GENERIC DECOMPOSITIONS OF DELIGNE–LUSZTIG REPRESENTATIONS

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Abstract. Let $G_0$ be a reductive group over $\mathbb{F}_p$ with simply connected derived subgroup, (geometrically) connected center and Coxeter number $h+1$. We extend Jantzen’s generic decomposition pattern from $(2h-1)$-generic to $h$-generic Deligne–Lusztig representations, which is optimal. We also prove several results on the “obvious” Jordan–Hölder factors of general Deligne–Lusztig representations. As an application we improve the weight elimination result of [LLHL19].

1. Introduction

Let $G_0$ be a reductive group over $\mathbb{F}_p$ with simply connected derived subgroup. An important problem is to understand the representations of the finite group of Lie type $G_0(\mathbb{F}_p)$ in characteristic 0, Deligne–Lusztig gave a beautiful geometric construction of characters of $G_0(\mathbb{F}_p)$ which effectively describes all irreducibles. It is natural to ask how characteristic 0 irreducibles decompose modulo a prime $\ell$. When $\ell \neq p$, this problem has been studied extensively see e.g. [BR03, BDR17]. In contrast, the defining characteristic $\ell = p$ case seems underdeveloped, despite its connections to number theory and more specifically the study of congruences of automorphic forms. The main result here, due to Jantzen [Jan81], describes the mod $p$ reduction of sufficiently generic Deligne–Lusztig representations.

Let $T$ be a maximally split $\mathbb{F}_p$-rational maximal torus of $G_0 \cong G_0 \otimes_{\mathbb{F}_p} \mathbb{F}_p$, $\mu$ a character of $T$ and $s$ an element in the Weyl group of $W$ (with respect to $T$). To this data we can associate an $\mathbb{F}_p$-rational maximal torus $T_s$ and a $W(\mathbb{F}_p)$-valued character $\theta(s, \mu)$ of $T_s(\mathbb{F}_p)$ and thus a Deligne–Lusztig representation $R_s(\mu)$ (see §3). Recall that in characteristic $p$, after choosing a Borel subgroup, $p$-restricted highest weights $\lambda$ parametrize irreducible representations $F(\lambda)$ of $G_0(\mathbb{F}_p)$. We refer to these as Serre weights. Let $h + 1$ denote the Coxeter number of $G_0$ and $\eta$ be (a lift of) the half of the sum of the positive roots. When $\mu - \eta$ is $2h-1$ deep in the base $p$-alcove $C_0$ (anchored at $-\eta$), Jantzen [Jan81] (and Gee, Herzig and Savitt [Her09, GHS18] for reductive groups) gives a formula for the reduction $\overline{R}_s(\mu)$ of $R_s(\mu)$ in terms of Frobenius kernel multiplicities.

A basic feature of Jantzen’s formula is that the highest weights of the Jordan–Hölder factors is given by universal combinatorial formulas in $s, \mu$, which in particular implies that the length of the reduction is independent of $s$ and $\mu$. In order for these universal formulas (as $s$ varies in $W$) to produce $p$-restricted weights, $\mu - \eta$ must be $h$-deep (see Remark [L3]). Thus the best one can hope for is that Jantzen’s formula always holds when $\mu - \eta$ is $h$-deep. Our first main result confirms this when the center of $G$ is connected.

Theorem 1.1. Suppose that $G$ has connected center, $\mu - \eta$ is $h$-deep in the base $p$-alcove, and $\lambda$ is a $p$-restricted dominant weight.

Then $[\overline{R}_s(\mu) : F(\lambda)] \neq 0$ if and only if there exist $\bar{w}$ and $\bar{w}_\lambda$ in the extended affine Weyl group $\hat{W}$ such that:

- $\bar{w} \cdot C_0$ is dominant and $\bar{w}_\lambda \cdot C_0$ is $p$-restricted (where $\cdot$ denotes the $p$-dot action);
• $\tilde{w} \uparrow \hat{w}_h \hat{w}_\lambda$ (where $\uparrow$ is the semi-infinite Bruhat order defined as in [Jan03 II.6.5] and $\hat{w}_h \overset{\text{def}}{=} w_0 t^{-\eta}$); and
• $\lambda = \hat{w}_\lambda : (\mu - \eta + s\pi(\hat{w}^{-1}(0)))$ (where $\pi$ denotes the automorphism of $X^*(T)$ corresponding to Frobenius; see §1.3).

Moreover, in this case:

$$[\hat{R}_s(\mu) : F(\lambda)]_{G_0(\mathfrak{p}_\mu)} = [\hat{Z}(1, \mu + (s\pi - p)(\hat{w}^{-1}(0)) + (p - 1)\eta) : \hat{L}(1, \lambda)]_{G_1T}.$$ 

Here, $\hat{Z}(1, -)$ and $\hat{L}(1, -)$ are the baby Verma/standard and simple modules, respectively, for the augmented Frobenius kernel $G_1T$ (see §3).

Remark 1.2. (1) The hypothesis on connected center is necessary for the second conclusion to hold—without it the right hand side should be replaced by a sum of Frobenius kernel multiplicities. The (generic) decomposition problem in the general reductive case can often be reduced to Theorem [1.1] by an analysis of isogenies.

(2) The multiplicity $[\hat{Z}(1, \mu') : \hat{L}(1, \lambda)]_{G_1T}$ can be nonzero only if $\mu', \lambda$ are in the same $p$-dot orbit of the affine Weyl group, in which case it depends combinatorially on $p$-alcoves of $\mu'$ and $\lambda$. For instance, when $p \gg h$ it is controlled by periodic Kazhdan–Lusztig polynomials.

(3) In a different direction, Pillen [Pil93] analyzes the contribution of the $p$-singular weights when $\mu$ lies in exactly one wall of $C_0$ and is $2h - 1$ away from the other walls.

(4) This result was first suggested by considerations in the theory of local models for potentially crystalline Emerton–Gee stacks. Specifically, the hypothesis on $\mu$ in Theorem [1.1] is the range where the special fiber cycles (which are expected to reflect the mod $p$ reduction of Deligne–Lusztig representations by the Breuil–Mézard conjecture) have uniform behavior.

It is also natural to contemplate the dual question, i.e. given a Serre weight $F(\lambda)$, for which $(s, \mu)$ is $F(\lambda)$ a Jordan–Hölder factor of $\hat{R}_s(\mu)$? This problem is essentially equivalent to decomposing the characteristic zero lift of a projective cover $P_\lambda$ of $F(\lambda)$ into irreducibles (the bulk of which are of the form $R_s(\mu)$). When $\lambda$ is $2h$-deep in its alcove, the complete decomposition can be obtained from Theorem [1.1]. In particular there are always $|W|$ “obvious” $R_s(\mu)$ which contain $F(\lambda)$ with multiplicity one. However, when $\lambda$ is not $2h$-deep the decomposition of $P_\lambda$ becomes considerably more complicated; for instance some $R_s(\mu)$ factors that appear generically may disappear.

Nevertheless, we show that the “obvious” $R_s(\mu)$ factors of $P_\lambda$ persist up to essentially the optimal threshold:

**Theorem 1.3.** Suppose that $G$ has connected center. Let $\lambda$ be a $p$-restricted dominant weight which is $h$-deep in its $p$-alcove. Then for all $s \in W$, $F(\lambda)$ is a Jordan–Hölder factor of $\hat{R}_s(\hat{w}_h \cdot \lambda + \eta)$ with multiplicity one.

Remark 1.4. In fact we prove the theorem under weaker hypotheses on $\lambda$ depending on its alcove, see Theorem [5.4].

Theorem [1.3] gives a large supply of characteristic zero irreducible representations containing $F(\lambda)$ when $\lambda$ is $h$-deep. For the number theoretic application discussed below, we would like to construct for any $(p$-dot regular) $\lambda$ an $R_s(\mu)$ containing $F(\lambda)$ such that $\mu$ is in the base alcove with essentially the same depth as $\lambda$. We establish such a statement in Theorem [5.4] under a mild “smallness” hypothesis which can always be arranged in type $A$. Note that this is rather subtle because when $\lambda$ is not $h$-deep the most obvious Deligne–Lusztig induction $R_1(\lambda)$ containing $F(\lambda)$ usually fails the depth requirement (because of a small translation when expressing $R_1(\lambda)$ as $R_s(\mu)$ with $\mu - \eta \in C_0$).
1.1. A number theoretic application. We now explain how the above results allow us to improve the main theorem of [LLHL19]. Recall the global setting of loc. cit. Let $F/F^+$ be a totally imaginary extension of a totally real field $F^+ \not\subseteq Q$ such that $p$ is inert in $F^+$ and splits in $F$. Given a reductive group $G_{F^+}$ which is an outer form for $GL_n$ and splits over $F$, and such that $G(F^+ \otimes Q_{\mathbb{R}})$ is compact, and given a compact open subgroup of the form $U = U^pG(O_{F^+_p}) \leq G(\mathbb{A}_{F^+}^\infty)$ and a $G(O_{F^+_p})$-module $M$, we define a space

$$S(U, M) \overset{\text{def}}{=} \{ f : G(F^+)/G(\mathbb{A}_{F^+}^\infty)/U \to M \mid f(gu) = u_p^{-1} f(g) \forall g \in G(\mathbb{A}_{F^+}^\infty), u \in U \}$$

of algebraic modular forms. It is endowed with a faithful action of a Hecke algebra $\mathbb{T}$ (with generators indexed by an infinite set of "good primes" for $U$, cf. [LLHL19 §4.2.2]) for which each maximal ideal $m \subseteq \mathbb{T}$ has an associated continuous semisimple representation $\tau_m : G_F \to GL_n(F)$ (cf. [CHT08 §3.4]). We further assume that $\tau_m$ is absolutely irreducible. In [Her09] (later generalized in [GHS18]), Herzig made a remarkable conjecture predicting that the set $W(\tau_m)$ of $p$-regular Serre weights $V$ such that $S(U,V)_m \neq 0$ is given by a combinatorially defined set $W^p(\tau_m|G_{F^+_p})$ when $\tau_m|G_{F^+_p}$ is semisimple. We remark that $W^p(\tau_m|G_{F^+_p})$ is given in terms of the Jordan–Hölder factors of a Deligne–Lusztig representation associated to $\tau_m|G_{F^+_p}$.

