Contents

Introduction générale

Cette thèse s'inscrit dans le cadre du programme de Langlands local modulo p. Le terme générique de "programme de Langlands" désigne habituellement une vaste famille de conjectures développées depuis 1967 ([Lan67]) qui peuvent se résumer de la manière suivante :

CONJECTURE 0.1. *Il existe une bijection "naturelle" entre*

$$\left\{\begin{array}{l}\text{représentations automorphes}\\ \text{paraboliques algébriques}\\ \text{de } \mathrm{GL}_n(\mathbf{A}_F)\end{array}\right\}_{/\sim} \longleftrightarrow \left\{\begin{array}{l}\text{représentations continues irréductibles}\\ \rho : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}_p}) \text{ nonramifiées enpresque}\\ \text{toutes les places et potentiellement}\\ \text{semi-stables aux places au-dessus de } p\end{array}\right\}_{/\sim}$$

où F désigne un corps de nombres et $\mathbf{A}_F$ son anneau des adèles.

La naturalité énoncée dans la conjecture 0.1 entraîne la compatibilité à la correspondance de Langlands locale ℓ-adique, démontrée pour GL_2 par Kutzko ([Kut80]) et pour GL_n par Harris et Taylor [HT] et par Henniart ([Hen00]). Il s'agit d'une bijection entre certaines représentations du groupe de Galois $\mathrm{Gal}(\overline{\mathbf{Q}_p}/F_\mathfrak{p})$ sur des $\mathbf{Q}_\ell$-espaces vectoriels de dimension n et certaines représentations de $\mathrm{GL}_n(F_\mathfrak{p})$ où $\mathfrak{p}$ désigne une place au-dessus de p et $F_\mathfrak{p}$ la complétion $\mathfrak{p}$-adique de F. La compatibilité de cette bijection à la réduction modulo ℓ a été ensuite démontrée par Vignéras ([Vig01]).

La difficulté essentielle pour formuler un analogue p-adique de cette correspondance (la "correspondance de Langlands locale p-adique") est la suivante : il est bien connu que la composante locale $\pi_\mathfrak{p}$ en $\mathfrak{p}|p$ de π ne détermine *pas* la restriction $\rho|_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/F_\mathfrak{p})}$ de la représentation galoisienne globale ρ bien que $\rho|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\mathfrak{p})}$, en vue de la conjecture 0.1, soit déterminée par π. Donc :

> "À l'origine du programme de Langlands p-adique est la volonté de comprendre ce qu'il faut rajouter à $\pi_\mathfrak{p}$ pour reconstruire $\rho|_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/F_\mathfrak{p})}$. Autrement dit, on veut élucider l'apparition de la théorie de Hodge p-adique côté Galois en termes de théorie des représentations côté automorphe."
>
> (C. Breuil, [Bre10b])

La correspondance pour $\mathrm{GL}_2(\mathbf{Q}_p)$.

La naissance du programme de Langlands local p-adique et modulo p peut se retrouver dans l'étude des multiplicités modulaires de [BM]. En généralisant les calculs qui ont mené à la preuve de la conjecture de Taniyama-Weil ([BCDT]), les auteurs proposent une conjecture ([BM], conjecture 1.1) qui relie la taille de certains anneaux de déformations d'une représentation modulo p de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q})$ avec certaines représentations de $\mathrm{GL}_2(\mathbf{Z}_p)$.

Cette conjecture, recemment prouvée par Kisin ([Kis]) grâce à l'etablissement de la correspondance p-adique de Langlands pour $\mathrm{GL}_2(\mathbf{Q}_p)$, fut interpretée par les auteurs comme une conséquence lointaine d'une hypothétique (à l'époque) correspondance de Langlands locale p-adique (ou, plus exactement, de la "réduction modulo p" d'une telle correspondance).

À l'époque de [BM], l'étude systématique des représentations p-adiques et modulo p de $\mathrm{GL}_2(F_\mathfrak{p})$ n'en était qu'à ses débuts : l'absence d'analogue p-adique de la mesure de Haar, ou des modèles de Whittaker (il n'y a pas de $\overline{\mathbf{F}}_p$-caractères non triviaux pour un p-groupe!) rendent inutilisables les constructions classiques pour la classification des représentations ℓ-adiques

([BK]) ou modulo ℓ ([Vig89]). C'est en fait à partir de l'œuvre de Schneider et Teitelbaum [ST] pour le cas p-adique et dans les travaux de Barthel et Livné [BL94], [BL95] que l'on retrouve les objets et les catégories adaptés à une formulation correcte d'une correspondance de Langlands p-adique, c'est ce qui a permis à Breuil ([Bre03a], [Bre03b] et [Bre04]) de fournir les premiers résultats.

En particulier Barthel et Livné ([BL94]) proposent une classifiaction pour $n = 2$ des représentations lisses admissibles absolument irréductibles, sur la base de l'immeuble de Bruhat-Tits de $\mathrm{GL}_2(F_{\mathfrak{p}})$ (cette méthode a été recemment géneralisée à $\mathrm{GL}_n(F_{\mathfrak{p}})$ par Herzig [Her] en donnant une classification à la Bernshtein-Zhelevinskii [BZ]). Si $\mathscr{O}_{F_{\mathfrak{p}}}$ désigne l'anneau des entiers de $F_{\mathfrak{p}}$ la classification de [BL94] permet alors de retrouver les caractères, les séries principales et spéciales, en termes de représentations lisses irréductibles σ de $\mathrm{GL}_2(\mathscr{O}_{F_{\mathfrak{p}}})$ (les poids de Serre) et des opérateurs convenables définis sur l'algèbre des endomorphismes des induites compactes $c\mathrm{-Ind}_{\mathrm{GL}_2(\mathscr{O}_{F_{\mathfrak{p}}})F_{\mathfrak{p}}^{\times}}^{\mathrm{GL}_2(F_{\mathfrak{p}})}\sigma$ (opérateurs "à la Hecke"). Cela met en évidence la présence d'une classe nouvelle d'objets, baptisés "supersinguliers", et dont l'existence était à l'époque assurée par un argument à la Zorn ([BL95], proposition 11). Les représentations supersingulières sont définies, à torsion par un caractère près, comme les quotients irréductibles admissibles de certaines représentations $\pi(\sigma, 0, 1)$: pour ce fait et étant donné le rôle crucial des objets supersingulières pour la détermination d'un programme de Langlands modulo p, on a décidé d'appeler les représentations $\pi(\sigma, 0, 1)$ comme les "*représentations universelles de* $\mathrm{GL}_2(F)$ *modulo* p". Elle font l'objet du titre de cette thèse.

Brièvement après la première incarnation de la correspondance de Langlands p-adique donnée par la conjecture Breuil-Mézard ([BM], conjecture 1.1), Breuil réussit ([Bre03a]) à compléter la classification de Barthel-Livné pour $F = \mathbf{Q}$, notamment en classifiant les objets supersinguliers. Pour cela, en étudiant l'action du pro-p Iwahori de $\mathrm{GL}_2(\mathbf{Q}_p)$ sur des vecteurs soigneusement choisis et à l'aune de calculs explicites sur les vecteurs de Witt de $\mathrm{W}(\mathbf{F}_p)$, il démontre que les représentations universelles sont en fait irréductibles pour $F = \mathbf{Q}$. Ceci a mis en évidence une correspondance "naturelle" ([Bre03a], définition 4.2.4) entre les représentations supersingulières de $\mathrm{GL}_2(\mathbf{Q}_p)$ et les représentations galoisiennes provenant des courbes elliptiques à réduction supersingulière : c'est la "correspondance de Langlands mod p".

Peu après, l'évolution de la correspondance locale p-adique connut des progrès rapides. Les premiers exemples historiques, découverts par Breuil ([Bre03b] et [Bre04]) mirent en évidence un lien précis entre les distributions p-adiques associées à une forme automorphe globale ([MTT]) et certaines fonctions sur $\mathbf{P}^1(\mathbf{Q}_p)$ ([Bre10a], corollaire 5.2.5). Inspiré par une telle relation -une sorte de dualité entre (certaines) fonctions et (certaines) mesures- Colmez découvrit le rôle décisif joué par la théorie des (ϕ, Γ)-modules pour la realisation d'un foncteur reliant les representations de $\mathrm{GL}_2(\mathbf{Q}_p)$ aux representations galoisiennes : ses travaux ([Col], [Col10]) ainsi que le travail de Kisin ([Kis10]), ont ensuite permis un étude fine des déformations des objets galoisiens et automorphes. C'est alors grâce aux études récents de Paskunas ([Pas]) concernant la reduction modulo p des $\mathrm{GL}_2(\mathbf{Q}_p)$-Banach, que les résultats de déformations de ([Col10]), ([Kis10]) permettent de disposer de la correspondence locale p-adique en toute generalité.

Cette correspondance est aujourd'hui presque complètement comprise : elle est compatible à la réduction modulo p ([Ber10a]), et permet de retrouver (presque) toutes les $\mathrm{GL}_2(\mathbf{Q}_p)$-représentations de Banach unitaires admissibles absolument irréductibles ([Pas]). De plus, l'article d'Emerton ([Eme10], théorèmes 1.2.1 et 1.2.6) achève la réalisation géométrique du programme de Langlands pour $\mathrm{GL}_2(\mathbf{Q}_p)$ (c'est la compatibilité locale-globale conjecturée dans [Eme06]) en montrant comment la correspondance se réalise dans le complété p-adique de la cohomologie d'une famille de courbes modulaires. C'est à l'aide de la correspondance p-adique pour $\mathrm{GL}_2(\mathbf{Q}_p)$ qu'Emerton ([Eme10]) et Kisin ([Kis]) ont pu démontrer presque tous les cas de la conjecture de Fontaine-Mazur ([FM]) pour $\mathrm{GL}_2(\mathbf{Q})$.

Toutefois, reposant de manière cruciale sur la théorie des (ϕ, Γ)-modules, la réalisation de la correspondance p-adique est loin d'être explicite. Par exemple, la détermination des vecteurs localement algébriques de la représentation p-adique associée à une représentation galoisienne passe par des méthodes globales ([Col], remarque **VI**.6.51) et il est difficile d'extraire les facteurs epsilon de ces constructions par des méthodes locales. De plus, si V_{k,a_p} est une représentation cristalline à poids de Hodge-Tate $(0, k-1)$ ayant a_p comme trace de Frobenius, la classe d'isomorphisme de sa réduction modulo p, $\overline{V}_{k,a_p}$ n'est connue que dans certaines cas (la situation à l'heure actuelle est résumée dans [Ber10b] théorème 5.2.1).

Cela suggère la nécessité d'une étude plus fine de certains objets apparaissant dans la correspondance locale.

Les parties I et II de cette thèse se placent dans le cadre de la correspondance modulo p pour $\mathrm{GL}_2(\mathbf{Q}_p)$. Dans ce qui suit $p \geqslant 3$ est un nombre premier impair.

Les calculs explicites de [Bre03a] prouvent l'irréductibilité des représentations universelles [1] pour $\mathrm{GL}_2(\mathbf{Q}_p)$. Si $r \in \{0, \ldots, p-1\}$ on utilise la notation $\pi(r, 0, 1)$ pour désigner une telle représentation à torsion près par un caractère lisse. On démontre que leur structure se décrit de manière détaillée via une filtration $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$-équivariante "naturelle" et une $\overline{\mathbf{F}}_p$-base adaptée compatible à la filtration. Le premier résultat que l'on déduit est la filtration par le $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$ socle des représentations supersingulières :

Théorème 0.2 [I, 9.1]. *Soit $r \in \{0, \ldots, p-1\}$, p impair. La restriction à $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$ de la représentation universelle $\pi(r, 0, 1)$ se décompose en la somme directe de deux termes dont la filtration par le $\mathrm{GL}_2(zp)\mathbf{Q}_p^\times$-socle est décrite par*

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 \text{---} \mathrm{SocFil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi^s_{-r-2}\det{}^{r+1}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi^s_{-r-4}\det{}^{r+2}) \text{---} \ldots$$

et

$$\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2 \otimes \det{}^r \text{---} \mathrm{SocFil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi^s_{r-2}\det) \text{---} \mathrm{SocFil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi^s_{r-4}\det{}^2) \text{---} \ldots$$

respectivement.

Précisons brièvement les notations de cet énoncé : pour $n \in \mathbf{N}$ on désigne par χ^s_n le caractère du Borel $B(\mathbf{F}_p)$ défini par $\chi^s_n(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}) = d^n$; la notation "socfil" signifie les facteurs gradués de la filtration par le socle des induites finies. En d'autres termes, si l'on ne se préoccupe pas de la valeur de r, le théorème 0.2 nous dit que la filtration par le socle de $\pi(r, 0, 1)|_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}$ est

[1] on remarque que Berger ([Ber10c]) et, independemment, Emerton ([Eme08]) ont achevé la preve de l'irreducibilité des representations universelles pour $\mathrm{GL}_2(\mathbf{Q}_p)$ par des methodes différentes de ceux de ([Bre03a]).

donnée par :

$$\begin{array}{ccccccccc}
\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 & \text{———} & \mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det^{r+1} & \text{—} & \mathrm{Sym}^{r+2}\overline{\mathbf{F}}_p^2\otimes\det^{p-2} & \text{—} & \mathrm{Sym}^{p-5-r}\overline{\mathbf{F}}_p^2\otimes\det^{r+2} & \text{—} & \cdots\\
\bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & & \cdots\\
\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2\otimes\det^r & \text{——} & \mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det & \text{——} & \mathrm{Sym}^{p+1-r}\overline{\mathbf{F}}_p^2\otimes\det^{r-1} & \text{——} & \mathrm{Sym}^{r-4}\overline{\mathbf{F}}_p^2\otimes\det^2 & \text{——} & \cdots
\end{array}$$

On esquisse les idées principales de la démonstration du théorème 0.2. Grâce à la filtration $\mathrm{GL}_2(\mathbf{Z}_p)$-équivariante sur $\pi(r,0,1)$ (§I-4 et §I-7) on réduit d'abord l'étude au cas des induites *finies* $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi$ (pour χ caractère convenable du sous-groupe de Borel fini $B(\mathbf{F}_p)$), dont la structure est complètement connue par les travaux de Bardoe et Sin [BS00]. Le point clef est alors de "recoller" les induites finies ainsi obtenues. Pour cela il s'agit de déterminer une $\overline{\mathbf{F}}_p$-base "naturelle" sur $\pi(r,0,1)$ qui est compatible à la filtration précédente (et aux calculs sur les vecteurs de Witt de $\mathbf{F}_p$ et à l'action de l'opérateur de Hecke canonique). On remarque que dans la $\overline{\mathbf{F}}_p$-base naturelle on retrouve les éléments X_n^0, X_n^1 de [Bre03a], qui étaient l'outil incontournable au prélude de l'irréductibilité des représentations universelles de $\mathrm{GL}_2(\mathbf{Q}_p)$.

Une variante de cette méthode permet de déterminer la $\mathrm{GL}_2(\mathbf{Z}_p)$-structure des séries principales modérément ramifiées :

Théorème 0.3 [I, 10.4]. *Soit $\lambda\in\overline{\mathbf{F}}_p^\times$ et $r\in\{0,\dots,p-1\}$. La filtration par le $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$-socle pour la série principale modérément ramifiée $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(un_\lambda\otimes\omega_1^r un_{\lambda^{-1}})$ est décrite par*

$$\mathrm{SocFil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_r^s)\text{—}\mathrm{SocFil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_{r-2}^s\det)\text{—}\mathrm{SocFil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_{r-4}^s\det^2)\text{—}\dots$$

Cette description explicite permet ensuite de calculer l'espace des invariants sous certains sous-groupes de congruence classiques de $\mathrm{GL}_2(\mathbf{Z}_p)$: c'est l'objet de la partie II. Soit $t\in\mathbf{N}$. On note par K_t le noyau du morphisme de réduction modulo p^t et par I_t l'image réciproque des matrices unipotentes supérieures $U(\mathbf{Z}_p/(p^t))$ de $\mathrm{GL}_2(\mathbf{Z}_p/(p^t))$ par le morphisme de réduction modulo p^t défini sur K_{t-1}. Le résultat suivant détermine l'espace des K_t-invariants pour les représentations supersingulières de $\mathrm{GL}_2(\mathbf{Q}_p)$:

Théorème 0.4 [II, 3.9]. *Soit $t\geqslant 1$. La filtration par le $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$-socle de $\pi(r,0,1)^{K_t}$ est décrite par*

$$\begin{array}{c}
\mathrm{Sym}^r\overline{\mathbf{F}}_p^2\text{—}\mathrm{socfil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_{-r-2}^s\det^{r+1})\text{—}\dots\text{—}\mathrm{socfil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_{-r}^s\det^r)\text{—}\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det^{r+1}\\
\bigoplus\\
\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2\otimes\det^r\text{—}\mathrm{socfil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_{r-2}^s\det)\text{—}\dots\text{—}\mathrm{socfil}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_r^s)\text{—}\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det
\end{array}$$

où l'on a $p^{t-1}-1$ inductions paraboliques sur chaque ligne et le poids $\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det^{r+1}$ *dans la première ligne (resp.* $\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det$ *dans la deuxième ligne) apparaît seulement si $p-3-r\geqslant 0$ (resp. $r-2\geqslant 0$).*

En particulier on dispose de la dimension de ces espaces

Corollaire 0.5 (II, 3.8). *Soit $t\geqslant 1$ et $r\in\{0,\dots,p-1\}$. La dimension des K_t-invariants pour* *une représentation supersingulière est*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{K_t}) = (p+1)(2p^{t-1}-1) + \begin{cases} p-3 & \text{if } r \notin \{0,p-1\} \\ p-2 & \text{if } r \in \{0,p-1\} \end{cases}$$

Bien évidemment, on dispose aussi de résultats analogues pour les séries principales modérément ramifiées (cf. §II, théorème 5.1).

La stratégie de la démonstration du théorème 0.4 se résume de la manière suivante. La filtration naturelle $\{\mathrm{Fil}^n\}_{n\in\mathbf{N}}$ (rappellée en II §2) nous permet de disposer d'une famille de suites exactes

$$0 \to \mathrm{Fil}^{n-1} \to \mathrm{Fil}^n \to \mathrm{Fil}^n/\mathrm{Fil}^{n-1} \to 0.$$

Une vérification directe, à partir de la connaissance de $(\mathrm{Fil}^n/\mathrm{Fil}^{n-1})^{K_t}$ et de l'exactitude à gauche du foncteur des K_t-invariants, montre l'existence d'un entier n_0 tel que le morphisme naturel $0 \to (\mathrm{Fil}^{n-1})^{K_t} \to (\mathrm{Fil}^n)^{K_t}$ soit un isomorphisme si $n \geqslant n_0$. De façon similaire, on détermine $n_1 \in \mathbf{N}$ avec $(\mathrm{Fil}^n)^{K_t} = \mathrm{Fil}^n$ pour $n \leqslant n_1$. Ainsi, on ramène l'étude des K_t-invariants à une petite partie de la représentation $\pi(r,0,1)$, à savoir $\mathrm{Fil}^{n_0}/\mathrm{Fil}^{n_1}$, ce qui rend possible d'établir la preuve du théorème 0.4 par des calculs directs.

La même technique s'applique à d'autres sous-groupes de congruence que K_t. Par exemple dans §II-4 on établit une description détaillée de l'espace des invariants selon les sous-groupes I_t, en donnant une $\overline{\mathbf{F}}_p$-base de vecteurs propres sous l'action du tore fini $\mathbf{F}_p^\times \times \mathbf{F}_p^\times$. Ces résultats reposent sur une étude combinatoire lourde. Toutefois on peut donner la dimension de ces espaces

Théorème 0.6 (II, 4.15). *Soient $r \in \{0,\dots,p-1\}$ et $t \in \mathbf{N}_>$. Alors*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{I_t}) = 2(2p^{t-1}-1).$$

La partie III de cette thèse est consacrée à l'étude des restrictions des supersingulières de $\mathrm{GL}_2(\mathbf{Q}_p)$ aux sous-groupes provenant des extensions $L/\mathbf{Q}_p$ de dégré 2. L'intéret demeure alors dans la recherche d'un analogue “modulo p” du théorème classique de Tunnel et Saito reliant le signe des facteurs epsilon du changement de base d'une représentation complexe locale supercuspidale $BC_{L/\mathbf{Q}_p}(\tau)$ à la structure de la restriction $\tau|_{L^\times}$. Le résultat est le suivant :

Théorème 0.7 [III, 1.2]. *Soit $L/\mathbf{Q}_p$ une extension quadratique, $\pi(\sigma,0,1)$ une représentation supersingulière, de caractère central ω^r (où ω désigne la réduction modulo p du caractère cyclotomique p-adique). Écrivons $\mathrm{soc}_{L^\times}^{(j)} \overset{\mathrm{def}}{=} \mathrm{soc}_{L^\times}^{(j)}(\pi(r,0,1)|_{L^\times})$ pour le j-ième facteur de composition de la filtration par le $L^\times$-socle de $\pi(\sigma,0,1)|_{L^\times}$.*

Alors

i) si $L/\mathbf{Q}_p$ est non ramifiée on a un isomorphisme de $\mathbf{F}_{p^2}^\times$-représentations

$$\mathrm{soc}_{L^\times}^{(j)}/\mathrm{soc}_{L^\times}^{(j-1)} \cong (\oplus_{i=0}^{p}\eta_i)^2$$

où η_i, pour $i=0,\dots,p$, sont les $(p+1)$ caractères distincts de $\mathbf{F}_{p^2}^\times$ qui étendent le caractère $x \mapsto x^r$ sur $\mathbf{F}_p$;

ii) si $L/\mathbf{Q}_p$ est totalement ramifiée on a un isomorphisme de $L^\times$-représentations

$$\mathrm{soc}_{L^\times}^{(j)}/\mathrm{soc}_{L^\times}^{(j-1)} \cong (V)^{2-\delta_{0,j}}$$

où V est un espace de dimension 2 muni d'une action de $\mathscr{O}_L^\times$ obtenue par inflation du $\mathbf{F}_p$-caractère $x \mapsto x^r$ et par l'action de l'uniformisante via une involution non triviale.

La technique de la preuve repose encore une fois sur la filtration $\mathrm{GL}_2(\mathbf{Z}_p)$-équivariante sur $\pi(\sigma, 0, 1)$ qui est déterminée dans la partie I. Cette filtration permet de nous ramener au cas fini (ce qui va ensuite décrire la structure des facteurs gradués de la filtration par le socle) et de recoller les morceaux ainsi obtenus par un calcul explicite sur certains éléments du sous groupe d'Iwahori.

Le cas de $\mathrm{GL}_2(F_{\mathfrak{p}})$, $F_{\mathfrak{p}} \neq \mathbf{Q}$.

Supposons maintenant que $F_{\mathfrak{p}}$ soit une extension *non triviale* de $\mathbf{Q}_p$. Dans cette situation la méthode de Breuil [Bre03a] -qui avait permis de démontrer l'irréducibilité des objets universels- ne permet actuellement pas de conclure. Nous ne disposons pas à ce jour de descriptions satisfaisantes des objets supersinguliers de $\mathrm{GL}_2(F_{\mathfrak{p}})$: puisse le lecteur penser que l'on ne dispose même *pas* d'un exemple donné par des équations explicites, d'un sous-quotient irréductible d'une représentation universelle. Cette lacune se révèle particulièrement grave dans le cadre d'un programme de Langlands modulo p et p-adique : en fait, comme pour $\mathrm{GL}_2(\mathbf{Q}_p)$, on est tenté d'associer à une représentation galoisienne modulo p provenant d'une courbe elliptique à réduction supersingulière, une représentation supersingulière de $\mathrm{GL}_2(F_{\mathfrak{p}})$ au sens de Barthel-Livné.

Une première description -du moins conjecturelle- des "bonnes" représentations supersingulières provient des méthodes globales issues des travaux récents de Buzzard, Diamond et Jarvis [BDJ] et de la compatibilité locale-globale étudiée par Emerton ([Eme10]). Ces arguments sont reliés à la conjecture de Serre ([Ser87], démontrée par Khare et Winterberger [KW1], [KW2]) qui prévoit qu'une représentation galoisienne globale $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$, irréductible continue et impaire provient d'une forme propre de Hecke dans l'espace $S_k(\Gamma_1(N))$. Emerton ([Eme10]) relie cette conjecture au programme de Langlands p-adique pour $\mathrm{GL}_2(\mathbf{Q}_p)$ grâce au principe de compatibilité locale-globale : le poids k de la forme de Hecke est réinterprété comme représentation algébrique irréductible de $\mathrm{GL}_2(\mathbf{Z}_p)$ apparaissant dans le facteur local en p de la représentation automorphe associée à ρ.

Le travail de Buzzard, Diamond et Jarvis [BDJ] est consacrée à une généralisation de la conjecture de Serre. À une représentation globale ρ (modulo p et de dimension 2) de $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ ils associent une représentation automorphe convenable π dont les facteurs locaux vérifient certaines conditions explicites ([BDJ], conjecture 4.7). En particulier si $F_{\mathfrak{p}}$ est non ramifiée, ils fournissent une description explicite du $\mathrm{GL}_2(F_{\mathfrak{p}})$-socle du facteur locale en $\mathfrak{p}$, à l'aide d'une famille de représentations algébriques irréductibles de $\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$ (les *poids de Diamond-Serre*) associée à la restriction $\rho|_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/F_{\mathfrak{p}})}$. Cette description a été ensuite généralisée au cas totalement ramifiée par Schein ([Sch08]).

Inspirés par ces constructions et animés par le désir de déterminer les "bonnes" représentations supersingulières de $\mathrm{GL}_2(F_{\mathrm{p}})$ par des moyens purement locaux, Breuil et Paskunas ont cherché à construire des objets supersinguliers ayant un $\rho|_{\mathrm{Gal}(\overline{\mathbf{Q}_p}/F_{\mathfrak{p}})}$-socle qui respecte la combinatoire dictée par la théorie des poids modulaires. En utilisant la théorie des diagrammes de base -introduite dans [Pas04]- les auteurs associent à une représentation galoisienne locale $\rho_{\mathfrak{p}}$ de $\mathrm{Gal}(\overline{\mathbf{Q}_p}/F_{\mathfrak{p}})$ (dans le cas $F_{\mathfrak{p}}/\mathbf{Q}_p$ non ramifiée, étant $\rho_{\mathfrak{p}}$ irréductible, générique) une famille *infinie* $\Pi(\rho_{\mathfrak{p}})$ de représentations supersingulières dont le socle est fixé par les poids modulaires associés à $\rho_{\mathfrak{p}}$. Toutefois, la possibilité de détecter un objet "naturel" parmi les éléments de $\Pi(\rho_{\mathfrak{p}})$ n'est pas claire, surtout à la lumière des contre-exemples de Hu [Hu]. Hu propose une variante des techniques de [Pas04], grâce à la notion des diagrammes canoniques ([Hu2]), qui permet de classifier les représentations supersingulières de $\mathrm{GL}_2(F_{\mathfrak{p}})$. L'aspect clef est le suivant : les informations nécessaires à caractériser une représentation supersingulière $\pi_{\mathfrak{p}}$ ne se trouvent *pas* dans l'espace des invariants sous l'action du pro-p-Iwahori, mais dans un espace qui est strictement (au moins si $F_{\mathfrak{p}} \neq \mathbf{Q}_p$ est non ramifiée) plus grand, notée $D_1(\pi_{\mathfrak{p}})$. Cet espace est, en géneral, difficile à comprendre.

Pendant la même periode, Schein a énoncé ([Sch]) un critère d'irréducibilité pour certaines sous-quotients de représentations universelles $\pi(\sigma, 0, 1)$ de $\mathrm{GL}_2(F_{\mathfrak{p}})$ lorsque $F_{\mathfrak{p}}/\mathbf{Q}_p$ est totalement ramifiée. En particulier il a obtenu un quotient naturel V_{e-1} associé à $\pi(\sigma, 0, 1)$ qui jouit d'une proprieté universelle relativement aux représentations supersingulières dont le $\mathrm{GL}_2(\mathscr{O}_{F_{\mathfrak{p}}})$-socle est décrit par la famille des poids modulaires. Ce résultat est achevé à l'aune de calculs directs assez laborieux sur l'anneau des entiers, qui généralisent les techniques utilisées dans [Bre03a].

La partie IV de cette thèse se place dans le cadre d'une recherche des représentations supersingulières pour une extension finie non ramifiée de $\mathbf{Q}_p$. On se propose de démontrer que la structure Iwahori (i.e. la filtration par le Iwahori-socle et la détermination des extentions entre deux facteurs gradués adjacents) des représentations universelles $\pi(\sigma, 0, 1)$ admet une description simple en termes de certaines données euclidiennes. Cela met en évidence les raisons et les modalités pour lesquelles les représentations universelles cessent d'être irréductibles lorsque $F_{\mathfrak{p}} \neq \mathbf{Q}_p$ en une vaste généralisation des méthodes de [Bre03a]. On obtient en corollaire la structure Iwahori pour les séries principales modérément ramifiées et la détermination d'injections naturelles $c{-}\mathrm{Ind}_{\mathrm{GL}_2(F_{\mathfrak{p}})}^{B(F_{\mathfrak{p}})}\sigma' \hookrightarrow \pi(\sigma, 0, 1)$. Ce phénomène a déjà été découvert par Paskunas, dans une note personnelle non publiée.

Le résultat principal de la partie IV se résume de la manière suivante. On détermine une $\overline{\mathbf{F}}_p$-base $\mathscr{B}$ sur $\pi(\sigma, 0, 1)$ ainsi qu'une injection $\mathscr{B} \hookrightarrow \mathbf{Z}^f$. L'image $\mathfrak{R}$ de cette injection est décrite de manière tout à fait explicite (il s'agit de la *structure euclidiene* associée à $\pi(\sigma, 0, 1)$ selon la terminologie de (IV)-§4 et§5). Alors on a :

Théorème 0.8 (IV, 5.18). *La structure Iwahori de la représentation universelle $\pi(\sigma, 0, 1)$ est décrite par $\mathfrak{R}$: la filtration par le Iwahori-socle est obtenue à partir de $\mathfrak{R}$ en éliminant successivement les points d'antécédent vide et les extensions entre deux facteurs gradués adjacents sont réalisées (lorsque σ est générique) entre les points adjacents de $\mathfrak{R}$.*

Précisons brièvement la terminologie du théorème 0.8 (on refère alors le lecteur à la page 118 pour la description précise du formalisme). On convient que l'*antécédent* d'un point $P \in \mathfrak{R}$ soit constitué par les $Q \in \mathfrak{R}$ vérifiant $P = Q + e_s$ (étant $\{e_0, \ldots, e_{f-1}\}$ la base canonique de $\mathbf{Z}^f$).
La locution "en éliminant successivement les points d'antécédent vide" admet ainsi la signification plus précise suivante: une $\overline{\mathbf{F}}_p$-base pour l'Iwahori socle soc_0 de $\pi(\sigma, 0, 1)$ est consitué par l'ensemble $\mathscr{P}_0$ des points de $\mathfrak{R}$ ayant antécédent vide; on associe alors l'ensemble $\mathfrak{R}_1 \overset{\text{def}}{=} \mathfrak{R} \setminus \mathscr{P}_0$ à la représentation $\pi(\sigma, 0, 1)/\mathrm{soc}_0$. Par récurrence, l'ensemble $\mathfrak{R}_N$ associé à $\pi(\sigma, 0, 1)/\mathrm{soc}_{N-1}$ étant donné, une $\overline{\mathbf{F}}_p$-base pour l'Iwahori socle $\mathrm{soc}_N/\mathrm{soc}_{N-1}$ de $\pi(\sigma, 0, 1)/\mathrm{soc}_{N-1}$ est consitué par l'ensemble $\mathscr{P}_N$ des points de $\mathfrak{R}_N$ ayant antécédent vide; on associe alors l'ensemble $\mathfrak{R}_{N+1} \overset{\text{def}}{=} \mathfrak{R}_N \setminus \mathscr{P}_N$ à la représentation $\pi(\sigma, 0, 1)/\mathrm{soc}_N$.

Pour avoir une idée de l'aspect fractal de la structure euclidienne $\mathfrak{R}$ le lecteur pourra se réferer aux figures IV.6 ou IV.9.

Des différentes étapes de la preuve du théorème 0.8, on déduit les deux résultats suivants :

Théorème 0.9 (IV, 4.16). *La structure Iwahori des séries principales modérément ramifiées est décrite par deux copies de* $\mathbf{N}^f$.

et

Théorème 0.10 (IV, 5.12). *Supposons* $\dim_{\overline{\mathbf{F}}_p}(\sigma) \notin \{1, q\}$ *et soit* χ^s *le caractère de l'Iwahori associé à* $(\sigma)^{U(\mathbf{F}_q)}$. *Il existe une sous-*$\mathrm{GL}_2(\mathscr{O}_{F_\mathfrak{p}})F_\mathfrak{p}^\times$*-représentation* V *de* $\pi(\underline{r}, 0, 1)|_{\mathrm{GL}_2(\mathscr{O}_{F_\mathfrak{p}})F_\mathfrak{p}^\times}$ *isomorphe au noyau du morphisme naturel*

$$\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)}\chi^s/\mathrm{soc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)}\chi^s) \twoheadrightarrow \mathrm{cosoc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)}\chi^s)$$

et telle que le morphisme naturel induit par reciprocite de Frobenius

$$c\text{−}\mathrm{Ind}_{KZ}^{G}V \to \pi(\sigma, 0, 1)$$

soit injectif.

La démonstration du théorème 0.8 repose sur les trois faits suivants.

i) On détermine d'abord la base $\mathscr{B}$ (§IV, lemma 5.1 et proposition 3.5). Cette base se prête agréablement aux manipulations avec les vecteurs de Witt de $\mathbf{F}_q$ et est de plus compatible à l'action de l'opérateur de Hecke canonique T_σ associé par Barthel et Livné au poids σ.

ii) L'action de l'Iwahori sur les éléments de $\mathscr{B}$ admet une interprétation simple en termes de certains polynômes universels de Witt et de leurs degrés homogènes.

iii) L'injection $\mathscr{B} \hookrightarrow \mathbf{Z}^f$, qui se déduit de la définition des éléments de $\mathscr{B}$, est compatible au degré homogène des polynômes universels apparaissant en *ii*).

La donnée euclidienne $\mathfrak{R}$ admet une structure fractale régulière obtenue à partir d'une famille $\{\mathscr{R}_n^+, \mathscr{R}_n^-\}_{n\in\mathbf{N}}$ de f-parallélépipoïdes emboîtés. Ces parallélépipoïdes représentent l'image dans $\pi(\sigma, 0, 1)$ des éléments de l'induite compacte $\mathrm{Ind}_{\mathrm{GL}_2(\mathscr{O}_{F_\mathfrak{p}})F_\mathfrak{p}^\times}^{\mathrm{GL}_2(F_\mathfrak{p})}\sigma$ ayant support sur une boule de rayon fixé et de l'immeuble de Bruhat-Tits de $\mathrm{GL}_2(F_\mathfrak{p})$. Ils sont ensuite recollés d'une manière liée au comportement de l'opérateur de Hecke canonique T_σ. Dit grossièrement, à partir du $n+2$-ième bloc (qui a un côté de longueur environ $p^{n+2} - 1$), on enlève le $n + 1$-ième bloc (qui a un côté de longueur environ $p^{n+1} - 1$). Cela fournit f sommets "libres" (i.e. vecteurs invariants sous l'action du pro-p-Iwahori), et on recolle dans la lacune ainsi obtenue l'n-ième bloc (dont le côté est de longueur environ $p^n - 1$). Les processus d'enlevement/recollement des blocs proviennent de l'operateur de Hecke T_σ. Ceci justifie la non-admissibilité des représentations universelles pour $F_\mathfrak{p} \neq \mathbf{Q}_p$: lorsque $f > 1$ les sommets libres obtenus ci-dessous ne sont pas concernés par le phénomène de recollement !

Part I. Explicit description of irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$

Abstract. Let p be an odd prime number. The classification of irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$ is known thanks to the works of Barthel-Livné [BL95] and Breuil [Bre03a]. In the present chapter we illustrate an exhaustive description of such irreducible representations, through the study of certain functions on the Bruhat-Tits tree of $\mathrm{GL}_2(\mathbf{Q}_p)$. In particular, we are able to detect the socle filtration for the KZ-restriction of supersingular representations, principal series and special series.

1. Introduction

Let p be a prime number. If F is a non-Archimedean local field, with finite residue field of characteristic p and cardinality $q = p^f$, the ℓ-adic Local Langlands correspondence (for $\ell \neq p$) let us dispose of a well understood "dictionary" between suitable representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$, n dimensional over $\mathbf{Q}_\ell$, and suitable representations of $\mathrm{GL}_n(\mathbf{F})$ (two independent proofs due to Harris and Taylor in [HT] and Henniart in [Hen00]). Moreover, via a process of "reduction of coefficients modulo ℓ", Vignéras deduces a semi-simple "mod ℓ" Local Langlands correspondence, as it results from her study in [Vig01].

The theory, in the p-adic case, is dramatically more complicated (e.g. Grothendieck's ℓ-adic monodromy theorem collapses, there are not reasonable analogues of the Haar measure, etc...). After a first conjectural approach pointed out by Breuil in [Bre04] and [Bre03b], we dispose nowadays of a "p-adic local langlands correspondence" in the 2-dimensional case for $F = \mathbf{Q}_p$ by the works of Berger-Breuil [BB] and Colmez [Col]. This correspondence is compatible with the "reduction of coefficients modulo p" and enable us to establish a semi-simple "mod p"-Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ (again, such a process has been conjectured and proved in few cases by Breuil in [Bre03b] and in generality (for $F = \mathbf{Q}_p$) by Berger in [Ber10a]).

A major problem in for a (conjectural) mod p-Langlands correspondence is represented by the lack of a complete classification for smooth irreducible admissible $\mathrm{GL}_2(\mathbf{Q}_p)$ representations over $\overline{\mathbf{F}}_p$. In [BL94] and [BL95], Barthel and Livné detect four families of such irreducible objects: besides a detailed study of principal and special series (and characters), the authors discover another class of smooth irreducible admissible representations, referred to as "supersingular", non-isomorphic to the previous ones. Supersingular representations can be characterised as subquotients of the cokernel of a "canonical Hecke operator" T, and their nature is still very mysterious. For instance, if $F \neq \mathbf{Q}_p$, the aforementioned cokernels are not even admissible and the works of Paskunas [Pas04], Breuil-Paskunas [BP] and Hu [Hu] show the existence of a huge number of supersingular representations with respect to Galois representations (whose classification is indeed well known).

The case $F = \mathbf{Q}_p$ is far different. The cokernels of the Hecke operators are indeed irreducible, therefore giving a complete description of supersingular representations for $\mathrm{GL}_2(\mathbf{Q}_p)$. The first proof of this phenomenon, due to Breuil, appears in [Bre03a]: the author is there able to compute explicitly the space of I_1-invariants (and therefore the socle) for $\mathrm{coker}(T)$, studying the behaviour of certain functions (denoted as X_n^0 and X_n^1) on the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$. Nowadays others paths to prove the irreducibility of $\mathrm{coker}(T)$ have been discovered (for instance, Ollivier in [Oll], Emerton in [Eme08]). Nevertheless, we show in the present chapter -deeply in debt to the works of Breuil [Bre03a] and [Bre]- that suitable modifications on the functions X_n^0, X_n^1 let us describe completely the socle filtration for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$: to some extent, we see that such representations can be obtained by a process of "glueing" parabolic inductions of characters defined on a filter of neighbourhood of the Iwahori subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. As a byproduct, we are also able to detect the socle filtration for principal and special series for $\mathrm{GL}_2(\mathbf{Q}_p)$.

Using the notations of §2.2 for the characters χ_r^s and $\mathfrak{a}$ and the formalism presented in the end of this § concerning the socle filtration, the main result of the chapter is the following:

Theorem 1.1. *Let $r \in \{0, \dots, p-1\}$, p odd. Then the $K\mathbf{Q}_p^\times$ restriction of the supersingular representation* $\mathrm{coker}(T_r)$ *(where K is the maximal compact subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$) consists of two direct summands, whose socle filtration is described by*

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}^{r+1}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}^{r+2}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}^{r+3}) \text{---} \dots$$

and

$$\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2 \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}^{2}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}^{3}) \text{---} \dots$$

respectively.

If we do not bother too much about the value of r, proposition 1.1 shows that the socle filtration for $\pi(r,0,1)|_{KZ}$ looks as follow:

$$\begin{array}{ccccccccc}
\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 & \text{———} & \mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det^{r+1} & \text{—} & \mathrm{Sym}^{r+2}\overline{\mathbf{F}}_p^2\otimes\det^{p-2} & \text{—} & \mathrm{Sym}^{p-5-r}\overline{\mathbf{F}}_p^2\otimes\det^{r+2} & \text{—} & \cdots \\
\bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & & \cdots \\
\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2\otimes\det^{r} & \text{——} & \mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det & \text{——} & \mathrm{Sym}^{p+1-r}\overline{\mathbf{F}}_p^2\otimes\det^{r-1} & \text{——} & \mathrm{Sym}^{r-4}\overline{\mathbf{F}}_p^2\otimes\det^{2} & \text{——} & \cdots
\end{array}$$

If moreover we write un_λ for the unramified character of $\mathbf{Q}_p$ sending the arithmetic Frobenius to $\lambda \in \overline{\mathbf{F}}_p$ and ω_1 for the cyclotomic character, the are able to prove (with the same techniques of proposition 1.1)

Theorem 1.2. *For p an odd prime number, let $\lambda \in \overline{\mathbf{F}}_p^\times$, $r \in \{0, \dots, p-1\}$ and assume $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$. The socle filtration for the KZ-restriction of the* $\mathrm{GL}_2(\mathbf{Q}_p)$*-principal series* $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(un_\lambda \otimes \omega_1^r un_\lambda)$ *is*

$$\mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \chi_r^s\mathfrak{a}^{2}) \text{---} \dots$$

The socle filtration for the KZ restriction of the Steinberg representation for $\mathrm{GL}_2(\mathbf{Q}_p)$ *is*

$$\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2 \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \mathfrak{a}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^K \mathfrak{a}^{2}) \text{---} \dots$$

The strategy of the proof -largely inspired by Breuil's notes [Bre]- is quite elementary and it can be sketched as follow. For a given $r \in \{0, \dots, p-1\}$ we can write the KZ-restriction of $\mathrm{coker}(T_r)$ as a direct sum of inductive limits of the form $\varinjlim_{2n+i,\, n\in\mathbf{N}} (R_i/R_{i-1} \oplus_{i+1} \cdots \oplus_{R_{2n+i-1}} R_{2n+i})$ (for $i \in \{0,1\}$) for suitable K-representations R_n (essentially, obtained as a K-induction from a family of representations defined on a filter of neighbourhood of the Iwahori subgroup); the

Hecke operator T_r translates then to a family of operators on the R_n's. We underline that such a decomposition is absolutely general (we do not require $F = \mathbf{Q}_p$).

Next, we give the inductive limit a natural filtration (whose graded pieces will be suitable quotients of the R_n's), and a filtration $\mathrm{Fil}^t(R_n)$ on the R_n's as well: the graded pieces of the latter let us reduce the problem to K-parabolic inductions $\mathrm{Ind}_{K_0(p^n)}^{K}\chi$ of characters on a neighbourhood of the identity in the Iwahori subgroup. Thanks to the description of parabolic inductions for $\mathrm{GL}_2(\mathbf{F}_p)$ we are able to extract the socle filtration for the $\mathrm{Ind}_{K_0(p^n)}^{K}\chi$'s (which will lead us to proposition 1.2), and the successive step consist in glueing the socle filtrations for the graded pieces of $\mathrm{Fil}^t(R_n)$ (via a descending induction by consecutive quotients). Finally, using the operator T_r we appropriately glue together the socle filtrations of the R_n's to get the statement of proposition 1.1.

The plan of the chapter is then the following.

In §2 we recall the structure of compact inductions Ind_{KZ}^{G}, their relations with the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$ and the structure of the Hecke algebra for compact inductions. We summarise the main properties of the parabolic induction for the finite case in §2.2, in particular recalling the description of the socle filtration.

Section 3 is devoted to the description of the KZ-restriction of supersingular representations in terms of "simpler" object as the representations R_n's (§3.1) and their amalgamed sums (cf. (4)) by means of convenient Hecke operators $T_n^{\pm}$ on R_n (defined in §3.2). Such objects will be endowed with filtrations in §4.

Sections §5, §6, §7 and §8 are devoted to the study, and the glueing, of the socle filtations on the graded pieces if the filtrations introduced in 4; in particular, in §8, such glueing are made by means of the Hecke operator T.

Finally, in 9, we make explicit how the right exactness of $\varinjlim$ makes possible to deduce the socle filtration for supersingular from the results in §8. The final section §10 shows how we can deduce easily the socle filtration for principal and special series using the techniques in §6.

We wish to outline that such an explicit nature for the description of supersingular $\mathrm{GL}_2(\mathbf{Q}_p)$-representations (as well as principal and special series) let us describe in greatest detail the K_t and I_t invariant elements. Such a study has been pursued in [Mo2].

We introduce now the main notations, convention and structure of the chapter.

We fix a prime number p. We write $\mathbf{Q}_p$ (resp. $\mathbf{Z}_p$) for the p-adic completion of $\mathbf{Q}$ (resp. $\mathbf{Z}$) and $\mathbf{F}_p$ the field with p elements; $\overline{\mathbf{F}}_p$ is then a fixed algebraic closure of $\mathbf{F}_p$. For any $\lambda \in \mathbf{F}_p$ (resp. $x \in \mathbf{Z}_p$) we write $[\lambda]$ (resp. $\overline{x}$) for the Teichmüller lift (resp. for the reduction modulo p), defining $[0] \overset{\text{def}}{=} 0$.

We write $G \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Q}_p)$, $K \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Z}_p)$ the maximal compact subgroup, I the Iwahori subgroup of K (i.e. the elements of K whose reduction modulo p is upper triangular) and I_1 for the pro-p-iwahori (i.e. the elements of I whose reduction is unipotent). Moreover, let $Z \overset{\text{def}}{=} Z(G) \cong \mathbf{Q}_p^{\times}$ be the centre of G and $B(\mathbf{Q}_p)$ (resp. $B(\mathbf{F}_p)$) the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ (resp. $\mathrm{GL}_2(\mathbf{F}_p)$).

For $r \in \{0, \dots, p-1\}$ we denote by σ_r the algebraic representation $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2$ (endowed with the natural action of $\mathrm{GL}_2(\mathbf{F}_p)$). Explicitly, if we consider the identification $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 \cong \overline{\mathbf{F}}_p[X,Y]_r^h$ (where $\overline{\mathbf{F}}_p[X,Y]_r^h$ means the graded component of degree r for the natural grading on $\overline{\mathbf{F}}_p[X,Y]$)

then

$$\sigma_r(\begin{bmatrix} a & b \\ c & d \end{bmatrix})X^{r-i}Y^i \stackrel{\text{def}}{=} (aX+cY)^{r-i}(bX+dY)^i$$

for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{F}_p)$, $i \in \{0,\dots,r\}$. We then endow σ_r with the action of K obtained by inflation $K \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_p)$ and, by imposing a trivial action of $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$, we get a smooth KZ-representation. Such a representation is still noted as σ_r, not to overload the notations.

If H stands for the maximal torus of $\mathrm{GL}_2(\mathbf{F}_p)$ and $\chi : H \to \overline{\mathbf{F}}_p^\times$ is a multiplicative character we will write χ^s for the conjugate character defined by $\chi^s(h) \stackrel{\text{def}}{=} \chi(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} h \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ for $h \in H$. Characters of H will be seen as characters of $B(\mathbf{F}_p)$ or (by inflation) of (a filter of neighbourhood of 1 in) I without any commentary.

With "representation" we always mean a smooth representations with central character with coefficients in $\overline{\mathbf{F}}_p^\times$. If V is a $\widetilde{K}$-representation, for $\widetilde{K}$ a subgroup of K, and $v \in V$, we write $\langle \widetilde{K} \cdot v \rangle$ to denote the sub-$\widetilde{K}$ representation of V generated by v. For a $\widetilde{K}$-representation V we write $\mathrm{soc}_{\widetilde{K}}(V)$ (or $\mathrm{soc}(V)$, or $\mathrm{soc}^1(V)$ if $\widetilde{K}$ is clear from the context) to denote the maximal semisimple sub-representation of V. Inductively, the subrepresentation $\mathrm{soc}^i(V)$ of V being defined, we define $\mathrm{soc}^{i+1}(V)$ as the inverse image of $\mathrm{soc}^1(V/\mathrm{soc}^i(V))$ via the projection $V \twoheadrightarrow V/\mathrm{soc}^i(V)$. We therefore obtain an increasing filtration $\{\mathrm{soc}^n(V)\}_{n\in\mathbf{N}_>}$ which will be referred to as the socle filtration for V; we will say that a subrepresentation W of V "comes from the socle filtration" if we have $W = \mathrm{soc}^n(V)$ for some $n \in \mathbf{N}_>$ (with the convention that $\mathrm{soc}^0(V) \stackrel{\text{def}}{=} 0$). The sequence of the graded pieces of the socle filtration for V will be shortly denoted by

$$\mathrm{SocFil}(V) \stackrel{\text{def}}{=} \mathrm{soc}^1(V)\text{—}\mathrm{soc}^1(V)/\mathrm{soc}^0(V)\text{—}\dots\text{—}\mathrm{soc}^{n+1}(V)/\mathrm{soc}^n(V)\text{—}\dots$$

We finally recall the Kroneker delta: if S is any set, and $s_1, s_2 \in S$ we define

$$\delta_{s_1,s_2} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad s_1 \neq s_2 \\ 1 & \text{if} \quad s_1 = s_2. \end{cases}$$

2. Preliminaries and definitions

The aim of this section is to recall some classical facts concerning (compact) induction of p-adic representations (§2.1 and §2.2), and to give some explicit computation in the ring of p-adic integers $\mathbf{Z}_p$ (§2.3): such computations will play a key role in the rest of the chapter.

2.1 Compact induction of KZ-representations

For the details and proofs, the reader is invited to see [Ser77] or ([Bre03a], §2).

We write $\mathscr{T}$ for the tree of $\mathrm{GL}_2(\mathbf{Q}_p)$. It is well known that we have an explicit G-equivariant bijection (with respect to the natural left G-action defined on the two sets) between the vertices $\mathscr{V}$ of $\mathscr{T}$ and the right side classes of G/KZ. We define the following elements of G:

$$\alpha \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} w \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and recall the Cartan decomposition

$$G = \coprod_{n\in\mathbf{N}} KZ\alpha^{-n}KZ;$$

then, for all $n \in \mathbf{N}$, the side classes in $KZ\alpha^{-n}KZ/KZ$ correspond to the vertices of the tree at distance n from the central vertex.

We set $I_0 \stackrel{\text{def}}{=} \{0\}$ and for $n \in \mathbf{N}_>$ we define the following subset of $\mathbf{Z}_p$:

$$I_n \stackrel{\text{def}}{=} \{\sum_{j=0}^{n-1} p^j[\mu_j] \quad \text{for}\, \mu_j \in \mathbf{F}_p\}.$$

Moreover for $n \in \mathbf{N}$, $\mu \in I_n$ we put

$$g^0_{n,\mu} \stackrel{\text{def}}{=} \begin{bmatrix} p^n & \mu \\ 0 & 1 \end{bmatrix}$$

$$g^1_{n,\mu} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ p\mu & p^{n+1} \end{bmatrix}.$$

We have then the following family of representatives for G/KZ:

$$G = \coprod_{n\in\mathbf{N},\,\mu\in I_n} g^0_{n,\mu}KZ \coprod \coprod_{n\in\mathbf{N},\,\mu\in I_n} g^1_{n,\mu}KZ; \tag{1}$$

more precisely, we have

$$KZ\alpha^{-n}KZ = \coprod_{\mu\in I_n} g^0_{n,\mu}KZ \coprod \coprod_{\mu\in I_{n-1}} g^1_{n,\mu}KZ$$

for $n \in \mathbf{N}_>$. Heuristically, the $g^0_{n,\mu}$'s correspond to the vertices at distance n from the central vertex, located in the "positive part" of the tree, while the $g^1_{n-1,\mu}$'s correspond to the vertices at distance n from the central vertex, located in the "negative" part of the tree.

Let σ be a smooth KZ-representation over $\overline{\mathbf{F}}_p$, V_σ the underlying $\overline{\mathbf{F}}_p$-vector space. The *induced representation from* σ, noted by

$$\mathrm{Ind}_{KZ}^G\sigma,$$

is defined as the $\overline{\mathbf{F}}_p$-vector space of functions $f : G \to V_\sigma$, compactly supported modulo Z and verifying the condition $f(\kappa g) = \sigma(\kappa)\cdot f(g)$ for any $\kappa \in KZ$, $g \in G$, this space being endowed with a left G-action defined by right translation of functions (i.e. $(g\cdot f)(t) \stackrel{\text{def}}{=} f(tg)$ for any $g, t \in G$). It turns out that $\mathrm{Ind}_{KZ}^G\sigma$ is again a smooth representation of G over $\overline{\mathbf{F}}_p$. For $g \in G$, $v \in V_\sigma$, we define the element $[g, v] \in \mathrm{Ind}_{KZ}^G\sigma$ as follow:

$$[g,v](t) \stackrel{\text{def}}{=} \sigma(tg)\cdot v \quad \text{if} \quad t \in KZg^{-1}$$

$$[g,v](t) \stackrel{\text{def}}{=} 0 \quad \text{if} \quad t \notin KZg^{-1}.$$

Then we have the equalities $g_1\cdot[g_2, v] = [g_1g_2, v]$ and $[g\kappa, v] = [g, \sigma(\kappa)\cdot v]$ for $g_1, g_2, g \in G$ and $\kappa \in KZ$. Moreover:

Proposition 2.1. *Let $\mathscr{B}$ an $\overline{\mathbf{F}}_p$-basis of V_σ, and $\mathscr{G}$ a system of representatives for the left side classes of G/KZ. Then, the family*

$$\mathscr{I} \stackrel{\text{def}}{=} \{[g,v],\, \text{for}\, g \in \mathscr{G},\, v \in \mathscr{B}\}$$

is an $\overline{\mathbf{F}}_p$-basis for the induced representation $\mathrm{Ind}_{KZ}^G\sigma$.

Proof: Omissis (cf. [BH], lemma 2.5 or [Bre], lemma 3.5). ♯

If $f \in \mathrm{Ind}_{KZ}^{G}\sigma$, the $\mathscr{T}-$*support* (or simply the support) of f is defined as the set of vertices gKZ of the tree $\mathscr{T}$ such that $f(g^{-1}) \neq 0$; this notion does not depend on the chosen representative g of the vertex gKZ. We define for $n \in \mathbf{N}$ the following subspace of $\mathrm{Ind}_{KZ}^{G}\sigma$:

$$W(n) \stackrel{\text{def}}{=} \{f \in \mathrm{Ind}_{KZ}^{G}\sigma, \quad \text{the support of } f \text{ is contained in } KZ\alpha^{-n}KZ\}.$$

We see (by Cartan decomposition) that the subspaces $W(n)$ are KZ-stable, for all $n \in \mathbf{N}$, and therefore

Lemma 2.2. *There is a natural KZ-equivariant isomorphism*

$$\mathrm{Ind}_{KZ}^{G}\sigma \xrightarrow{\sim} \bigoplus_{n\in\mathbf{N}} W(n).$$

Proof: Obvious.♯

Some Hecke Operators. The Hecke algebra for the induced representation from σ is defined by

$$\mathcal{H} \stackrel{\text{def}}{=} \mathrm{End}_G(\mathrm{Ind}_{KZ}^{G}\sigma).$$

It is an $\overline{\mathbf{F}}_p$ algebra; moreover it exists a canonical operator $T \in \mathcal{H}$ which induces an isomorphism of $\overline{\mathbf{F}}_p$-algebras

$$\mathcal{H} \xrightarrow{\sim} \overline{\mathbf{F}}_p[T]$$

(cf. [BL95], §3). If we specialise to the case $\sigma = \sigma_r$ for $0 \leqslant r \leqslant p-1$ we have the following explicit description of the Hecke operator T:

Lemma 2.3. *For $n \in \mathbf{N}_{>}$, $\mu \in I_n$ and $0 \leqslant j \leqslant r$ we have:*

$$T([g_{n,\mu}^{0}, X^{r-j}Y^{j}]) = \sum_{\mu_n\in\mathbf{F}_p} [g_{n+1,\mu+p^n[\mu_n]}^{0}, (-\mu_n)^{j}X^{r}] + [g_{n-1,[\mu]_{n-1}}^{0}, \delta_{j,r}(\mu_{n-1}X+Y)^{r}]$$
$$T([g_{n,\mu}^{1}, X^{r-j}Y^{j}]) = \sum_{\mu_n\in\mathbf{F}_p} [g_{n+1,\mu+p^n[\mu_n]}^{1}, (-\mu_n)^{r-j}Y^{r}] + [g_{n-1,[\mu]_{n-1}}^{0}, \delta_{j,0}(X+\mu_{n-1}Y)^{r}].$$

For $n = 0$, $0 \leqslant j \leqslant r$ we have

$$T([1_G, X^{r-j}Y^{j}]) = \sum_{\mu_0\in\mathbf{F}_p} [g_{1,[\mu_0]}^{0}, (-\mu_0)^{j}X^{r}] + [\alpha, \delta_{j,r}Y^{r}]$$
$$T([\alpha, X^{r-j}Y^{j}]) = \sum_{\mu_1\in\mathbf{F}_p} [g_{1,[\mu_1]}^{1}, (-\mu_1)^{r-j}Y^{r}] + [1_G, \delta_{j,0}X^{r}]$$

.

