GENERIC DECOMPOSITIONS OF DELIGNE–LUSZTIG REPRESENTATIONS

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Abstract. Let G_0 be a reductive group over $\mathbb{F}_p$ with simply connected derived subgroup, connected center and Coxeter number $h+1$. We extend Jantzen's generic decomposition pattern from $(2h-1)$-generic to h-generic Deligne–Lusztig representations, which is optimal. We also prove several results on the "obvious" Jordan–Hölder factors of general Deligne–Lusztig representations. As an application we improve the weight elimination result of [LLHL19].

1. Introduction

Let G_0 be a reductive group over $\mathbb{F}_p$ with simply connected derived subgroup. An important problem is to understand the representations of the finite group of Lie type $G_0(\mathbb{F}_p)$. In characteristic 0, Deligne–Lusztig gave a beautiful geometric construction of characters of $G_0(\mathbb{F}_p)$ which effectively describes all irreducibles. It is natural to ask how characteristic 0 irreducibles decompose modulo a prime ℓ. When $\ell \neq p$, this problem has been studied extensively see e.g. [BR03, BDR17]. In contrast, the defining characteristic $\ell = p$ case seems underdeveloped, despite its connections to number theory and more specifically the study of congruences of automorphic forms. The main result here, due to Jantzen [Jan81], describes the mod p reduction of *sufficiently generic* Deligne–Lusztig representations.

Let T be a maximally split $\mathbb{F}_p$-rational maximal torus of $G \overset{\text{def}}{=} G_0 \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$, μ a character of T and s an element in the Weyl group of W (with respect to T). To this data we can associate an $\mathbb{F}_p$-rational maximal torus T_s and a $W(\overline{\mathbb{F}}_p)^\times$-valued character $\theta(s,\mu)$ of $T_s(\mathbb{F}_p)$ and thus a Deligne–Lusztig representation $R_s(\mu)$ (see §3). Recall that in characteristic p, after choosing a Borel subgroup, p-restricted highest weights λ parametrize irreducible representations $F(\lambda)$ of $G_0(\mathbb{F}_p)$. We refer to these as Serre weights. Let $h+1$ denote the Coxeter number of G_0 and η be (a lift of) the half of the sum of the positive roots. When $\mu-\eta$ is $(2h-1)$-deep in the base p-alcove C_0 (anchored at $-\eta$), Jantzen [Jan81] (and Gee, Herzig and Savitt [Her09, GHS18] for reductive groups) gives a formula for the reduction $\overline{R}_s(\mu)$ of $R_s(\mu)$ in terms of Frobenius kernel multiplicities.

A basic feature of Jantzen's formula is that the highest weights of the Jordan–Hölder factors is given by universal combinatorial formulas in s, μ, which in particular implies that the length of the reduction is independent of s and μ. In order for these universal formulas (as s varies in W) to produce p-restricted weights, $\mu-\eta$ must be h-deep (see Remark 4.3). Thus the best one can hope for is that Jantzen's formula always holds when $\mu-\eta$ is h-deep. Our first main result confirms this when the center of G is connected.

Theorem 1.1. *Suppose that G has connected center, $\mu-\eta$ is h-deep in the base p-alcove, and λ is a p-restricted dominant weight.*

Then $[\overline{R}_s(\mu) : F(\lambda)] \neq 0$ if and only if there exist $\widetilde{w}$ and $\widetilde{w}_\lambda$ in the extended affine Weyl group $\widetilde{W}$ such that:

- *$\widetilde{w} \cdot C_0$ is dominant and $\widetilde{w}_\lambda \cdot C_0$ is p-restricted (where $\cdot$ denotes the p-dot action);*

- $\widetilde{w} \uparrow \widetilde{w}_h \widetilde{w}_\lambda$ *(where* $\uparrow$ *is the semi-infinite Bruhat order defined as in* [Jan03, II.6.5], $\widetilde{w}_h \stackrel{def}{=} w_0 t_{-\eta}$ *and* w_0 *is the longest element of* W*); and*
- $\lambda = \widetilde{w}_\lambda \cdot (\mu - \eta + s\pi(\widetilde{w}^{-1}(0)))$ *(where* π *denotes the automorphism of* $X^*(T)$ *corresponding to Frobenius; see §1.3).*

Moreover, in this case:

$$[\overline{R}_s(\mu) : F(\lambda)]_{G_0(\mathbb{F}_p)} = [\widehat{Z}(1, \mu + (s\pi - p)(\widetilde{w}^{-1}(0)) + (p-1)\eta) : \widehat{L}(1,\lambda)]_{G_1 T}.$$

Here, $\widehat{Z}(1,-)$ *and* $\widehat{L}(1,-)$ *are the baby Verma/standard and simple modules, respectively, for the augmented Frobenius kernel* $G_1 T$ *(see §3).*

Remark 1.2.
1. The hypothesis on connected center is necessary for the second conclusion to hold—without it the right hand side should be replaced by a sum of Frobenius kernel multiplicities. The (generic) decomposition problem in the general reductive case can often be reduced to Theorem 1.1 by an analysis of isogenies.
2. The multiplicity $[\widehat{Z}(1,\mu') : \widehat{L}(1,\lambda)]_{G_1 T}$ can be nonzero only if μ', λ are in the same p-dot orbit of the affine Weyl group, in which case, for p sufficiently large, it depends combinatorially on the p-facets containing μ' and λ. For instance, when $p \gg h$ it is controlled by periodic Kazhdan–Lusztig polynomials.
3. In a different direction, Pillen [Pil93] analyzes the contribution of the p-singular weights when μ lies in exactly one wall of C_0 and is $2h-1$ away from the other walls.
4. This result was first suggested by considerations in the theory of local models for potentially crystalline Emerton–Gee stacks. Specifically, the hypothesis on μ in Theorem 1.1 is the range where the special fiber cycles (which are expected to reflect the mod p reduction of Deligne–Lusztig representations by the Breuil–Mézard conjecture) have uniform behavior.

It is also natural to contemplate the dual question, i.e. given a Serre weight $F(\lambda)$, for which (s,μ) is $F(\lambda)$ a Jordan–Hölder factor of $\overline{R}_s(\mu)$? This problem is essentially equivalent to decomposing the characteristic zero lift of a projective cover P_λ of $F(\lambda)$ into irreducibles (the bulk of which are of the form $R_s(\mu)$). When λ is $2h$-deep in its alcove, the complete decomposition can be obtained from Theorem 1.1. In particular there are always $|W|$ "obvious" $R_s(\mu)$ which contain $F(\lambda)$ with multiplicity one. However, when λ is not $2h$-deep the decomposition of P_λ becomes considerably more complicated; for instance some $R_s(\mu)$ factors that appear generically may disappear.

Nevertheless, we show that the "obvious" $R_s(\mu)$ factors of P_λ persist up to essentially the optimal threshold:

Theorem 1.3. *Suppose that* G *has connected center. Let* λ *be a* p*-restricted dominant weight which is* h*-deep in its* p*-alcove. Then for all* $s \in W$*,* $F(\lambda)$ *is a Jordan–Hölder factor of* $\overline{R}_s(\widetilde{w}_h \cdot \lambda + \eta)$ *with multiplicity one.*

Remark 1.4. In fact we prove the theorem under weaker hypotheses on λ depending on its alcove, see Theorem 5.4.

Theorem 1.3 gives a large supply of characteristic zero irreducible representations containing $F(\lambda)$ when λ is h-deep. For the number theoretic application discussed below, we would like to construct for *any* (p-dot regular) λ an $R_s(\mu)$ containing $F(\lambda)$ such that μ is in the base alcove with essentially the same depth as λ. We establish such a statement in Theorem 5.4 under a mild "smallness" hypothesis which can always be arranged in type A. Note that this is rather subtle because when λ is not h-deep the most obvious Deligne–Lusztig induction $R_1(\lambda)$ containing $F(\lambda)$ usually fails the depth requirement (because of a small translation when expressing $R_1(\lambda)$ as $R_s(\mu)$ with $\mu - \eta \in C_0$).

1.1. **A number theoretic application.** We now explain how the above results allow us to improve the main theorem of [LLHL19]. Recall the global setting of *loc. cit.* Let F/F^+ be a totally imaginary extension of a totally real field $F^+ \neq \mathbb{Q}$ such that p is inert in F^+ and splits in F. Given a reductive group $G_{/F^+}$ which is an outer form for GL_n and splits over F, and such that $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, and given a compact open subgroup of the form $U = U^p G(\mathcal{O}_{F^+_p}) \leq G(\mathbb{A}^{\infty}_{F^+})$ and a $G(\mathcal{O}_{F^+_p})$-module M, we define a space

$$S(U, M) \stackrel{\text{def}}{=} \{f : G(F^+)\backslash G(\mathbb{A}^{\infty}_{F^+})/U \to M \mid f(gu) = u_p^{-1} f(g)\, \forall g \in G(\mathbb{A}^{\infty}_{F^+}),\, u \in U\}$$

of algebraic modular forms. It is endowed with a faithful action of a Hecke algebra $\mathbb{T}$ (with generators indexed by an infinite set of "good primes" for U, cf. [LLHL19, §4.2.2]) for which each maximal ideal $\mathfrak{m} \subseteq \mathbb{T}$ has an associated continuous semisimple representation $\overline{r}_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ (cf. [CHT08, §3.4]). We further assume that $\overline{r}_{\mathfrak{m}}$ is absolutely irreducible. In [Her09] (later generalized in [GHS18]), Herzig made a remarkable conjecture predicting that the set $W(\overline{r}_{\mathfrak{m}})$ of *p-regular* Serre weights V such that $S(U, V)_{\mathfrak{m}} \neq 0$ is given by a combinatorially defined set $W^?(\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}})$ when $\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}}$ is semisimple. We remark that $W^?(\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}})$ is given in terms of the Jordan–Hölder factors of a Deligne–Lusztig representation associated to $\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}}$.

The weight elimination statement which we obtain is the following:

Theorem 1.5. *Assume that $\overline{r}_{\mathfrak{m}}$ is absolutely irreducible, and that $\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}}$ is $(2n+1)$-generic. Then $W(\overline{r}_{\mathfrak{m}}) \subseteq W^?((\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}})^{\mathrm{ss}})$.*

This result was proven in [LLHL19] with the assumption that $\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}}$ is $(6n-2)$-generic instead of $(2n+1)$-generic. As in *loc. cit.* the main mechanism to show $F(\lambda) \notin W(\overline{r}_{\mathfrak{m}})$ is to find sufficiently many $\overline{R}_s(\mu)$ containing it and use p-adic Hodge theory constraints implied by the condition $S(U, R_s(\mu))_{\mathfrak{m}} \neq 0$. In turn, these constraints translate to combinatorial admissibility conditions which exactly match Jantzen's generic pattern for $W^?((\overline{r}_{\mathfrak{m}}|_{G_{F^+_p}})^{\mathrm{ss}})$. Our representation theoretic results show that we can find all the necessary Deligne–Lusztig representations under weaker genericity hypotheses.

Strategy: Jantzen gives a very general character formula which describes the multiplicity of $R_s(\mu)$ in a certain projective $G_0(\mathbb{F}_p)$-module Q_λ containing $F(\lambda)$ in terms of Frobenius kernels multiplicities. As long as μ is h-generic in the lowest p-alcove, those multiplicities are controlled by the principal block and hence are independent of μ. Under Jantzen's stronger assumption that μ is $(2h-1)$-generic, any $F(\lambda)$ that can contribute to $R_s(\mu)$ has the property that $Q_\lambda = P_\lambda$ is indecomposable, thus one gets the formula for the multiplicity $[R_s(\mu) : F(\lambda)]$ of $F(\lambda)$ in $R_s(\mu)$. In contrast, the key difficulty when μ is not $2h$-generic is that Q_λ can be decomposable and Jantzen's character formula only gives a formula for certain weighted sums $\sum [R_s(\mu) : F(\lambda')][Q_\lambda : P_{\lambda'}]$ over "packets" of Serre weights. Our key observation is that if μ is h-generic and $R_s(\mu)$ occurs in Q_λ then $R_s(\mu)$ does not occur in $Q_{\lambda'}$ for any other λ' in the packet, and hence the above sum collapses.

The complication arising from packets also occurs in Theorem 1.3, and we resolve it in the same way. It is clear from Jantzen's general formula that $\mathrm{Hom}_{G_0(\mathbb{F}_p)}(Q_\lambda, \overline{R}_s(\widetilde{w}_h \cdot \lambda + \eta)) \neq 0$. By a series of delicate estimates in alcove geometry, we show that if λ is h-deep in its p-alcove then $\mathrm{Hom}_{G_0(\mathbb{F}_p)}(Q_{\lambda'}, \overline{R}_s(\widetilde{w}_h \cdot \lambda + \eta)) = 0$ for any other λ' in the packet.

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1.3. **Notation.** Let p be a prime. Let G_0 be a reductive group over $\mathbb{F}_p$. Let $\mathbb{F}/\mathbb{F}_p$ denote a finite extension such that $G \stackrel{\text{def}}{=} G_0 \otimes_{\mathbb{F}_p} \mathbb{F}$ is split. We assume throughout that G has simply connected derived subgroup. Let $T \subset B \subset G$ denote a maximal torus and a Borel subgroup. Let $G_1 \subset G$ denote the kernel of the relative (p-)Frobenius isogeny F on G. Let $G_1T \subset G$ denote the subgroup scheme generated by G_1 and T. Let $\Gamma \stackrel{\text{def}}{=} G_0(\mathbb{F}_p)$.

