k^+	k^+	k^+	k^+	k^+	k^+	k^+	k^+
Entries below this line only valid for $\mathrm{char}(k) = p$ **as indicated**										
$\mathrm{char}(k) = p$			$p \neq 2, 3$	$p \neq 2$	$p \neq 3$	$p \neq 2$	$p \neq 2$	$p \neq 3$	$p \neq 2$	$p \neq 2, 3$
$v(\mathcal{D}_{E/K})$ (discriminant)	0	n	2	3	4	6	$6+n$	8	9	10
$f(E/K)$ (conductor)	0	1	2	2	2	2	2	2	2	2
behavior of j	$v(j) \geq 0$	$v(j) = -n$	$\tilde{j} = 0$	$\tilde{j} = 1728$	$\tilde{j} = 0$	$v(j) \geq 0$	$v(j) = -n$	$\tilde{j} = 0$	$\tilde{j} = 1728$	$\tilde{j} = 0$

Local arithmetic of hyperelliptic curves, p **odd** (joint with T. and V. Dokchitser and A. Morgan)

Cluster Picture							
$\overline{\mathscr{C}}$							
Number of components	1	$r+1$	n	$n+r$	$n+m-1$	$n+m+k-1$	$n+m+r-1$
$\overline{\mathscr{J}}(k)/\overline{\mathscr{J}}^0(k)$	(0)	(0)	$\frac{\mathbb{Z}}{n\mathbb{Z}}$	$\frac{\mathbb{Z}}{n\mathbb{Z}}$	$\frac{\mathbb{Z}}{n\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}$	$\frac{\mathbb{Z}}{d\mathbb{Z}} \times \frac{\mathbb{Z}}{t\mathbb{Z}}$ $d = gcd(n,m,k)$ $t = (nm+nk+mk)/d$	$\frac{\mathbb{Z}}{n\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}$
c_p	1	1	n	n	nm	$nm+nk+km$	nm
$v(\Delta_{min})$	0	$12r$	n	$12r+n$	$n+m$	$n+m+k$	$12r+n+m$
$f(C/K)$	0	0	1	1	2	2	2

Parity conjecture for principally polarized abelian surfaces

Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Types of p.p. abelian surfaces
- Reduce to Jacobians with specific 2-torsions
  - $C : y^2 = f(x)$ with $Gal(f) \subseteq C_2 \times D_4$
- Use BSD invariance under isogeny to compute parity of rank
  - $Gal(f) \subseteq C_2 \times D_4 \Rightarrow$ Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(J)} = \prod_v (-1)^{ord_2(\frac{c_v \mu_v}{c'_v \mu'_v})}$,
- Compute λ_v for all v
  - Ω_J, c_ℓ, μ_v
- Compare λ_v and $w_v(J)$
  - $$(-1)^{rk(J)} = \prod_v \lambda_v \qquad (-1)^{rk(J)} = \prod_v w_v(J) = w(J)$$

Parity conjecture for principally polarized abelian surfaces

Strategy

- Types of p.p. abelian surfaces

Theorem (see Gonzales-Guàrdia-Rotger)

Let A/K be a principally polarized abelian surface defined over a number field K. Then A is one of the following three types:

- $A \simeq_K J(C)$, where C/K is a smooth curve of genus 2,
- $A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K,
- $A \simeq_K Res_{F/K} E$, where $Res_{F/K} E$ is the Weil restriction of an elliptic curve defined over a quadratic extension F/K.

Parity conjecture for principally polarized abelian surfaces

Strategy

$A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K

- Use BSD invariance under isogeny to compute parity of rank
  - $E_1[2] \simeq E_2[2] \Rightarrow$ Singular Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(JE_1 \times E_2)} = \prod_v \lambda_v$
- Compute λ_v for all v
  - $\Omega_{E_1 \times E_2}, c_\ell, \mu_v$
- Compare λ_v and $w_v(E_1 \times E_2)$
  - $(-1)^{rk(E_1 \times E_2)} = \prod_v \lambda_v \qquad \prod_v \lambda_v = \prod_v w_v(E_1 \times E_2) = w(E_1 \times E_2)$

Parity conjecture for principally polarized abelian surfaces

Strategy

$A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K

- Use BSD invariance under isogeny to compute parity of rank
  - $E_1[2] \simeq E_2[2] \Rightarrow$ Singular Richelot isogeny

Let $f(x)$ be a separable monic cubic polynomial with $f(0) \neq 0$. Then (up to quadratic twists)

- $E_1 \simeq y^2 = f(x), \quad E_2 \simeq y^2 = xf(x),$
- there exists $\phi : E \times \mathrm{Jac}E' \to \mathrm{Jac}C$, where $C : y^2 = f(x^2)$; such that $\phi\phi^\vee = [2]$.