Proof: Cf. [Bre03a], §2.5 and lemme 3.1.1♯

We are going to fix the notations for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$: if $r \in \{0,\dots,p-1\}$ we write

$$\pi(r,0,1) \stackrel{\text{def}}{=} \mathrm{coker}(T : \mathrm{Ind}_{KZ}^{G}\sigma_r \to \mathrm{Ind}_{KZ}^{G}\sigma_r).$$

2.2 Induction of $B(\mathbf{F}_p)$-representations

For details and proofs we invite the reader to see §1 and §2 in Breuil and Paskunas's article [BP].

Let η be an $\overline{\mathbf{F}}_p$-character of the Borel subgroup $B(\mathbf{F}_p)$; it is by inflation a character of the Iwahori subgroup $K_0(p)$ of K and we have a natural isomorphism

$$\mathrm{Ind}_{K_0(p)}^{K}\eta \xrightarrow{\sim} \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\eta.$$

For $i \in \mathbf{N}$ we define the following $\overline{\mathbf{F}}_p$-characters of the Borel subgroup $B(\mathbf{F}_p)$:

$$\begin{aligned}\chi_i^s : B(\mathbf{F}_p) &\to \overline{\mathbf{F}}_p\\ \begin{bmatrix} a & b\\ 0 & d\end{bmatrix} &\mapsto d^i\end{aligned}$$

and

$$\begin{aligned}\mathfrak{a} : B(\mathbf{F}_p) &\to \overline{\mathbf{F}}_p\\ \begin{bmatrix} a & b\\ 0 & d\end{bmatrix} &\mapsto ad^{-1}.\end{aligned}$$

If e_η is an $\overline{\mathbf{F}}_p$-basis of η, the element $[1_K, e_\eta]$ is a K-generator of $\mathrm{Ind}_{K_0(p)}^{K}\eta$. The structure of the induced representations $\mathrm{Ind}_{K_0(p)}^{K}\eta$ is completely known, and the following proposition collects the main results which will be needed in the rest of the chapter. We introduce the following notation: for any $x \in \mathbf{Z}$, define $\lceil x\rceil \in \{1, \dots, p-1\}$ (resp. $\lfloor x\rfloor \in \{0, \dots, p-2\}$) by $x \equiv \lceil x\rceil \bmod p-1$ (resp. $x \equiv \lfloor x\rfloor \bmod p-1$).

Proposition 2.4. *Let $i, j \in \{0, \dots, p-1\}$, $\chi \overset{\mathrm{def}}{=} \chi_i^s\mathfrak{a}^j$. Then the induction $\mathrm{Ind}_{K_0(p)}^{K}\chi$ has length 2, with components:*

i) $\mathrm{Sym}^{\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes\det^j$*, which is isomorphic to the K-subrepresentation generated by* $\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, e_\chi]$*;*

ii) $\mathrm{Sym}^{p-1-\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes det^{i-j}$.

Moreover

i′) if $\chi \neq \chi^s$ the short exact sequence

$$0 \to \mathrm{Sym}^{\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes\det^j \to \mathrm{Ind}_{K_0(p)}^{K}\chi \to \mathrm{Sym}^{p-1-\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes det^{i-j} \to 0$$

is nonsplit;

ii′) if $\chi = \chi^s$ (i.e. $i-2j \equiv 0 \bmod [p-1]$) then $\mathrm{Ind}_{K_0(p)}^{K}\chi$ is semisimple and $\mathrm{Sym}^{p-1-\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes det^{i-j}$ (i.e. $\det^j$) is the K-subrepresentation of $\mathrm{Ind}_{K_0(p)}^{K}\chi$ generated by

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, e_\chi] + (-1)^j[1_K, e_\chi].$$

Proof: It is a well known result about representations of $\mathrm{GL}_2(\mathbf{F}_p)$ over $\overline{\mathbf{F}}_p$. See also [BP], lemmas 2.2, 2.6, 2.7 $\sharp$

The next lemma will play a crucial role in the sequel.

Lemma 2.5. *Let $0 \leqslant r \leqslant p-1$, $0 \leqslant t \leqslant p-2$ be integers, and consider the projection*

$$\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^t \overset{\pi}{\twoheadrightarrow} \mathrm{Sym}^{p-1-\lfloor r-2t\rfloor}\overline{\mathbf{F}}_p^2\otimes\det^{r-t}.$$

If $f \in \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^t$ *is such that*

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} f = a^{r-(t+1)} d^{t+1}$$

for any $a, d \in \mathbf{F}_p^\times$ *then* $\pi(f)$ *is of the following form:*

i) *if* $r - 2t \not\equiv 0, 1\,[p-1]$ *then* $\pi(f) = 0$*;*

ii) *if* $r - 2t \equiv 1\,[p-1]$ *then* $\pi(f) = X^{p-2}$*;*

iii) *if* $r - 2t \equiv 0\,[p-1]$ *then* $\pi(f) = X^{p-2}Y$*. More precisely, the image of* f *via the isomorphism*

$$\mathrm{Ind}_{K_0(p)}^K \det{}^t \xrightarrow{\sim} \det{}^t \oplus \mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2 \otimes \det{}^t$$

is $(0, X^{p-2}Y)$*.*

Proof: The H-eigencharacters of $\mathrm{Sym}^{p-1-\lfloor r-2t\rfloor}\overline{\mathbf{F}}_p^2 \otimes \det^{r-t}$ are

$$a^{p-1-(r-2t)+r-t-j} d^{r-t+j}$$

for $j \in \{0, \ldots, p-1-\lfloor r-2t\rfloor\}$, each of them corresponding respectively to the H-eigenvector $X^{p-1-\lfloor r-2t\rfloor-j}Y^j$. Therefore, the condition on $\pi(f)$ to be an H-eigencharacter gives

$$a^{t-j} d^{r-t+j} = a^{r-t-1} d^{t+1}$$

for a suitable $j \in \{0, \ldots, p-1-\lfloor r-2t\rfloor\}$ and for all $a, d \in \mathbf{F}_p^\times$; in other words

$$p - 1 - \lfloor r - 2t\rfloor \equiv j - 1\,[p-1]$$

for some $j \in \{0, \ldots, p-1-\lfloor r-2t\rfloor\}$. This is possible iff $j = 0$ and $r - 2t \equiv 1\,[p-1]$ or $j = 1$ and $r - 2t \equiv 0\,[p-1]$. $\sharp$

2.3 Computations on Witt vectors

In this section we are going to describe the p-adic expansion of some elements in $\mathbf{Z}_p$. The explicit description of lemma 2.6 and 2.7 is one of the key arguments to describe the socle filtration for the KZ-restriction of supersingular. The main reference for this section is [Ser63], Ch. II.

For $\lambda, \mu \in \mathbf{F}_p$ we define the following element of $\mathbf{F}_p$:

$$-P_\lambda(\mu) \stackrel{\mathrm{def}}{=} \sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} \lambda^{p-j}\mu^j.$$

Note that $P_\lambda(\mu)$ is a polynomial in μ, of degree $p-1$ and whose leading coefficient is $-\lambda$. We have the

Lemma 2.6. *Let* $\lambda, \mu \in \mathbf{F}_p$*. Then*

i) *the following equality holds in* $\mathbf{Z}_p$*:*

$$[\lambda] + [\mu] = [\lambda + \mu] + p[P_\lambda(\mu)] + p^2 t_{\lambda,\mu}$$

where $t_{\lambda,\mu} \in \mathbf{Z}_p$ *is a suitable* p*-adic integer depending only on* λ, μ*;*

ii) *the following equality holds in* $\mathbf{F}_p$

$$P_\lambda(\mu - \lambda) = -P_{-\lambda}(\mu).$$

Proof: Omissis.$\sharp$

We can use lemma 2.6 to deduce more general results.

LEMMA 2.7. *Let $\lambda \in \mathbf{F}_p$, $\sum_{j=0}^{n} p^j[\mu_j] \in I_{n+1}$. Then the following equality holds in $\mathbf{Z}_p/(p^{n+1})$:*

$$[\lambda] + \sum_{j=0}^{n} p^j[\mu_j] \equiv [\lambda + \mu_0] + p[\mu_1 + P_\lambda(\mu_0)] + \cdots + p^n[\mu_n + P_{\lambda,\ldots,\mu_{n-2}}(\mu_{n-1})]$$

where, for all $j = 1, \ldots, n-2$, the $P_{\lambda,\ldots,\mu_j}(X)$'s (resp. $P_{\lambda,\mu_0}(X)$, resp. $P_\lambda(X)$) are suitable polynomials in $\mathbf{F}_p[X]$, of degree $p-1$, depending only on $\lambda, \ldots, \mu_j$ (resp. on λ, μ_0, resp. on λ), and whose dominant coefficient is $-P_{\lambda,\ldots,\mu_{j-1}}(\mu_j)$ (resp. $-P_\lambda(\mu_0)$, resp. $-\lambda$).

Proof: It is an immediate induction using lemma 2.6-*i*). $\sharp$

LEMMA 2.8. *Let $\lambda \in \mathbf{F}_p$, $z \overset{\text{def}}{=} \sum_{j=1}^{n} p^j[\mu_j]$ and let $k \geqslant 0$. It exists a p-adic integer $z' = \sum_{j=1}^{n} p^j[\mu'_j] \in \mathbf{Z}_p$ such that*

$$z \equiv z'(1 + zp^k[\lambda]) \bmod p^{n+1}.$$

Furthermore, for $j = k+3, \ldots, n$ (resp. $j = k+2$, resp. $j \leqslant k+1$) we have the following equality in $\mathbf{F}_p$:

$$\mu_j = \mu'_j + \mu_{j-k-1}\mu'_1\lambda + \cdots + \mu_1\mu_{j-k-1}\lambda + S_{j-2}(\mu_{j-1})$$

(resp. $\mu_{k+2} = \mu'_{k+2} + \mu'_1\mu_1\lambda$ if $j = k+2$, resp. $\mu_j = \mu'_j$ if $j \leqslant k+1$) where $S_{j-2}(X) \in \mathbf{F}_p[X]$ is a polynomial of degree $p-1$, depending only on $\lambda, \ldots, \mu_{j-2}$ and leading coefficient $-s_{\lambda,\ldots,\mu_{j-2}} \overset{\text{def}}{=} \mu'_{j-1} - \mu_{j-1}$.

Proof: Exercise on Witt vectors.$\sharp$

To conclude this section we recall two elementary results which will be used in the rest of the chapter:

LEMMA 2.9. *i) For $0 \leqslant j \leqslant p-1$ we have the equality in $\mathbf{F}_p$:*

$$\sum_{\mu \in \mathbf{F}_p} \mu^j = -\delta_{j,p-1}.$$

ii) Let V be an $\overline{\mathbf{F}}_p$-vector space and let $v_0, \ldots, v_{p-1} \in V$ be any p-tuple of elements of V. The sub $\overline{\mathbf{F}}_p$-vector space of V generated by $\sum_{j=0}^{p-1} \mu^j v_j$ for μ varying in $\mathbf{F}_p$ coincide with the $\overline{\mathbf{F}}_p$-subvector space of V generated by the elements $v_0, \ldots, v_{p-1}$.

Proof: The assertions are both elementary; the second comes from the fact that the Vandermonde matrix

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 2^2 & \ldots & 2^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p-1 & (p-1)^2 & \ldots & (p-1)^{p-1} \end{bmatrix}$$

is invertible modulo p. $\sharp$

3. Reinterpret the KZ-restriction of supersingular representations: the KZ-representations R_n's

The goal of this section is to give a precise description of the KZ-restriction of supersingular representations $\pi(r, 0, 1)|_{KZ}$; the main result is then proposition 3.9, whose formulation is due to Breuil ([Bre], §4.2). To be more precise, the first step is to introduce, in §3.1, the K-representations R_n, from which we get an alternative description of the compact induction $\mathrm{Ind}_{KZ}^G\sigma$ (cf. proposition 3.5). Subsequently, we endow the R_n's with suitable "hecke" operators $T_n^{\pm}: R_n \to R_{n\pm 1}$ which let us define the amalgamed sums in (4); proposition 3.9 will then be a formal consequence.

3.1 Defining the K-representations R_n

For all $n \in \mathbf{N}$ we define the following subgroup of K:

$$K_0(p^n) \overset{\text{def}}{=} \{ \begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K, \text{ where } c \in \mathbf{Z}_p \}$$

(in particular, $K_0(p^0) = K$ and $K_0(p)$ is the Iwahori subgroup). For $0 \leqslant r \leqslant p-1$ and $n \in \mathbf{N}$ we define the following $K_0(p^n)$-representation σ_r^n over $\overline{\mathbf{F}}_p$: the associated $\overline{\mathbf{F}}_p$-vector space of σ_r^n is $\mathrm{Sym}^r \overline{\mathbf{F}}_p^2$, while the left action of $K_0(p^n)$ is given by

$$\sigma_r^n(\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix}) \cdot X^{r-j}Y^j \overset{\text{def}}{=} \sigma_r(\begin{bmatrix} d & c \\ p^n b & a \end{bmatrix}) \cdot X^{r-j}Y^j$$

for any $\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K_0(p^n)$, $0 \leqslant j \leqslant r$; in particular, σ_r^0 is isomorphic to σ_r. Finally, we define

$$R_r^n \overset{\text{def}}{=} \mathrm{Ind}_{K_0(p^n)}^K \sigma_r^n.$$

If r is clear from the context, we will write simply R_n instead of R_r^n.

In order to establish the relation between the R_r^n's and the compact induction $\mathrm{Ind}_{KZ}^G\sigma_r$ we need the following elementary lemma:

Lemma 3.1. *Fix $n \in \mathbf{N}$. Right translation by $\alpha^n w$ induces a bijection*

$$K/K_0(p^n) \xrightarrow{\sim} KZ\alpha^{-n}KZ/KZ.$$

Proof: Elementary, noticing that $(\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} KZ \begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix}) \cap K = K_0(p^n)$. ♯

For any $n \in \mathbf{N}_>$, $\mu \in I_n$ and $\mu' \in I_{n-1}$ we see that

$$g_{n,\mu}^0 = \begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix} \alpha^n w g_{n-1,\mu'}^1 w = \begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix} \alpha^n w$$

from which we deduce the following corollaries.

Corollary 3.2. *Let $n \in \mathbf{N}_>$. We have the following decomposition for K:*

$$K = \coprod_{\mu \in I_n} \begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix} K_0(p^n) \coprod \coprod_{\mu' \in I_{n-1}} \begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix} K_0(p^n).$$

Proof: Immediate from the decomposition given in (1). ♯

Corollary 3.3. *Let $0 \leqslant r \leqslant p-1$, $n \in \mathbf{N}_>$. The family*

$$\mathcal{R}_r^n \overset{\text{def}}{=} \{[\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j], [\begin{bmatrix} 1 & 0 \\ p\mu & 1 \end{bmatrix}, X^{r-j}Y^j] \text{ for } \mu \in I_n,\ \mu' \in I_{n-1}, 0 \leqslant j \leqslant r\}$$

is an $\overline{\mathbf{F}}_p$*-basis for the representations* R_n*. Moreover, the element* R_n

$$[1_{KZ}, Y^r] \in R_r^n$$

is a K*-generator for the representation* R_r^n*.*

Proof: Immediate from proposition 2.1 and corollary 3.2. ♯

The following result is the key to establish the relation between the compact induction $\mathrm{Ind}_{KZ}^G \sigma_r$ and the R_n's.

PROPOSITION 3.4. *Let* $0 \leqslant r \leqslant$, $n \in \mathbf{N}$ *and let* $W(n)$ *be the* KZ *subrepresentation of* $\mathrm{Ind}_{KZ}^G \sigma_r$ *defined in §2.1. We have a* KZ*-equivariant isomorphism*

$$\Phi_n : W(n) \xrightarrow{\sim} R_n$$

such that

$$\Phi_n([g^0_{n,\mu}, X^{r-j}Y^j]) = [\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]$$
$$\Phi_n([g^1_{n-1,\mu'}, X^{r-j}Y^j]) = [\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^{j}Y^{r-j}]$$

for $n > 0$ *and*

$$\Phi_0([1_G, X^{r-j}Y^j]) = X^jY^{r-j}$$

for $n = 0$.

Proof: We fix an index $n \geqslant 1$ (the case $n = 0$ is immediately verified). Thanks to proposition 2.1 it is clear that Φ_n is an $\overline{\mathbf{F}}_p$-linear isomorphism. Concerning the KZ-equivariance, we fix $\kappa \in K$, $l \in \mathbf{N}$ and, for $i \in \{0, 1\}$, $g^i_{n-i,\mu}$ and $\mu \in I_{n-i}$. Then $\kappa p^l g^i_{n-i,\mu} = g^{i(\kappa)}_{n-i(\kappa),\mu(\kappa)} \kappa_1 p^{l_1}$ for some $\kappa_1 \in K$, $l_1 \in \mathbf{N}$ while $i(\kappa) \in \{0, 1\}$ and $\mu(\kappa) \in I_{n-i(\kappa)}$ depend only on κ. If $g_{i,\mu}$ (resp. $g_{i(\kappa),\mu(\kappa)}$) is the representative of $K/K_0(p^n)$ corresponding to $g^i_{n-i,\mu}$ (resp. $g^{i(\kappa)}_{n-i(\kappa),\mu(\kappa)}$) via the bijection of lemma 3.1 we get:

$$\begin{cases} \kappa g_{i,\mu} = g_{i(\kappa),\mu(\kappa)} \kappa_2 \\ \kappa p^l g^i_{n-i,\mu} = g_{i(\kappa),\mu(\kappa)} \kappa_1 p^{l_1} \end{cases}$$

for some $\kappa_2 \in K_0(p^n)$ and since $g^i_{n-i,\mu} = g_{i,\mu} \begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} w^i$ (and similarly for $g^{i(\kappa)}_{n-i(\kappa),\mu(\kappa)}$, $g_{i(\kappa),\mu(\kappa)}$) we conclude

$$\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} \kappa_2 \begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} w^i = w^{i(\kappa)} \kappa_1 p^{n+l_1-l}.$$

We finally need the equality

$$\sigma_r(\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} \kappa_2 \begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix}) = \sigma_r^n(\kappa_2).$$

to see that

$$\Phi_n(\kappa p^l \cdot [g^i_{n,\mu}, v]) = \kappa \cdot \Phi_n([g_{i,n}, w \cdot v])$$

and the proof is complete. ♯

We deduce immediately the main result of this section:

COROLLARY 3.5. *Let $r \in \{0, \dots, p-1\}$. We have a KZ equivariant isomorphism*

$$\mathrm{Ind}_{KZ}^{G}\sigma_r \xrightarrow{\sim} \bigoplus_{n\in\mathbf{N}} R_r^n$$

3.2 Hecke operators on the R_n's, description of $\pi(r,0,1)|_{KZ}$

In this section we are going to define some "Hecke" operators T_n^+, T_n^- on the representations R_n's which allow us to give a description of the KZ-restriction of a supersingular representation $\pi(r,0,1)|_{KZ}$ in terms of the R_n, T_n^+, T_n^-. The main result will be proposition 3.9.

We start from the definition of the Hecke operators on the R_n's.

DEFINITION 3.6. *Let $n \in \mathbf{N}_>$. We define the $\overline{\mathbf{F}}_p$-linear morphism $T_n^+ : R_n \to R_{n+1}$ by the conditions*

$$T_n^+([\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]) \stackrel{\text{def}}{=} \sum_{\mu_n\in\mathbf{F}_p}[\begin{bmatrix} \mu+p^n[\mu_n] & 1 \\ 1 & 0 \end{bmatrix}, (-\mu_n)^j X^r]$$

$$T_n^+([\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^{j}Y^{r-j}]) \stackrel{\text{def}}{=} \sum_{\mu_n\in\mathbf{F}_p}[\begin{bmatrix} 1 & 0 \\ p(\mu'+[\mu_n]p^{n-1}) & 1 \end{bmatrix}, (-\mu_n)^{r-j} X^r]$$

for $\mu \in I_n$, $\mu' \in I_{n-1}$ and $0 \leqslant j \leqslant r$.
We define the $\overline{\mathbf{F}}_p$-linear morphism $T_0^+ : R_0 \to R_1$ by the condition:

$$T_0^+([1_K, X^{r-j}Y^j]) \stackrel{\text{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}[\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix}, (-\mu_0)^{r-j} X^r] + [1_K, \delta_{j,0}X^r]$$

for $0 \leqslant j \leqslant r$.

Identifying R_n with $W(n)$ via the isomorphism described in proposition 3.4 and using the results of §2.1 we see that

$$T_n^+([g,v]) = T([g,v]) \cap W(n+1) \tag{2}$$

for all $g \in KZ\alpha^{-n}KZ$, $v \in \sigma_r$.

Similarly, we have

DEFINITION 3.7. *Let $n \in \mathbf{N}$, $n \geqslant 2$. We define the $\overline{\mathbf{F}}_p$-linear morphism $T_n^- : R_n \to R_{n-1}$ by the conditions:*

$$T_n^-([\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]) \stackrel{\text{def}}{=} [\begin{bmatrix} [\mu]_{n-1} & 1 \\ 1 & 0 \end{bmatrix}, \delta_{j,r}(\mu_{n-1}X+Y)^r]$$

$$T_n^-([\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^{j}Y^{r-j}]) \stackrel{\text{def}}{=} [\begin{bmatrix} 1 & 0 \\ p[\mu']_{n-2} & 1 \end{bmatrix}, \delta_{j,0}(\mu_{n-2}X+Y)^r]$$

for $\mu \in I_n$, $\mu' \in I_{n-1}$ and $0 \leqslant j \leqslant r$.
For $n = 1$ we define $T_1^- : R_1 \to R_0$ by the conditions:

$$T_1^-([\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]) \stackrel{\text{def}}{=} \delta_{j,r}(X+\mu_0 Y)^r$$

$$T_1^-([1_K, X^{j}Y^{r-j}]) \stackrel{\text{def}}{=} \delta_{j,0}Y^r.$$

for $\mu_0 \in \mathbf{F}_p$, $0 \leqslant j \leqslant r$.

Again, identifying R_n with $W(n)$ via the isomorphism described in proposition 3.4 and using the results of §2.1 we see

$$T_n^-([g,v]) = T([g,v]) \cap W(n-1) \tag{3}$$

for all $g \in KZ\alpha^{-n}KZ$, $v \in \sigma_r$ and $n \in \mathbf{N}_>$.

Thanks to the isomorphism of proposition 3.4, we deduce the following properties of the Hecke operators $T_n^\pm$:

LEMMA 3.8. *The operators $T_n^\pm$ enjoy the following properties:*

1) *for all $n \in \mathbf{N}_>$, the morphisms is T_n^+, T_n^- are K-equivariant; for $n = 0$, the morphism T_0^+ is K-equivariant;*
2) *for all $n \geqslant 0$ the morphism T_n^+ is injective;*
3) *for all $n \geqslant 1$ the morphism T_n^- is surjective.*

Proof: *i*). We recall that the KZ-action on the tree preserves the distances from the central vertex. The assertion is then clear from the KZ-equivariance of T and the equalities (2), (3).

ii) and *iii*). We recall that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r & r^2 & \dots & r^r \end{bmatrix}$$

is invertible modulo p. This implies, for any fixed $i \in \{0,1\}$, the following facts:

-) by support reasons the condition $T_n^+([g_{i,\mu}, v]) = 0$ forces $v = 0$ for any choice $\mu \in I_{n-i}$;
-) if $n \geqslant 1+i$ and $\mu \in I_{n-1-i}$ the $\overline{\mathbf{F}}_p$-subvector space of R_{n-1} generated by $T_n^-([g_{i,p^i\mu+p^{n-1}[\mu_{n-1}]}, Y^r])$ for $\mu_{n-1} \in \mathbf{F}_p$ coincide with the $\overline{\mathbf{F}}_p$-subvector space of R_{n-1} generated by $[g_{i,p^i\mu}, X^{r-j}Y^j]$ for $j \in \{0, \dots, r\}$.

This ends the proof. ♯

From now onwards we will consider R_n as a K-subrepresentation of R_{n+1} via the monomorphism T_n^+, for any $n \in \mathbf{N}$, without further comment.

We can use the Hecke operators $T_n^\pm$ in order to construct a sequence of amalgamed sums of the R_n's. We define $R_0 \oplus_{R_1} R_2$ as the amalgamed sum

$$\begin{tikzcd} R_1 \arrow[r, hook, "T_1^+"] \arrow[d, "-T_1^-"] & R_2 \arrow[d, dotted, "pr_2"] \\ R_0 \arrow[r, dotted] & R_0 \oplus_{R_1} R_2 \end{tikzcd}$$

where the second projection pr_2 is epi by base change. For any odd integer $n \in \mathbf{N}_>$ we define inductively the amalgamed sum $R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$ as:

$$\begin{tikzcd} R_n \arrow[r, hook, "T_n^+"] \arrow[d, "-pr_{n-1}\circ T_n^-"'] & R_{n+1} \arrow[d, dotted, two heads, "pr_{n+1}"] \\ R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, dotted] & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}; \end{tikzcd} \tag{4}$$

once again, the second projection pr_{n+1} is epi by base change.

For any even positive integer $m \in \mathbf{N}_>$ we define the amalgamed sum $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1}$ in the evident similar way.

We are now ready to state the main result of this section

Proposition 3.9. *Let $0 \leqslant r \leqslant p-1$. We have a KZ equivariant isomorphism*

$$\pi(r,0,1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{m\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1}).$$

Proof: We have the following commutative diagram, with KZ-equivariant arrows:

$$\begin{tikzcd} (\mathrm{Ind}_{KZ}^G \sigma_r)|_{KZ} \arrow[r, "T|_{KZ}"] \arrow[d, "\wr"] & (\mathrm{Ind}_{KZ}^G \sigma_r)|_{KZ} \arrow[d, "\wr"] \\ \bigoplus_{n\in\mathbf{N}} R_n \arrow[r, "T_0^+ + \sum_{n\geqslant 1}(T_n^+ + T_n^-)"] & \bigoplus_{n\in\mathbf{N}} R_n; \end{tikzcd}$$

as the restriction functor is exact, we deduce that the isomorphism of corollary 3.5 induces an isomorphism $\pi(r,0,1)|_{KZ} \cong \mathrm{coker}(T_0^+ + \sum_{n\geqslant 1}(T_n^+ + T_n^-))$. We dispose of the evident inductive systems:

$$\left\{\sum_{j=1,\,j\,\mathrm{odd}}^{n} T_j^+ + T_j^- : \bigoplus_{j=1,\,j\,odd}^{n} R_j \to \bigoplus_{i=0,\,i\,even}^{n+1} R_i\right\}_{n\in\mathbf{N},\,n\,\mathrm{odd}}$$

$$\left\{T_0^+ + \sum_{j=1,\,j\,\mathrm{even}}^{n} T_j^+ + T_j^- : \bigoplus_{j=0,\,j\,even}^{n} R_j \to \bigoplus_{i=0,\,i\,odd}^{n+1} R_i\right\}_{n\in\mathbf{N},\,n\,\mathrm{even}}$$

so that, by the right exactness of the functor $\varinjlim$, the isomorphism of corollary 3.5 gives

$$\pi(r,0,1)|_{KZ} \cong \varinjlim_{n,\,\mathrm{odd}} \left(\mathrm{coker}\big(\sum_{j=1,\,j\,\mathrm{odd}} T_j^+ + T_j^-\big)\right) \oplus \varinjlim_{n,\,\mathrm{even}} \left(\mathrm{coker}\big(T_0^+ + \sum_{j=1,\,j\,\mathrm{even}}^{n} T_j^+ + T_j^-\big)\right).$$

It follows finally from the definitions of the amalgamed sum (and an immediate induction) that

$$\varinjlim_{n,\,\mathrm{odd}} (\mathrm{coker}(\sum_{j=1,\,j\,\mathrm{odd}} T_j^+ + T_j^-)) = R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$$

$$\varinjlim_{n,\,\mathrm{even}} (\mathrm{coker}(T_0^+ + \sum_{j=1,\,j\,\mathrm{even}}^{n} T_j^+ + T_j^-)) = R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$$

and the proof is complete. $\sharp$

4. Defining the filtrations on the spaces R_n, $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$

In this section, we fix once for all an integer $r \in \{0, \dots, p-1\}$. Our aim is to to point out, in definition 4.3, a filtration on $\varinjlim\limits_{n\,\text{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ (resp. $\varinjlim\limits_{n\,\text{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$) which will let us describe explicitly the socle filtration for the KZ-restriction the supersingular representation $\pi(r, 0, 1)|_{KZ}$.

Proposition 4.1. *For any odd integer $n \in \mathbf{N}_>$ we have a natural commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d, two heads, "-pr_{n-1}\circ T_n^-"] & R_{n+1} \arrow[r] \arrow[d, two heads] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r] & R_{n+1}/R_n \arrow[r] & 0
\end{tikzcd}$$

with exact lines. We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N} \setminus \{0\}}.$$

Proof: The proof is by induction. We dispose of the commutative diagram:

$$\begin{tikzcd}
R_n \arrow[r, hook, "T_n^+"] \arrow[d, "-pr_{n-1}\circ T_n^-"] & R_{n+1} \arrow[d, dotted, two heads, "pr_{n+1}"] \\
R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, dotted] & R_0 \oplus_{R_1} \oplus_{R_n} R_{n+1}
\end{tikzcd}$$

where the morphism $-pr_{n-1} \circ T_n^-$ is epi by the inductive hypothesis; it follows then from the universal property of the amalgamed sum that the morphism pr_{n+1} is epi too. Moreover, since the forgetful functor $For : \mathrm{Rep}_K \to \mathrm{Vect}_{\overline{\mathbf{F}}_p}$ is right exact we deduce, by the injectivity of T_n^+ and base change in the category $\mathrm{Vect}_{\overline{\mathbf{F}}_p}$ that the morphism $R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \to R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ is injective too.

From the universal property of the amalgamed sum we get the natural commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r] \arrow[d, two heads] & R_{n+1} \arrow[r] \arrow[d, two heads] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\
& R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, hook] \arrow[rr, bend right, "0"'] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r, dotted, "\exists !"] & R_{n+1}/R_n &
\end{tikzcd}$$

where the first line is exact. The exactness of the second line is then an immediate diagram chase. $\sharp$

From the proof of proposition 4.1 we see that we have actually a much stronger result: if $0 \leqslant j \leqslant n-2$ is odd and Q_{j+1} is any quotient of R_{j+1} we can still define the amalgamed sums $Q_{j+1} \oplus_{R_{j+2}} \cdots \oplus_{R_n} R_{n+1}$ as in 4; then

Corollary 4.2. *Let $0 \leqslant j \leqslant n-2$ be odd, Q_{j+1} be a quotient of R_{j+1}. We have a natural* *commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d] & R_{n+1} \arrow[r] \arrow[d] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q_{j+1} \oplus_{R_{j+2}} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & Q_{j+1} \oplus_{R_{j+2}} \cdots \oplus_{R_n} R_{n+1} \arrow[r] & R_{n+1}/R_n \arrow[r] & 0
\end{tikzcd}$$

with exact lines (and with the obvious convention $Q_{j+1} \oplus_{R_j} R_{j+1} \overset{\text{def}}{=} Q_{j+1}$*).*

We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N}\setminus\{0\}}.$$

For each $n \in \mathbf{N}$ we look at a natural filtration on R_{n+1}. The definition is the following:

Definition 4.3. *Let* $n \in \mathbf{N}$, $0 \leqslant t \leqslant r$. *We define* $\mathrm{Fil}^t(R_{n+1})$ *as the* K*-subrepresentation of* R_{n+1} *generated by* $[1_K, X^{r-t}Y^t]$. *For* $t = -1$, *we define* $\mathrm{Fil}^{-1}(R_{n+1}) \overset{\text{def}}{=} 0$.

We note that

Lemma 4.4. *Let* $n \in \mathbf{N}$. *The family*

$$\{\mathrm{Fil}^t(R_{n+1})\}_{t=-1}^{t=r}$$

defines a separated and exhaustive decreasing filtration on R_{n+1}. *Moreover, for each* $t \in \{0, \ldots, r\}$, *the family*

$$\mathscr{B}_{n+1,t} \overset{\text{def}}{=} \left\{ [\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-i}Y^i],\ [\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^{r-i}Y^i],\ \mu \in I_{n+1},\ \mu' \in I_n,\ 0 \leqslant i \leqslant t \right\}$$

is an $\overline{\mathbf{F}}_p$ *basis for* $\mathrm{Fil}^t(R_{n+1})$; *in particular* $\mathrm{Fil}^t(R_{n+1})$ *has dimension* $(p+1)p^n(t+1)$ *over* $\overline{\mathbf{F}}_p$.

Proof: It is immediate from corollary 3.3 and the definition of the σ_r^{n+1}'s. $\sharp$

By Frobenius reciprocity, we have an explicit description of the graded pieces of the filtration defined in 4.3:

Lemma 4.5. *Let* $n \in \mathbf{N}$, *and fix* $-1 \leqslant t \leqslant r$. *Then, we have a* K*-equivariant isomorphism:*

$$\mathrm{Fil}^t(R_{n+1})/\mathrm{Fil}^{t-1}(R_{n+1}) \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t.$$

where the characters χ_r^s, $\mathfrak{a}$, *defined in §2.2, are seen as characters on* $K_0(p^{n+1})$ *by inflation* $K_0(p^{n+1}) \twoheadrightarrow B(\mathbf{F}_p)$.

Proof: As (the image of) the element $[1_K, X^{r-t}Y^t]$ is a K-generator of the graded piece $\mathrm{Fil}^t(R_{n+1})/\mathrm{Fil}^{t-1}(R_{n+1})$, and $K_0(p^{n+1})$ acts on it by the character $\chi_r^s\mathfrak{a}^t$ we deduce by Frobenius reciprocity a K-equivariant epimorphism:

$$\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t \twoheadrightarrow \mathrm{Fil}^t(R_{n+1})/\mathrm{Fil}^{t-1}(R_{n+1}).$$

As the two spaces have the same $\overline{\mathbf{F}}_p$-dimension, the latter is indeed an isomorphism. $\sharp$

We then see that the first step to understand the nature of $\pi(r, 0, 1)|_{KZ}$ consists in the study of the induced representations $\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t$ for $n \in \mathbf{N}$, $0 \leqslant t \leqslant r$; such a study will be the object of the following two sections (§5, §6).

5. Study of an Induction-I

In this section, we will fix two integers $1 \leqslant m \leqslant n+1$ and η a character of $B(\mathbf{F}_p)$ (which will be considered as a continuous character of $K_0(p^{n+1})$ by inflation), and we will fix a basis $\{e_\eta\}$ for η. The object of this section is then (cf. proposition 5.10) to describe explicitly the socle filtration for

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta$$

and the proof will be essentially an induction on the length $n+1-m$ (§5.1, §5.2).

For $1 \leqslant m \leqslant n+1$ define a subset I_{n+1}/I_m of $\mathbf{Z}_p$:

$$I_{n+1}/I_m \stackrel{\mathrm{def}}{=} \{\sum_{j=m}^{n} p^j[\mu_j],\ \mu_j \in \mathbf{F}_p\}.$$

We have the following elementary lemmas.

Lemma 5.1. *For $1 \leqslant m \leqslant n+1$ we have the decomposition*

$$K_0(p^m)/K_0(p^{n+1}) = \coprod_{x \in I_{n+1}/I_m} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} K_0(p^{n+1}).$$

In particular, the family

$$\mathscr{I}_{m,n+1} \stackrel{\mathrm{def}}{=} \{[\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, e_\eta],\ x \in I_{n+1}/I_m\}$$

is an $\overline{\mathbf{F}}_p$-basis for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta$ and $\dim_{\overline{\mathbf{F}}_p}\left(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta\right) = p^{n+1-m}$.

Proof: Immediate from corollary 3.3. $\sharp$

Lemma 5.2. *Let $1 \leqslant m \leqslant n+1$ be integers and η a character of $B(\mathbf{F}_p)$. Then we have a $K_0(p^m)$-equivariant canonical isomorphism:*

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta \xrightarrow{\sim} (\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1) \otimes \eta$$

where η is seen (by inflation) as a character of $K_0(p^{n+1})$ and $K_0(p^m)$ in the left hand side and in the right hand side respectively.

Proof: The assignment, for $x \in I_{n+1}/I_m$,

$$[\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, e_\eta] \mapsto [\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, e_1] \otimes e_\eta$$

defines an $\overline{\mathbf{F}}_p$-isomorphism which is actually $K_0(p^m)$-equivariant, as $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \in K_1$ for all $x \in I_{n+1}/I_m$. $\sharp$

In particular, by lemma 5.2, we can assume $\eta = 1$.

5.1 The case $m = n$

We establish here the first step concerning the inductive description of the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$; fix once for all an $\overline{\mathbf{F}}_p$-basis $\{e\}$ for the underlying vector space of the trivial character 1. We introduce the objects:

Definition 5.3. *Let $n \in \mathbf{N}_>$ and $0 \leqslant l_n \leqslant p-1$. Then:*

i) we define the following element of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)} 1$:

$$F_{l_n}^{(n)} \stackrel{\text{def}}{=} \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} [\begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix}, e];$$

we define formally $F_{-1}^{(n)}, F_p^{(n)} \stackrel{\text{def}}{=} 0$;

ii) we define the following quotient of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)} 1$:

$$Q_{l_n}^{(n,n+1)} \stackrel{\text{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)} 1 / \langle F_0^{(n)}, \dots, F_{l_n-1}^{(n)} \rangle_{\overline{\mathbf{F}}_p};$$

we define formally $Q_p^{(n,n+1)} \stackrel{\text{def}}{=} 0$.

For any $0 \leqslant l_n, l'_n \leqslant p-1$ we will often commit the abuse to use the same notation for $F_{l_n}^{(n)}$ and its image in the quotient $Q_{l'_n}^{(n,n+1)}$. The meaning will be clear according to the contest.

The next computation is the main tool to describe the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)} 1$.

Lemma 5.4. *Let $g \in K_0(p^{n+1})$, $\lambda \in \mathbf{F}_p$ and $0 \leqslant l_n \leqslant p-1$. Then we have the equalities in* $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)} 1$:

i) $g \cdot F_{l_n}^{(n)} = \mathfrak{a}^{l_n}(g) F_{l_n}^{(n)}$;

ii) $\begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} F_{l_n}^{(n)} = \sum_{j=0}^{l_n} \binom{l_n}{j} (-\lambda)^j F_{l_n-j}^{(n)}$.

Proof: *i*). If $g = \begin{bmatrix} a & b \\ p^{n+1}c & d \end{bmatrix}$, then we can write

$$g \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p^n[\mu_n \overline{a}^{-1} \overline{d}] & 1 \end{bmatrix} \begin{bmatrix} a' & b \\ p^{n+1}c' & d' \end{bmatrix}$$

where $a', c', d' \in \mathbf{Z}_p$ and $a' \equiv a\,[p]$, $d' \equiv d\,[p]$. Thus,

$$gF_{l_n}^{(n)} = \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} [\begin{bmatrix} 1 & 0 \\ p^n[\mu_n \overline{a}^{-1} \overline{d}] & 1 \end{bmatrix}, e] = (\overline{a}\overline{d}^{-1})^{l_n} F_{l_n}^{(n)}.$$

Since $[\lambda] + [\mu_n] \equiv [\lambda + \mu_n]$ modulo p, we deduce

$$\begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} F_{l_n}^{(n)} = \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} [\begin{bmatrix} 1 & 0 \\ p^n[\mu_n + \lambda] & 1 \end{bmatrix}, e].$$

The result follows. $\sharp$

As a consequence, we get the corollaries:

Corollary 5.5. *For any $0 \leqslant l_n \leqslant p-1$, the sub-$K_0(p^n)$ representation of $Q_l^{(n,n+1)}$ generated by $F_{l_n}^{(n)}$ is isomorphic to $\mathfrak{a}^{l_n}$.*

Proof: For any $g \in K_0(p^n)$ we can write $g = \begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} \kappa$ with suitables elements $\lambda \in \mathbf{F}_p$, $\kappa \in K_0(p^{n+1})$ (lemma 5.1). The result comes from lemma 5.4 and the definition of $Q_{l_n}^{(n,n+1)}$. $\sharp$

Corollary 5.6. *For any $0 \leqslant l_n \leqslant p-1$ we have $K_0(p^n)$-equivariant exact sequence*

$$0 \to \langle F_{l_n}^{(n)} \rangle \to Q_{l_n}^{(n,n+1)} \to Q_{l_n+1}^{(n,n+1)} \to 0$$

which is nonsplit if $l_n \leqslant p-2$. Moreover,

$$\dim_{\overline{\mathbf{F}}_p}(Q_{l_n}^{(n,n+1)}) = p - l_n.$$

Proof: The exact sequence is clear. Furthermore, if $\phi : Q_{l_n}^{(n,n+1)} \to \langle F_{l_n}^{(n)} \rangle$ is any $K_0(p^n)$-equivariant morphism, we see that

$$\phi(F_{l_n}^{(n)}) = \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} \phi([1_{K_0(p^n)}, e]) = \phi([1_{K_0(p^n)}, e]) \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n}.$$

Thus, there cannot be any $K_0(p^n)$ equivariant sections for $\langle F_{l_n}^{(n)} \rangle \to Q_{l_n}^{(n,n+1)}$ if $0 \leqslant l_n \leqslant p-2$. The assertion concerning the dimension is immediate by induction.♯

Corollary 5.7. *Let $0 \leqslant l_n \leqslant p-1$. Then the socle of $Q_{l_n}^{(n,n+1)}$ is given by:*

$$\mathrm{soc}(Q_{l_n}^{(n,n+1)}) = \langle F_{l_n}^{(n)} \rangle.$$

Proof: We have $Q_{p-1}^{(n,n+1)} \cong \langle F_{p-1}^{(n)} \rangle$, as the two spaces are 1-dimensional. By a decreasing induction, assume $\mathrm{soc}(Q_{l_n+1}^{(n,n+1)}) = \langle F_{l_n+1}^{(n)} \rangle$ for $l_n \leqslant p-2$ and consider the exact sequence

$$0 \to \langle F_{l_n}^{(n)} \rangle \to Q_{l_n}^{(n,n+1)} \to Q_{l_n+1}^{(n,n+1)} \to 0.$$

If τ is an irreducible $K_0(p^n)$-subrepresentation of $Q_{l_n}^{(n,n+1)}$ such that $\tau \cap \langle F_{l_n}^{(n)} \rangle = 0$, we deduce that $F_{l_n+1}^{(n)} + c_1 F_{l_n}^{(n)} \in \tau$ for a suitable $c_1 \in \overline{\mathbf{F}}_p$. From the equality

$$\begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} (F_{l_n+1}^{(n)} + c_1 F_{l_n}^{(n)}) = F_{l_n+1}^{(n)} - (l_n+1)(\lambda) F_{l_n}^{(n)} + c_1 F_{l_n}^{(n)}$$

in $Q_{l_n}^{(n,n+1)}$ (where $\lambda \in \mathbf{F}_p^\times$), we find $F_{l_n}^{(n)} \in \tau$, contradiction. ♯

5.2 The general case

Fix two integers $1 \leqslant m \leqslant n+1$. In this section we establish the inductive step which let us describe the socle filtration for the representation $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$. We recall the following result:

Proposition 5.8. *Let $1 \leqslant m \leqslant n+1$. For any $m \leqslant j \leqslant n+1$ we have a canonical isomorphism:*

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^j)}^{K_0(p^m)} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^j)} 1.$$

For any two $(n+1-m)$-tuple $(j_m, \ldots, j_n)$, $(l_m, \ldots, l_n) \in \{0, \ldots, p-1\}^{n-m+1}$ we define inductively

$$(j_m, \ldots, j_n) \prec (l_m, \ldots, l_n)$$

if either $(j_{m+1}, \ldots, j_n) \prec (l_{m+1}, \ldots, l_n)$ or $(j_{m+1}, \ldots, j_n) = (l_{m+1}, \ldots, l_n)$ and $j_m < l_m$. We can therefore introduce the objects:

Definition 5.9. *Let $(l_m, \ldots, l_n) \in \{0, \ldots, p-1\}^{n-m+1}$ be an $(n+1-m)$-tuple. Then:*

i) *we define inductively the following element of* $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$:

$$F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)} \overset{\text{def}}{=} \sum_{\mu_m \in \mathbf{F}_p} \mu_m^{l_m} \begin{bmatrix} 1 & 0 \\ p^m[\mu_m] & 1 \end{bmatrix} [1_{K_0(p^m)}, F_{l_{m+1}}^{(m+1)} * \cdots * F_{l_n}^{(n)}]$$

where we adopt the convention $F_{l_m+1}^{(m)} * \cdots * F_{l_n}^{(n)} \overset{\text{def}}{=} F_0^{(m)} * F_{l_{m+1}+1}^{(m+1)} * \cdots * F_{l_n}^{(n)}$ *if* $l_m = p-1$.

ii) *We define the following quotient of* $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$:

$$Q_{l_m,\dots,l_n}^{(m,n+1)} \overset{\text{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 / \langle F_{j_m}^{(m)} * \cdots * F_{j_n}^{(n)} \text{ for } (j_m, \dots, j_n) \prec (l_m, \dots, l_n) \rangle_{\overline{\mathbf{F}}_p}$$

where we adopt the convention $Q_{l_m+1,\dots,l_n}^{(m,n+1)} \overset{\text{def}}{=} Q_{0,l_{m+1}+1,\dots,l_n}^{(m,n+1)}$ *if* $l_m = p-1$.

We give here the statement of the main result.

Proposition 5.10. *Let* $1 \leqslant m \leqslant n+1$ *be integers, and* $(l_m, \dots, l_n) \in \{0, \dots, p-1\}^{n-m+1}$ *a* $(n-m+1)$*-tuple. Then*

i) *The* $K_0(p^m)$*-subrepresentation of* $Q_{l_m,\dots,l_n}^{(m,n+1)}$ *generated by* $F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}$ *is isomorphic to* $\mathfrak{a}^{l_m} \otimes \cdots \otimes \mathfrak{a}^{l_n}$;

ii) *we have a* $K_0(p^m)$*-equivariant exact sequence:*

$$0 \to \langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)} \rangle \to Q_{l_m,\dots,l_n}^{(m,n+1)} \to Q_{l_m+1,\dots,l_n}^{(m,n+1)} \to 0 \tag{5}$$

which is nonsplit if $(l_m, \dots, l_n) \neq (p-1, \dots, p-1)$. *Moreover*

$$Q_{0,l_{m+1},\dots,l_n}^{(m,n+1)} = \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q_{l_{m+1},\dots,l_n}^{(m+1,n+1)}$$

and

$$\dim_{\overline{\mathbf{F}}_p}(Q_{l_m,\dots,l_n}^{(m,n+1)}) = p^{n-m+1} - \sum_{j=0}^{n-m} p^{n-m-j} l_{n-j}.$$

iii) *The socle of* $Q_{l_m,\dots,l_n}^{(m,n+1)}$ *is given by*

$$\mathrm{soc}(Q_{l_m,\dots,l_n}^{(m,n+1)}) = \langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)} \rangle.$$

As we said, the proof is an induction on the length $n+1-m$, the case $m = n$ being proved in the previous section; in what follows, we will therefore assume proposition 5.10 for any length l with $l < n+1-m$. We first need the following tools.

Lemma 5.11. *Let* $(l_m, \dots, l_n) \in \{0, \dots, p-1\}^{n-m+1}$ *be an* $(n-m+1)$*-tuple. The following diagrams are commutative with exact lines*

i)

$$\begin{tikzcd}
0 \arrow[r] & \langle F_{l_m}^{(m)} * \cdots * F_{l_{n-1}}^{(n-1)} \rangle \otimes \mathfrak{a}^{l_n} \arrow[r] \arrow[d, "\wr"] & Q_{l_m,\dots,l_{n-1}}^{(m,n)} \otimes \mathfrak{a}^{l_n} \arrow[r] \arrow[d, hook] & Q_{l_m+1,\dots,l_{n-1}}^{(m,n)} \otimes \mathfrak{a}^{l_n} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r] & Q_{l_m,\dots,l_n}^{(m,n+1)} \arrow[r] & Q_{l_m+1,\dots,l_n}^{(m,n+1)} \arrow[r] & 0;
\end{tikzcd}$$

ii)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} F_{l_{m+1}}^{(m+1)} * \cdots * F_{l_n}^{(n)} \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q_{l_{m+1},\dots,l_n}^{(m+1,n+1)} \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q_{l_{m+1}+1,\dots,l_n}^{(m+1,n+1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q_{l_m}^{(m,m+1)} \otimes \mathfrak{a}^{l_{m+1}} \otimes \cdots \otimes \mathfrak{a}^{l_n} \arrow[r] & Q_{l_m,\dots,l_n}^{(m,n+1)} \arrow[r] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q_{l_{m+1}+1,\dots,l_n}^{(m+1,n+1)} \arrow[r] & 0.
\end{tikzcd}$$

Proof: The proof will be an induction on the $(n+1-m)$-tuple $(l_m,\dots,l_n) \in \{0,\dots,p-1\}^{n+1-m}$.

i) From corollary 5.6 and the exactness of the induction functor we dispose of the following exact sequence for any $0 \leqslant l_n \leqslant p-1$:

$$0 \to \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} \langle F_{l_n}^{(n)} \rangle \to \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} Q_{l_n}^{(n,n+1)} \to \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} Q_{l_n+1}^{(n,n+1)} \to 0$$

and $\langle F_{l_n}^{(n)} \rangle \cong \mathfrak{a}^{l_n}$. We assume, inductively, to have the commutative diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} 1 \otimes \mathfrak{a}^{l_n} \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} Q_{l_n}^{(n,n+1)} \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} Q_{l_n+1}^{(n,n+1)} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q_{l_m,\dots,l_{n-1}}^{(m,n)} \otimes \mathfrak{a}^{l_n} \arrow[r] & Q_{l_m,\dots,l_n}^{(m,n+1)} \arrow[r] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} Q_{l_n+1}^{(n,n+1)} \arrow[r] & 0.
\end{tikzcd}$$

We can invoke proposition 5.10 for $\mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} 1 \otimes \mathfrak{a}^{l_n}$ deducing the diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\langle F_{l_m}^{(m)} * \cdots * F_{l_{n-1}}^{(n-1)} \rangle \otimes \mathfrak{a}^{l_n} \arrow[r, hook] \arrow[d, "\wr"] & Q_{l_m,\dots,l_{n-1}}^{(m,n)} \otimes \mathfrak{a}^{l_n} \arrow[d] \\
\langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r, hook] \arrow[d] & Q_{l_m,\dots,l_n}^{(m,n+1)} \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} Q_{l_n+1}^{(n,n+1)} \arrow[d] \\
0 & 0
\end{tikzcd}$$

and we are left to use the snake lemma to conclude the induction (notice that if $(l_m,\dots,l_{n-1}) = (p-1,\dots,p-1)$ we just deduce the isomorphism $\mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)} Q_{l_n+1}^{(n,n+1)} \cong Q_{0,\dots,0,l_n+1}^{(m,n+1)}$).

ii). It is similar to *i)*. The details are left to the reader. ♯

Lemma 5.12. *Fix two integers $1 \leqslant m \leqslant n+1$, let $(l_m,\dots,l_n) \in \{0,\dots,p-1\}^{n-m+1}$ be an $(n-m+1)$-tuple and assume $(l_m,\dots,l_n) \prec (p-1,\dots,p-1)$. Moreover, let $\lambda \in \mathbf{F}_p$ and $t = \sum_{j\in\mathbf{N}} p^j[t_j] \in \mathbf{Z}_p$ be a p-adic integer.*

Then, the action of $\begin{bmatrix} 1 & 0 \\ p^m[\lambda] + p^{m+1}t & 1 \end{bmatrix}$ *on* $F_{l_{m+1}}^{(m)} * F_{l_{m+1}}^{(m+1)} * \cdots * F_{l_n}^{n}$ *inside* $Q_{l_m,\dots,l_n}^{(m,n+1)}$ *is described* *by*

$$\begin{bmatrix} 1 & 0 \\ p^m[\lambda]+p^{m+1}t & 1 \end{bmatrix} \cdot F^{(m)}_{l_m+1} * \cdots * F^{(n)}_{l_n} =$$
$$= F^{(m)}_{l_m+1} * \cdots * F^{(n)}_{l_n} + (l_j+1)(-1)^{j-m+1}\lambda F^{(m)}_{l_m} * \cdots * F^{(n)}_{l_n}$$

where $j \in \{m, \ldots, n\}$ is minimal with respect to the property that $l_j + 1 \not\equiv 0 \bmod p$.

Proof: The case $m = n$ is an immediate computation, and it is left to the reader. In order to establish the general step, we need to distinguish two situation:

Situation A). Assume $l_m \leqslant p-2$. It follows from proposition 5.10 applied to $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^{m+1})}1$ that $\begin{bmatrix} 1 & 0 \\ p^{m+1}\mathbf{Z}_p & 1 \end{bmatrix}$ acts trivially on $F^{(m+1)}_{l_{m+1}} * \cdots * F^{(n)}_{l_n}$ in $Q^{(m+1,n+1)}_{l_{m+1},\ldots,l_n}$, and we deduce the following equalities in $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1},\ldots,l_n}$:

$$\begin{bmatrix} 1 & 0 \\ p^m[\lambda]+p^{m+1}t & 1 \end{bmatrix} \sum_{\mu_m \in \mathbf{F}_p} \mu_m^{l_m+1} \begin{bmatrix} 1 & 0 \\ p^m[\mu_m] & 1 \end{bmatrix} [1, F^{(m+1)}_{l_{m+1}} * \cdots * F^{(n)}_{l_n}] =$$
$$= \sum_{\mu_m \in \mathbf{F}_p} \mu_m^{l_m} \begin{bmatrix} 1 & 0 \\ p^m[\lambda+\mu_m] & 1 \end{bmatrix} [1, F^{(m+1)}_{l_{m+1}} * \cdots * F^{(n)}_{l_n}] =$$
$$= \sum_{j=0}^{l_m+1} \binom{l_m+1}{j} (-\lambda)^j [1, F^{(m+1)}_{l_{m+1}} * \cdots * F^{(n)}_{l_n}].$$

We conclude using the projection $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1},\ldots,l_n} \twoheadrightarrow Q^{(m,n+1)}_{l_m,\ldots,l_n}$.

Situation B). Assume $l_m = p-1$; therefore $F^{(m)}_{l_m+1} * \cdots * F^{(n)}_{l_n} = F^{(m)}_0 * F^{(m+1)}_{l_{m+1}+1} * \cdots * F^{(n)}_{l_n}$. Lemma 2.6 and the inductive hypothesis applied to $F^{(m+1)}_{l_{m+1}+1} * \cdots * F^{(n)}_{l_n} \in Q^{(m+1,n+1)}_{l_{m+1},\ldots,l_n}$ let us deduce the following equalities inside $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1},\ldots,l_n}$:

$$\begin{bmatrix} 1 & 0 \\ p^m[\lambda]+p^{m+1}t & 1 \end{bmatrix} \sum_{\mu_m \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^m[\mu_m] & 1 \end{bmatrix} [1, F^{(m+1)}_{l_{m+1}+1} * \cdots * F^{(n)}_{l_n}] =$$
$$= \sum_{\mu_m \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^m[\mu_m+\lambda] & 1 \end{bmatrix} [1, F^{(m+1)}_{l_{m+1}+1} * \cdots * F^{(n)}_{l_n}] +$$
$$+(l_j+1)(-1)^{j-m} \sum_{\mu_m \in \mathbf{F}_p} (P_\lambda(\mu_m)+t_0) \begin{bmatrix} 1 & 0 \\ p^m[\lambda+\mu_m] & 1 \end{bmatrix} [1, F^{(m+1)}_{l_{m+1}+1} * \cdots * F^{(n)}_{l_n}]$$
$$= F^{(m)}_{l_m+1} * \cdots * F^{(n)}_{l_n} + (l_j+1)(-1)^{j-m}(t_0 F^{(m)}_0 * F^{(m+1)}_{l_{m+1}} * \cdots * F^{(n)}_{l_n} +$$
$$+ \sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} (-\lambda)^{p-j} F^{(m)}_j * F^{(m+1)}_{l_{m+1}} * \cdots * F^{(n)}_{l_n})$$

where $j \in \{m+1, \ldots, n\}$ is minimal with respect to the property that $l_j < p-1$. The conclusion comes using the projection $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1},\ldots,l_n} \twoheadrightarrow Q^{(m,n+1)}_{l_m,\ldots,l_n}$. ♯

We are now able to deduce easily proposition 5.10.