The weight elimination statement which we obtain is the following:

**Theorem 1.5.** Assume that $\tau_m$ is absolutely irreducible, and that $\tau_m|G_{F^+_p}$ is $(2n + 1)$-generic. Then $W(\tau_m) \subseteq W^p((\tau_m|G_{F^+_p})^{ss})$.

This result was proven in [LLHL19] with the assumption that $\tau_m|G_{F^+_p}$ is $(6n - 2)$-generic instead of $(2n + 1)$-generic. As in loc. cit. the main mechanism to show $F(\lambda) \not\in W(\tau_m)$ is to find sufficiently many $\bar{R}_s(\mu)$ containing it and use $p$-adic Hodge theory constraints implied by the condition $S(U,R_s(\mu))_m \neq 0$. In turn, these constraints translate to combinatorial admissibility conditions which exactly match Jantzen’s generic pattern for $W^p((\tau_m|G_{F^+_p})^{ss})$. Our representation theoretic results show that we can find all the necessary Deligne–Lusztig representations under weaker genericity hypotheses.

**Strategy:** Jantzen gives a very general character formula which describes the multiplicity of $R_s(\mu)$ in a certain projective $G_0(F_p)$-module $Q_\lambda$ containing $F(\lambda)$ in terms of Frobenius kernels multiplicities. As long as $\mu$ is $h$-generic in the lowest $p$-alcove, those multiplicities are controlled by the principal block and hence are independent of $\mu$. Under Jantzen’s stronger assumption that $\mu$ is $(2h - 1)$-generic, any $F(\lambda)$ that can contribute to $R_s(\mu)$ has the property that $Q_\lambda = P_\lambda$ is indecomposable, thus one gets the formula for the multiplicity $[R_s(\mu) : F(\lambda)]$ of $F(\lambda)$ in $R_s(\mu)$. In contrast, the key difficulty when $\mu$ is not $2h$-generic is that $Q_\lambda$ can be decomposable and Jantzen’s character formula only gives a formula for certain weighted sums $\sum [R_s(\mu) : F(\lambda)]|Q_\lambda : P_\lambda|$ over “packets” of Serre weights. Our key observation is that if $\mu$ is $h$-generic and $R_s(\mu)$ occurs in $Q_\lambda$ then $R_s(\mu)$ does not occur in $Q_{\lambda'}$ for any other $\lambda'$ in the packet, and hence the above sum collapses.

The complication arising from packets also occurs in Theorem 1.3 and we resolve it in the same way. It is clear from Jantzen’s general formula that $\text{Hom}_{G_0(F_p)}(Q_{\lambda'}, \bar{R}_s(\bar{w}_h \cdot \lambda + \eta)) \neq 0$. By a series of delicate estimates in alcove geometry, we show that if $\lambda$ is $h$-deep in its $p$-alcove then $\text{Hom}_{G_0(F_p)}(Q_{\lambda'}, \bar{R}_s(\bar{w}_h \cdot \lambda + \eta)) = 0$ for any other $\lambda'$ in the packet.

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1.3. Notation. Let \( p \) be a prime. Let \( \mathbb{Q}_p \subset E \subset \overline{\mathbb{Q}}_p \) be a sufficiently large finite extension of \( \mathbb{Q}_p \). Let \( \mathcal{O} \) be the ring of integers of \( E \) and \( \mathbb{F} \) the residue field.

Let \( G \) be a reductive group over \( \mathbb{F}_p \). Let \( \mathbb{F}/\mathbb{F}_p \) denote a finite extension so that \( G \) is split. We assume throughout that \( \mathfrak{g} \) is simply connected derived subgroup. Let \( T \subset B \subset G \) denote a maximal torus and a Borel subgroup. Let \( G_1 \subset G \) denote the kernel of the relative \((p)\)-Frobenius isogeny \( F \) on \( G \). Let \( G_1T \subset G \) denote the subgroup scheme generated by \( G_1 \) and \( T \). Let \( \Gamma \equiv G_0(\mathbb{F}_p) \).

Recall the following standard notations:

- the character group \( \mathfrak{g}^{\ast} \) of \( T \times_{\mathbb{F}} \mathbb{F}_p \);
- \( R \subset \mathfrak{g}^{\ast} \) the set of roots of \( G \) with respect to \( T \);
- the root lattice \( \mathbb{Z}R \subset \mathfrak{g}^{\ast} \) generated by \( R \);
- \( R^+ \subset R \) the subset of positive roots with respect to \( B \), i.e. the roots occurring in \( \text{Lie}(B) \); note that this is the convention in [Jan81] but opposite to [Jan03];
- \( \Delta \subset R^+ \) the subset of simple roots;
- \( X(T)^+ \subset \mathfrak{g}^{\ast} \) the dominant weights with respect to \( R^+ \);
- the \( p \)-restricted set \( X_1(T) \subset \mathfrak{g}^{\ast} \) of dominant weights \( \lambda \) such that \( \langle \lambda, \alpha^\vee \rangle \leq p - 1 \) for all \( \alpha \in \Delta \);
- the partial order \( \leq \) on \( \mathfrak{g}^{\ast} \) defined by \( \lambda \geq \mu \) if \( \lambda - \mu \in \mathbb{Z}_{\geq 0}R \);
- for \( \nu \in \mathfrak{g}^{\ast} \), let \( h_{\nu} \equiv \max_{\alpha \in R} \langle \nu, \alpha^\vee \rangle \);
- the automorphism \( \pi \) of \( \mathfrak{g}^{\ast} \) such that \( F = p\pi^{-1} \) on \( \mathfrak{g}^{\ast} \);
- a choice of \( \pi \)-invariant \( \eta \in \mathfrak{g}^{\ast} \) such that \( \langle \eta, \alpha^\vee \rangle = 1 \) for all \( \alpha \in \Delta \);
- the Weyl group \( W \) of \((G,T)\);
- the extended affine Weyl group \( \widetilde{W} \equiv \mathfrak{g}^{\ast} \times W \), which acts on \( \mathfrak{g}^{\ast} \) on the left by affine transformations; for \( \nu \in \mathfrak{g}^{\ast} \) we write \( t_{\nu} \in \widetilde{W} \) for the corresponding element;
- the affine Weyl group \( W_a \equiv \mathbb{Z}R \rtimes W \subset \widetilde{W} \);
- for \( \kappa \in X^\ast(T) \), we write \( \kappa_+ \in W\kappa \) for the unique dominant element in its \( W \)-orbit, and \( \text{Conv}(\kappa) \subset \mathfrak{g}^{\ast} \otimes \mathbb{R} \) for the convex hull of \( W\kappa \);
- the set of alcoves of \( \mathfrak{g}^{\ast}(T) \otimes \mathbb{Z} \mathbb{R} \), i.e. the set of connected components of

\[
\mathfrak{g}^{\ast}(T) \otimes \mathbb{Z} \mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}, \alpha \in R} \{ \lambda \in \mathfrak{g}^{\ast}(T) \otimes \mathbb{Z} \mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle = n \},
\]

which has a (transitive) left action of \( \widetilde{W} \);
- the dominant alcoves, i.e. alcoves \( A \) such that \( 0 < \langle \lambda, \alpha^\vee \rangle \) for all \( \alpha \in \Delta, \lambda \in A \);
- the lowest (dominant) alcove \( A_0 = \{ \lambda \in \mathfrak{g}^{\ast}(T) \otimes \mathbb{Z} \mathbb{R} \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in R^+ \} \);
- \( \Omega \subset \widetilde{W} \) the stabilizer of the base alcove;
- the restricted alcoves, i.e. alcoves \( A \) such that \( 0 < \langle \lambda, \alpha^\vee \rangle < 1 \) for all \( \alpha \in \Delta, \lambda \in A \);
- the set \( \widetilde{W}^+ \subset \widetilde{W} \) of elements \( \tilde{w} \) such that \( \tilde{w}(A_0) \) is dominant;
- the set \( \widetilde{W}_1 \subset \widetilde{W}^+ \) of elements \( \tilde{w} \) such that \( \tilde{w}(A_0) \) is restricted;
Lemma 2.2. Let $L$ be the Borel opposite to $W$. Suppose that $\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e., the set of connected components of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}, \alpha \in R} \{ \lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \eta, \alpha^\vee \rangle = np \}$:

- a left $p$-dot action of $\tilde{W}$ on $X^*(T)$ defined by $(t_{\nu}, w) \cdot \lambda \overset{\text{def}}{=} p\nu + w(\lambda + \eta) - \eta$; this induces a $p$-dot action of $\tilde{W}$ on $X^*(T)$ whose restriction to $W_a$ is simply transitive;
- the dominant $p$-aloves, i.e., alcoves $C$ such that $0 < \langle \lambda, \alpha^\vee \rangle$ for all $\alpha \in \Delta, \lambda \in C$;
- the lowest (dominant) $p$-alcove $C_0 \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ characterized by $\lambda \in C_0$ if $0 < \langle \lambda + \eta, \alpha^\vee \rangle < p$ for all $\alpha \in R^+$;
- the $p$-restricted alcoves, i.e., alcoves $C$ such that $0 < \langle \lambda + \eta, \alpha^\vee \rangle < p$ for all $\alpha \in \Delta, \lambda \in C$;

- the Bruhat order $\leq$ on $W_a$ with respect to $A_0$ (i.e., using the reflections across walls of $A_0$ as a set of Coxeter generators);

- the $\uparrow$ order on the set of $p$-aloves defined in [Jan03, II.6.5];
- the $\uparrow$ order on $W_a$ induced from the ordering $\uparrow$ on the set of $p$-aloves (via the bijection $\tilde{w} \mapsto \tilde{w} \cdot C_0$);
- the Bruhat order on $\tilde{W} = W_a \times \Omega$ defined by $\tilde{w} \delta \leq \tilde{w}' \delta'$ if and only if $\tilde{w} \leq \tilde{w}'$ and $\delta = \delta'$ where $\delta, \delta' \in \Omega$ and $\tilde{w}, \tilde{w}' \in W_a$;
- the $\uparrow$ order on $\tilde{W}$ defined by $\tilde{w} \delta \uparrow \tilde{w}' \delta'$ if and only if $\tilde{w} \uparrow \tilde{w}'$ and $\delta = \delta'$ where $\delta, \delta' \in \Omega$ and $\tilde{w}, \tilde{w}' \in W_a$.