Parity conjecture for principally polarized abelian surfaces

Strategy

$A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K

- Use BSD invariance under isogeny to compute parity of rank
  - $E_1[2] \simeq E_2[2] \Rightarrow$ Singular Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(E_1 \times E_2)} = \prod_v \lambda_v$
- Compute λ_v for all v
  - $\Omega_{E_1 \times E_2}, c_\ell, \mu_v$
- Compare λ_v and $w_v(E_1 \times E_2)$
  - $(-1)^{rk(E_1 \times E_2)} = \prod_v \lambda_v \qquad \prod_v \lambda_v = \prod_v w_v(E_1 \times E_2) = w(E_1 \times E_2)$

Parity conjecture for principally polarized abelian surfaces

Strategy

$A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K

- Use BSD invariance under isogeny to compute parity of rank
  - $E_1[2] \simeq E_2[2] \Rightarrow$ Singular Richelot isogeny
- Express the parity as a product of local terms
  -

Computing the parity of the rank

$$(-1)^{rk(E_1 \times E_2)} = (-1)^{ord_2\left(\frac{\Omega_{E_1 \times E_2}}{\Omega_{\mathrm{Jac}C}} \prod_\ell \frac{c_\ell(E_1 \times E_2)}{c_\ell(\mathrm{Jac}C)} \frac{1}{|\text{Ш}(\mathrm{Jac}C)|}\right)}$$

Parity conjecture for principally polarized abelian surfaces

Strategy

$A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K

- Use BSD invariance under isogeny to compute parity of rank
  - $E_1[2] \simeq E_2[2] \Rightarrow$ Singular Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(E_1 \times E_2)} = \prod_v \lambda_v$
- Compute λ_v for all v
  - $\Omega_{E_1 \times E_2}, c_\ell, \mu_v$
- Compare λ_v and $w_v(E_1 \times E_2)$
  - $(-1)^{rk(E_1 \times E_2)} = \prod_v \lambda_v \qquad \prod_v \lambda_v = \prod_v w_v(E_1 \times E_2) = w(E_1 \times E_2)$

Local comparison : Elliptic curves

Let $E/\mathbb{Q}$ and $E'/\mathbb{Q}$ be two elliptic curves related by a 2-isogeny

$E : y^2 = x(x^2 + ax + b)$ $\qquad E' : y^2 = x(x^2 - 2ax + (a^2 - 4b))$

Theorem (Dokchitser-Dokchitser)

$$(-1)^{ord_2(\frac{c_\ell}{c'_\ell})} = (-2a, a^2 - 4b)_\ell (a, -b)_\ell w_\ell$$

- $\prod_v (a, b)_v = 1$ (product formula for Hilbert symbols)
- non-split multiplicative reduction where $v(\Delta(E))$ is odd
- need "discriminant and field of definition of tangents"
- consider the real place to find right invariants

Local comparison : Elliptic curves

Let $E/\mathbb{Q}$ and $E'/\mathbb{Q}$ be two elliptic curves related by a 2-isogeny

$E : y^2 = x(x^2 + ax + b) \qquad E' : y^2 = x(x^2 - 2ax + (a^2 - 4b))$

Theorem (Dokchitser-Dokchitser)

$$(-1)^{ord_2(\frac{c_\ell}{c'_\ell})} = (-2a, a^2 - 4b)_\ell (a, -b)_\ell w_\ell$$

- $\prod_v (a, b)_v = 1$ (product formula for Hilbert symbols)
- non-split multiplicative reduction where $v(\Delta(E))$ is odd
- need "discriminant and field of definition of tangents"
- consider the real place to find right invariants