Proof of proposition 5.10:

i) From lemma 5.11-*i*) we have an isomorphism $\langle F^{(m)}_{l_m} * \cdots * F^{(n-1)}_{l_{n-1}} \rangle \otimes \mathfrak{a}^{l_n} \xrightarrow{\sim} \langle F^{(m)}_{l_m} * \cdots * F^{(n)}_{l_n} \rangle$

and we have $\langle F_{l_m}^{(m)} * \cdots * F_{l_{n-1}}^{(n-1)}\rangle \cong \mathfrak{a}^{l_m} \otimes \cdots \otimes \mathfrak{a}^{l_{n-1}}$ by the inductive hypothesis.

ii) As in corollary 5.6, we see that for any $K_0(p^m)$-equivariant morphism $\phi : Q_{l_m,\dots,l_n}^{(m,n+1)} \rightarrow \langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}\rangle$ we have

$$\phi(F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}) = (-\delta_{p-1,l_m}) \dots (-\delta_{p-1,l_n})\phi([1_{K_0(p^n)}, e])$$

so that there cannot be any splitting for $\langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}\rangle \rightarrow Q_{l_m,\dots,l_n}^{(m,n+1)}$ if $(l_m,\dots,l_n) \prec (p-1,\dots,p-1)$. The identity

$$\dim_{\overline{\mathbf{F}}_p}(Q_{l_m,\dots,l_n}^{(m,n+1)}) = p^{n-m+1} - \sum_{j=0}^{n-m} p^{n-m-j} l_{n-j}$$

is now an immediate induction.

iii) The case $(l_m,\dots,l_n) = (p-1,\dots,p-1)$ is trivial. We will prove the general case by a descending induction on the $(n+1-m)$-tuple $(l_m,\dots,l_n)$. Consider the exact sequence

$$0 \rightarrow \langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}\rangle \rightarrow Q_{l_m,\dots,l_n}^{(m,n+1)} \rightarrow Q_{l_m+1,\dots,l_n}^{(m,n+1)} \rightarrow 0$$

and let $\tau \leqslant Q_{l_m,\dots,l_n}^{(m,n+1)}$ be an irreducible subrepresentation such that $\tau \cap \langle F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}\rangle = 0$. The inductive hypothesis $\mathrm{soc}(Q_{l_m+1,\dots,l_n}^{(m,n+1)}) = \langle F_{l_m+1}^{(m)} * \cdots * F_{l_n}^{(n)}\rangle$ let us conclude that

$$\tau = \langle F_{l_m+1}^{(m)} * \cdots * F_{l_n}^{(n)} + c_1 F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}\rangle \cong \mathfrak{a}^{l_m+1} \otimes \dots \mathfrak{a}^{l_n}$$

for a suitable $c_1 \in \overline{\mathbf{F}}_p$. But by lemma 5.12 we have the equalities in $Q_{l_m,\dots,l_n}^{(m,n+1)}$:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ p^m[\lambda] & 1 \end{bmatrix} (F_{l_m+1}^{(m)} * \cdots * F_{l_n}^{(n)} + c_1 F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}) &= \\ = (F_{l_m+1}^{(m)} * \cdots * F_{l_n}^{(n)} + c_1 F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)}) &+ \\ +\lambda(l_j+1)(-1)^{j-m+1} F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)} & \end{aligned}$$

(where $j \in \{m,\dots,n\}$ is defined as in lemma 5.12) from which $F_{l_m}^{(m)} * \cdots * F_{l_n}^{(n)} \in \tau$ if $\lambda \neq 0$, contradiction. $\sharp$.

6. Study of an Induction -II

Throughout this section we consider integers r, t with $0 \leqslant r \leqslant p-1$, $0 \leqslant t \leqslant p-2$ and $n \in \mathbf{N}_>$. Our aim is to describe the socle filtration of the induction

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^t$$

using the result of section §5; the main result is then proposition 6.6.

We start by fixing the following elements of $\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^t$.

Definition 6.1. *Let $(l_1,\dots,l_n) \in \{0,\dots,p-1\}^n$ be an n-tuple, and let $t' \overset{\text{def}}{=} \sum_{i=1}^{n} l_i$. We define*

$$F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \overset{\text{def}}{=} \begin{cases} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1_K, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \\ \qquad \text{if } r - 2(t+t') \not\equiv 0\,[p-1]; \\ \\ \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1_K, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] + (-1)^{t+t'}[1_K, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \\ \qquad \text{if } r - 2(t+t') \equiv 0\,[p-1] \end{cases}$$

$$F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \overset{\text{def}}{=} \begin{cases} [1_K, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \\ \qquad \text{if } r - 2(t+t') \not\equiv 0\,[p-1]; \\ \\ \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1_K, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \\ \qquad \text{if } r - 2(t+t') \equiv 0\,[p-1]. \end{cases}$$

If $(j_1, \ldots, j_n)$, $(j'_1, \ldots, j'_n) \in \{0, \ldots, p-1\}^n$ are two n-tuples and $i, i' \in \{0,1\}$ we define

$$(i, j_1, \ldots, j_n) \prec (i', j'_1, \ldots, j'_n)$$

iff either $(j_1, \ldots, j_n) \prec (j'_1, \ldots, j'_n)$ or $(j_1, \ldots, j_n) = (j'_1, \ldots, j'_n)$ and $i < i'$. Finally

Definition 6.2. *Let $(l_1, \ldots, l_n) \in \{0, \ldots, p-1\}^n$ be an n-tuple, $i \in \{0,1\}$ and let $t' \overset{\text{def}}{=} \sum_{j=1}^n l_j$. We define the following quotient of $\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t$*

$$Q_{i,l_1,\ldots,l_n}^{(0,n+1)} \overset{\text{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t / (\langle K \cdot F_j^{(0)} * \ldots F_{j_n}^{(n)} \rangle, \text{ for } (j, j_0, \ldots, j_n) \prec (i, l_1, \ldots, l_n)).$$

As usual, we adopt the convention

$$Q_{i+1,l_1,\ldots,l_n}^{(0,n+1)} \overset{\text{def}}{=} Q_{0,l_1+1,\ldots,l_n}^{(0,n+1)}$$

if $i = 1$. We remark that in the previous definitions we do not keep track of the integers r, t: we adopted this choice in order not to overload the notations. We believe the values of r, t will be clear from the context (cf. §7, §8).

The study of the socle filtration start from the following elementary lemma

Lemma 6.3. *If $(l_1, \ldots, l_n) \in \{0, \ldots, p-1\}^n$ is an n-tuple, we have the following commutative diagrams with exact rows:*

i)

$$\begin{tikzcd} 0 \arrow[r] & \langle K \cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r] \arrow[d, equal] & \mathrm{Ind}_{K_0(p)}^K \langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r] \arrow[d, two heads] & \langle K \cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \langle K \cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r] & Q_{0,l_1,\ldots,l_n}^{(0,n+1)} \arrow[r] & Q_{1,l_1,\ldots,l_n}^{(0,n+1)} \arrow[r] & 0; \end{tikzcd}$$

ii)

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ind}_{K_0(p)}^K \langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r] \arrow[d, two heads] & Q_{0,l_1,\ldots,l_n}^{(0,n+1)} \arrow[r] \arrow[d, two heads] & Q_{0,l_1+1,\ldots,l_n}^{(0,n+1)} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \langle K \cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r] & Q_{1,l_1,\ldots,l_n}^{(0,n+1)} \arrow[r] & Q_{0,l_1+1,\ldots,l_n}^{(0,n+1)} \arrow[r] & 0. \end{tikzcd}$$

Proof: It is an induction on the n-tuple $(l_1, \ldots, l_n)$. By proposition 5.10 and the exactness of the induction functor we have the exact sequence

$$0 \to \mathrm{Ind}_{K_0(p)}^K \langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \to \mathrm{Ind}_{K_0(p)}^K Q_{l_1,\ldots,l_n}^{(1,n+1)} \to \mathrm{Ind}_{K_0(p)}^K Q_{l_1+1,\ldots,l_n}^{(1,n+1)} \to 0$$

and we dispose of the exact sequence (cf. lemma 2.4)

$$0 \to \langle K \cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \to \mathrm{Ind}_{K_0(p)}^K \langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \to \langle K \cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \to 0.$$

The conclusion comes applying the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\langle K \cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r, hook] \arrow[d, equal] & \mathrm{Ind}_{K_0(p)}^K \langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[d] \\
\langle K \cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \arrow[r, hook] \arrow[d] & \mathrm{Ind}_{K_0(p)}^K Q_{l_1,\ldots,l_n}^{(1,n+1)} \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p)}^K Q_{l_1+1,\ldots,l_n}^{(1,n+1)} \arrow[d] \\
0 & 0
\end{tikzcd}$$

assuming inductively that $\mathrm{Ind}_{K_0(p)}^K Q_{l_1,\ldots,l_n}^{(1,n+1)} = Q_{0,l_1,\ldots,l_n}^{(0,n+1)}$. ♯

We deduce the following two corollaries:

Corollary 6.4. *Let $(l_1, \ldots, l_n) \in \{0, \ldots, p-1\}^n$ be an n-tuple. Then:*

i) the K-subrepresentation of $Q_{0,l_1,\ldots,l_n}^{(0,n+1)}$ generated by $F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ is isomorphic to

$$\begin{aligned} \langle K F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle &\xrightarrow{\sim} \mathrm{Sym}^{\lfloor r-2(t+t') \rfloor} \overline{\mathbf{F}}_p^2 \otimes \det^{t+t'} \\ F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} &\mapsto X^{\lfloor r-2(t+t') \rfloor}. \end{aligned}$$

If, moreover, $r - 2(t+t') \equiv 0[p-1]$, then the K-subrepresentation of $Q_{0,l_1,\ldots,l_n}^{(0,n+1)}$ generated by $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ is isomorphic to

$$\begin{aligned} \langle K F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle &\xrightarrow{\sim} \mathrm{Sym}^{p-1} \overline{\mathbf{F}}_p^2 \otimes \det^{t+t'} \\ F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} &\mapsto X^{p-1}. \end{aligned}$$

ii) The K-subrepresentation of $Q_{1,l_1,\ldots,l_n}^{(0,n+1)}$ generated by $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ is isomorphic to

$$\begin{aligned} \langle K F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle &\xrightarrow{\sim} \mathrm{Sym}^{p-1-\lfloor r-2(t+t') \rfloor} \overline{\mathbf{F}}_p^2 \otimes \det^{r-(t+t')} \\ F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} &\mapsto X^{p-1-\lfloor r-2(t+t') \rfloor}. \end{aligned}$$

Proof: As $\langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle \cong \chi_r^s \mathfrak{a}^{t+t'}$ the statement is an immediate consequence of lemma 6.3 and proposition 2.4. ♯

Corollary 6.5. *Let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple. Then:*

i) If $(l_1,\dots,l_n)\neq(p-1,\dots,p-1)$ the exact sequences:

$$0\to\langle KF_0^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle\to Q^{(0,n+1)}_{0,l_1,\dots,l_n}\to Q^{(0,n+1)}_{1,l_1,\dots,l_n}\to 0;$$
$$0\to\langle KF_1^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle\to Q^{(0,n+1)}_{1,l_1,\dots,l_n}\to Q^{(0,n+1)}_{0,l_1+1,\dots,l_n}\to 0$$

are non split.

ii) If $(l_1,\dots,l_n)=(p-1,\dots,p-1)$ the exact sequence

$$0\to\langle KF_0^{(0)}*F_{p-1}^{(1)}*\cdots*F_{p-1}^{(n)}\rangle\to Q^{(0,n+1)}_{0,p-1,\dots,p-1}\to Q^{(0,n+1)}_{1,p-1,\dots,p-1}\to 0$$

is nonsplit iff $r-2t\equiv 0[p-1]$.

iii) The dimension of the quotients $Q^{(0,n+1)}_{i,l_1,\dots,l_n}$ for $i\in\{0,1\}$ is:

$$\dim_{\overline{\mathbf{F}}_p}(Q^{(0,n+1)}_{0,l_1,\dots,l_n})=(p+1)p^n-(p+1)(\sum_{j=1}^n p^{j-1}l_j)$$
$$\dim_{\overline{\mathbf{F}}_p}(Q^{(0,n+1)}_{1,l_1,\dots,l_n})=(p+1)p^n-(p+1)(\sum_{j=1}^n p^{j-1}l_j)-(\lfloor r-2(t+t')\rfloor+1).$$

Proof: *i) and ii)* As the action of K_1 on $\langle K\cdot F_i^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle$ is trivial (for $i\in\{0,1\}$), we deduce as in proposition 5.10-*ii*) that

$$\phi(F_i^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)})=0$$

for any K-equivariant morphism $Q^{(0,n+1)}_{i,l_1,\dots,l_n}\to\langle K\cdot F_i^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle$ and for any $(n+1)$-tuple $(i,l_1,\dots,l_n)\in\{0,1\}\times\{0,\dots,p-1\}^n$ such that $(l_1,\dots,l_n)\prec(p-1,\dots,p-1)$. The assertion *ii*) is then immediate from proposition 2.4.

The proof on *iii*) is finally an obvious induction. ♯

6.2 Study of the socle filtration

The present section is devoted to the proof of the following result:

Proposition 6.6. *Assume p is odd; let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple, and let $t'\overset{def}{=}\sum_{i=1}^n l_i$. Then*

i) the socle of $Q^{(0,n+1)}_{1,l_1,\dots,l_n}$ is described by

$$\mathrm{soc}(Q^{(0,n+1)}_{1,l_1,\dots,l_n})=\langle KF_1^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle$$

ii) the socle of $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$ is described by

$$\mathrm{soc}(Q^{(0,n+1)}_{0,l_1,\dots,l_n})=\begin{cases}\langle KF_0^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle\text{ if } r-2(t+t')\not\equiv 0[p-1];\\[2ex] \langle KF_0^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle\oplus\langle KF_1^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle\\ \text{if } r-2(t+t')\equiv 0[p-1].\end{cases}$$

The proof is a descending induction on the n-tuple $(l_1,\dots,l_n)$, the statement being clear if $(l_1,\dots,l_n)=(p-1,\dots,p-1)$.

We prove the result for a fixed n-tuple $(l_1,\dots,l_n)$, assuming it true for $Q^{(0,n+1)}_{0,l_1+1,\dots,l_n}$ (resp. for $Q^{(0,n+1)}_{1,l_1,\dots,l_n}$).

Study of $\mathrm{soc}(Q^{(0,n+1)}_{1,l_1+1,\dots,l_n})$. We dispose of the following commutative diagram with exact lines (cf. lemma 6.3):

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_{K_0(p)}^{K}\langle F^{(1)}_{l_1}*\cdots*F^{(n)}_{l_n}\rangle \arrow[r] \arrow[d, "pr_1"] & Q^{(0,n+1)}_{0,l_1,\dots,l_n} \arrow[r] \arrow[d, "pr_2"] & Q^{(0,n+1)}_{0,l_1+1,\dots,l_n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \langle K\cdot F^{(0)}_1 * F^{(1)}_{l_1}*\cdots*F^{(n)}_{l_n}\rangle \arrow[r] & Q^{(0,n+1)}_{1,l_1,\dots,l_n} \arrow[r] & Q^{(0,n+1)}_{0,l_1+1,\dots,l_n} \arrow[r] & 0.
\end{tikzcd}$$

We define the elements of $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$:

$$x \overset{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, F^{(1)}_{l_1+1}*\cdots*F^{(n)}_{l_n}]$$
$$x' \overset{\mathrm{def}}{=} [1_K, F^{(1)}_{l_1+1}*\cdots*F^{(n)}_{l_n}]$$
$$y \overset{\mathrm{def}}{=} x+(-1)^{t+t'+1}x';$$

the behaviour of the elements x, x' in $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$ is the object of the next

LEMMA 6.7. *We have the following equalities in* $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$ *for* p *odd* [2]*:*

i) *if* $a,d\in\mathbf{F}_p^\times$ *then*

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x = a^{r-(t+t'+1)}d^{t+t'+1}x;$$
$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x' = a^{t+t'+1}d^{r-(t+t'+1)}x'.$$

ii) *Let* $j\in\{1,\dots,n\}$ *be minimal with respect to the property that* $l_j\leqslant p-2$ *and let* $\lambda\in\mathbf{F}_p$*. Then*

$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x = x+(l_j+1)(-1)^j\sum_{\mu_0\in\mathbf{F}_p}-P_{-\lambda}(\mu_0)\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, F^{(1)}_{l_1}*\cdots*F^{(n)}_{l_n}];$$
$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x' = x'+(l_j+1)(-1)^j\delta_{p,3}(1-\delta_{1,j})\lambda[1_K, F^{(1)}_{l_1}*\cdots*F^{(n)}_{l_n}].$$

Proof: *i)* Follows easily from the definition of the elements x, x' and the equalities

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}\begin{bmatrix}z & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}z[ad^{-1}] & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}[d] & 0\\ 0 & [a]\end{bmatrix}$$
$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}\begin{bmatrix}1 & 0\\ z & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ z[a^{-1}d] & 1\end{bmatrix}\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}$$

for $z\in\mathbf{Z}_p$, $a,d\in\mathbf{F}_p^\times$

[2]this is required only for the equality concerning x' in *ii)*

ii) The first equality is immediately deduced from lemma 5.12 and the relation:

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix}\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} [\lambda+\mu_0] & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ p[P_\lambda(\mu_0)]+p^2h & 1 \end{bmatrix}$$

for $\lambda, \mu_0 \in \mathbf{F}_p$ and $h \in \mathbf{Z}_p$ a suitable p-adic integer.

The second equality is more delicate. From lemma 2.8 we deduce

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ p[\mu_1]+\cdots+p^n[\mu_n] & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p[\mu_1']+\cdots+p^n[\mu_n'] & 1 \end{bmatrix}\begin{bmatrix} 1+pt_1 & [\lambda] \\ p^{n+1}t_2 & 1-pt_3 \end{bmatrix}$$

where $t_1, t_2, t_3 \in \mathbf{Z}_p$ are suitable p-adic integers and, for $i \geqslant 3$ we have

$$\mu_i = \mu_i' + \mu_{i-1}'\mu_1\lambda + \cdots + \mu_1'\mu_{i-1}\lambda + S_{i-2}(\mu_{i-1})$$

where $S_{i-2} \in \mathbf{F}_p[X]$ is a polynomial of degree $p-1$ and leading coefficient $-s_{i-2} \overset{\text{def}}{=} \mu_{i-1}' - \mu_{i-1}$, while, for $i \in \{1,2\}$ we have

$$\mu_2 = \mu_2' + \mu_1\mu_1'\lambda, \mu_1 = \mu_1'.$$

If $j \in \{1, \ldots, n\}$ is as in the statement we can write

$$F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} = F_0^{(1)} * \cdots * F_0^{(j-1)} * F_{l_j+1}^{(j)} * \cdots * F_{l_n}^{(n)}.$$

(with the obvious convention if $j = 1$) and a direct computation in $\mathrm{Ind}_{K_0(p^n)}^{K_0(p^1)}\chi_r^s\mathfrak{a}^t$ gives:

$$\begin{aligned} v &\overset{\text{def}}{=} \begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} F_{l_1+1}^{(1)} * \cdots * F_{l_n}^{(n)} \\ &= \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \cdots \sum_{\mu_{j-1} \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{j-1}[\mu_{j-1}'] & 1 \end{bmatrix} \sum_{\mu_j \in \mathbf{F}_p} \mu_j^{l_j+1} \begin{bmatrix} 1 & 0 \\ p^j[\mu_j'] & 1 \end{bmatrix} \cdots \\ &\quad \cdots \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n'] & 1 \end{bmatrix} [1, e]. \end{aligned}$$

If $j < n$ we can now use the recursive property of the s_{i-1}'s for $i = j, \ldots, n-1$ and project v successively via the epimorphisms

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_r^s\mathfrak{a}^t \twoheadrightarrow \mathrm{Ind}_{K_0(p^n)}^{K_0(p)}Q_{l_n}^{(n,n+1)} \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{Ind}_{K_0(p^{j+1})}^{K_0(p)}Q_{l_{j+1},\ldots,l_n}^{(j+1,n+1)}$$

and we see that v is sent to the following element $\widetilde{v}$ of $\mathrm{Ind}_{K_0(p^{j+1})}^{K_0(p)}Q_{l_{j+1},\ldots,l_n}^{(j+1,n+1)}$ (with the convention that if $j = n$, we just have $v = \widetilde{v}$ and $Q_{l_{j+1},\ldots,l_n}^{(j+1,n+1)} \overset{\text{def}}{=} \chi_r^s\mathfrak{a}^t$):

$$\begin{aligned} \widetilde{v} &= \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \cdots \\ &\quad \cdots \sum_{\mu_{j-1} \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{j-1}[\mu_{j-1}'] & 1 \end{bmatrix} \sum_{\mu_j \in \mathbf{F}_p} (\mu_j' + s_{j-1})^{l_j+1} \begin{bmatrix} 1 & 0 \\ p^j[\mu_j'] & 1 \end{bmatrix} \sum_{\mu_{j+1} \in \mathbf{F}_p} \mu_{j+1}^{l_{j+1}} \begin{bmatrix} 1 & 0 \\ p^{j+1}[\mu_{j+1}] & 1 \end{bmatrix} \cdots \\ &\quad \cdots \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1, e]. \end{aligned}$$

This let us deduce the statement if $j = 1$, while, if $j \geqslant 2$ we map $\widetilde{v}$ in $\mathrm{Ind}_{K_0(p^j)}^{K_0(p)}Q_{l_j,\ldots,l_n}^{(j,n+1)}$ via the epimorphism $\mathrm{Ind}_{K_0(p^{j+1})}^{K_0(p)} Q_{l_{j+1},\dots,l_n}^{(j+1,n+1)} \twoheadrightarrow \mathrm{Ind}_{K_0(p^j)}^{K_0(p)} Q_{l_j,\dots,l_n}^{(j,n+1)}$ to get:

$$F_{l_1+1}^{(1)} * \cdots * F_{l_n}^{(n)} + (l_j+1) \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \cdots$$
$$\cdots \sum_{\mu_{j-1} \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{j-1}[\mu_{j-1}'] & 1 \end{bmatrix} s_{j-1} \sum_{\mu_j \in \mathbf{F}_p} \mu_j^{l_j} \begin{bmatrix} 1 & 0 \\ p^j[\mu_j] & 1 \end{bmatrix} \cdots$$
$$\cdots \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1, e].$$

We use again the recursive property of the s_{i-1}'s for $i = 2, \dots, j$ and the chain of epimorphisms

$$\mathrm{Ind}_{K_0(p^j)}^{K_0(p)} Q_{l_j,\dots,l_n}^{(j,n+1)} \twoheadrightarrow \mathrm{Ind}_{K_0(p^{j-1})}^{K_0(p)} Q_{p-1,\dots,l_n}^{(j-1,n+1)} \twoheadrightarrow \cdots \twoheadrightarrow Q_{l_1,\dots,l_n}^{(1,n+1)}$$

to see that the image of v in $Q_{l_1,\dots,l_n}^{(1,n+1)}$ is

$$F_{l_1+1}^{(1)} * \cdots * F_{l_n}^{(n)} + (l_j+1)(-1)^j \lambda \delta_{p,3} F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}.$$

This let us conclude the proof. $\sharp$

Let τ be an irreducible K-subrepresentation of $Q_{1,l_1,\dots,l_n}^{(0,n+1)}$ such that $\tau \cap \langle K \cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \rangle = 0$. Therefore the natural projection $Q_{1,l_1,\dots,l_n}^{(0,n+1)} \twoheadrightarrow Q_{0,l_1+1,\dots,l_n}^{(0,n+1)}$ induces an isomorphism on τ onto an irreducible summand of $\mathrm{soc}(Q_{0,l_1+1,\dots,l_n}^{(0,n+1)})$. We distinguish the situations:

A) the subrepresentation τ maps isomorphically in the K-subrepresentation of $Q_{0,l_1+1,\dots,l_n}^{(0,n+1)}$ generated by (the image of) x.

B) We have $r - 2(t+t'+1) \equiv 0\,[p-1]$ and the subrepresentation τ maps isomorphically in the K-subrepresentation of $Q_{0,l_1+1,\dots,l_n}^{(0,n+1)}$ generated by (the image of) y.

Study of case *A*. Let $f \in \mathrm{Ind}_{K_0(p)}^{K} F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ be such that $pr_2(x+f) \in \tau$. The induced isomorphism $\tau \xrightarrow{\sim} \langle Kx \rangle$ and the behaviour of x in $\mathrm{soc}(Q_{0,l_1+1,\dots,l_n}^{(0,n+1)})$ let us deduce the necessary conditions:

1) for all $a, d \in \mathbf{F}_p^\times$,

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (x+f) - a^{r-(t+t'+1)} d^{t+t'+1} (x+f) \in \ker(pr_2);$$

2) for all $\lambda \in \mathbf{F}_p$

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (f+x) - (f+x) \in \ker(pr_2).$$

Condition 1) and lemma 6.7-i) give $\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} f - a^{r-(t+t'+1)}d^{t+t'+1}f \in \ker pr_1$ so that, by lemma 2.5, we deduce

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} pr_1(f) - pr_1(f) = \begin{cases} 0 \text{ if } r-2(t+t') \not\equiv 0\,[p-1] \\ \\ c_1\lambda \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \\ \text{if } r-2(t+t') \equiv 0\,[p-1] \end{cases}$$

for some $c_1 \in \overline{\mathbf{F}}_p$. Thus, condition 2) and lemma 6.7-ii) let us conclude that

$$(l_j+1)(-1)^j \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p}(-\lambda)^{p-i} \sum_{\mu_0\in\mathbf{F}_p} \mu_0^i \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \in \ker pr_1$$
$$c_1\delta_{0,r-2(t+t')}\lambda \sum_{\mu_0\in\mathbf{F}_p} \mu_0^i \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \in \ker pr_1$$

for any $\lambda \in \mathbf{F}_p$, and by lemma 2.9-ii) we can deduce

$$\sum_{\mu_0\in\mathbf{F}_p} \mu_0^{p-1} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \in \ker pr_1$$
$$\sum_{\mu_0\in\mathbf{F}_p} \mu_0 \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \in \ker pr_1.$$

Both conditions are absurds, for the case $r-2(t+t') \not\equiv 0\,[p-1]$ and $r-2(t+t') \equiv 0\,[p-1]$ respectively. ♯

Study of case B***.*** Let $f \in \mathrm{Ind}_{K_0(p)}^K F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ be such that $pr_2(y+f) \in \tau$. The induced isomorphism $\tau \xrightarrow{\sim} \langle Ky\rangle$ and the behaviour of y in $\mathrm{soc}(Q_{0,l_1+1,\dots,l_n}^{(0,n+1)})$ let us deduce the necessary conditions:

1) for all $a, d \in \mathbf{F}_p^{\times}$,

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (y+f) - (ad)^{t+t'+1}(y+f) \in \ker(pr_2);$$

2) for all $\lambda \in \mathbf{F}_p$

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (f+y) - (f+y) \in \ker(pr_2).$$

We deduce from condition 1) and lemma 6.7-i) that $pr_1(f)$ is an H-eigenvector for $\langle K \cdot F_1^0 * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle$ with associated eigencharacter $a^{r-(t+t'+1)}d^{t+t'+1}$. Thus, by lemma 2.9, we have

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} pr_1(f) = \begin{cases} 0 \text{ if } r-2(t+t') \not\equiv 0\,[p-1] \text{ i.e. } p \neq 3 \\ \\ c_1\lambda \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \\ \text{if } r-2(t+t') \equiv 0\,[p-1] \text{ i.e. } p = 3. \end{cases}$$

for some $c_1 \in \overline{\mathbf{F}}_p$. The conclusion follows again from lemma 6.7-ii), similarly to case A). ♯

Study of $\mathrm{soc}(Q^{(0,n+1)}_{0,l_1,\dots,l_n})$**.** We have the following commutative diagram with exact lines (cf. lemma 6.3):

$$\begin{tikzcd}
0 \arrow[r] & \langle K\cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \arrow[r] \arrow[d, equal] & \mathrm{Ind}_{K_0(p)}^K \langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \arrow[r] \arrow[d, hook] & \langle K\cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \langle K\cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \arrow[r] & Q^{(0,n+1)}_{0,l_1,\dots,l_n} \arrow[r] & Q^{(0,n+1)}_{1,l_1,\dots,l_n} \arrow[r] & 0;
\end{tikzcd}$$

Let τ be an irreducible K-subrepresentation of $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$ and assume

$$\tau \cap \mathrm{Ind}_{K_0(p)}^K \langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle = 0.$$

In particular, the natural projection $Q^{(0,n+1)}_{0,l_1,\dots,l_n} \twoheadrightarrow Q^{(0,n+1)}_{1,l_1,\dots,l_n}$ induces an isomorphism $\tau \xrightarrow{\sim} \mathrm{soc}(Q^{(0,n+1)}_{1,l_1,\dots,l_n})$. By the inductive hypothesys, we deduce that it exists $f \in \langle K\cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle$ such that

$$f + F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \in \tau$$

is a K-generator of τ, contradiction.

7. Socle filtration for the spaces R_n

In this section we will use the results of §6 to give an exhaustive description of the socle filtration for the R_n's, for any $n \in \mathbf{N}$. The precise statement is the following:

PROPOSITION 7.1. *Assume p odd; let $1 \leqslant r \leqslant p-1$, $n \in \mathbf{N}_>$ and $1 \leqslant t \leqslant r$ be integers. Then*

$$\mathrm{soc}(\mathrm{Fil}^{t-1}(R_{n+1})) = \mathrm{soc}(\mathrm{Fil}^{t}(R_{n+1})).$$

More generally, we have

$$\mathrm{soc}(\mathrm{Fil}^{t-1}(R_{n+1})/Q) = \mathrm{soc}(\mathrm{Fil}^{t}(R_{n+1})/Q)$$

for any subrepresentation Q of $\mathrm{Fil}^j(R_{n+1})$, $0 \leqslant j \leqslant t-1$ coming from the socle filtration of $\mathrm{Fil}^j(R_{n+1})$.

The rest of the paragraph is devoted to its proof, which is very similar to the proof of proposition 6.6.

We fix integers $0 \leqslant r \leqslant p-1$, $n \in \mathbf{N}$, $1 \leqslant t \leqslant r$, and we define the elements of $\mathrm{Fil}^t(R_{n+1})$:

$$x \stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_n\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1_K, X^{r-t}Y^t] \in \mathrm{Fil}^t(R_{n+1});$$

$$x' \stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_n\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1_K, X^{r-t}Y^t] \in \mathrm{Fil}^t(R_{n+1});$$

$$y \stackrel{\mathrm{def}}{=} x + (-1)^t x'.$$

Moreover, we consider the map

$$pr : \mathrm{Fil}^{t-1}(R_{n+1}) \twoheadrightarrow \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^{t-1} \twoheadrightarrow Q^{(0,n+1)}_{0,p-1,\dots,p-1} \xrightarrow{\sim} \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t-1}$$

where the first arrow is the natural projection given by the reduction modulo $\mathrm{Fil}^{t-2}(R_{n+1})$ and the second arrow is more precisely described by the commutative diagram (cf. also lemma 5.11)

$$\begin{tikzcd}
\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^{t-1} \arrow[r] \arrow[d] & Q^{(0,n+1)}_{0,p-1,\dots,p-1} \\
Q^{(0,n+1)}_{0,\dots,0,p-1}\cong \mathrm{Ind}_{K_0(p^{n})}^{K}\chi_r^s\mathfrak{a}^{t-1} \arrow[d] & \\
\vdots \arrow[d] & \\
Q^{(0,2)}_{0,p-1}\cong \mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{t-1}. \arrow[uuur] &
\end{tikzcd}$$

We finally set

$$pr_{\mathrm{tot}}:\mathrm{Fil}^{t-1}(R_{n+1})\overset{pr}{\twoheadrightarrow}\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{t-1}\overset{\pi}{\twoheadrightarrow}\mathrm{Sym}^{p-1-\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{r-(t-1)}$$

where π is the natural projection defined in lemma 2.5. We start from the following computational lemma.

LEMMA 7.2. *We have the following equalities in* $\mathrm{Fil}^t(R_{n+1})$ *for* p *odd* [3]*:*

i) For all $a,d\in\mathbf{F}_p^\times$,

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x=a^{r-t}d^tx$$
$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x'=a^td^{r-t}x'.$$

ii) For all $\lambda\in\mathbf{F}_p$ *then* $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x-x$ *and* $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x'-x'$ *are in* $\mathrm{Fil}^{t-1}(R_{n+1})$ *and*

$$pr(\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x-x)=t(-1)^n\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}(-P_{-\lambda,}(\mu_0))[1_K,X^{r-(t-1)}Y^{t-1}]$$
$$pr(\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x'-x')=t(-1)^n\lambda\delta_{p,3}[1_K,X^{r-(t-1)}Y^{t-1}]$$

(where $P_{-\lambda}(\mu_0)$ *has been defined in §2.3)*

[3] the requirement p odd is used for the equality concerning x' in ii)

Proof: *i*) It is analogous to the proof of proposition 6.6-*i*). *ii*) From lemma 2.7 we deduce

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x =$$
$$= \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\lambda+\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1})] & 1 \end{bmatrix} [1, X^{r-t}(P_{\lambda,\dots,\mu_{n-1}}(\mu_n)X+Y)^t] =$$
$$= x + t \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\lambda+\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1})] & 1 \end{bmatrix} P_{\lambda,\dots,\mu_{n-1}}(\mu_n)[1, X^{r-(t-1)}Y^{t-1}] + q$$

for a suitable $q \in \mathrm{Fil}^{t-2}(R_{n+1})$ and where the elements $P_{\lambda,\dots,\mu_{j-1}}(\mu_j)$ for $j \in \{1,\dots,n\}$ (resp. $P_\lambda(\mu_0)$) are defined in lemma 2.7. We are now left to map the element $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x - x \in \mathrm{Fil}^{t-1}(R_{n+1})$ in $\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^{t-1}$ to get

$$t \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\lambda+\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}] & 1 \end{bmatrix} (\mu_{n-1}) P_{\lambda,\dots,\mu_{n-1}}(\mu_n)[1, X^{r-(t-1)}Y^{t-1}]$$

and the result follows using the chain of epimorphisms

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^{t-1} \twoheadrightarrow Q_{0,\dots,0,p-1}^{(0,n+1)} \twoheadrightarrow \dots \twoheadrightarrow Q_{0,p-1,\dots,p-1}^{(0,n+1)}$$

and the recursive property of the polynomials $P_{\lambda,\dots,\mu_{j-1}}(X) \in \mathbf{F}_p[X]$ for $j \in \{1,\dots,n\}$.

Similarly, from lemma 2.8 we deduce the following equality in $\mathrm{Fil}^t(R_{n+1})$:

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x' =$$
$$= x' + t \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} (-s_{\lambda,\dots,\mu_n})[1, X^{r-(t-1)}Y^{t-1}] + q'$$

for some $q' \in \mathrm{Fil}^{t-2}(R_{n+1})$. We map the element $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x' - x' \in \mathrm{Fil}^{t-1}(R_{n+1})$ in $\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^{t-1}$ to get

$$t \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \dots \sum_{\mu_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n'] & 1 \end{bmatrix} (-s_{\lambda,\dots,\mu_n})[1, X^{r-(t-1)}Y^{t-1}]$$

and the result follows using the chain of epimorphisms

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^{t-1} \twoheadrightarrow Q_{0,\dots,0,p-1}^{(0,n+1)} \twoheadrightarrow \dots \twoheadrightarrow Q_{0,p-1,\dots,p-1}^{(0,n+1)}$$

and the recursive property of the s_i for $i \in \{1,\dots,n\}$ (here we need $p \geqslant 3$). $\sharp$

Let now τ be an irreducible K-subrepresentation of $\mathrm{Fil}^t(R_{n+1})$, and assume $\tau \cap \mathrm{Fil}^{t-1}(R_{n+1}) =$

0. Therefore the natural projection $\mathrm{Fil}^t(R_{n+1}) \twoheadrightarrow \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t$ induces an isomorphism of τ onto an irreducible factor of $\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t)$, and the latter is completely described by proposition 6.6. We distinguish two situations:

A) the subrepresentation τ maps isomorphically in the K-subrepresentation of $\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t$ generated by (the image of) x.

B) We have $r-2t \equiv 0\,[p-1]$ and the subrepresentation τ maps isomorphically in the K-subrepresentation of $\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t$ generated by (the image of) y.

Study of case *A***.** Let $f \in \mathrm{Fil}^{t-1}(R_{n+1})$ be such that $x+f \in \tau$. From the induced isomorphism $\tau \xrightarrow{\sim} \langle Kx \rangle$ and the behaviour of x in $\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t)$ we deduce the following necessary conditions:

1) for all $a, d \in \mathbf{F}_p^\times$ we have

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (x+f) - a^{r-t} d^t (x+f) = 0$$

inside $\mathrm{Fil}^t(R_{n+1})$;

2) for all $\lambda \in \mathbf{F}_p$ we have

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (x+f) - (x+f) = 0$$

inside $\mathrm{Fil}^t(R_{n+1})$.

Condition 1) and lemma 7.2-*i*) imply in particular that $pr_{\mathrm{tot}}(f)$ is an H-eigenvector of

$$\mathrm{Sym}^{p-1-\lfloor r-2(t-1)\rfloor} \overline{\mathbf{F}}_p^2 \otimes \det{}^{r-(t-1)} \cong \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t-1} / \mathrm{Sym}^{\lfloor r-2(t-1)\rfloor} \overline{\mathbf{F}}_p^2 \otimes \det{}^{t-1}$$

of associated eigencharacter $a^{r-t} d^t$. It follows then from lemma 2.5 that

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} pr_{\mathrm{tot}}(f) - pr_{\mathrm{tot}}(f) = \begin{cases} 0 \text{ if } r-2(t-1) \not\equiv 0\,[p-1] \\ \\ c_1 \displaystyle\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-(t-1)} Y^{t-1}] \\ \text{if } r-2(t-1) \equiv 0\,[p-1] \end{cases}$$

for a suitable $c_1 \in \overline{\mathbf{F}}_p$. We conclude from condition 2) and lemma 7.2-*ii*)

$$t(-1)^n \sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} (-\lambda)^{p-j} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^j \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-(t-1)} Y^{t-1}] +$$
$$+\delta_{0, r-2(t-1)} c_1 \lambda \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-(t-1)} Y^{t-1}] = 0$$

inside $\mathrm{Sym}^{p-1-\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det^{r-(t-1)}$, and this is clearly impossible: by lemma 2.9-*ii*) we would get in particular

$$\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{p-1}\left[\begin{array}{cc}[\mu_0] & 1\\ 1 & 0\end{array}\right][1,X^{r-(t-1)}Y^{t-1}]=0$$
$$\sum_{\mu_0\in\mathbf{F}_p}\mu_0\left[\begin{array}{cc}[\mu_0] & 1\\ 1 & 0\end{array}\right][1,X^{r-(t-1)}Y^{t-1}]=0$$

which gives an absurd for $r-2(t-1)\not\equiv 0\,[p-1]$ and $r-2(t-1)\equiv 0\,[p-1]$ respectively. $\sharp$

***Study of case** B.* Let $f\in\mathrm{Fil}^{t-1}(R_{n+1})$ be such that $y+f\in\tau$. From the induced isomorphism $\tau\overset{\sim}{\to}\langle Ky\rangle$ and the behaviour of y in $\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t)$ we deduce the following necessary conditions:

1) for all $a,d\in\mathbf{F}_p^{\times}$ we have
$$\left[\begin{array}{cc}[a] & 0\\ 0 & [d]\end{array}\right](y+f)-a^{r-t}d^t(y+f)=0$$
inside $\mathrm{Fil}^t(R_{n+1})$;
2) for all $\lambda\in\mathbf{F}_p$ we have
$$\left[\begin{array}{cc}1 & [\lambda]\\ 0 & 1\end{array}\right](y+f)-(y+f)=0$$
inside $\mathrm{Fil}^t(R_{n+1})$.

We deduce from condition 1) and lemma 7.2 that $pr_{\mathrm{tot}}(f)$ is an H-eigenvector of
$$\mathrm{Sym}^{p-1-\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det^{r-(t-1)}\cong\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{t-1}/\mathrm{Sym}^{\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det^{t-1}$$
of associated eigencharacter $a^{r-t}d^t$ and therefore, by lemma 2.5
$$\left[\begin{array}{cc}1 & [\lambda]\\ 0 & 1\end{array}\right]pr_{\mathrm{tot}}(f)-pr_{\mathrm{tot}}(f)=\begin{cases}0 \text{ if } r-2(t-1)\not\equiv 0\,[p-1]\,(\text{i.e.}\, p\neq 3)\\[2ex] c_1\displaystyle\sum_{\mu_0\in\mathbf{F}_p}\left[\begin{array}{cc}[\mu_0] & 1\\ 1 & 0\end{array}\right][1,X^{r-(t-1)}Y^{t-1}]\\ \text{if } r-2(t-1)\equiv 0\,[p-1]\,(\text{i.e.}\, p=3)\end{cases}$$
for a suitable $c_1\in\overline{\mathbf{F}}_p$. The conclusion follows from lemma 7.2, similarly to the previous case. $\sharp$

8. Socle filtration for the spaces $R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}$

We are finally ready to describe the socle filtration for the K-representations
$$\varinjlim_{n\,\text{even}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}),\qquad \varinjlim_{m\,\text{odd}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_m}R_{m+1}).$$
The main statement is the following:

Proposition 8.1. *Assume p is odd; let $n\in\mathbf{N}_{>}$ (resp. $m\in\mathbf{N}_{>}$) be an odd (resp. even) integer, $0\leqslant r\leqslant p-2$. Then :*

i)

$$\mathrm{soc}(R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1}) = \mathrm{soc}(R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$$
$$(\mathrm{soc}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{m-2}} R_{m-1}) = \mathrm{soc}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1}) \quad \text{resp.})$$

where we formally define $R_0 \oplus_{R_{-1}} R_0 \stackrel{\text{def}}{=} R_0$ (resp. $R_1/R_0 \oplus_{R_0} R_1 \stackrel{\text{def}}{=} R_1/R_0$).

ii) More generally, if $0 \leqslant j \leqslant n-1$ is even (resp. $1 \leqslant j' \leqslant m-1$ is odd) and Q is a K-subrepresentation of R_j/R_{j-1} (resp. $R_{j'}/R_{j'-1}$) coming from the socle filtration of R_j/R_{j-1} (resp. $R_{j'}/R_{j'-1}$), then

$$\mathrm{soc}((R_j/Q) \oplus_{R_{j+1}} \cdots \oplus_{R_{n-2}} R_{n-1}) = \mathrm{soc}((R_j/Q) \oplus_{R_{j+1}} \cdots \oplus_{R_n} R_{n+1})$$
$$(\mathrm{soc}((R_{j'}/Q) \oplus_{R_{j'+1}} \cdots \oplus_{R_{m-2}} R_{m-1}) = \mathrm{soc}((R_{j'}/Q) \oplus_{R_{j'+1}} \cdots \oplus_{R_m} R_{m+1}) \quad \text{resp.})$$

where we formally define $(R_j/Q) \oplus_{R_{n-2}} R_{n-1} \stackrel{\text{def}}{=} (R_j/Q)$ if $j = n-1$ (resp. $(R_{j'}/Q) \oplus_{R_{m-2}} R_{m-1}$ if $j' = m-1$).

The rest of the paragraph is devoted to its proof, starting with the following lemmas.

Lemma 8.2. *Let $n \geqslant 2$ be an integer and $0 \leqslant r \leqslant p-1$. The composite map $T_2^- \circ \cdots \circ T_n^- : R_n \twoheadrightarrow R_1$ induces an isomorphism:*

$$\begin{tikzcd}
R_n \arrow[r, two heads, "T_2^-\circ\cdots\circ T_n^-"] \arrow[d] & R_1/\mathrm{Fil}^{r-1}(R_1) \\
R_n/\mathrm{Fil}^{r-1}(R_n) \cong \mathrm{Ind}_{K_0(p^n)}^K \chi_r \arrow[d] & \\
Q_{0,p-1,\ldots,p-1}^{(0,n)} \cong \mathrm{Ind}_{K_0(p)}^K \chi_r \arrow[uur, "\cong"'] &
\end{tikzcd}$$

Moreover, if $r \neq 0, p-1$ the composite map $T_1^- \circ \cdots \circ T_n^- : R_n \twoheadrightarrow R_0$ induces an isomorphism:

$$\begin{tikzcd}
R_n \arrow[r, two heads, "T_1^-\circ\cdots\circ T_n^-"] \arrow[d] & R_0 \\
R_n/\mathrm{Fil}^{r-1}(R_n) \cong \mathrm{Ind}_{K_0(p^n)}^K \chi_r \arrow[d] & \\
Q_{1,p-1,\ldots,p-1}^{(0,n)} \cong \mathrm{Sym}^r \overline{\mathbf{F}}_p^2 \arrow[uur, "\cong"'] &
\end{tikzcd}$$

Proof: First of all, notice that for any $m \geqslant 1$ we have a factorisation:

$$\begin{tikzcd}
R_m \arrow[r, two heads, "T_m^-"] \arrow[d] & R_{m-1} \\
R_m/\mathrm{Fil}^{r-1}(R_m) \arrow[ur, dotted, two heads] &
\end{tikzcd}$$

Thus, by the very definition of the operators T_j^-'s and lemma 2.9-i) we deduce

$$\begin{aligned} R_n/\mathrm{Fil}^{r-1}(R_n) &\twoheadrightarrow R_1/\mathrm{Fil}^{r-1}(R_1)\\ [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] &\mapsto (-\delta_{p-1,l_1})\dots(-\delta_{p-1,l_n}) \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1\\ 1 & 0\end{bmatrix} [1, Y^r]\end{aligned}$$

(where we put

$$[1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \overset{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0\\ p[\mu_1] & 1\end{bmatrix} \cdots \sum_{\mu_n\in\mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0\\ p^n[\mu_n] & 1\end{bmatrix} [1, Y^r]\,).$$

The previous epimorphism factorise then through

$$R_n/\mathrm{Fil}^{r-1}(R_n) \cong \mathrm{Ind}_{K_0(p^n)}^K \chi_r \twoheadrightarrow Q_{0,p-1,\dots,p-1}^{(0,n)}$$

and such a factorisation is indeed an isomorphism as the spaces $Q_{0,p-1,\dots,p-1}^{(0,n)}$ and $R_1/\mathrm{Fil}^{r-1}(R_1)$ have the same dimension.

Moreover, if $r \neq 0, p-1$, we see that

$$T_1^-(\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1\\ 1 & 0\end{bmatrix} [1, Y^r]) = 0$$

and therefore the morphism

$$R_n/\mathrm{Fil}^{r-1}(R_n) \twoheadrightarrow R_1/\mathrm{Fil}^{r-1}(R_1) \overset{T_1^-}{\twoheadrightarrow} R_0$$

factorise through

$$R_n/\mathrm{Fil}^{r-1}(R_n) \cong \mathrm{Ind}_{K_0(p^n)}^K \chi_r \twoheadrightarrow Q_{1,p-1,\dots,p-1}^{(0,n)};$$

again such a factorisation is an isomorphism by dimensional reasons. ♯

Lemma 8.3. *Let $n \geqslant 1$ (resp. $n = 0$), and $0 \leqslant r \leqslant p-2$. Then the natural map* $\mathrm{Fil}^0(R_{n+1}) \overset{\sim}{\to} \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s$ *induces an isomorphism*

$$\begin{aligned} &\mathrm{Fil}^0(R_{n+1})/R_n \overset{\sim}{\to} Q_{0,\dots,0,r+1}^{(0,n+1)}\\ &(\mathrm{Fil}^0(R_1)/R_0 \overset{\sim}{\to} \mathrm{Sym}^{p-1-\lfloor r\rfloor}\overline{\mathbf{F}}_p^2 \qquad \text{resp.})\end{aligned}$$

Proof: Assume $n \geqslant 1$. For any $(n-1)$-tuple $(l_1, \dots, l_{n-1}) \in \{0, \dots, p-1\}^{n-1}$ and any $j \in \{0, \dots, r\}$ we have

$$\begin{aligned} &T_n^+(\sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0\\ p[\mu_1] & 1\end{bmatrix} \cdots \sum_{\mu_{n-1}\in\mathbf{F}_p} \mu_{n-1}^{l_{n-1}} \begin{bmatrix} 1 & 0\\ p^{n-1}[\mu_{n-1}] & 1\end{bmatrix} [1, X^{r-j}Y^j]) =\\ &(-1)^j \sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0\\ p[\mu_1] & 1\end{bmatrix} \cdots \sum_{\mu_{n-1}\in\mathbf{F}_p} \mu_{n-1}^{l_{n-1}} \begin{bmatrix} 1 & 0\\ p^{n-1}[\mu_{n-1}] & 1\end{bmatrix} \sum_{\mu_n\in\mathbf{F}_p} \mu_n^j \begin{bmatrix} 1 & 0\\ p^n[\mu_n] & 1\end{bmatrix} [1, X^r].\end{aligned}$$

We thus conclude that the natural map

$$\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^r \twoheadrightarrow \mathrm{Fil}^0(R_{n+1})/R_n$$

factors through $\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \twoheadrightarrow Q_{0,\dots,0,r+1}^{(0,n+1)}$. Such a factorisation is indeed an isomorphism by dimensional reasons. The case $n = 0$ is similar and left to the reader. ♯

We are now ready to proof proposition 8.1 and the strategy will be analogous to the one used in the proof of proposition 7.1. Let us fix integers $n \geqslant 3$, n odd, $0 \leqslant r \leqslant p-2$; the case $n=1$ or $m \geqslant 2$, m even will be treated exactly in the same manner and will be left to the reader. We recall the commutative diagram with exact lines (cf. proposition 4.1):

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[rr, "T_n^+"] \arrow[d, "-T_n^-"] & & R_{n+1} \arrow[r] \arrow[dd, "pr_{n+1}"] & R_{n+1}/R_n \arrow[r] \arrow[dd, equal] & 0 \\
& R_{n-1} \arrow[d, "pr_{n-1}"] & & & & \\
0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[rr] & & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r] & R_{n+1}/R_n \arrow[r] & 0;
\end{tikzcd}$$

we write then π_{n-1} for the natural epimorphism

$$\pi_{n-1} : R_{n-1} \twoheadrightarrow R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}) \twoheadrightarrow Q^{(0,n-1)}_{0,p-1,\dots,p-1} \xrightarrow{\sim} R_1/\mathrm{Fil}^{r-1}(R_1)$$

where the last isomorphism is the one described in lemma 8.2. As we did in §7 we define the following elements in R_{n+1}:

$$x \overset{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\cdots\sum_{\mu_{n-1}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{n-1}[\mu_{n-1}] & 1\end{bmatrix}\sum_{\mu_n\in\mathbf{F}_p}\mu_n^{r+1}\begin{bmatrix}1 & 0\\ p^n[\mu_n] & 1\end{bmatrix}[1_K, X^r]$$

$$x' \overset{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\cdots\sum_{\mu_{n-1}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{n-1}[\mu_{n-1}] & 1\end{bmatrix}\sum_{\mu_n\in\mathbf{F}_p}\mu_n^{r+1}\begin{bmatrix}1 & 0\\ p^n[\mu_n] & 1\end{bmatrix}[1_K, X^r]$$

$$y \overset{\mathrm{def}}{=} x + (-1)^{r+1}x'.$$

A direct computation gives the key result:

Lemma 8.4. *Assume p is odd* [4]*; let $a, d \in \mathbf{F}_p^\times$, $\lambda \in \mathbf{F}_p$. Then:*

i) we have the following equalities in R_{n+1}:

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x = a^{-1}d^{r+1}x$$

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x' = a^{r+1}d^{-1}x'$$

ii) the elements $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x - x$ and $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x' - x'$ are in R_n and we have:

$$\pi_{n-1}\circ(-T_n^-)(\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x - x) = (r+1)(-1)^{r+1}\sum_{\mu_0\in\mathbf{F}_p}P_{-\lambda}(\mu_0)\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, Y^r]$$

$$\pi_{n-1}\circ(-T_n^-)(\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x' - x') = (r+1)(-1)^{r+1}(-\lambda)\delta_{p,3}[1_K, Y^r]$$

(where $P_{-\lambda}(\mu_0)$ has been defined in §2.3).

Proof: *i*) It is analogous to the proof of lemma 7.2-*i*).

ii). First of all, we study the action of $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}$ on x inside R_{n+1}. As $\begin{bmatrix}1 & 0\\ p^{n+1}\mathbf{Z}_p & 1\end{bmatrix}$ acts

[4] such a requirement is needed for the equality concerning x' in *ii*)

trivially on $[1, X^r] \in R_{n+1}$ we deduce from lemma 2.7:

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x =$$

$$= \sum_{j=0}^{r+1} \binom{r+1}{j} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0 + \lambda] & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

$$\cdots \sum_{\mu_{n-1} \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1} + P_{\lambda,\dots,\mu_{n-3}}(\mu_{n-2})] & 1 \end{bmatrix} (-P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1}))^j$$

$$\sum_{\mu_n \in \mathbf{F}_p} \mu_n^{r-(j-1)} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1_K, X^r]$$

and therefore

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x - x = T_n^+(v)$$

where $v \in R_n$ is defined as

$$v \stackrel{\text{def}}{=} \sum_{j=1}^{r+1} \binom{r+1}{j} (-1)^{r+(j-1)} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0 + \lambda] & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

$$\cdots \sum_{\mu_{n-1} \in \mathbf{F}_p} (-P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1}))^j \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1} + P_{\lambda,\dots,\mu_{n-3}}(\mu_{n-2})] & 1 \end{bmatrix} [1_K, X^{j-1}Y^{r-(j-1)}].$$

We are now left to study the image of $-T_n^-(v) \in R_{n-1}$ via the epimorphism π_{n-1}: a direct computation using the recursive property of the Witt polynomials $P_{\lambda,\dots,\mu_{j-2}}(X) \in \mathbf{F}_p[X]$ (for $j \in \{2, \dots, n\}$) together with lemma 2.9-*i*) yields finally the result.

The behaviour of the element $x' \in R_{n+1}$ can be described in a similar way, using now lemma 2.8 and the recursive property of the $s_{\lambda,\dots,\mu_{j-1}}$'s for $j \in \{2, \dots, n\}$. The details are left to the reader. $\sharp$

We now fix an irreducible K-subrepresentation τ of $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ such that $\tau \cap R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} = 0$; therefore the natural projection $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \twoheadrightarrow R_{n+1}/R_n$ induces an isomorphism of τ onto an irreducible factor of $\mathrm{soc}(R_{n+1}/R_n)$, which is completely known thanks to lemma 8.3. We distinguish two situations:

A) the subrepresentation τ maps isomorphically in the K-subrepresentation of R_{n+1}/R_n generated by (the image of) x.

B) We have $r = p-3$ and the subrepresentation τ maps isomorphically in the K-subrepresentation of R_{n+1}/R_n generated by (the image of) y.

Study of case *A***.** Let $f \in R_n$ be such that $pr_{n+1}(x + T_n^+(f)) \in \tau$. From the induced isomorphism $\tau \xrightarrow{\sim} \langle Kx \rangle$ and the behaviour of x in R_{n+1}/R_n we deduce the following necessary conditions:

1) for all $a, d \in \mathbf{F}_p^\times$ we have

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (x + T_n^+(f)) - a^{-1}d^{r+1}(x + T_n^+(f)) \in \ker(pr_{n+1})$$

2) for all $\lambda \in \mathbf{F}_p$ we have
$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (x + T_n^+(f)) - (x + T_n^+(f)) \in \ker(pr_{n+1}).$$

From condition 1) and lemma 8.4-*ii*) we see that $\pi_{n-1} \circ (-T_n^-)(f)$ is an H-eigenvector of $R_1/\mathrm{Fil}^{r-1}(R_1) \cong \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^r$ of associated eigencharacter $a{-}1d^{r+1}$. We then deduce from lemma 2.5 that

- if $r \neq 0$ the image of $\pi_{n-1} \circ (-T_n^-)(f)$ through the epimorphism
$$\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^r \overset{\pi}{\twoheadrightarrow} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2$$
is $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix}$-invariant;
- if $r = 0$, then
$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} \pi_{n-1} \circ (-T_n^-)(f) - \pi_{n-1} \circ (-T_n^-)(f) = c_1 \lambda \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, e]$$
inside $\mathrm{Ind}_{K_0(p)}^K 1$, for a suitable $c_1 \in \overline{\mathbf{F}}_p$.

It follows then from condition 2) and lemma 8.4 that for any $\lambda \in \mathbf{F}_p$ the element
$$\sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} (-\lambda)^{p-j} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^j \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] +$$
$$+ \delta_{0,r} c_1 \lambda \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] \in R_1/\mathrm{Fil}^{r-1}(R_1)$$
maps to zero via
$$\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^r \overset{\pi}{\twoheadrightarrow} \mathrm{Sym}^{\lceil r \rceil} \overline{\mathbf{F}}_p^2.$$
Thus, lemma 2.9-*ii*) implies in particular that
$$\sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{p-1} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] \in \ker(\pi)$$
$$\sum_{\mu_0 \in \mathbf{F}_p} \mu_0 \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] \in \ker(\pi)$$
giving an absurd for $r \neq 0$ and $r = 0$ respectively. $\sharp$

Study of case B**.** Let $f \in R_n$ be such that $pr_{n+1}(y + T_n^+(f)) \in \tau$. From the induced isomorphism $\tau \xrightarrow{\sim} \langle Ky \rangle (\cong \det^{-1})$ and the behaviour of y in R_{n+1}/R_n we deduce the following necessary conditions:

1) for all $a, d \in \mathbf{F}_p^\times$ we have
$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (y + T_n^+(f)) - (ad)^{-1}(y + T_n^+(f)) \in \ker(pr_{n+1})$$
2) for all $\lambda \in \mathbf{F}_p$ we have
$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (y + T_n^+(f)) - (y + T_n^+(f)) \in \ker(pr_{n+1}).$$

We then argue as in the previous case to get an absurd. The details are left to the reader. $\sharp$

This acheives the proof of proposition 8.1 for $n \geqslant 3$, n odd, and we leave it to the reader to check (by the explicit description of T_1^-) that the same procedure applies also for $n = 1$. It is then obvious that the same proof applies for the case $m \in \mathbf{N}_>$ is even and for part *ii*).