Recall the following standard notations:

- the character group $X^*(T)$ of $T \times_{\mathbb{F}} \overline{\mathbb{F}}_p$;
  - $R \subset X^*(T)$ the set of roots of G with respect to T;
  - $X^0(T) \subset X^*(T)$ the set of elements ν with $\langle \nu, \alpha^\vee \rangle = 0$ for all $\alpha \in R$;
  - the root lattice $\mathbb{Z}R \subset X^*(T)$ generated by R;
  - $R^+ \subset R$ the subset of positive roots with respect to B, i.e. the roots occurring in Lie(B); note that this is the convention in [Jan81] but opposite to [Jan03];
  - $\Delta \subset R^+$ the subset of simple roots;
  - $X(T)^+ \subset X^*(T)$ the dominant weights with respect to R^+;
  - the p-restricted set $X_1(T) \subset X(T)^+$ of dominant weights λ such that $\langle \lambda, \alpha^\vee \rangle \leq p-1$ for all $\alpha \in \Delta$;
  - the partial order $\leq$ on $X^*(T)$ and $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ defined by $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{R}_{\geq 0} R^+$;
  - for $\nu \in X^*(T)$ or $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, let $h_\nu \stackrel{\text{def}}{=} \max_{\alpha \in R} \langle \nu, \alpha^\vee \rangle$;
  - the automorphism π of $X^*(T)$ such that $F = p\pi^{-1}$ on $X^*(T)$;
  - a choice of π-invariant $\eta \in X^*(T)$ such that $\langle \eta, \alpha^\vee \rangle = 1$ for all $\alpha \in \Delta$;
- the Weyl group W of (G,T) and $w_0 \in W$ its longest element;
  - the extended affine Weyl group $\widetilde{W} \stackrel{\text{def}}{=} X^*(T) \rtimes W$, which acts on $X^*(T)$ on the left by affine transformations; for $\nu \in X^*(T)$ we write $t_\nu \in \widetilde{W}$ for the corresponding element;
  - the affine Weyl group $W_a \stackrel{\text{def}}{=} \mathbb{Z}R \rtimes W \subset \widetilde{W}$;
  - for $\kappa \in X^*(T)$, we write $\kappa_+ \in W\kappa$ for the unique dominant element in its W-orbit, and $\mathrm{Conv}(\kappa) \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ for the convex hull of $W\kappa$; note that this operation is subadditive;
- the set of alcoves of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e. the set of connected components of
$$X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}, \alpha \in R} \{\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda, \alpha^\vee \rangle = n\},$$
which has a (transitive) left action of $\widetilde{W}$;
  - the dominant alcoves, i.e. alcoves A such that $0 < \langle \lambda, \alpha^\vee \rangle$ for all $\alpha \in \Delta, \lambda \in A$;
  - the lowest (dominant) alcove $A_0 = \{\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in R^+\}$;
  - $\Omega \subset \widetilde{W}$ the stabilizer of the base alcove;
  - the restricted alcoves, i.e. alcoves A such that $0 < \langle \lambda, \alpha^\vee \rangle < 1$ for all $\alpha \in \Delta, \lambda \in A$;
  - the set $\widetilde{W}^+ \subset \widetilde{W}$ of elements $\widetilde{w}$ such that $\widetilde{w}(A_0)$ is dominant;
  - the set $\widetilde{W}_1 \subset \widetilde{W}^+$ of elements $\widetilde{w}$ such that $\widetilde{w}(A_0)$ is restricted;
  - $\widetilde{w}_h = w_0 t_{-\eta} \in \widetilde{W}_1$; note that $\widetilde{W}_1 = \widetilde{W}^+ \cap \widetilde{w}_h \widetilde{W}^+$;

- the set of p-alcoves of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e. the set of connected components of
$$X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \setminus \bigcup_{n\in\mathbb{Z},\alpha\in R} \{\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \eta, \alpha^\vee \rangle = np\};$$
  - a left p-dot action of $\widetilde{W}$ on $X^*(T)$ defined by $(t_\nu w)\cdot\lambda \stackrel{\text{def}}{=} p\nu + w(\lambda+\eta) - \eta$; this induces a p-dot action of $\widetilde{W}$ on the set of p-alcoves whose restriction to W_a is simply transitive;
  - the dominant p-alcoves, i.e. alcoves C such that $0 < \langle \lambda + \eta, \alpha^\vee \rangle$ for all $\alpha \in \Delta, \lambda \in C$;
  - the lowest (dominant) p-alcove $C_0 \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ characterized by $\lambda \in C_0$ if $0 < \langle \lambda + \eta, \alpha^\vee \rangle < p$ for all $\alpha \in R^+$;
  - the p-restricted alcoves, i.e. alcoves C such that $0 < \langle \lambda+\eta, \alpha^\vee \rangle < p$ for all $\alpha \in \Delta, \lambda \in C$;
- the Bruhat order $\leq$ on W_a with respect to A_0 (i.e. using the reflections across walls of A_0 as a set of Coxeter generators);
  - the $\uparrow$ order on the set of p-alcoves defined in [Jan03, II.6.5];
  - the $\uparrow$ order on W_a induced from the ordering $\uparrow$ on the set of p-alcoves (via the bijection $\widetilde{w} \mapsto \widetilde{w} \cdot C_0$);
  - the Bruhat order on $\widetilde{W} = W_a \rtimes \Omega$ defined by $\widetilde{w}\delta \leq \widetilde{w}'\delta'$ if and only if $\widetilde{w} \leq \widetilde{w}'$ and $\delta = \delta'$ where $\delta, \delta' \in \Omega$ and $\widetilde{w}, \widetilde{w}' \in W_a$;
  - the $\uparrow$ order on $\widetilde{W}$ defined by $\widetilde{w}\delta \uparrow \widetilde{w}'\delta'$ if and only if $\widetilde{w} \uparrow \widetilde{w}'$ and $\delta = \delta'$ where $\delta, \delta' \in \Omega$ and $\widetilde{w}, \widetilde{w}' \in W_a$;

We will assume throughout that $h_\eta < p$ so that C_0 is nonempty.

2. Lemmata

In this section, we collect several lemmata, mostly of a root-theoretic nature, that will be used in later sections.

Lemma 2.1. *Suppose that $\widetilde{s}, \widetilde{w} \in \widetilde{W}$ such that*
- $\widetilde{s} \in \widetilde{W}^+$;
- $\widetilde{s} \uparrow \widetilde{w}$;
- $\widetilde{s}(0) = \widetilde{w}(0)$; *and*
- *the closure of some Weyl chamber contains both $\widetilde{s}^{-1}(0)$ and $\widetilde{w}^{-1}(0)$.*

Then $\widetilde{s} = \widetilde{w}$.

Proof. Since $\widetilde{s}(0) = \widetilde{w}(0)$, $W\widetilde{s}^{-1}(0) = W\widetilde{w}^{-1}(0)$. Since the closure of some Weyl chamber contains $\widetilde{s}^{-1}(0)$ and $\widetilde{w}^{-1}(0)$, we have $\widetilde{s}^{-1}(0) = \widetilde{w}^{-1}(0)$. This implies that $W\widetilde{s} = Wt_{-\widetilde{s}^{-1}(0)} = Wt_{-\widetilde{w}^{-1}(0)} = W\widetilde{w}$. Then $\widetilde{s} \in \widetilde{W}^+$ and $\widetilde{s} \uparrow \widetilde{w}$ imply that $\widetilde{s} = \widetilde{w}$. □

Lemma 2.2. *Let $C \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ be an open chamber for the action of W (in particular its closure $\overline{C}$ is a fundamental domain for the action of W). Let $x \in \overline{C}$ and $\varepsilon \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. If $x + \varepsilon'$ is the unique element in $\overline{C} \cap W(x+\varepsilon)$, then $\varepsilon' \in \mathrm{Conv}(\varepsilon)$.*

Proof. Applying an element of $w \in W$ to C, x, and ε, we can and do assume without loss of generality that $\overline{C}$ is the dominant chamber containing $X(T)^+$. Suppose that $x + \varepsilon' = w(x+\varepsilon)$ for $w \in W$. We will induct on the length of w (allowing ε to vary). If $\ell(w) = 0$, then $\varepsilon' = \varepsilon$, and we are done. Suppose that $w = w's_\alpha$ for $\alpha \in \Delta$ and $w' \in W$ with $\ell(w') = \ell(w) - 1$. Then $\langle x+\varepsilon, \alpha^\vee \rangle \leq 0 \leq \langle x, \alpha^\vee \rangle$ so that $\langle x + r\varepsilon, \alpha^\vee \rangle = 0$ for some $r \in [0,1]$. Then $x + \varepsilon' = w(x+\varepsilon) = w'(x + r\varepsilon + (1-r)s_\alpha(\varepsilon))$. Note that $r\varepsilon + (1-r)s_\alpha(\varepsilon) \in \mathrm{Conv}(\varepsilon)$. By the inductive hypothesis, $\varepsilon' \in \mathrm{Conv}(r\varepsilon + (1-r)s_\alpha(\varepsilon)) \subset \mathrm{Conv}(\varepsilon)$. □

Lemma 2.3. *Let $x \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ be in the closure of A_0. Then either $h_x \leq \frac{h_\eta}{h_\eta+1}$ or*

$$\langle x, \alpha^\vee \rangle \geq \frac{1}{h_\eta + 1} \tag{2.1}$$

for some simple root $\alpha \in \Delta$.

Proof. Suppose that (2.1) does not hold for all simple roots α. Let $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ be a positive root. Then $\langle x, \beta^\vee \rangle = \sum_{\alpha \in \Delta} n_\alpha \langle x, \alpha^\vee \rangle < \frac{1}{h_\eta+1} \sum_{\alpha \in \Delta} n_\alpha$. On the other hand, $h_\eta \geq \langle \eta, \beta^\vee \rangle = \sum_{\alpha \in \Delta} n_\alpha \langle \eta, \alpha^\vee \rangle = \sum_{\alpha \in \Delta} n_\alpha$. Thus $h_x \leq \frac{h_\eta}{h_\eta+1}$. □

Lemma 2.4. *Suppose that $x, \varepsilon \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ with $h_\varepsilon \leq \frac{1}{h_\eta+1}$. If the closure of an alcove A_1 contains x, then there exists an alcove A_2 whose closure contains $x + \varepsilon$ such that $\overline{A}_1$ and $\overline{A}_2$ intersect.*

Proof. We immediately reduce to the case of an irreducible root system. Applying $\widetilde{w}(-)$ for $\widetilde{w} \in \widetilde{W}_a$ with $\widetilde{w}(A_1) = A_0$, we can assume without loss of generality that A_1 is A_0. Suppose that $h_x \leq \frac{h_\eta}{h_\eta+1}$. Then $h_{x+\varepsilon} \leq h_x + h_\varepsilon \leq 1$ so that $x + \varepsilon$ is in the closure of $W(A_0)$ and we can take A_2 in $W(A_0)$.

If $h_x > \frac{h_\eta}{h_\eta+1}$, then Lemma 2.3 implies that (2.1) holds for some $\alpha \in \Delta$ which we now fix. Let $\alpha_0 = \sum_{\beta \in \Delta} c_\beta \beta$ be the highest root (recall that we are assuming that the root system is irreducible). Consider the facet F defined as the intersection of the hyperplanes of the form

$$\{\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda, \beta^\vee \rangle = 0\}$$

for all $\beta \in \Delta$ with $\beta \neq \alpha$ and the hyperplane

$$\{\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda, \alpha_0^\vee \rangle = 1\}.$$

Then F is contained in the closure of A_0. Let $W' \subset W_a$ be the stabilizer of F. Recall from [Bou08, Ch. 5, §3, Proposition 1] that the stabilizer of F in W_a is generated by reflections along hyperplanes passing through F. In particular, a conjugate W'' of W' by a translation in $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a subgroup of W corresponding to a root subsystem.

We claim that

$$C \stackrel{\text{def}}{=} \{\lambda \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda, \beta^\vee \rangle > 0 \, \forall \, \alpha \neq \beta \in \Delta, \, \langle \lambda, \alpha_0^\vee \rangle < 1\}$$

is a chamber for the action of W'. By [Bou08, Ch. 5, §1, Proposition 5] it suffices to show that any hyperplane passing through F does not intersect C. Let $\gamma \in R^+$ such that $\langle y, \gamma^\vee \rangle \in \mathbb{Z}$ for some $y \in F$. If $\gamma = \sum_{\beta \in \Delta} n_\beta \beta$, then $\langle y, \gamma^\vee \rangle = \langle y, n_\alpha \alpha^\vee \rangle = \frac{n_\alpha}{c_\alpha} \langle y, \alpha_0^\vee \rangle = \frac{n_\alpha}{c_\alpha}$ so that $n_\alpha = 0$ or c_α. If $n_\alpha = 0$, then $\langle y, \gamma^\vee \rangle = 0$ for all $y \in F$ and $\langle x, \gamma^\vee \rangle > 0$ for all $x \in C$. If $n_\alpha = c_\alpha$, then $\langle y, \gamma^\vee \rangle = 1$ for all $y \in F$ and $\langle x, \gamma^\vee \rangle < 1$ for all $x \in C$. This establishes the claim. We conclude from [Bou08, Ch. 5, §3, Theorem 2] that the closure $\overline{C}$ is a fundamental domain for the action of W'.

Let $x + \varepsilon'$ denote the unique element of the W'-orbit of $x + \varepsilon$ in $\overline{C}$. By Lemma 2.2, $\varepsilon' \in \operatorname{Conv}(\varepsilon)$ (note that the convex hull of $W(\varepsilon)$ contains that of $W''(\varepsilon)$) so that in particular $h_{\varepsilon'} \leq h_\varepsilon$. Then $\langle x + \varepsilon', \alpha^\vee \rangle \geq \langle x, \alpha^\vee \rangle - h_{\varepsilon'} \geq \frac{1}{h_\eta+1} - h_\varepsilon \geq 0$. This implies that $x + \varepsilon'$ is in the closure of A_0. We can then take A_2 so be $w'(A_0)$ where $w' \in W'$ and $w'(x + \varepsilon') = x + \varepsilon$. □

Given $\lambda \in X^*(T)$ we let $W(\lambda)$ be the virtual representation $\sum_i (-1)^i R^i \operatorname{Ind}_{B^-}^G \lambda$ where B^- denotes the Borel opposite to B. If λ is dominant then $W(\lambda)$ is the representation $\operatorname{Ind}_{B^-}^G \lambda$, and we write $L(\lambda)$ for its (irreducible) socle. Recall from § 1 that we defined $\widetilde{w}_h = w_0 t_{-\eta}$ where w_0 is the longest element of W.