Local comparison : Jacobians of $C_2 \times D_4$ genus 2 curves

Theorem

If $Gal(f) \subseteq C_2 \times D_4$ and C is semistable at v (and good ordinary above 2) then

$$(-1)^{ord_2\left(\frac{c_v \mu_v}{c'_v \mu'_v}\right)} = E_v \cdot w_v.$$

For each place v of K, define the following Hilbert symbols at v

$$\begin{aligned} E_v = \ & (\delta_2 + \delta_3, -\ell_1^2 \delta_2 \delta_3) \cdot \\ & (\delta_2 \eta_2 + \delta_3 \eta_3, -\ell_1^2 \eta_2 \eta_3 \delta_2 \delta_3) \cdot \\ & (\hat{\delta}_2 \eta_3 + \hat{\delta}_3 \eta_2, -\ell_1^2 \eta_2 \eta_3 \hat{\delta}_2 \hat{\delta}_3) \cdot \\ & (\xi, -\delta_1 \hat{\delta}_2 \hat{\delta}_3) \cdot (\eta_2 \eta_3, -\delta_2 \delta_3 \hat{\delta}_2 \hat{\delta}_3) \cdot (\eta_1, -\delta_2 \delta_3 \Delta^2 \hat{\delta}_1) \cdot \\ & (c, \delta_1 \delta_2 \delta_3 \hat{\delta}_2 \hat{\delta}_3) \cdot (\hat{\delta}_1, \tfrac{\ell_1}{\Delta}) \cdot (\ell_1^2, \ell_2 \ell_3) \cdot (2, -\ell_1^2) \cdot (\hat{\delta}_2 \hat{\delta}_3, -2) \end{aligned}$$

Local comparison : Jacobians of $C_2 \times D_4$ genus 2 curves

Theorem

If $Gal(f) \subseteq C_2 \times D_4$ and C is semistable at v (and good ordinary above 2) then

$$(-1)^{ord_2\left(\frac{c_v \mu_v}{c'_v \mu'_v}\right)} = E_v \cdot w_v.$$

For each place v of K, define the following Hilbert symbols at v

$$\begin{aligned} E_v \;=\; & (\delta_2 + \delta_3, -\ell_1^2 \delta_2 \delta_3) \cdot \\ & (\delta_2 \eta_2 + \delta_3 \eta_3, -\ell_1^2 \eta_2 \eta_3 \delta_2 \delta_3) \cdot \\ & (\hat{\delta}_2 \eta_3 + \hat{\delta}_3 \eta_2, -\ell_1^2 \eta_2 \eta_3 \hat{\delta}_2 \hat{\delta}_3) \cdot \\ & (\xi, -\delta_1 \hat{\delta}_2 \hat{\delta}_3) \cdot (\eta_2 \eta_3, -\delta_2 \delta_3 \hat{\delta}_2 \hat{\delta}_3) \cdot (\eta_1, -\delta_2 \delta_3 \Delta^2 \hat{\delta}_1) \cdot \\ & (c, \delta_1 \delta_2 \delta_3 \hat{\delta}_2 \hat{\delta}_3) \cdot (\hat{\delta}_1, \tfrac{\ell_1}{\Delta}) \cdot (\ell_1^2, \ell_2 \ell_3) \cdot (2, -\ell_1^2) \cdot (\hat{\delta}_2 \hat{\delta}_3, -2) \end{aligned}$$

Parity conjecture for principally polarized abelian surfaces

Strategy

- Reduce to Jacobians of hyperelliptic curves of genus 2
  - Types of p.p. abelian surfaces
- Reduce to Jacobians with specific 2-torsions
  - $C : y^2 = f(x)$ with $Gal(f) \subseteq C_2 \times D_4$
- Express the parity as a product of local terms
  - $(-1)^{rk(J)} = \prod_v (-1)^{ord_2(\frac{c_v \mu_v}{c'_v \mu'_v})}$,
- Compute λ_v for all v
  - Ω_J, c_ℓ, μ_v
- Compare λ_v and $w_v(J)$
  - $$(-1)^{rk(J)} = \prod_v \lambda_v = \prod_v w_v(J) = w(J)$$

Theorem (Dokchitser V., M.)

Let K be a number field. Assuming finiteness of Ш, the Birch and Swinnerton-Dyer conjecture correctly predicts the parity of the rank of all semistable* principally polarized abelian surfaces over K.

*good ordinary reduction a places above 2.

Local comparison : $E_1 \times E_2$

Theorem

Let $f(x) = x^3 + ax^2 + bx + c \in K[x]$ such that $c \neq 0$ and write $L = ab - 9c$. Then

$$(-1)^{ord_2\left(\frac{c_v(E)c_v(\mathrm{Jac}E')}{c_v(\mathrm{Jac}C)\mu_v(C)}\right)} = E_v \cdot w_v(E) w_v(\mathrm{Jac}(E')).$$

For each place v of K, define the following Hilbert symbols at v

$$E_v = (b, -c)(-2L, \Delta_f)(L, -b)$$

Invariants were found using Sturm polynomials.