9. Conclusion

We are now ready to describe the socle filtration for the KZ-restriction of supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$: it will be a formal consequence of the explicit computations given in paragraphs §6, 7, 8.

THEOREM 9.1. *Assume p is odd; let r be an integer, with $0 \leqslant r \leqslant p-2$. The socle filtration for* $\varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ *is described by*

$$R_0 \text{---} \mathrm{SocFil}(R_2/R_1) \text{---} \ldots \text{---} \mathrm{SocFil}(R_{n+1}/R_n) \text{---} \ldots$$

while the socle filtration for $\varinjlim_{m\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1})$ *is described by*

$$\mathrm{SocFil}(R_1/R_0) \text{---} \mathrm{SocFil}(R_3/R_2) \text{---} \ldots \text{---} \mathrm{SocFil}(R_{m+1}/R_m) \text{---} \ldots$$

Proof: The proof is by induction; we will treat the case n odd (the other is analogous). Fix an odd integer $n \in \mathbf{N}_{\geqslant 1}$ and let Q be a quotient coming from the socle filtration of R_{n-1}/R_{n-2}. Assume (by inductive hypothesis) we dispose of an inductive system

$$\{Q \oplus_{R_n} R_{n+1} \cdots \oplus_{R_m} R_{m+1}\}_{m \geqslant n-2,\, m\,\mathrm{odd}}$$

(with the convention $Q \oplus_{R_{n-2}} R_{n-1} \overset{\mathrm{def}}{=} Q$) and where the amalgamed sums are defined through the Hecke operators $T_j^{\pm}$ for $j \geqslant n$ as in §3.2, as well as natural exact sequences:

$$0 \to Q \oplus_{R_n} \cdots \oplus_{R_{m-2}} R_{m-1} \to Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1} \to R_{m+1}/R_m \to 0$$

for $m \geqslant n$, m odd. If we set

$$\tau \overset{\mathrm{def}}{=} \mathrm{soc}(Q)$$

we formally verify that for $\tau \neq Q$

$$Q/\tau \oplus_Q (Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1}) = \mathrm{coker}(\tau \to Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})$$

for any $m \geqslant n$, m odd, while, if $\tau = Q$,

$$R_{n+1}/R_n \oplus_{\tau \oplus_{R_n} R_{n+1}} (\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1}) = \mathrm{coker}(\tau \to Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})$$

for any $m > n$, m odd. We therefore get an inductive system:

$$\{Q/\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1}\}_{m \geqslant n-2,\, m\,\mathrm{odd}}$$

and natural exacts sequences

$$0 \to Q/\tau \oplus_{R_n} \cdots \oplus_{R_{m-2}} R_{m-1} \to Q/\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1} \to R_{m+1}/R_m \to 0$$

for $m \geqslant n$, m odd (where we write R_{n+1} instead of $Q/\tau \oplus_{R_n} R_{n+1}$ in the case $\tau = Q$). As $\varinjlim$ is right exact, we deduce that

$$\mathrm{coker}(\tau \to \varinjlim_{m \geqslant n,\, m\,\mathrm{odd}} (Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})) = \varinjlim_{m \geqslant n,\, m\,\mathrm{odd}} (Q/\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})$$

and the statement is now clear from proposition 8.1 $\sharp$

The socle filtration for $\pi(r,0,1)|_{KZ}$, with $0 \leqslant r \leqslant p-1$ and p odd is then immediate from proposition 3.9 and from the isomorphism $\pi(0,0,1) \cong \pi(p-1,0,1)$.

We give now the idea of the socle filtration for $\varinjlim_{n,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$:

$$\begin{aligned}\mathrm{SocFil}(\varinjlim_{n,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})) &= \\ = R_0 \text{—} \mathrm{SocFil}(R_2/R_1) \text{—} \mathrm{SocFil}(R_4/R_3) \text{—} \dots\end{aligned}$$

which gives, developing the socle filtration of the quotients R_{n+1}/R_n,

$$R_0\text{—}\mathrm{SocFil}(\mathrm{Fil}^0(R_2/R_1))\text{—}\mathrm{SocFil}(\mathrm{Fil}^1(R_2)/\mathrm{Fil}^0(R_2))\text{—}\mathrm{SocFil}(\mathrm{Fil}^2(R_2)/\mathrm{Fil}^1(R_2))\text{—}\dots$$

and, using proposition 7.1,

$$R_0\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{r+1})\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{r+2})\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{r+3})\text{—}\dots$$

To be even more explicit, if we suppose $1 \leqslant r \leqslant p-6$ the beginning of the socle filtration for $\varinjlim_{n,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ looks as follow:

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2\text{—}\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det{}^{r+1}\text{—}\mathrm{Sym}^{r+2}\overline{\mathbf{F}}_p^2\otimes\det{}^{p-2}\text{—}\mathrm{Sym}^{p-5-r}\overline{\mathbf{F}}_p^2\otimes\det{}^{r+2}\text{—}\dots$$

10. The principal series and the Steinberg

In this section we want to describe the socle filtration for the K-restriction of principal series and Steinberg representation for $\mathrm{GL}_2(\mathbf{Q}_p)$. The techniques are very closed to those of §6 and therefore will be mainly left to the reader. If $\lambda \in \overline{\mathbf{F}}_p^\times$ and $r \in \{0,\dots,p-1\}$ we recall the parabolic induction

$$\mathrm{Ind}_B^G(\mathrm{un}_\lambda \otimes \omega^r \mathrm{un}_{\lambda^{-1}}). \tag{6}$$

If $V_{\lambda,r}$ is the underlying vector space associated to the B-representation $\mathrm{un}_\lambda \otimes \omega^r \mathrm{un}_{\lambda^{-1}}$, the induction (6) is the $\overline{\mathbf{F}}_p$-vector space of locally constant functions $f : G \to V_{\lambda,r}$ such that $f(bg) = b \cdot f(g)$ for any $b \in B$, $g \in G$; the left G-action defined by right translation of functions gives (6) a structure of smooth G-representation.

We recall also that, for $(\lambda,r) \notin \{(0,\pm1),\,(p-1,\pm1)\}$, the representations (6) are irreducible (referred to as "principal series"), otherwise they fit a short exact sequence

$$0 \to 1 \to \mathrm{Ind}_B^G 1 \to \mathrm{St} \to 0$$

and the quotient St is referred to as the "Steinberg" representation.

We turn our attention to the K-restriction of inductions (6).

Lemma 10.1. *For any $\lambda \in \overline{\mathbf{F}}_p^\times$ and $r \in \{0,\dots,p-1\}$ we have a K-equivariant isomorphism*

$$(\mathrm{Ind}_B^G(\mathrm{un}_\lambda \otimes \omega^r \mathrm{un}_{\lambda^{-1}}))|_K \cong \mathrm{Ind}_{K\cap B}^K \chi_r^s$$

where χ_r^s, which is a character of $B(\mathbf{F}_p)$, is seen as a smooth character of $B \cap K$ by inflation.

Proof: It is an immediate consequence of Mackey theorem and the Iwasawa decomposition $G = KB$.$\sharp$

We have a natural homeomorphism

$$K/K\cap B \xrightarrow{\sim} \mathbf{P}^1_{\mathbf{Z}_p}$$

(coming from the natural left action of K on $[1:0]\in\mathbf{P}^1_{\mathbf{Z}_p}$) and the decomposition of corollary 3.2 let us deduce an open disjoint covering of $\mathbf{P}^1_{\mathbf{Z}_p}$ with balls of radius $(\frac{1}{p})^n$ (for the normalised norm on $\mathbf{Z}_p$: $|p|\overset{\text{def}}{=}\frac{1}{p}$). The following result is then clear

LEMMA 10.2. *Let $n\in\mathbf{N}$, $r\in\{0,\dots,p-2\}$; we fix a basis $\{e\}$ of the underlying vector space of χ_r^s. We have K-equivariant monomorphisms*

$$\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s \overset{\iota_{n+1}}{\hookrightarrow} \mathrm{Ind}_{K\cap B}^K\chi_r^s, \mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s \overset{\iota_{n+1,n+2}}{\hookrightarrow} \mathrm{Ind}_{K_0(p^{n+2})}^K\chi_r^s$$

characterzed by

i) $\iota_{n+1}([1,e])$ *is the unique function* $f\in\mathrm{Ind}_{K\cap B}^K\chi_r^s$ *such* $\mathrm{Supp}(f)=K_0(p^{n+1})$ *and* $f(1)=e$;

ii)

$$\iota_{n+1,n+2}([1,e])=\sum_{\mu_{n+1}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{n+1}[\mu_{n+1}] & 1\end{bmatrix}[1,e]$$

Proof: It is a standard verification that the conditions in $i)$ and $ii)$ let define K-equivariant morphisms ι_{n+1}, $\iota_{n+1,n+2}$. Such morphisms are then injective by support reasons. ♯

From the monomorphisms defined in lemma 10.2 we deduce then a natural monomorphism:

$$\varinjlim_{n\in\mathbf{N}}(\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s)\hookrightarrow\mathrm{Ind}_{K\cap B}^K\chi_r^s; \tag{7}$$

as K is compact and all functions $f\in\mathrm{Ind}_{K\cap B}^K\chi_r^s$ are locally constant, we conclude that (7) is actually an isomorphism. Moreover:

LEMMA 10.3. *Let $n\in\mathbf{N}$, $r\in\{0,\dots,p-2\}$. Then*

$$\mathrm{coker}(\iota_{n+1,n+2})=Q_{0,\dots,0,1}^{(0,n+2)}.$$

Proof: From the definitions of $Q_{0,\dots,0,1}^{(0,n+2)}$ and $\iota_{n+1,n+2}$ we deduce a natural epimorphism $\mathrm{coker}(\iota_{n+1,n+2})\twoheadrightarrow Q_{0,\dots,0,1}^{(0,n+2)}$. We conclude, as the two spaces have the same dimension.♯

We dispose now of K-equivariant exact sequences, where $n\in\mathbf{N}$:

$$0\to\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\to\mathrm{Ind}_{K_0(p^{n+2})}^K\chi_r^s\to Q_{0,\dots,0,1}^{(0,n+2)}\to 0.$$

Thanks to the explicit description of $\mathrm{soc}(Q_{(0,\dots,0,1)}^{0,n+2})$ we deduce, with arguments which are very similar to those of proposition 8.1, the following result

THEOREM 10.4. *Let $n\in\mathbf{N}$, $r\in\{0,\dots,p-2\}$. Then*

$$\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s)=\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+2})}^K\chi_r^s).$$

More generally, if $Q\leqslant\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s$ is a K-subrepresentation coming from the socle filtration of $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s$, we have

$$\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s/Q)=\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+2})}^K\chi_r^s/\iota_{n+1,n+2}(Q)).$$

Proof: It suffices to use the same arguments of the proof of proposition 8.1, and similar explicit computations. The details are left to the reader. $\sharp$

Once again, we can use proposition 10.4 to describe the behaviour of the socle filtration for $\mathrm{Ind}_{K\cap B}^{K}\chi_r^s$. The graded pieces of such a filtration look as follow:

$$\mathrm{SocFil}(\mathrm{Ind}_{K\cap B}^{K}\chi_r^s) = \mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s)\text{—}\mathrm{SocFil}(Q_{0,1}^{(0,2)})\text{—}\mathrm{SocFil}(Q_{0,0,1}^{(0,3)})\text{—}\dots$$

and, developing the socle filtration of $Q_{0,\dots,0,1}^{(0,n+2)}$,

$$\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a})\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^2)\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^3)\text{—}\dots$$

Part II. Invariant elements under some congruence subgroups for irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$ representations over $\overline{\mathbf{F}}_p$

Abstract. Let p be an odd prime number. Using the explicit description for irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$ made in [Mo1], we determine all invariant elements of such representations under the actions of the congruence subgroups K_t, I_t, for any integer $t \geqslant 1$. In particular, we have the dimension of the K_t-invariants for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, for any $t \geqslant 1$.

1. Introduction

Let p be a prime number. The efforts to describe a "p-adic analogue" of the classical local Langlands correspondence met great progresses in the last few years. After a first, conjectural approach studied by Breuil in [Bre04] and [Bre03b], the works of Berger-Breuil [BB] and Colmez [Col] establish a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. Moreover, such a correspondence is compatible with respect to the reduction of coefficients modulo p: we get a semisimple mod p Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ (again, conjectured by Breuil in [Bre03b] and proved by Berger in [Ber10a]).

But, if the local field is different from $\mathbf{Q}_p$ the situation is far from being defined. In the direction of a semisimple Langland correspondence for $\mathrm{GL}_2(F)$ for F a non-Archimedean local field, we find the works of Barthel and Livné [BL94] and [BL95]. In those papers the authors classify the smooth irreducible admissible $\mathrm{GL}_2(F)$-representations into four classes: besides characters, principal series and special series, they find a new family of irreducible objects, referred to as "supersingular" whose nature is still very mysterious. Supersingular representations are actually characterised as the subquotients of the cokernel of some "canonical Hecke operator" T, but for $F \neq \mathbf{Q}_p$ such cokernels are not even admissible (cf. [Bre03a], Remarque 4.2.6); moreover the works of Paskunas [Pas04], Breuil-Paskunas [BP] and Hu [Hu] show that for $F \neq \mathbf{Q}_p$ there exists a huge number of supersingular representations with respect to Galois representations (whose classification is indeed well known).

We focus here on the case $F = \mathbf{Q}_p$ where $p$is an odd prime. In this situation the work of Breuil [Bre03a] (followed later by other proofs by Ollivier in [Oll], Emerton in [Eme08]) show that the cokernels of the aforementioned Hecke operators T are actually irreducible, completing the classification for smooth irreducible admissible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$. In the work [Mo1] we develop an explicit approach to the description of irreducible representations for $\mathrm{GL}_2(\mathbf{Q}_p)$: studying the action of T on some privileged elements we are able to describe in great detail supersingular representations (and principal and special series as well), in particular detecting the socle filtrations for their $K\mathbf{Q}_p^{\times}$-restriction.

In the present work, we pursue the study of such explicit elements of irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations in order to describe their invariants under some congruence subgroups of K.

If we first focus on K_t, i.e. the kernel of the mod p^t-reduction map on K (where $t \geqslant 1$) we see in §3 that taking K_t-invariants of a supersingular representation π comes down, roughly speaking, to "cut" its socle filtration.

The main result (corollary 3.9) is that we can detect *precisely* where such a cutting occurs: if we refer to the socle filtration of a supersingular representation $\pi(r,0,1)$ as "two lines of weights" we get

THEOREM 1.1. *Let $t \geqslant 1$ be an integer. The socle filtration of $\pi(r,0,1)^{K_t}$ is described by*

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2\text{—socfil}(\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^{r+1})\text{—}\ldots\text{—socfil}(\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^{r})\text{—}\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det{}^{r+1}$$
$$\bigoplus$$
$$\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2\otimes\det{}^{r}\text{—socfil}(\mathrm{Ind}_I^K\chi_r^s\mathfrak{a})\text{—}\ldots\text{—socfil}(\mathrm{Ind}_I^K\chi_r^s)\text{—}\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det$$

where we have $p^{t-1}-1$ parabolic induction in each line and the weight $\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det^{r+1}$ in the fist line (resp. $\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det$ in the second line) appears only of $p-3-r\geqslant 0$ (resp. $r-2\geqslant 0$).

In particular, we have the dimension of the spaces of K_t-invariants (corollary 3.8):

COROLLARY 1.2. *Let $t\geqslant 1$ be an integer and $r\in\{0,\ldots,p-1\}$. The dimension of K_t invariant for a supersingular representation is*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{K_t})=(p+1)(2p^{t-1}-1)+\begin{cases} p-3 & \text{if } r\notin\{0,p-1\}\\ p-2 & \text{if } r\in\{0,p-1\}\end{cases}$$

Moreover, if we write I_t for the subgroup of K_{t-1} whose elements are upper unipotent mod p^t, we are able, by similar techniques, to describe in greatest detail the space of I_t-invariant of any supersingular representations π of $\mathrm{GL}_2(\mathbf{Q}_p)$. Again, we can roughly say that taking I_t-invariants comes down to "cut" the socle filtration of π, but this time some "reminders elements" appear.

The results of section 4 tells us *exactly* where such cutting occurs and who the reminders elements are. As the combinatoric of such result is a bit heavy, we prefere to omit the statements here, refering the interested reader directly to propositions 4.4, 4.8, 4.11, 4.14 in §4.

We can anyway remark that an immediate corollary is then the $\overline{\mathbf{F}}_p$ dimension of such invariant spaces:

COROLLARY 1.3. *Let $r\in\{0,\ldots,p-1\}$ and $t\in\mathbf{N}_>$ be integers. Then:*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{I_t})=2(2p^{t-1}-1).$$

The proof of such results relies on the explicit description made in [Mo1] and can be sketched as follow.

We reduce of course to the direct sum decomposition of $\pi|_{K\mathbf{Q}_p^\times}$ (for π a supersingular representation) in terms of the inductive limits of the amalgamed sums $R_i/R_{i-1}\oplus_{R_{i+1}}\cdots\oplus_{R_n}R_{n+1}$ ($i\in\{0,1\}$), treating each summand separately.

We are then able (lemmas 3.2 and 3.3) to give a first estimate of the behaviour of K_t-invariants in terms of the filtrations $\{R_i/R_{i-1}\oplus_{R_{i+1}}\cdots\oplus_{R_n}R_{n+1}\}_{n\in\mathbf{N}}$; for instance for t and n odd we get the following exact sequence:

$$0\to R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}\to(\varinjlim_{n\,\mathrm{odd}}R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})^{K_t}\to(R_{t+1}/R_t)^{K_t}.$$

Finally (proposition 3.7) in order to extract the K_t-invariants from the previous exact sequence, we exploit the description of the generators of the socle filtration for R_{t+1}/R_t: we get some explicit nullity conditions of certains elements of the amalgamed sums, conditions which can easily be translated into a condition inside $\mathrm{soc}(\pi)$ (where we are able to do direct computations) via an inductive process by means of the operator T (cf. lemma 2.12).

The proof concerning the I_t invariants is similar. For instance, for n odd, we get a first estimate by an exact sequence of the form

$$0 \to \mathbf{V}_0 \to (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{I_t} \to \mathbf{V}^{I_t}$$

where $\mathbf{V}_0$ is a suitable subobject of $(\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_{t-1}}$ and

$$\mathbf{V} = (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_t}/\mathbf{V}_0$$

(cf. §4.1 and §4.2). We then describe the I_t-invariants of the spaces of the form $\mathbf{V}$, via a decomposition into stable subspaces (cf. for instance propositions 4.3 and 4.7), from which we deduce the I_t-invariants of the inductive limits through some nullity conditions completely analogous to those of proposition 3.7 (cf. propositions 4.4, 4.8, 4.11, 4.14).

We outline here that by similar techniques we are able to describe the space of $\Gamma_0(p^k)$ and $\Gamma_1(p^k)$ invariants for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$ (cf. [Mo3]). Such spaces appear naturally in the study of torsion points in the cohomology of certain modular curves.

The plan of the chapter is the following.

Section 2 is devoted to a brief summary of the results in [Mo1], [Bre] and [BP], in order to handle the computational techniques for the rest of the chapter. More precisely, in 2.1 we re-interpret the $K\mathbf{Q}_p^\times$-restriction of a supersingular representations π in terms of certains induced representations R_n endowed with Hecke operators $T_n^\pm$; we give then a precise description of the socle filtration of π in 2.2 (cf. proposition 2.7) using "certains explicit elements" $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$. We recall also (§2.2.1) some classical results concerning $\mathrm{GL}_2(\mathbf{F}_p)$-parabolic induction for $B(\mathbf{F}_p)$-representation. Finally, we deal with some explicit computations on Witt vectors (lemmas 2.10 and 2.11) and study a nullity condition for some elements of the amalgamed sums introduced in 2.1 (cf. lemma 2.12).

Section 3 is devoted to an exhaustive description of K_t-invariants for supersingular representations. After a first estimate (cf. lemmas 3.2 and 3.3) we introduce in definition 3.4 the elements $x_{l_1,\dots,l_{t-1}}^{(')}$, $y_{l_1,\dots,l_{t-1}}^{(')}$ $z_{l_1,\dots,l_{t-1}}^{(')}$. Their behaviour let us refine the previous estimates. They indeed lead us to introduce the subobjects $\sigma(p-2)$, $\sigma(p-3)$, etc.. of definition 3.6, which let us complete the analysis of K_t-invariants stated in proposition 3.7. As a byproduct, we compute the $\overline{\mathbf{F}}_p$-dimension of such spaces.

Section 4 is concerned on the I_t-invariants and is divided into four numbers (completely analogous to each other) §4.1.1, §4.1.2, §4.2.1 and §4.2.2 (according to the parity of t and the direct summand in the decomposition of $\pi|_{K\mathbf{Q}_p^\times}$). In each number we start from a first estimate of such invariants by means of an exact sequence issued from the results in 3; we then introduce some explicit elements (cf. definitions 4.1, 4.5, 4.9, 4.12) the study of which let us describe precisely the space of I_t-invariants in each term of the aforementioned exact sequences (cf. propositions 4.3, 4.4, 4.7, 4.8, etc..).

Finally, in section §5 we describe precisely the spaces of K_t and I_t-invariants for principal and special series (where the computations are much simpler than in the supersingular case!).

We introduce now the main notations, convention and structure of the chapter.

We fix a prime number p, which will always be assumed to be odd. We write $\mathbf{Q}_p$ (resp. $\mathbf{Z}_p$) for the p-adic completion of $\mathbf{Q}$ (resp. $\mathbf{Z}$) and $\mathbf{F}_p$ the field with p elements; $\overline{\mathbf{F}}_p$ is then a fixed algebraic closure of $\mathbf{F}_p$. For any $\lambda \in \mathbf{F}_p$ (resp. $x \in \mathbf{Z}_p$) we write $[\lambda]$ (resp. $\overline{x}$) for the Teichmüller lift (resp. for the reduction modulo p), defining $[0] \overset{\text{def}}{=} 0$.

We write $G \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Q}_p)$, $K \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Z}_p)$ the maximal compact subgroup, I the Iwahori subreoup of K (i.e. the elements of K whose reduction modulo p is upper triangular) and I_1 for the pro-p-iwahori (i.e. the elements of I whose reduction is unipotent). For an integer $t \geqslant 1$ we define $K_t \overset{\text{def}}{=} \ker K \twoheadrightarrow \mathrm{GL}_2(\mathbf{Z}_p/p^t\mathbf{Z}_p)$ and

$$I_t \overset{\text{def}}{=} \{\begin{bmatrix} 1+p^ta & p^{t-1}b \\ p^tc & 1+p^td \end{bmatrix} \in K \quad a,b,c,d \in \mathbf{Z}_p\}, U_t \overset{\text{def}}{=} \{\begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix} \in K \quad b \in \mathbf{Z}_p\}.$$

Moreover, let $Z \overset{\text{def}}{=} Z(G) \cong \mathbf{Q}_p^\times$ be te center of G and $B(\mathbf{Q}_p)$ (resp. $B(\mathbf{F}_p)$) the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ (resp. $\mathrm{GL}_2(\mathbf{F}_p)$).

For $r \in \{0, \dots, p-1\}$ we denote by σ_r the algebraic representation $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2$ (endowed with the natural action of $\mathrm{GL}_2(\mathbf{F}_p)$). Explicitely, if we consider the identification $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 \cong \overline{\mathbf{F}}_p[X,Y]_r^h$ (where $\overline{\mathbf{F}}_p[X,Y]_r^h$ means the graded component of degree r for the natural grading on $\overline{\mathbf{F}}_p[X,Y]$) then

$$\sigma_r(\begin{bmatrix} a & b \\ c & d \end{bmatrix})X^{r-i}Y^i \overset{\text{def}}{=} (aX+cY)^{r-i}(bX+dY)^i$$

for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{F}_p)$, $i \in \{0, \dots, r\}$. We then endow σ_r with the action of K obtained by inflation $K \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_p)$ and, by imposing a trivial action of $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$, we get a smooth KZ-representation. Such a representation is still noted as σ_r, not to overload the notations.

If $r \in \{0, \dots, p-1\}$ it follows from the results in [BL95] that we have an isomorphism of $\overline{\mathbf{F}}_p$-algebras

$$\mathrm{End}_G(\mathrm{Ind}_{KZ}^G\sigma_r) \xrightarrow{\sim} \overline{\mathbf{F}}_p[T]$$

for a suitable endomorphism T, which depends on r, and where $\mathrm{Ind}_{KZ}^G\sigma_r$ is the usual compact induction (cf. [Bre], §3.2 for a detailed description of compact inductions). We then write $\pi(r,0,1)$ to mean the cokernel $\mathrm{coker}(\mathrm{Ind}_{KZ}^G\sigma_r \xrightarrow{T} \mathrm{Ind}_{KZ}^G\sigma_r$; such representations exhaust all supersingular representations for $\mathrm{GL}_2(\mathbf{Q}_p)$ (cf. Breuil's [Bre03a], Corollaire 4.1.1 et 4.1.4).

If H stands for the maximal torus of $\mathrm{GL}_2(\mathbf{F}_p)$ and $\chi : H \to \overline{\mathbf{F}}_p^\times$ is a multiplicative character we will write χ^s for the conjugate character defined by $\chi^s(h) \overset{\text{def}}{=} \chi(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} h \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ for $h \in H$. Characters of H will be seen as characters of $B(\mathbf{F}_p)$ or (by inflation) of (a filter of neighborhood of 1 in) I withouth any commentary.

With "representation" we always mean a smooth representations with central character with coefficients in $\overline{\mathbf{F}}_p^\times$. If V is a $\widetilde{K}$-representation, for $\widetilde{K}$ a subgroup of K, and $v \in V$, we write $\langle \widetilde{K} \cdot v \rangle$ to denote the sub-$\widetilde{K}$ representation of V generated by v. For a $\widetilde{K}$-representation V we write $\mathrm{soc}_{\widetilde{K}}(V)$ (or $\mathrm{soc}(V)$, or $\mathrm{soc}^1(V)$ if $\widetilde{K}$ is clear from the context) to denote the maximal semisimple sub-representation of V. Inductively, the subrepresentation $\mathrm{soc}^i(V)$ of V being defined, we define $\mathrm{soc}^{i+1}(V)$ as the inverse image of $\mathrm{soc}^1(V/\mathrm{soc}^i(V))$ via the projection $V \twoheadrightarrow V/\mathrm{soc}^i(V)$. We therefore obtain an increasing filtration $\{\mathrm{soc}^n(V)\}_{n\in\mathbf{N}_>}$ which will be referred to as the socle filtration for V; we will say that a subrepresentation W of V "comes from the socle filtration" if we have $W = \mathrm{soc}^n(V)$ for some $n \in \mathbf{N}_>$ (with the convention that $\mathrm{soc}^0(V) \overset{\mathrm{def}}{=} 0$). The sequence of the graded pieces of the socle filtration for V will be shortly denoted by

$$\mathrm{SocFil}(V) \overset{\mathrm{def}}{=} \mathrm{soc}^1(V)\text{---}\mathrm{soc}^1(V)/\mathrm{soc}^0(V)\text{---}\dots\text{---}\mathrm{soc}^{n+1}(V)/\mathrm{soc}^n(V)\text{---}\dots$$

We recall the Kroneker delta: if S is any set, and $s_1, s_2 \in S$ we define

$$\delta_{s_1,s_2} \overset{\mathrm{def}}{=} \begin{cases} 0 & \text{if} \quad s_1 \neq s_2 \\ 1 & \text{if} \quad s_1 = s_2. \end{cases}$$

Moreover, for $x \in \mathbf{Z}$, we define $\lfloor x \rfloor \in \{0, \dots, p-2\}$ by the condition $\lfloor x \rfloor \equiv x \,\mathrm{mod} p - 1$.

2. Preliminaries and definitions

The aim of this section is to give the necessary tools to deal with the explicit computations needed for the description of K_t and I_t-invariants of supersingular representations $\pi(r, 0, 1)$. In §2.1 and §2.2 we recall the socle filtration of the KZ-representations $\pi(r, 0, 1)|_{KZ}$ made in [Mo1], together with the generators for the irreducible factors of the graded pieces of such filtration. Some classical results concerning $\mathrm{GL}_2(\mathbf{F}_p)$-parabolic induction for $B(\mathbf{F}_p)$-representations will be recalled in §2.2 as well, while §2.3 is devoted to some explicit computations on Witt vectors and elements of the amalgamed sums $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$. These computations will be a key tool in §3 and §4

2.1 On the KZ restriction of supersingular representations

We fix $r \in \{0, \dots, p-1\}$ and consider the supersingular representation $\pi(r, 0, 1)$; our goal is to give an exhaustive description of the objects involved in proposition 2.3. For this purpose, we recall the definition of the K-representations R_n, where $n \in \mathbf{N}$ as well as the "Hecke" operators $T_n^\pm : R_n \to R_{n\pm1}$, leading us to the decomposition of proposition 2.3. The reader is invited to refer to [Mo1] for the omitted details.

For any $n \in \mathbf{N}$ we define the following subgroup of K:

$$K_0(p^n) \overset{\mathrm{def}}{=} \{\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K, \text{ where } c \in \mathbf{Z}_p\}$$

(in particular, $K_0(p^0) = K$ and $K_0(p)$ is the Iwahori subgroup). For $0 \leqslant r \leqslant p-1$ and $n \in \mathbf{N}$ we define the $K_0(p^n)$-representation σ_r^n over $\overline{\mathbf{F}}_p$ as follow. The associated $\overline{\mathbf{F}}_p$-vector space of σ_r^n is $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2$, while the left action of $K_0(p^n)$ is given by

$$\sigma_r^n(\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix}) \cdot X^{r-j}Y^j \overset{\mathrm{def}}{=} \sigma_r(\begin{bmatrix} d & c \\ p^n b & a \end{bmatrix}) \cdot X^{r-j}Y^j$$

for any $\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K_0(p^n)$, $0 \leqslant j \leqslant r$; in particular, the σ_r^n's are smooth and σ_r^0 is isomorphic to σ_r. Finally, we define

$$R_r^n \overset{\mathrm{def}}{=} \mathrm{Ind}_{K_0(p^n)}^K \sigma_r^n.$$

If r is clear from the context, we will write simply R_n instead of R_r^n.

We recall that an $\overline{\mathbf{F}}_p$-basis for R_n is then described by

$$\mathscr{B}_n \stackrel{\text{def}}{=} \{[\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j], [\begin{bmatrix} 1 & 0 \\ p\mu & 1 \end{bmatrix}, X^{r-j}Y^j] \,\text{for}\, \mu \in I_n,\, \mu' \in I_{n-1}, 0 \leqslant j \leqslant r\}$$

Each of the K-representations R_n is endowed with "natural" Hecke operators $T_n^{\pm}$. Their definitions and properties are summed up in the next

Proposition 2.1. *For all $n \in \mathbf{N}$ we have a K-equivariant monomorphism $T_n^+ : R_n \hookrightarrow R_{n+1}$ characterised by:*

$$T_n^+([1_K, X^{r-j}Y^j]) = \sum_{\mu_n \in \mathbf{F}_p} (-\mu_n)^j \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1_K, X^r] \,\text{if}\, n > 0$$
$$T_0^+([1_K, X^{r-j}Y^j]) = \sum_{\mu_0 \in \mathbf{F}_p} (-\mu_0)^{r-j} [\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix}, X^r] + [1_K, \delta_{j,0}X^r] \,\text{if}\, n = 0.$$

For all $n \in \mathbf{N}_>$ we have a K-equivariant epimorphism $T_n^- : R_n \twoheadrightarrow R_{n-1}$; such a morphism is characterised by the conditions:

$$T_n^-([\begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1}] & 1 \end{bmatrix}, X^{r-j}Y^j]) = [1_K, \delta_{r,j}(\mu_{n-1}X + Y)^r] \,\text{if}\, n \geqslant 2$$
$$T_1^-([1_K, X^{r-j}Y^j]) = \delta_{r,j}Y^r \,\text{if}\, n = 1$$

for $\mu_{n-1} \in \mathbf{F}_p$.

Proof: Omissis. Cf. [Mo1] §3.2.♯

We identify R_n as a K-subrepresentation of R_{n+1} via the monomorphism T_n^+ without any further commentary. For any odd integer $n \geqslant 1$ we use the hecke operators $T_n^{\pm}$ to define (inductively) the amalgamed sum $R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$ via the following co-cartesian diagram

$$\begin{tikzcd} R_n \arrow[r, hook, "T_n^+"] \arrow[d, two heads, "-pr_{n-1}\circ T_n^-"'] & R_{n+1} \arrow[d, dotted, two heads, "pr_{n+1}"] \\ R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, dotted] & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}. \end{tikzcd}$$

Similarly we define the amalgamed sums $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ for any positive even integer $n \in \mathbf{N}_>$. Then

Proposition 2.2. *For any odd integer $n \in \mathbf{N}$, $n \geqslant 1$ we have a natural commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d, two heads, "-pr_{n-1}\circ T_n^-"] & R_{n+1} \arrow[r] \arrow[d, two heads, "pr_{n+1}"] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r, "\pi"] & R_{n+1}/R_n \arrow[r] & 0 \end{tikzcd}$$

with exact lines.

We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N}\setminus\{0\}}.$$

Proof: Omissis. Cf. [Mo1], proposition 4.1 $\sharp$

As claimed at the beginning of the paragraph, we can translate the KZ-restriction of $\pi(r,0,1)|_{KZ}$ in terms of the R_n's and $T_n^{\pm}$:

PROPOSITION 2.3. *We have a KZ-equivariant isomorphism:*

$$\pi(r,0,1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{m\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1})$$

where we define an action of Z on the left hand side by making $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ *act trivially.*

Proof: Omissis. Cf. [Mo1], proposition 3.9.$\sharp$

2.2 Socle filtration for $\pi(r,0,1)|_{KZ}$ and parabolic inductions

Let us fix an integer $r \in \{0,\dots,p-1\}$. In this paragraph we are going to define a filtration (definition 2.4) on the inductive limits of proposition 2.3. Such filtration is rather finer than the one which can be deduced from proposition 2.2 and will let us describe the socle filtration for $\pi(r,0,1)|_{KZ}$. In what follows, we will assume the obvious conventions $R_0 \oplus_{R_{-1}} R_0 \overset{\mathrm{def}}{=} R_0$ and $R_1/R_0 \oplus_{R_0} R_1 \overset{\mathrm{def}}{=} R_1/R_0$.

DEFINITION 2.4. *Let $n \in \mathbf{N}$, $0 \leqslant h \leqslant r$. We define $\mathrm{Fil}^h(R_{n+1})$ as the K-subrepresentation of R_{n+1} generated by $[1_K, X^{r-h}Y^h]$. For $h=-1$, we define $\mathrm{Fil}^{-1}(R_{n+1}) \overset{\mathrm{def}}{=} 0$.*

The family $\{\mathrm{Fil}^h(R_{n+1})\}_{h=-1}^{r}$ defines a separated and exhaustive filtration on R_{n+1}, and for each $h \in \{0,\dots,r\}$ the family

$$\mathscr{B}_{n+1,t} \overset{\mathrm{def}}{=} \left\{ [\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-i}Y^i],\ [\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^{r-i}Y^i]\, \mu \in I_{n+1},\ \mu' \in I_n,\ 0 \leqslant i \leqslant h \right\}$$

is an $\overline{\mathbf{F}}_p$ basis for $\mathrm{Fil}^h(R_{n+1})$. By Frobenius reciprocity we get a K-isomorphism

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^h \xrightarrow{\sim} \mathrm{Fil}^h(R_{n+1})/\mathrm{Fil}^{h-1}(R_{n+1})$$

(cf. [Mo1], lemma 4.4).

To give explicit description for the socle filtration of the induced representation $\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_r^s \mathfrak{a}^h$ needs the introduction of the following elements.

DEFINITION 2.5. *Fix $n \in \mathbf{N}$ and let $(l_1,\dots,l_n) \in \{0,\dots,p-1\}^n$ be an n-tuple. We define then*

$$F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} \overset{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1,e]$$

where e is an $\overline{\mathbf{F}}_p$-basis for the underlying vector space associated to the $K_0(p^{n+1})$-representation χ_r^s.

For a fixed n-tuple $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ we set $h'\overset{\text{def}}{=}\sum_{j=1}^n l_j$. Then

$$F_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\overset{\text{def}}{=}\begin{cases}\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\not\equiv 0\,[p-1]; \\ \\ \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}]+(-1)^{h+h'}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\equiv 0\,[p-1]\end{cases}$$

$$F_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\overset{\text{def}}{=}\begin{cases}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\not\equiv 0\,[p-1]; \\ \\ \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\equiv 0\,[p-1].\end{cases}$$

Such definitions look a bit awkward, but they come essentially from the description of the socle filtration for $\mathrm{GL}_2(\mathbf{F}_p)$-parabolic inductions (proposition 2.9)

We provide the set $\{0,1\}\times\{0,\dots,p-1\}^n$ with the antilexicographic ordering, writing $(i+1,l_1,\dots,l_n)$ for the $n+1$-tuple immediately succeeding $(i,l_1,\dots,l_n)$. We introduce then the quotients

$$Q(h)_{i,l_1,\dots,l_n}^{(0,n+1)}\overset{\text{def}}{=}\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^h/(\langle K\cdot F_j^{(0)}*\dots F_{j_n}^{(n)}\rangle,\ \text{for}\,(j,j_0,\dots,j_n)\prec(i,l_1,\dots,l_n)).$$

We remark that such notations do not keep track of the integer r; moreover if there will not be any ambiguities on h, we will simply write $Q_{i,l_1,\dots,l_n}^{(0,n+1)}$ instead of $Q(h)_{i,l_1,\dots,l_n}^{(0,n+1)}$. We believe such notations will not arise any confusion: the meaning will be clear from the context (cf. §3, §4). We are now able to give a complete description for the socle filtration of $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^h$:

Proposition 2.6. *Let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple, and let $h'\overset{\text{def}}{=}\sum_{i=1}^n l_i$. Then*

i) the socle of $Q_{1,l_1,\dots,l_n}^{(0,n+1)}$ is described by

$$\mathrm{soc}(Q_{1,l_1,\dots,l_n}^{(0,n+1)})=\langle KF_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\cong\mathrm{Sym}^{p-1-\lfloor r-2(h+h')\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{r-(h+h')}$$

moreover, $F_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}$ is an H-eigenvector whose associated eigencharacter is

$$\chi_{2(h+h')-r}\det{}^{r-(h+h')}.$$

ii) the socle of $Q_{0,l_1,\dots,l_n}^{(0,n+1)}$ is described by

$$\mathrm{soc}(Q_{0,l_1,\dots,l_n}^{(0,n+1)})=\begin{cases}\langle KF_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\cong\mathrm{Sym}^{\lfloor r-2(h+h')\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{h+h'} \\ \text{if } r-2(h+h')\not\equiv 0[p-1]; \\ \\ \langle KF_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\oplus\langle KF_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\cong \\ \cong\det{}^{h+h'}\oplus\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2\otimes\det{}^{h+h'} \\ \text{if } r-2(h+h')\equiv 0[p-1].\end{cases}$$

Further, $F_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}$ (and moreover $F_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}$ if $r-2(h+h')\equiv 0\,[p-1]$) is an H-eigenvector whose associated eigencharacter is $\chi_{r-2(h+h')}\det^{h+h'}$.

Proof: Omissis. Cf. [Mo1], proposition 6.6.♯

The filtration $\{\mathrm{Fil}^h(R_{n+1})\}_{h=-1}^r$ induces a natural filtration on the quotient R_{n+1}/R_n such that $\mathrm{Fil}^h(R_{n+1}/R_n)/\mathrm{Fil}^{h-1}(R_{n+1}/R_n) \cong \mathrm{Fil}^h(R_{n+1})/\mathrm{Fil}^{h-1}(R_{n+1})$ for all $h > 0$; concerning $h = 0$ we see (cf. [Mo1] lemma 8.3) that $\mathrm{Fil}^0(R_{n+1}/R_n) \cong Q^{(0,n+1)}_{0,\dots,0,r+1}$. The main result of [Mo1] (cf. proposition 9.1) is that we can describe the socle filtration of $\pi(r,0,1)|_{KZ}$ in terms of the socle filtration of the quotients R_{n+1}/R_n. Precisely:

Proposition 2.7. *Let $r \in \{0,\dots,p-1\}$, $n \in \mathbf{N}$. Then*

i) The socle filtration for R_{n+1}/R_n is described by

$$\begin{aligned}&\mathrm{SocFil}(R_{n+1}/R_n) =\\ &= \mathrm{SocFil}(Q^{(0,n+1)}_{0,\dots,r+1}) \text{—} \mathrm{SocFil}(\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a}) \text{—} \dots \text{—} \mathrm{SocFil}(\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a}^r)\end{aligned}$$

(where, if $r = p-1$, we forget about $\mathrm{SocFil}(Q^{(0,n+1)}_{0,\dots,r+1})$ and the socle filtration starts from $\mathrm{SocFil}(\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a})$)

ii) If n is odd, the socle filtration for $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ is described by

$$\begin{aligned}&\mathrm{SocFil}(R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) =\\ &= R_0 \text{—} \mathrm{SocFil}(R_2/R_1) \text{—} \mathrm{SocFil}(R_4/R_3) \text{—} \dots \text{—} \mathrm{SocFil}(R_{n+1}/R_n).\end{aligned}$$

iii) If n is even, the socle filtration for $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ is described by

$$\begin{aligned}&\mathrm{SocFil}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}) =\\ &= \mathrm{SocFil}(R_1/R_0) \text{—} \mathrm{SocFil}(R_3/R_2) \text{—} \mathrm{SocFil}(R_5/R_4) \text{—} \dots \text{—} \mathrm{SocFil}(R_{n+1}/R_n).\end{aligned}$$

Proof: Omissis. Cf. [Mo1], proposition 7.1 and 9.1.♯

In particular, we are able to compute the dimension of some subquotients of $\pi(r,0,1)|_{KZ}$.

Lemma 2.8. *Let $r \in \{0,\dots,p-1\}$.*

i) Let $t \in \mathbf{N}_>$; then

$$\dim_{\overline{\mathbf{F}}_p}(\mathrm{Fil}^0(R_t/R_{t-1})) = \begin{cases} (p-1-r)(p+1)p^{t-2} & \text{if } t \geqslant 2 \\ p-r & \text{if } t = 1. \end{cases}$$

ii) Let $t \in \mathbf{N}_>$; then

$$\dim_{\overline{\mathbf{F}}_p}(R_t/R_{t-1}) = \begin{cases} (r+1)(p^2-1)p^{t-2} & \text{if } t \geqslant 2 \\ p(r+1) & \text{if } t = 1. \end{cases}$$

iii) If n is odd then

$$\dim_{\overline{\mathbf{F}}_p}(R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) = (r+1)p^{n+1}.$$

iv) If n is even, then

$$\dim_{\overline{\mathbf{F}}_p}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}) = (r+1)p^{n+1}.$$

Proof: It is an elementary computation, using [Mo1], Corollary 6.5-*iii*) and the decompositions of proposition 2.7.♯

2.2.1 *Induced representations for* $B(\mathbf{F}_p)$. Let us consider the $B(\mathbf{F}_p)$-character $\chi_i^s\mathfrak{a}^j$. If e is a fixed $\overline{\mathbf{F}}_p$-basis for the underlying vector space associated to $\chi_i^s\mathfrak{a}^j$, we define the following elements of the induced representations $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$:

$$f_k \stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\mu_0^k\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,e]$$

where $k\in\{0,\dots,p-1\}$. We can give an explicit description of the socle filtration for $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ in terms of the functions f_k:

Proposition 2.9. *Let $i,j\in\{0,\dots,p-1\}$. Then*

i) for $k\in\{0,\dots,p-1\}$, f_k is an H-eigenvector, whose associated eigencharacter is $\chi_{i-2j}\det^j\mathfrak{a}^{-k}$, and the family

$$\mathscr{B}\stackrel{\mathrm{def}}{=}\{f_k,\,0\leqslant k\leqslant p-1,\,[1,e]\}$$

is an $\overline{\mathbf{F}}_p$-basis for $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$.

ii) If $i-2j\not\equiv 0\,[p-1]$ then we have a nontrivial extention

$$0\to\mathrm{Sym}^{\lfloor i-2j\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^j\to\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j\to\mathrm{Sym}^{p-1-\lfloor i-2j\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{i-j}\to 0.$$

The families

$$\{f_0,\dots,f_{\lfloor i-2j\rfloor-1},f_{\lfloor i-2j\rfloor}+(-1)^{i-j}[1,e]\},\{f_{i-2j},\dots,f_{p-1}\}$$

induce a basis for the socle and the cosocle of $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ respectively.

iii) If $i-2j\equiv 0\,[p-1]$ then $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ is semisimple and

$$\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)\chi_i^s\mathfrak{a}^j}\xrightarrow{\sim}\det{}^j\oplus\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2\otimes\det{}^j;$$

The families

$$\{f_0+(-1)^j[1,e]\},\{f_0,f_1,\dots,f_{p-2},f_{p-1}+(-1)^j[1,e]\}$$

induce an $\overline{\mathbf{F}}_p$-basis for $\det^j$ and $\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2\otimes\det^j$ respectively.

Proof: Omissis. Cf. [BP], lemmas 2.5, 2.6, 2.7.♯

2.3 Computations on Witt vectors.

In this paragraph we collect all the technical computations needed for the study of K_t and I_t-invariants. For $\mu,\lambda\in\mathbf{F}_p$ we define

$$P_\lambda(\mu)\stackrel{\mathrm{def}}{=}-\sum_{j=1}^{p-1}\frac{\binom{p}{j}}{p}\lambda^{p-j}\mu^j\in\mathbf{F}_p.$$

Then, we have the following results concerning the sum of Witt vectors in $\mathbf{Z}_p$:

Lemma 2.10. *Let $\lambda\in\mathbf{F}_p$, $\sum_{j=0}^n p^j[\mu_j]\in I_{n+1}$. Then the following equality holds in $\mathbf{Z}_p/(p^{n+1})$:*

$$[\lambda]+\sum_{j=0}^n p^j[\mu_j]\equiv[\lambda+\mu_0]+p[\mu_1+P_\lambda(\mu_0)]+\cdots+p^n[\mu_n+P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1})]$$

where, for all $j = 1, \dots, n-2$, the $P_{\lambda,\dots,\mu_j}(X)$'s (resp. $P_{\lambda,\mu_0}(X)$, resp. $P_\lambda(X)$) are suitable polynomials in $\mathbf{F}_p[X]$, of degree $p-1$, depending only on $\lambda, \dots, \mu_j$ (resp. on λ, μ, resp. on λ), and whose dominant coefficient is $-P_{\lambda,\dots,\mu_{j-1}}(\mu_j)$ (resp. $-P_\lambda(\mu_0)$, resp. $-\lambda$).

Proof: Immediate exercise on Witt vectors in $\mathbf{Z}_p$. $\sharp$

LEMMA 2.11. *Let $\lambda \in \mathbf{F}_p$, $z \stackrel{\text{def}}{=} \sum_{j=1}^n p^j[\mu_j]$ and $k \geqslant 0$. Then it exists a p-adic integer $z' = \sum_{j=1}^n p^j[\mu'_j] \in \mathbf{Z}_p$ such that*

$$z \equiv z'(1 + zp^k[\lambda]) \bmod p^{n+1}.$$

Furthermore, for $j = k+3, \dots, n$ (resp. $j = k+2$, resp. $j \leqslant k+1$) we have the following equality in $\mathbf{F}_p$:

$$\mu_j = \mu'_j + \mu_{j-k-1}\mu'_1\lambda + \cdots + \mu_1\mu_{j-k-1}\lambda + S_{j-2}(\mu_{j-1})$$

(resp. $\mu_{k+2} = \mu'_{k+2} + \mu'_1\mu_1\lambda$ for $j = k-2$, resp. $\mu_j = \mu'_j$ if $j \leqslant k+1$) where $S_{j-2}(X) \in \mathbf{F}_p[X]$ is a polynomial of degree $p-1$, depending only on $\lambda, \dots, \mu_{j-2}$ and leading coefficient $-s_{\lambda,\dots,\mu_{j-2}} \stackrel{\text{def}}{=} \mu'_{j-1} - \mu_{j-1}$.

Proof: Exercise on Witt vectors.$\sharp$

To conclude, we give a criterion to detect wether a certain element (which naturally appears in the study of K_t and I_t-invariants) is zero in the amalgamed sums $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ (n odd) and $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ (n even).

LEMMA 2.12. *Let $k \geqslant 2$ and fix an $(k-1)$-tuple $(l_1, \dots, l_{k-1})$. If we set*

$$x_{l_1,\dots,l_{k-1}} \stackrel{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots$$
$$\cdots \sum_{\mu_{k-1} \in \mathbf{F}_p} \mu_{k-2}^{l_{k-2}} \begin{bmatrix} 1 & 0 \\ p^{k-2}[\mu_{k-2}] & 1 \end{bmatrix} \sum_{\mu_{k-1} \in \mathbf{F}_p} \mu_{k-1}^{l_{k-1}}[1, (\mu_{k-1}X + Y)^r] \in R_{k-1}.$$

i) Assume k odd. We describe the image of $x_{l_1,\dots,l_{k-1}}$ in the amalgamed sum $R_0 \oplus_{R_1} \cdots \oplus_{R_{k-2}} R_{k-1}$ as follow:

- *a) $x_{l_1,\dots,l_{k-1}} \equiv 0$ if $(l_1, \dots, l_{k-1}) \prec (r, p-1-r, \dots, r, p-1-r)$;*
- *b) if $(r, p-1-r, \dots, r, p-1-r) \prec (l_1, \dots, l_{k-1})$, then the image of $x_{l_1,\dots,l_k-1}$ induces a I_1-invariant generator in a subquotient of $R_0 \oplus_{R_1} \cdots \oplus_{R_{k-2}} R_{k-1}$ of the form $\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t'}$ for some suitable $t' \in \mathbf{N}$;*
- *c) equal to (the image of) $(-1)^{(r+2)(\frac{k-1}{2})} Y^r \in R_0$ if $(l_1, \dots, l_{k-1}) = (r, p-1-r, \dots, r, p-1-r)$.*

ii) Assume k even. We describe the image of $x_{l_1,\dots,l_{k-1}}$ in the amalgamed sum $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{k-2}} R_{k-1}$ as follow:

- *a′) $x_{l_1,\dots,l_k-1} \equiv 0$ if $(l_1, \dots, l_{k-1}) \prec (p-1-r, r, \dots, r, p-1-r)$;*
- *b′) if $(p-1-r, r, \dots, r, p-1-r) \prec (l_1, \dots, l_{k-1})$, then the image of $x_{l_1,\dots,l_k-1}$ induces a $I_!$-invariant generator of a subquotient of $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{k-2}} R_{k-1}$ of the form $\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t'}$ for some suitable $t' \in \mathbf{N}$;*
- *c′) equal to (the image of) $(-1)^{(r+2)(\frac{k-2}{2})}(-1)[1, X^r] \in \mathrm{Fil}^0(R_1/R_0)$ if $(l_1, \dots, l_{k-1}) = (p-1-r, r, \dots, r, p-1-r)$.*

Proof: It is an induction on k; we treat the case k even, the other being completely similar. The result is clearly true for $k=2$. For the general case, we consider the image of the element

$$u \stackrel{\text{def}}{=} \sum_{j=0}^{r}\binom{r}{j}\sum_{\mu_{k-2}\in\mathbf{F}_p}\mu_{k-2}^{l_{k-2}}\left[\begin{array}{cc}1 & 0\\ p^{k-2}[\mu_{k-2}] & 1\end{array}\right][1,X^{r-j}Y^j]\sum_{\mu_{k-1}\in\mathbf{F}_p}\mu_{k-1}^{l_{k-1}+r-j}\in R_{k-1}$$

in R_{k-1}/R_{k-2} via the natural projection $R_{k-1}\twoheadrightarrow R_{k-1}/R_{k-2}$. We see then if $(r+1,p-1-r)\preceq(l_{k-2},l_{k-1})$ such an image is nonzero in R_{k-1}/R_{k-2}; we deduce that the image of $x_{l_1,\ldots,l_{k-1}}$ in R_{k-1}/R_{k-2} is a K-generator of a subquotient of the form $\mathrm{Ind}_{K_0(p^{k-1})}^{K}\chi_r^s\mathfrak{a}^{t'}$, for a suitable $t'\in\mathbf{N}$. If $l_{k-1}=p-1-r$ and $l_{k-2}\leqslant r$ we see that u is in the image of T_{k-2}^{+}:

$$u=T_{k-2}^{+}((-1)^{l_{k-2}+1}[1,X^{r-l_{k-2}}Y^{l_{k-2}}]).$$

If $l_{k-2}<r$, then $T_{k-2}^{-}([1,X^{r-l_{k-2}}Y^{l_{k-2}}])=0\in R_{k-3}$, while for $l_{k-2}=r$ we get

$$-T_{k-2}^{-}((-1)^{r+1}[1,X^{r-l_{k-2}}Y^{l_{k-2}}]=(-1)^{r+2}[1,Y^r]\in R_{k-3}.$$

This let us establish the inductive step and the proof is complete.♯

3. Study of K_t-invariants

Fix an integer $r\in\{0,\ldots,p-2\}$; in this section we use the explicit description of the socle filtration of $\pi(r,0,1)|_{KZ}$ to deduce the space of K_t-invariants $\pi(r,0,1)^{K_t}$.

We start from rough estimates of such spaces in terms of the filtrations $R_i/(R_{i-1})\oplus_{R_{i+1}}\cdots\oplus_{R_n}R_{n+1}$ in lemmas 3.2 and 3.3: they let us rule out a wide range of possibilities for the K_t invariants. For those cases which are not covered by the previous estimates, we pursue a detailed (and, unfortunately, rather technical) analysis, by means of the elements introduced in definition 3.4. Such analysis lead us to refine the results of lemmas 3.2 and 3.3 in proposition 3.5 from which we deduce the exact description of the K_t-invariants given in proposition 3.7.