Lemma 2.5. *Let $\lambda_0 \in C_0$, $\widetilde{w}_\lambda \in \widetilde{W}_1$, $\lambda' \in X_1(T)$, and $\nu \in X(T)^+$. Set $\lambda \stackrel{def}{=} \widetilde{w}_\lambda \cdot \lambda_0$.*

(1) If $[L(\lambda')\otimes L(\pi(\nu)) : L(\lambda+p\nu)]_G \neq 0$*, then* $\lambda+p\nu\uparrow\lambda'+\pi(\nu')$ *for some* $\nu'\in\mathrm{Conv}(\nu)$*.*
(2) If $\lambda+p\nu\uparrow\lambda'+\pi(\nu')$ *for some* $\nu'\in\mathrm{Conv}(\nu)$*, then* $h_\nu\leq\max_{v\in\overline{A}_0} h_{\widetilde{w}_h\widetilde{w}_\lambda(v)}\leq h_\eta$*.*

Proof. Suppose that $L(\lambda+p\nu)\in\mathrm{JH}(L(\lambda')\otimes L(\pi(\nu)))\subset\mathrm{JH}(W(\lambda')\otimes L(\pi(\nu)))$. Then $L(\lambda+p\nu)\in\mathrm{JH}(W(\lambda'+\pi(\nu')))$ for some $\nu'\in\mathrm{Conv}(\nu)$. Moreover, we can assume without loss of generality that $\lambda'+\pi(\nu')\in X(T)^+$ by [LLHLM23, Lemma 2.2.2]. By the linkage principle, we conclude that

$$\lambda+p\nu\uparrow\lambda'+\pi(\nu') \tag{2.2}$$

for some $\nu'\in\mathrm{Conv}(\nu)$.

Suppose now that (2.2) holds. If we let α_0 be a dominant root such that $\langle\nu,\alpha_0^\vee\rangle=h_\nu$, then (2.2) implies that

$$\begin{aligned}(p-1)h_\nu&\leq\langle p\nu-\pi(\nu'),\alpha_0^\vee\rangle\\&\leq\langle\lambda'-\lambda,\alpha_0^\vee\rangle\\&=\langle\lambda'-\widetilde{w}_\lambda\cdot\lambda_0,\alpha_0^\vee\rangle\\&\leq\langle(p-1)\eta-\widetilde{w}_\lambda\cdot\lambda_0,\alpha_0^\vee\rangle\\&=\langle p\eta-(\widetilde{w}_\lambda\cdot\lambda_0+\eta),\alpha_0^\vee\rangle\\&=\langle -pw_0\eta+(w_0\widetilde{w}_\lambda)\cdot\lambda_0+\eta,-w_0\alpha_0^\vee\rangle\\&=\langle\widetilde{w}_h\widetilde{w}_\lambda\cdot\lambda_0+\eta,-w_0\alpha_0^\vee\rangle\end{aligned}$$

where

- the second inequality uses that α_0 is dominant; and
- the third inequality uses α_0 is positive and $\lambda'\in X_1(T)$.

We deduce that

$$\begin{aligned}\frac{p-1}{p}h_\nu&\leq\langle\widetilde{w}_h\widetilde{w}_\lambda(\frac{1}{p}(\lambda_0+\eta)),-w_0\alpha_0^\vee\rangle\\&<\max_{v\in\overline{A}_0}\langle\widetilde{w}_h\widetilde{w}_\lambda(v),-w_0\alpha_0^\vee\rangle\\&\leq\max_{v\in\overline{A}_0}h_{\widetilde{w}_h\widetilde{w}_\lambda(v)}\\&\leq h_{-w_0\eta}=h_\eta\\&\leq p-1\end{aligned}$$

where

- the strict inequality uses that $\lambda\in C_0$; and
- the fourth inequality uses that $-w_0\eta-\widetilde{w}_h\widetilde{w}_\lambda(v)\in X(T)^+$ noting that $\widetilde{w}_h^{-1}(-w_0\eta-\widetilde{w}_h\widetilde{w}_\lambda(v))=\widetilde{w}_\lambda(v)$ is in the closure of a restricted alcove.

This implies that $h_\nu<p$ so that $h_\nu=\lceil\frac{p-1}{p}h_\nu\rceil\leq\max_{v\in\overline{A}_0}h_{\widetilde{w}_h\widetilde{w}_\lambda(v)}\leq h_\eta$. (The last inequality follows from the fact that $\eta-\widetilde{w}_h\widetilde{w}_\lambda(v)$ is dominant for all $v\in\overline{A}_0$.) □

Given $m\in\mathbb{Z}$ and a p-alcove

$$C=\{\mu\in X^*(T)\otimes_{\mathbb{Z}}\mathbb{R}\mid n_\alpha p<\langle\mu+\eta,\alpha^\vee\rangle<(n_\alpha+1)p,\alpha\in R^+\},$$

we say that $\lambda\in X^*(T)$ is m-deep in the p-alcove C if for all $\alpha\in R^+$, $n_\alpha p+m<\langle\lambda+\eta,\alpha^\vee\rangle<(n_\alpha+1)p-m$.

Remark 2.6. For an m-deep weight in an alcove to exist, one must have $p \geq (m+1)(h_\eta+1)$. Indeed, suppose that λ is m-deep in C_0. Then $\lambda - m\eta$ is dominant. Then $\langle (m+1)\eta, \alpha^\vee\rangle < p-m$ for all $\alpha \in R$ from which we deduce the desired inequality.

Lemma 2.7. *Let $s \in W$, $\mu-\eta \in C_0$, and $\lambda \in X_1(T)$. If $\widetilde{w} \in \widetilde{W}$ is such that $\widetilde{w}\cdot(\mu-\eta+s\pi\widetilde{w}^{-1}(0))+\eta \in X(T)^+$ and $\widetilde{w}\cdot(\mu-\eta+s\pi\widetilde{w}^{-1}(0)) \leq \widetilde{w}_h\cdot\lambda$, then $h_{\widetilde{w}(0)} = h_{\widetilde{w}^{-1}(0)} \leq h_\eta + 1$. If $\mu-\eta$ or λ is 1-deep in their respective p-alcoves, then $h_{\widetilde{w}(0)} = h_{\widetilde{w}^{-1}(0)} \leq h_\eta$.*

Proof. Let $\sigma \in W$ be such that $\sigma\widetilde{w} \in \widetilde{W}^+$, and let τ be the image of $\sigma\widetilde{w}$ in W. Letting α_0 be the highest root so that $h_{\widetilde{w}(0)} = \langle \sigma\widetilde{w}(0), \alpha_0^\vee\rangle$, the hypotheses imply that

$$\begin{aligned}
(p-1)(h_{\widetilde{w}(0)}-1) &= ph_{\widetilde{w}(0)} - h_{\widetilde{w}(0)} - (p-1) \\
&\leq p\langle \widetilde{w}(0), \sigma^{-1}\alpha_0^\vee\rangle + \langle \widetilde{w}^{-1}(0), \pi^{-1}((\tau s)^{-1}\alpha_0^\vee)\rangle + \langle \mu, \tau^{-1}\alpha_0^\vee\rangle \qquad (2.3)\\
&= \langle p\sigma\widetilde{w}(0) + \tau(\mu + s\pi(\widetilde{w}^{-1}(0)), \alpha_0^\vee\rangle \\
&= \langle \sigma\widetilde{w}\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0))) + \eta, \alpha_0^\vee\rangle \\
&\leq \langle \widetilde{w}_h\cdot\lambda + \eta, \alpha_0^\vee\rangle \\
&\leq (p-1)h_\eta, \qquad (2.4)
\end{aligned}$$

from which we deduce that $h_{\widetilde{w}(0)} \leq h_\eta + 1$. If $\mu-\eta$ (resp. λ) is 1-deep in its p-alcove, then the inequality in (2.3) (resp. (2.4)) is strict. The result follows. ☐

Lemma 2.8. *Let $m \geq 0$. Suppose that $\mu - \eta \in X^*(T)$ is m-deep in C_0 and $\sigma(\mu) + p\nu - s\pi\nu - \eta$ is $(-m+1)$-deep in C_0 for $\sigma, s \in W$ and $\nu \in X^*(T)$. Then $t_\nu\sigma \in \Omega$.*

Proof. Let $\mu-\eta, \sigma, s$, and ν be as in the statement of the lemma. We first claim that $h_\nu \leq 2$. For $\alpha \in R^+$, we have that

$$-m+1 < \langle \sigma(\mu) + (p-s\pi)\nu, \alpha^\vee\rangle < p+m-1. \qquad (2.5)$$

Using that $-p+m < \langle \sigma(\mu), \alpha^\vee\rangle < p-m$ for all $\alpha \in R^+$, we have that $|p\langle \nu, \alpha^\vee\rangle - \langle s\pi\nu, \alpha^\vee\rangle| \leq 2p-2$ for all $\alpha \in R^+$. There exists $\alpha \in R^+$ such that $|\langle \nu, \alpha^\vee\rangle| = h_\nu$ so that $(p-1)h_\nu \leq |p\langle \nu, \alpha^\vee\rangle - \langle s\pi\nu, \alpha^\vee\rangle| \leq 2p-2$. The claim follows.

Since $\mu - \eta \in X^*(T)$ is m-deep in C_0, for each $\alpha \in R^+$ we have

$$n_\alpha p + m < \langle \sigma(\mu) + p\nu, \alpha^\vee\rangle < (n_\alpha+1)p - m$$

for a unique $n_\alpha \in \mathbb{Z}$. On the other hand, (2.5) and that $h_\nu \leq 2$ imply that

$$-m-1 < \langle \sigma(\mu) + p\nu, \alpha^\vee\rangle < p+m+1$$

for each $\alpha \in R^+$. Together, these imply that $n_\alpha = 0$ for all $\alpha \in R^+$ so that $\sigma(\mu) + p\nu - \eta \in C_0$. Equivalently, we have $t_\nu\sigma \in \Omega$. ☐

Lemma 2.9. *Suppose that the center Z of G is connected. Suppose also that $\lambda_0, \mu_0 \in X^*(T)$ are in the closure of C_0 and $\widetilde{w}_\lambda, \widetilde{w}_\mu \in \widetilde{W}$ such that*

(1) $\pi^{-1}(\widetilde{w}_\lambda)\cdot\lambda_0 \uparrow \pi^{-1}(\widetilde{w}_\mu)\cdot\mu_0$; and
(2) $t_{\lambda_0}\widetilde{w}_\lambda W_a = t_{\mu_0}\widetilde{w}_\mu W_a$.

Then $\lambda_0 = \mu_0$. Furthermore, let F be the facet of C_0 determined by λ_0. Then $\pi^{-1}(\widetilde{w}_\lambda)\cdot F \uparrow \pi^{-1}(\widetilde{w}_\mu)\cdot F$. In particular, if $\lambda_0 \in C_0$, then $\widetilde{w}_\lambda \uparrow \widetilde{w}_\mu$.

Proof. (1) implies that

$$\pi^{-1}(\widetilde{w}\widetilde{w}_\lambda)\cdot\lambda_0 = \pi^{-1}(\widetilde{w}_\mu)\cdot\mu_0$$

for some $\widetilde{w} \in W_a$ so that $\mu_0 - \lambda_0 \equiv p\pi^{-1}(\widetilde{w}_\mu^{-1}\widetilde{w}_\lambda)(0) \pmod{\mathbb{Z}R}$. On the other hand, (2) implies that $\mu_0 - \lambda_0 \cong \widetilde{w}_\mu^{-1}\widetilde{w}_\lambda(0) \pmod{\mathbb{Z}R}$. We conclude that $(1 - p\pi^{-1})\widetilde{w}_\mu^{-1}\widetilde{w}_\lambda(0) \in \mathbb{Z}R$. If π has order f, then we conclude that

$$(1-p^f)\widetilde{w}_\mu^{-1}\widetilde{w}_\lambda(0) = (1 + p\pi^{-1} + \ldots + p^{f-1}\pi^{-f+1})(1-p\pi^{-1})\widetilde{w}_\mu^{-1}\widetilde{w}_\lambda(0) \in \mathbb{Z}R.$$

As Z is connected, $X^*(Z) = X^*(T)/\mathbb{Z}R$ is $(1-p^f)$-torsion-free so that $\widetilde{w}_\mu^{-1}\widetilde{w}_\lambda(0) \in \mathbb{Z}R$ or in other words $\widetilde{w}_\lambda W_a = \widetilde{w}_\mu W_a$. Recall that the closure of C_0 is a fundamental domain for the p-dot action of W_a on $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ in a strong sense: $\overline{C}_0$ intersects any W_a-orbit exactly once (see [Bou08, Ch. 5, §3, Theorem 2]). Then the fact that λ_0, μ_0 are in $\overline{C}_0$ implies that $\lambda_0 = \mu_0$.

(1) implies that there is a sequence of affine reflections $\widetilde{r}_1, \ldots, \widetilde{r}_m$ such that

- for each $1 \leq k \leq m$, $\widetilde{r}_{k-1} \cdots \widetilde{r}_1 \pi^{-1}(\widetilde{w}_\lambda) \cdot \lambda_0$ (resp. $\widetilde{r}_k \cdots \widetilde{r}_1 \pi^{-1}(\widetilde{w}_\lambda) \cdot \lambda_0$) is in the negative (resp. positive) halfspace defined by $\widetilde{r}_k$; and
- $\widetilde{r}_m \cdots \widetilde{r}_1 \pi^{-1}(\widetilde{w}_\lambda) \cdot \lambda_0 = \pi^{-1}(\widetilde{w}_\mu) \cdot \mu_0$.

These properties hold after replacing λ_0 and μ_0 with F. If $\lambda_0, \mu_0 \in C_0$, then $F = C_0$ and $\widetilde{w}_\lambda \uparrow \widetilde{w}_\mu$. □

We say that $\mu \in X^*(T)$ is p-regular if μ is 0-deep in its p-alcove. Recall from [Jan03, II.7.2] the notion of blocks for G. By the linkage principle ([Jan03, II.2.12(1) and II.6.17]) if $L(\lambda)$ and $L(\mu)$ are in the same block, then λ is p-regular if and only if μ is, in which case we say that the block is p-regular. Given a G-module V, let V_{reg} be the projection of V to the p-regular blocks.