- E_v recovers the error term for elliptic curves with a 2-isogeny
- E_v generalizes for $\deg(f) > 4$ (H. Green)

Local comparison : $E_1 \times E_2$

Theorem

Let $f(x) = x^3 + ax^2 + bx + c \in K[x]$ such that $c \neq 0$ and write $L = ab - 9c$. Then

$$(-1)^{ord_2\left(\frac{c_v(E)c_v(\mathrm{Jac}E')}{c_v(\mathrm{Jac}C)\mu_v(C)}\right)} = E_v \cdot w_v(E) w_v(\mathrm{Jac}(E')).$$

For each place v of K, define the following Hilbert symbols at v

$$E_v = (b, -c)(-2L, \Delta_f)(L, -b)$$

Invariants were found using Sturm polynomials.

- E_v recovers the error term for elliptic curves with a 2-isogeny
- E_v generalizes for $\deg(f) > 4$ (H. Green)

Parity conjecture for principally polarized abelian surfaces

Strategy

$A \simeq_K E_1 \times E_2$, where E_1, E_2 are two elliptic curves defined over K

- Use BSD invariance under isogeny to compute parity of rank
  - $E_1[2] \simeq E_2[2] \Rightarrow$ Singular Richelot isogeny
- Express the parity as a product of local terms
  - $(-1)^{rk(J)} = \prod_v \lambda_v$
- Compute λ_v for all v
  - Ω_J, c_ℓ, μ_v
- Compare λ_v and $w_v(E_1 \times E_2)$
  - $$(-1)^{rk(E_1 \times E_2)} = \prod_v \lambda_v = \prod_v w_v(E_1 \times E_2) = w(E_1 \times E_2)$$

Theorem (Green H, M.)

Let K be a number field. Assuming finiteness of Ш, the Birch and Swinnerton-Dyer conjecture correctly predicts the parity of the rank of $E_1 \times E_2/K$, for elliptic curves E_1, E_2 with isomorphic 2-torsion groups.

Thank you for your attention

Precise resutls

Theorem (Dokchitser V., M.)

The parity conjecture holds for all principally polarized abelian surfaces over number fields A/K such that $Ш_{A/K(A[2])}$ has finite $2-$, $3-$, $5-$primary part that are either

- the Jacobian of a semistable genus 2 curve with good ordinary reduction above 2, or
- semistable and not isomorphic to the Jacobian of a genus 2 curve.

Theorem (Green H, M.)

Let K be a number field and $E_1, E_2/K$ be elliptic curves. If $E_1[2] \simeq E_2[2]$ as Galois modules, then the 2-parity conjecture holds for E_1/K if and only if it holds for E_2/K.

Regulator constants

Theorem (T. and V. Dokchitser)

Suppose
- A semistable p.p. abelian variety,
- $F = K(A[2])$,
- $Ш(A/F)[p^\infty]$ is finite for odd primes p dividing $[F:K]$,
- Parity holds for A/L for all $K \subseteq L \subseteq F$ with $Gal(F/L)$ a 2-group.

Then the parity conjecture holds for A/K.

Remark

The Sylow 2-subgroup of S_6 is $C_2 \times D_4$.
Hence if $Gal(K_f/L)$ is a 2-group then $Gal(K_f/L) \subseteq C_2 \times D_4$.
By Theorem 2.ii: if $Gal(K_f/L) \subseteq C_2 \times D_4$, C semistable and good ordinary at 2-adic places then the 2-parity conjecture holds for J/L.
Thus if $|Ш(J/K_f)[2^\infty]| < \infty$ then the parity conjecture holds for J/L.

Complete local formula

Theorem

Fix an exterior form Ω' of J' and denote $\Omega_v'^o$, Ω_v^o the Néron exterior forms at the place v of K associated to Ω' and $\phi^*\Omega'$ respectively. Then $(-1)^{rk_2(J)} = \prod_v (-1)^{\lambda_v}$ with

$$\lambda_{v|\infty} = ord_2\left(\frac{n \cdot m_v}{|ker(\alpha)| \cdot n' \cdot m_v'}\right), \quad \lambda_{v\nmid\infty} = ord_2\left(\frac{c_v \cdot m_v}{c_v' \cdot m_v'}\left|\frac{\phi^*\Omega_v'^o}{\Omega_v^o}\right|_v\right),$$

where n, n' are the number of K_v-connected components of J and J', α is the restriction of ϕ to the identity component of $J(K_v)$, c_v and c_v' the Tamagawa numbers of J and J', and $m_v = 2$ if C is deficient at v, $m_v = 1$ otherwise.