First of all, we have

$$K_t=\left[\begin{array}{cc}1 & 0\\ p^t\mathbf{Z}_p & 1\end{array}\right]\left[\begin{array}{cc}1+p^t\mathbf{Z}_p & 0\\ 0 & 1+p^t\mathbf{Z}_p\end{array}\right]\left[\begin{array}{cc}1 & p^t\mathbf{Z}_p\\ 0 & 1\end{array}\right].$$

Furthermore

LEMMA 3.1. *If σ is a smooth K-representation over $\overline{\mathbf{F}}_p$ and $t\in\mathbf{N}$, then*

$$\mathrm{soc}_K(\sigma)=\mathrm{soc}_{K/K_t}(\sigma^{K_t}).$$

Proof: It is enough to recall that for any irreducible smooth K representation τ we have $\tau^{K_1}=\tau$. ♯

LEMMA 3.2. *Let $t\geqslant 1$. Then*

i) If t is odd then

$$(\varinjlim_{n,\,\mathrm{odd}} R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})^{K_t}=(R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1})^{K_t}$$

$$(\varinjlim_{n,\,\mathrm{even}} R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1})^{K_t}=\begin{cases}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-1}}R_t)^{K_t} & \text{if}\quad r\neq 0\\ (R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t+1}}R_{t+2})^{K_t} & \text{if}\quad r=0.\end{cases}$$

ii) If t is even, then

$$(\varinjlim_{n,\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_t} = \begin{cases} (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-1}} R_t)^{K_t} & \text{if} \quad r \neq 0 \\ (R_0 \oplus_{R_1} \cdots \oplus_{R_{t+1}} R_{t+2})^{K_t} & \text{if} \quad r = 0. \end{cases}$$

$$(\varinjlim_{n,\,\mathrm{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})^{K_t} = (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_t} R_{t+1})^{K_t}$$

Proof: We first prove the statement for $r \neq 0$. Let $n > t$ and assume we have $z \in (\cdots \oplus_{R_n} R_{n+1})^{K_t}$ such that $\pi(z) \neq 0$ in R_{n+1}/R_n (where π denotes the natural projection of proposition 2.2). As K_t is normal in K, we conclude that $\pi(\langle K \cdot z \rangle) \leqslant (R_{n+1}/R_n)^{K_t}$ and, by lemma 3.1, $\mathrm{soc}_K(\pi(\langle K \cdot z\rangle)) \cap \mathrm{soc}_K(R_{n+1}/R_n) \neq 0$. By the explicit description of $\mathrm{soc}_K(R_{n+1}/R_n)$ we deduce that it exists $y \in (\cdots \oplus_{R_{n-2}} R_{R-1})$ such that we are in one of the following situations:

i) the element
$$\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x' + y$$
is K_t-invariant (in the amalgamed sum);

ii) we have $p - 3 - r = 0$ and the element
$$\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x' + (-1)^{r+1} x' + y$$
is K_t-invariant (in the amalgamed sum);

where we put
$$x' \overset{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{n-1} \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1}] & 1 \end{bmatrix} \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{r+1} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1, X^r].$$

Consider now the projection
$$(\cdots \oplus_{R_n} R_{n+1}) \twoheadrightarrow R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}) \oplus_{R_n} R_{n+1}.$$

As the space $(R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}))$ is fixed under the action of $\begin{bmatrix} 1 & p^n \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$, it follows that the elements $\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x'$ (resp. $\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x' + (-1)^{r+1} x'$) should be fixed under the action of $\begin{bmatrix} 1 & p^n \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$ inside $R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}) \oplus_{R_n} R_{n+1}$, which is absurde. Indeed, a computation using lemma 2.10 shows
$$\begin{bmatrix} 1 & p^{n-1}[\lambda] \\ 0 & 1 \end{bmatrix} x - x = \sum_{j=1}^{r+1} \binom{r+1}{j} (-1)^{r+1-j} T_n^+(v_j)$$
where
$$v_j \overset{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{n-1} \in \mathbf{F}_p} (-P_\lambda(\mu_{n-1}))^j \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1} + \lambda] & 1 \end{bmatrix} [1, X^{j-1} Y^{r-(j-1)}].$$

Using the operator $-T_n^-$ and the natural projection $R_n \twoheadrightarrow R_n/\mathrm{Fil}^{r-1}(R_n)$ we get

$$-T_n^-(\sum_{j=1}^{r+1}\binom{r+1}{j}(-1)^{r+1-j}v_j) \equiv$$
$$\equiv (r+1)(-1)^{r+2}\lambda\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots\sum_{\mu_{n-2}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{n-2}[\mu_{n-2}] & 1\end{bmatrix}[1,Y^r] \mod \mathrm{Fil}^{r-1}(R_{n-1})$$
$$(\text{resp. } \equiv (r+1)(-1)^{r+2}\lambda\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,Y^r] \quad \text{if } n=2)$$

and such an element is nonzero in $R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1})$ if $r\neq 0$. As x' is anyway $\begin{bmatrix}1 & p^{n-1}[\lambda]\\ 0 & 1\end{bmatrix}$-invariant in R_{n+1} we deduce that the elements in $i)$, $ii)$ can be K_t-invariant only if $n-1<k$: this let us conclude the case $r\neq 0$.

We pass to the the case $r=0$ and and let $n>t+1$. Using the same arguments for the case $r\neq 0$ we see that if we have $z\in(\cdots\oplus_{R_n}R_{n+1})^{K_t}$ such that $\pi(z)\neq 0$ in R_{n+1}/R_n it would exists an element $\overline{y}\in R_{n-1}/R_{n-2}$ such that $w\stackrel{\text{def}}{=}\overline{y}+\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}x'$ (resp. $w\stackrel{\text{def}}{=}\overline{y}+\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}x'+(-1)^{r+1}x'$ if $r=p-3$) is $\begin{bmatrix}1 & p^{n-2}\mathbf{Z}_p\\ 0 & 1\end{bmatrix}$ invariant inside $R_{n-1}/R_{n-2}+\langle w\rangle$. On the other hand, we have a decomposition of R_{n-1}/R_{n-2} in $\begin{bmatrix}1 & p^{n-2}\\ 0 & 1\end{bmatrix}$-stable subspaces. Indeed R_{n-1}/R_{n-2} is a quotient of $\mathrm{Ind}_{K_0(p^{n-1})}^{K}1$ and if we put

$$w'_{l_1,\dots,l_{n-2}}(0)\stackrel{\text{def}}{=}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\dots\sum_{\mu_{n-1}\in\mathbf{F}_p}\mu_{n-1}^{l_{n-1}}\begin{bmatrix}1 & 0\\ p^{n-1}[\mu_{n-1}] & 1\end{bmatrix}[1,e]$$

the latter admits the following decomposition in $\begin{bmatrix}1 & p^{n-2}\mathbf{Z}_p\\ 0 & 1\end{bmatrix}$-stable subspaces:

a) for a fixed -tuple $(l_0,\dots,l_{n-3})\in\{0,\dots,p-1\}^{n-2}$ the $\overline{\mathbf{F}}_p$-subspace generated by the elements

$$\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}w_{l_1,\dots,l_{n-3},j}(0)$$

where $j\in\{0,\dots,p-1\}$;

b) the subspace generated by the elements $w'_{l_1,\dots,l_{n-2}}(0)$ for $(l_1,\dots,l_{n-3})\in\{0,\dots,p-1\}^{n-3}$.

For $r\neq p-3$ (resp. $r=p-3$) we study, analogously to the case $r\neq 0$, the action of $\begin{bmatrix}1 & p^{n-2}\mathbf{Z}_p\\ 0 & 1\end{bmatrix}$ on the element $\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}x'+(-1)^{r+1}x'$ (resp. $\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}x'+(-1)^{r+1}x'+(-1)^{r+1}x'$); using the previous decomposition in stable subspaces for R_{n-1}/R_{n-2} we deduce again a contraddiction with the assumption $\pi(z)\neq 0$ (the computational details are left to the reader). $\sharp$

On the other hand, we have

LEMMA 3.3. *Let $t\geqslant 1$. Then:*

i) for t odd we have

$$\begin{aligned}(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1})^{K_t} &= R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} \\ (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2})^{K_t} &= R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} \quad \text{if } r \neq 0 \\ (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-1}} R_t)^{K_t} &= R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-1}} R_t \quad \text{if } r = 0\end{aligned}$$

ii) for t even we have

$$\begin{aligned}(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2})^{K_t} &= R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2} \quad \text{if } r \neq 0 \\ (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-1}} R_t)^{K_t} &= R_0 \oplus_{R_1} \cdots \oplus_{R_{t-1}} R_t \quad \text{if } r = 0 \\ (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1})^{K_t} &= R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1}\end{aligned}$$

where we convene that $R_1/R_0 \oplus_{R_{-2}} R_{-1} \overset{\text{def}}{=} 0$.

Proof: If $\kappa \in K$ and $z \in I_t$ then

$$\kappa \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} \kappa_1 ; \kappa \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \kappa_2$$

for suitable κ_1, $\kappa_2 \in K_t$. As the action of K_t is trivial on σ_r^j for $j < t$ (resp for $j \leqslant t$ if $r = 0$) we get the desired result. ♯

We are thus able to insert the K_t invariants into a natural exact sequence coming from the filtrations of proposition 2.2. For instance, for t odd we have

$$\begin{tikzcd} 0 \arrow[r] & (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1})^{K_t} \arrow[r] \arrow[d, equal] & (R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1})^{K_t} \arrow[r] \arrow[d, hook] & (R_{t+1}/R_t)^{K_t} \arrow[d, hook] & \\ 0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1} \arrow[r] & R_{t+1}/R_t \arrow[r] & 0 \end{tikzcd}$$

and the reader can deduce similar diagrams, according to lemmas 3.2, 3.3. In particular, we are lead to the study of the K_t invariants of the quotients R_{n+1}/R_n, which is the object of the next proposition. We introduce the following notations:

Definition 3.4. *Let $t \geqslant 2$ be an integer, $(l_1, \ldots, l_{t-1}) \in \{0, \ldots, p-1\}^{k-1}$ be an $(t-1)$-tuple. We define:*

i)

$$\begin{aligned} x_{l_1,\ldots,l_{t-1}} &\overset{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \\ &\cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r]; \\ x'_{l_1,\ldots,l_{t-1}} &\overset{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \\ &\cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} [1, X^r] \end{aligned}$$

which will be seen as elements of R_{t+1}, R_{t+1}/R_t or in the amalgamed sum, accordingly to the context.

ii) *if* $r \neq 0$ *we define*

$$y_{l_1,\dots,l_{t-1}} \overset{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{r-1}Y];$$

$$y'_{l_1,\dots,l_{t-1}} \overset{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{r-1}Y]$$

which will be seen as elements of R_t, R_t/R_{t-1} *or in the amalgamed sum, accordingly to the context.*

iii) *if* $r = 0$ *and* X^r *is a fixed* $\overline{\mathbf{F}}_p$ *basis of* $\mathrm{Sym}^0 \overline{\mathbf{F}}_p^2$, *we define*

$$z_{l_1,\dots,l_{t-1}} \overset{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} \sum_{\mu_{t+1} \in \mathbf{F}_p} \mu_{t+1} \begin{bmatrix} 1 & 0 \\ p^{t+1}[\mu_{t+1}] & 1 \end{bmatrix} [1, X^r];$$

$$z'_{l_1,\dots,l_{t-1}} \overset{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} \sum_{\mu_{t+1} \in \mathbf{F}_p} \mu_{t+1} \begin{bmatrix} 1 & 0 \\ p^{t+1}[\mu_{t+1}] & 1 \end{bmatrix} [1, X^r]$$

which will be seen as elements of R_{t+2}, R_{t+2}/R_{t+1} *or in the amalgamed sum, accordingly to the context.*

The result concerning the K_t-invariants of R_{t+1}/R_t is the following

Proposition 3.5. *Let* $t \geqslant 1$, $r \in \{0, \dots, p-2\}$. *Then*

i) *the* K_t *invariants of* R_{t+1}/R_t *are described by*

$$(R_{t+1}/R_t)^{K_t} = \mathrm{Ind}_{K_0(p^t)}^{K} F_{r+1}^{(t)} \hookrightarrow \mathrm{Fil}^{(0)}(R_{t+1}/R_t);$$

ii) *if* $r \neq 0$ *and* $t \geqslant 2$ *the* K_t*-invariants of* R_t/R_{t-1} *are described by*

$$(R_t/R_{t-1})^{K_t} = (\mathrm{Fil}^0(R_t/R_{t-1}) + \tau_t) \hookrightarrow \mathrm{Fil}^1(R_t/R_{t-1})$$

where τ_t *is the* K*-subrepresentation of* $\mathrm{Fil}^1(R_t/R_{t-1})$ *generated by (the image of) the elements* $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ *with* $(l_1, \dots, l_{t-1}) \prec (0, \dots, 0, r+1)$. *If* $t = 1$ *then the* K_1*-invariants of* R_1/R_0 *are described by*

$$(R_1/R_0)^{K_1} = (\mathrm{Fil}^0(R_1/R_0) + \tau_1) \hookrightarrow \mathrm{Fil}^1(R_1/R_0)$$

where $\tau_1 = 0$ *if* $r \in \{0, 1\}$ *and* τ_1 *is the* K*-subrepresentation of* $\mathrm{Fil}^1(R_1/R_0)$ *generated by* $\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-1}Y]$ *if* $r \geqslant 3$

(resp. by $\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-1}Y]-[1,X^{r-1}Y]$ *if* $r=2$*).*

iii) If $r=0$ *then the* K_t*-invariants of* R_{t+2}/R_{t+1} *are described by*

$$(R_{t+2}/R_{t+1})^{K_t}=\mathrm{Ind}_{K_0(p^k)}^{K}F_0^{(t)}*F_1^{(t+1)}\hookrightarrow Q_{0,\dots,0,1}^{(0,t+2)}$$

Proof: First, let $z\in(R_{t+1}/R_t)^{K_t}$, say $z\in\mathrm{Fil}^t(R_{t+1}/R_t)\setminus\mathrm{Fil}^{t-1}(R_{t+1}/R_t)$. We deduce, as in the proof of lemma 3.2 that one of the following condition must hold:

a) the element

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}v'_t$$

is K_t-invariant;

b) we have $r-2t\equiv 0\,[p-1]$ and the element

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}v'_t+(-1)^tv'_t$$

is K_t-invariant,

where we put

$$v'_t\stackrel{\mathrm{def}}{=}\sum_{\mu_1\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\cdots\sum_{\mu_t\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^t[\mu_t] & 1\end{bmatrix}[1,X^{r-t}Y^t].$$

For $t\geqslant 1$ we study the action of $\begin{bmatrix}1 & p^t[\lambda]\\ 0 & 1\end{bmatrix}$ on the elements in *a*), *b*) modulo Fil^{t-2} to deduce that such elements can not be K_t-invariant: we conclude that

$$(R_{t+1}/R_t)^{K_t}=\mathrm{Fil}^0(R_{t+1}/R_t).$$

A similar argument, using the exact sequence

$$0\to\mathrm{Ind}_{K_0(p^t)}^{K}F_{r+1}^{(t)}\to Q_{0,\dots,0,r+1}^{(0,t+1)}\to Q_{0,\dots,0,r+2}^{(0,t+1)}\to 0$$

shows that

$$(Q_{0,\dots,0,r+1}^{(0,t+1)})^{K_t}=(\mathrm{Ind}_{K_0(p^t)}^{K}F_{r+1}^{(t)})^{K_t}.$$

As the latter is K_t-invariant we get the desired result.

ii) Assume $t\geqslant 2$. With the same arguments of *i*) we can check that

$$(R_t/R_{t-1})^{K_t}=(\mathrm{Fil}^1(R_t/R_{t-1}))^{K_t}$$

and, using the definition of $Q(1)_{0,\dots,0,r+1}^{(0,t)}$ and the fact that $\mathrm{Fil}^0(R_t/R_{t-1})$ is a quotient of $\mathrm{Ind}_{K_0(p^t)}^{K}\chi_r^s$, we have

$$(\mathrm{Fil}^1(R_t/R_{t-1}))^{K_t}=(\mathrm{Fil}^0(R_t/R_{t-1})+\tau_t)^{K_t}.$$

We can now check directly that the action of

$$\begin{bmatrix}1+p^t\mathbf{Z}_p & 0\\ 0 & 1+p^t\mathbf{Z}_p\end{bmatrix},\begin{bmatrix}1 & 0\\ p^t\mathbf{Z}_p & 1\end{bmatrix}$$

on $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}\in R_t/R_{t-1}$ is trivial, provided that $l_{t-1}\leqslant r$, and we conclude.

The case of $t=1$ is a similar computation, and it is left to the reader.

iii) It is similar to the previous one and left to the reader. ♯

Thanks to proposition 3.5 it suffices to study the behaviour of the elements of the form $x^{(\prime)}_{l_1,\dots,l_{k-1}}$, $y^{(\prime)}_{l_1,\dots,l_{k-1}}$, $z^{(\prime)}_{l_1,\dots,l_{k-1}}$ in order to describe completely the K_t-invariants of supersingular representations. First of all we introduce the objects:

Definition 3.6. *Let $t \geqslant 1$. We define the following subrepresentations of R_{t+1}, R_t:*

i) for $t \geqslant 2$, t odd, let

a_1) $\sigma(p-2)$ as the K-subrepresentation of R_{t+1} generated by $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$ with $(l_1,\dots,l_{t-1}) \prec (r, p-1-r, \dots, r, p-1-r)$;

a_2) $\sigma(p-3)$ as the K-subrepresentaiton of R_{t+1} generated by $\sigma(p-2)$ and the element $x_{r,p-1-r,\dots,r,p-1-r} + (-1)^{r+1}x'_{r,p-1-r,\dots,r,p-1-r}$;

a_3) $\sigma(<p-3)$ as the K-subrepresentation of R_{t+1} generated by $\sigma(p-2)$ and the element $x_{r,p-1-r,\dots,r,p-1-r}$.

b_1) if $r \neq 0$, $\sigma^s_y(1)$ (resp. $\sigma^s_z(0)$ if $r=0$) as the K-subrepresentation of R_t generated by $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ (resp. $z_{l_1,\dots,l_{t-1}}$, $z'_{l_1,\dots,l_{t-1}}$) with $(l_1,\dots,l_{t-1}) \prec (p-1-r, r, \dots, p-1-r, r)$ if $r \neq 0$ (resp. if $r=0$).

b_2) if $r \neq 0$, $\sigma^s_y(2)$ as the K-subrepresentaiton of R_t generated by $\sigma^s_y(1)$ and the element $y_{p-1-r,r,\dots,p-1-r,r} + (-1)^{(r-2)+1}y'_{p-1-r,r,\dots,p-1-r,r}$;

b_3) if $r \neq 0$, $\sigma^s_y(>2)$ as the K-subrepresentation of R_t generated by $\sigma^s_y(1)$ and the element $y_{p-1-r,r,\dots,r,p-1-r,r}$.

ii) For $t \geqslant 2$, t even, let

a'_1) $\sigma_y(p-2)$ (resp. $\sigma_z(p-2)$) the K subrepresentation of R_t generated by $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ (resp. $z_{l_1,\dots,l_{t-1}}$, $z'_{l_1,\dots,l_{t-1}}$) with $(l_1,\dots,l_{t-1}) \prec (r, p-1-r, \dots, p-1-r, r)$ if $r \neq 0$ (resp. if $r=0$);

a'_2) $\sigma_y(p-3)$ (resp. $\sigma_z(p-3)$) as the K-subrepresentaiton of R_t generated by $\sigma_y(p-2)$ (resp. $\sigma_z(p-2)$) and the element $y_{r,p-1-r,\dots,r,p-1-r} + (-1)^{r+1}y'_{r,p-1-r,\dots,p-1-r,r}$ if $r \neq 0$ (resp. $z_{r,p-1-r,\dots,r,p-1-r} + (-1)^{r+1}z'_{r,p-1-r,\dots,p-1-r,r}$ if $r=0$);

a'_3) $\sigma_y(<p-3)$ (resp. $\sigma_z(<p-3)$) as the K-subrepresentation of R_t generated by $\sigma_y(p-2)$ (resp. $\sigma_z(p-2)$) and the element $y_{r,p-1-r,\dots,p-1-r,r}$ (resp. $z_{r,p-1-r,\dots,p-1-r,r}$).

b'_1) $\sigma^s(0)$ and $\sigma^s(1)$ as the K-subrepresentation of R_{t+1} generated by $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$ with $(l_1,\dots,l_{t-1}) \prec (p-1-r, r, \dots, r, p-1-r)$;

b'_2) $\sigma^s(2)$ as the K-subrepresentaiton of R_{t+1} generated by $\sigma^s(1)$ and the element

$$x_{p-1-r,r,\dots,r,p-1-r} + (-1)^{(r-2)+1}x'_{r,p-1-r,\dots,r,p-1-r};$$

b'_3) $\sigma^s(>2)$ as the K-subrepresentation of R_{t+1} generated by $\sigma^s(1)$ and the element

$$x_{p-1-r,r,\dots,r,p-1-r}.$$

iii) Assume $t=1$. We define:

a''_1) $\sigma(p-2)=0$;

a''_2) $\sigma(p-3)$ as the K-subrepresentaiton of R_2 generated by

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{r+1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}[1,X^r] + (-1)^{r+1}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{r+1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}[1,X^r]$$

a''_3) $\sigma(<p-3)$ as the K-subrepresentation of R_2 generated by the element

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{r+1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}[1,X^r].$$

b_1'') $\sigma_y^s(1) = \sigma_z^s(0) = 0$;
b_2'') $\sigma_y^s(2)$ *as the* K*-subrepresentation of* R_1 *generated by:*
$\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-1}Y] + (-1)^{r+1}[1, X^{r-1}Y]$ *for* $r \geqslant 1$;
b_3'') $\sigma_y^2(>2)$ *as the* K*-subrepresentation of* R_1 *generated by:*
$\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-1}Y]$ *for* $r \geqslant 1$.

With the above formalism, we are ready to describe completely the K_t-invariants of supersingular representations $\pi(r, 0, 1)$ with $r \in \{0, \dots, p-2\}$ (and therefore also for $r = p-1$ since $\pi(0, 0, 1) \cong \pi(p-1, 0, 1)$.

Proposition 3.7. *Let* $t \geqslant 1$ *and* $r \in \{0, \dots, p-2\}$*; then*

i) Assume t *odd. Then*

a_1) *the* K_t*-invariants of* $\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ *are precisely:*

$$R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \begin{cases} \sigma(p-2) & \text{if } r = p-2 \\ \sigma(p-3) & \text{if } r = p-3 \\ \sigma(<p-3) & \text{if } r < p-3. \end{cases}$$

b_1) *the* K_t*-invariants of* $\varinjlim_{n,\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ *for* $r \neq 0$ *are precisely:*

$$R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) + \begin{cases} \sigma_y^s(1) & \text{if } r = 1 \\ \sigma_y^s(2) & \text{if } r = 2 \\ \sigma_y^s(>2) & \text{if } r > 2. \end{cases}$$

while, if $r = 0$

$$R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-1}} R_t + \sigma_z^s(0).$$

ii) Assume t *even. Then*

a_2) *the* K_t*-invariants of* $\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ *for* $r \neq 0$ *are precisely:*

$$R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) + \begin{cases} \sigma_y(p-2) & \text{if } r = p-2 \\ \sigma_y(p-3) & \text{if } r = p-3 \\ \sigma_y(<p-3) & \text{if } r < p-3. \end{cases}$$

while, for $r = 0$

$$R_0 \oplus_{R_1} \cdots \oplus_{R_t-1} R_t + \begin{cases} \sigma_z(p-3) & \text{if } p = 3 \\ \sigma_z(<p-3) & \text{if } p \neq 3. \end{cases}$$

b_2) *the* K_t*-invariants of* $\varinjlim_{n,\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ *are precisely:*

$$R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1} + \begin{cases} \sigma^s(0) & \text{if } r = 0 \\ \sigma^s(1) & \text{if } r = 1 \\ \sigma^s(2) & \text{if } r = 2 \\ \sigma^s(>2) & \text{if } r > 2. \end{cases}$$

Note that $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) = pr_t(\mathrm{Fil}^0(R_t))$ and $R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) = pr_t(\mathrm{Fil}^0(R_t))$; we believe that the redoundant notation of proposition 3.7 is more expressive.

We warn the reader that the proof of proposition 3.7 is rather technical and lenghty, relying on a detailed computations in the amalgamed sums by means of the Hecke operators $T_n^{\pm}$; we inserted it for sake of completeness. We apologize with the reader for its length and technicity.

Proof: i), $\mathbf{a_1}$). For $t \geqslant 2$ we fix an $(t-1)$-tuple $(l_1, \dots, l_{t-1}) \in \{0, \dots, p-1\}^{t-1}$ and we consider the action of $\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix}$ on the element $x_{l_1,\dots,l_{t-1}}$ (the case $t = 1$ is similar and left to the reader). We have the following equality in R_{t+1}:

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} x_{l_1,\dots,l_{t-1}} = x_{l_1,\dots,l_{t-1}} + \sum_{j=0}^{r+1} \binom{r+1}{j} (-\lambda)^j (-1)^{r+1-j} T_n^+(v_j)$$

where we put

$$v_j \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{j-1} Y^{r-(j-1)}] \in R_t.$$

Since

$$\begin{aligned} -T_n^-(\sum_{j=1}^{r+1} \binom{r+1}{j} (-\lambda)^j (-1)^{r+1-j} v_j = \\ = (r+1)(-1)^{r+2}(\lambda) \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} [1, (\mu_{t-1} X + Y)^r] \end{aligned}$$

we conclude by lemma 2.12 (using of course the definition of T_1^- and $r \leqslant p-2$ in the case $(l_1, \dots, l_{t-1}) = (r, p-1-r, \dots, p-1-r)$) that

-) the element $x_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant in the amalgamed sum if $(l_1, \dots, l_{t-1}) \preceq (r, p-1-r, \dots, p-1-r)$;

-) the element $x_{l_1,\dots,l_{t-1}}$ is not $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant in the amalgamed sum if $(r, p-1-r, \dots, p-1-r) \prec (l_1, \dots, l_{t-1})$.

Moreover, as

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} \begin{bmatrix} 1 + p^t * & p^t[\lambda] \\ p^{t+1} * & 1 + p^t * \end{bmatrix}$$

for $z' = \sum_{j=1}^t p^j[\mu_j]$, we see that $x'_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant (already inside R_{t+1}).

If we define σ as the K-subrepresentation of R_{k+1} generated by the elements $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$ with $(l_1, \dots, l_{t-1}) \preceq (r+1, p-1-r, r, \dots, p-1-r)$ we deduce from the diagram:

$$\begin{tikzcd} 0 \arrow[r] & (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma)^{K_t} \arrow[r] \arrow[d, hook] & (R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1})^{K_t} \arrow[r] \arrow[d, hook] & (R_{t+1}/(R_t + \sigma))^{K_t} \arrow[d, hook] & \\ 0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1} \arrow[r] & R_{t+1}/(R_t + \sigma) \arrow[r] & 0 \end{tikzcd}$$

that

$$(R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1})^{K_t} = (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma)^{K_t},$$

as the K-socle of $R_{t+1}/(R_t + \sigma)$ is generated by $x_{r+1,p-1-r,r,\dots,p-1-r}$ (and $x_{r+1,p-1-r,r,\dots,p-1-r} + (-1)^{r+2}$ if the K-socle is semisimple).

Similarly, we study the action of $\begin{bmatrix} 1+p^t a & 0 \\ 0 & 1+p^t d \end{bmatrix}$ on $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$, for $(l_1,\dots,l_{t-1}) \preceq (r+1, p-1-r, r, \dots, p-1-r)$ and $a, d \in \mathbf{Z}_p^\times$. From the equality

$$\begin{bmatrix} 1+p^t a & 0 \\ 0 & 1+p^t d \end{bmatrix}\begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z(1+p^t d)^{-1}(1+p^t a) & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+p^t d & 0 \\ 0 & 1+p^t a \end{bmatrix}$$

we deduce the following equality in R_{t+1}:

$$\begin{bmatrix} 1+p^t a & 0 \\ 0 & 1+p^t d \end{bmatrix} x_{l_1,\dots,l_{t-1}} = x_{l_1,\dots,l_{t-1}} + \sum_{j=1}^{r+1}\binom{r+1}{j}(\overline{d-a})^j(-1)^{r+1-j}T_t^+(v'_j)$$

where

$$v'_j \stackrel{\text{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\mu_0^j\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix}\cdots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix}[1, X^{j-1}Y^{r-(j-1)}] \in R_t.$$

We deduce from lemma 2.12 (using again the definition of T_1^- and $r \leqslant p-2$ in the case $(l_1,\dots,l_{t-1}) = (r, p-1-r, \dots, p-1-r)$) that

- -) the element $x_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$-invariant in the amalgamed sum if $(l_1,\dots,l_{t-1}) \prec (r, p-1-r, \dots, p-1-r)$ or if $(l_1,\dots,l_{t-1}) = (r, p-1-r, \dots, p-1-r)$ and $r \leqslant p-3$;
- -) the element $x_{l_1,\dots,l_{t-1}}$ is not $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$-invariant in the amalgamed sum if $(r, p-1-r, \dots, p-1-r) = (l_1,\dots,l_{t-1})$ and $r = p-2$.

Moreover, the equality

$$\begin{bmatrix} 1+p^t a & 0 \\ 0 & 1+p^t d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix}\begin{bmatrix} 1+p^t * & 0 \\ p^{t+1}* & 1+p^t * \end{bmatrix}$$

for $z' = \sum_{j=1}^t p^j[\mu_j]$ shows that the action of $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$ on $x'_{l_1,\dots,l_{t-1}}$ is trivial (already in R_{t+1}). As above, we conclude that

- -) if $r = p-2$ then
$$(R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1})^{K_t} = (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma(p-2))^{K_t};$$
- -) if $r \leqslant p-3$ then
$$(R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1})^{K_t} = (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma)^{K_t}.$$

We are now left to study the action of $\begin{bmatrix} 1 & 0 \\ p^t\mathbf{Z}_p & 1 \end{bmatrix}$ on $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$, for $(l_1,\dots,l_{t-1}) \preceq (r+1, p-1-r, r, \dots, p-1-r)$. For $z = \sum_{j=0}^t p^j[\mu_j]$ we have the equality

$$\begin{bmatrix} 1 & 0 \\ p^t[\lambda] & 1 \end{bmatrix}\begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z_1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+p^t * & p^t[\lambda] \\ p^{t+1}* & 1+p^t * \end{bmatrix}$$

where $z_1 = \sum_{j=0}^{t-1} p^j[\mu_j]+p^t[\mu_t+\mu_0^2\lambda]$. We can use lemma 2.12 (and the definition of T_1^- in the case $(r, p-1-r, \dots, p-1-r) = (l_1, \dots, l_{t-1})$) to deduce the following equality in $R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1}$:

$$\begin{bmatrix} 1 & 0 \\ p^t[\lambda] & 1 \end{bmatrix} x_{l_1,\dots,l_{t-1}} - x_{l_1,\dots,l_{t-1}} =$$

$$= \begin{cases} 0 & \text{if either } (l_1,\dots,l_{t-1}) \prec (r, p-1-r, \dots, p-1-r) \text{ or} \\ & \quad (l_1,\dots,l_{t-1}) = (r, p-1-r,\dots,p-1-r) \text{ and } r < p-3 \\ & \\ (r+1)\lambda(-1)^{(r+2)\frac{k-1}{2}} Y^r & \text{if } (l_1,\dots,l_{t-1}) = (r, p-1-r,\dots,p-1-r) \text{ and } r = p-3. \end{cases}$$

On the other hand we have

$$\begin{bmatrix} 1 & 0 \\ p^t[\lambda] & 1 \end{bmatrix} x'_{l_1,\dots,l_{t-1}} - x'_{l_1,\dots,l_{t-1}} =$$

$$= \begin{cases} 0 & \text{if either } (l_1,\dots,l_{t-1}) \prec (r, p-1-r, \dots, p-1-r) \\ (r+1)\lambda(-1)^{(r+2)\frac{k-1}{2}} Y^r & \text{if } (l_1,\dots,l_{t-1}) = (r, p-1-r,\dots,p-1-r). \end{cases}$$

As

$$\mathrm{soc}(R_{t+1}/R_t + \sigma(\bullet)) = \mathrm{soc}(Q^{(0,t+1)}_{1,r,p-1-r,\dots,p-1-r,r+1})$$

(where $\sigma(\bullet) \in \{\sigma(p-3), \sigma(< p-3)\}$ according to r as in the statement of i)-a_1)) we conclude, as above, that

$$(R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1})^{K_t} = (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma(\bullet))^{K_t}$$

and the result follows, as $\sigma(\bullet)$ are generated by K_t-invariant elements.

i), $\mathbf{b_1}$) The proof is similar to the previous. First of all, notice that $\mathrm{Fil}^0(R_t) \cong \mathrm{Ind}_{K_0(p^t)}^K \chi_r^s$ is K_t-invariant. Therefore, we focus on the action of K_t on the elements $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ if $r \neq 0$ (resp. $z_{l_1,\dots,l_{t-1}}$, $z'_{l_1,\dots,l_{t-1}}$ if $r = 0$). We notice moreover that, if $r \neq 0$ and $t = 1$, the K_1 invariants of $\varinjlim_{n \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ are described by lemma 3.2 and 3.5: we can exclude this situation in the reminder of the proof of $i), b_1)$.

Assume now $r \neq 0$. As above, we have the following equality in R_t for $l_{t-1} \leqslant r$:

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} y_{l_1,\dots,l_{t-1}} - y_{l_1,\dots,l_{t-1}} = \mu(-1)^{l_{t-1}} T_{t-1}^+(w)$$

where

$$w \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\mu_{t-2} \in \mathbf{F}_p} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-l_{t-1}} Y^{l_{t-1}}] \in R_{t-1}$$

Using lemma 2.12 see that

- -) the element $y_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant in the amalgamed sum if $(l_1,\dots,l_{t-1}) \prec (p-r, r, \dots, p-1-r, r)$;
- -) the element $y_{l_1,\dots,l_{t-1}}$ is not $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant in the amalgamed sum if $(p-r, r, \dots, p-1-r, r) \preceq (l_1,\dots,l_{t-1})$.

We see again that $y'_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant (already in R_t), and we conclude by the usual argument that

$$(\varinjlim_{n\,\text{even}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}))^{K_t}=(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}+pr_t(\mathrm{Fil}^0(R_t))+\sigma)^{K_t}$$

where σ is the (image of the) K-subrepresentation of R_t generated by the elements $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ where $(l_1,\dots,l_{t-1})\prec(p-r,r,p-1-r,\dots,p-1-r,r)$.

We pass to the action of $\begin{bmatrix}1+p^ta&0\\0&1+p^td\end{bmatrix}$, with $a,d\in\mathbf{Z}_p^{\times}$. Exactly as in the proof of $i)$-$a_1)$ we use the matrix relation

$$\begin{bmatrix}1+p^ta&0\\0&1+p^td\end{bmatrix}\begin{bmatrix}z&1\\1&0\end{bmatrix}=\begin{bmatrix}z(1+p^td)^{-1}(1+p^ta)&1\\1&0\end{bmatrix}\begin{bmatrix}1+p^td&0\\0&1+p^ta\end{bmatrix}$$

and lemma 2.12 to see that

-) the element $y_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix}1+p^t\mathbf{Z}_p&0\\0&1+p^t\mathbf{Z}_p\end{bmatrix}$-invariant in the amalgamed sum if $(l_1,\dots,l_{t-1})\prec(p-1-r,r,\dots,p-1-r,r)$ or if $(l_1,\dots,l_{t-1})=(p-1-r,r,\dots,p-1-r,r)$ and $r\geqslant 2$;

-) the element $x_{l_1,\dots,l_{t-1}}$ is not $\begin{bmatrix}1+p^t\mathbf{Z}_p&0\\0&1+p^t\mathbf{Z}_p\end{bmatrix}$-invariant in the amalgamed sum if $(p-1-r,r,\dots,p-1-r,r)=(l_1,\dots,l_{t-1})$ and $r=p-2$.

Moreover, as $y'_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix}1+p^t\mathbf{Z}_p&0\\0&1+p^t\mathbf{Z}_p\end{bmatrix}$-invariant (already in R_t), we deduce

$$(\varinjlim_{n\,\text{even}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}))^{K_t}=\begin{cases}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}+pr_t(\mathrm{Fil}^0(R_t))+\sigma_y^s(1))^{K_t}\\ \quad\text{if } r=1\\[2ex] (R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}+pr_t(\mathrm{Fil}^0(R_t))+\sigma)^{K_t}\\ \quad\text{if } r\geqslant 2.\end{cases}$$

We are left to study the action of $\begin{bmatrix}1&0\\p^t\mathbf{Z}_p&1\end{bmatrix}$ on $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ in the situation $(l_1,\dots,l_{t-1})\preceq(p-r,r,p-1-r,\dots,p-1-r,r)$. For $z\in I_{t-1}$ we have the equality

$$\begin{bmatrix}1&0\\p^t[\lambda]&1\end{bmatrix}\begin{bmatrix}z&1\\1&0\end{bmatrix}=\begin{bmatrix}z&1\\1&0\end{bmatrix}\begin{bmatrix}1+p^t*&p^t[\lambda]\\p^t[-\lambda\mu_0^2]&1+p^t*\end{bmatrix}.$$

We can use lemma 2.12 to deduce the following equality in $R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-1}}R_t$:

$$\begin{bmatrix}1&0\\p^t[\lambda]&1\end{bmatrix}y_{l_1,\dots,l_{t-1}}-y_{l_1,\dots,l_{t-1}}=$$
$$=\begin{cases}0\quad\text{if }(l_1,\dots,l_{t-1})\prec(p-1-r,r\dots,p-1-r,r)\text{ or}\\ \qquad(l_1,\dots,l_{t-1})=(p-1-r,r,\dots,p-1-r,r)\text{ and } r\geqslant 3\\[2ex] -\lambda(-1)^{(r+2)\frac{t-1}{2}}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^2\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1,X^r]\quad\text{if}\\ \quad(l_1,\dots,l_{t-1})=(p-1-r,r,\dots,p-1-r,r)\text{ and } r=2.\end{cases}$$

We compute now

$$\begin{bmatrix} 1 & 0 \\ p^t[\lambda] & 1 \end{bmatrix} x'_{l_1,\dots,l_{t-1}} - x'_{l_1,\dots,l_{t-1}} =$$
$$= \begin{cases} 0 \quad \text{if either } (l_1,\dots,l_{t-1}) \prec (p-1-r, r, \dots, p-1-r, r) \\ \lambda(-1)^{(r+2)\frac{t-1}{2}}[1, X^r] \quad \text{if } (l_1,\dots,l_{t-1}) = (p-1-r, r, \dots, p-1-r, r). \end{cases}$$

As

$$\mathrm{soc}(R_t/(\mathrm{Fil}^0(R_t) + \sigma_y^s(\bullet))) = \mathrm{soc}(Q^{(0,t)}(1)_{1,p-1-r,r\dots,p-1-r,r})$$

(where $\sigma(\bullet) \in \{\sigma_y^s(2), \sigma_s^y(>2)\}$ according to r) we conclude, as above, that

$$(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-1}} R_t)^{K_t} = (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) + \sigma_y^s(\bullet))^{K_t}$$

and the result follows, as $\sigma_y^s(\bullet)$ are generated by K_t-invariant elements.

We consider the case $r = 0$. We see that the following equalities hold in the amalgamed sum (with the obvious conventions if $t = 1$):

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} z_{l_1,\dots,l_{t-1}} - z_{l_1,\dots,l_{t-1}} =$$
$$= \lambda \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, e];$$
$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} z'_{l_1,\dots,l_{t-1}} - z'_{l_1,\dots,l_{t-1}} = 0$$

and therefore, the study of $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariance can be recovered from the formalism of the case $r \neq 0$, $t \geqslant 3$. Notice that, if $t = 1$ and

$$z \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \sum_{\mu_1} \mu_1 \begin{bmatrix} 1 & 0 \\ p^2[\mu_2] & 1 \end{bmatrix} [1, e]$$

we get

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} z - z = \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, e];$$

as the latter is nonzero in R_1/R_0 we can conclude that the K_1-invariants of the inductive limit are simply the elements of R_1/R_0.

We can now assume $t \geqslant 2$; from the the action of $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ p^t\mathbf{Z}_p & 1 \end{bmatrix}$ on $z_{l_1,\dots,l_{t-1}}$, $z'_{l_1,\dots,l_{t-1}}$ we see as above that their K_t-invariance can be reduced to the formalism for the case $r \neq 0$. The conclusion follows.

ii) The proof is completely analogous to the case *i*), without any new ideas. It is therefore left to the reader. ♯

The formalism of proposition 3.7 may look a bit heavy, but we can use the description of the socle filtration for $\pi(r, 0, 1)|_{KZ}$ to have an immediate idea of what is going on. Roughly speaking, when we extract the K_t-invariants from $\pi(r, 0, 1)$ we are "cutting" the socle filtration, and proposition 3.7 tells us precisely where such a "cutting" occurs. For instance, the description of $(\varinjlim_{n,\mathrm{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}))^{K_t}$, for t odd, in terms of the socle filtration is the following:

$$\mathrm{SocFil}(R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1})\text{—}\begin{cases}\mathrm{SocFil}Q^{(0,t+1)}_{0,\dots,0,r+1}\setminus\mathrm{SocFil}Q^{(0,t+1)}_{0,r,p-1-r,\dots,p-1-r,r+1}\\ \text{if } r=p-2\\ \\ \mathrm{SocFil}Q^{(0,t+1)}_{0,\dots,0,r+1}\setminus\mathrm{SocFil}Q^{(0,t+1)}_{1,r,p-1-r,\dots,p-1-r,r+1}\\ \text{if } r\leqslant p-3\end{cases}$$

where we used the notation "$\mathbf{V}_1\setminus\mathbf{V}_2$" to mean that we have to rule out the factors of the socle filtration of $\mathbf{V}_2$ from the socle filtration of $\mathbf{V}_1$ (or, more scientifically but less immaginative, to mean the socle filtration of the kernel $\ker(\mathbf{V}_1\twoheadrightarrow\mathbf{V}_2)$ of the natural projection).

Corollary 3.8. *Let $r\in\{0,\dots,p-1\}$, $t\geqslant 1$. The $\overline{\mathbf{F}}_p$-dimension of $(\pi(r,0,1))^{K_t}$ is then:*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{K_t})=(p+1)(2p^{t-1}-1)+\begin{cases}p-3 & \text{if } r\notin\{0,p-1\}\\ p-2 & \text{if } r\in\{0,p-1\}\end{cases}$$

Proof: Thanks to the isomorphism $\pi(0,0,1)\cong\pi(p-1,0,1)$ we can assume $r\leqslant p-2$. Let us assume t odd (the case t even is analogous). Using [Mo1], corollary 6.5 we get

$$\dim_{\overline{\mathbf{F}}_p}(\sigma(\bullet)/R_t)=(p+1)p^t-(p+1)p^{t-1}(r+1)-((p+1)p^t-(p+1)\sum_{j=1}^{t}p^{j-1}l_j-(p-2-r))$$

where $(l_1,\dots,l_t-1)=(r,p-1-r,\dots,r,p-1-r,r+1)$ if $t\geqslant 2$; thus

$$\dim_{\overline{\mathbf{F}}_p}(\sigma(\bullet)/R_t)=(p-r)(p^{t-1}-1)+(p-2-r).$$

Similarly we find, for $r\neq 0$,

$$\begin{aligned}\dim_{\overline{\mathbf{F}}_p}(\sigma^s_y(\bullet)/\mathrm{Fil}^0(R_t))&=p^{t-1}(p+1)-(p^{t-1}(p+1)-(p+1)\sum_{j=1}^{t-1}p^{j-1}l_j-(r-1))\\&=(r+1)(p^t-1)+(r-1)\end{aligned}$$

where $(l_1,\dots,l_{t-1})=(p-1-r,r,\dots,p-1-r,r)$ if $t\geqslant 2$; if $r=0$ we similarly get

$$\dim_{\overline{\mathbf{F}}_p}(\sigma^s_z(\bullet)/\mathrm{Fil}^0(R_t))=(r+1)(p^t-1).$$

As

$$\begin{aligned}&\dim_{\overline{\mathbf{F}}_p}(R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1})+\dim_{\overline{\mathbf{F}}_p}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2})+\\&\quad+\dim_{\overline{\mathbf{F}}_p}(\mathrm{Fil}^0(R_t/R_{t-1}))=(p+1)p^{t-1}\end{aligned}$$

the result follows. ♯

With respect to the description of the socle filtration of $\pi(r,0,1)|_K$ as "two lines of weights", proposition 3.7 let us deduce the following result:

Corollary 3.9. *Let $t\geqslant 1$ be an integer, $r\in\{0,\dots,p-1\}$.*

1) *The socle filtration for $(\varinjlim_{n,\mathrm{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}))^{K_t}$ is described by:*

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2\text{—}\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^{r+1}\text{—}\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^{r+2}\text{—}\dots\text{—}\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^{r}\text{—}\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2\otimes\det{}^{r+1}$$

where the number of parabolic inductions $\mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^j$ *is* $p^{t-1}-1$ *and last weight* $\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2 \otimes \det^{r+1}$ *appears only if* $p-3-r \geqslant 0$.

2) *The socle filtration for* $(\varinjlim_{n,\,\mathrm{even}} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}))^{K_t}$ *is described by:*

$$\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2 \otimes \det^r \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a} \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^2 \text{—} \ldots \text{—} \mathrm{Ind}_I^K \chi_r^s \text{—} \mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2 \otimes \det$$

where the number of parabolic inductions $\mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^j$ *is* $p^{t-1}-1$ *and last weight* $\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2 \otimes \det$ *appears only if* $r-2 \geqslant 0$.

Proof: We sketch here the proof for t odd, $r \in \{0, \ldots, p-2\}$. Using the computations in the proof of corollary 3.8 and the result in lemma 2.8 we see that

$$\dim_{\overline{\mathbf{F}}_p}(\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}))^{K_t} = (r+1) + (p+1)(p^{t-1}-1) + (p-2-r)$$

and

$$\dim_{\overline{\mathbf{F}}_p}(\varinjlim_{n,\,\mathrm{even}} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}))^{K_t} = (p-r) + (p+1)(p^{t-1}-1) + (r-1) + \delta_{r,0}$$

As the dimension of the parabolic inductions $\mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^j$ is $p+1$ we conclude from proposition 3.7. $\sharp$

4. Study of I_t-invariants.

Let $t \geqslant 1$ be an integer and $r \in \{0, \ldots, p-1\}$. The aim of this section is to study in detail the space of I_t-invariant of supersingular representations $\pi(r, 0, 1)$; thanks to the isomorphism $\pi(0,0,1) \cong \pi(p-1,0,1)$ we will assume $r \leqslant p-2$, unless otherwise specified. Moreover the relations

$$K_{t-1} \leqslant I_t \leqslant K_t, I_t = \begin{bmatrix} 1 & p^{t-1}\mathbf{Z}_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^t\mathbf{Z}_p & 1 \end{bmatrix}$$

show that the hard task consist in studying the $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$-invariants (for $\lambda \in \mathbf{F}_p$) of $\pi(r,0,1)^{K_t}$, the latter being completely described in proposition 3.7. We distingush two cases, accordingly to the parity of t.

4.1 The case t odd.

In the present section, we assume $t \geqslant 1$, t odd. We then can write, accordingly to the value of r,

$$(\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}))^{I_t} \leqslant R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \begin{cases} \sigma(p-2) & \text{if } r = p-2 \\ \sigma(p-3) & \text{if } r = p-3 \\ \sigma(<p-3) & \text{if } r < p-3. \end{cases}$$
$$(\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}))^{I_t} \geqslant pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$$

with the obvious convention that $pr_{t-1}(\mathrm{Fil}^0(R_{t-1})) = R_0$ if $t=1$. Notice that all vectors in the spaces $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ are $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$-invariants. Similarly, we get

$$(\varinjlim_{n,\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}))^{I_t} \leqslant pr_t(\mathrm{Fil}^0(R_t)) + \begin{cases} \sigma_z^s(0) & \text{if } r=0 \\ \sigma_y^s(1) & \text{if } r=1 \\ \sigma_y^s(2) & \text{if } r=2 \\ \sigma_y^s(>2) & \text{if } r>2. \end{cases}$$

$$(\varinjlim_{n,\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}))^{I_t} \geqslant R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2}$$

with the obvious convention that $pr_1(\mathrm{Fil}^0(R_1/R_0)) = \mathrm{Fil}^0(R_1/R_0)$. Notice that all vectors in the spaces $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2}$ are $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$-invariants.

From now onwards we assume $t>1$: the case $t=1$ it is well known (cf. [Bre03a], Théoréme 3.2.4) and can anyway be treated with analogous techniques.

4.1.1 ***Concerning*** $R_0 \oplus_1 \cdots \oplus_{R_n} R_{n+1}$, n ***odd.*** We conside the K-equivariant exact sequence

$$0 \to pr_{t-1}(\mathrm{Fil}^0(R_{t-1})) \to R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma(\bullet) \to \mathbf{V} \to 0$$

where $\bullet$ depends on r accordingly to proposition 3.7-a_1). We introduce the following elements:

Definition 4.1. *Let $t>1$ be odd, $(l_0,\ldots,l_{t-2}) \in \{0,\ldots,p-1\}^{t-1}$ a $(t-1)$-tuple.*

i) For $j \in \{0,\ldots,r\}$ we define the following elements of R_{t-1}

$$\mathfrak{x}_{l_0,\ldots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

$$\mathfrak{x}'_{l_1,\ldots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

ii) For $l_{t-1} \in \{0,\ldots,p-1\}$, we define the following elements of R_{t+1}

$$\mathfrak{y}_{l_0,\ldots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r]$$

$$\mathfrak{y}'_{l_1,\ldots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} [\mu_1] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

The rôle of such elements is explained by the next

Lemma 4.2. *An $\overline{\mathbf{F}}_p$-basis for $\mathbf{V}$ is described as follow:*

i) the elements $\mathfrak{y}_{l_0,\ldots,l_{t-1}}$, $\mathfrak{y}'_{l_1,\ldots,l_{t-1}}$ where $(l_1,\ldots,l_{t-1}) \prec (r, p-1-r, \ldots, r, p-1-r)$, $l_0 \in \{0,\ldots,p-1\}$;

ii) if $r \leqslant p-3$, the elements $\mathfrak{y}_{j,r,p-1-r,\ldots,r,p-1-r}$, $j \in \{0,\ldots,p-3-r-1\}$ and $\mathfrak{y}_{p-3-r,r,p-1-r,\ldots,r,p-1-r} + (-1)^{(r+1)+p-3-r}\mathfrak{y}'_{p-3-r,r,p-1-r,\ldots,r,p-1-r}$

iii) *if* $r \neq 0$, *the elements*

$$\mathfrak{x}_{l_0,\dots,l_{t-2}}(j) \overset{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j} Y^j]$$

$$\mathfrak{x}'_{l_1,\dots,l_{t-2}}(j) \overset{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j} Y^j]$$

where $j \in \{1, \dots, r\}$.

Proof: It is a formal fact to verify that we have a K-equivariant exact sequence

$$0 \to R_{t-1}/\mathrm{Fil}^1(R_{t-1}) \to \mathbf{V} \to \sigma(\bullet)/R_t \to 0;$$

the the assertion is then an immediate consequence. $\sharp$

Thanks to lemma 4.2 we can describe the structure of $\mathbf{V}^{I_t}$.

PROPOSITION 4.3. *An* $\overline{\mathbf{F}}_p$*-basis for* $\mathbf{V}^{U_t}$ *is given by*

a) *the elements* $\mathfrak{x}'_{l_1,\dots,l_{t-2}}(j)$ *for* $(l_1, \dots, l_{t-2}) \in \{0, \dots, p-1\}^{t-2}$, $j \in \{1, \dots, r\}$ *if* $r \geqslant 1$;

b) *the elements* $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$ *where* $(l_1, \dots, l_{t-1}) \prec (r, p-1-r, \dots, r, p-1-r)$

c) *if* $r \neq 0$ *the elements*

$$\mathfrak{x}_{l_0,\dots,l_{t-2}}(1)$$

where $(l_0, \dots, l_{t-2}) \in \{0, \dots, p-1\}^{t-1}$, *while, if* $r = 0$, *the elements*

$$\mathfrak{y}_{l_0,\dots,l_{t-2},0}$$

where $(l_1, \dots, l_{t-2}) \preceq (r, \dots, p-1-r, r)$ *and* $l_0 \in \{0, \dots, p-1\}$.

Proof: Assume $r \neq 0$ (the case $r = 0$ is strictly analogous). First of all, we look for a decomposition of $\mathbf{V}$ into $\begin{bmatrix} 1 & p^{t-1}\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-stable subspaces. We deduce immediately the following equalities (in R_{t-1}):

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{x}_{l_0,\dots,l_{t-2}}(j) = \sum_{i=0}^{j} \binom{j}{i} \lambda^{j-i} \mathfrak{x}_{l_0,\dots,l_{t-2}}(j-i); \tag{8}$$

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{x}'_{l_1,\dots,l_{t-2}} = \mathfrak{x}'_{l_1,\dots,l_{t-2}};$$

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{y}'_{l_1,\dots,l_{t-2}} = \mathfrak{y}'_{l_1,\dots,l_{t-2}}.$$

Using the operators $T_t^{\pm}$, we get the following equality inside $R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1}$:

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{y}_{l_0,\dots,l_{t-1}} = \sum_{i=0}^{l_{t-1}} \binom{l_{t-1}}{i} (-\lambda)^i \mathfrak{y}_{l_0,\dots,l_{t-2},(l_{t-1}-i)} + \tag{9}$$
$$+ \sum_{i=0}^{l_{t-1}} \binom{l_{t-1}}{i} (-\lambda)^i (r+1)(-1)^{r+1} v_i$$

where

$$v_i \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

$$\cdots \sum_{\mu_{t-2} \in \mathbf{F}_p} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}-i} P_{-\lambda}(\mu_{t-1})[1, (\lambda_{t-1}X + Y)^r].$$

In particular, each v_i belongs to the linear space generated by the elements $\mathfrak{x}_{l_0 \ldots, l_{t-2}}(j)$ and, for $l_{t-1} = 0$, we see that the coefficient of $[1, Y^r]$ in $\sum\limits_{\mu_{t-1} \in \mathbf{F}_p} P_{-\lambda}(\mu_{t-1})[1, (\lambda X + Y)^r]$ is $-\lambda$.

We deduce that the following spaces are $\begin{bmatrix} 1 & p^{t-1}\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-stable: the spaces

$$\langle \mathfrak{x}'_{l_1, \ldots, l_{t-2}}(j) \rangle_{\overline{\mathbf{F}}_p}; \langle \mathfrak{y}'_{l_1, \ldots, l_{t-1}} \rangle_{\overline{\mathbf{F}}_p}$$

and, for a given $(t-1)$-tuple $(l_0, \ldots, l_{t-2}) \in \{0, \ldots, p-1\}^{t-1}$, the space $\mathbf{V}_{l_0, \ldots, l_{t-2}}$ which is defined as the $\overline{\mathbf{F}}_p$-vector subspace of $\mathbf{V}$ generated by $\mathfrak{x}_{l_0, \ldots, l_{t-2}}(j)$, for $j \in \{1, \ldots, r\}$ and

-) the elements $\mathfrak{y}_{l_0, \ldots, l_{t-2}, i}$ with $i \in \{0, \ldots, p-1-r\}$ if either $(l_1, \ldots, l_{t-2}) \prec (r, p-1-r, \ldots, p-1-r, r)$ or $(l_1, \ldots, l_{t-2}) = (r, p-1-r, \ldots, p-1-r, r)$ and $l_0 < p-3-r$;

-) the elements $\mathfrak{y}_{l_0, \ldots, l_{t-2}, i}$ with $i \in \{0, \ldots, p-2-r\}$ if $(l_1, \ldots, l_{t-2}) = (r, p-1-r, \ldots, p-1-r, r)$ and $l_0 > p-3-r$;

-) the elements $\mathfrak{y}_{l_0, \ldots, l_{t-2}, i}$ with $i \in \{0, \ldots, p-2-r\}$ and the element

$$\mathfrak{y}_{l_0, \ldots, l_{t-2}, p-1-r} + (-1)^{p-3-r+(r+1)} \mathfrak{y}'_{l_1, \ldots, l_{t-2}, p-1-r}$$

if $(l_1, \ldots, l_{t-2}) = (r, p-1-r, \ldots, p-1-r, r)$ and $l_0 = p-3-r$.

For a fixed $(t-1)$-tuple $(l_0, \ldots, l_{t-2}) \in \{0, \ldots, p-1\}^{t-1}$ we deduce from the equalities (8) and (9) that there exists an $\overline{\mathbf{F}}_p$-basis of $\mathbf{V}_{l_0, \ldots, l_{t-2}}$ such that the matrix associated to the action of $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ is unipotent, and the elements on the superdiagonal are all nonzero. In other words, the $\mathbf{V}_{l_0, \ldots, l_{t-2}}$-restriction of the $\mathbf{V}$-endomorphism associated to $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ has a unique eigenvalue (equal to 1) and the associated eigenspace has dimension 1. We see that a generator of such eigenspace is $\mathfrak{x}_{l_0, \ldots, l_{t-2}}(1)$ and the proof is complete. The statement concerning the case $r = 0$ can be proved with the same techniques and it is left to the reader. $\sharp$

We remark that the elements in *a*), *b*) of proposition 4.3 are already U_t invariant inside the amalgamed sum $R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1}$. Together with the elements inside $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ they are denoted as the *trivial I_t-invariants* We therefore are left to study the U_t-invariance of the elements of the form *c*) inside $\varinjlim\limits_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ to complete the description of I_t-invariants.

Proposition 4.4. *An $\overline{\mathbf{F}}_p$-basis for the space of nontrivial I_t-invariants of $\varinjlim\limits_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ modulo the trivial invariants is described as follow:*

i) *if $r \neq 0$, by the family*

$$\mathfrak{x}(1)_{l_0, \ldots, l_{t-2}}$$

where $(l_0, l_1, \ldots, l_{t-2}) \prec (p-1-r, r, \ldots, p-1-r, r)$.

ii) if $r = 0$, by the family

$$\mathfrak{y}_{l_0,\ldots,l_{t-2},0}$$

where $(l_0,\ldots,l_{t-2}) \prec (p-1-r, r, \ldots, p-1-r, r)$

Proof: i) The proof is an induction on t, and follows closely the computations of lemma 2.12. Let $t = 3$, and consider a I_t-invariant vector v which we can assume of the following form:

$$v = \sum_{(l_0,l_1)\in\{0,\ldots,p-1\}^2} c_{l_0,l_1}\mathfrak{x}(1)_{l_0,l_1}$$

for suitable $c_{l_0,l_1} \in \overline{\mathbf{F}}_p$. We have

$$\begin{bmatrix} 1 & p^2[\lambda] \\ 0 & 1 \end{bmatrix} v - v = \lambda \sum_{(l_0,l_1)\in\{0,\ldots,p-1\}^2} c_{l_0,l_1}\mathfrak{x}(0)_{l_0,l_1};$$

it is now clear that $\mathfrak{x}(0)_{l_0,l_1} \equiv 0$ inside R_2/R_1 if $l_1 \leqslant r$ while the (image of the) elements $\mathfrak{x}(0)_{l_0,l_1}$ inside R_2/R_1 induce a free family for $l_1 \geqslant r+1$: the I_3-invariance of v shows that $c_{l_0,l_1} = 0$ if $(0, r+1) \preceq (l_0, l_1)$. Therefore, using the operators $T_1^{\pm}$, we get the following equality in the amalgamed sum

$$\sum_{(l_0,l_1)\prec(0,r+1)} c_{l_0,l_1}\mathfrak{x}(0)_{l_0,l_1} = (-1)^{r+1}\sum_{l_0=0}^{p-1} c_{l_0,r}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}(X+\mu_0 Y)^r$$

which shows that $c_{l_0,r} = 0$ for $l_0 \geqslant p-1-r$. This let us establish the case $t = 3$.