Lemma 2.10. *Let $\mu \in X(T)^+$ be 0-deep in its alcove, and suppose that $\nu \in X(T)^+$ such that $\mu + \kappa$ is in the closure of the p-alcove containing μ for all $\kappa \in \mathrm{Conv}(\nu)$. Then*

$$(L(\mu) \otimes L(\nu))_{\mathrm{reg}} \cong \bigoplus_{\substack{\kappa \in \mathrm{Conv}(\nu) \\ \mu+\kappa \text{ is 0-deep}}} L(\mu+\kappa)^{\oplus [L(\nu)|_T : \kappa]_T}$$

(and each summand that appears on the RHS has highest weight $\mu + \kappa$ in the same p-alcove as μ).

Proof. The proof is as in [Hum89, Lemma], except that we project to p-regular blocks. As the linkage principle ensures that there are no nontrivial G-extensions between the constituents of $(L(\mu) \otimes L(\nu))_{\mathrm{reg}}$, it suffices to prove an equality at the level of formal characters. We have

$$\begin{aligned} \mathrm{ch}\,(W(\mu) \otimes L(\nu))_{\mathrm{reg}} &= \sum_{\kappa \in X^*(T)} [L(\nu)|_T : \kappa]_T \,\mathrm{ch}\, W(\mu+\kappa)_{\mathrm{reg}} \\ &= \sum_{\substack{\kappa \in \mathrm{Conv}(\nu) \\ \mu+\kappa \text{ is 0-deep}}} [L(\nu)|_T : \kappa]_T \,\mathrm{ch}\, W(\mu+\kappa), \end{aligned} \tag{2.6}$$

where the first equality follows from a formula of Brauer and the second equality follows from the fact that $W(\mu+\kappa)_{\mathrm{reg}} = 0$ if $\mu+\kappa$ is not p-regular by the linkage principle and that $[L(\nu)|_T : \kappa]_T \neq 0$ implies that $\kappa \in \mathrm{Conv}(\nu)$. By assumption, the highest weight of each $W(\mu+\kappa)$ appearing in (2.6) is 0-deep in the same p-alcove as μ. If $\mu = \widetilde{w} \cdot \lambda$ for $\widetilde{w} \in W_a$ and $\lambda \in C_0$ and $\mu + \kappa$ is 0-deep in the same p-alcove as μ, then the linkage principle implies that there are nonnegative integers $a(\widetilde{w}, \widetilde{w}')$ for each $\widetilde{w}' \in W_a$ such that

$$\mathrm{ch}\, W(\mu+\kappa) = \sum_{\widetilde{w}' \in W_a} a(\widetilde{w}, \widetilde{w}')\,\mathrm{ch}\, L(\widetilde{w}' \cdot (\lambda + w^{-1}(\kappa))) \tag{2.7}$$

where $w \in W$ is the image of $\widetilde{w}$, the integers $a(\widetilde{w}, \widetilde{w}')$ are independent of μ and κ by the translation principle [Jan03, II.7.5], $a(\widetilde{w}, \widetilde{w}) = 1$, and $a(\widetilde{w}, \widetilde{w}') \neq 0$ implies that $\widetilde{w}' \uparrow \widetilde{w}$. Then (2.6), (2.7), and induction using the partial ordering $\uparrow$ yields

$$\mathrm{ch}\,(L(\mu) \otimes L(\nu))_{\mathrm{reg}} = \sum_{\substack{\kappa \in \mathrm{Conv}(\nu) \\ \mu+\kappa \text{ is 0-deep}}} [L(\nu)|_T : \kappa]_T \,\mathrm{ch}\, L(\mu + \kappa).$$

□

We have the following immediate corollary of Lemma 2.10.

Corollary 2.11. *Let $\mu \in X(T)^+$ be 0-deep in its alcove, and suppose that $\nu \in X(T)^+$ such that $\mu + \kappa$ is in closure of the p-alcove containing μ for all $\kappa \in \mathrm{Conv}(\nu)$. If $\lambda \in X(T)^+$ is 0-deep in its p-alcove and $[L(\mu) \otimes L(\nu) : L(\lambda)]_G \neq 0$, then $\lambda = \mu + \kappa$ for some $\kappa \in \mathrm{Conv}(\nu)$ and λ and μ are in the same p-alcove.*

3. Ingredients

In this section, we summarize key results that we will use to investigate reductions of Deligne–Lusztig representations. We assume from now on that $p \geq 2h_\eta$. Given $\lambda \in X_1(T)$ we let $\widehat{Q}_1(\lambda)$ be the G-representation constructed in [Jan03, II.11.11] and $Q_\lambda \overset{\mathrm{def}}{=} \widehat{Q}_1(\lambda)|_\Gamma$. Then Q_λ is a projective $\mathbb{F}[\Gamma]$-module ([Jan80, § 4.5], see also [Hum06, Theorem 10.4]). Let P_λ denote a $\mathbb{F}[\Gamma]$-projective cover of $F(\lambda) \overset{\mathrm{def}}{=} L(\lambda)|_\Gamma$. We first record a result of Chastkofsky and Jantzen (see [Cha81, Theorem 1] and [Jan81, Corollar 2] and also [Her09, Appendix A] for the generalization to reductive groups with simply connected derived subgroup).

Proposition 3.1. *For $\lambda \in X_1(T)$, we have*

$$Q_\lambda \cong \bigoplus_{\lambda' \in X_1(T)/(p-\pi)X^0(T)} \bigoplus_{\nu \in X(T)^+} P_{\lambda'}^{\oplus [L(\lambda') \otimes L(\pi(\nu)) : L(\lambda + p\nu)]_G}$$

Moreover, $[L(\lambda) \otimes L(\pi(\nu)) : L(\lambda + p\nu)]_G$ is 1 if $\nu = 0$ and is 0 otherwise.

Given $\lambda \in X^*(T)$, let $\widehat{Z}(1, \lambda)$ be the baby Verma G_1T-module of highest weight λ as defined in [Jan81, §2.5] and write $\widehat{L}(1, \lambda)$ for its irreducible cosocle. Following [Jan81, §3.1], given $(s, \mu) \in W \times X^*(T)$ we let T_s be the F-stable maximal torus $g_s T g_s^{-1}$ where $g_s \in G(\overline{\mathbb{F}}_p)$ is any element such that $g_s^{-1} F(g_s) \in N_G(T)$ is a lift of s and $\theta(s, \mu) : \Gamma \cap T_s \to E^\times$ be a character defined by $\theta(s, \mu)(t) \overset{\mathrm{def}}{=} [\mu(g_s^{-1} t g_s)]$ where $[\cdot]$ denotes the Teichmüller lift. This data gives rise to the (signed) Deligne–Lusztig induction $R_s(\mu) \overset{\mathrm{def}}{=} \varepsilon_G \varepsilon_{T_s} R_{T_s}^{\theta(s,\mu)}$; this is the (occasionally virtual) Γ-representation denoted $R_s(1, \mu)$ in [Jan81, §3.1]. We implicitly assume $R_s(\mu)$ is defined over E and write $\overline{R}_s(\mu)$ for the semisimplification of the reduction of a Γ-stable $\mathcal{O}$-lattice in $R_s(\mu)$. The following is a convenient reformulation of [Jan81, 3.2] which describes the decomposition of Q_λ into reductions of Deligne–Lusztig representations.

Theorem 3.2. *Let $\mu \in X^*(T)$, $s \in W$, and $\lambda \in X_1(T)$. Then*

$$\dim \mathrm{Hom}_\Gamma(Q_\lambda, \overline{R}_s(\mu)) = \sum_{\nu \in X^*(T)} [\widehat{Z}(1, \mu + (s\pi - p)\nu + (p-1)\eta) : \widehat{L}(1, \lambda)]_{G_1T}$$

where the left hand side is suitably interpreted for virtual representations $R_s(\mu)$.

Moreover, $[\widehat{Z}(1,\mu+(s\pi-p)\nu+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T}\neq 0$ *if and only if there exists* $\widetilde{w}\in\widetilde{W}$ *such that* $\widetilde{w}^{-1}(0)=\nu$, $\widetilde{w}\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))+\eta$ *is dominant, and*

$$\widetilde{w}\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))\uparrow\widetilde{w}_h\cdot\lambda.$$

Proof. [Jan81, 3.2, (3) and (4)] give that

$$\begin{aligned}\dim\operatorname{Hom}_\Gamma(Q_\lambda,\overline{R}_s(\mu)) &= \sum_{\nu\in X^*(T)}[\widehat{Z}(1,\mu+s\pi\nu+(p-1)\eta):\widehat{L}(1,p\nu+\lambda)]_{G_1T}\\ &= \sum_{\nu\in X^*(T)}[\widehat{Z}(1,\mu+(s\pi-p)\nu+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T}.\end{aligned}$$

By [GHS18, Lemma 10.1.5],

$$[\widehat{Z}(1,\mu+(s\pi-p)\nu+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T}\neq 0$$

if and only if $\sigma\cdot(\mu-\eta+(s\pi-p)\nu)\uparrow w_0\cdot(\lambda-p\eta)$ for all $\sigma\in W$. This is equivalent to $\widetilde{w}\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))\uparrow\widetilde{w}_h\cdot\lambda$ where $\widetilde{w}=wt_{-\nu}$ and $w\in W$ is any element such that $w\cdot(\mu-\eta+(s\pi-p)\nu)+\eta$ is dominant. □

The following is an immediate corollary of Proposition 3.1 and Theorem 3.2.

Corollary 3.3. *Suppose that* $\mu\in X^*(T)$, $s\in W$, *and* $\lambda\in X_1(T)$. *Then*

$$\begin{aligned}\dim\operatorname{Hom}_\Gamma(Q_\lambda,\overline{R}_s(\mu)) &= \sum_{\substack{\lambda'\in X_1(T)/(p-\pi)X^0(T)\\ \nu\in X(T)^+}}[\overline{R}_s(\mu):F(\lambda')]_\Gamma[L(\lambda')\otimes L(\pi(\nu)):L(\lambda+p\nu)]_G \\ &= \sum_{\nu\in X^*(T)}[\widehat{Z}(1,\mu+(s\pi-p)\nu+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T}.\end{aligned}\tag{3.1}$$

4. Generic decompositions of Deligne–Lusztig representations

In this section, we prove our main result on the reductions of generic Deligne–Lusztig representations. We begin with a corollary of the results from the last section.

Corollary 4.1. *Let* $\mu-\eta\in C_0$ *and* $\lambda\in X_1(T)$. *Suppose that* $\mu-\eta$ *or* λ *is* h_η*-deep in its* p*-alcove. Then* $\dim\operatorname{Hom}_\Gamma(Q_\lambda,\overline{R}_s(\mu))\neq 0$ *if and only if there exist* $\widetilde{w}\in\widetilde{W}^+$ *and* $\widetilde{w}_\lambda\in\widetilde{W}_1$ *such that* $\widetilde{w}\uparrow\widetilde{w}_h\widetilde{w}_\lambda$ *and* $\lambda=\widetilde{w}_\lambda\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))$. *In fact,* $\dim\operatorname{Hom}_\Gamma(Q_\lambda,\overline{R}_s(\mu))$ *equals*

$$\sum_{\substack{\widetilde{w}\in\widetilde{W}^+,\widetilde{w}_\lambda\in\widetilde{W}_1\\ \widetilde{w}\uparrow\widetilde{w}_h\widetilde{w}_\lambda\\ \lambda=\widetilde{w}_\lambda\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))}}[\widehat{Z}(1,\mu+(s\pi-p)(\widetilde{w}^{-1}(0))+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T},\tag{4.1}$$

where every term in (4.1) *is nonzero and for each* $\widetilde{w}$ *that appears in the sum,* $\mu-\eta+s\pi(\widetilde{w}^{-1}(0))$ *is in* C_0. *If the center* Z *of* G *is connected, then there is only one term in* (4.1).

Proof. Let μ and λ be as in the statement of the corollary. Suppose that $\nu\in X^*(T)$ so that $[\widehat{Z}(1,\mu+(s\pi-p)\nu+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T}\neq 0$. We claim that $h_\nu\leq h_\eta$. As in the proof of Theorem 3.2, $\sigma\cdot(\mu-\eta+(s\pi-p)\nu)\uparrow w_0\cdot(\lambda-p\eta)$ for all $\sigma\in W$. Let $w\in W$ be the unique element such that $\widetilde{w}\stackrel{\text{def}}{=}wt_{-\nu}\in\widetilde{W}^+$. Setting $\sigma=w$, Lemma 2.7 implies that $h_\nu=h_{\widetilde{w}^{-1}(0)}\leq h_\eta$.

We claim that $\mu-\eta+s\pi\nu\in C_0$ using that $h_\nu\leq h_\eta$. This is clear if $\mu-\eta$ is h_η-deep in C_0. If λ is h_η-deep in its p-alcove, then so is $\mu-\eta+s\pi\nu$ as it is in the same $\widetilde{W}$-orbit under the p-dot action, in which case $\mu-\eta+s\pi\nu$ and $\mu-\eta$ must lie in the same p-alcove which is C_0.

Theorem 3.2 implies that $[\widehat{Z}(1,\mu+(s\pi-p)\nu+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T}\neq 0$ is equivalent to $\widetilde{w}\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))\uparrow\widetilde{w}_h\cdot\lambda$ where $\widetilde{w}$ is defined in terms of ν as before. This is in turn equivalent to the fact that $\widetilde{w}\uparrow\widetilde{w}_h\widetilde{w}_\lambda$ and $\lambda=\widetilde{w}_\lambda\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))$ for some $\widetilde{w}_\lambda\in\widetilde{W}_1$.