2. LEMMATA

In this section, we collect several lemmata, mostly of a root-theoretic nature, that will be used in later sections.

Lemma 2.1. Suppose that $\tilde{s}, \tilde{w} \in \tilde{W}$ such that

- $\tilde{s} \in \tilde{W}^+$;
- $\tilde{s} \uparrow \tilde{w}$;
- $\tilde{s}(0) = \tilde{w}(0)$; and
- the closure of some Weyl chamber contains both $\tilde{s}^{-1}(0)$ and $\tilde{w}^{-1}(0)$.

Then $\tilde{s} = \tilde{w}$.

Proof. Since $\tilde{s}(0) = \tilde{w}(0)$, $W\tilde{s}^{-1}(0) = W\tilde{w}^{-1}(0)$. Since the closure of some Weyl chamber contains $\tilde{s}^{-1}(0)$ and $\tilde{w}^{-1}(0)$, we have $\tilde{s}^{-1}(0) = \tilde{w}^{-1}(0)$. This implies that $W\tilde{s} = W\tilde{w} = Wt_{-\tilde{s}^{-1}(0)} = Wt_{-\tilde{w}^{-1}(0)} = W\tilde{w}$. Then $\tilde{s} \in \tilde{W}^+$ and $\tilde{s} \uparrow \tilde{w}$ imply that $\tilde{s} = \tilde{w}$.

Given $\lambda \in X(T)$ we let $W(\lambda)$ be the virtual representation $\sum_i (-1)^i R^i \text{Ind}_{\tilde{B}_{-\lambda}}^{\tilde{B}} 1$ where $B^-$ denotes the Borel opposite to $B$. If $\lambda$ is dominant then $W(\lambda)$ is the representation $\text{Ind}_{\tilde{B}_{-\lambda}}^{\tilde{B}} 1$ and we write $L(\lambda)$ for its (irreducible) socle.

Lemma 2.2. Let $\lambda_0 \in C_0$, $\tilde{w}_\lambda \in \tilde{W}_1$, $\lambda' \in X_1(T)$, and $\nu \in X(T)^+$. Set $\lambda \overset{\text{def}}{=} \tilde{w}_\lambda \cdot \lambda_0$.

1. If $|L(\lambda') \otimes L(\pi(\nu))| \neq 0$, then $\lambda + \nu \uparrow \lambda' + \pi(\nu')$ for some $\nu' \in \text{Conv}(\nu)$.
2. If $\lambda + \nu \uparrow \lambda' + \pi(\nu')$ for some $\nu' \in \text{Conv}(\nu)$, then $h_\nu \leq h_{\tilde{w}_\lambda \tilde{w}_\lambda(\nu)} \leq h_{\eta}$ where $\nu$ is any vertex of the dominant base alcove.

Proof. Suppose that $L(\lambda + \nu) \in \text{JH}(L(\lambda') \otimes L(\pi(\nu))) \subset \text{JH}(W(\lambda') \otimes L(\pi(\nu)))$. Then $L(\lambda + \nu) \in \text{JH}(W(\lambda' + \pi(\nu')))$ for some $\nu' \in \text{Conv}(\nu)$. Moreover, we can assume without loss of generality that $\lambda' + \pi(\nu') \in X(T)^+$ by [LHLM23, Lemma 2.2.2]. By the linkage principle, we conclude that

$$
\lambda + \nu \uparrow \lambda' + \pi(\nu')
$$
for some $\nu' \in \text{Conv}(\nu)$.

Suppose now that (2.1) holds. If we let $\alpha_0$ be a dominant root such that $\langle \nu, \alpha_0^\vee \rangle = h_\nu$, then (2.1) implies that $(p - 1)h_\nu \leq \langle p\nu - \pi(\nu'), \alpha_0^\vee \rangle \leq \langle \lambda' - \lambda, \alpha_0^\vee \rangle$ so that $h_\nu \leq \lfloor \frac{1}{p - 1}(\lambda' - \lambda, \alpha_0^\vee) \rfloor \leq h_{\tilde{w}_h \tilde{w}_\lambda (v)}$ for any vertex $v$ of the dominant base alcove.

Given $m \in \mathbb{Z}$ and a $p$-alcove

$$C = \{ \mu \in X^*(T) \otimes \mathbb{Z} \mathbb{R} \mid n_\alpha p < \langle \mu + \eta, \alpha^\vee \rangle < (n_\alpha + 1)p, \alpha \in R^+ \},$$

we say that $\lambda \in X^*(T)$ is $m$-deep in the $p$-alcove $C$ if for all $\alpha \in R^+$, $n_\alpha p + m < \langle \lambda + \eta, \alpha^\vee \rangle < (n_\alpha + 1)p - m$.

Lemma 2.3. Let $s \in W$, $\mu - \eta \in C_0$, and $\lambda \in X_1(T)$. If $\tilde{w} \in \tilde{W}$ is such that $\tilde{w} \cdot (\mu - \eta + s\pi \tilde{w}^{-1}(0)) + \eta \in X(T)^+$ and $\tilde{w} \cdot (\mu - \eta + s\pi \tilde{w}^{-1}(0)) \leq \tilde{w}_h \cdot \lambda$, then $h_{\tilde{w}(0)} = h_{\tilde{w}^{-1}(0)} \leq h_\eta + 1$. If $\mu - \eta$ or $\lambda$ is 1-deep in their respective $p$-alcoves, then $h_{\tilde{w}(0)} = h_{\tilde{w}^{-1}(0)} \leq h_\eta$.

Proof. Let $\sigma \in W$ be such that $\sigma \tilde{w} \in \tilde{W}^+$. Letting $\alpha_0$ be the highest root so that $h_{\tilde{w}(0)} = \langle \sigma \tilde{w}(0), \alpha_0^\vee \rangle$, the hypotheses imply that

$$\langle \sigma \tilde{w} \cdot (\mu - \eta + s\pi \tilde{w}^{-1}(0)) + \eta, \alpha_0^\vee \rangle \leq \langle \tilde{w}_h \cdot \lambda + \eta, \alpha_0^\vee \rangle \leq (p - 1)h_\eta,$$

from which we deduce that $h_{\tilde{w}(0)} \leq h_\eta + 1$. If $\mu - \eta$ (resp. $\lambda$) is 1-deep in its $p$-alcove, then the first (resp. third) inequality in (2.2) is strict. The result follows.

Lemma 2.4. Let $m \geq 0$. Suppose that $\mu - \eta \in X^*(T)$ is $m$-deep in $C_0$ and $\sigma(\mu) + p\nu - s\pi \nu - \eta$ is $(-m + 1)$-deep in $C_0$ for $s \in W$ and $\nu \in X^*(T)$. Then $t_{\nu} \sigma \in \Omega$.

Proof. Let $\mu - \eta, \sigma, s$, and $\nu$ be as in the statement of the lemma. We first claim that $h_\nu \leq 2$. For $\alpha \in R^+$, we have that

$$-m + 1 < \langle \sigma(\mu) + (p - s\pi)\nu, \alpha^\vee \rangle < p + m - 1.$$

Using that $-p + m < \langle \sigma(\mu), \alpha^\vee \rangle < p - m$ for all $\alpha \in R^+$, we have that $|p\langle \nu, \alpha^\vee \rangle - (s\pi \nu, \alpha^\vee)\rangle| \leq 2p - 2$ for all $\alpha \in R^+$. There exists $\alpha \in R^+$ such that $|\langle \nu, \alpha^\vee \rangle| = h_\nu$ so that $(p - 1)h_\nu \leq |p\langle \nu, \alpha^\vee \rangle - (s\pi \nu, \alpha^\vee)\rangle| \leq 2p - 2$. The claim follows.

Since $\mu - \eta \in X^*(T)$ is $m$-deep in $C_0$, for each $\alpha \in R^+$ we have

$$n_\alpha p + m < \langle \sigma(\mu) + p\nu, \alpha^\vee \rangle < (n_\alpha + 1)p - m$$

for a unique $n_\alpha \in \mathbb{Z}$. On the other hand, (2.3) and that $h_\nu \leq 2$ imply that

$$-m - 1 < \langle \sigma(\mu) + p\nu, \alpha^\vee \rangle < p + m + 1$$

for each $\alpha \in R^+$. Together, these imply that $n_\alpha = 0$ for all $\alpha \in R^+$ so that $\sigma(\mu) + p\nu - \eta \in C_0$. Equivalently, we have $t_{\nu} \sigma \in \Omega$.

Lemma 2.5. Suppose that the center $Z$ of $G$ is geometrically connected. Suppose also that $\lambda_0, \mu_0 \in X^*(T)$ are in the closure of $C_0$ and $\tilde{w}_\lambda, \tilde{w}_\mu \in \tilde{W}$ such that

1. $\pi^{-1}(\tilde{w}_\lambda) \cdot \lambda_0 \uparrow \pi^{-1}(\tilde{w}_\mu) \cdot \mu_0$; and
2. $t_{\lambda_0} \tilde{w}_\lambda W_\alpha = t_{\mu_0} \tilde{w}_\mu W_\alpha$.

Then $\lambda_0 = \mu_0$. Furthermore, let $F$ be the facet of $C_0$ determined by $\lambda_0$. Then $\pi^{-1}(\tilde{w}_\lambda) \cdot F \uparrow \pi^{-1}(\tilde{w}_\mu) \cdot F$. In particular, if $\lambda_0 \in C_0$, then $\tilde{w}_\lambda \uparrow \tilde{w}_\mu$. 