p^∞-Selmer rank and p-parity conjecture

p^∞ Selmer rank

For a prime p, define the p^∞ Selmer rank as

$$rk_p(A) = rk(A) + \delta_p, \text{ where}$$

$$Ш[p^\infty] = (\mathbb{Q}_p/\mathbb{Z}_p)^{\delta_p} \times Ш_0[p^\infty], \quad |Ш_0[p^\infty]| < \infty.$$

Assuming finiteness of $Ш(A)$; for all prime p

$$rk(A) = rk_p(A).$$

p-parity conjecture

For all prime p,

$$(-1)^{rk_p(A)} = w(A).$$

Error term and arithmetic invariants of the variety

- E/K with multiplicative reduction and 2-isogeny

$$\Rightarrow -\frac{c_4}{c_6} \equiv b_2 \equiv a \bmod K^{\times 2}$$

- Let A/K be an abelian variety. For any prime ℓ, write $\phi_o(\ell)$ for the ℓ-primary component of

$$\phi_o(k) \simeq A(K)/A(K)^0.$$

Then for $\ell \neq p$

$$\phi_o(\ell) \simeq \frac{\left(T_\ell(A(\bar{K})) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell\right)^{I_K}}{\left(T_\ell(A(\bar{K}))\right)^{I_K} \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)}$$

The error term (except the contribution of Ш) is Galois theoretic

Deficiency

Definition : Deficiency

If X is a curve of genus g over a local field $\mathcal{K}$, we say that X is deficient if X has no $\mathcal{K}$-rational divisor of degree $g-1$. If X is a curve of genus g over a global field K, then a place v of K is called deficient if X/K_v is deficient.

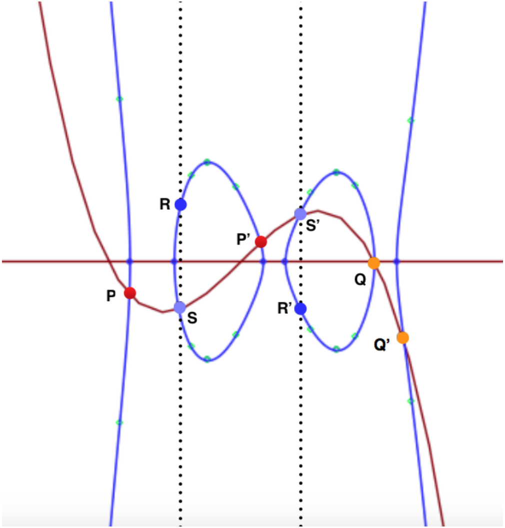
2-isogeny equivalent for Jacobians : Richelot isogeny

Existence

$$\{\text{Galois stable subgroup of } J_{tors} \text{ of order d}\}$$
$$\leftrightarrow$$
$$\{\text{Isogeny of degree } d\}$$

$\Rightarrow$ Look at $J[2]$ and find a Galois stable subgroup of order 4

Points on $J(K)$ and $J(K)[2]$



Points on $J(K)$:
$D = P + Q - P_\infty^+ - P_\infty^- = [P, Q]$,
where $P, Q \in C(K)$ or
$P = \overline{Q} \in C(F), \quad [F : K] = 2$

Adding points on $J(K)$:
$[P, P'] + [Q, Q'] = [R, R']$

2 torsion: $J(\overline{K})[2] = \{[T_i, T_k], i \neq k\} \cup \{0\}$, where $T_i = (x_i, 0) \in C(\overline{K})$.

Richelot isogeny

- $\mathrm{Gal}(f) \subseteq C_2^3 \rtimes S_3 \implies$ Richelot isogeny

$$f(x) = q_1(x)q_2(x)q_3(x) \text{ with roots } \alpha_i, \beta_i.$$

$$D_1 = [(\alpha_1, 0), (\beta_1, 0)], \quad D_2 = [(\alpha_2, 0), (\beta_2, 0)], \quad D_3 = [(\alpha_3, 0), (\beta_3, 0)]$$

lie in $J(\overline{K})[2]$ and $\{0, D_1, D_2, D_3\}$ is a Galois stable subgroup of $J(K)[2]$.

Proposition

If $Gal(f) \subseteq C_2^3 \rtimes S_3$ then J admits a **Richelot isogeny** Φ s.t. $\Phi\Phi^* = [2]$.

Remark : Explicit construction

There is an explicit model for the curve C' underlying the isogenous Jacobian J'.