Finally, we show that only one term in (4.1) is nonzero when the center Z of G is connected. Suppose that $\widetilde{w},\widetilde{w}'\in\widetilde{W}^+$ and $\widetilde{w}_\lambda,\widetilde{w}'_\lambda\in\widetilde{W}_1$ such that $\widetilde{w}\uparrow\widetilde{w}_h\widetilde{w}_\lambda$, $\widetilde{w}'\uparrow\widetilde{w}_h\widetilde{w}'_\lambda$ and $\widetilde{w}_\lambda\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))=\widetilde{w}'_\lambda\cdot(\mu-\eta+s\pi(\widetilde{w}'^{-1}(0)))$. We apply Lemma 2.9 with $\lambda_0=\mu-\eta+s\pi(\widetilde{w}^{-1}(0))$, $\mu_0=\mu-\eta+s\pi(\widetilde{w}'^{-1}(0))$, $\widetilde{w}_\lambda=\pi(\widetilde{w}_\lambda)$ and $\widetilde{w}_\mu=\pi(\widetilde{w}'_\lambda)$ (resp. with $\lambda_0=\mu-\eta+s\pi(\widetilde{w}'^{-1}(0))$, $\mu_0=\mu-\eta+s\pi(\widetilde{w}^{-1}(0))$, $\widetilde{w}_\lambda=\pi(\widetilde{w}'_\lambda)$ and $\widetilde{w}_\mu=\pi(\widetilde{w}_\lambda)$) to obtain $\widetilde{w}_\lambda\uparrow\widetilde{w}'_\lambda$ (resp. $\widetilde{w}'_\lambda\uparrow\widetilde{w}_\lambda$). Hence $\widetilde{w}_\lambda=\widetilde{w}'_\lambda$ and thus $\widetilde{w}^{-1}(0)=\widetilde{w}'^{-1}(0)$ from which we deduce that $\widetilde{w}=\widetilde{w}'$. (Note that condition (2) in Lemma 2.9 is satisfied since $\widetilde{w}\uparrow\widetilde{w}_h\widetilde{w}_\lambda$, $\widetilde{w}'\uparrow\widetilde{w}_h\widetilde{w}'_\lambda$ imply $\widetilde{w}(0)+\eta\equiv\widetilde{w}_\lambda$, $\widetilde{w}'(0)+\eta\equiv\widetilde{w}'_\lambda$ modulo W_a, so that $\mu-\eta+s\pi(\widetilde{w}^{-1}(0))+\pi(\widetilde{w}_\lambda)\equiv\mu\equiv\mu-\eta+s\pi(\widetilde{w}'^{-1}(0))+\pi(\widetilde{w}'_\lambda)$ modulo W_a. Similar arguments will be used to check condition (2) whenever we invoke Lemma 2.9.) □

The following is our main result on the reduction of generic Deligne–Lusztig representations.

Theorem 4.2. *Suppose that the center Z of G is connected, $\mu-\eta$ is h_η-deep in C_0, and $\lambda\in X_1(T)$. Then $[\overline{R}_s(\mu):F(\lambda)]_\Gamma\neq 0$ if and only if there exist $\widetilde{w}\in\widetilde{W}^+$ and $\widetilde{w}_\lambda\in\widetilde{W}_1$ such that $\widetilde{w}\uparrow\widetilde{w}_h\widetilde{w}_\lambda$ and $\lambda=\widetilde{w}_\lambda\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))$. Moreover, in this case*

$$[\overline{R}_s(\mu):F(\lambda)]_\Gamma=[\widehat{Z}(1,\mu+(s\pi-p)(\widetilde{w}^{-1}(0))+(p-1)\eta):\widehat{L}(1,\lambda)]_{G_1T}.$$

Proof. If $\mathrm{Hom}(Q_\lambda,\overline{R}_s(\mu))=0$, then the result holds by Corollary 4.1, and so we assume otherwise. Suppose now that $\lambda,\lambda'\in X_1(T)$ so that

(1) $[L(\lambda')\otimes L(\pi(\nu)):L(\lambda+p\nu)]_G\neq 0$ for some $\nu\in X(T)^+$ and
(2) $[\overline{R}_s(\mu):F(\lambda')]_\Gamma\neq 0$.

We will show that $\lambda-\lambda'\in(p-\pi)X^0(T)$. Then the result follows from Corollaries 3.3 and 4.1.

By Proposition 3.1 and (2), we have that $\mathrm{Hom}_\Gamma(Q_{\lambda'},\overline{R}_s(\mu))\neq 0$. Corollary 4.1 implies that there are $\widetilde{w}'\in\widetilde{W}^+$ and $\widetilde{w}_{\lambda'}\in\widetilde{W}_1$ such that $\widetilde{w}'\uparrow\widetilde{w}_h\widetilde{w}_{\lambda'}$ and $\lambda'=\widetilde{w}_{\lambda'}\cdot(\mu-\eta+s\pi(\widetilde{w}'^{-1}(0)))$. In particular, $h_{\widetilde{w}'^{-1}(0)}=h_{\widetilde{w}'(0)}\le h_{\widetilde{w}_h\widetilde{w}_{\lambda'}(0)}\le h_\eta$. Similarly, there are $\widetilde{w}\in\widetilde{W}^+$ and $\widetilde{w}_\lambda$ such that $\widetilde{w}\uparrow\widetilde{w}_h\widetilde{w}_\lambda$ and $\lambda=\widetilde{w}_\lambda\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))$ (and $h_{\widetilde{w}^{-1}(0)}\le h_\eta$). Since $\mu-\eta$ is h_η-deep in C_0, we conclude that

$$\lambda \text{ and } \lambda' \text{ are 0-deep in } \widetilde{w}_\lambda\cdot C_0 \text{ and } \widetilde{w}_{\lambda'}\cdot C_0\text{, respectively.} \tag{4.2}$$

By (1) and Lemma 2.5(1), we have $\lambda+p\nu\uparrow\lambda'+\pi(\nu')$ for some $\nu\in X(T)^+$ and $\nu'\in\mathrm{Conv}(\nu)$ so that

$$(t_\nu\widetilde{w}_\lambda)\cdot(\mu-\eta+s\pi(\widetilde{w}^{-1}(0)))\uparrow\widetilde{w}_{\lambda'}\cdot(\mu-\eta+s\pi(\widetilde{w}'^{-1}(0))+\pi(\nu')) \tag{4.3}$$

for some $\nu\in X(T)^+$ and some (possibly different) $\nu'\in\mathrm{Conv}(\nu)$. (4.3) implies that $\mu-\eta+s\pi(\widetilde{w}'^{-1}(0))+\pi(\nu')$, which is in the same p-dot orbit as $\mu-\eta+s\pi(\widetilde{w}^{-1}(0))$, is 0-deep in its p-alcove. Let $w\in W_a$ be the unique element such that $w^{-1}\cdot(\mu-\eta+s\pi(\widetilde{w}'^{-1}(0))+\pi(\nu'))$ is in C_0. (4.3) and Lemma 2.9 then imply that $t_\nu\widetilde{w}_\lambda\uparrow\widetilde{w}_{\lambda'}w$ and $w^{-1}\cdot(\mu-\eta+s\pi(\widetilde{w}'^{-1}(0))+\pi(\nu'))=\mu-\eta+s\pi(\widetilde{w}^{-1}(0))$. (The fact that condition (2) in Lemma 2.9 holds is checked by a similar argument as in the proof of Corollary 4.1, using moreover $\nu'\equiv\nu$ modulo W_a from $\nu'\in\mathrm{Conv}(\nu)$.) In particular, we have

$$\nu+\widetilde{w}_\lambda(v)\le\widetilde{w}_{\lambda'}w(v)$$

for any $v\in\overline{A}_0$. We assume without loss of generality that $p>h_\eta(h_\eta+1)$ by Remark 2.6. As $\nu'\in\mathrm{Conv}(\nu)$ and $h_\nu\le h_\eta<\frac{p}{h_\eta+1}$ by Lemma 2.5(2), the closures of the alcoves A_0 and $w(A_0)$ intersect by Lemma 2.4 (taking $x=\frac{1}{p}(\mu+s\pi(\widetilde{w}'^{-1}(0)))$ and $\varepsilon=\frac{\pi(\nu')}{p}$, and noting that $x+\varepsilon$ is in the interior of $w(\overline{A_0})$, so that $\overline{A}_2 = w(\overline{A_0})$ in the notation of Lemma 2.4), say at $v \in \overline{A}_0$. Thus, w stabilizes $v \in \overline{A}_0$ (by [Bou08, Ch. 5, §3, Proposition 1]), and we have

$$\nu + \widetilde{w}_\lambda(v) \leq \widetilde{w}_{\lambda'}(v). \tag{4.4}$$

We now claim that

$$\langle s\pi(\widetilde{w}'^{-1}(0)) + \pi(\kappa), \alpha^\vee \rangle \leq h_\eta + 1 \tag{4.5}$$

for any root α and $\kappa \in \mathrm{Conv}(\nu)$. Using that $\mathrm{Conv}(\nu)$ is W-invariant and that α_0 is a highest root if and only if $\pi^{-1}(\alpha_0)$ is, it suffices to show that $\langle \sigma\widetilde{w}'^{-1}(0) + \kappa, \alpha_0^\vee \rangle \leq h_\eta + 1$ for any $\sigma \in W$, any highest (and thus dominant) root α_0, and any $\kappa \in \mathrm{Conv}(\nu)$. We in fact claim the following series of inequalities:

$$\begin{aligned}
\langle \sigma\widetilde{w}'^{-1}(0) + \kappa, \alpha_0^\vee \rangle &\leq \langle \sigma\widetilde{w}'^{-1}(0) + \nu, \alpha_0^\vee \rangle \\
&\leq \langle \sigma\widetilde{w}'^{-1}(0) + \widetilde{w}_{\lambda'}(v) - \widetilde{w}_\lambda(v), \alpha_0^\vee \rangle \\
&\leq \langle \eta - \widetilde{w}_{\lambda'}(0) + \widetilde{w}_{\lambda'}(v) - \widetilde{w}_\lambda(v), \alpha_0^\vee \rangle \\
&\leq h_\eta + \langle w_{\lambda'}(v) - \widetilde{w}_\lambda(v), \alpha_0^\vee \rangle \\
&\leq h_\eta + \langle w_{\lambda'}(v), \alpha_0^\vee \rangle \\
&\leq h_\eta + \langle v, \alpha_0^\vee \rangle \\
&\leq h_\eta + 1,
\end{aligned}$$

where $w_{\lambda'} \in W$ is the image of $\widetilde{w}_{\lambda'}$ under the projection $\widetilde{W} \twoheadrightarrow W$. Indeed,

- the first inequality follows from the fact that $\kappa \leq \nu$ for $\kappa \in \mathrm{Conv}(\nu)$ and α_0 is dominant;
- the second inequality follows from $\nu \leq \widetilde{w}_{\lambda'}(v) - \widetilde{w}_\lambda(v)$ by (4.4) and $\alpha_0 \in X(T)^+$;
- the third inequality follows from $\alpha_0 \in X(T)^+$ and $\sigma\widetilde{w}'^{-1}(0) \leq \eta - \widetilde{w}_{\lambda'}(0)$ which uses $\sigma\widetilde{w}'^{-1}(0) \leq -w_0\widetilde{w}'(0)$ (since both sides are in $W(-\widetilde{w}'(0))$ and the RHS is dominant) and $\widetilde{w}' \uparrow \widetilde{w}_h\widetilde{w}_{\lambda'}$;
- the fifth inequality uses $\alpha_0 \in R^+$ and $\widetilde{w}_\lambda(v) \in X(T)^+$;
- the sixth inequality uses $w_{\lambda'}(v) \leq v$ and $\alpha_0 \in X(T)^+$; and
- the final inequality uses $v \in \overline{A}_0$.

We will use Corollary 2.11 with μ, ν, and λ taken to be λ', $\pi\nu$, and $\lambda + p\nu$, respectively, to show that $\lambda + p\nu$ and λ' are in the same p-alcove. It suffices to check that the hypotheses apply. (4.2) gives that (the dominant) λ' and $\lambda + p\nu$ are 0-deep in their p-alcoves. Using that $\mu - \eta$ is h_η-deep in C_0, (4.5) implies that $\lambda' + \pi(\kappa) = \widetilde{w}_{\lambda'} \cdot (\mu - \eta + s\pi(\widetilde{w}'^{-1}(0)) + w_{\lambda'}^{-1}\pi(\kappa))$ is (-1)-deep in $\widetilde{w}_{\lambda'} \cdot C_0$ for any $\kappa \in \mathrm{Conv}(\nu)$, i.e. that $\lambda' + \pi(\kappa)$ is in the closure of the p-alcove containing λ' for any $\kappa \in \mathrm{Conv}(\nu)$. (1) gives the final hypothesis.

From the previous paragraph, $\lambda + p\nu$ is p-restricted so that $\nu \in X^0(T)$. Then $L(\lambda') \otimes L(\pi(\nu)) \cong L(\lambda' + \pi(\nu))$ so that (1) implies that $L(\lambda' + \pi(\nu)) \cong L(\lambda + p\nu)$ which implies that $\lambda - \lambda' = (p - \pi)\nu$. ☐

Remark 4.3. In fact, the bound in Theorem 4.2 is sharp. If $G_0 = \mathrm{GL}_{2/\mathbb{F}_p}$, then $\overline{R}_s(\mu)$ has 2 Jordan–Hölder factors if $\mu - \eta$ is 1-deep, but $\overline{R}_{(12)}(1, 0)$ has 1 Jordan–Hölder factor.

5. Deligne–Lusztig reductions containing a simple module

In this section, we prove Theorem 1.3 which exhibits Deligne–Lusztig representations whose reductions contain a fixed simple module (see Theorem 5.4).

Lemma 5.1. *Suppose that the center Z of G is connected. Let $s \in W$, $\lambda_0 \in C_0$, $\lambda' \in X_1(T)$, $\widetilde{w}_\lambda \in \widetilde{W}_1$, $\widetilde{w} \in \widetilde{W}$, $\nu \in X(T)^+$, and $\nu' \in \mathrm{Conv}(\nu)$ such that*

(1) $t_\nu \widetilde{w}_\lambda \cdot \lambda_0 \uparrow \lambda' + \pi(\nu')$;
(2) *if* $\widetilde{w}_{\lambda'} \in t_\nu \widetilde{w}_\lambda W_a \cap \widetilde{W}_1$ *such that* $\lambda' \in \widetilde{w}_{\lambda'} \cdot \overline{C}_0$, *then* $\lambda' + \pi(\nu') \in \widetilde{w}_{\lambda'} W \cdot C_0$ *and* $t_\nu \widetilde{w}_\lambda \uparrow \widetilde{w}_{\lambda'}$;
(3) $\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0) \in C_0$; *and*
(4) $\widetilde{w} \cdot (\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0)) + \eta \in X(T)^+$ *and* $\widetilde{w} \cdot (\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0)) \uparrow \widetilde{w}_h \cdot \lambda'$.