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Proof. (1) implies that
\[(2.4) \quad \pi^{-1}(\tilde{w}_\lambda) \cdot \lambda_0 = \pi^{-1}(\tilde{w}_\mu) \cdot \mu_0\]
for some \(\tilde{w} \in W_a\) so that \(\mu_0 - \lambda_0 \equiv p\pi^{-1}(\tilde{w}_\mu - \tilde{w}_\lambda)(0) \pmod{ZR}\). On the other hand, (2) implies that \(\mu_0 - \lambda_0 \equiv \tilde{w}_\mu^{-1}\tilde{w}_\lambda(0) \pmod{ZR}\). We conclude that \((1 - p\pi^{-1})\tilde{w}_\mu^{-1}\tilde{w}_\lambda(0) \in ZR\). If \(\pi\) has order \(f\), then we conclude that
\[(1 - pf)\tilde{w}_\mu^{-1}\tilde{w}_\lambda(0) = (1 + p\pi^{-1} + \ldots + pf^{-1}\pi^{-f+1})(1 - p\pi^{-1})\tilde{w}_\mu^{-1}\tilde{w}_\lambda(0) \in ZR.\]
As \(Z\) is geometrically connected, \(X^*(T)/ZR\) was torsion-free so that \(\tilde{w}_\mu^{-1}\tilde{w}_\lambda(0) \in ZR\) or in other words \(\tilde{w}_\lambda W_a = \tilde{w}_\mu W_a\). As \(\lambda_0, \mu_0\) are in the closure of \(C_0\), we conclude that \(\lambda_0 = \mu_0\).

(2) implies that there is a sequence of affine reflections \(r_1, \ldots, r_m\) such that
\[
\begin{align*}
&\text{for each } 1 \leq k \leq m, r_{k-1} \cdots r_1 \pi^{-1}(\tilde{w}_\lambda) \cdot \lambda_0 \text{ (resp. } r_k \cdots r_1 \pi^{-1}(\tilde{w}_\lambda) \cdot \lambda_0) \text{ is in the negative (resp. positive) halfspace defined by } r_k; \\
&\text{and } r_m \cdots r_1 \pi^{-1}(\tilde{w}_\lambda) \cdot \lambda_0 = \pi^{-1}(\tilde{w}_\lambda) \cdot \mu_0.
\end{align*}
\]
These properties hold after replacing \(\lambda_0\) and \(\mu_0\) with \(F\). If \(\lambda_0, \mu_0 \in C_0\), then \(F = C_0\) and \(\tilde{w}_\lambda \uparrow \tilde{w}_\mu\).

We say that \(\mu \in X^*(T)\) is \(p\)-regular if \(\mu\) is 0-deep in its \(p\)-alcove. Recall from [Jan03, II.7.2] the notion of blocks for \(G\). By the linkage principle ([Jan03, II.2.12(1) and II.6.17]) if \(L(\lambda)\) and \(L(\mu)\) are in the same block, then \(\lambda\) is \(p\)-regular if and only if \(\mu\) is, in which case we say that the block is \(p\)-regular. Given a \(G\)-module \(V\), let \(V_{\text{reg}}\) be the projection of \(V\) to the \(p\)-regular blocks.

**Lemma 2.6.** Let \(\mu \in X(T)^+\) be 0-deep in its alcove, and suppose that \(\nu \in X(T)^+\) such that \(\mu + \kappa\) is in the closure of the \(p\)-alcove containing \(\mu\) for all \(\kappa \in \text{Conv}(\nu)\). Then
\[
(L(\mu) \otimes L(\nu))_{\text{reg}} \cong \bigoplus_{\kappa \in \text{Conv}(\nu), \mu + \kappa \text{ is } 0\text{-deep}} L(\mu + \kappa) \otimes [L(\nu)|_{T : \kappa}]_T
\]
(and each summand that appears on the RHS has highest weight \(\mu + \kappa\) in the same \(p\)-alcove as \(\mu\)).

**Proof.** The proof is as in [Hum89, Lemma], except that we project to \(p\)-regular blocks. By the linkage principle, it suffices to prove an equality at the level of formal characters. We have
\[
\text{ch}(W(\mu) \otimes L(\nu))_{\text{reg}} = \sum_{\kappa \in X^*(T)} [L(\nu)|_{T : \kappa}]_T \text{ch} W(\mu + \kappa)_{\text{reg}}
\]
for all \(\kappa \in \text{Conv}(\nu)\) such that \(\mu + \kappa\) is 0-deep.

where the first equality follows from a formula of Brauer and the second equality follows from the fact that \(W(\mu + \kappa)_{\text{reg}} = 0\) if \(\mu + \kappa\) is not \(p\)-regular by the linkage principle and that \([L(\nu)|_{T : \kappa}]_T \neq 0\) implies that \(\kappa \in \text{Conv}(\nu)\). By assumption, the highest weight of each \(W(\mu + \kappa)\) appearing in (2.5) is 0-deep in the same \(p\)-alcove as \(\mu\). If \(\mu = \tilde{w} \cdot \lambda\) for \(\tilde{w} \in W_a\) and \(\lambda \in C_0\) and \(\mu + \kappa\) is 0-deep in the same \(p\)-alcove as \(\mu\), then the translation principle ([Jan03, II.7.5]) implies that there are nonnegative integers \(a(\tilde{w}, \tilde{w}')\) for each \(\tilde{w}' \in W_a\) such that
\[
\text{ch} W(\mu + \kappa) = \sum_{\tilde{w}' \in W_a} a(\tilde{w}, \tilde{w}') \text{ch} L(\tilde{w}' \cdot (\lambda + w^{-1}(\kappa))}
\]

where \( w \in W \) is the image of \( \tilde{w} \), \( a(\tilde{w}, \tilde{w}) = 1 \), and \( a(\tilde{w}, \tilde{w}') \neq 0 \) implies that \( \tilde{w}' \uparrow \tilde{w} \). Then (2.5), (2.6), and induction using the partial ordering \( \uparrow \) yields

\[
\text{ch}(L(\mu) \otimes L(\nu))_{\text{reg}} = \sum_{\kappa \in \text{Conv}(\nu)} [L(\nu)|_{\mathcal{T}} : \kappa]|_{\mathcal{T}} \text{ch}(\mu + \kappa).
\]

\[\square\]

We have the following immediate corollary of Lemma 2.6.

**Corollary 2.7.** Let \( \mu \in X(T)^+ \) be 0-deep in its alcove, and suppose that \( \nu \in X(T)^+ \) such that \( \mu + \kappa \) is in closure of the \( p \)-alcove containing \( \mu \) for all \( \kappa \in \text{Conv}(\nu) \). If \( \lambda \in X(T)^+ \) is 0-deep in its \( p \)-alcove and \( [L(\mu) \otimes L(\nu) : L(\lambda)]_G \neq 0 \), then \( \lambda = \mu + \kappa \) for some \( \kappa \in \text{Conv}(\nu) \) and \( \lambda \) and \( \mu \) are in the same \( p \)-alcove.

### 3. Ingredients

In this section, we summarize key results that we will use to investigate reductions of Deligne–Lusztig representations. We assume from now on that \( p \geq 2h_\eta \). Given \( \lambda \in X_1(T) \) we let \( \hat{Q}_1(\lambda) \) be the \( G \)-representation constructed in [Jan03 II.11.11] and \( Q_{\lambda} \equiv \hat{Q}_1(\lambda)|_\Gamma \). Then \( Q_{\lambda} \) is a projective \( F(\Gamma) \)-module. Let \( P_\lambda \) denote a \( F(\Gamma) \)-projective cover of \( F(\lambda) \equiv L(\lambda)|_\Gamma \). We first record a result of Chastkofsky and Jantzen (see [Cha81, Theorem 1] and [Jan81, Corollary 2] and also [Her09, Appendix A] for the generalization to reductive groups with simply connected derived subgroup).

**Proposition 3.1.** For \( \lambda \in X_1(T) \), we have

\[
Q_{\lambda} \cong \bigoplus_{\lambda' \in X_1(T) / (p - \pi)X_0(T)} \bigoplus_{\nu \in X(T)^+} P_{\lambda'}^{[L(\lambda') \otimes L(\pi(\nu)) : L(\lambda + p\nu)]_G}
\]

Moreover, \([L(\lambda) \otimes L(\pi(\nu)) : L(\lambda + p\nu)]_G = 1 \) if \( \nu = 0 \) and is 0 otherwise.

Given \( \lambda \in X^*(T) \) let \( \hat{Z}(1, \lambda) \) be the \( G_1T \)-module defined in [Jan81 §2.5] and write \( \hat{L}(1, \lambda) \) for its irreducible cosocle. Following [Jan81 §3.1], given \((s, \mu) \in W \times X^*(T)\) we let \( T_s \) be the \( F \)-stable maximal torus \( g_sTg_{s}^{-1} \) where \( g_s \in G(F_p) \) is any element such that \( g_s^{-1}F(g_s) \in N_G(T) \) is a lift of \( s \) and \( \theta(s, \mu) : \Gamma \cap T_s \rightarrow E^\times \) be a character defined by \( \theta(s, \mu)(t) \equiv [\mu(g_s^{-1}tg_s)] \) where \([\cdot]\) denotes the Teichmüller lift. This data gives rise to the (signed) Deligne–Lusztig induction \( R_s(\mu) \equiv \varepsilon_{G \in T_s} R_{\theta(s, \mu)}^\h \); this is the (occasionally virtual) \( \Gamma \)-representation denoted \( R_s(1, \mu) \) in [Jan81 §3.1]. We implicitly assume \( R_s(\mu) \) is defined over \( E \) and write \( \overline{R}_s(\mu) \) for the semisimplification of the reduction of a \( \Gamma \)-stable \( O \)-lattice in \( R_s(\mu) \). The following is a convenient reformulation of [Jan81 §3.2] which describes the decomposition of \( Q_{\lambda} \) into reductions of Deligne–Lusztig representations.