Concerning the general case, let v be a I_t-invariant element which we can assume of the form

$$v = \sum_{(l_0,\ldots,l_{t-2})\in\{0,\ldots,p-1\}^{t-1}} c_{l_0,\ldots,l_{t-2}}\mathfrak{x}(1)_{l_0,\ldots,l_{t-2}}$$

for suitable $c_{l_0,\ldots,l_{t-2}} \in \overline{\mathbf{F}}_p$. We have

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} v - v = \lambda \sum_{(l_0,\ldots,l_{t-2})\in\{0,\ldots,p-1\}^{t-1}} c_{l_0,\ldots,l_{t-2}}\mathfrak{x}(0)_{l_0,\ldots,l_{t-2}};$$

since $\mathfrak{x}(0)_{l_0,\ldots,l_{t-2}} \equiv 0$ inside R_{t-1}/R_{t-2} if $l_{t-2} \leqslant r$ and the family

$$\{\mathfrak{x}(0)_{l_0,\ldots,l_{t-2}}\}_{l_{t-2}\geqslant r+1}$$

is linearly independent in R_{t-1}/R_{t-2} we conclude that $c_{l_0,\ldots,l_{t-2}} = 0$ as soon as $l_{t-2} \geqslant r+1$. Using the operators $T_{t-2}^{\pm}$ and a similar argument (i.e. studying the image of the sum inside R_{t-3}/R_{t-4}) we deduce that we must have $c_{l_0,\ldots,l_{t-3},r} = 0$ if $l_{t-3} > p-1-r$, therefore getting the following equality in the amalgamed sum:

$$\sum_{(l_0,\ldots,l_{t-2})\in\{0,\ldots,p-1\}^{t-1}} c_{l_0,\ldots,l_{t-2}}\mathfrak{x}(0)_{l_0,\ldots,l_{t-2}} =$$
$$= (-1)^{r+2}\sum_{(l_0,\ldots,l_{t-4})\in\{0,\ldots,p-1\}^{t-3}} c_{l_0,\ldots,l_{t-4},p-1-r,r}\mathfrak{x}(0)_{l_0,\ldots,l_{t-4}}.$$

The conclusion follows by induction.

ii) It is completely analogous and left to the reader. ♯

4.1.2 ***Concerning*** $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$, n ***even***. We consider now the K-equivariant exact sequence for $r \neq 0$

$$0 \to R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} \to pr_t(\mathrm{Fil}^0(R_t)) + \sigma_y^s(\bullet) \to \mathbf{W} \to 0$$

and, for $r = 0$,

$$0 \to R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} \to pr_t(\mathrm{Fil}^0(R_t)) + \sigma_z^s(0) \to \mathbf{W} \to 0$$

where $\sigma_y^s(\bullet)$ depends on r accordingly to proposition 3.7i)-b_1). As in the previous section, we introduce the elements

Definition 4.5. *Let $t > 1$ be odd, $(l_0, \ldots, l_{t-1}) \in \{0, \ldots, p-1\}^t$ a t-tuple.*

i) *we define the following elements of R_t*

$$\mathfrak{w}_{l_0,\ldots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^r];$$

$$\mathfrak{w}'_{l_1,\ldots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^r]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

ii) *For $r \neq 0$, we define the following elements of R_t*

$$\mathfrak{z}_{l_0,\ldots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{r-1}Y];$$

$$\mathfrak{z}'_{l_1,\ldots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} [\mu_1] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{r-1}Y]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

iii) *For $r = 0$ we define the following elements of R_{t+2}*

$$\mathfrak{h}_{l_0,\ldots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

$$\cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} \sum_{\mu_{t+1} \in \mathbf{F}_p} \mu_{t+1} \begin{bmatrix} 1 & 0 \\ p^{t+1}[\mu_{t+1}] & 1 \end{bmatrix} [1, X^r];$$

$$\mathfrak{h}'_{l_1,\ldots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} [\mu_1] & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

$$\sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} \sum_{\mu_{t+1} \in \mathbf{F}_p} \mu_{t+1} \begin{bmatrix} 1 & 0 \\ p^{t+1}[\mu_{t+1}] & 1 \end{bmatrix} [1, X^r]$$

where X^r is a fixed $\overline{\mathbf{F}}_p$-basis of $\mathrm{Sym}^0\overline{\mathbf{F}}_p^2$; such elements will be also seen as elements of the amalgamed sums accordingly to the context.

As for lemma 4.2, we are able to describe an $\overline{\mathbf{F}}_p$-basis for $\mathbf{W}$ in terms of the elements defined in 4.5

Lemma 4.6. *An $\overline{\mathbf{F}}_p$-basis for $\mathbf{W}$ is described by:*

a) *the elements* $\mathfrak{w}_{l_0,\dots,l_{t-1}}$, $\mathfrak{w}'_{l_1,\dots,l_{t-1}}$ *where* $l_{t-1}\geqslant r+1$, $(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}$;

b) *if* $r\neq 0$, *the elements* $\mathfrak{z}_{l_0,\dots,l_{t-1}}$, $\mathfrak{z}'_{l_0,\dots,l_{t-1}}$ *with* $(l_1,\dots,l_{t-1})\prec(p-1-r,r,\dots,p-1-r,r)$ *and* $l_0\in\{0,\dots,p-1\}$ *as well as the elements* $\mathfrak{z}_{j,p-1-r,r,\dots,p-1-r,r}$ *for* $j\in\{0,\dots,(r-2)-1\}$ *and* $\mathfrak{z}_{r-2,p-1-r,\dots,p-1-r,r}+(-1)^{(r-2)+1}\mathfrak{z}'_{p-1-r,\dots,p-1-r,r}$;

c) *if* $r=0$, *the elements* $\mathfrak{h}_{l_0,\dots,l_{t-1}}$, $\mathfrak{h}'_{l_0,\dots,l_{t-1}}$ *with* $(l_1,\dots,l_{t-1})\prec(p-1-r,r,\dots,p-1-r,r)$ *and* $l_0\in\{0,\dots,p-1\}$.

Proof: As in lemma 4.2, it is a formal verification that $\mathbf{W}$ fits into a K-equivariant exact sequence:

$$0\to\mathrm{Fil}^0(R_t/R_{t-1})\to\mathbf{W}\to\sigma^s_{\bullet}(\bullet)/\mathrm{Fil}^0(R_t)\to 0$$

where $\sigma^s_{\bullet}(\bullet)$ is defined according to the value of r as in proposition 3.7*i*)-b_1). The result follows. $\sharp$.

Again, we are going to describe the I_t-invariants of $\mathbf{W}$:

Proposition 4.7. *An* $\overline{\mathbf{F}}_p$*-basis for* $\mathbf{W}^{U_t}$ *is given by*

a) *the elements* $\mathfrak{w}'_{l_1,\dots,l_{t-1}}$ *for* $(l_1,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-2}$, *and* $l_{t-1}\geqslant r+1$

b) *if* $r\neq 0$, *the elements* $\mathfrak{z}'_{l_1,\dots,l_{t-1}}$ *where* $(l_1,\dots,l_{t-1})\prec(p-1-r,r\dots,p-1-r,r)$ *while, if* $r=0$, *the elements* $\mathfrak{h}'_{l_1,\dots,l_{t-1}}$ *where* $(l_1,\dots,l_{t-1})\prec(p-1-r,r\dots,p-1-r,r)$

c) *the elements*

$$\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$$

where $(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}$.

Proof: Assume $r\neq 0$ (the case $r=0$ is analogous). For $\lambda\in\mathbf{F}_p$ we easily get the following equalities in $\mathbf{W}$:

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{w}_{l_0,\dots,l_{t-1}}=\sum_{j=r+1}^{l_{t-1}}\binom{l_{t-1}}{j}(-\lambda)^j\mathfrak{w}_{l_0,\dots,(l_{t-1}-j)};\tag{10}$$

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{w}'_{l_0,\dots,l_{t-1}}=\mathfrak{w}'_{l_0,\dots,l_{t-1}};$$

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{z}'_{l_0,\dots,l_{t-1}}=\mathfrak{z}'_{l_0,\dots,l_{t-1}}.\tag{11}$$

Moreover, using lemma 2.10, we get the following equality in $\mathbf{W}$:

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{z}_{l_0,\dots,l_{t-1}}=\sum_{j=0}^{l_{t-1}}\binom{l_{t-1}}{j}(-\lambda)^j\mathfrak{z}_{l_0,\dots,(l_{t-1}-j)}+\sum_{j=0}^{l_{t-1}}\binom{l_{t-1}}{j}(-\lambda)^j w_j\tag{12}$$

where we set

$$w_j\stackrel{\mathrm{def}}{=}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots$$

$$\dots\sum_{\mu_{t-2}\in\mathbf{F}_p}\mu_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\mu_{t-2}] & 1\end{bmatrix}\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}-j}(-P_{-\lambda}(\mu_{t-1}))\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,X^r].$$

We notice that w_j belongs to the linear space generated by $\mathfrak{w}_{l_0,\ldots,l_{t-2},i}$ for $i \in \{r+1,\ldots,p-1\}$ and, for $l_{t-1}=0$, the coefficient of $\mathfrak{w}_{l_0,\ldots,l_{t-2},p-1}$ in w_0 is $-\lambda$.

We deduce that the following subspaces of $\mathbf{W}$ give a decomposition of $\mathbf{W}$ in $\begin{bmatrix} 1 & p^{t-1}\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-stable subspaces:

$$\langle \mathfrak{w}'_{l_1,\ldots,l_{t-1}} \rangle_{\overline{\mathbf{F}}_p}; \langle \mathfrak{z}'_{l_1,\ldots,l_{t-1}} \rangle_{\overline{\mathbf{F}}_p};$$

and, for any fixed $(t-1)$-tuple $(l_0,\ldots,l_{t-2}) \in \{0,\ldots,p-1\}^{t-1}$, the subspace $\mathbf{W}_{l_0,\ldots,l_{t-2}}$, which is defined as the $\overline{\mathbf{F}}_p$-vector subspace of $\mathbf{W}$ generated by $\mathfrak{w}_{l_0,\ldots,l_{t-2},j}$ with $j \in \{r+1,\ldots,p-1\}$ and

-) the elements $\mathfrak{z}_{l_0,\ldots,l_{t-2},i}$ with $i \in \{0,\ldots,r\}$ if either $(l_1,\ldots,l_{t-2}) \prec (p-1-r, r, \ldots, r, p-1-r)$ or $(l_1,\ldots,l_{t-2}) = (p-1-r, r, \ldots, r, p-1-r)$ and $l_0 < r-2$;
-) the elements $\mathfrak{z}_{l_0,\ldots,l_{t-2},i}$ with $i \in \{0,\ldots,r-1\}$ and $\mathfrak{z}_{l_0,\ldots,l_{t-2},r} + (-1)^{(r-2)+1}\mathfrak{z}'_{l_1,\ldots,l_{t-2},r}$ if $(l_1,\ldots,l_{t-2}) = (p-1-r, r, \ldots, r, p-1-r)$ and $l_0 = r-2$;
-) the elements $\mathfrak{z}_{l_0,\ldots,l_{t-2},i}$ with $i \in \{0,\ldots,r-1\}$ if $(l_1,\ldots,l_{t-2}) = (p-1-r, r, \ldots, r, p-1-r)$ and $l_0 > r-2$.

As in proposition 4.3, we see that we can find a basis of $\mathbf{W}_{l_0,\ldots,l_{t-2}}$ such that the matrix associated to the action of $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ on $\mathbf{W}_{l_0,\ldots,l_{t-2}}$ (for $\lambda \in \mathbf{F}_p^\times$) is upper unipotent with nonzero scalars in the superdiagonal. In other words, the $\mathbf{W}_{l_0,\ldots,l_{t-2}}$-restriction of the $\mathbf{W}$-endomorphism associated to $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ (for $\lambda \in \mathbf{F}_p^\times$) has a unique eigenvalue (equal to 1) and the associated eigenspace has dimension 1. Since such eigenspace is generated by $\mathfrak{w}_{l_0,\ldots,l_{t-2},r}$, the conclusion follows.

The case $r=0$ is strictly analogous; we point out anyway that the equalities of type (11), (12) for $\mathfrak{h}'_{l_1,\ldots,l_{t-1}}$, $\mathfrak{h}_{l_0,\ldots,l_{t-1}}$ are now established via the operators $T_t^\pm$ $\sharp$

We remark that the elements in a), b) of proposition 4.7 are already U_t invariant inside the amalgamed sum $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t+1}} R_{t+2}$. Together with the elements inside $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2}$ they will be denoted as the *trivial I_t-invariants* We therefore are left to study the U_t-invariance of the elements of the form c) inside $\varinjlim_{n \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ to complete the description of I_t-invariants.

Proposition 4.8. *An $\overline{\mathbf{F}}_p$-basis for the space of nontrivial I_t-invariants of $\varinjlim_{n \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ modulo the trivial invariants is described by the elements $\mathfrak{w}_{l_0,\ldots,l_{t-2},r+1}$ where $(l_0,\ldots,l_{t-2}) \prec (r, p-1-r, \ldots, r, p-1-r)$*

Proof: The proof is an induction on t, analogous to proposition 4.4. Assume $t=3$ and consider a I_t-invariant vector which we can assume of the following form:

$$v = \sum_{(l_0,l_1) \in \{0,\ldots,p-1\}^2} c_{l_0,l_1} \mathfrak{w}_{l_0,l_1,r+1}$$

for suitable $c_{l_0,l_1} \in \overline{\mathbf{F}}_p$. Using the operators $T_2^\pm$ we deduce the following equality in R_1/R_0:

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} v - v = \sum_{(l_0,l_1) \in \{0,\ldots,p-1\}^2} c_{l_0,l_1} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} [1, (\mu_1 X + Y)^r].$$

We therefore see (thanks to proposition 2.7-*i*)) that $c_{l_0,l_1}=0$ as soon as $l_1>p-1-r$, while we can use proposition 2.9-*ii*) and *iii*) to deduce that $c_{l_0,p-1-r}=0$ for $l_0\geqslant r$. This establish the case $t=3$.

Concerning the general case, let v be a I_t-invariant vector, which we may assume of the form

$$v=\sum_{(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}} c_{l_0,\dots,l_{t-2}}\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$$

for suitable $c_{l_0,\dots,l_{t-2}}\in\overline{\mathbf{F}}_p$. Using the operators $T^{\pm}_{t-1}$ we get the following equality in the amalgamed sum

$$\begin{aligned}&\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}v-v=\\ &\quad=(r+1)(-1)^{r+1}(-\lambda)\sum_{(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}} c_{l_0,\dots,l_{t-2}}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots\\ &\qquad\dots\sum_{\mu_{t-3}\in\mathbf{F}_p}\mu_{t-3}^{l_{t-3}}\begin{bmatrix}1 & 0\\ p^{t-3}[\mu_{t-3}] & 1\end{bmatrix}\sum_{\mu_{t-2}\in\mathbf{F}_p}\mu_{t-2}^{l_{t-2}}[1,(\mu_{t-2}X+Y)^r].\end{aligned}$$

We map the latter in R_{t-2}/R_{t-3} to deduce that $c_{l_0,\dots,l_{t-2}}=0$ if $(r+1,p-1-r)\preceq(l_{t-3},l_{t-2})$ and therefore we get the following equality in the amalgamed sum:

$$\begin{aligned}&\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}v-v=\\ &\quad=(r+1)(-1)^{r+2}(-1)^{r+1}\sum_{(l_0,\dots,l_{t-4})\in\{0,\dots,p-1\}^{t-3}} c_{l_0,\dots,l_{t-4},r,p-1-r}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots\\ &\qquad\dots\sum_{\mu_{t-4}\in\mathbf{F}_p}\mu_{t-4}^{l_{t-4}}\begin{bmatrix}1 & 0\\ p^{t-4}[\mu_{t-4}] & 1\end{bmatrix}[1,(\lambda_{t-4}X+Y)^r].\end{aligned}$$

This let us conclude the inductive step and the proof is complete. $\sharp$

4.2 The case t even

We assume now t even. The study of I_t-invariants for the inductive limits $\varinjlim_{n\,\mathrm{odd}}(R_0\oplus_{R_1}\dots\oplus_{R_n}R_{n+1})$ and $\varinjlim_{n\,\mathrm{even}}(R_1/R_0\oplus_{R_2}\dots\oplus_{R_n}R_{n+1})$ is treated in a completely analogous way as we did in paragraph 4.1. We therefore content ourselves to give the results, leaving the computational efforts to the reader.

4.2.1 ***Concerning*** $R_0\oplus_1\dots\oplus_{R_n}R_{n+1}$, n ***odd.*** We now should consider the K-equivariant short exact sequence

$$0\to R_0\oplus_{R_1}\dots\oplus_{R_{t-3}}R_{t-2}\to pr_t(\mathrm{Fil}^0(R_t))+\begin{cases}\sigma_y(\bullet)\text{ if }r\neq 0\\ \sigma_z(\bullet)\text{ if }r=0\end{cases}\to\mathbf{V}'\to 0$$

where $\sigma_y(\bullet)$, $\sigma_z(\bullet)$ are defined accordingly to proposition 3.7-a_2). We then introduce the following elements of R_t, R_{t+2}:

Definition 4.9. *Let $t\geqslant 2$, t even.*

i) *For any* $(t-1)$*-tuple* $(l_0,\dots,l_{t-1})\in\{0,\dots,p-1\}^{t-1}$, $l_{t-1}\in\{r+1,\dots,p-1\}$ *define*

$$\mathfrak{x}_{l_0,\dots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}[t-1] & 1\\ 1 & 0\end{bmatrix}\mu_{t-1}[1,X^r];$$

$$\mathfrak{x}'_{l_1,\dots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\dots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}[t-1] & 1\\ 1 & 0\end{bmatrix}\mu_{t-1}[1,X^r];$$

ii) *if* $r\neq 0$, *define*

$$\mathfrak{y}'_{l_1,\dots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\dots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,X^{r-1}Y]$$

where $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,p-1-r,r)$.

iii) *if* $r=0$, *define*

$$\mathfrak{z}'_{l_1,\dots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\dots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}$$
$$\sum_{\mu_t\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{t}[\mu_{t}] & 1\end{bmatrix}\sum_{\mu_{t+1}\in\mathbf{F}_p}\mu_{t+1}\begin{bmatrix}1 & 0\\ p^{t+1}[\mu_{t+1}] & 1\end{bmatrix}[1,X^r]$$

where $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,p-1-r,r)$ *and* X^r *is a fixed* $\overline{\mathbf{F}}_p$*-basis for* $\mathrm{Sym}^0\overline{\mathbf{F}}_p^2$.

The element defined in 4.9 will be seen also as elements of the amalgamed sums, according to the context. We see as above that

Lemma 4.10. *Let* $t\geqslant 2$, t *even. The following elements are* I_t *invariant in the inductive limit* $\varinjlim_{n\,\text{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})$:

i) *the elements* $\mathfrak{x}'_{l_1,\dots,l_{t-1}}$ *for* $l_{t-1}\geqslant r+1$, $(l_1,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-2}$;

ii) *if* $r\neq 0$, *the elements* $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$,
where $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,p-1-r,r)$;

iii) *if* $r=0$, *the elements* $\mathfrak{z}'_{l_1,\dots,l_{t-1}}$,
where $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,p-1-r,r)$;

Proof: Omissis. ♯

As in §4.1.1, the elements of lemma 4.10, as well as the elements of $R_0\oplus_{R_1}\cdots\oplus_{R_{t-3}}R_{t-2}$ will be referred to as the *trivial* I_t*-invariants*. The result is then the following:

Proposition 4.11. *An* $\overline{\mathbf{F}}_p$*-basis for the space of nontrivial* I_t*-invariants of* $\varinjlim_{n\,\text{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})$ *modulo the trivial invariants is described by the elements* $\mathfrak{x}_{l_0,\dots,l_{t-2},r+1}$ *where* $(l_0,\dots,l_{t-2})\prec(p-1-r,r,\dots,r,p-1-r)$

Proof: Omissis.♯

4.2.2 ***Concerning*** $R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}$, *n even.* We have now to consider the K-equivariant short exact sequence

$$0\to pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))\to R_1/R_0\oplus_{R_2}\cdots\oplus_{t-2}R_{t-1}+\left\{\begin{array}{l}\sigma_z^s(0)\text{ if }r=0\\ \sigma_y^s(\bullet)\text{ if }r\neq 0\end{array}\right.\to\mathbf{W}'\to 0$$

where $\sigma_y^s(\bullet)$ are defined accordingly to proposition 3.7-b_2). We introduce the following elements:

Definition 4.12. *Let $t \geqslant 2$, t even.*

i) for $l_0, \dots, l_{t-2} \in \{0, \dots, p-1\}$ and $j \in \{0, \dots, r\}$ define

$$\mathfrak{w}_{l_0,\dots,l_{t-2}}(j) \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\mu_{t-2} \in \mathbf{F}_p} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

$$\mathfrak{w}'_{l_1,\dots,l_{t-2}}(j) \stackrel{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_{t-2} \in \mathbf{F}_p} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

where we set, for $t = 2$,

$$\mathfrak{w}'(j) \stackrel{\text{def}}{=} [1, X^{r-j}Y^j].$$

Note that such elements are in $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ (which is K_{t-1}-invariant) iff $j = 0$.

ii) for $l_0, \dots, l_{t-1} \in \{0, \dots, p-1\}$ define

$$\mathfrak{z}_{l_0,\dots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r]$$

$$\mathfrak{z}'_{l_1,\dots,l_{t-1}} 1 \stackrel{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r].$$

The element defined in 4.12 will be seen also as elements of the amalgamed sums, according to the context. We see as above that

Lemma 4.13. *Let $t \geqslant 2$, t even. The following elements are I_t invariant in the inductive limit $\varinjlim_{n\,\text{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$:*

i) the elements $\mathfrak{w}'_{l_1,\dots,l_{t-2}}(j)$ for $j \in \{1, \dots, r\}$, $(l_0, \dots, l_{t-2}) \in \{0, \dots, p-1\}^{t-1}$;

ii) the elements $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$,
where $(l_1, \dots, l_{t-1}) \prec (p-1-r, r, \dots, r, p-1-r)$.

Proof: Omissis. $\sharp$

As in §4.1.1, the elements of lemma 4.13, as well as the elements of $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ will be referred to as the *trivial I_t-invariants*. The result is then the following:

Proposition 4.14. *An $\overline{\mathbf{F}}_p$-basis for the space of nontrivial I_t-invariants of $\varinjlim_{n\,\text{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ modulo the trivial invariants is described by*

i) the elements $\mathfrak{w}_{l_0,\dots,l_{t-2}}(1)$ where $(l_0, \dots, l_{t-2}) \prec (r, p-1-r \dots, p-1-r, r)$ if $r \neq 0$;

ii) the elements $\mathfrak{z}_{l_0,\dots,l_{t-2},0}$ where $(l_0, \dots, l_{t-2}) \prec (r, p-1-r \dots, p-1-r, r)$ if $r = 0$.

Proof: Omissis.$\sharp$

We are finally in the position to compute the dimension of I_t-invariants for $\pi(r, 0, 1)$:

Corollary 4.15. *Let $r \in \{0, \dots, p-1\}$, $t \in \mathbf{N}_>$. Then*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r, 0, 1))^{I_t}) = 2(2p^{t-1} - 1).$$

Proof: We assume $t \geqslant 2$ and we will prove the result for t odd (the case t even is similar and left to the reader). We deduce from propositions 4.4 and 4.3 that

$$\dim_{\overline{\mathbf{F}}_p}((\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{I_t}/pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))) = rp^{t-2} + \sum_{j=1}^{t-1} p^{j-1} l_j + \sum_{j=0}^{t-2} p^j l'_j$$

where $(l_1, \dots, l_{t-1}) = (r, p-1-r, \dots, r, p-1-r)$ and $(l'_0, \dots, l'_{t-2}) = (p-1-r, r, \dots, p-1-r, r)$: they correspond to the invariants of type $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{x}_{l_0,\dots,l_{t-2}}(1)$ if $r \neq 0$ (resp. $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{y}_{l_0,\dots,l_{t-2},0}$ if $r = 0$).

Similarly, propositions 4.8 and 4.7 give

$$\dim_{\overline{\mathbf{F}}_p}((\varinjlim_{n\,\mathrm{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})^{I_t}/pr_{t-2}(R_{t-2})) = (p-1-r)p^{t-2} + \sum_{j=1}^{t-1} p^{j-1} l_j + \sum_{j=0}^{t-2} p^j l'_j$$

where $(l_1, \dots, l_{t-1}) = (p-1-r, r \dots, p-1-r, r)$ and $(l'_0, \dots, l'_{t-2}) = (r, p-1-r, \dots, r, p-1-r)$: they correspond to the invariants of type $\mathfrak{z}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$ if $r \neq 0$ (resp. $\mathfrak{h}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$ if $r = 0$). As

$$\dim_{\overline{\mathbf{F}}_p}(pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))) + \dim_{\overline{\mathbf{F}}_p}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2}) = p^{t-2}(p+1)$$

(lemma 2.8) an elementary computation yields the desired result for $t \geqslant 2$, t odd.

Since $\pi(r, 0, 1)^{I_1}$ is 2 dimensional (cf. [Bre03a] Théoréme 3.2.4) the conclusion follows. $\sharp$.

5. The case of principal series and the Steinberg.

We are going to describe briefly the K_t and I_t-invariants of principal series and Steinberg for $\mathrm{GL}_2(\mathbf{Q}_p)$; by Mackey's theorem and the Iwasawa decomposition for $\mathrm{GL}_2(\mathbf{Q}_p)$ it will be enough to study the inductions $\mathrm{Ind}_{K\cap B}^K \chi_r^s$ for $r \in \{0, \dots, p-2\}$. As the techniques involved are completely similar to what we have seen for the supersingular case, we will content ourselves to state the results, leaving the proofs to the reader.

Concerning the K_t-invariants. Let $t \in \mathbf{N}_>$. From the exact sequences

$$0 \to \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \to \mathrm{Ind}_{K_0(p^{n+2})}^K \chi_r^s \to Q_{0,\dots,0,1}^{(0,n+2)} \to 0$$

we see (as in the proof of lemma 3.2) that all K_t-invariants for $\mathrm{Ind}_{K\cap B}^K \chi_r^s$ must be inside $\mathrm{Ind}_{K_0(p^k)}^K \chi_r^s$. More precisely, we have

Theorem 5.1. *Let $t \in \mathbf{N}_>$. Then*

$$(\mathrm{Ind}_{K\cap B}^K \chi_r^s)^{K_t} = \mathrm{Ind}_{K_0(p^t)}^K \chi_r^s.$$

In particular, $\dim_{\overline{\mathbf{F}}_p}(\mathrm{Ind}_{K\cap B}^K \chi_r^s) = p^{t-1}(p+1)$.

Proof: Omissis. $\sharp$

Concerning the I_t-invariants. Fix $t \in \mathbf{N}_>$. As $I_t \geqslant K_t$ we see that $(\mathrm{Ind}_{K\cap B}^K \chi_r^s)^{I_t} = (\mathrm{Ind}_{K_0(p^t)}^K \chi_r^s)^{I_t}$. We can therefore use the $\overline{\mathbf{F}}_p$-basis of $\mathrm{Ind}_{K_0(p^t)}^K \chi_r^s$ given by

$$\mathfrak{x}_{l_0,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1,e]$$

$$\mathfrak{x}'_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1,e]$$

(where, if $t=1$, we set $\mathfrak{x}' \stackrel{\mathrm{def}}{=} [1,e]$) to describe completely the space $(\mathrm{Ind}_{K_0(p^t)}^K \chi_r^s)^{I_t}$. We find that

PROPOSITION 5.2. *Let $t \geqslant 2$. Then an $\overline{\mathbf{F}}_p$-basis for the space $(\mathrm{Ind}_{K_0(p^t)}^K \chi_r^s)^{I_t}/(\mathrm{Ind}_{K_0(p^{t-1})}^K \chi_r^s)$ is given by the elements $\mathfrak{x}'_{l_1,\dots,l_{t-1}}$ with $l_{t-1} \geqslant 1$. In particular we have*

$$\dim_{\overline{\mathbf{F}}_p}((\mathrm{Ind}_{K\cap B}^K \chi_r^s)^{I_t}) = 2p^{t-1}$$

for any $t \in \mathbf{N}_>$.

Proof: It follows the arguments in the proofs of propositions 4.3, 4.7 and it is left to the reader.♯

Part III. On some restriction of supersingular representations for $\mathrm{GL}_2(\mathbf{Q}_p)$

Abstract. If L/F is a quadratic extension of local fields (of characteristic zero) and π a supercuspidal representation of $\mathrm{GL}_2(F)$ a theorem of Tunnel and Saito relates the epsilon local factor associated to π to the $L^\times$-socle of $\pi|_{L^\times}$. In this chapter we consider the problem of giving a detailed description of the $L^\times$-structure of supersingular mod p-representations for the case $F = \mathbf{Q}_p$, in the spirit of a theory of mod p epsilon factors.

1. Introduction, Notations and Preliminaries

Let F be a non-archimedean local field of characteristic 0 and π an admissible irreducible infinite dimensional representation of $\mathrm{GL}_2(F)$ over $\mathbf{C}$. For a quadratic field extension L/F we fix an embedding $L^\times \hookrightarrow \mathrm{GL}_2(F)$. By a theorem of Tunnel and Saito (cf. [Tun83], [Sai93]) it is possible to characterize the epsilon factors associated to the base change $\mathrm{BC}_{L/F}(\pi)$ to the $L^\times$-structure of the restriction $\pi|_{L^\times}$:

THEOREM 1.1 (Tunnel, Saito). *Let λ be a character of $L^\times$ extending the central character ω_π of π. The following are equivalent:*

i) the character λ occurs in the restriction $\pi|_{L^\times}$;

ii) the following equality is true:

$$\varepsilon(\mathrm{BC}_{L/F}(\pi) \otimes \lambda^{-1})\omega_\pi(-1) = 1$$

where $\varepsilon(\mathrm{BC}_{L/F}(\pi) \otimes \lambda^{-1})$ denotes the root number of the representation $\mathrm{BC}_{L/F}(\pi) \otimes \lambda^{-1}$.

Indeed, the problem of looking for multiplicities of $L^\times$-characters in $\pi|_{L^\times}$ goes back to a work of Silberger ([Sil69]) and has been approached again in the works of Tunnel [Tun83] and Prasad [Pras90]. In particular, the Tunnel-Saito theorem appears again in [Ragh], where Raghuram gives explicit sufficient conditions for a character to appear in the $\pi|_{L^\times}$ for π supercuspidal.

In this part we approach the "mod p"-analogue of such problem in the case $F = \mathbf{Q}_p$, describing the structure of $\pi(r, 0, 1)|_{L^\times}$ (where the representations $\pi(r, 0, 1)$ are, up to twist, the supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ classified by Breuil in [Bre03a]). We rely on the works [Mo1], [Mo5], where we gave a detailed description of the Iwahori and $\mathrm{GL}_2(\mathbf{Z}_p)$-structure for the representations $\pi(r, 0, 1)$. In all what follows, we will assume $p \geqslant 5$.

The results can be summed up in the following

THEOREM 1.2. *Let $L/\mathbf{Q}_p$ be a quadratic extension, $r \in \{0, \ldots, p-1\}$ and $\pi(r, 0, 1)$ a supersingular representation. Write $\mathrm{soc}^{(j)}_{L^\times} \overset{def}{=} \mathrm{soc}^{(j)}_{L^\times}(\pi(r, 0, 1)|_{L^\times})$ for the j-th composition factor of the $L^\times$-socle filtration for $\pi(r, 0, 1)|_{L^\times}$.*

Then:

i) if $L/\mathbf{Q}_p$ is unramified we have an isomorphism of $k_L^\times$-representations

$$\mathrm{soc}^{(j)}_{L^\times}/\mathrm{soc}^{(j-1)}_{L^\times} \cong (\oplus_{i=0}^{p} \eta_i(r))^2$$

where $\eta_i(r)$, for $i = 0, \ldots, p$, are the $(p+1)$ distincts characters of $k_L^\times$ extending the character $x \mapsto x^r$ on $k_{\mathbf{Q}_p}$;

ii) if $L/\mathbf{Q}_p$ is totally ramified we have an isomorphism of $L^\times$-representations

$$\mathrm{soc}^{(j)}_{L^\times}/\mathrm{soc}^{(j-1)}_{L^\times} \cong (V)^{2-\delta_{0,j}}$$

where V is a two dimensional vector space, endowed with the $\mathscr{O}_L^\times$ action inflated from the $k_{\mathbf{Q}_p}$-character $x \mapsto x^r$ and the action of the uniformiser given by a nontrivial involution.

The plan of the chapter and the strategy of the proof is the following.

We treat first the unramified case (§2). As announced, thanks to the $\mathrm{GL}_2(\mathbf{Z}_p)$-equivariant filtration introduced in [Mo1] we can treat first the finite case (i.e. representations of $\mathbf{F}_{p^2}^\times$) in section 2.1. The main tool is then lemma 2.3 which let us glue together the representations obtained in the finite case.

The totally ramified case is much easier and follows (almost immediately) from the Iwahori description of the universal representations given in [Mo5]

We briefly introduce the essential notation for the chapter (see also [Bre03a]).

For a finite extension L of $\mathbf{Q}_p$ we write $\mathscr{O}_L$ to denote its ring of integers and k_L the residue field. We write $[\cdot] : \mathbf{F}_p^\times \to W(\mathbf{F}_p)^\times$ for the Teichmüller character. We put as usual $G \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Q}_p)$, $K \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Z}_p)$ and $Z \overset{\text{def}}{=} Z(G)$ for the center of G.

We fix for the rest of the chapter an integer $r \in \{0, \dots, p-1\}$. The algebraic representation $\sigma_r \overset{\text{def}}{=} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2$ (endowed with the modular action of $\mathrm{GL}_2(\mathbf{F}_p)$) is seen as a representation of KZ (by inflation and making p act trivially) so that we can consider the compact induction

$$c\text{--}\mathrm{Ind}_{KZ}^G \sigma_r.$$

In [BL94], proposition 8-(1) Barthel and Livné showed the existence of a canonical Hecke operator $T \in \mathrm{End}_G(c\text{--}\mathrm{Ind}_{KZ}^G \sigma_r)$ (depending on r) which realizes an isomorphism of the $\overline{\mathbf{F}}_p$-algebra of endomorphisms $\mathrm{End}_G(c\text{--}\mathrm{Ind}_{KZ}^G \sigma_r)$ with the ring of polynomials in one variable over $\overline{\mathbf{F}}_p$. We then define the universal representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ as the cokernel of the canonical operator T:

$$\pi(r, 0, 1) \overset{\text{def}}{=} \mathrm{coker}(T).$$

As recalled above, the computations of Breuil ([Bre03a]) show that all such representations are irreducible and, up to twist by a character of $\mathbf{Q}_p$, exhaust all the supersingular representations for $\mathrm{GL}_2(\mathbf{Q}_p)$.

Let H be the maximal torus of $\mathrm{GL}_2(\mathbf{F}_p)$ and $t \in \{0, \dots, p-2\}$. We have the characters

$$\begin{array}{ccc} \chi_t : H \to \overline{\mathbf{F}}_p & \qquad & \mathfrak{a} : H \to \overline{\mathbf{F}}_p \\ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mapsto a^t & \qquad & \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mapsto ad^{-1}. \end{array}$$

If χ is a character of H we will write χ^s to denote its conjugate. Finally any character of H will be considered as a character of $B(\mathbf{F}_p)$ or (by inflation) as a character of $K_0(p) = I$, without any commentary.

We recall (see [Mo1]) that the $\mathrm{GL}_2(\mathbf{Z}_p)$-restriction of the supersingular representation $\pi(r, 0, 1)$ admits a decomposition

$$\pi(r, 0, 1)|_{\mathrm{GL}_2(\mathbf{Z}_p)} \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{m\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1}) \qquad (13)$$

(see [Mo1], §2 for the precise definition of the representations R_n).

For any $n \geqslant 0$ (resp. $n \geqslant 1$) we define the following elements of $\varinjlim(\cdots \oplus_{R_n} R_{n+1})$.

If $i \in \{1, \ldots, r\}$ and $(l_0, \ldots, l_n) \in \{0, \ldots, p-1\}^{n+1}$ is an $(n+1)$-tuple (resp. $(l_1, \ldots, l_n) \in \{0, \ldots, p-1\}^n$ an n-tuple) we set

$$F^{(0,n)}_{l_0,\ldots,l_n}(i) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \lambda_0^{l_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, X^{r-i}Y^i]$$

and

$$F^{(1,n)}_{l_1,\ldots,l_n}(i) \overset{\text{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, X^{r-i}Y^i].$$

For $i = 0$ we define similarly $F^{(0,n)}_{l_0,\ldots,l_n}(i)$ and $F^{(1,n)}_{l_1,\ldots,l_n}(i)$, where we add the additional condition $l_n \geqslant r+1$ (in particular for $r = p-1$ such functions are not considered).

We formally define

$$\begin{aligned} F^{(0,-1)}_{\emptyset}(i) &\overset{\text{def}}{=} X^i Y^{r-i}; \\ F^{(1,0)}_{\emptyset}(i) &\overset{\text{def}}{=} [1, X^{r-i}Y^i]; \\ F^{(-1,0)}_{\emptyset} &\overset{\text{def}}{=} Y^r. \end{aligned}$$

From the results of [Mo1] it follows that:

Proposition 1.3. *For $n \in \mathbf{N}$, $j \in \{0, 1\}$, the elements $F^{(j,n)}_{l_j,\ldots,l_n}(i)$ defined above describe an $\overline{\mathbf{F}}_p$-basis for the representation $\pi(r, 0, 1)$.*

Notice that for $j \in \{0, 1\}$ the injective map

$$F^{(j,n)}_{l_j,\ldots,l_n}(i) \mapsto \sum_{s=j}^{n} p^{s-j} l_s + p^{n+1-j} i \in \mathbf{N}$$

provides the set of functions $F^{(0,n)}_{l_0,\ldots,l_n}(i)$ (resp. $F^{(1,n)}_{l_1,\ldots,l_n}(i)$) with a natural linear ordering; we will then write $F^{(j,n)}_{l_j-1,\ldots,l_n}(i)$ to mean the antecedent of $F^{(j,n)}_{l_j,\ldots,l_n}(i)$.

Each direct summand in the decomposition (13) admits a $\mathrm{GL}_2(\mathbf{Z}_p)$-filtration with respect to which the extensions between two consecutive graded pieces can be summarized as follow:

$$\ldots \text{—}\mathrm{Ind}^K_{K_0(p)}\langle [1, F^{1,n}_{l_1-1,\ldots,l_n}(i)]\rangle \text{—} \mathrm{Ind}^K_{K_0(p)}\langle [1, F^{1,n}_{l_1,\ldots,l_n}(i)]\rangle \text{—} \ldots \tag{14}$$

where

$$\mathrm{Ind}^K_{K_0(p)}\langle [1, F^{1,n}_{l_1,\ldots,l_n}(i)]\rangle \cong \mathrm{Ind}^{\mathrm{GL}_2(\mathbf{F}_p)}_{B(\mathbf{F}_p)} \chi_r^s \mathfrak{a}^{i+\sum_{j=1}^n l_j}.$$

Moreover we have, for $a, b, c, d \in \mathbf{Z}_p$,

$$\begin{bmatrix} 1+pa & pb \\ pc & 1+pd \end{bmatrix} [1, F^{1,n}_{l_1,\ldots,l_n}(i)] = F^{1,n}_{l_1,\ldots,l_n}(i) - \overline{c}\kappa(l_1, \ldots, l_n, i)(F^{1,n}_{l_1-1,\ldots,l_n}(i)) + y \tag{15}$$

where y is a suitable sum of functions strictly preceeding $F^{1,n}_{l_1-1,\ldots,l_n}(i)$ in the natural ordering on $F^{(1,n)}_{l_1,\ldots,l_n}(i)$ and $\kappa(l_1, \ldots, l_n, i) \in \mathbf{F}_p^{\times}$.

Let L be a quadratic extension of $\mathbf{Q}_p$. Fixing a $\mathbf{Z}_p$-base $\mathscr{B}$ of $\mathscr{O}_L$ gives us the embeddings groups:

$$\begin{tikzcd} L^\times \cong \mathrm{Aut}_L(L) \arrow[r, hook, "\iota_{\mathscr{B}}"] & \mathrm{GL}_2(\mathbf{Q}_p) \\ \mathscr{O}_L^\times \cong \mathrm{Aut}_{\mathscr{O}_L}(\mathscr{O}_L) \arrow[u, hook] \arrow[r, hook] & \mathrm{GL}_2(\mathbf{Z}_p) \arrow[u, hook] \end{tikzcd}$$

and therefore we can study the structure of $\pi(r,0,1)|_{L^\times}$. Note that such a structure does not depend on the choice of the basis as the subgroups $\iota_{\mathscr{B}}(L^\times)$, $\iota_{\mathscr{B}'}(L^\times)$, for $\mathscr{B}$, $\mathscr{B}'$ two $\mathbf{Z}_p$-bases of $\mathscr{O}_L$, are conjugated in $\mathrm{GL}_2(\mathbf{Q}_p)$.

2. The unramified case

Throughout this section, we will assume $L/\mathbf{Q}_p$ unramified. The main result is proposition 2.10, which gives the $L^\times$ structure for the representation $\pi(r,0,1)$. After an analysis of the finite case in §2.1 (which is made possible by the filtration (14)) we give the key result (lemma 2.3) which let us glue together the characters appearing between two consecutive graded pieces of the filtration (14). This will enable us to detect the socle filtration for $\pi(r,0,1)|_{L^\times}$, and an elementary observation (lemma 2.5) gives us a full description of the extensions between two two consecutive graded pieces of the socle filtration.

Fix $\alpha \in \mathbf{F}_q$, a (p^2-1)-th primitive root of unity; its minimal polynomial over $\mathbf{F}_p$ is $X^2 - Tr(\alpha) + N(\alpha)$ where Tr, N denotes respectively the trace and norm of $\mathbf{F}_q$ over $\mathbf{F}_p$. Thus, we get a $\mathbf{Z}_p$-basis $\mathscr{B} \overset{\text{def}}{=} \{1, [\alpha]\}$ of $\mathscr{O}_L$ and a $\mathbf{Z}_p$-linear isomorphism

$$\mathscr{O}_L \cong \mathbf{Z}_p \otimes \mathbf{Z}_p[\alpha].$$

We see that $\iota_{\mathscr{B}}(p) = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ and such element acts trivially on $\pi(r,0,1)$: in order to study $\pi(r,0,1)|_{L^\times}$ we can therefore content ourself to study the restriction

$$\pi(r,0,1)|_{\mathscr{O}_L^\times}.$$

Finally, let $x, y \in \mathbf{Z}_p$ be such that

$$[\alpha^2] = [-N(\alpha)] + [\alpha][Tr(\alpha)] + px + p[\alpha]y.$$

It follows, for $a, b \in \mathbf{Z}_p$ such that $a + [\alpha]b \in \mathscr{O}_L^\times$, that

$$\iota_{\mathscr{B}}(a + [\alpha]b) = \begin{bmatrix} a & b[-N(\alpha)] + pxb \\ b & a + b[Tr(\alpha)] + pby \end{bmatrix}.$$

2.1 The finite case

Let $l, m \in \mathbf{Z}$ be integers. We consider

$$V_{l,m} \overset{\text{def}}{=} \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_l^s \otimes \det{}^m$$

as a K-representation via the inflation map $K \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_p)$. One verifies that the restriction

$$V_{l,m}|_{\mathscr{O}_L^\times}$$

is naturally isomorphic to the $\mathscr{O}_L^\times$-representation obtained -via the inflation map $\mathscr{O}_L^\times \twoheadrightarrow \mathbf{F}_q^\times$- from

$$V_{l,m}|_{\mathbf{F}_q^\times}.$$

The object of this subsection is to describe the representation $V_{l,m}|_{\mathbf{F}_q^\times}$.

As the group $\mathbf{F}_q^\times$ is abelian, and $|\mathbf{F}_q^\times|$ is coprime with p it follows that $V_{l,m}|_{\mathbf{F}_q^\times}$ decomposes in a direct sum of characters. Moreover, Mackey decomposition gives us an isomorphism of $\mathbf{F}_q^\times$-representations

$$V_{l,m}|_{\mathbf{F}_q^\times} \xrightarrow{\sim} \mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_q^\times}(\cdot)^{-l} \otimes N^{m+l}.$$

We give below the explicit description of such isomorphism.

Define the following permutation σ of $\{0,\dots,p-1,\infty\}$. For $\lambda_0 \in \{0,\dots,p-1\}$ we set

$$\begin{aligned}\sigma(\lambda_0) &\stackrel{\mathrm{def}}{=} -\frac{N(\alpha)}{\lambda_0 + Tr(\alpha)} \quad \text{if } \lambda_0 \neq -Tr(\alpha);\\ \sigma(\lambda_0) &\stackrel{\mathrm{def}}{=} \infty \quad \text{if } \lambda_0 = -Tr(\alpha);\end{aligned}$$

and

$$\sigma(\infty) \stackrel{\mathrm{def}}{=} 0$$

In other words, we are considering the projective transformation on $\mathbf{P}^1(\mathbf{F}_p)$ associated to the matrix $\begin{bmatrix} Tr(\alpha) & 1 \\ -N(\alpha) & 0 \end{bmatrix}$. We moreover define a map $x(\cdot) : \{0,\dots,p-1,\infty\} \to \mathbf{F}_p$ by

$$\begin{aligned}x(\lambda_0) &\stackrel{\mathrm{def}}{=} \lambda_0 + Tr(\alpha) \quad \text{if } \lambda_0 \notin \{-Tr(\alpha),\infty\};\\ x(-Tr(\alpha)) &\stackrel{\mathrm{def}}{=} -N(\alpha);\\ x(\infty) &\stackrel{\mathrm{def}}{=} 1.\end{aligned}$$

Recall that a $\overline{\mathbf{F}}_p$-basis for $\mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_q^\times}(\cdot)^{-l} \otimes N^{m+l}$ is described by

$$\mathscr{B} = \Big\{[\lambda_0 + \alpha, e] \quad \text{for } \lambda_0 \in \mathbf{F}_p; \quad [1,e]\Big\}.$$

We finally consider the lemma

Lemma 2.1. *We have an $\mathbf{F}_q^\times$-equivariant isomorphism defined by:*

$$\begin{aligned} V_{l,m}|_{\mathbf{F}_q^\times} &\xrightarrow{\sim} \mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_q^\times}(\cdot)^{-l} \otimes N^{m+l}\\ [\begin{bmatrix} \lambda_0 & 1 \\ 1 & 0 \end{bmatrix}, e] &\mapsto [\lambda_0 + \alpha, e];\\ [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e] &\mapsto [\alpha, e](-1)^l.\end{aligned}$$

Proof. The $\overline{\mathbf{F}}_p$-linear morphism of the statement is clearly an isomorphism and we claim it is $\mathbf{F}_q^\times$-equivariant. It is enough to check the compatibility of the isomorphism with the action of α, on a fixed base of $V_{l,m}|_{\mathbf{F}_q^\times}$. A direct computation gives

$$\left[\begin{array}{cc} 0 & -N(\alpha) \\ 1 & Tr(\alpha) \end{array}\right][\left[\begin{array}{cc} \lambda_0 & 1 \\ 1 & 0 \end{array}\right], e] = \begin{cases} (N(\alpha))^m(-\sigma(\lambda_0))^l[\left[\begin{array}{cc} \sigma(\lambda_0) & 1 \\ 1 & 0 \end{array}\right], e] \text{ if } \lambda_0 \neq -Tr(\alpha); \\ (N(\alpha))^m[1,e] \text{ if } \lambda_0 = -Tr(\alpha); \end{cases}$$

$$\left[\begin{array}{cc} 0 & -N(\alpha) \\ 1 & Tr(\alpha) \end{array}\right][1,e] = (N(\alpha))^m(-N(\alpha))^l[\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], e]$$

and

$$\alpha[\lambda_0 + \alpha, e] = (N(\alpha))^{m+l}(x(\lambda_0))^{-l}[\sigma(\lambda_0) + \alpha, e].$$

The conclusion follows. □

2.1.1 *Study of* $\mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_q^\times}(\cdot)^l$. Let $l \in \{0, \ldots, p-1\}$. The $\mathbf{F}_q^\times$-representation $\mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_q^\times}(\cdot)^l$ decomposes into a direct sums of $p+1$-characters, and these characters are precisely all the $p+1$-possible extensions of $\lambda \mapsto \lambda^l$ to $\mathbf{F}_q^\times$.

If $s_0, s_1 \in \{0, \ldots, p-1\}$ are such that $(s_0, s_1) \neq (p-1, p-1)$ then the $\mathbf{F}_q^\times$-character defined by

$$\alpha \mapsto \alpha^{s_0 + ps_1}$$

extends $(\cdot)^l$ if and only if

$$(s_0 + s_1) + p(s_0 + s_1) \equiv l + pl \mod p^2 - 1$$

that is if and only if the couple (s_0, s_1) verify one of the following relations:

$$s_0 + s_1 = l \qquad s_0 + s_1 = p - 1 + l.$$

We will say that (s_0, s_1) is an *admissible couple for* l. This can be effectively summed up by figure III.1.

For an admissible couple (s_0, s_1) we let

$$v^{(s_0,s_1)} \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \mu_{\lambda_0}^{(s_0,s_1)}[\lambda_0 + \alpha, e] + \mu_\infty^{(s_0,s_1)}[1,e]$$

be an eigenvector for the action of $\mathbf{F}_q^\times$, of associated eigencharacter $(\cdot)^{s_0+ps_1}$. The scalars $\mu_{\lambda_0}^{(s_0,s_1)}$, $\mu_\infty^{(s_0,s_1)}$ admit the following description:

LEMMA 2.2. *Let* (s_0, s_1) *be an admisibble couple for* l *and let* $n \in \{0, \ldots, p\}$. *Then*

$$\mu_{\sigma^{-n}(0)}^{(s_0,s_1)} = \mu_0^{(s_0,s_1)} \alpha^{n(s_0+ps_1)} \big(x(\sigma^{-1}(0)) \cdot \cdots \cdot x(\sigma^{-n}(0))\big)^{-l}.$$

Proof. It is enough to study the action of α on $v^{(s_0,s_1)}$. A computation gives:

$$\alpha v^{(s_0,s_1)} = \sum_{\lambda_0 \notin \{-Tr(\alpha), \infty\}} \mu_{\lambda_0}^{(s_0,s_1)}(x(\lambda_0))^l[\sigma(\lambda_0) + \alpha, e] + \\ + \mu_{-Tr(\alpha)}^{(s_0,s_1)}(x(-Tr(\alpha)))^l[1,e] + \mu_\infty^{(s_0,s_1)}[\alpha, e]$$

so that, assuming that $v^{(s_0,s_1)}$ is an eigenvector of associated eigencaracter $(\cdot)^{s_0+ps_1}$ we get the

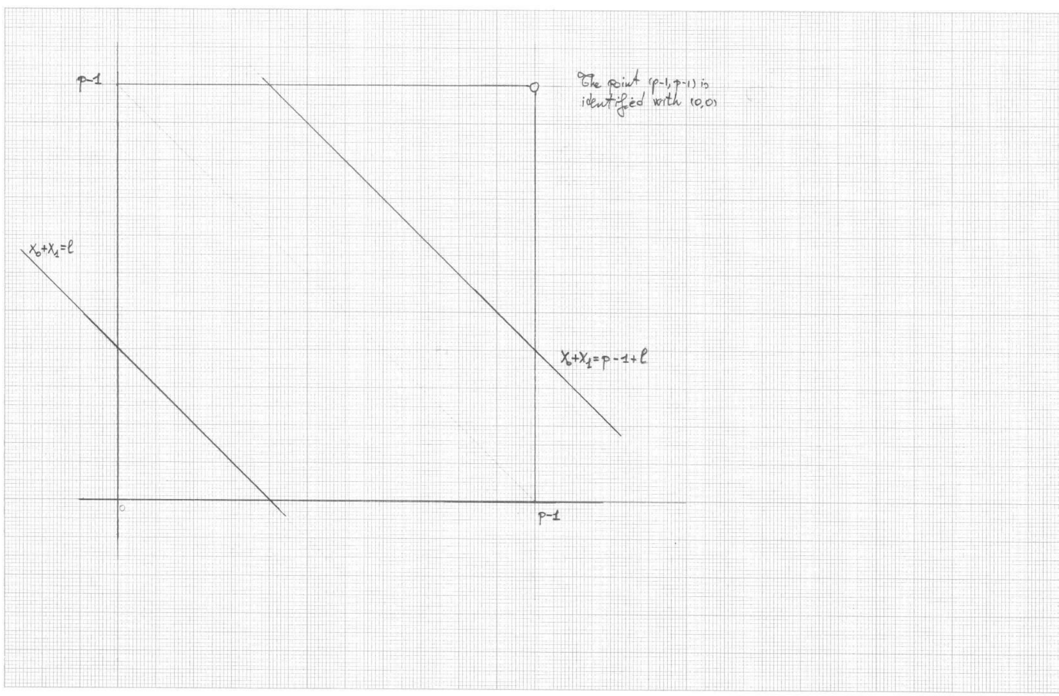


FIGURE III.1. The combinatoric of admissible couples for l.

relations:

$$\begin{aligned}\mu_{\lambda_0}^{(s_0,s_1)}(x(\lambda_0))^l &= \alpha^{s_0+ps_1}\mu_{\sigma(\lambda_0)}^{s_0+ps_1} \text{ if } \lambda_0 \notin \{-Tr(\alpha),\infty\};\\ \mu_{-Tr(\alpha)}^{(s_0,s_1)}(x(-Tr(\alpha)))^l &= \alpha^{s_0+ps_1}\mu_{\infty}^{(s_0,s_1)};\\ \mu_{\infty}^{(s_0,s_1)} &= \alpha^{s_0+ps_1}\mu_0.\end{aligned}$$

The result is trivial for $n=0$ and follows immediately for $n=1$ (as $\sigma^{-1}(0)=\infty$, and $x(\infty)=1$). The general case follows by induction. □

2.2 Extensions inside the supersingular representation

In this section we are going to study the extensions of two successive graded pieces

$$S_j/S_{j-1}\text{---}S_{j+1}/S_j \tag{16}$$

appearing in the filtration (14) (for each of the directed summands appearing in the decomposition (13)). The case $j=0$ -i.e. where S_j is indeed the socle of the directed summand under study- needs additional care and will be treated in §2.3. We then consider the extension

$$\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1,F^{(1,n)}_{l_1-1,\dots,l_n}(i)]\text{---}\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1,F^{(1,n)}_{l_1,\dots,l_n}(i)] \tag{17}$$

where we recall that

$$\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1,F^{(1,n)}_{l_1-\epsilon,\dots,l_n}(i)]\cong \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi^s_{r-2m+2\epsilon}\otimes \det{}^{m-\epsilon}\xrightarrow{\sim}\mathrm{Ind}_{\mathbf{F}_p^\times}^{\mathbf{F}_q^\times}(\cdot)^{2m-2\epsilon-r}\otimes N^{r-n+\epsilon}$$

for $\epsilon\in\{0,1\}$, $m\overset{\text{def}}{=}i+\sum_{j=1}^n l_j$. Let (s_0,s_1) an admissible couple for $2m-r$ and let

$$v^{(s_0,s_1)}(l_1,\dots,l_n,i)\overset{\text{def}}{=}v^{(s_0,s_1)}(F^{(1,n)}_{l_1,\dots,l_n}(i))\in \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1,F^{(1,n)}_{l_1,\dots,l_n}(i)]$$

be an eigenvector for the action of $\mathbf{F}_q^\times$, of associated eigencharacter $(\cdot)^{s_0+ps_1}N^{r-m}$. We have the following result:

LEMMA 2.3. *Inside the extension (17) we have the following equality, for any $x',y'\in\mathbf{Z}_p$:*

$$\begin{bmatrix}1 & p[N(\alpha)]+p^2x'\\ p & 1+p[Tr(\alpha)]+p^2y'\end{bmatrix}v^{(s_0,s_1)}(l_1,\dots,l_n,i)=v^{(s_0,s_1)}(l_1,\dots,l_n,i)+$$
$$+N(\alpha)\kappa(l_1-1,\dots,l_n,i)v^{(s_0-1,s_1-1)}(l_1,\dots,l_n,i)$$

where s_i-1 are the cyphers of the p-adic development of $s_0+ps_1-(p+1)\bmod p^2-1$ and $\kappa(l_1,\dots,l_n,i)\in\mathbf{F}_p^\times$.