Then $\nu \in X^0(T)$ *and* $t_\nu \widetilde{w}_\lambda \cdot \lambda_0 = \lambda' + \pi(\nu)$.

Proof. Let $\widetilde{w}_{\lambda'}$ be as in (2), and let $\lambda'_0 \overset{\text{def}}{=} \widetilde{w}_{\lambda'}^{-1} \cdot \lambda' \in \overline{C}_0$. Then (2) implies that $\lambda'_0 + w_{\lambda'}^{-1}\pi(\nu') = \widetilde{w}_{\lambda'}^{-1} \cdot (\lambda' + \pi(\nu')) \in w \cdot C_0$ for some $w \in W$. By [LLHLM23, Lemma 2.2.2], $w^{-1} \cdot (\lambda'_0 + w_{\lambda'}^{-1}\pi(\nu')) = \lambda'_0 + \pi(\nu'')$ for some $\nu'' \in \mathrm{Conv}(\nu)$. Furthermore, (1) and Lemma 2.9 with λ_0, μ_0, $\widetilde{w}_\lambda$, and $\widetilde{w}_\mu$ taken to be λ_0, $w^{-1} \cdot (\lambda'_0 + w_{\lambda'}^{-1}\pi(\nu')) = \lambda'_0 + \pi(\nu'')$, $\pi(t_\nu \widetilde{w}_\lambda)$, and $\pi(\widetilde{w}_{\lambda'} w)$, respectively imply that $t_\nu \widetilde{w}_\lambda \uparrow \widetilde{w}_{\lambda'} w$ and $\lambda_0 = \lambda'_0 + \pi(\nu'')$.

(3), (4), and Lemma 2.9 with λ_0, μ_0, $\widetilde{w}_\lambda$, and $\widetilde{w}_\mu$ taken to be $\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0)$, λ'_0, $\pi(\widetilde{w})$ and $\pi(\widetilde{w}_h \widetilde{w}_{\lambda'})$, respectively, imply that $\widetilde{w} \in \widetilde{W}^+$, $\widetilde{w} \uparrow \widetilde{w}_h \widetilde{w}_{\lambda'}$, and $\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0) = \lambda'_0 = \lambda_0 - \pi(\nu'')$. Thus, we have $\nu'' = \pi^{-1}(s)((\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) - \widetilde{w}^{-1}(0))$. From $\widetilde{w} \uparrow \widetilde{w}_h \widetilde{w}_{\lambda'} \uparrow \widetilde{w}_h t_\nu \widetilde{w}_\lambda$ using (2), we also have that $\widetilde{w}(0) \leq w_0\nu + w_0 \widetilde{w}_\lambda(0) - w_0\eta$.

Let $w_\lambda \in W$ be the image of $\widetilde{w}_\lambda$. We have the inequalities

$$\begin{aligned} \nu &\leq \eta - \widetilde{w}_\lambda(0) + w_0 \widetilde{w}(0) \\ &\leq \eta - \widetilde{w}_\lambda(0) - w_\lambda \widetilde{w}^{-1}(0) \\ &= w_\lambda \pi^{-1}(s^{-1})(\nu'') \\ &\leq \nu \end{aligned}$$

(the second inequality follows from the fact that $w_0\widetilde{w}(0)$ is antidominant and $-w_\lambda \widetilde{w}^{-1}(0) \in W w_0 \widetilde{w}(0)$). Thus, these inequalities are all equalities. In particular, $\widetilde{w}(0) = t_{w_0\nu} \widetilde{w}_h \widetilde{w}_\lambda(0)$ and $w_\lambda \widetilde{w}^{-1}(0) = -w_0 \widetilde{w}(0) \in X(T)^+$. The first of these equalities also implies that $w_\lambda (t_{w_0\nu} \widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) = -w_0 \widetilde{w}(0) \in X(T)^+$. Lemma 2.1 with $\widetilde{s}$ and $\widetilde{w}$ taken to be $\widetilde{w}$ and $t_{w_0\nu} \widetilde{w}_h \widetilde{w}_\lambda = \widetilde{w}_h t_\nu \widetilde{w}_\lambda$, respectively, implies that $\widetilde{w} = \widetilde{w}_h t_\nu \widetilde{w}_\lambda$ (recall that $\widetilde{w} \uparrow \widetilde{w}_h \widetilde{w}_{\lambda'} \uparrow \widetilde{w}_h t_\nu \widetilde{w}_\lambda$ by (2)). In particular, $\widetilde{w}_h t_\nu \widetilde{w}_\lambda = \widetilde{w} \in \widetilde{W}^+$. Since also $t_\nu \widetilde{w}_\lambda \in \widetilde{W}^+$ (as $\widetilde{w}_\lambda \in \widetilde{W}^+$ and $\nu \in X(T)^+$), we deduce that $t_\nu \widetilde{w}_\lambda \in \widetilde{W}^+ \cap \widetilde{w}_h^{-1} \widetilde{W}^+ = \widetilde{W}_1$. Since $\widetilde{w}_\lambda \in \widetilde{W}_1$, we must have $\nu \in X^0(T)$. This further implies that $\lambda' + \pi(\nu') \in \widetilde{w}_{\lambda'} \cdot \overline{C}_0$ so that we can take $w = 1$ above and $t_\nu \widetilde{w}_\lambda \uparrow \widetilde{w}_{\lambda'}$. We now have inequalities $\widetilde{w} \uparrow \widetilde{w}_h \widetilde{w}_{\lambda'} \uparrow t_{w_0\nu} \widetilde{w}_h \widetilde{w}_\lambda = \widetilde{w}$. Thus all these inequalities are equalities and so $t_\nu \widetilde{w}_\lambda = \widetilde{w}_{\lambda'}$. We conclude that $t_\nu \widetilde{w}_\lambda \cdot \lambda_0$ and $\lambda' + \pi(\nu')$ are in the same alcove and must be equal by (1). □

In practice, the hypotheses (2) and (3) in Lemma 5.1 are sometimes implied by other hypotheses.

Lemma 5.2. *Suppose that the center Z of G is connected and $p > (h_\eta + 1)^2$. Let $s \in W$, $\widetilde{w} \in \widetilde{W}$, $\widetilde{w}_\lambda \in \widetilde{W}_1$, λ_0 be $\max_{v \in \overline{A}_0} h_{\widetilde{w}_h \widetilde{w}_\lambda(v)}$-deep in C_0, $\lambda' \in X_1(T)$, $\nu \in X(T)^+$, and $\nu' \in \mathrm{Conv}(\nu)$ such that*

- $t_\nu \widetilde{w}_\lambda \cdot \lambda_0 \uparrow \lambda' + \pi(\nu')$; *and*
- $\widetilde{w} \cdot (\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0)) + \eta \in X(T)^+$; *and*
- $\widetilde{w} \cdot (\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0)) \uparrow \widetilde{w}_h \cdot \lambda'$.

Then, letting $\widetilde{w}_{\lambda'} \in t_\nu \widetilde{w}_\lambda W_a \cap \widetilde{W}_1$ such that $\lambda' \in \widetilde{w}_{\lambda'} \cdot \overline{C}_0$, we have

- λ' *and* $\lambda' + \pi(\nu')$ *lie in the same p-alcove;*
- $\lambda_0 - s\pi(\widetilde{w}_h \widetilde{w}_\lambda)^{-1}(0) + s\pi \widetilde{w}^{-1}(0) \in C_0$; *and*
- $t_\nu \widetilde{w}_\lambda \uparrow \widetilde{w}_{\lambda'}$.

Proof. Let $m \stackrel{\text{def}}{=} \max_{v\in\overline{A}_0} h_{\widetilde{w}_h\widetilde{w}_\lambda(v)}$. As λ_0 is m-deep in C_0, we have that $\lambda' + \pi(\nu')$ is m-deep in its p-alcove. Lemma 2.5(2) implies that $h_\nu \leq m$ hence λ' and $\lambda' + \pi(\nu')$ lie in the same p-alcove. (The inequality $h_\nu \leq m$, coming from Lemma 2.5(2), will be used multiple times in this proof.)

The third bullet point in the statement of the lemma gives

$$\widetilde{w}\cdot(\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0) + \pi(\nu'')) \uparrow \widetilde{w}_h\cdot(\lambda' + \pi(\nu')) \tag{5.1}$$

for $\nu'' = \pi^{-1}(w^{-1}w_0)(\nu') \in \mathrm{Conv}(\nu)$ with $w \in W$ the image of $\widetilde{w}$. Moreover, since the LHS of (5.1) lies in the same p-alcove as $\widetilde{w}\cdot(\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0))$ by the depth bound in the previous paragraph and Lemma 2.5(2), $\widetilde{w}\cdot(\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0) + \pi(\nu''))$ is dominant by the second item in the statement of the lemma. Moreover it is m-deep in its p-alcove. Since $h_{s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)} \leq m$, $\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0) + \pi(\nu'')$ and $\lambda_0 + s\pi\widetilde{w}^{-1}(0) + \pi(\nu'')$ are in the same p-alcove, which we denote by $\widetilde{u}\cdot C_0$ for $\widetilde{u} \in W_a$. As $\widetilde{w}\cdot(\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0) + \pi(\nu''))$ is dominant, we conclude that $\widetilde{w}\widetilde{u} \in \widetilde{W}^+$. The second and third bullet points and Lemma 2.7 imply that $h_{\widetilde{w}(0)} \leq h_\eta + 1$.

We claim that $\widetilde{u}$ fixes a $v_0 \in \overline{A}_0$. Since $h_{\nu''} \leq m$ by Lemma 2.5(2), $\lambda_0 + \pi(\nu'') \in C_0$. Then Lemma 2.4, taking $x = \frac{1}{p}(\lambda_0 + \pi(\nu'') + \eta)$ and $\varepsilon = \frac{1}{p}s\pi\widetilde{w}^{-1}(0)$ and using $p > (h_\eta + 1)^2$, implies that $\widetilde{u}(\overline{A}_0)$ and $\overline{A}_0$ intersect (note that $x + \varepsilon$ above is in the interior of an alcove so that $\overline{A}_2 = \widetilde{u}(\overline{A}_0)$ in the notation of Lemma 2.4). We take $v_0 \in \widetilde{u}(\overline{A}_0) \cap \overline{A}_0$. Then $\widetilde{u}(v_0) = v_0$ by [Bou08, Ch. 5, §3, Theorem 2].

We now summarize our key conclusions. (5.1) implies that $\widetilde{w}\widetilde{u} \uparrow \widetilde{w}_h\widetilde{w}_{\lambda'}$ for some $\widetilde{w}_{\lambda'} \in \widetilde{W}$ with $\lambda' \in \widetilde{w}_{\lambda'}\cdot C_0$. In particular, $\widetilde{w}(v_0) \leq \widetilde{w}_h\widetilde{w}_{\lambda'}(v_0)$ for some $v_0 \in \overline{A}_0$. Additionally, Lemma 2.9 with λ_0, μ_0, $\widetilde{w}_\lambda$, and $\widetilde{w}_\mu$ taken to be λ_0, $\widetilde{w}_{\lambda'}^{-1}\cdot(\lambda' + \pi(\nu'))$, $\pi(t_\nu\widetilde{w}_\lambda)$, and $\pi(\widetilde{w}_{\lambda'})$ implies that $t_\nu\widetilde{w}_\lambda \uparrow \widetilde{w}_{\lambda'}$.

We claim that

$$\langle s\pi\widetilde{w}^{-1}(0) + \pi(\kappa), \alpha^\vee\rangle \leq h_{\widetilde{w}_h\widetilde{w}_\lambda(v_0)} + 1 \tag{5.2}$$

for all roots α and $\kappa \in \mathrm{Conv}(\nu)$. We will use that for any $\sigma \in W$,

$$\begin{aligned}
\sigma\widetilde{w}^{-1}(0) &\leq -w_0(\widetilde{w}(0)_+) \\
&= -w_0(\widetilde{w}(v_0) - w(v_0))_+ \\
&\leq -w_0(\widetilde{w}(v_0)_+ + (-v_0)_+) \\
&= -w_0(\widetilde{w}(v_0) - w_0v_0) \\
&\leq -w_0\widetilde{w}_h\widetilde{w}_{\lambda'}(v_0) + v_0 \\
&\leq -w_0\widetilde{w}_h t_\nu\widetilde{w}_\lambda(v_0) + v_0 \\
&= -\nu + \eta - \widetilde{w}_\lambda(v_0) + v_0.
\end{aligned} \tag{5.3}$$

Here,

- the first inequality uses that both sides are in the same W-orbit and the RHS is dominant;
- the third inequality uses that $\widetilde{w}(v_0) \leq \widetilde{w}_h\widetilde{w}_{\lambda'}(v_0)$; and
- the fourth inequality uses that $\widetilde{w}_h\widetilde{w}_{\lambda'} \uparrow \widetilde{w}_h t_\nu\widetilde{w}_\lambda$ and $v_0 \in \overline{A}_0$.

Then for $\kappa \in \mathrm{Conv}(\nu)$,

$$\begin{aligned}\langle s\pi(\widetilde{w}^{-1}(0)) + \pi(\kappa), \alpha^\vee\rangle &= \langle \pi^{-1}(s)\widetilde{w}^{-1}(0) + \kappa, \pi^{-1}(\alpha)^\vee\rangle \\ &\leq \langle (\pi^{-1}(s)\widetilde{w}^{-1}(0) + \kappa)_+, \alpha_0^\vee\rangle \\ &\leq \langle \pi^{-1}(s)\widetilde{w}^{-1}(0)_+ + \kappa_+, \alpha_0^\vee\rangle \\ &\leq \langle -\nu + \eta - \widetilde{w}_\lambda(v_0) + v_0 + \nu, \alpha_0^\vee\rangle \\ &= \langle \eta - \widetilde{w}_\lambda(v_0) + v_0, \alpha_0^\vee\rangle \\ &\leq h_{\widetilde{w}_h\widetilde{w}_\lambda(v_0)} + 1\end{aligned}$$

where α_0 is some highest root, the third inequality follows from (5.3), and the last inequality follows from the fact that $v_0 \in \overline{A}_0$.