**Theorem 3.2.** Let \( \mu \in X^*(T) \), \( s \in W \), and \( \lambda \in X_1(T) \). Then

\[
(3.1) \quad \dim \text{Hom}_{\Gamma}(Q_{\lambda}, \overline{R}_s(\mu)) = \sum_{\nu \in X^*(T)} [\hat{Z}(1, \mu + (s\pi - p)\nu + (p - 1)\eta) : \hat{L}(1, \lambda)]_{G_1T}
\]

where the left hand side is suitably interpreted for virtual representations \( R_s(\mu) \).

Moreover, \( \hat{Z}(1, \mu + (s\pi - p)\nu + (p - 1)\eta) : \hat{L}(1, \lambda)]_{G_1T} \neq 0 \) if and only if there exists \( \tilde{w} \in \tilde{W} \) such that \( \tilde{w}^{-1}(0) = \nu \), \( \tilde{w} \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) + \eta \) is dominant, and

\[
(3.2) \quad \tilde{w} \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) \uparrow \tilde{w}_h \cdot \lambda.
\]
Corollary 3.3. Suppose that $\mu - \eta$ is dominant.

\[
\begin{aligned}
\dim \Hom_{G}(Q_\lambda, \mathcal{R}_s(\mu)) &= \sum_{\nu \in X^*(T)} \dim \Hom_{G}(Q_\lambda, \mathcal{R}_s(\mu)) = \sum_{\nu \in X^*(T)} [Z(1, \mu + s\pi \nu + (p-1)\eta) : \hat{L}(1, \lambda)]_{G_1T} \\
&= \sum_{\nu \in X^*(T)} [Z(1, \mu + (s\pi - p)\nu + (p-1)\eta) : \hat{L}(1, \lambda)]_{G_1T}.
\end{aligned}
\]

By [GHS18] Lemma 10.1.5,

\[(3.3) \quad [Z(1, \mu + (s\pi - p)\nu + (p-1)\eta) : \hat{L}(1, \lambda)]_{G_1T} \neq 0\]

if and only if $\sigma \cdot (\mu - \eta + (s\pi - p)\nu) \uparrow w_0 \cdot (\lambda - p\eta)$ for all $\sigma \in W$. This is equivalent to $\tilde{w} \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) \uparrow \tilde{w}_h \cdot \lambda$ where $\tilde{w} = wt_{-\mu}$ and $w \in W$ is any element such that $w \cdot (\mu - \eta + (s\pi - p)\nu) + \eta$ is dominant.

The following is an immediate corollary of Proposition 3.1 and Theorem 3.2.

Corollary 3.3. Suppose that $\mu \in X^*(T)$, $s \in W$, and $\lambda \in X_1(T)$. Then

\[
\dim \Hom_{G}(Q_\lambda, \mathcal{R}_s(\mu)) = \sum_{\nu \in X^*(T)} [\mathcal{R}_s(\mu) : F(\lambda')_{G(T)}] L(\lambda') \otimes L(\pi(\nu)) : L(\lambda + \nu)]_{G_1T} = \sum_{\nu \in X^*(T)} [Z(1, \mu + (s\pi - p)\nu + (p-1)\eta) : \hat{L}(1, \lambda)]_{G_1T}.
\]

4. Generic decompositions of Deligne–Lusztig representations

In this section, we prove our main result on the reductions of generic Deligne–Lusztig representations. We begin with a corollary of the results from the last section.

Corollary 4.1. Let $\mu - \eta \in C_0$ and $\lambda \in X_1(T)$. Suppose that $\mu - \eta$ or $\lambda$ is $h_\eta$-deep in its $p$-alcove. Then $\dim \Hom_{G}(Q_\lambda, \mathcal{R}_s(\mu)) \neq 0$ if and only if there exist $\tilde{w} \in W^+$ and $\tilde{w}_\lambda \in W_1$ such that $\tilde{w} \uparrow \tilde{w}_h \tilde{w}_\lambda$ and $\lambda = \tilde{w}_\lambda \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0)))$. In fact, $\dim \Hom_{G}(Q_\lambda, \mathcal{R}_s(\mu))$ equals

\[
\sum_{\tilde{w} \in W^+, \tilde{w}_\lambda \in W_1} [Z(1, \mu + (s\pi - p)(\tilde{w}^{-1}(0))) + (p-1)\eta) : \hat{L}(1, \lambda)]_{G_1T},
\]

where every term in (4.1) is nonzero and for each $\tilde{w}$ that appears in the sum, $\mu - \eta + s\pi(\tilde{w}^{-1}(0))$ is in $C_0$. If the center $Z$ of $G$ is geometrically connected, then there is only one term in (4.1).

Proof. Let $\mu$ and $\lambda$ be as in the statement of the corollary. Suppose that $\nu \in X^*(T)$ so that $[Z(1, \mu + (s\pi - p)\nu + (p-1)\eta) : \hat{L}(1, \lambda)]_{G_1T} \neq 0$. We claim that $h_\nu \leq h_\eta$. As in the proof of Theorem 3.2, $\sigma \cdot (\mu - \eta + (s\pi - p)\nu) \uparrow w_0(\lambda - p\eta)$ for all $\sigma \in W$. Let $w \in W$ be the unique element such that $\tilde{w} = wt_{-\mu} \in W^+$. Setting $\sigma = w$, Lemma 2.3 implies that $h_\nu = h_{\tilde{w}^{-1}(0)} \leq h_\eta$.

We claim that $\mu - \eta + s\pi \nu \in C_0$ using that $h_\nu \leq h_\eta$. This is clear if $\mu - \eta$ is $h_\eta$-deep in $C_0$. If $\lambda$ is $h_\eta$-deep in its $p$-alcove, then so is $\mu - \eta + s\pi \nu$ as it is in the same $W$-orbit under the $p$-dot action, in which case $\mu - \eta + s\pi \nu$ and $\mu - \eta$ must lie in the same $p$-alcove which is $C_0$.

Theorem 3.2 implies that $[Z(1, \mu + (s\pi - p)\nu + (p-1)\eta) : \hat{L}(1, \lambda)]_{G_1T} \neq 0$ is equivalent to $\tilde{w} \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) \uparrow \tilde{w}_h \cdot \lambda$ where $\tilde{w}$ is defined in terms of $\nu$ as before. This is in turn equivalent to the fact that $\tilde{w} \uparrow \tilde{w}_h \tilde{w}_\lambda$ and $\lambda = \tilde{w}_\lambda \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0)))$ for some $\tilde{w}_\lambda \in W_1$. 

Finally, we show that only one term in (4.1) is nonzero when the center \( Z \) of \( G \) is geometrically connected. Suppose that \( \tilde{w}, \tilde{w}' \in \tilde{W}^+ \) and \( \tilde{w}_\lambda, \tilde{w}'_\lambda \in \tilde{W}_1 \) such that \( \tilde{w} \uparrow \tilde{w}_h \tilde{w}_\lambda \) and \( \tilde{w}_\lambda \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) = \tilde{w}'_\lambda \cdot (\mu - \eta + s\pi(\tilde{w}'^{-1}(0))) \). Lemma 2.5 implies that \( \tilde{w}_\lambda = \tilde{w}'_\lambda \) and that \( \tilde{w}^{-1}(0) = \tilde{w}'^{-1}(0) \) from which we deduce that \( \tilde{w} = \tilde{w}' \).

The following is our main result on the reduction of generic Deligne–Lusztig representations.

**Theorem 4.2.** Suppose that the center \( Z \) of \( G \) is geometrically connected, \( \mu - \eta \) is \( h_\eta \)-deep in \( C_0 \), and \( \lambda \in X_1(T) \). Then \( [\overline{R}_*(\mu) : F(\lambda)]_\Gamma \neq 0 \) if and only if there exist \( \tilde{w} \in \tilde{W}^+ \) and \( \tilde{w}_\lambda \in \tilde{W}_1 \) such that \( \tilde{w} \uparrow \tilde{w}_h \tilde{w}_\lambda \) and \( \lambda = \tilde{w}_\lambda \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) \). Moreover, in this case

\[
[\overline{R}_*(\mu) : F(\lambda)]_\Gamma = [\tilde{Z}(1, \mu + (s\pi - p)(\tilde{w}^{-1}(0)) + (p - 1)\eta) : \tilde{L}(1, \lambda)]_{G_1T}.
\]

**Proof.** If \( \text{Hom}(Q_\lambda, \overline{R}_*(\mu)) = 0 \), then the result holds by Corollary 4.1 and so we assume otherwise. Suppose now that \( \lambda, \lambda' \in X_1(T) \) so that

1. \( [L(\lambda') \otimes L(\pi(\nu)) : L(\lambda + \nu)]_G \neq 0 \) for some \( \nu \in X^*(T)^+ \) and
2. \( [\overline{R}_*(\mu) : F(\lambda')]_\Gamma \neq 0 \).

We will show that \( \lambda - \lambda' \in (p - \pi)X^0(T) \). Then the result follows from Corollaries 3.3 and 4.1.