Proof. We rely crucially on the behaviour of the functions $F^{(1,n)}_{l_1,\dots,l_n}(i)$ described in (15). Thanks to the isomorphism of lemma 2.1 we can write:

$$v^{(s_0,s_1)}(l_1,\dots,l_n,i))=\sum_{\lambda_0\in\mathbf{F}_p}\mu_{\lambda_0}^{(s_0,s_1)}[\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix},F^{(1,n)}_{l_1,\dots,l_n}(i)]+(-1)^{r-2m}\mu_\infty^{(s_0,s_1)}[1,F^{(1,n)}_{l_1,\dots,l_n}(i)]$$

where we may assume, without loss of generality, that $\mu_0^{(s_0,s_1)}=1$; moreover we have the matrix equality

$$\begin{bmatrix}1 & p[N(\alpha)]+p^2x'\\ p & 1+p[Tr(\alpha)]+p^2y'\end{bmatrix}\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1+p* & p*\\ p[-\lambda_0^2-\lambda_0Tr(\alpha)-N(\alpha)]+p^2* & 1+p*\end{bmatrix}$$

where we do not care about the value of $* \in \mathbf{Z}_p$ thanks to the relation given in (15). As the roots of $P(X) \stackrel{\text{def}}{=} X^2 + Tr(\alpha)X + N(\alpha)$ are $-\alpha, -\alpha^p$, we notice that $-P(\lambda_0) \neq 0$ and we get

$$\begin{bmatrix} 1 & p[N(\alpha)] + p^2x' \\ p & 1 + p[Tr(\alpha)] + p^2y' \end{bmatrix} v^{(s_0,s_1)}(l_1, \ldots, l_n, i) = v^{(s_0,s_1)}(l_1, \ldots, l_n, i) +$$
$$\kappa(l_1, \ldots, l_n, i)(\sum_{\lambda_0 \in \mathbf{F}_p} \mu_{\lambda_0}^{(s_0,s_1)} P(\lambda_0)[\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}, F_{l_1-1,\ldots,l_n}^{(1,n)}(i)] + (-1)^{r-2m}\mu_\infty^{(s_0,s_1)}[1, F_{l_1-1,\ldots,l_n}^{(1,n)}(i)]).$$

We have of course

$$P(0) = N(\alpha);$$
$$(-1)^{r-2m}\mu_\infty^{(s_0,s_1)} = (-1)^{r-2m}\alpha^{s_0+ps_1} = (-1)^{r-2m}\alpha^{s_0-1+p(s_1-1)}N(\alpha) = (-1)^{r-2m}\mu_\infty^{(s_0-1,s_1-1)}N(\alpha)$$

and we are left to prove that

$$\mu_{\sigma^{-n}(0)}^{(s_0,s_1)} P(\sigma^{-n}(0)) = N(\alpha)\mu_{\sigma^{-n}(0)}^{(s_0-1,s_1-1)}$$

where $2 \leqslant n \leqslant p$.

This will be done by induction on n, the case $n = 1$ being proved; for a notational convenience, we put $P(\infty) \stackrel{\text{def}}{=} 1$. Assume the result true for $n - 1$; if $i \stackrel{\text{def}}{=} \sigma^{-(n-1)}(0)$ we then have

$$\begin{aligned}
\mu_{\sigma^{-n}(0)}^{(s_0,s_1)} P(\sigma^{-n}(0)) &= \mu_{\sigma^{-1}(i)}^{(s_0,s_1)} P(\sigma^{-1}(i)) \\
&= \mu_i^{(s_0,s_1)} \alpha^{(s_0+ps_1)} P(\sigma^{-1}(i))(x(\sigma^{-1}(i)))^{r-2m} \\
&= \mu_i^{(s_0-1,s_1-1)} N(\alpha)(P(i))^{-1}\alpha^{(s_0+ps_1)} P(\sigma^{-1}(i))(x(\sigma^{-1}(i)))^{r-2m+2}(x(\sigma^{-1}(i)))^{-2} \\
&= N(\alpha) \underbrace{\mu_i^{(s_0-1,s_1-1)} x(\sigma^{-1}(i))^{r-2m+2}\alpha^{s_0-1+p(s_1-1)}}_{\mu_{\sigma^{-1}(i)}(s_0-1,s_1-1)} N(\alpha)P(i)^{-1}P(\sigma^{-1}(i))(x(\sigma^{-1}(i)))^{-2}.
\end{aligned}$$

To conclude the induction is then enough to show that

$$N(\alpha)(x(i))^{-2}P(i) = P(\sigma(i)).$$

A direct computation gives, for $\sigma(i) \notin \{0, \infty\}$:

$$P(\sigma(i)) = \left(-\frac{N(\alpha)}{i + Tr(\alpha)}\right)^2 - \frac{Tr(\alpha)N(\alpha)}{i + Tr(\alpha)} + N(\alpha) = N(\alpha)x(i)^{-2}P(i).$$

The remaining are formal: if $\sigma(i) = \infty$ we get $i = -Tr(\alpha)$, $x(i) = -N(\alpha)$, $P(\infty) = 1$ and $P(-Tr(\alpha)) = N(\alpha)$; if finally $\sigma(i) = 0$ then $i = \infty$, $x(\infty) = 1$, $P(\infty) = 1$ and $P(0) = N(\alpha)$. This ends the inductive step and the proof is complete. □

Remark 2.4. *It is immediate to see that the satement of lemma 2.3 can be improved as follow: if $a \in \mathscr{O}_L$, $b \in \mathscr{O}_L^\times$ and*

$$g \equiv \begin{bmatrix} 1 + p[\overline{a}] & p[-\overline{b}N(\alpha)] \\ p[\overline{b}] & 1 + p[\overline{b}T(\alpha) + \overline{a}] \end{bmatrix} \mod p^2$$

then

$$gv^{(s_0,s_1)}(l_1, \ldots, l_n, i) = v^{(s_0,s_1)}(l_1, \ldots, l_n, i) +$$
$$-\kappa(l_1, \ldots, l_n, i)N(\alpha)\overline{b}v_{l_1-1,\ldots,l_n,i}^{(s_0-1,s_1-1)}.$$

The next lemma shows that there cannot be other non-trivial extensions inside (16), i.e. that the extension

$$0 \to S_j/S_{j-1} \to S_{j+1}/S_{j-1} \to S_{j+1}/S_j \to 0$$

between two consecutive graded pieces in the filtration (14) splits into the direct sum of $(p+1)$ $\mathscr{O}_L^\times$-extensions (at least if $j \geqslant 1$).

LEMMA 2.5. *For $i \in \{0,1\}$ we consider the couples $(s_0^{(i)}, s_1^{(i)}) \in \{0,\dots,p-1\}^2 \setminus \{(p-1,p-1)\}$. Then*

$$\mathrm{Ext}^1_{\mathscr{O}_L^\times}((s_0^{(0)}, s_1^{(0)}),(s_0^{(1)}, s_1^{(1)})) \neq 0$$

if and only if $(s_0^{(0)}, s_1^{(0)}) = (s_0^{(1)}, s_1^{(1)})$.

Proof. We consider the following functors

$$\begin{aligned} \mathcal{R}ep_{\mathscr{O}_L^\times} &\longrightarrow \mathcal{R}ep_{\mathbf{F}_q^\times} \\ V &\longmapsto V^{1+p\mathscr{O}_L} \end{aligned}$$

and

$$\begin{aligned} \mathcal{R}ep_{\mathbf{F}_q^\times} &\longmapsto \mathcal{A}b \\ W &\longmapsto \mathrm{Hom}_{\mathbf{F}_q^\times}((s_0^{(0)}, s_1^{(0)}), W). \end{aligned}$$

For $V \in \mathcal{R}ep_{\mathscr{O}_L^\times}$ we have the following isomorphism (functorial in V)

$$\mathrm{Hom}_{\mathbf{F}_q^\times}((s_0^{(0)}, s_1^{(0)}), V^{1+\mathscr{O}_L}) \cong \mathrm{Hom}_{\mathscr{O}_L^\times}((s_0^{(0)}, s_1^{(0)}), V);$$

therefore, the first level of Grothendieck's spectral sequence gives

$$\begin{aligned} 0 \to \mathrm{Ext}^1((s_0^{(0)}, s_1^{(0)}),(s_0^{(1)}, s_1^{(1)})) \to \mathrm{Ext}^1_{\mathscr{O}_L^\times}((s_0^{(0)}, s_1^{(0)}),(s_0^{(1)}, s_1^{(1)})) \to \\ \to \mathrm{Hom}_{\mathbf{F}_q^\times}((s_0^{(0)}, s_1^{(0)}), H^1(1+p\mathscr{O}_L,((s_0^{(1)}, s_1^{(1)})))). \end{aligned}$$

We recall (cf. [Wil], §10.2) that the action of $\mathscr{O}_L^\times$ on

$$H^1(1+p\mathscr{O}_L,((s_0^{(0)}, s_1^{(1)}))) \cong \mathrm{Hom}(1+p\mathscr{O}_L,(s_0^{(1)}, s_1^{(1)}))$$

is described as

$$(g\cdot f)(t) \stackrel{\mathrm{def}}{=} (s_0^{(1)}, s_1^{(1)})(g) f(gtg^{-1}) = (s_0^{(1)}, s_1^{(1)})(g) f(t)$$

for any $g \in \mathscr{O}_L^\times$, $t \in 1+\mathscr{O}_L$ and $f \in \mathrm{Hom}(1+\mathscr{O}_L,(s_0^{(1)}, s_1^{(1)}))$. In particular, $\mathbf{F}_q^\times$ acts on $\mathrm{Hom}(1+\mathscr{O}_L,(s_0^{(1)}, s_1^{(1)}))$ by the character $((s_0^{(1)}, s_1^{(1)}))$. We deduce that

$$\mathrm{Hom}_{\mathbf{F}_q^\times}((s_0^{(0)}, s_1^{(0)}), H^1(1+p\mathscr{O}_L,((s_0^{(1)}, s_1^{(1)})))) = 0$$

if and only if $(s_0^{(0)}, s_1^{(0)}) = (s_0^{(1)}, s_1^{(1)})$.

As $(p, p^2-1)=1$, the category $\mathcal{R}ep_{\mathbf{F}_q^\times}$ is semisimple and the result follows. $\square$

REMARK 2.6. *We can actually prove more. Let χ_1, χ_2 be the $\mathscr{O}_L^\times$-inflation of distinct characters of $\mathbf{F}_{p^2}^\times$ and let T be a uniserial $\mathscr{O}_L^\times$ smooth representation over $\overline{\mathbf{F}}_p$ of finite length, having Jordan Hölder factors isomorphic to χ_1. Then an immediat dévissage (on the length of T) shows that*

$$\mathrm{Ext}^n_{\mathscr{O}_L^\times}(\chi_2, T) = 0 = \mathrm{Ext}^n_{\mathscr{O}_L^\times}(T, \chi_2)$$

for any $n \in \mathbf{N}$.

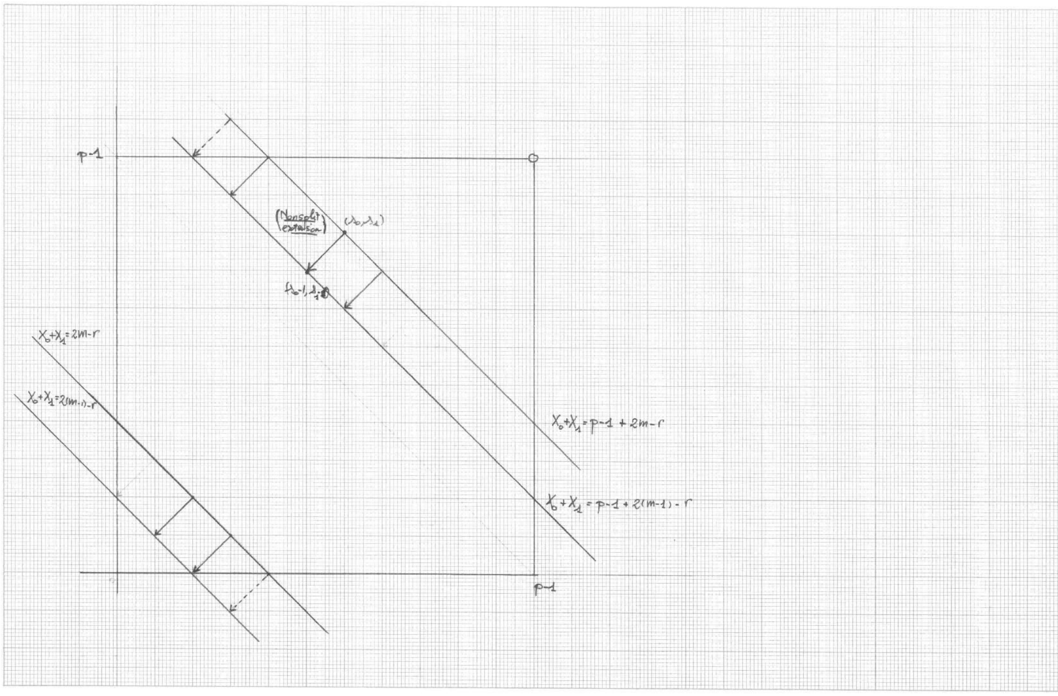


FIGURE III.2. A graphic gloss of proposition 2.7.

We can summarize the results of lemma 2.3 and 2.5 in the following:

PROPOSITION 2.7. *Consider the extension*

$$\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, F^{(1,n)}_{l_1-1,\dots,l_n}(i)] \text{——} \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, F^{(1,n)}_{l_1,\dots,l_n}(i)] \tag{18}$$

of two successive graded pieces in the filtration (14), and assume

$$F^{(1,n)}_{l_1-1,\dots,l_n}(i) \notin \{Y^r, [1, X^r]\}.$$

The $\mathscr{O}_L^\times$ restriction of (18) decomposes as a direct sum of $p+1$ non trivial extensions

$$\bigoplus_{(s_0,s_1)} \left(v^{(s_0-1,s_1-1)}(l_1-1,\dots,l_n,i) \text{——} v^{(s_0,s_1)}(l_1,\dots,l_n,i) \right)$$

where (s_0,s_1) varies between all the admissible couples for $2(i+\sum_{j=1}^n l_j)-r$.

In terms of figure III.1 the meaning of proposition 2.7 is clear, illustrated in figure III.2.

2.3 Conclusion

We are now left to treat the extensions between the first two graded pieces of the filtration (14). More precisely, for $r \in \{1,\dots,p-2\}$ such extensions are described by

$$\mathrm{Cosoc}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} Y^r|_{B(\mathbf{F}_p)}) \text{——} \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, F^{(1)}_{r+1}(0)] \tag{19}$$

and

$$\mathrm{Cosoc}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, X^r])\text{——}\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, X^{r-1}Y]. \tag{20}$$

respectively for the K-representations $\varinjlim_{n\,\mathrm{odd}}(R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ and $\varinjlim_{m\,\mathrm{even}}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1})$. For $r = 0$ we have analogously

$$(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} e|_{B(\mathbf{F}_p)})/\mathrm{Sym}^{p-1}\text{——}\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, F_{r+1}^{(1)}(0)] \tag{21}$$

$$(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, e])/\mathbf{1}\text{——}\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, F_{0,1}^{(1,2)}(e)]. \tag{22}$$

We start with the following lemma.

Lemma 2.8. *Let $l, m \in \{0, \dots, p-2\}$ and let $V \overset{\mathrm{def}}{=} \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_l^s \det^m$. Then*

i) the $\mathbf{F}_q^\times$-restriction of the socle $\mathrm{soc}(V)|_{\mathbf{F}_q^\times}$ (resp. $\mathrm{St} \otimes \det^m|_{\mathbf{F}_q^\times}$ if $l = 0$) decomposes as the direct sum of the characters $(\cdot)^{s_0+ps_1}$, where (s_0, s_1) are the admissible couple for l lying on the line $X_0 + X_1 = (p-1) + l$;

ii) the $\mathbf{F}_q^\times$-restriction of the cosocle $\mathrm{cosoc}(V)|_{\mathbf{F}_q^\times}$ (resp. $1 \otimes \det^m|_{\mathbf{F}_q^\times}$ if $l = 0$) decomposes as the direct sum of the characters $(\cdot)^{s_0+ps_1}$, where the (s_0, s_1) are the admissible couple for l lying on the line $X_0 + X_1 = l$.

Proof. Up to twist by powers of det we may assume $V = \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} \chi_{p-1-r}^s \det^r$ for a suitable $r \in \{1, \dots, p-1\}$. It is now enough to show that

$$\mathrm{cosoc}(V)|_{\mathbf{F}_q^\times} \cong \mathrm{Sym}^r(\overline{\mathbf{F}}_p^2)$$

decomposes as the direct sum of the characters $(\cdot)^{s_0+ps_1}$ where $s_0 + ps_1 = r$; this will imply that $\mathrm{soc}(V)|_{\mathbf{F}_q^\times}$ decomposes as the direct sum of the characters $(\cdot)^{s_0+ps_1}$ where $s_0 + ps_1 = (p-1) + r$.

For $r = 1$, the action of $\mathrm{GL}_2(\mathbf{F}_p)$ on $\mathrm{Sym}^1(\overline{\mathbf{F}}_p) \cong \overline{\mathbf{F}}_p^2$ is the natural one. The linear automorphism $\phi_\alpha \in \mathrm{End}(\mathrm{Sym}^1(\overline{\mathbf{F}}_p))$ associated to the action of $\alpha \in \mathbf{F}_q^\times$ has spectrum $\mathscr{S} = \{\alpha, \alpha^p\}$ (indeed, in a $\overline{\mathbf{F}}_p$-basis of $\mathrm{Sym}^1(\overline{\mathbf{F}}_p)$, the associated matrix is $\phi_\alpha \overset{\mathrm{def}}{=} \begin{bmatrix} 0 & -N(\alpha) \\ 1 & Tr(\alpha) \end{bmatrix}$). This gives the case $r = 1$.

Let now $\mathscr{B} \overset{\mathrm{def}}{=} \{v_1, v_2\}$ be a basis of eigenvectors for ϕ_α. If we define, for $j \in \{0, \dots, r\}$,

$$v_j \overset{\mathrm{def}}{=} \bigvee_{i=1}^{j} v_1 \vee \bigvee_{i=j+1}^{r} v_2 \in \mathrm{Sym}^r(\overline{\mathbf{F}}_p^2),$$

then the family $\mathscr{B}_r \overset{\mathrm{def}}{=} \{v_j,\, 0 \leqslant j \leqslant r\}$ is obviously a basis of eigenvectors for the automorphism $\mathrm{Sym}(\phi_\alpha)$ such that the eigenvalue associated to v_j is $\alpha^{j+p(r-j)}$. This gives the result. □

Combining proposition 2.7 and lemma 2.8 we get the behaviour of the extensions (19), (20), (21), (22)

Proposition 2.9. *Let $r \in \{0, \dots, p-2\}$.*

A) The $\mathbf{F}_q^\times$-restriction of the extension

$$\mathrm{Cosoc}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)} Y^r|_{B(\mathbf{F}_p)})\text{——}\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1, F_{r+1}^{(1)}(0)]$$

splits as

$$\left(\bigoplus_{s_0+s_1=r} v^{(s_0-1,s_1-1)}(Y^r)\text{———}v^{(s_0,s_1)}(F^{(1)}_{r+1}(0))\right)\bigoplus\bigoplus_{s_0+s_1=p-1-r} v^{(s_0,s_1)}(F^{(1)}_{r+1}(0)).$$

B) The $\mathbf{F}_q^\times$*-restriction of the extension*

$$\mathrm{Cosoc}(\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1,X^r])\text{———}\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}[1,X^{r-1}Y]$$

splits as

$$\left(\bigoplus_{s_0+s_1=p-1-r} v^{(s_0-1,s_1-1)}([1,X^r])\text{———}v^{(s_0,s_1)}([1,X^{r-1}Y])\right)\bigoplus\bigoplus_{s_0+s_1=r} v^{(s_0,s_1)}([1,X^{r-1}Y]).$$

We now sum up the structure of the $L^\times$ restriction for a supersingular representation $\pi(r,0,1)|_{L^\times}$:

Proposition 2.10. *In the previous notations it exists a* $L^\times$*-equivariant filtration* $\{U_j\}_{j\in\mathbf{N}}$ *on* $\varinjlim_{n\,\mathrm{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})$ *such that for all* $j\geqslant 1$ *the following holds:*

i) we have an isomorphism of $k_L^\times$ *representations:*

$$U_j/U_{j-1}\cong\oplus_{i=0}^p\eta_i$$

where η_i *are the* $(p+1)$*-distinct characters of* $k_L^\times$ *extending the* $k_{\mathbf{Q}_p}^\times$*-character* $x\mapsto x^r$*;*

ii) the $L^\times$*-extension*

$$0\to U_j/U_{j-1}\to U_{j+1}/U_{j-1}\to U_{j+1}/U_j\to 0$$

decomposes into $p+1$ *nontrivial extensions*

$$0\to\eta_i\to *\to\eta_i\to 0.$$

Moreover $U_0\cong\oplus_{i=0}^r\eta_i$ *(where the* η_i *are as above) and the extension*

$$0\to U_0\to U_1\to U_1/U_0\to 0$$

decomposes into the direct sum of $r+1$ *nontrivial extensions*

$$0\to\eta_i\to *\to\eta_i\to 0$$

for $0\leqslant i\leqslant r$ *and the characters* η_i *for* $r+1\leqslant i\leqslant p$*.*

We have an analogous result for $\varinjlim_{n\,\mathrm{even}}(R_1/R_0)\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}$*.*

Proof. This is deduced from propositions 2.9 and 2.7. □

We notice that we could strengthen the result of proposition 2.10. Indeed, let S_n de the filtration (14) on $\varinjlim_{n\,\mathrm{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})$. By remark 2.6 we deduce that S_n (for $n\geqslant 1$) splits into a direct sum of $(p+1)$ terms $\oplus_{i=0}^p S_n(i)$ and each of $S_n(i)$ is a uniserial representation of $\mathscr{O}_L^\times$ having Jordan Hölder factors isomorphic to η_i. Moreover, by the same remark, the exact sequence

$$0\to\oplus_{i=0}^p S_n(i)\to S_{n+1}\to\oplus_{i=0}^p\eta_i\to 0$$

splits. This let us define, inductively, an $\overline{\mathbf{F}}_p$-basis on the limit $\varinjlim_{n\,\mathrm{even}}(R_1/R_0)\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}$ which realise a splitting as a direct sum of $(p+1)$-terms $S_\infty(i)$ and each of $S_\infty(i)$ is a uniserial $\mathscr{O}_L^\times$-representation having Jordan Hölder factors isomorphic to η_i. Notice that the $S_\infty(i)$ is deduced to be uniserial from lemma 2.3. We have obviously an analogous result for the limit $\varinjlim_{n\,\text{even}} (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$

COROLLARY 2.11. *The statement* $i)$ *of theorem 1.2 holds.*

3. The ramified case

We assume now $L/\mathbf{Q}_p$ totally ramified. The structure of $\pi(r,0,1)|_{L^\times}$ is much easier to deduce and actually follows almost immediately as a particular case of the results of [Mo5] (which describes the Iwahori structure of the universal representations of $\mathrm{GL}_2(F)$ for $F/\mathbf{Q}_p$ unramified). The main result is proposition 3.5, giving the $L^\times$-structure for the supersingular representation $\pi(r,0,1)$.

We consider the $\mathbf{Z}_p$-base $\mathscr{B} \overset{\text{def}}{=} \{\varpi, 1\}$ of $\mathscr{O}_L$, where $\varpi \in \mathscr{O}_L$ is a fixed uniformiser (as remarked before the choice of a basis is ininfluential for the statement of proposition 3.5). Since $\varpi^2 = p$ and p acts trivially on $\pi(r,0,1)$ we see that $\iota_{\mathscr{B}}(\varpi)$ is an involution: we can first content ourself to study the restriction

$$\pi(r,0,1)|_{\mathscr{O}_L^\times}.$$

We notice that

$$\iota_{\mathscr{B}}(\mathscr{O}_L^\times) = \{\begin{bmatrix} a & b \\ pb & a \end{bmatrix}, \quad a \in \mathbf{Z}_p^\times, b \in \mathbf{Z}_p\}$$

is a subgroup of the Iwahori I of $\mathrm{GL}_2(\mathbf{Z}_p)$. The structure of $\pi(r,0,1)|_I$ is known (cf. [Mo5]) and is described by the following two propositions.

PROPOSITION 3.1. *Let* $r \in \{0,\dots,p-1\}$ *and consider the restriction* $\pi(r,0,1)|_I$*. We have the following* I*-equivariant exact sequences*

$$0 \to \langle (Y^r, -Y^r) \rangle \to \varinjlim_{n\,\text{odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus \varinjlim_{n\,\text{odd}} (\langle Y^r \rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \to$$
$$\to \varinjlim_{n\,\text{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \to 0$$

and

$$0 \to \langle (F_r^{(0)}(0), [1, X^r]) \rangle \to \varinjlim_{n\,\text{even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \oplus \varinjlim_{n\,\text{even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \to$$
$$\to \varinjlim_{n\,\text{even}} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}) \to 0.$$

Proof. It is a particular case of [Mo5]. □

PROPOSITION 3.2. *We have an* I*-equivariant filtration on each of*

$$\varinjlim_{n\,\text{odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+); \qquad \varinjlim_{n\,\text{even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \tag{23}$$
$$\varinjlim_{n\,\text{odd}} (\langle Y^r \rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-); \qquad \varinjlim_{n\,\text{even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-). \tag{24}$$

The extensions between two consecutive graded pieces for the representations (23) (resp. (24)) is described as

$$\langle F_{l_0-1,\dots,l_n}^{(0,n)}(i) \rangle \text{———} \langle F_{l_0,\dots,l_n}^{(0,n)}(i) \rangle \tag{25}$$

(resp.

$$\langle F^{(1,n)}_{l_0-1,\dots,l_n}(i)\rangle \text{———} \langle F^{(1,n)}_{l_0,\dots,l_n}(i)\rangle) \tag{26}$$

where $\langle F^{(0,n)}_{l_0,\dots,l_n}(i)\rangle \cong \chi_r(\mathfrak{a}^{-1})^{i+\sum_{j=0}^n l_j}$ *(resp.* $\langle F^{(1,n)}_{l_0-1,\dots,l_n}(i)\rangle \cong \chi_r^s \mathfrak{a}^{i+\sum_{j=1}^n l_j}$*).*

Finally, for $g \overset{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in I$ *and* $p \geqslant 5$ *we have the following equality in the extension (25) and (26):*

$$\begin{bmatrix} a & b \\ pc & d \end{bmatrix} F^{(0,n)}_{l_0,\dots,l_n}(i) = \chi_r(\mathfrak{a}^{-1})^{i+\sum_{j=0}^n l_j}(g)(F^{(0,n)}_{l_0,\dots,l_n}(i) - \bar{b}\kappa_0(l_1,\dots,l_n,i)F^{(0,n)}_{l_0-1,\dots,l_n}(i));$$

$$\begin{bmatrix} a & b \\ pc & d \end{bmatrix} F^{(1,n)}_{l_1,\dots,l_n}(i) = \chi_r^s(\mathfrak{a})^{i+\sum_{j=1}^n l_j}(g)(F^{(1,n)}_{l_1,\dots,l_n}(i) - \bar{c}\kappa_1(l_1,\dots,l_n,i)F^{(1,n)}_{l_1-1,\dots,l_n}(i))).$$

where $\kappa_0(l_0,\dots,l_n,i), \kappa_1(l_1,\dots,l_n,i) \in \mathbf{F}_p^\times$.

Proof. Using the Iwahori decomposition for I we get

$$\begin{aligned}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ pz & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ pz' & 1 \end{bmatrix}\begin{bmatrix} \alpha' & 0 \\ 0 & \beta' \end{bmatrix}\begin{bmatrix} 1 & x' \\ 0 & 1 \end{bmatrix}\end{aligned}$$

where $\overline{\alpha} = \overline{\alpha'} = \overline{a}$, $\overline{\beta} = \overline{\beta'} = \overline{d}$, $\overline{x} = bd^{-1}$, $\overline{x'} = cd^{-1}$. The result follows from [Mo5]. □

As we can always find $\begin{bmatrix} a & b \\ pb & a \end{bmatrix} \in \iota_{\mathscr{B}}(\mathscr{O}_L^\times)$ with $b \in \mathbf{Z}_p^\times$ we deduce immediately from propositions 3.1 and 3.2 the required structure of $\pi(r,0,1)|_{\mathscr{O}_L^\times}$:

Proposition 3.3. *Let* $r \in \{0,\dots,p-1\}$*; the restriction* $\pi(r,0,1)|_{\mathscr{O}_L^\times}$ *is described as follow. We have two exact,* $\mathscr{O}_L^\times$*-equivariant sequences:*

$$0 \to \langle (Y^r, -Y^r)\rangle|_{\mathscr{O}_L^\times} \to \varinjlim_{n\,\text{odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+)|_{\mathscr{O}_L^\times} \oplus \varinjlim_{n\,\text{odd}} (\langle Y^r\rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-)|_{\mathscr{O}_L^\times} \to$$
$$\to \varinjlim_{n\,\text{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{\mathscr{O}_L^\times} \to 0$$

and

$$0 \to \langle (F_r^{(0)}(0), [1, X^r])\rangle|_{\mathscr{O}_L^\times} \to \varinjlim_{n\,\text{even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-)|_{\mathscr{O}_L^\times} \oplus \varinjlim_{n\,\text{even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+)|_{\mathscr{O}_L^\times} \to$$
$$\to \varinjlim_{n\,\text{even}} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})|_{\mathscr{O}_L^\times} \to 0.$$

We have an $\mathscr{O}_L^\times$*-equivariant filtration on each of*

$$\varinjlim_{n\,\text{odd}} (R_0 \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+)|_{\mathscr{O}_L^\times}; \qquad \varinjlim_{n\,\text{even}} ((R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+)|_{\mathscr{O}_L^\times} \tag{27}$$

$$\varinjlim_{n\,\text{odd}} (\langle Y^r\rangle \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-)|_{\mathscr{O}_L^\times}; \qquad \varinjlim_{n\,\text{even}} (R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-)|_{\mathscr{O}_L^\times} \tag{28}$$

such that the extension determined by two consecutive graded pieces of the representations in (27) and (28) is nonsplit and admits a description

$$\langle F^{(0,n)}_{l_0-1,\dots,l_n}(i)\rangle \text{———} \langle F^{(0,n)}_{l_0,\dots,l_n}(i)\rangle \qquad \text{where} \qquad \langle F^{(0,n)}_{l_0,\dots,l_n}(i)\rangle \cong \chi_r(\mathfrak{a}^{-1})^{i+\sum_{j=0}^n l_j}|_{\mathbf{F}_p^\times};$$

$$\langle F^{(1,n)}_{l_0-1,\dots,l_n}(i)\rangle \text{———} \langle F^{(1,n)}_{l_0,\dots,l_n}(i)\rangle \qquad \text{where} \qquad \langle F^{(1,n)}_{l_1,\dots,l_n}(i)\rangle \cong \chi_r^s(\mathfrak{a})^{i+\sum_{j=1}^n l_j}|_{\mathbf{F}_p^\times}).$$

3.1 Conclusion

We are now left to study the involution $\iota_{\mathscr{B}}(\varpi) = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$. As

$$\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \begin{bmatrix} p^{n+1} & \sum_{j=0}^n p^j[\lambda_j] \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \sum_{j=0}^n p^{j+1}[\lambda_j] & p^{n+2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sum_{j=1}^n p^j[\lambda_j] & p^{n+1} \end{bmatrix} = p \begin{bmatrix} p^n & \sum_{j=0}^{n-1} p^j[\lambda_{j+1}] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

the action of $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$ is deduced from [Mo1] -proposition 3.4:

PROPOSITION 3.4. *For $i \in \{0,1\}$ consider a function*

$$F^{(0,n)}_{l_0,\dots,l_n}(i) \in \varinjlim_{n\,\mathrm{odd}} ((R_i/R_{i-1})^+ \oplus_{R^+_{i+1}} \cdots \oplus_{R^+_{n+i}} R^+_{n+i+1})$$

(resp.

$$F^{(1,n)}_{l_1,\dots,l_n}(i) \in \varinjlim_{n\,\mathrm{odd}} ((R_i/R_{i-1})^- \oplus_{R^-_{i+1}} \cdots \oplus_{R^-_{n+i}} R^-_{n+i+1})).$$

Then we have

$$\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} F^{(0,n)}_{l_0,\dots,l_n}(i) = F^{(1,n+1)}_{l'_1,\dots,l'_{n+1}}(i) \qquad (\text{resp.} \quad \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} F^{(1,n)}_{l_1,\dots,l_n}(i) = F^{(0,n-1)}_{l'_0,\dots,l'_{n-1}} i\,)$$

where $l'_j = l_{j-1}$ for all $j \in \{1,\dots,n+1\}$ (resp. $l'_j = l_{j+1}$ for all $j \in \{0,\dots,n-1\}$).

We can sum up the previous result, getting the $L^\times$ structure for $\pi(r,0,1)|_{L^\times}$:

PROPOSITION 3.5. *Keep the notations of §3. There exsists two $L^\times$ sub-representations $U(0)$, $U(r) \leqslant \pi(r,0,1)|_{L^\times}$, each of them being moreover $K_0(p^0)$-stable, such that we have an exact sequence*

$$0 \to U_0 \to U(0) \oplus U(r) \to \pi(r,0,1)|_{L^\times} \to 0.$$

For $\epsilon_2 \in \{0,r\}$ the $L^\times$-representation $U(\epsilon_2)$ admits an $L^\times$ (and $K_0(p)$) equivariant filtration $\{U(\epsilon_2)_j\}_{n\in\mathbf{N}}$ such that for all $j \geqslant 0$ the space $U_j(\epsilon_2)/U_{j-1}(\epsilon_2)$ is two dimensional, admitting a basis $\mathscr{U}(\epsilon_2)_j = \{v_{j,1}(\epsilon_2), v_{j,s}(\epsilon_2)\}$, and

i) for $\epsilon_1 \in \{1,s\}$, $K_0(p)$ acts on $v_{j,\epsilon_1}(\epsilon_2)$ by the character $(\chi_r\alpha^{-j-\epsilon_2})^{\epsilon_1}$ and the uniformiser ϖ by the involution $\varpi v_{j,1}(\epsilon_2) = v_{j,s}(\epsilon_2)$;

ii) the $L^\times$ extension

$$0 \to U_j(\epsilon_2)/U_{j-1}(\epsilon_2) \to U_{j+1}(\epsilon_2)/U_{j-1}(\epsilon_2) \to U_{j+1}(\epsilon_2)/U_j(\epsilon_2) \to 0$$

is non split.

Moreover, $U_0(0) = U_0(r)$.

Proof. It follows from propositions 3.3 and 3.4. $\square$

Corollary 3.6. *Part* ii) *of theorem 1.2 holds.*

Part IV. On some representations of the Iwahori subgroup

Abstract. Let $p \geqslant 5$ be a prime number. In [BL94] Barthel and Livné described a classification for irreducible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$, for F a p-adic field, discovering some objects, referred as "supersingular", which appear as subquotients of a universal representations $\pi(\underline{r}, 0, 1)$. In this chapter we give a detailed description the Iwahori structure of such universal representations for F an unramified extension of $\mathbf{Q}_p$. We determine a fractal structure which shows how and why the thechniques used for $\mathbf{Q}_p$ fail and which let us determine"natural" subrepresentations of the universal object $\pi(\underline{r}, 0, 1)$. As a corollary, we get the Iwahori structure of tamely ramified principal series.

1. Introduction

Let p be a prime number and F a p-adic field. In their works [BL94], [BL95] Barthel and Livné studied a classification (recently generalized for general $\mathrm{GL}_n(F)$ by Herzing in [Her]) for the representations of $\mathrm{GL}_2(F)$ with coefficients in an algebraic closure of $\mathbf{F}_p$. Besides characters, principal unramified series and special series, they found a new class of irreducible objects referred as "supersingular", which are defined, up to twist, as subquotients of a universal representation, which we will note $\pi(\underline{r}, 0, 1)$ (and $\underline{r} = (r_0, \ldots, r_{f-1})$ if f is the residual degree of F). The existence of supersingular representations is assured by a Zorn-type argument (see [BL95], proposition 11) and a complete exhaustive study for supersingular representations is a relevant open problem in the emerging p-adic Langlands program. Indeed, in a conjectural mod p-Langlands correspondence it is expected that the supersingular object are those $\mathrm{GL}_2(F)$ representations which should naturally be attached to Galois representations arising from elliptic curves with supersingular reduction.

This is actually the case if $F = \mathbf{Q}_p$ (when the universal representations are indeed irreducible). Such result is due to Breuil [Bre03a] where he reaches a complete classification of supersingular representations thanks to direct computations on the ring of Witt vectors of $\mathbf{F}_p$. If $F \neq \mathbf{Q}_p$ the situation is not clear. For the time being, the problem of classifying supersingular representations looks to be infinitely more involved compared to its Galois analogue (known from the works of Serre [Ser72]). The methods of Paskunas [Pas04] and Breuil-Paskunas [BP] let us associate an *infinite* family $\Pi(\rho)$ of supersingular representations to a single Galois object ρ, are a major progress in this direction, but it is not clear, especially after the work of Hu [Hu], how to distingush in a canonical way a privileged supersingular representation inside $\Pi(\rho)$. We remark that the methods of [Pas04] and [BP] have been improved by Hu's canonical diagrams in [Hu2]; unfortunately canonical diagrams are difficult to calculate explicitely.

Another approach to the problem has been treated by Schein in [Sch] where he studies the universal representations for a totally ramified extension $F/\mathbf{Q}_p$. He detects a natural quotient V_{e-1} of $\pi(\underline{r}, 0, 1)$ which enjoys an universal property with respect to supersingular representations whose $\mathrm{GL}_2(\mathscr{O}_F)$-socle respects a certain combinatoric conjecturally associated to suitable Galois representations arising from elliptic curves with supersingular reduction (the modular weights introduced in [BDJ] and generalised in [Sch08])

In this chapter we describe the Iwahori structure for the universal representation $\pi(\underline{r}, 0, 1)$ in the case where $F/\mathbf{Q}_p$ is unramified generalizing Breuil's method (in particular, our result give the irreducibility for $F = \mathbf{Q}_p$ and shows how and why the universal representations fail to be irreducible otherwise). With "Iwahori structure" we mean that we are able to detect the Iwahori-socle filtration for $\pi(\underline{r}, 0, 1)$ as well as the extension between two consecutive graded pieces. As a byproduct we will deduce the Iwahori structure of principal and special series and the presence of a natural injection $c\text{--}\mathrm{Ind}_{KZ}^{G} V \hookrightarrow \pi(\underline{r}, 0, 1)$. The reader will find out that, as soon as $F \neq \mathbf{Q}_p$, the Iwahori-socle filtration for the universal representation relies on an extremely complicated combinatoric.

The main result of this chapter is to show that such combinatoric can be handled with the help of some simple euclidean data; such a method can be briefly described as follow. We detect a natural $\overline{\mathbf{F}}_p$-basis $\mathscr{B}$ of $\pi(\underline{r}, 0, 1)$ as well as an injection:

$$\mathscr{B} \hookrightarrow \mathbf{Z}^{[F:\mathbf{Q}_p]};$$

as we will show, its image $\mathfrak{R}$ is explicitely known. For $v \in \mathscr{B}$ we define the set of antecedents $\mathfrak{S}_v$ of v as the set of $v' \in \mathscr{B}$ such that $v' = v - e_s$ where e_s is the s-th element of the canonical base of $\mathbf{Z}^{[F:\mathbf{Q}_p]}$. When we claim that the Iwahori structure for $\pi(\underline{r}, 0, 1)$ is described by $\mathfrak{R}$ we mean the following facts:

i) the Iwahori-socle filtration is obtained from $\mathfrak{R}$ by successively removing the points with empty antecedents;

ii) if $v_0, v_1 \in \mathscr{B}$ and $J \in \mathbf{N}$ is such that v_i is an eigenvector for the $J-i$-th graded piece $(\pi(\underline{r}, 0, 1))_{J-i}$ of the socle filtration of the universal representation then we have a nontrivial extension *inside the quotient* $\pi(\underline{r}, 0, 1)/(\pi(\underline{r}, 0, 1))_{J-1}$ if and only if v_0 is an antecedent of v_1.

According to this terminology the main result is the following (see proposition 5.18):

THEOREM 1.1. *The Iwahori structure of the universal representations is described by $\mathfrak{R}$.*

We give in figure IV.1 the idea of such structure for the quadratic unramified extension of $\mathbf{Q}_p$.

As annonced, we get some other byproducts as

THEOREM 1.2. *The Iwahori structure of tamely ramified principal series is described by two copies of $\mathbf{N}^{[F:\mathbf{Q}_p]}$.*

and

THEOREM 1.3. *Let $\underline{r} \notin \{(0, \ldots, 0), (p-1, \ldots, p-1)\}$ and let χ^s be the conjugate character of $(\sigma_{\underline{r}})^{U(\mathbf{F}_q)}$. There is a sub KZ-representation $V \leqslant \pi(\underline{r}, 0, 1)|_{KZ}$ isomorphic to the kernel of the natural map*

$$\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)} \chi^s / \mathrm{soc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)} \chi^s) \twoheadrightarrow \mathrm{cosoc}(\mathrm{Ind}_{B(\mathbf{F}_q)}^{\mathrm{GL}_2(\mathbf{F}_q)} \chi^s)$$

and such that the map (induced by Frobenius reciprocity)

$$c\text{--}\mathrm{Ind}_{KZ}^{G} V \to \pi(\underline{r}, 0, 1)$$

is injective.

We remark that a similar phenomenon has already been discovered by Paskunas in an unpublished draft.

Such results rely on an heavy formalism and they need preparation to be handled. In particular, from section §4 we start using the euclidean dictionary as a key tool to manage the

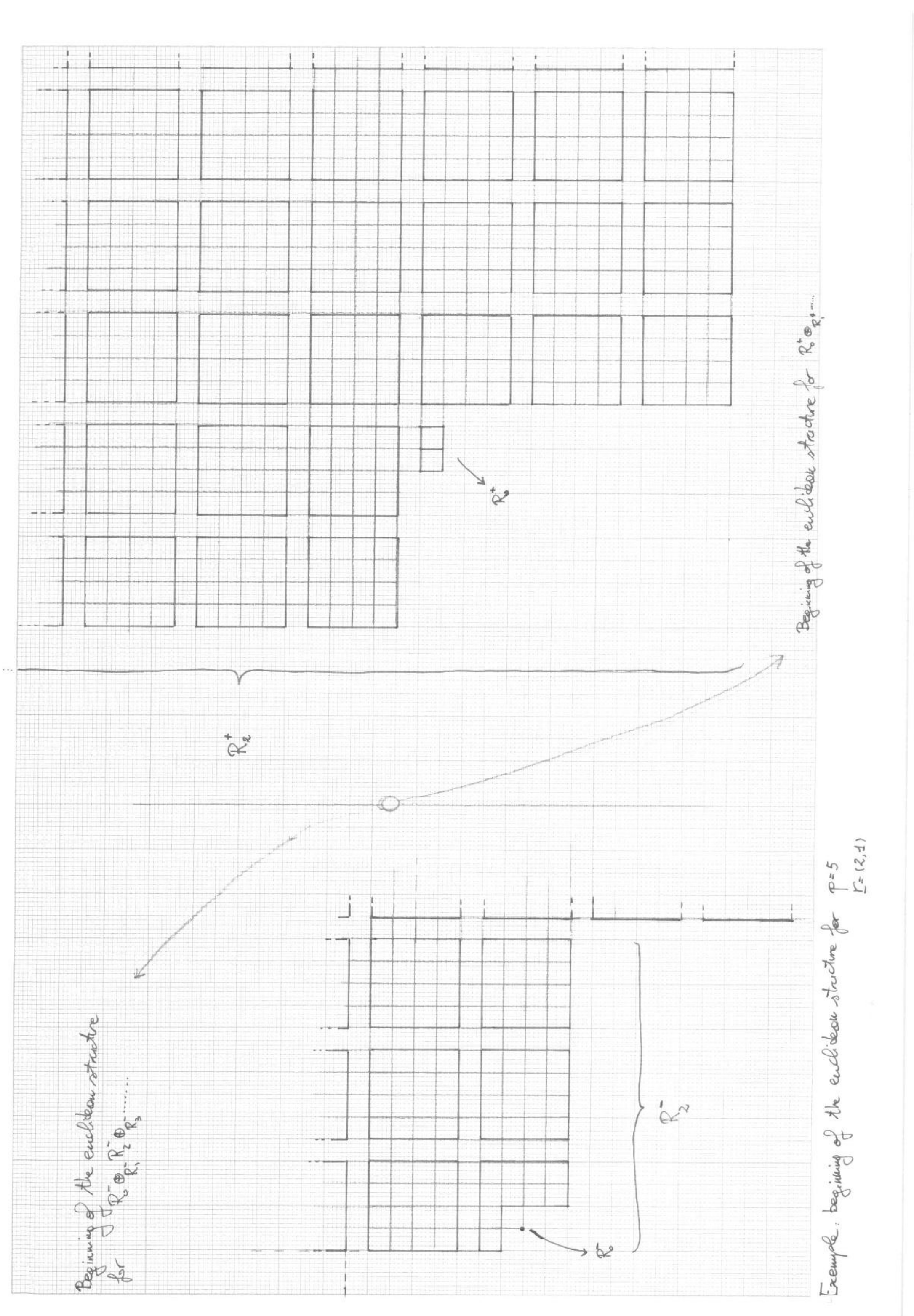


FIGURE IV.1. Part of the euclidean structure for $f = 2$, $\underline{r} = (2, 1)$.

combinatoric of the representation under study. In order to guide the reader the statements are preceeded by a detailed translation in geometric terms (otherwise they would sound as empty exercices of combinatoric) and each section opens with an exhaustive description of the euclidean strategy adopted to reach our aims.

The reasons which make such strategy work are essentially three:

i) we detect a suitable basis $\mathscr{B}$ of the universal representation which is well behaved with respect to the action of the Iwahory subgroup and the canonical Hecke operator $T \in \mathrm{End}_G(c\text{-}\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}})$;

ii) the action of the Iwahori subgroup on the elements of $\mathscr{B}$ can be read through certains universal Witt polynomials whose homogeneous degree is known;

iii) the correspondence between the elements of the basis $\mathscr{B}$ and integers points in $\mathbf{R}^{[F:\mathbf{Q}_p]}$ is compatible with the homogeneous degree of the polynomials of *ii*).

The structure of the chapter is then the following.

First two sections §2 and §3 are formal and do not need the hypothesys $F/\mathbf{Q}_p$ unramified. Section §2 is essentially a dictionary which let us detect a natural KZ-filtration on the KZ-restriction of the universal representation. We first introduce a family of KZ-representations $\{R_n\}_{n\in\mathbf{N}}$. Through some convenient Hecke operators $T_n^{\pm}: R_n \to R_{n\pm1}$ we define inductively a direct system of amalgamed sums (each of them endowed with a natural filtration) which leads to explicit isomorphism (proposition 2.9):

$$\pi(\sigma_{\underline{r}},0,1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{n\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}).$$

We remark that such isomorphism was already draft by Breuil in [Bre].

In section 3 we start from an Iwahori-splitting $R_{n+1} = R_{n+1}^{+} \oplus R_{n+1}^{-}$ to deduce, in the same flavour of the preceeding section, an inductive system of amalgamed sums $\cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$. Such amalgamed sums are endowed with a natural Iwahori-filtration revealed by a short exact sequence

$$0 \to \cdots \oplus_{R_{n-2}^{\pm}} R_{n-1}^{\pm} \to \cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm} \to R_{n+1}^{\pm}/R_n^{\pm} \to 0. \tag{29}$$

The resulting inductive limits are related to the universal representation by the following

Proposition 1.4. *We have an exact Iwahori-equivariant sequence*

$$0 \to \langle (v_+, v_-)\rangle_{\overline{\mathbf{F}}_p} \to (\varinjlim_{n\,\mathrm{odd}} R_0^{+} \oplus_{R_1^{+}} \cdots \oplus_{R_n^{+}} R_{n+1}^{+}) \oplus (\varinjlim_{n\,\mathrm{odd}} R_0^{-} \oplus_{R_1^{-}} \cdots \oplus_{R_n^{-}} R_{n+1}^{-}) \to$$
$$\to (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0$$

where $v_{\pm} \in \varinjlim_{n\,\mathrm{odd}} R_0^{\pm} \oplus_{R_1^{\pm}} \cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$ *(and are explicitely known).*

We have an analogous result in the even case.

It will therefore be enough to focus our attention on the inductive limits of section §3.

The euclidean dictionary is developed in section 4. Thanks to the natural filtration on the inductive limits, we are primarly concerned with the Iwahory structure of the representations $R_{n+1}^{\pm}$. We detect a convenient $\overline{\mathbf{F}}_p$-basis $\mathscr{B}_{n+1}^{\pm}$ (lemma 2.6) and determine a natural way to identify the elements of $\mathscr{B}_{n+1}^{\pm}$ to integer valued points of $\mathbf{R}^{[F:\mathbf{Q}_p]}$ (see section 4.1.1 for details). If we write $\mathscr{R}_{n+1}^{\pm}$ to denote the image of $\mathscr{B}_{n+1}^{\pm}$ in the $[F:\mathbf{Q}_p]$-dimensional real euclidean space (such an image looks as a parallelepipoid of side $p^{n+\epsilon}(\underline{r}+\underline{1})$ for $\epsilon \in \{0,1\}$ according to the cases R_{n+1}^{+}, R_{n+1}^{-}) then

PROPOSITION 1.5. *The Iwahori structure of $R_{n+1}^{\pm}$ is described by $\mathscr{R}_{n+1}^{\pm}$.*

Because of the geometry of the polytope $\mathscr{R}_{n+1}^{\pm}$ we indeed see that the socle filtration can be detected by successive cuttings by a suitable hyperplanes (parallel to the antidiagonal $X_0+\cdots+X_{f-1}=0$).

We similarly deduce the structure of tamely ramified principal series given in proposition 1.2 Unfortunately, these results rely on a careful analysis of the behaviour of some universal Witt polynomials, contained in the two appendices A and B.

Section §5 deals finally with the universal representaiton $\pi(\underline{r},0,1)$. We are first concerned with the graded pieces of the natural filtrations introduced in §3: it is the object of §5.1. Thanks to the behaviour of the canonical basis $\mathscr{B}_{n}^{\pm}$ with respect to the Hecke operators of §3 we easily determine a natural basis $\mathscr{B}_{n+1/n}^{\pm}$ for each $R_{n+1}^{\pm}/R_{n}^{\pm}$ and associate an euclidean structure $\mathscr{R}_{n+1}^{\pm}$ to it. Such a structure is more complicated than the previous $\mathscr{R}_{n+1}^{\pm}$ and can not be determined directely by proposition 1.5 but a suitable decomposition of $\mathscr{R}_{n+1/n}^{\pm}$ as a union of inreasing polytopes enable us to state the

PROPOSITION 1.6. *The Iwahori structure of $R_{n+1}^{\pm}/R_{n}^{\pm}$ is described by $\mathscr{R}_{n+1/n}^{\pm}$.*

The euclidean image of $\mathscr{R}_{n+1/n}^{\pm}$ is more or less given in figure IV.2.

As a byproduct, the natural filtrations of section §3 and the previous description of the basis $\mathscr{B}_{n+1}^{\pm}$ let us deduce proposition 1.3.

The conclusion is in section §5.2 where we study the amalgamed sums $\cdots\oplus_{R_{n}^{\pm}} R_{n+1}^{\pm}$. Again, the behaviour of the canonical base $\mathscr{B}_{n}^{\pm}$ with respect to the Hecke operators let us deduce, by induction on the exact sequence (29), an euclidean structure, say $\mathfrak{R}_{\text{even,odd}}^{\pm}$. Such a structure has a regular fractal nature, due to a convenient glueing of the bloks $\mathscr{R}_{n+1/n}^{\pm}$ and *simple* remarks on the geometry of $\mathfrak{R}_{\text{even,odd}}^{\pm}$, as well as the fact that $\cdots\oplus_{R_{n-2}^{\pm}} R_{n-1}^{\pm}$ is a Iwahori-*subrepresentation* of $\cdots\oplus_{R_{n}^{\pm}} R_{n+1}^{\pm}$, let us deduce the main result of proposition 1.1.

We introduce now the basic conventions and notations of the chapter (we essentially use the formalism and notations of [Bre03a]).

Fix a prime $p\geqslant 5$ and let F be a finite unramified extension of $\mathbf{Q}_p$; let $f\overset{\text{def}}{=}[F:\mathbf{Q}_p]$ be the residue degree. We write $\mathscr{O}_F$ to denote the ring of integers of F and fix the uniformizer $p\in\mathscr{O}_F$: let k_F be the residue field; it is a finite field with $q\overset{\text{def}}{=}p^f$ elements. We fix an isomorphism $k_F\cong\mathbf{F}_q$; as F is unramified, we deduce an isomorphism $\mathscr{O}_F\cong W(\mathbf{F}_q)$ where $W(\mathbf{F}_q)$ denote the ring of Witt vectors of $\mathbf{F}_q$. We will write $[\cdot]:\mathbf{F}_q^{\times}\to W(\mathbf{F}_q)^{\times}$ to denote the Teichmüller character (putting $[0]\overset{\text{def}}{=}0$). We finally fix an algebraic closure $\overline{\mathbf{F}}_p$ of $\mathbf{F}_q$.

For any $k\in\mathbf{N}$ the natural action of $\mathrm{GL}_2(\mathbf{F}_q)$ on $\mathbf{F}_q^2$ let us determine, by functoriality of the k-th symmetric power, the $\mathrm{GL}_2(\mathbf{F}_q)$-representation $\mathrm{Sym}^k\mathbf{F}_q^2$. It is isomorphic (up to a choice of an $\mathbf{F}_q$-basis for $\mathbf{F}_q^2$) to $\mathbf{F}_q[X,Y]_k^h$, the homogeneous component of degree k of the ring $\mathbf{F}_q[X,Y]$, endowed with the usual modular action:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} X^{k-i}Y^i=(aX+cY)^{k-i}(bX+dY)^i.$$

We recall that for $s\in\mathbf{N}$ $(\mathbf{F}_q[X,Y]_k^h)^{Frob^s}$ is the representation obtained by functoriality, in the

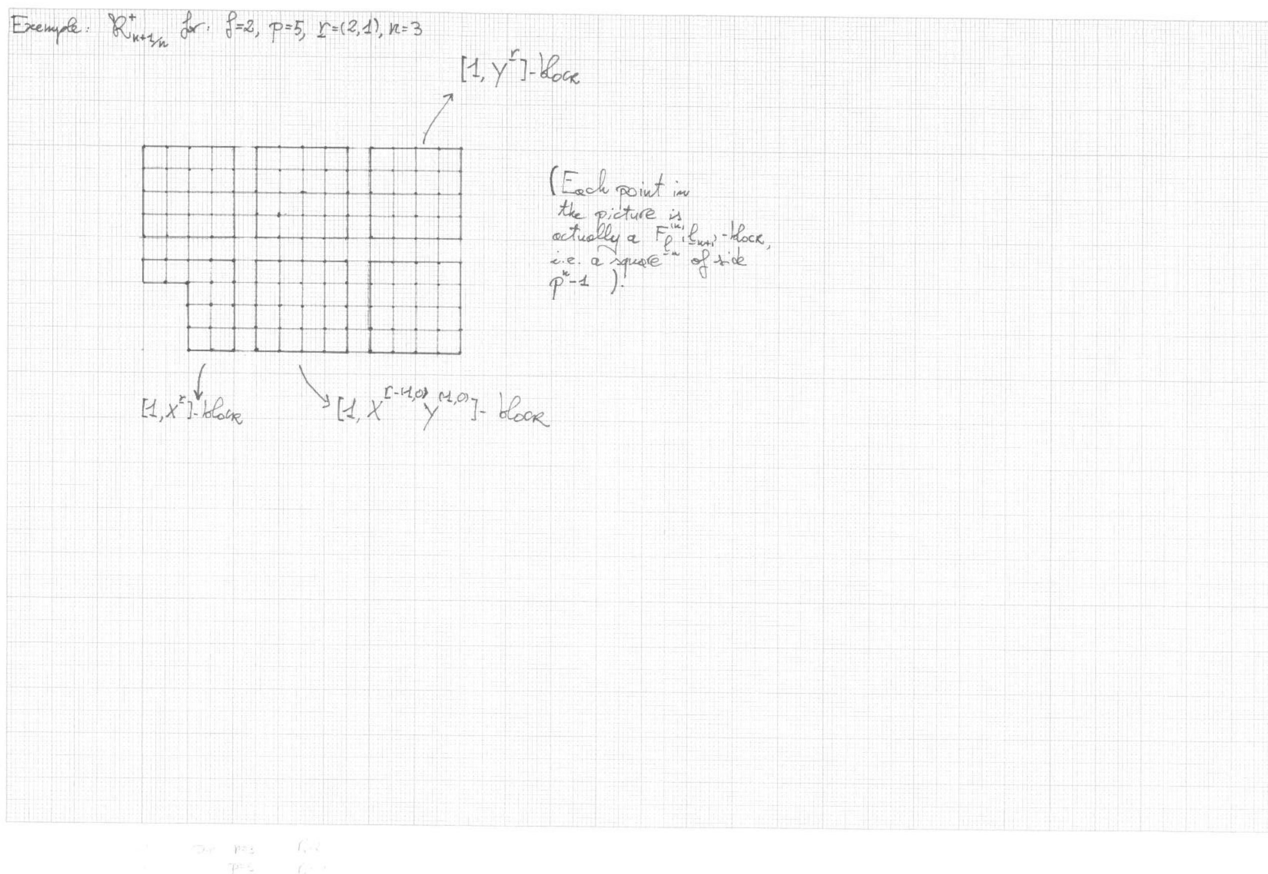


FIGURE IV.2. The structure of the quotients $R^{\pm}_{n+1}/R^{\pm}_{n}$.

evident way, from the field automorphism $x \mapsto x^{p^s}$ defined on $\mathbf{F}_q$.