As $\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0) + \pi(\nu'')$ is m-deep in its p-alcove, (5.2) implies that $\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)$ is (-1)-deep in the same p-alcove. On the other hand, we know that $\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)$ is 0-deep in C_0 so that $\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0) + \pi(\nu'')$ is m-deep in C_0. Then $\lambda_0 - s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0) + s\pi\widetilde{w}^{-1}(0)$ is in C_0 by Lemma 2.5. □

Lemma 5.3. *Suppose that the center Z of G is connected. Let $\lambda_0 \in C_0$, $\lambda' \in X_1(T)$, $\widetilde{w}_\lambda \in \widetilde{W}_1$ with image $w_\lambda \in W$, $\widetilde{w} \in \widetilde{W}$, $\nu \in X(T)^+$, and $\nu' \in \mathrm{Conv}(\nu)$ such that*

1. *$\langle \lambda_0 + \eta, \alpha^\vee\rangle < p - h_\eta - \max_{v\in\overline{A}_0} h_{\widetilde{w}_h\widetilde{w}_\lambda(v)}$ for all roots α;*
2. *$t_\nu\widetilde{w}_\lambda \cdot \lambda_0 \uparrow \lambda' + \pi(\nu')$;*
3. *$\widetilde{w} \cdot (\lambda_0 + \pi\widetilde{w}_h\widetilde{w}_\lambda(0) + \pi w_0 w_\lambda \widetilde{w}^{-1}(0)) + \eta \in X(T)^+$; and*
4. *$\widetilde{w} \cdot (\lambda_0 + \pi\widetilde{w}_h\widetilde{w}_\lambda(0) + \pi w_0 w_\lambda \widetilde{w}^{-1}(0)) \uparrow \widetilde{w}_h \cdot \lambda'$.*

Then, letting $\widetilde{w}_{\lambda'} \in t_\nu\widetilde{w}_\lambda W_a \cap \widetilde{W}_1$ such that $\lambda' \in \widetilde{w}_{\lambda'} \cdot \overline{C}_0$, we have

- *$\lambda' + \pi(\nu') \in \widetilde{w}_{\lambda'}W \cdot C_0$;*
- *$\lambda_0 + \pi\widetilde{w}_h\widetilde{w}_\lambda(0) + \pi w_0 w_\lambda \widetilde{w}^{-1}(0) \in C_0$; and*
- *$t_\nu\widetilde{w}_\lambda \uparrow \widetilde{w}_{\lambda'}$.*

Proof. Let $w_\lambda \in W$ be the image of $\widetilde{w}_\lambda$. Then

$$\langle \lambda_0 - w_\lambda^{-1}\pi(\nu') + \eta, \alpha^\vee\rangle < p - h_\eta - \max_{v\in\overline{A}_0} h_{\widetilde{w}_h\widetilde{w}_\lambda(v)} + h_\nu \leq p$$

for all roots α by (1), (2), and Lemma 2.5, so that $\sigma_1 \cdot (\lambda_0 - w_\lambda^{-1}\pi(\nu')) \in \overline{C}_0$ for some $\sigma_1 \in W$. Letting $\lambda_0' \stackrel{\text{def}}{=} \widetilde{w}_{\lambda'}^{-1} \cdot \lambda' \in \overline{C}_0$, Lemma 2.9 with λ_0, μ_0, $\widetilde{w}_\lambda$, and $\widetilde{w}_\mu$ taken to be $\sigma_1 \cdot (\lambda_0 - w_\lambda^{-1}\pi(\nu'))$, λ_0', $\pi(t_\nu\widetilde{w}_\lambda\sigma_1^{-1})$, and $\pi(\widetilde{w}_{\lambda'})$ implies that $\lambda_0' = \sigma_1 \cdot (\lambda_0 - w_\lambda^{-1}\pi(\nu'))$. Since

$$t_\nu\widetilde{w}_\lambda\sigma_1^{-1} \cdot \lambda_0' \uparrow \widetilde{w}_{\lambda'} \cdot \lambda_0', \tag{5.4}$$

we also have that $t_\nu\widetilde{w}_\lambda(0) \leq \widetilde{w}_{\lambda'}(0)$.

Next, we claim that $h_{\widetilde{w}(0)} \le h_\eta$. Let $w \in W$ be the image of $\widetilde{w}$ and suppose that α is a root so that $\langle \widetilde{w}(0), \alpha^\vee \rangle = h_{\widetilde{w}(0)}$. Then we claim that

$$\begin{aligned}(p-1)h_{\widetilde{w}(0)} - (p-1-h_\eta) &\le \langle p\widetilde{w}(0), \alpha^\vee\rangle + \langle \pi w_0 w_\lambda \widetilde{w}^{-1}(0), w^{-1}\alpha^\vee\rangle \\ &\quad + \langle \lambda_0 + \eta, w^{-1}\alpha^\vee\rangle + \langle \pi \widetilde{w}_h \widetilde{w}_\lambda(0), w^{-1}\alpha^\vee\rangle \\ &= \langle p\widetilde{w}(0) + w(\lambda_0 + \eta + \pi\widetilde{w}_h\widetilde{w}_\lambda(0) + \pi w_0 w_\lambda \widetilde{w}^{-1}(0)), \alpha^\vee\rangle \\ &= \langle \widetilde{w}\cdot(\lambda_0 + \pi\widetilde{w}_h\widetilde{w}_\lambda(0) + \pi w_0 w_\lambda \widetilde{w}^{-1}(0)) + \eta, \alpha^\vee\rangle \\ &\le \langle \widetilde{w}\cdot(\lambda_0 + \pi\widetilde{w}_h\widetilde{w}_\lambda(0) + \pi w_0 w_\lambda \widetilde{w}^{-1}(0)) + \eta, \alpha_0^\vee\rangle \\ &\le \langle \widetilde{w}_h \cdot \lambda' + \eta, \alpha_0^\vee\rangle \\ &\le (p-1)h_\eta\end{aligned}$$

where α_0 is some highest root. Indeed,

- the first inequality follows from (1);
- the second inequality follows from (3);
- the third inequality follows from (4) and the dominance of α_0; and
- the final inequality follows from the fact that $\lambda' \in X_1(T)$.

Thus, $h_{\widetilde{w}(0)} \le \frac{p-2}{p-1}h_\eta + 1 < h_\eta + 1$, and the claim follows.

From the previous claim and (1), we have $\langle \lambda_0+\eta+\pi\widetilde{w}_h\widetilde{w}_\lambda(0)+\pi w_0 w_\lambda\widetilde{w}^{-1}(0), \alpha^\vee\rangle < p$ for all roots α. Then $\sigma_2\cdot(\lambda_0+\pi\widetilde{w}_h\widetilde{w}_\lambda(0)+\pi w_0 w_\lambda\widetilde{w}^{-1}(0)) \in \overline{C}_0$ for some $\sigma_2 \in W$ and in particular $\widetilde{w}(0) \in X(T)^+$. (4) and Lemma 2.9 with λ_0, μ_0, $\widetilde{w}_\lambda$, and $\widetilde{w}_\mu$ taken to be $\sigma_2\cdot(\lambda_0+\pi\widetilde{w}_h\widetilde{w}_\lambda(0)+\pi w_0 w_\lambda\widetilde{w}^{-1}(0))$, λ_0', $\pi(\widetilde{w}\sigma_2^{-1})$, and $\pi(\widetilde{w}_h\widetilde{w}_{\lambda'})$ then imply that $\lambda_0' = \sigma_2\cdot(\lambda_0+\pi\widetilde{w}_h\widetilde{w}_\lambda(0)+\pi w_0 w_\lambda\widetilde{w}^{-1}(0))$ and $\widetilde{w}(0) \le \widetilde{w}_h\widetilde{w}_{\lambda'}(0)$ (as 0 lies in the closure of the facet determined by λ_0'). Putting things together, we have $\lambda_0+\pi\widetilde{w}_h\widetilde{w}_\lambda(0)+\pi w_0 w_\lambda\widetilde{w}^{-1}(0) = \sigma_2^{-1}\sigma_1\cdot(\lambda_0 - w_\lambda^{-1}\pi(\nu'))$, or equivalently that

$$\lambda_0+\pi\widetilde{w}_h\widetilde{w}_\lambda(0)+\pi w_0 w_\lambda\widetilde{w}^{-1}(0) + \sigma_2^{-1}\sigma_1 w_\lambda^{-1}\pi(\nu') = \sigma_2^{-1}\sigma_1\cdot\lambda_0. \tag{5.5}$$

Now

$$\begin{aligned}\widetilde{w}_h\widetilde{w}_\lambda(0) + w_0 w_\lambda\widetilde{w}^{-1}(0) + \pi^{-1}(\sigma_2^{-1}\sigma_1 w_\lambda^{-1})(\nu') &\ge \widetilde{w}_h\widetilde{w}_\lambda(0) + w_0 w_\lambda \widetilde{w}^{-1}(0) + w_0\nu \\ &\ge w_0\nu + \widetilde{w}_h\widetilde{w}_\lambda(0) - \widetilde{w}(0) \\ &= \widetilde{w}_h t_\nu \widetilde{w}_\lambda(0) - \widetilde{w}(0) \\ &\ge 0\end{aligned} \tag{5.6}$$

where

- the second inequality uses that $\widetilde{w}(0) \ge -w_0 w_\lambda \widetilde{w}^{-1}(0)$ since both sides are in the same W-orbit and $\widetilde{w}(0) \in X(T)^+$; and
- the last inequality uses the inequalities $t_\nu\widetilde{w}_\lambda(0) \le \widetilde{w}_{\lambda'}(0)$ and $\widetilde{w}(0) \le \widetilde{w}_h\widetilde{w}_{\lambda'}(0)$ proven in the first and third paragraphs, respectively.

(5.5) and (5.6) imply that $\lambda_0 \le \sigma_2^{-1}\sigma_1\cdot\lambda_0$ so that $\sigma_1 = \sigma_2$ and $w_\lambda^{-1}\pi(\nu') = \pi w_0\nu$. In particular, $\lambda_0 - w_\lambda^{-1}\pi(\nu') = \lambda_0 - \pi w_0 \nu \in C_0$ by (1) and Lemma 2.5(2) (which applies by hypothesis (2)). By definition of σ_1, we conclude that $\sigma_1 = 1$ and thus $\sigma_2 = 1$.

We now deduce the three desired conclusions. First, we have $\lambda_0' = \lambda_0 - w_\lambda^{-1}\pi(\nu') = \lambda_0 + \pi\widetilde{w}_h\widetilde{w}_\lambda(0) + \pi w_0 w_\lambda \widetilde{w}^{-1}(0) \in C_0$. In conjunction with (5.4), this gives the inequality $t_\nu\widetilde{w}_\lambda \uparrow \widetilde{w}_{\lambda'}$.

Finally, $\lambda' + \pi(\nu') = \widetilde{w}_{\lambda'}\cdot(\lambda_0 - w_\lambda^{-1}\pi(\nu') + w_{\lambda'}^{-1}\pi(\nu'))$. Then $\lambda_0 - w_\lambda^{-1}\pi(\nu') + w_{\lambda'}^{-1}\pi(\nu') \in W\cdot\overline{C}_0$ by (1) and Lemma 2.5 so that $\lambda' + \pi(\nu')$, which is linked to $\lambda_0 \in C_0$, is in $\widetilde{w}_{\lambda'}W\cdot C_0$. □

Theorem 5.4. *Suppose that the center Z of G is connected. Let $\widetilde{w}_\lambda \in \widetilde{W}_1$, $s \in W$, and $\lambda_0 \in C_0$. Suppose further that either*

(1) $p > (h_\eta+1)^2$ and that λ_0 is $\max_{v\in\overline{A}_0} h_{\widetilde{w}_h\widetilde{w}_\lambda(v)}$-deep in C_0; or
(2) $\langle \lambda_0+\eta, \alpha^\vee\rangle < p - h_\eta - \max_{v\in\overline{A}_0} h_{\widetilde{w}_h\widetilde{w}_\lambda(v)}$ for all roots α and $s = \pi(w_0 w_\lambda)$.

Then $F(\widetilde{w}_\lambda\cdot\lambda_0)$ is a Jordan–Hölder factor of $\overline{R}_s(\lambda_0+\eta-s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0))$ with multiplicity one.

Proof. Let $\lambda \stackrel{\text{def}}{=} \widetilde{w}_\lambda\cdot\lambda_0$ and $\mu \stackrel{\text{def}}{=} \lambda_0+\eta-s\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)$. We apply Corollary 4.1 with $\widetilde{w} \stackrel{\text{def}}{=} \widetilde{w}_h\widetilde{w}_\lambda$ to get

$$\begin{aligned}\dim \mathrm{Hom}_\Gamma(Q_\lambda, \overline{R}_s(\mu)) &= [\widehat{Z}(1, \lambda_0 - p(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)) + p\eta) : \widehat{L}(1,\lambda)]_{G_1T} \\ &= [\widehat{Z}(1, w_\lambda(\lambda_0 - p(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)+\eta) + (p-1)\eta) : \widehat{L}(1,\lambda)]_{G_1T} \\ &= [\widehat{Z}(1,\lambda) : \widehat{L}(1,\lambda)]_{G_1T} \\ &= 1\end{aligned}$$

where $w_\lambda \in W$ is the image of $\widetilde{w}_\lambda$, the second equality follows from [Jan03, 9.16(5)] and the fourth equality follows for instance by using that $[\widehat{Z}(1,\lambda) : \widehat{L}(1,\lambda)]_{G_1T}$ is nonzero and λ appears with multiplicity one in both $\widehat{Z}(1,\lambda)|_T$ and $\widehat{L}(1,\lambda)|_T$. We claim that if $\lambda' \in X_1(T)$ and $[L(\lambda')\otimes L(\pi(\nu)) : L(\lambda+p\nu)]_G \neq 0$ for some $\nu \in X(T)^+$ and $\mathrm{Hom}_\Gamma(Q_{\lambda'}, \overline{R}_s(\mu)) \neq 0$, then $\nu \in X^0(T)$. This will finish the proof: first note that $[\overline{R}_s(\mu) : F(\lambda')]_\Gamma \neq 0$ implies $\mathrm{Hom}_\Gamma(Q_{\lambda'}, \overline{R}_s(\mu)) \neq 0$. Thus the claim shows that the only term in the first sum in equation (3.1) of Corollary 3.3 that contributes is the term where $\lambda' \equiv \lambda$ modulo $(p-\pi)X^0(T)$ and that both factors of this term are 1.