By Proposition 3.1 and (2), we have that \( \text{Hom}_\Gamma(Q_{\lambda'}, \overline{R}_*(\mu)) \neq 0 \). Corollary 4.1 implies that there are \( \tilde{w}' \in \tilde{W}^+ \) and \( \tilde{w}_{\lambda'} \in \tilde{W}'_1 \) such that \( \tilde{w}' \uparrow \tilde{w} h \tilde{w}_{\lambda'} \) and \( \lambda' = \tilde{w}_{\lambda'} \cdot (\mu - \eta + s\pi(\tilde{w}'^{-1}(0))). \)

In particular, \( h_{\tilde{w}^{-1}(0)} \leq h_\eta \). Similarly, there are \( \tilde{w} \in \tilde{W}^+ \) and \( \tilde{w}_\lambda \) such that \( \tilde{w} \uparrow \tilde{w} h \tilde{w}_\lambda \) and \( \lambda = \tilde{w}_\lambda \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) \) (and \( h_{\tilde{w}^{-1}(0)} \leq h_\eta \)). By (1) and Lemma 2.2, we have \( \lambda + \nu \uparrow \lambda' + \pi(\nu') \) so that

\[
(t_{\nu} \tilde{w}_\lambda) \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) \uparrow \tilde{w}_{\lambda'} \cdot (\mu - \eta + s\pi(\tilde{w}'^{-1}(0)) + \pi(\nu'))
\]

for some \( \nu \in X(T)^+ \) and \( \nu' \in \text{Conv}(\nu) \). Since \( \mu - \eta \) is \( h_\eta \)-deep in \( C_0 \) and \( h_{\tilde{w}^{-1}(0)} \leq h_\eta \), \( \mu - \eta + s\pi(\tilde{w}^{-1}(0)) \) is \( 0 \)-deep in \( C_0 \). (4.2) implies that \( \mu - \eta + s\pi(\tilde{w}'^{-1}(0)) + \pi(\nu') \), which is in the same \( p \)-dot orbit as \( \mu - \eta + s\pi(\tilde{w}^{-1}(0)) \), is \( 0 \)-deep in its \( p \)-alcove. Let \( w \in W_a \) be the unique element such that \( w^{-1} \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0))) \) (and \( \pi(\nu') \)) is in \( C_0 \). (4.2) and Lemma 2.5 then imply that \( t_{\nu} \tilde{w}_\lambda \uparrow \tilde{w}_{\lambda} w \) and \( w^{-1} \cdot (\mu - \eta + s\pi(\tilde{w}^{-1}(0)) + \pi(\nu')) = \mu - \eta + s\pi(\tilde{w}^{-1}(0)). \) In particular, we have

\[
\nu + \tilde{w}_{\lambda}(v) \leq \tilde{w}_{\lambda'} w(v)
\]

for any \( v \) of the dominant base alcove. As \( \nu' \in \text{Conv}(\nu) \) and \( h_\nu \leq h_\eta \), the alcoves corresponding to \( 1 \) and \( w \) share a vertex \( v \). Thus, \( w \) stabilizes a vertex \( v \) of the dominant base alcove, and we have

\[
\nu + \tilde{w}_\lambda(v) \leq \tilde{w}_{\lambda'}(v).
\]

We now claim that

\[
<s\pi(\tilde{w}'^{-1}(0)) + \pi(\kappa), \alpha^\vee> \leq h_\eta + 1
\]

for any root \( \alpha \) and \( \kappa \in \text{Conv}(\nu) \). Using that \( \text{Conv}(\nu) \) is \( W \)-invariant and that \( \alpha_0 \) is a highest root if and only if \( \pi^{-1}(\alpha_0) \) is, it suffices to show that \( <s\pi(\tilde{w}'^{-1}(0)) + \kappa, \alpha_0^\vee> \leq h_\eta \) for any \( \sigma \in W \), any highest root \( \alpha_0 \), and any \( \kappa \in \text{Conv}(\nu) \). To show this, we will need that \( \sigma \tilde{w}'^{-1}(0) \leq \eta - \tilde{w}_{\lambda'}(0) \) for any \( \sigma \in W \), which follows from the inequalities \( \sigma \tilde{w}'^{-1}(0) \leq -w_0 \tilde{w}'(0) \) and \( \tilde{w}' \uparrow \tilde{w}_h \tilde{w}_{\lambda'}. \) Using further
Lemma 5.1. Suppose that the center $Z$ of $G$ is geometrically connected. Let $s \in W$, $\lambda_0 \in C_0$, $\lambda' \in X_1(T)$, $\tilde{w}_\lambda \in \tilde{W}_1$, $\tilde{w} \in \tilde{W}$, $\nu \in X(T)^+$, and $\nu' \in \text{Conv}(\nu)$ such that

1. $t_s \tilde{w}_\lambda \cdot \lambda_0 \uparrow \lambda' + \pi(\nu')$;
2. if $\tilde{w}_{\lambda'} \in t_s \tilde{w}_\lambda W_0 \cap W_1$, then $\lambda' + \pi(\nu') \in \tilde{w}_{\lambda'} W \cdot C_0$;
3. $\lambda_0 - s\pi(\tilde{w}_h \tilde{w}_\lambda)^{-1}(0) + s\pi\tilde{w}^{-1}(0) \in C_0$; and
4. $\tilde{w} \cdot (\lambda_0 - s\pi(\tilde{w}_h \tilde{w}_\lambda)^{-1}(0) + s\pi\tilde{w}^{-1}(0)) + \eta \in X(T)^+$ and $\tilde{w} \cdot (\lambda_0 - s\pi(\tilde{w}_h \tilde{w}_\lambda)^{-1}(0) + s\pi\tilde{w}^{-1}(0)) \uparrow \tilde{w}_h \cdot \lambda'$.

Then $\nu \in X^0(T)$ and $t_s \tilde{w}_\lambda \cdot \lambda_0 = \lambda' + \pi(\nu)$.

Proof. Let $\tilde{w}_{\lambda'}$ be as in (2) and let $\lambda_0' \overset{\text{def}}{=} \tilde{w}_{\lambda'}^{-1} \cdot \lambda' \in \tilde{C}_0$. Then (2) implies that $\lambda_0' + w_{\lambda'}^{-1} \cdot \pi(\nu') = \tilde{w}_{\lambda'}^{-1} \cdot (\lambda' + \pi(\nu')) \in w \cdot C_0$ for some $w \in W$. By [LLHLM23 Lemma 2.2.2], $w^{-1} \cdot (\lambda_0' + w_{\lambda'}^{-1} \cdot \pi(\nu')) = \lambda_0' + \pi(\nu'')$ for some $\nu'' \in \text{Conv}(\nu)$. Furthermore, (1) and Lemma 2.5 imply that $t_s \tilde{w}_{\lambda'} \uparrow \tilde{w}_{\lambda'} W$ and $\lambda_0 = \lambda_0' + \pi(\nu'')$. In particular, $t_s \tilde{w}_{\lambda}(0) \leq \tilde{w}(0)$.

(2), (3), and Lemma 2.5 imply that $\tilde{w} \in \tilde{W}^+$, $\tilde{w} \uparrow \tilde{w}_h \tilde{w}_{\lambda'}$, and $\lambda_0 - s\pi(\tilde{w}_h \tilde{w}_{\lambda})^{-1}(0) + s\pi\tilde{w}^{-1}(0) = \lambda_0' = \lambda_0 - \pi(\nu'')$. We deduce that $\tilde{w}(0) \leq w_0 \nu + w_0 \tilde{w}_{\lambda}(0) - w_0 \eta$ and $\nu'' = \pi^{-1}(s)((\tilde{w}_h \tilde{w}_{\lambda})^{-1}(0) - \tilde{w}^{-1}(0))$.

Let $w_{\lambda} \in W$ be the image of $\tilde{w}_{\lambda}$. We have the inequalities

$$\begin{align*}
\nu & \leq \eta - \tilde{w}_{\lambda}(0) + w_0 \tilde{w}(0) \\
& \leq \eta - \tilde{w}_{\lambda}(0) - w_0 \tilde{w}^{-1}(0) \\
& = w_{\lambda} \pi^{-1}(s^{-1})(\nu'') \\
& \leq \nu.
\end{align*}$$

where $w_{\lambda'} \in W$ is the image of $\tilde{w}_{\lambda}$ under the projection $\tilde{W} \rightarrow W$ and the last inequality follows from the fact that $\nu$ is in the dominant base alcove.

Using that $\mu - \eta$ is $\eta_p$-deep in $C_0$, $\lambda' + \nu'$ is in the closure of the $p$-alcove containing $\lambda'$ for any $\nu' \in \text{Conv}(\nu)$ by (4.5). The fact that $\lambda$ is 0-deep, (1), and Corollary 2.7 imply that $\lambda + p\nu$ and $\nu'$ are in the same $p$-alcove. In particular, $\lambda + p\nu$ is $p$-restricted so that $\nu \in X^0(T)$. Then $\lambda - \lambda' \in (p - \pi)\nu$ for some $\nu \in X^0(T)$.

$\square$

Remark 4.3. In fact, the bound in Theorem 4.2 is sharp. If $G_0 = GL_{2/p}$, then $\tilde{R}_s(\mu)$ has 2 Jordan–Hölder factors if $\mu - \eta$ is 1-deep, but $\tilde{R}_{12}(1, 0)$ has 1 Jordan–Hölder factor.

5. Deligne–Lusztig reductions containing a simple module

In this section, we prove Theorem 1.3 which exhibits Deligne–Lusztig representations whose reductions contain a fixed simple module (see Theorem 5.4).
Thus, these inequalities are all equalities. In particular, \( \tilde{w}(0) = t_w(0)\tilde{w}(0) = w(0)\tilde{w}(0) - w(0)\tilde{w}(0) \in X(T)^+ \). The first of these equalities also implies that \( w(0)(t_w(0)\tilde{w}(0))^{-1}(0) = w(0)\tilde{w}(0) = X(T)^+ \). Lemma 2.1 then implies that \( \tilde{w} = t_w(0)\tilde{w}(0) \). As both \( t_w(0)\tilde{w}(0) \) and \( \tilde{w}(0) \tilde{w}(0) \) are in \( \tilde{W}^+ \), we have \( t_w(0) \tilde{w}(0) \in \tilde{W}_1 \). Since \( \tilde{w}(0) \tilde{w}(0) \in \tilde{W}_1 \), we must have \( \nu \in \tilde{X}^0(T) \). This further implies that \( \lambda + \pi(\nu') \in \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w}(0) \tilde{w(}
where \( v \) is a vertex of \( A_0 \) stabilized by \( \bar{w} \). Then for \( \kappa \in \text{Conv}(\nu) \),
\[
(s \pi(\tilde{w}^{-1}(0))) \in \pi(\kappa), \alpha \gamma) = (\pi^{-1}(s \tilde{w}^{-1}(0)) + \kappa, \pi^{-1}(\alpha \gamma)) \\
\leq (-\nu + \eta - \tilde{w}_\lambda(\nu) + v + \nu, \alpha \gamma) \\
\leq (\eta - \tilde{w}_\lambda(\nu) + v, \alpha_0) \\
\leq h_{\tilde{w}_\lambda\tilde{w}_\lambda}(v) + 1
\]
where \( v \) is a vertex of \( A_0 \) as before and \( \alpha_0 \) is some highest root.

As \( \lambda_0 - s \pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) \) is 0-deep in \( C_0 \) and \( \lambda_0 - s \pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) + s \pi \tilde{w}^{-1}(0) \) is \( h_{\tilde{w}_h\tilde{w}_\lambda}(\nu) \)-deep in its\( p \)-alcove, Lemma 2.2 implies that \( \lambda_0 - s \pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) + s \pi \tilde{w}^{-1}(0) + \pi(\nu') \) is \( h_{\tilde{w}_h\tilde{w}_\lambda}(\nu) \)-deep in \( C_0 \) so that \( \lambda_0 - s \pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) + s \pi \tilde{w}^{-1}(0) \) is in \( C_0 \) by Lemma 2.2.

**Lemma 5.3.** Suppose that the center \( Z \) of \( G \) is geometrically connected. Let \( \lambda_0 \in C_0 \), \( \lambda' \in X_1(T) \), \( \tilde{w}_\lambda \in \tilde{W}_1 \) with image \( \bar{w}_\lambda \in W \), \( \bar{w} \in \tilde{W} \), \( \nu \in X(T)^+ \), and \( \nu' \in \text{Conv}(\nu) \) such that

(1) \( \langle \lambda_0 + \eta, \alpha \gamma \rangle < p - 2h_\eta - h_{\tilde{w}_h\tilde{w}_\lambda}(0) \) for all roots \( \alpha \);

(2) \( t_\nu\tilde{w}_\lambda \cdot \lambda_0 \uparrow \lambda' + \nu(\nu') \);

(3) \( \tilde{w} \cdot (\lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0)) + \eta \in X(T)^+ \) and \( \tilde{w} \cdot (\lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0)) \uparrow \tilde{w}_h \cdot \lambda' \).

Then letting \( \tilde{w}_\lambda \in t_\nu\tilde{w}_\lambda w_a \cap \tilde{W}_1 \) such that \( \lambda' \in \tilde{w}_\lambda, C_0, \lambda' + \nu(\nu') \in \tilde{w}_\lambda W \cdot C_0 \) and \( \lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0) \in C_0 \).

**Proof.** Let \( w_\lambda \in W \) be the image of \( \tilde{w}_\lambda \). Then \( \langle \lambda_0 - w_\lambda^{-1}\pi(\nu'), \eta, \alpha \gamma \rangle < p - 2h_\eta - h_{\tilde{w}_h\tilde{w}_\lambda}(0) \) for all roots \( \alpha \) by (1), (2), and Lemma 2.2. So that \( \sigma_1 \cdot (\lambda_0 - w_\lambda^{-1}\pi(\nu')) \in C_0 \) for some \( \sigma_1 \in W \). Letting \( \lambda'_0 \overset{\text{def}}{=} \tilde{w}_\lambda^{-1} \cdot \lambda' \in C_0 \), Lemma 2.5 implies that \( \lambda'_0 = \sigma_1 \cdot (\lambda_0 - w_\lambda^{-1}\pi(\nu')) \) Since \( t_\nu\tilde{w}_\lambda \sigma_1^{-1} \cdot \lambda'_0 \uparrow \tilde{w}_\lambda \cdot \lambda'_0 \), we also have that \( t_\nu\tilde{w}_\lambda(0) \leq w_\lambda(0) \).

Next, we claim that \( h_{\tilde{w}_\lambda(0)} \leq h_\eta \). Suppose that \( \alpha \) is a root so that \( \langle \tilde{w}(0), \alpha \gamma \rangle = h_{\tilde{w}(0)} \). Then (1), (2), and (3) imply that \( \tilde{w}(0) \in X^*(T)^+ \) and
\[
(p - 1)h_{\tilde{w}(0)} - (p - 1 - h_\eta) \leq \langle \tilde{w} \cdot (\lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0)) + \eta, \alpha \gamma \rangle \\
\leq \langle \tilde{w}_h \cdot \lambda', \eta, \alpha \gamma \rangle \\
\leq (p - 1)h_\eta
\]
where \( \alpha_0 \) is some highest root. Thus, \( h_{\tilde{w}(0)} \leq \frac{p - 2}{p - 1}h_\eta + 1 < h_\eta + 1 \), and the claim follows.

From the previous claim and (1), we have \( \langle \lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0), \alpha \gamma \rangle < p \) for all roots \( \alpha \). Then \( \sigma_2 \cdot (\lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0)) \in C_0 \) for some \( \sigma_2 \in W \). (4) then imply that \( \lambda'_0 = \sigma_2 \cdot (\lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0)) \) and \( \tilde{w}(0) \leq \tilde{w}_h \tilde{w}_\lambda(0) \) (as 0 lies in the closure of the facet determined by \( \lambda'_0 \)). Putting things together, we have \( \lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0) = \sigma_2^{-1} \sigma_1 \cdot (\lambda_0 - w_\lambda^{-1}\pi(\nu')) \), or equivalently that
\[
\lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0) + \sigma_2^{-1} \sigma_1 w_\lambda^{-1}(\nu) = \sigma_2^{-1} \sigma_1 \cdot \lambda_0,
\]
and
\[
\tilde{w}_h \tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0) + \pi^{-1}(\sigma_2^{-1} \sigma_1 w_\lambda^{-1}(\nu)) \geq \tilde{w}_h \tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0) + w_0 \nu \geq 0
\]
using that \( t_\nu \tilde{w}_\lambda(0) \leq \tilde{w}_\lambda(0) \), \( \tilde{w}(0) \leq \tilde{w}_h \tilde{w}_\lambda(0) \), and \( \tilde{w}(0) \in X^*(T)^+ \). (5.3) and (5.4) imply that \( \lambda_0 \leq \sigma_2^{-1} \sigma_1 \cdot \lambda_0 \) so that \( \sigma_1 = \sigma_2 \) and \( w_\lambda^{-1}(\nu) = \pi w_0 \nu \). In particular, \( \lambda'_0 = \lambda_0 - \pi w_0 \nu \in X^+ \) so that \( \lambda'_0 \in C_0 \) by (1) and Lemma 2.2. This implies that \( \sigma_1 = 1 \) and thus \( \sigma_2 = 1 \). We conclude that \( \lambda_0 + \eta + \pi\tilde{w}_h\tilde{w}_\lambda(0) + \pi w_0 w_\lambda \tilde{w}^{-1}(0) = \lambda'_0 \in C_0 \).
Finally, \( \lambda' + \pi(\nu') = \tilde{w}_\lambda' \cdot (\lambda_0 - w_\lambda^{-1} \pi(\nu') + w_\lambda^{-1} \pi(\nu')) \). Then \( \lambda_0 - w_\lambda^{-1} \pi(\nu') + w_\lambda^{-1} \pi(\nu') \in W \cdot \mathcal{C}_0 \) by (1) so that \( \lambda' + \pi(\nu') \), which is linked to \( \lambda_0 \in C_0 \), is in \( \tilde{w}_\lambda W \cdot C_0 \). 

\( \square \)

**Theorem 5.4.** Suppose that the center \( Z \) of \( G \) is geometrically connected. Let \( \tilde{w}_\lambda \in \tilde{W}_1 \), \( s \in W \), and \( \lambda_0 \in C_0 \). Suppose that either

1. \( \lambda_0 \) is \( h_{\tilde{w}_h\tilde{w}_\lambda}(v) \)-deep in \( C_0 \) for every vertex \( v \) of the dominant base alcove; or
2. \( \langle \lambda_0 + \eta, \alpha^\vee \rangle < p - 2h_\eta - h_{\tilde{w}_h\tilde{w}_\lambda(0)} \) for all roots \( \alpha \) and \( s = \pi(\tilde{w}_h\tilde{w}_\lambda) \).

Then \( F(\tilde{w}_\lambda \cdot \lambda_0) \) is a Jordan–Hölder factor of \( \tilde{R}_s(\lambda_0 + \eta - s\pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0)) \) with multiplicity one.

**Proof.** Let \( \lambda \overset{\text{def}}{=} \tilde{w}_\lambda \cdot \lambda_0 \). Then setting \( \tilde{w} \overset{\text{def}}{=} \tilde{w}_h\tilde{w}_\lambda \) in Corollary 4.1 shows that

\[
\dim \text{Hom}_G(Q_\lambda, \tilde{R}_s(\lambda_0 + \eta - s\pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0))) = [\tilde{Z}(1, \lambda_0 - p(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) + \eta)] : \tilde{L}(1, \lambda)|_{G_1T} = [\tilde{Z}(1, \lambda_0 - p(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) + \eta + (p - 1)\eta)] : \tilde{L}(1, \lambda)|_{G_1T} = 1 \]

where \( w_h \in W \) is the image of \( \tilde{w}_h \), the second equality follows from [2003, 9.16(5)] and the fourth equality follows for instance by using that \( [\tilde{Z}(1, \lambda) : \tilde{L}(1, \lambda)]|_{G_1T} \) is nonzero and \( \lambda \) appears with multiplicity one in both \( \tilde{Z}(1, \lambda)|_{T} \) and \( \tilde{L}(1, \lambda)|_{T} \). We claim that if \( \lambda' \in X_1(T) \) and \( L(\lambda') \otimes L(\pi(\nu)) : L(\lambda + p\nu)|_{G_1T} \neq 0 \) for some \( \nu \in X^*(T)^+ \) and \( \text{Hom}_G(Q_{\lambda'}, R_s(\lambda_0 + \eta - s\pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0))) \neq 0 \), then \( \nu \in X^0(T) \). Then the result follows from Corollary 3.3.