We now prove the claim. Suppose that $\lambda' \in X_1(T)$, $[L(\lambda')\otimes L(\pi(\nu)) : L(\lambda+p\nu)]_G \neq 0$ for some $\nu \in X(T)^+$, and $\mathrm{Hom}_\Gamma(Q_{\lambda'}, \overline{R}_s(\mu)) \neq 0$. Then Lemma 2.5 implies that there exists $\nu' \in \mathrm{Conv}(\nu)$ such that $\lambda + p\nu \uparrow \lambda' + \pi(\nu')$, and Theorem 3.2 implies that there exists $\widetilde{w} \in \widetilde{W}$ such that $\widetilde{w}\cdot(\mu-\eta+s\pi\widetilde{w}^{-1}(0))+\eta \in X(T)^+$ and $\widetilde{w}\cdot(\mu-\eta+s\pi\widetilde{w}^{-1}(0)) \uparrow \widetilde{w}_h\cdot\lambda'$. Lemmas 5.2 and 5.3 imply conditions (2) and (3) of Lemma 5.1. Finally, Lemma 5.1 implies that $\nu \in X^0(T)$. □

6. Applications to weight elimination

Let q be a power of p and $K = W(\mathbb{F}_q)[p^{-1}]$. Let $\mathbb{Q}_p \subset E \subset \overline{\mathbb{Q}}_p$ be a sufficiently large finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}$ be the ring of integers of E and $\mathbb{F}$ the residue field. Assume that any homomorphism $K \to \overline{\mathbb{Q}}_p$ factors through E. We now take G_0 to be $\mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p}\mathrm{GL}_{n/\mathbb{F}_q}$. Let ${}^LG \stackrel{\text{def}}{=} \prod_{K\to E}\mathrm{GL}_{n/\mathbb{Z}} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p)$. Let $G_{\mathbb{Q}_p} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{K}/K)$ with inertia subgroups $I_{\mathbb{Q}_p}$ and I_K, respectively. Restriction and projection gives a bijection between conjugacy classes of continuous L-homomorphisms ${}^L\overline{\rho} : G_{\mathbb{Q}_p} \to {}^LG(\mathbb{F})$ and conjugacy classes of continuous homomorphisms $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$.

Let $X^*_{\mathrm{reg}}(T) \subset X_1(T)$ denote the subset of $\lambda \in X_1(T)$ such that $\langle\lambda,\alpha^\vee\rangle < p-1$ for all simple roots α. We say that a simple $\mathbb{F}[\mathrm{GL}_n(\mathbb{F}_q)]$-module is *regular* if its highest weight is in $X^*_{\mathrm{reg}}(T)$. The p-dot action of $\widetilde{w}_h$ defines a self-bijection $X^*_{\mathrm{reg}}(T) \to X^*_{\mathrm{reg}}(T)$. Let $\mathcal{R}$ be the corresponding self-bijection on the set isomorphism classes of regular simple $\mathbb{F}[\mathrm{GL}_n(\mathbb{F}_q)]$-modules, that is $\mathcal{R}(F(\lambda)) = F(\widetilde{w}_h\cdot\lambda)$. For a conjugacy class of tame continuous homomorphisms $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$, let $W^?(\overline{\rho})$ be the set $W^?({}^L\overline{\rho}|_{I_{\mathbb{Q}_p}})$ in [GHS18, Definition 9.2.5] where ${}^L\overline{\rho}$ is the L-parameter corresponding to $\overline{\rho}$. Outside degenerate cases which are irrelevant in our context, the set $W^?(\overline{\rho})$ has the following concrete description [GHS18, Proposition 9.2.3]: we can write ${}^L\overline{\rho}|_{I_{\mathbb{Q}_p}}$ (in possibly several ways) as an explicit representation $\tau(s,\mu)$ depending on $(s,\mu-\eta) \in W\times(C_0\cap X^*(T))$ [GHS18, Proposition 9.2.3], and the set $W^?(\overline{\rho})$ is $\mathcal{R}(\mathrm{JH}(\overline{R}_s(\mu)))$ (in this case we say that $\overline{\rho}$ is m-generic if we can choose $\mu-\eta$ to be m-deep in C_0).

It is conjectured [GHS18] that the set $W^?(\overline{\rho})$ controls weights of mod p automorphic forms for any globalization of $\overline{\rho}$ e.g. mod p Langlands parameters contributing to spaces of mod p algebraic modular forms on definite unitary groups as in §1.1.

In any such context, establishing the upper bound given by $W^?(\overline{\rho})$ is referred to as "weight elimination". In our previous work [LLHL19], we establish weight elimination in an axiomatic framework that applies to many global contexts (for instance the one in Theorem 1.5) under the hypothesis that $\overline{\rho}$ is $(6n-2)$-generic.

The method of *loc. cit.* was to combine constraints from p-adic Hodge theory with generic decomposition patterns of Deligne–Lusztig representations. Our new results on the latter allow us to improve our earlier axiomatic weight elimination results to the following

Theorem 6.1. *Let $\overline{\rho}: G_K \to \mathrm{GL}_n(\mathbb{F})$ be a continuous homomorphism. Suppose that $W(\overline{\rho})$ is a set of isomorphism classes of simple $\mathbb{F}[\mathrm{GL}_n(\mathbb{F}_q)]$-modules, $w\in W$, and $\nu\in X^*(T)$ such that if $W(\overline{\rho})\cap \mathrm{JH}(\overline{R}_w(\nu))\neq\emptyset$ and either*

- *$\tau(w,\nu)$ is regular (i.e. multiplicity free); or*
- *$w=1$;*

then $\overline{\rho}$ has a potentially semistable lift of type $(\eta,\tau(w,\nu))$. Assume further that $\overline{\rho}^{\mathrm{ss}}|_{I_K}$ is $(2n+1)$-generic.

Then $W(\overline{\rho})\subset W^?(\overline{\rho}^{\mathrm{ss}})$.

Proof. We follow the general outline of [LLHL19, §4.2].

By [Enn19, Lemma 5], if $\overline{\rho}$ has a potentially semistable lift of type $(\eta,\tau(w,\nu))$, so does $\overline{\rho}^{\mathrm{ss}}$. We then reduce to the case where $\overline{\rho}$ is semisimple.

Suppose that $\lambda\in X_1(T)$ with $F(\lambda)\in W(\overline{\rho})$. First, $\overline{\rho}$ is $(2n+1)$-generic in the sense of [Enn19, Definition 2] so that λ is p-regular by [Enn19, Theorem 8]. Then we can write $\lambda=\widetilde{w}_\lambda\cdot\lambda_0$ with $\lambda_0\in C_0$ (and $\widetilde{w}_\lambda\in\widetilde{W}_1$).

In order for a $(2n+1)$-generic Galois representation to exist at all, one must have $p>(2n+1)n$ by Remark 2.6. Thus, we assume without loss of generality that $p>(2n+1)n$. Using this, after possibly replacing λ_0 by an element in $\Omega\cdot\lambda_0$, we can assume that $\langle\lambda_0+\eta,\alpha^\vee\rangle<p-2h_\eta$ for all $\alpha\in R$. We claim that λ_0 is $(h_{\widetilde{w}_h\widetilde{w}_\lambda(0)}+1)$-deep in C_0. Suppose otherwise. Then Theorem 5.4(2) implies that $F(\lambda)\in\mathrm{JH}(\overline{R}_w(\nu))$ with $w=\pi(w_0w_\lambda)$ and $\nu=\lambda_0+\eta-\pi(w_0w_\lambda(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0))$. Since $\widetilde{w}_\lambda\in\widetilde{W}_1$, we have $0\le\langle -w_0w_\lambda(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0),\alpha^\vee\rangle\le 1$ for all $\alpha\in\Delta$ and $\langle\lambda_0+\eta-\pi(w_0w_\lambda(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)),\alpha^\vee\rangle<p-h_\eta$ for all $\alpha\in R$. We conclude that $\overline{\rho}$ has a potentially semistable (and thus potentially crystalline) lift of type (η,τ) where τ is the (regular) inertial type $\tau(w,\nu)$ and $\nu-\eta$ is 1-deep, but not $(h_{\widetilde{w}_h\widetilde{w}_\lambda(0)}+2)$-deep. We claim that $\overline{\rho}|_{I_K}\cong\overline{\tau}(s,\mu)$ for some $s\in W$ and $\mu\in X^*(T)$ such that

$$(t_\nu w)^{-1}t_\mu s\in\mathrm{Adm}(\eta)\overset{\mathrm{def}}{=}\{\widetilde{w}\in\widetilde{W}\mid\widetilde{w}\le t_{\sigma(\eta)}\text{ for some }\sigma\in W\}.\tag{6.1}$$

Given this claim, we deduce that $\mu-\eta$ is not $(2h_\eta+2)$-deep so that $\overline{\rho}$ is not $(2h_\eta+3)$-generic. (Indeed, if $\mu-\eta$ is $(2h_\eta+2)$-deep, then (6.1) implies that $\nu-\eta$ is $(h_\eta+2)$-deep and thus $(h_{\widetilde{w}_h\widetilde{w}_\lambda(0)}+2)$-deep.) This is a contradiction.

We now prove the claim in the previous paragraph. There is a unique lowest alcove presentation $(s,\mu-\eta)$ of $\overline{\rho}$ compatible in the sense of [LLHLM23, §2.4] with the 1-generic lowest alcove presentation $(w,\nu-\eta)$ of τ by [LLHLM23, Lemma 2.4.4] (though it should be assumed in the cited lemma that the center Z is connected). Moreover, $\mu-\eta$ is $2n$-deep by [LLHL19, Proposition 2.2.15]. Let K'/K be a finite unramified extension so that the restriction $\overline{\rho}'$ is a direct sum of characters and the base change inertial type τ' is principal series. Furthermore, $(w', \nu' - \eta')$ is a 1-generic lowest alcove presentation for τ' where $w'_{j'} = w_{j'|_K}$ and $\nu'_{j'} = \nu_{j'|_K}$ for any embedding $j' : K' \to E$, and similarly $(s', \mu' - \eta')$ is a compatible $2n$-generic lowest alcove presentation of $\overline{\rho}'$. [LLHL19, Theorem 3.2.1] implies that $\tau(s', \mu') \cong \tau(x', \xi')$ for some $\xi' \in (\mathbb{Z}^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(K',E)}$ and $x' \in S_n^{\mathrm{Hom}_{\mathbb{Q}_p}(K',E)}$ such that $(t_{\nu'}w')^{-1}t_{\xi'}x' \in \mathrm{Adm}(\eta')$. In particular, $\xi' - \eta'$ is $(-n+2)$-deep in C_0. Since $\mu' - \eta'$ is $2n$-deep in C'_0, the centralizer of the semisimple element corresponding to $\theta_{s',\mu'}$ is a maximal torus by the proof of [GHS18, Lemma 10.1.10]. Since the corresponding centralizer for $\theta_{x',\xi'}$ is conjugate, it is also a maximal torus. We conclude that both $(T_{s'}, \theta_{s',\mu'})$ and $(T_{x'}, \theta_{x',\xi'})$ are maximally split. By [GHS18, Proposition 9.2.1], $(T_{s'}, \theta_{s',\mu'})$ and $(T_{x'}, \theta_{x',\xi'})$ are $G'_0(\mathbb{F}_p)$-conjugate. By [Her09, Lemma 4.2], we have

$$(x', \xi') = (\sigma s' \pi' \sigma^{-1} \pi'^{-1}, \sigma(\mu') + (p - \sigma s' \pi' \sigma^{-1})\omega)$$

for some $\sigma \in S_n^{\mathrm{Hom}_{\mathbb{Q}_p}(K',E)}$ and $\omega \in (\mathbb{Z}^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(K',E)}$. By Lemma 2.8 taking $m = 2n$, we have that $t_\omega \sigma \in \Omega'$. Since $\mu' - \xi' \in \mathbb{Z}R'$ and $\widetilde{W}'/W'_a$ is torsion-free, $t_\omega\sigma \in \Omega' \cap W'_a = 1$. Thus, $s' = x'$ and $\mu' = \xi'$ so that $(t_{\nu'}w')^{-1}t_{\mu'}s' \in \mathrm{Adm}(\eta')$ which implies that $(t_\nu w)^{-1}t_\mu s \in \mathrm{Adm}(\eta)$.

Thus, λ_0 is $(h_{\widetilde{w}_h \widetilde{w}_\lambda(0)} + 1)$-deep in C_0. Then $F(\lambda) \in \mathrm{JH}(\overline{R}_w(\lambda_0 + \eta - w\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)))$ for all $w \in W$ by Theorem 5.4(1) (recall that we are assuming that $p > (2n+1)n \geq n^2 = (h_\eta + 1)^2$), which implies that $\overline{\rho}$ has a potentially semistable (and thus potentially crystalline) lift of type $\tau_w \stackrel{\mathrm{def}}{=} \tau(w, \lambda_0 + \eta - w\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0))$ for all $w \in W$. For all $w \in W$, $(w, \lambda_0 - w\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0))$ is a 1-generic lowest alcove presentation for τ_w. The argument in the previous paragraph shows that $\overline{\rho}|_{I_K} \cong \tau(s, \mu)$ for some $s \in W$ and $\mu \in X^*(T)$ with $(t_{\lambda_0 + \eta - w\pi(\widetilde{w}_h\widetilde{w}_\lambda)^{-1}(0)}w)^{-1}t_\mu s \in \mathrm{Adm}(\eta)$ for all $w \in W$. Then the proof of [LLHL19, Lemma 4.1.10] implies that $F(\lambda) \in W^?(\overline{\rho})$. □

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