Suppose that \( \lambda' \in X_1(T) \), \( L(\lambda') \otimes L(\pi(\nu)) : L(\lambda + p\nu)|_{G_1T} \neq 0 \) for some \( \nu \in X^*(T)^+ \), and \( \text{Hom}_G(Q_{\lambda'}, R_s(\mu)) \neq 0 \). Then Lemma 2.2 implies that there exists \( \nu \in \text{Conv}(\nu) \) such that \( \lambda + p\nu \uparrow \lambda' + \pi(\nu') \), and Theorem 3.2 implies that there exists \( \tilde{w} \in W \) such that \( \tilde{w} \cdot (\lambda_0 - s\pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) + s\pi\tilde{w}^{-1}(0) + \eta) \in X(T)^+ \) and \( \tilde{w} \cdot (\lambda_0 + \pi(\tilde{w}_h\tilde{w}_\lambda)^{-1}(0) + s\pi\tilde{w}^{-1}(0)) \uparrow \tilde{w}_h \cdot \lambda' \). Lemmas 5.2 and 5.3 imply conditions (2) and (3) of Lemma 5.1. Finally, Lemma 5.1 implies that \( \nu \in X^0(T) \). 

\( \square \)

6. **Applications to weight elimination**

Let \( q \) be a power of \( p \) and \( K = W(F_q)[p^{-1}] \). Assume that any homomorphism \( K \to \overline{Q}_p \) factors through \( E \). We now take \( G_0 \) to be \( \text{Res}_{F_p/F_q} \text{GL}_n/F_q \). Let \( L(G) \overset{\text{def}}{=} \prod_{K \to E} \text{GL}_{n/2} \times \text{Gal}(E/F_q) \). Let \( G_{Q_p} \overset{\text{def}}{=} \text{Gal}(\overline{Q}_p/F_p) \) and \( G_K \overset{\text{def}}{=} \text{Gal}(\overline{K}/K) \) with inertia subgroups \( I_{Q_p} \) and \( I_K \), respectively. Restriction and projection gives a bijection between conjugacy classes of continuous \( L \)-homomorphisms \( L(\overline{\varphi}) : G_{Q_p} \to \overline{L}(F) \) and conjugacy classes of continuous homomorphisms \( \overline{\varphi} : G_K \to \text{GL}_n(F) \).

Let \( X_{reg}^*(T) \subset X_1(T) \) denote the subset of \( \lambda \in X_1(T) \) such that \( \langle \lambda, \alpha^\vee \rangle < p - 1 \) for all simple roots \( \alpha \). We say that a simple \( \overline{F} \text{GL}_n(F_q) \)-module is regular if its highest weight is in \( X_{reg}^*(T) \). The \( p \)-dot action of \( \tilde{w}_h \) defines a self-bijection \( X_{reg}^*(T) \rightarrow X_{reg}^*(T) \). Let \( R \) be the corresponding self-bijection on the set isomorphism classes of regular simple \( \overline{F} \text{GL}_n(F_q) \)-modules, that is \( R(F(\lambda)) = F(\tilde{w}_h \cdot \lambda) \). For a conjugacy class of tame continuous homomorphisms \( \overline{\varphi} : G_K \to \text{GL}_n(F) \), let \( W^2(\overline{\varphi}) \) be the set \( W^2(L(\overline{\varphi})|_{I_{Q_p}}) \) in [2018, Definition 9.2.5] where \( L(\overline{\varphi}) \) is the \( L \)-parameter corresponding to \( \overline{\varphi} \). Outside degenerate cases which are irrelevant in our context, the set \( W^2(\overline{\varphi}) \) has the following concrete description [2018, Proposition 9.2.3]: we can write \( L(\overline{\varphi}) \) (possibly several ways) as an explicit representation \( \tau(s, \mu) \) depending on \( (s, \mu - \eta) \in W \times (C_0 \cap X^+(T)) \) [2018, Proposition 9.2.3], and the set \( W^2(\overline{\varphi}) \) is \( R(JH(\overline{R}_s(\mu))) \) (in this case we say that \( \overline{\varphi} \) is \( m \)-generic if we can choose \( \mu - \eta \) to be \( m \)-deep in \( C_0 \) ).
It is conjectured [GHS18] that the set $W^2(\mathfrak{p})$ controls weights of mod $p$ automorphic forms for any globalization of $\mathfrak{p}$ e.g. mod $p$ Langlands parameters contributing to spaces of mod $p$ algebraic modular forms on definite unitary groups as in [11].

In any such context, establishing the upper bound given by $W^2(\mathfrak{p})$ is referred to as “weight elimination”. In our previous work [LLHL19], we establish weight elimination in an axiomatic framework that applies to many global contexts (for instance the one in Theorem [1.5] under the hypothesis that $\mathfrak{p}$ is $(6n-2)$-generic.

The method of loc. cit. was to combine constraints from $p$-adic Hodge theory with generic decomposition patterns of Deligne–Lusztig representations. Our new results on the latter allow us to improve our earlier axiomatic weight elimination results to the following

**Theorem 6.1.** Suppose that $p > 3n(n-1)$. Let $\mathfrak{p} : G_K \to \text{GL}_n(\mathbb{F})$ be a continuous homomorphism. Suppose that $W(\mathfrak{p})$ is a set of isomorphism classes of simple $\mathbb{F}[\text{GL}_n(\mathbb{F}_q)]$-modules such that if $W(\mathfrak{p}) \cap JH(\mathcal{R}_w(\nu)) \neq \emptyset$ and either

- $\tau(w, \nu)$ is regular (i.e. multiplicity free); or
- $w = 1$;

then $\mathfrak{p}$ has a potentially semistable lift of type $(\eta, \tau(w, \nu))$. Assume further that $\mathfrak{p}^{ss}|_{I_K}$ is $(2n+1)$-generic.

Then $W(\mathfrak{p}) \subset W^2(\mathfrak{p}^{ss})$.

**Proof.** We follow the general outline of [LLHL19] §4.2.

By [Enn19] Lemma 5, if $\mathfrak{p}$ has a potentially semistable lift of type $(\eta, \tau(w, \nu))$, so does $\mathfrak{p}^{ss}$. We then reduce to the case where $\mathfrak{p}$ is semisimple.

Suppose that $\lambda \in X_1(T)$ with $F(\lambda) \in W(\mathfrak{p})$. First, $\mathfrak{p}$ is $(2n+1)$-generic in the sense of [Enn19] Definition 2 so that $\lambda$ is $p$-regular by [Enn19] Theorem 8. Then we can write $\lambda = \tilde{w}_\lambda \cdot \lambda_0$ with $\lambda_0 \in C_0$ (and $\tilde{w}_\lambda \in \tilde{W}_1$).

Using that $p > 3n(n-1)$, after possibly replacing $\lambda_0$ by an element in $\Omega \cdot \lambda_0$, we can assume that $(\lambda_0 + \eta, \alpha^\lor) < p - 3h_\eta$ for all $\alpha \in R$. We claim that $\lambda_0 = (h_{\tilde{w}_\lambda} \tilde{w}_\lambda(0) + 1)$-deep in $C_0$. Suppose otherwise. Then Theorem [3.4] implies that $F(\lambda) \in JH(\mathcal{R}_w(\nu))$ with $w = \pi(w_0 w_\lambda)$ and $\nu = \lambda_0 + \eta - \pi(w_0 w_\lambda(\tilde{w}_\lambda \tilde{w}_\lambda)^{-1}(0))$. Since $\tilde{w}_\lambda \in \tilde{W}_1$, we have $0 \leq (\lambda_0 + \eta - \pi(w_0 w_\lambda(\tilde{w}_\lambda \tilde{w}_\lambda)^{-1}(0), \alpha^\lor) \leq p - 2h_\eta$ for all $\alpha \in R$. We conclude that $\mathfrak{p}$ has a potentially semistable (and thus potentially crystalline) lift of type $(\eta, \tau)$ where $\tau$ is the (regular) inertial type $\tau(w, \nu)$ and $\nu - \eta$ is 1-deep, but not $(h_{\tilde{w}_\lambda} \tilde{w}_\lambda(0) + 2)$-deep. We claim that $\mathfrak{p}|_{I_K} \cong \tau(s, \mu)$ for some $s \in W$ and $\mu \in X^*(T)$ such that

$$(t, w)^{-1} I_{\mu} s \in \text{Adm}(\eta) \overset{\text{def}}{=} \{ \tilde{w} \in \tilde{W} \mid \tilde{w} \leq t_{\sigma(\eta)} \text{ for some } \sigma \in W \}.$$  

Given this claim, we deduce that $\mu - \eta$ is not $(2h_\eta + 2)$-deep so that $\mathfrak{p}$ is not $(2h_\eta + 3)$-generic. This is a contradiction.

We now prove the claim in the previous paragraph. There is a unique lowest alcove presentation $(s, \mu - \eta)$ of $\mathfrak{p}$ compatible in the sense of [LLHLM23] §2.4 with the 1-generic lowest alcove presentation $(w, \nu - \eta)$ of $\tau$ by [LLHLM23] Lemma 2.4.4 (though it should be assumed in the cited lemma that the center $Z$ is geometrically connected). Moreover, $\mu - \eta$ is $2n$-deep by [LLHL19] Proposition 2.2.15. Let $K'/K$ be a finite unramified extension so that the restriction $\mathfrak{p}'$ is a direct sum of characters and the base change inertial type $\tau'$ is principal series. Furthermore, $(w', \nu' - \eta')$ is a 1-generic lowest alcove presentation for $\tau'$ where $w_j' = w_j|_{I_K}$ and $\nu_j' = \nu_j|_{I_K}$ for any embedding $j' : K' \to E$, and similarly $(s', \mu' - \eta')$ is a compatible $2n$-generic lowest alcove presentation of $\mathfrak{p}'$. [LLHL19] Theorem 3.2.1 implies that $\tau(s', \mu') \cong \tau(s', \xi')$ for some $\xi' \in (\mathbb{Z})^{\text{Hom}_{q_p}(K', E)}$ and...
The weight in a Serre-type conjecture for tame Generic Cartan invariants for Frobenius kernels and Chevalley groups [Hum89] J. E. Humphreys, [BDR17] Cédric Bonnafé, Jean-François Dat, and Raphaël Rouquier, [GHS18] Toby Gee, Florian Herzig, and David Savitt, (w, η = wπ(\bar{w}_h\bar{w}_\lambda)^{-1}(0)) for all w ∈ W. Then the proof of [LLHL19 Lemma 4.1.10] implies that \( F(\lambda) \in \hat{W}^e(\overline{\mathfrak{p}}) \).

**References**