For $\tau \in \mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ and $r_\tau, t_\tau \in \{0, \ldots, p-1\}$ we consider the $\mathrm{GL}_2(\mathbf{F}_q)$-representation

$$\sigma_{\{r_\tau\},\{t_\tau\}} \overset{\mathrm{def}}{=} \bigotimes_{\tau \in \mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)} (\det{}^{t_\tau} \otimes_{\mathbf{F}_q} \mathrm{Sym}^{r_\tau} \mathbf{F}_q^2) \otimes_\tau \overline{\mathbf{F}}_p;$$

such representations exhaust all irreducible $\mathrm{GL}_2(\mathbf{F}_q)$-representations with coefficients in $\overline{\mathbf{F}}_p$ (and they are pairwise non isomorphic if we impose $t_\tau < p-1$ for at least one element $\tau \in \mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$).

We fix once for all an immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$. Such a choice determines, up to twist, a manifest isomorphism

$$\sigma_{\{r_\tau\},\{t_\tau\}} \cong \sigma_{(r_0,\ldots,r_{f-1})} \overset{\mathrm{def}}{=} \bigotimes_{s=0\,\overline{\mathbf{F}}_p}^{f-1} (\overline{\mathbf{F}}_p[X_s, Y_s]^h_{r_s})^{Frob^s}$$

for a convenient $\underline{r} \overset{\mathrm{def}}{=} (r_0, \ldots, r_{f-1}) \in \{0, \ldots, p-1\}^f$; such an isomorphism will be assumed to be fixed once for all throughout the chapter. We notice that the choice of another immersion acts on the right hand side by a circular permutation on the indexes s in the obvious sense.

Write $G \overset{\mathrm{def}}{=} \mathrm{GL}_2(F)$, $K \overset{\mathrm{def}}{=} \mathrm{GL}_2(\mathscr{O}_F)$ and $Z \overset{\mathrm{def}}{=} Z(G)$. We write $K_0(p)$ to denote the Iwahori subgroup of K. The $\mathrm{GL}_2(\mathbf{F}_q)$-representation $\sigma_{\underline{r}}$ will be seen, by the inflation map $K \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_q)$, as a smooth representation of K. By imposing $p \in Z$ to act trivially, the smooth K-action on $\sigma_{\underline{r}}$ extends to a smooth action of KZ: by abuse of notation we will write $\sigma_{\underline{r}}$ to denote either the $\mathrm{GL}_2(\mathbf{F}_q)$ or the K or the KZ-representation obtained by this procedure (or, as usual, the underlying vector space of $\sigma_{\underline{r}}$).

Similarly, the character

$$\begin{aligned} \chi_{\underline{r}} : B(\mathbf{F}_q) &\to \overline{\mathbf{F}}_p^\times \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto a^{\sum_{s=0}^{f-1} p^s r_s} \end{aligned}$$

will be considered, by inflation as a character of any open subgroup of $K_0(p)$. We write then $\chi_{\underline{r}}^s$ to denote the conjugate character of $\chi_{\underline{r}}$. We denote by $\mathfrak{a}$ the character

$$\begin{aligned} B(\mathbf{F}_q) &\to \overline{\mathbf{F}}_p^\times \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto ad^{-1}. \end{aligned}$$

Recall the compact induction:

$$c\text{--}\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}}$$

defined as the $\overline{\mathbf{F}}_p$-linear space of functions $f : G \to \sigma_{\underline{r}}$, compactly supported modulo Z, verifying $f(kg) = k \cdot f(g)$ for any $k \in K$, $g \in G$; it is endowed with the smooth left action of G defined by right translations.

For $g \in G$, $v \in \sigma_{\underline{r}}$ we define $[g, v] \in c\text{--}\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}}$ as the unique function f supported in KZg^{-1} and such that $f(g) = v$. Then we have

$$\begin{aligned} g' \cdot [g, v] &= [g'g, v] &&\text{for } g' \in G \\ [gk, v] &= [g, k \cdot v] &&\text{for } k \in KZ. \end{aligned}$$

Each function $f \in c\text{--}\mathrm{Ind}_{KZ}^G \sigma_{\underline{r}}$ can be written as a $\overline{\mathbf{F}}_p$-linear combination of a finite family of functions $[g, v]$; if g varies in a fixed system of coset for G/KZ and v varies in a fixed $\overline{\mathbf{F}}_p$-basis of $\sigma\underline{r}$ the aforementioned writing is then unique.

We leave to the reader the task to adapt the previous definitions and remarks to such objects as

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\tau$$

where $K_0(p^{n+1}) \overset{\circ}{\leqslant} K_0(p^m) \overset{\circ}{\leqslant} K$ are open subgroups of K and τ is a smooth representation of $K_0(p^{n+1})$.

From [BL94], proposition 8-(1) there exists a canonical Hecke operator (depending on $\underline{r}$) $T \in \mathrm{End}_G(c\text{--}\mathrm{Ind}_{KZ}^G\sigma_{\underline{r}})$. It realizes an isomorphism of the $\overline{\mathbf{F}}_p$-algebra of endomorphisms $\mathrm{End}_G(c\text{--}\mathrm{Ind}_{KZ}^G\sigma_{\underline{r}})$ with the ring of polynomials in one variable over $\overline{\mathbf{F}}_p$. We then define the universal representation of $\mathrm{GL}_2(F)$ as the cokernel of the canonical operator T:

$$\pi(\underline{r}, 0, 1) \overset{\mathrm{def}}{=} \mathrm{coker}(T).$$

We recall some conventions on the multiindex notations. For $\alpha_s \in \mathbf{N}$ we write $\underline{\alpha} \overset{\mathrm{def}}{=} (\alpha_0, \dots, \alpha_{f-1})$ to denote an f-tuple $\underline{\alpha} \in \mathbf{N}^f$. If $\underline{\alpha}, \underline{\beta}$ are f-tuples we define

i) $\underline{\alpha} + \underline{\beta} \overset{\mathrm{def}}{=} (\alpha_s + \beta_s)_{s=0}^{f-1}$;

ii) $\underline{\alpha} \geqslant \underline{\beta}$ if and only if $\alpha_s \geqslant \beta_s$ for all $s \in \{0, \dots, f-1\}$;

iii) $\binom{\underline{\alpha}}{\underline{\beta}} \overset{\mathrm{def}}{=} \prod_{s=0}^{f-1}\binom{\alpha_s}{\beta_s}$.

For $n \in \mathbf{N}$ we will write $\underline{n} \overset{\mathrm{def}}{=} (n, \dots, n) \in \mathbf{N}^f$.

If $\underline{\alpha} + \underline{\beta} = \underline{r}$ we define the following element of $\sigma_{\underline{r}}$:

$$X^{\underline{\alpha}}Y^{\underline{\beta}} \overset{\mathrm{def}}{=} \otimes_{s=0}^{f-1} X_s^{\alpha_s}Y_s^{\beta_s};$$

for $\lambda \in \mathbf{F}_q$ and $\underline{\alpha} \in \{0, \dots, p-1\}^f$ we put

$$\lambda^{\underline{\alpha}} \overset{\mathrm{def}}{=} \lambda^{\sum_{s=0}^{f-1} p^s\alpha_s}.$$

For an integer $n \in \mathbf{N}$ we define $\lfloor n \rfloor \in \{0, \dots, f-1\}$ as the unique integer $m \in \{0, \dots, f-1\}$ congruent to n modulo f. Similarly, if $n \neq 0$ we define $\lceil n \rceil \in \{1, \dots, q-1\}$ as the unique integer $m \in \{1, \dots, q-1\}$ congruent to n modulo $q-1$; we set $\lceil 0 \rceil \overset{\mathrm{def}}{=} 0$.

Finally, for a smooth representation R of $K_0(p)$ over $\overline{\mathbf{F}}_p$ we write $\{\mathrm{soc}_N(R)\}_{N\in\mathbf{N}}$ to denote its socle filtration (with the convention $\mathrm{soc}(R)_0 \overset{\mathrm{def}}{=} \mathrm{soc}(R)$).

Let $\mathscr{B}$ be an $\overline{\mathbf{F}}_p$-basis of R and P a bijection of $\mathscr{B}$ onto a subset $\mathscr{R}$ in $\mathbf{Z}^f$. Let $\mathscr{B}' \subseteq \mathscr{B}$ be a subset and $\mathscr{R}'$ denotes its image through the bijection P; for $v \in \mathscr{B}'$ we define the set of antecedents of v in $\mathscr{R}'$ as:

$$\mathfrak{S}_v(\mathscr{B}') \overset{\mathrm{def}}{=} \big\{w \in \mathscr{B}' \text{ s.t. } P(w) = P(v) - e_s \,\mathrm{for}\, s \in \{0, \dots, f-1\}\big\}$$

(where $(e_s)_{s=0}^{f-1}$ is the canonical basis of $\mathbf{Z}^f$).

We say that the socle filtration $\{\mathrm{soc}_N(R)\}_{N\in\mathbf{N}}$ of R is described by $\mathscr{R}$ if the following holds: it exists an increasing family $\{\mathscr{B}_N\}_{N\in\mathbf{N}}$ of subsets of $\mathscr{B}$ such that

i) for all $N \in \mathbf{N}$ the family $\mathscr{B}_N$ is an $\overline{\mathbf{F}}_p$-basis of $\mathrm{soc}_N(R)$;

ii) for all $N \in \mathbf{N}$ an $\overline{\mathbf{F}}_p$-basis for $\mathrm{soc}(R/\mathrm{soc}_{N-1}(R))$ is described as

$$\big\{v \in \mathscr{B} \setminus \mathscr{B}_{N-1}, \quad \text{s.t. } \mathfrak{S}_v(\mathscr{B} \setminus \mathscr{B}_{N-1}) = \emptyset\big\}.$$

If the socle filtration of R is described by $\mathscr{R}$ we will say that the extensions between two graded pieces are described by $\mathscr{R}$ if the following holds true:

> for all $N \in \mathbf{N}$ and $v \in \mathscr{B}_{N+1}$ the $\overline{\mathbf{F}}_p$-linear subspace $E_{v,N}$ *of* $R/\mathrm{soc}_{N-1}(R)$ generated by $v, \mathfrak{S}_v(\mathscr{B} \setminus \mathscr{B}_{N-1})$ is $K_0(p)$-stable and for each $w \in \mathfrak{S}_v(\mathscr{B} \setminus \mathscr{B}_{N-1})$ the induced extension
>
> $$0 \to \overline{w} \to E_{v,N}/\langle \mathfrak{S}_v(\mathscr{B} \setminus \mathscr{B}_{N-1}) \setminus \{w\}\rangle_{\overline{\mathbf{F}}_p} \to \overline{v} \to 0$$
>
> is nonsplit (with the obvious meaning of $\overline{w}$, $\overline{v}$).

In euclidean terms the segments between v and the set of its antecedents let us detemines all the nonsplit extensions between two graded pieces of the socle filtration.

2. Preliminaries

As we outlined in the introduction, the main aim of this section is to describe the Iwahori-structure of the universal representations $\pi(\underline{r}, 0, 1)$ of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$.

Such representations have a completely explicit description in terms of the Bruhat-Tits tree and of the Hecke operator T given in [Bre03a], §2 and their Iwahory structure can indeed be found by direct methods. Nevertheless, the extremely involved combinatoric of such results lead us to introduce an intermediary step -namely a suitable KZ-filtration- which let us handle, in a reasonable way, the high amount of technical computations. Precisely, we start (cf. definition 2.3) by introducing the KZ-representations

$$R_{n+1} \overset{\mathrm{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{K} \sigma_{\underline{r}^{n+1}}$$

(where $\sigma_{\underline{r}^{n+1}}$ is a $K_0(p^{p^{n+1}})$-representations obtained by twisting the action of $K_0(p^{n+1})$ on $\sigma_{\underline{r}}|_{K_0(p^{n+1})}$). Such objects are endowed with an action of suitable "Hecke" operators $T_n^{\pm} : R_n \to R_{n\pm 1}$ (cf. lemma 2.7), with respect to which we are able to define (inductively) a direct system of amalgamed sums $\cdots \oplus_{R_n} R_{n+1}$ (cf. proposition 2.8). Such amalgamed sums fit in a natural commutative diagram (see proposition 2.8) which let us deduce a natural filtration on the resulting inductive limits. The final result is then the isomorphism of proposition 2.9, which relies the KZ-restriction of the universal representation $\pi(\underline{r}, 0, 1)|_{KZ}$ to the inductive limits constructed above; in particular, we have a natural KZ-equivariant filtration on the universal representation $\pi(\underline{r}, 0, 1)$.

In lemma 2.6 we introduce a "canonical" basis for the representations R_{n+1}. Such basis is well behaved with respect to both the action of the Hecke operators and the action of the Iwahori subgroup: this will be the key observation which lead us to the description of the Iwahory structure for $\pi(\underline{r}, 0, 1)$.

We remark that the isomorphism of proposition 2.9 does not rely on the fact that $F/\mathbf{Q}_p$ is unramified: the content of this section can be generalised in the evident manner for any finite extension F of $\mathbf{Q}_p$.

Reminders on the universal representations $\pi(\underline{r}, 0, 1)$. For $n \in \mathbf{N}_{\geqslant 1}$ we define

$$I_n \overset{\mathrm{def}}{=} \{\sum_{j=0}^{n-1} p^j[\lambda_j] \quad \text{for } \lambda_j \in \mathbf{F}_q\}$$

and we put $I_0 \overset{\text{def}}{=} \{0\}$. The sets I_n's let us describe the Bruhat-Tits tree in the following way: if $n, m \in \mathbf{N}$, $\lambda \in I_n$ and

$$g^0_{n,\lambda} \overset{\text{def}}{=} \begin{bmatrix} p^n & \lambda \\ 0 & 1 \end{bmatrix}, \qquad g^1_{n,\lambda} \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ p\lambda & p^{n+1} \end{bmatrix}$$

we get a decomposition

$$KZ\alpha^{-m}KZ = \coprod_{\lambda \in I_m} g^0_{m,\lambda}KZ \coprod \coprod_{\lambda \in I_{m-1}} g^1_{m,\lambda}KZ \tag{30}$$

thus describing the vertex of the tree having distance m from KZ (where we have written $\alpha \overset{\text{def}}{=} g^1_{0,0}$). The canonical Hecke operator $T \in \mathrm{End}_G(\mathrm{Ind}^G_{KZ}\sigma_{\underline{r}})$, defined in [Bre03a] §2.7, is then characterized as follow:

Lemma 2.1. *For $n \in \mathbf{N}_>$, $\lambda \in I_n$ and $0 \leqslant \underline{j} \leqslant \underline{r}$ we have:*

$$T([g^0_{n,\lambda}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = \sum_{\lambda_n \in \mathbf{F}_q} [g^0_{n+1,\lambda+p^n[\lambda_n]}, (-\lambda_n)^{\underline{j}}X^{\underline{r}}] + [g^0_{n-1,[\lambda]_{n-1}}, \delta_{\underline{j},\underline{r}}(\lambda_{n-1}X+Y)^{\underline{r}}]$$
$$T([g^1_{n,\lambda}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = \sum_{\lambda_n \in \mathbf{F}_q} [g^1_{n+1,\lambda+p^n[\lambda_n]}, (-\lambda_n)^{\underline{r}-\underline{j}}Y^{\underline{r}}] + [g^0_{n-1,[\lambda]_{n-1}}, \delta_{\underline{j},\underline{0}}(X+\lambda_{n-1}Y)^{\underline{r}}].$$

If $n = 0$ we have

$$T([1_G, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = \sum_{\lambda_0 \in \mathbf{F}_q} [g^0_{1,[\lambda_0]}, (-\lambda_0)^{\underline{j}}X^{\underline{r}}] + [\alpha, \delta_{\underline{j},\underline{r}}Y^{\underline{r}}]$$
$$T([\alpha, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = \sum_{\lambda_1 \in \mathbf{F}_q} [g^1_{1,[\lambda_1]}, (-\lambda_1)^{\underline{r}-\underline{j}}Y^{\underline{r}}] + [1_G, \delta_{\underline{j},\underline{0}}X^{\underline{r}}]$$

.

Proof. A computation shows that the statement of lemme 3.1.1 in [Bre03a] has an obvious generalisation for $f > 1$. The result follows then from *Ibid.*, §2.5. □

For $n \in \mathbf{N}$ we define the $\overline{\mathbf{F}}_p$-subspace of $\mathrm{Ind}^G_{KZ}\sigma_{\underline{r}}$:

$$W(n) \overset{\text{def}}{=} \{f \in \mathrm{Ind}^G_{KZ}\sigma_{\underline{r}}, \quad \text{s.t. the support of } f \text{ is contained in } KZ\alpha^{-n}KZ\}.$$

By Cartan decomposition the subspaces $W(n)$ are KZ-stable for all $n \in \mathbf{N}$ and therefore

Lemma 2.2. *There is a natural KZ-equivariant isomorphism*

$$\mathrm{Ind}^G_{KZ}\sigma_{\underline{r}} \xrightarrow{\sim} \bigoplus_{n \in \mathbf{N}} W(n).$$

The representations R_n's and the dictionary. Let $n \in \mathbf{Z}_{\geqslant -1}$; we define the open subgroups of K:

$$K_0(p^{n+1}) \overset{\text{def}}{=} \{g \in K, \text{ s.t. } g = \begin{bmatrix} a & b \\ p^{n+1}c & d \end{bmatrix} \text{ for } a,b,c,d \in \mathscr{O}_F\}.$$

As $\begin{bmatrix} 0 & 1 \\ p^{n+1} & 0 \end{bmatrix}$ normalizes $K_0(p^{n+1})$, the representation $\sigma_{\underline{r}}|_{K_0(p^n)}$ induces, by conjugation, a $K_0(p^{n+1})$-representation which will be denoted as $\sigma^{n+1}_{\underline{r}}$ (or simply $\sigma_{\underline{r}}$ if there is no risk of confusion). Explicitely, we have

$$\sigma^{n+1}_{\underline{r}}(\begin{bmatrix} a & b \\ p^{n+1}c & d \end{bmatrix}) = \sigma_{\underline{r}}(\begin{bmatrix} d & c \\ p^{n+1}b & a \end{bmatrix}).$$

We can therefore introduce the representations R_{n+1}'s:

Definition 2.3. *Let $n \in \mathbf{Z}_{\geqslant -1}$. The K-representation R_{n+1} is defined as*

$$R_{n+1} \stackrel{\mathrm{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{K} \sigma_{\underline{r}}^{n+1}.$$

We can extend the action of K on R_{n+1} to an action of KZ by letting $p \in Z$ act trivially; the resulting representation will be denoted again by R_{n+1} and we will pass from the one to the other without commentary.

Thanks to the decomposition (30) we get the following, elementary, description of the R_n's:

Lemma 2.4. *Let $n \in \mathbf{Z}_{\geqslant -1}$ Then:*

i) right translation by $\alpha^{n+1}w$ induces a bijection

$$K/K_0(p^{n+1}) \xrightarrow{\sim} KZ\alpha^{-n-1}KZ/KZ;$$

ii) we have a decomposition

$$K = \coprod_{\lambda \in I_{n+1}} \begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix} K_0(p^{n+1}) \coprod \coprod_{\lambda' \in I_n} \begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix} K_0(p^{n+1});$$

Moreover, if $1 \leqslant m \leqslant n$ we have a decomposition

$$K_0(p^m) = \coprod_{\lambda' \in I_{n+1-m}} \begin{bmatrix} 1 & 0 \\ p^m\lambda' & 1 \end{bmatrix} K_0(p^{n+1});$$

iii) the family

$$\left\{ [\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}], [\begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}] \text{ for } \lambda \in I_{n+1},\, \lambda' \in I_n, \underline{0} \leqslant \underline{j} \leqslant \underline{r} \right\}$$

defines an $\overline{\mathbf{F}}_p$-basis for the representation R_{n+1}. Moreover, if $1 \leqslant m \leqslant n$, the family

$$\{ [\begin{bmatrix} 1 & 0 \\ p^m\lambda' & 1 \end{bmatrix}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}] \text{ for } \lambda \in I_{n+1-m},\, \underline{0} \leqslant \underline{j} \leqslant \underline{r} \}$$

defines an $\overline{\mathbf{F}}_p$-basis for the representation $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} \sigma_{\underline{r}}$.

Proof. Omissis. □

The relation between the representations R_n's and the compact induction $\mathrm{Ind}_{KZ}^{G} \sigma_{\underline{r}}|_{KZ}$ is then described by the following

Proposition 2.5. *Let $n \in \mathbf{Z}_{\geqslant -1}$. We have a KZ-equivariant isomorphism*

$$\Phi_{n+1} : W(n+1) \xrightarrow{\sim} R_{n+1}$$

such that

$$\Phi_{n+1}([g^0_{n+1,\lambda}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = [\begin{bmatrix} \lambda & 1 \\ 1 & 0 \end{bmatrix}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]$$

$$\Phi_{n+1}([g^1_{n,\lambda'}, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = [\begin{bmatrix} 1 & 0 \\ p\lambda' & 1 \end{bmatrix}, X^{\underline{j}}Y^{\underline{r}-\underline{j}}]$$

for $n \geqslant 0$ and

$$\Phi_0([1_G, X^{\underline{r}-\underline{j}}Y^{\underline{j}}]) = X^{\underline{j}}Y^{\underline{r}-\underline{j}}$$

for $n = 0$.

In particular, we have a KZ*-equivariant isomorphism*

$$\mathrm{Ind}_{KZ}^{G}\sigma_{\underline{r}} \xrightarrow{\sim} \bigoplus_{n\in\mathbf{N}} R_n$$

Proof. Elementary (see for instance [Mo1], proposition 3.4, whose proof generalizes line by line). □

We introduce now a convenient $\overline{\mathbf{F}}_p$-basis for the representation R_{n+1}. Thanks to the transitivity

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\sigma_{\underline{r}} \cong \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)}\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^{m+1})}\sigma_{\underline{r}}$$

(where $0 \leqslant m \leqslant n$) we see that a Vandermonde argument together with an immediate induction give us the following:

Lemma 2.6 (Definition). *Let* $n \in \mathbf{N}$*. An* $\overline{\mathbf{F}}_p$ *basis for the* K*-representation* R_{n+1} *is described by the elements*

$$F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n}\sum_{\lambda_i\in\mathbf{F}_q}(\lambda_i^{\frac{1}{p^i}})^{\underline{l}_i}\begin{bmatrix} 1 & 0\\ p^i[\lambda_i^{\frac{1}{p^i}}] & 1\end{bmatrix}[1, X^{\underline{r}-\underline{l}_{n+1}}Y^{\underline{l}_{n+1}}]$$

$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \stackrel{\mathrm{def}}{=} \sum_{\lambda_0\in\mathbf{F}_q}\lambda_0^{\underline{l}_0}\begin{bmatrix} [\lambda_0] & 1\\ 1 & 0\end{bmatrix}[1, F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1})]$$

for $\underline{l}_i \in \{0,\dots,p-1\}^f$ *(where* $i \in \{0,\dots,n\}$*) and* $\underline{l}_{n+1} \leqslant \underline{r}$*, with the obvious conventions that if* $n = 0$ *we have*

$$F^{(1,0)}_{\emptyset}(\underline{l}_1) \stackrel{\mathrm{def}}{=} [1, X^{\underline{r}-\underline{l}_{n+1}}Y^{\underline{l}_n+1}].$$

For notational convenience we define

$$F^{(0,-1)}_{\emptyset}(\underline{l}_0) \stackrel{\mathrm{def}}{=} (-1)^{\underline{l}_0}X^{\underline{l}_0}Y^{\underline{r}-\underline{l}_0}$$
$$F^{(1,-1)}_{\emptyset}(\emptyset) \stackrel{\mathrm{def}}{=} Y^{\underline{r}}.$$

Such basis will be denoted by $\mathscr{B}_{n+1}$*.*

The subset $\mathscr{B}^+_{n+1} \subset \mathscr{B}_{n+1}$ *described by the elements of the form* $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$ *will be referred to as the set of* positive elements *of* R_{n+1}*; the* $\overline{\mathbf{F}}_p$*-linear subspace generated by the positive elements will be denoted as* R^+_{n+1}*.*

Similarly the subset $\mathscr{B}^-_{n+1} \subset \mathscr{B}_{n+1}$ *described by elements of the form* $F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1})$ *will be referred to as the set of* negative elements *of* R_{n+1}*; the* $\overline{\mathbf{F}}_p$*-linear subspace generated by the negative elements will be denoted as* R^-_{n+1}*.*

Hecke operators on the R_{n+1}***'s.*** Let $n \in \mathbf{N}$. Thanks to lemma 2.1 the $W(n)$-restriction of the operator T gives the $\overline{\mathbf{F}}_p$-linear morphism

$$T|_{W(n)} : W(n) \to W(n-1) \oplus W(n+1).$$

Such restriction is KZ-equivariant (by Cartan decomposition) so that composition by the natural projections gives us the KZ-equivariant operators

$$T^+_n : W(n) \to W(n+1) \qquad T^-_n : W(n) \to W(n-1);$$

so that the transport of structure (via the isomorphisms of lemma 2.5) gives

$$T_n^+ : R_n \to R_{n+1} \qquad T_n^- : R_n \to R_{n-1}$$

(using the same notations for the operators on $W(n)$ and R_n). Their description in terms of the canonical basis of R_{n+1} is immediate, following from lemmas 2.1 and 2.5:

LEMMA 2.7. *Let $n > 0 \in \mathbf{N}$. The KZ-equivariant operators T_n^+, T_n^- are characterized by*

$$T_n^+ : R_n \to R_{n+1}$$
$$[1, X^{\underline{r}-\underline{l}_n} Y^{\underline{l}_n}] \mapsto (-1)^{\underline{l}_n} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, X^{\underline{r}}]$$

$$T_n^- : R_n \to R_{n-1}$$
$$[1, X^{\underline{r}-\underline{l}_n} Y^{\underline{l}_n}] \mapsto \begin{cases} \delta_{\underline{r},\underline{l}_n}[1, Y^{\underline{r}}] & \text{if } n > 1 \\ \delta_{\underline{r},\underline{l}_n} Y^{\underline{r}} & \text{if } n = 1. \end{cases}$$

For $n = 0$ we have

$$R_0 \hookrightarrow R_1$$
$$X^{\underline{r}-\underline{l}_0} Y^{\underline{l}_0} \quad \mapsto \quad \sum_{\lambda_0 \in \mathbf{F}_q} (-1)^{\underline{r}-\underline{l}_0} \lambda_0^{\underline{r}-\underline{l}_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{\underline{r}}] + \delta_{\underline{l}_0,\underline{0}}[1, X^{\underline{r}}].$$

Moreover, the operators T_n^+ are monomorphisms for all $n \in \mathbf{N}$ and the operators T_n^- are epimorphisms for all $n \in \mathbf{N}_{\geqslant 1}$.

Proof. The characterisation of the operators $T_n^{\pm}$ follows by the explicit descriptions given in lemmas 2.1 and 2.5.

As T_n^+ maps the basis $\mathscr{B}_n$ into a subset of $\mathscr{B}_{n+1}$, the operator is injective for $n \geqslant 1$. As $[1, Y^{\underline{r}}]$ (resp. $Y^{\underline{r}}$) is a K-generator for R_{n-1} (resp. R_0) for $n \geqslant 2$ (resp. $n = 1$), the operator T_n^- is surjective. $\square$

We identify R_n as a K-subrepresentation of R_{n+1} via the monomorphism T_n^+ without any further commentary. For any odd integer $n \geqslant 1$ we use the Hecke operators $T_n^{\pm}$ to define (inductively) the amalgamed sum $R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$ via the following co-cartesian diagram

$$\begin{tikzcd} R_n \arrow[r, hook, "T_n^+"] \arrow[d, "-pr_{n-1} \circ T_n^-"'] & R_{n+1} \arrow[d, dotted, "pr_{n+1}"] \\ R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, dotted] & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1} \end{tikzcd}$$

(where we define pr_0 to be the identity map). Similarly we define the amalgamed sums $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ for any positive even integer $n \in \mathbf{N}_{>}$. The following result is then formal

PROPOSITION 2.8. *For any odd integer $n \in \mathbf{N}$, $n \geqslant 1$ we have a natural commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d, "-pr_{n-1} \circ T_n^-"] & R_{n+1} \arrow[r] \arrow[d, "pr_{n+1}"] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r, "\pi"] & R_{n+1}/R_n \arrow[r] & 0 \end{tikzcd}$$

with exact lines.

We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n\in 2\mathbf{N}\setminus\{0\}}.$$

Proof. Formal. See for instance [Mo1], proposition 4.1. □

The following result let us complete the dictionary

Proposition 2.9. *We have a KZ equivariant isomorphism*

$$\pi(\sigma_{\underline{r}},0,1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{n\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}).$$

Proof. The proof is formal and identical to [Mo1], proposition 3.9. □

Remark 2.10. *We can give analogous (in the evident way) definitions in the case where F is any finite extension of $\mathbf{Q}_p$: we would then get a statement completely analogous to proposition 2.9.*

3. First description of the Iwahori structure

The goal of this section is to give a first, general description for the $K_0(p)$-representation $\pi(\underline{r},0,1)|_{K_0(p)}$. The endpoint is proposition 3.7, which is the "Iwahori analogue" of proposition 2.9 of the preceeding section. More precisely, for each $n \in \mathbf{N}$ the block R_{n+1} has a natural $K_0(p)$-equivariant splitting

$$R_{n+1} = R^+_{n+1} \oplus R^-_{n+1}$$

which is compatible with the Hecke operators $T^\pm_n$ in the obvious sense (cf. lemma/definition 3.2). This will enable us to repeat the constructions of §2, i.e. the construction of the (inductive family of) amalgamed sums $\cdots \oplus_{R^\pm_n} R^\pm_{n+1}$, endowed with a natural filtration (cf. lemma 3.5) .

Thanks to proposition 3.7 we see that we can content ourselves to the study of the amalgames sums $\cdots \oplus_{R^\pm_n} R^\pm_{n+1}$: actually we have a $K_0(p)$-equivariant surjection

$$\begin{array}{c}(\varinjlim_{n\,\mathrm{odd}} \cdots \oplus_{R^+_n} R^+_{n+1}) \oplus (\varinjlim_{n\,\mathrm{odd}} \cdots \oplus_{R^-_n} R^-_{n+1}) \oplus (\varinjlim_{n\,\mathrm{even}} \cdots \oplus_{R^+_n} R^+_{n+1}) \oplus (\varinjlim_{n\,\mathrm{even}} \cdots \oplus_{R^-_n} R^-_{n+1})\\ \downarrow \\ \pi(\underline{r},0,1)|_{K_0(p)}\end{array}$$

whose kernel is "small" (and explicitely determined).

The following elementary result will be crucial.

Lemma 3.1. *Let $a \in \{0,\dots,q-1\}$. Then*

$$\sum_{\lambda\in\mathbf{F}_q} \lambda^a = \begin{cases} 0 & \text{if } a \neq q-1 \\ -1 & \text{if } a = q-1. \end{cases}$$

Proof. Omissis. □

The representations $R^\pm_{n+1}$ and the Hecke operators $(T^\pm_n)^{\mathrm{pos,\,neg}}$. Fix $n \in \mathbf{N}$; the $\overline{\mathbf{F}}_p$-linear decomposition

$$R_{n+1} \cong R^+_{n+1} \oplus R^-_{n+1} \tag{31}$$

is easily checked to be $K_0(p)$-equivariant (realising the Mackey decomposition for $R_{n+1}|_{K_0(p)}$) and we clearly have a $K_0(p)$-equivariant isomorphism

$$R_{n+1}^{-} \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \sigma_{\underline{r}}^{n+1}.$$

We moreover define the following $K_0(p)$-representations:

$$R_0^{+} \stackrel{\mathrm{def}}{=} R_0 \qquad R_0^{-} \stackrel{\mathrm{def}}{=} \langle Y^{\underline{r}} \rangle_{\overline{\mathbf{F}}_p}$$
$$(R_1/R_0)^{+} \stackrel{\mathrm{def}}{=} \mathrm{Im}(R_1^{+} \hookrightarrow R_1 \twoheadrightarrow R_1/R_0) \qquad Q \stackrel{\mathrm{def}}{=} \mathrm{Coker}\big((R_1/R_0)^{+} \hookrightarrow R_1/R_0\big).$$

The decomposition given in (31) and the description of lemma 2.7 let us define the Hecke operators $(T_n^{\pm})^{\mathrm{pos,\,neg}}$ on the representations $R_{n+1}^{\pm}$:

Lemma 3.2 (Definition). *Let $n \in \mathbf{N}_{\geqslant 1}$.*

i) The restriction of Hecke operator T_n^{+} on the $K_0(p)$-subrepresentations R_n^{+}, R_n^{-} of R_n induces two $K_0(p)$-equivariant monomorphisms,

$$(T_n^{+})^{\mathrm{pos}} : R_n^{+} \hookrightarrow R_{n+1}^{+}$$
$$(T_n^{+})^{\mathrm{neg}} : R_n^{-} \hookrightarrow R_{n+1}^{-}$$

ii) The restriction of Hecke operator T_n^{-} on the $K_0(p)$-subrepresentations R_n^{+}, R_n^{-} of R_n induces two $K_0(p)$-equivariant epimorphisms,

$$(T_n^{-})^{\mathrm{pos}} : R_n^{+} \twoheadrightarrow R_{n-1}^{+}$$
$$(T_n^{-})^{\mathrm{neg}} : R_n^{-} \twoheadrightarrow R_{n-1}^{-}$$

Proof. Except for the operator $(T_1^{-})^{\mathrm{pos}}$, the result follows immediately from the decomposition $R_n|_{K_0(p)} \cong R_n^{+} \oplus R_n^{-}$ and the properties and characterisations of the Hecke operators $T_n^{\pm}$.

Concerning $(T_1^{-})^{\mathrm{pos}} : R_1^{+} \to R_0$ we notice that

$$(T_1^{-})^{\mathrm{pos}}(F_{\underline{l}_0(\underline{r})}^{(0)}) = \sum_{\underline{i} \leqslant \underline{r}} X^{\underline{r}-\underline{i}} Y^{\underline{i}} \Big(\sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0 + \underline{i}} \Big)$$

and the result follows from lemma 3.1. □

Corollary 3.3. *The natural $K_0(p)$-equivariant maps*

$$R_2^{+} \to (R_1/R_0)^{+}$$
$$R_2^{-} \to Q$$

are epimorphisms.

Proof. Omissis. □

Remark 3.4. *The notation $(T_n^{\pm})^{\mathrm{pos,neg}}$ may look a bit awkard. We believe, though, that a notation of the kind $(T_n^{\pm})^{\pm}$, even if it could be more convenient for statements (see lemma 3.5), it can be disagreeable for the computations (and especially misprints!)*

Amalgamed sums and first description of the Iwahori structure. Using the Hecke operators defined in lemma 3.2 we can introduce the following amalgamed sums, analogously to the constructions of §2.

Let $n \in \mathbf{N}$ be odd and $\bullet \in \{+, -\}$. We can define inductively a natural $K_0(p)$-representation

$R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$ together with canonical morphisms $pr_{n+1}^\bullet, \iota_{n-1}^\bullet$ by the condition that the diagram

$$\begin{tikzcd}
R_n^\bullet \arrow[r, hook, "(T_n^+)^\bullet"] \arrow[d, "-(pr_{n-1})^\bullet \circ (T_n^-)^\bullet"'] & R_{n+1}^\bullet \arrow[d, dotted, "\exists!\ (pr_{n+1})^\bullet"] \\
R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet \arrow[r, dotted, "\iota_{n-1}^\bullet", "\exists!"'] & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet .
\end{tikzcd}$$

is co-cartesian (with the convention that $(T_j^\pm)^+ \overset{\text{def}}{=} (T_j^\pm)^{\text{pos}}$ and $(T_j^\pm)^- \overset{\text{def}}{=} (T_j^\pm)^{\text{neg}}$).

For $n \in \mathbf{N}$ even and $\bullet \in \{+,-\}$ we can define the amalgamed sums $(R_1/R_0)^\bullet \oplus_{R_2^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$, together with canonical morphisms $pr_{n+1}^\bullet, \iota_{n-1}^\bullet$ in the evident analogous way (with the convention that $(R_1/R_0)^- \in \{R_1^-, Q\}$.)

The following result is similar to proposition 2.8:

Lemma 3.5. *Let $n \in \mathbf{N}$ be odd, $\bullet \in \{+,-\}$. Then $\iota_{n-1}^\bullet$ is a monomorphism, $pr_{n+1}^\bullet$ is an epimorphism and we have a ($K_0(p)$-equivariant) commutative diagram with exact lines:*

$$\begin{tikzcd}
0 \arrow[r] & R_n^\bullet \arrow[r, "(T_n^+)^\bullet"] \arrow[d, two heads, "-(T_n^-)^\bullet"] & R_{n+1}^\bullet \arrow[r, "\pi_{n+1}"] \arrow[dd, two heads, "pr_{n+1}^\bullet"] & R_{n+1}^\bullet/R_n^\bullet \arrow[r] \arrow[dd, equal] & 0 \\
 & R_{n-1}^\bullet \arrow[d, two heads, "pr_{n-1}^\bullet"] & & & \\
0 \arrow[r] & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_{n-2}^\bullet} R_{n-1}^\bullet \arrow[r, "\iota_{n-1}"] & R_0^\bullet \oplus_{R_1^\bullet} \cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet \arrow[r, "\pi_{n+1}"] & R_{n+1}^\bullet/R_n^\bullet \arrow[r] & 0.
\end{tikzcd}$$

We have an analogous (in the evident way) result in the case $n \in \mathbf{N}_>$ is even.

Proof. The proof is identical to proposition 2.8, provided that the maps $R_1^\bullet \overset{(T_1^-)^\bullet}{\twoheadrightarrow} R_0^\bullet$ and $R_2^\bullet \overset{(T_2^-)^\bullet}{\twoheadrightarrow} (R_1/R_0)^\bullet$ are epimorphisms. □

In order to give a first description of the $K_0(p)$-representation $\pi(\underline{r}, 0, 1)|_{K_0(p)}$ we are now left to determine the relations between the amalgamed sums $\cdots \oplus_{R_n^\bullet} R_{n+1}^\bullet$ (where $\bullet \in \{+,-\}$) and the restriction $(\cdots \oplus_{R_n} R_{n+1})|_{K_0(p)}$.

We will treat in detail the analysis of the limit $(\varinjlim_{n,\,\text{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)}$. The case n even is proved in a similar way and is left to the reader.

Proposition 3.6. *The decomposition $R_n|_{K_0(p)} \cong R_n^+ \oplus R_n^-$ for $n \geqslant 1$ induces the following $K_0(p)$-equivariant exact sequences:*

$$0 \to \varinjlim_{n\,\text{even}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+ \to (\varinjlim_{n\,\text{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to$$
$$\to \varinjlim_{n\,\text{odd}} R_2^-/R_1^- \oplus_{R_3^-} \cdots \oplus_{R_n^-} R_{n+1}^- \to 0$$

and

$$0 \to \varinjlim_{n\,\mathrm{even}} (R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+ \to (\varinjlim_{n\,\mathrm{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to$$
$$\to \varinjlim_{n\,\mathrm{even}} Q \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^- \to 0.$$

Proof. Let us assume n odd, leaving the case n even to the reader (the proof is analogous). Since the functor $\varinjlim$ is exact if the index category is filtrant and since the forgetful functor $For : \mathcal{R}ep_{K_0(p)} \to \mathcal{V}ect_{\overline{\mathbf{F}}_p}$ commutes with $\varinjlim$ it is enough to show that we have an inductive system of exact sequence

$$0 \to R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+ \to (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to R_2^-/R_1^- \oplus_{R_3^-} \cdots \oplus_{R_n^-} R_{n+1}^- \to 0$$

for $n \in \mathbf{N}$ odd. More precisely, we claim we have a natural diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & R_{n+1}^+ \arrow[r] \arrow[d, "pr_{n+1}^-"] & R_{n+1} \arrow[r] \arrow[d, "pr_{n+1}"] & R_{n+1}^- \arrow[r] \arrow[d, "pr_{n+1}^-"] & 0 \\
0 \arrow[r] & R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+ \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r] & R_2^-/R_1^- \oplus_{R_3^-} \cdots \oplus_{R_n^-} R_{n+1}^- \arrow[r] & 0
\end{tikzcd}$$

for $n \in \mathbf{N}$ odd. The proof is now an induction on n.

Let $n = 1$. We recall that, by definition, the "Hecke" operators $(T_1^{\pm})^{\mathrm{pos,neg}}$ are the restrictions of the operators $T_1^{\pm}$; moreover the forgetful functor $For : \mathcal{R}ep_K \to \mathcal{V}ect_{\overline{\mathbf{F}}_p}$ commutes with the pushout. We deduce the following commutative diagram

$$\begin{tikzcd}
& & 0 \arrow[rr] & & R_1^+ \arrow[rrr] \arrow[dl] \arrow[dd, two heads] & & & R_2^+ \arrow[rrr] \arrow[dl] \arrow[dd, two heads] & & & R_2^+/R_1^+ \arrow[rr] \arrow[dl] \arrow[dd, equals] & & 0 \\
& 0 \arrow[rr] & & R_1 \arrow[rrr] \arrow[ddl] \arrow[ddd, two heads] & & & R_2 \arrow[rrr] \arrow[ddl] \arrow[ddd, two heads] & & & R_2/R_1 \arrow[rr] \arrow[ddl] \arrow[ddd, equals] & & 0 & \\
& & 0 \arrow[rr] & & R_0^+ \arrow[rrr] \arrow[ddl] & & & R_0^+ \oplus_{R_1^+} R_2^+ \arrow[rrr] \arrow[ddl, dotted, "f_1\ \exists!"'] & & & R_2^+/R_1^+ \arrow[rr] \arrow[ddl] & & 0 \\
0 \arrow[rr] & & R_1^- \arrow[rrr] \arrow[dd, two heads] & & & R_2^- \arrow[rrr] \arrow[dd, two heads] & & & R_2^-/R_1^- \arrow[rr] \arrow[dd, equals] & & 0 & & \\
& 0 \arrow[rr] & & R_0 \arrow[rrr] \arrow[dl] & & & R_0 \oplus_{R_1} R_2 \arrow[rrr] \arrow[dl, dotted, "f_2\ \exists!"'] & & & R_2/R_1 \arrow[rr] \arrow[dl] & & 0 & \\
0 \arrow[rr] & & 0 \arrow[rrr] & & & R_2^-/R_1^- \arrow[rrr, equals] & & & R_2^-/R_1^- \arrow[rr] & & 0 & &
\end{tikzcd}$$

where f_1 is a $K_0(p)$-equivariant morphism deduced deduced by the universal property of $R_0^+ \oplus_{R_1^+} R_2^+$ and f_2 by the universal property of $For(R_0 \oplus_{R_1} R_2|_{K_0(p)})$. Notice that, a priori, the morphism f_2 is only $\overline{\mathbf{F}}_p$-linear; the fact that it is $K_0(p)$-equivariant is immediately deduced as the maps $R_2 \twoheadrightarrow R_0 \oplus_{R_1} R_2$ and $R_2^- \twoheadrightarrow R_2^-/R_1^-$ are $K_0(p)$ equivariant epimorphism.

As $R_2^+ \twoheadrightarrow R_0^+ \oplus_{R_1^+} R_2^+$ is an epimorphism, we deduce that $f_2 \circ f_1 = 0$.

As the lower horizontal lines are exacts, we deduce (e.g. from the five lemma) that f_1 is a monomorphism and f_2 is an epimorphism.

Finally, we have the equalities

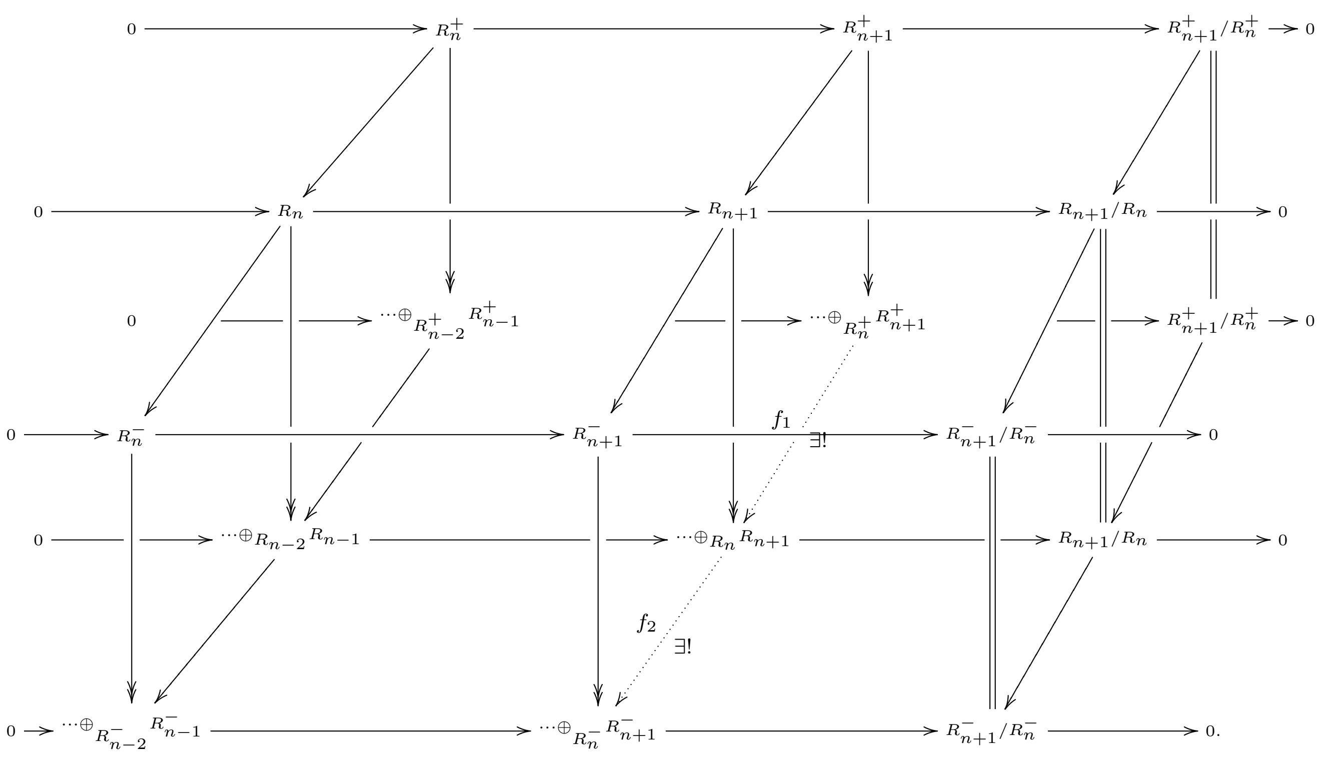
$$\begin{aligned}
\dim(\mathrm{Im}(f_1)) &= \dim(R_0^+ \oplus_{R_1^+} R_2^+) \\
&= \dim(R_2^+/R_1^+) + \dim(R_0^+) \\
&= \dim(R_2/R_1) - \dim(R_2^-/R_1^-) + \dim(\mathrm{R}_0) \\
&= \dim(R_0 \oplus_{R_1} R_2) - \dim(R_2^-/R_1^-) \\
&= \dim(\ker(f_2))
\end{aligned}$$

where we deduce the first (resp. the last) equality from the injectivity of f_1 (resp. surjectivity of f_2) and the others equalities are deduced from the exactness of all the bottom lines (except from the "central, vertical" line).

For the general case, we deduce from the inductive hypothesis and the definition of the "Hecke" operators the following commutative diagramm with exact lines

$$\begin{tikzcd}
0 \arrow[r] & R_n^+ \arrow[r] \arrow[d, "(T_n^-)^{\mathrm{pos}}"] & R_n \arrow[r] \arrow[d, "T_n^-"] & R_n^- \arrow[r] \arrow[d, "(T_n^-)^{\mathrm{neg}}"] & 0 \\
0 \arrow[r] & R_{n-1}^+ \arrow[r] \arrow[d, "pr_{n-1}^+"] & R_{n-1} \arrow[r] \arrow[d, "pr_{n-1}"] & R_{n-1}^- \arrow[r] \arrow[d, "pr_{n-1}^-"] & 0 \\
0 \arrow[r] & R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-2}^+} R_{n-1}^+ \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & R_2^-/R_1^- \cdots \oplus_{R_{n-2}^-} R_{n-1}^- \arrow[r] & 0.
\end{tikzcd}$$

Exactly as for the $n = 1$ case, it is easy to deduce a commutative $K_0(p)$-equivariant diagram



Again, the map f_2 is *a priori* just a $\overline{\mathbf{F}}_p$-morphism, and the $K_0(p)$-equivariance is immediately deduced using the surjectivity of pr_{n+1}; moreover all lines and rows are exact, except for

$$\cdots \oplus_{R_n^+} R_{n+1}^+ \xrightarrow{f_1} \cdots \oplus_{R_n} R_{n+1} \xrightarrow{f_2} \cdots \oplus_{R_n^-} R_{n+1}^-.$$

Exactly as for the $n = 1$ case we see that $f_2 \circ f_1 = 0$, that f_1 (resp. f_2) is a monomorphism (resp. epimorphism) and that $\mathrm{Im}(f_1) = \ker(f_2)$. □

With a similar "formal" argument, we deduce another result in the flavour of proposition 3.6

PROPOSITION 3.7. *The decomposition $R_n|_{K_0(p)} \cong R_n^+ \oplus R_n^-$ induces the following $K_0(p)$-equivariant exact sequences:*

$$0 \to \langle (F_{\emptyset}^{(0,-1)}(\underline{0}), F_{\emptyset}^{(1,-1)}(\emptyset))\rangle_{\overline{\mathbf{F}}_p} \to (\varinjlim_{n\,\mathrm{odd}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n\,\mathrm{odd}} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \to$$
$$\to (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0$$

and

$$0 \to \langle (F_{\underline{r}}^{(0)}(\underline{0}), F_{\emptyset}^{(1,0)}(\underline{0}))\rangle_{\overline{\mathbf{F}}_p} \to (\varinjlim_{n\,\mathrm{even}} (R_1/R_0)^+ \oplus_{R_2^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (\varinjlim_{n\,\mathrm{even}} R_1^- \oplus_{R_2^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \to$$
$$\to (\varinjlim_{n\,\mathrm{even}} (R_1/R_0) \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0.$$

Proof. Again, we prove the statement concerning the first exact sequence, leaving the other to the reader; the proof is similar to the proof of proposition 3.6. By the exactness of $\varinjlim_{n,\,\mathrm{odd}}$ (and commutativity with the forgetful functor) the statement is proved once we have shown that we have an exact sequence

$$0 \to \langle (F_{\emptyset}^{(0,-1)}(\underline{0}), F_{\emptyset}^{(1,-1)}(\emptyset))\rangle_{\overline{\mathbf{F}}_p} \to (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R_{n+1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-) \to$$
$$\to (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \to 0.$$

The proof is again an induction on n. Let $\bullet \in \{+, -\}$. By the universal property of the push out we deduce the following commutative diagramm

$$\begin{tikzcd}
0 \arrow[r] & R_1^\bullet \arrow[r] \arrow[d] \arrow[dd, two heads] & R_2^\bullet \arrow[r] \arrow[d] \arrow[dd, two heads] & R_2^\bullet/R_1^\bullet \arrow[r] \arrow[d] \arrow[dd, equal] & 0 \\
0 \arrow[r] & R_1 \arrow[r] \arrow[dd, two heads] & R_2 \arrow[r] \arrow[dd] & R_2/R_1 \arrow[r] \arrow[dd, equal] & 0 \\
0 \arrow[r] & R_0^\bullet \arrow[r] \arrow[d, hook] & R_0^\bullet \oplus_{R_1^\bullet} R_2^\bullet \arrow[r] \arrow[d, dotted, "f_\bullet"', "\exists!"] & R_2^\bullet/R_1^\bullet \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & R_0 \arrow[r] & (R_0 \oplus_{R_1} R_2)|_{K_0(p)} \arrow[r] & R_2/R_1 \arrow[r] & 0
\end{tikzcd}$$

from wich we deduce the commutative diagramm with exact lines

$$\begin{tikzcd}
0 \arrow[r] & R_0^+ \oplus R_0^- \arrow[r] \arrow[d] & (R_0^+ \oplus_{R_1^+} R_2^+) \oplus (R_0^- \oplus_{R_1^-} R_2^-) \arrow[r] \arrow[d] & (R_2^+/R_1^+) \oplus (R_2^-/R_1^-) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R_0 \arrow[r] & (R_0 \oplus_{R_1} R_2)|_{K_0(p)} \arrow[r] & R_2/R_1 \arrow[r] & 0.
\end{tikzcd} \tag{32}$$

As we have an isomorphism $(R_2^+/R_1^+) \oplus (R_2^-/R_1^-) \xrightarrow{\sim} R_2/R_1$ and an exact sequence

$$0 \to \langle (F_{\emptyset}^{(0,-1)}(\underline{0}), -F_{\emptyset}^{(1,-1)}(\emptyset))\rangle \to R_0^+ \oplus R_0^- \to R_0 \to 0$$

we get the result by applying the snake lemma to the diagramm (32).

We treat now the inductive step. Then, by the inductive hypothesis and the definition od the "Hecke" operators $(T_n^{\pm})^{\mathrm{pos,neg}}$, we dispose of the commutative diagrams

$$\begin{tikzcd}
R_n^{\bullet} \arrow[r, hook] \arrow[d, two heads] & R_n \arrow[d, two heads] \\
R_{n-1}^{\bullet} \arrow[r, hook] \arrow[d] & R_{n-1} \arrow[d] \\
R_0^{\bullet} \oplus_{R_1^{\bullet}} \cdots \oplus_{R_{n-2}^{\bullet}} R_{n-1}^{\bullet} \arrow[r, hook] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1}
\end{tikzcd}$$

from which we deduce the commutative diagram (with exact rows)

$$\begin{tikzcd}
0 \arrow[rr] & & R_n^{\bullet} \arrow[rr] \arrow[dl] \arrow[dd] & & R_{n+1}^{\bullet} \arrow[rr] \arrow[dl] \arrow[dd] & & R_{n+1}^{\bullet}/R_n^{\bullet} \arrow[r] \arrow[dl] \arrow[dd, equal] & 0 \\
0 \arrow[r] & R_n \arrow[rr] \arrow[dd] & & R_{n+1} \arrow[rr] \arrow[dd] & & R_{n+1}/R_n \arrow[rr] \arrow[dd, equal] & & 0 \\
0 \arrow[rr] & & \cdots \oplus_{R_{n-2}^{\bullet}} R_{n-1}^{\bullet} \arrow[rr] \arrow[dl, hook] & & \cdots \oplus_{R_n^{\bullet}} R_{n+1}^{\bullet} \arrow[rr] \arrow[dl, dotted, "f_{\bullet}", "\exists!"'] & & R_{n+1}^{\bullet}/R_n^{\bullet} \arrow[r] \arrow[dl, hook] & 0 \\
0 \arrow[r] & (\cdots \oplus_{R_{n-2}} R_{n-1})|_{K_0(p)} \arrow[rr] & & (\cdots \oplus_{R_{n2}} R_{n+1})|_{K_0(p)} \arrow[rr] & & R_{n+1}/R_n \arrow[rr] & & 0
\end{tikzcd}$$

which lead us to the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
(R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-1}^+} R_{n-1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-1}^-} R_{n-1}^-) \arrow[r] \arrow[d] & (R_0 \oplus_{R_1} \cdots \oplus_{R_{n-1}} R_{n-1})|_{K_0(p)} \arrow[d] \\
(R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n+1}^+} R_{n+1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n+1}^-} R_{n+1}^-) \arrow[r] \arrow[d] & (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})|_{K_0(p)} \arrow[d] \\
(R_{n+1}^+/R_n^+) \oplus (R_{n+1}^-/R_n^-) \arrow[r] \arrow[d] & R_{n+1}/R_n \arrow[d] \\
0. & 0
\end{tikzcd}$$

As the natural morphism $(R_{n+1}^+/R_n^+) \oplus (R_{n+1}^-/R_n^-) \to R_{n+1}/R_n$ is an isomorphism, the conclusion follows by applying the snake lemma and using the exact sequence

$$0 \to \langle (F_{\emptyset}^{(0,-1)}(\underline{0}), F_{\emptyset}^{(1,-1)}(\emptyset))\rangle_{\overline{\mathbf{F}}_p} \to (R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_{n-2}^+} R_{n-1}^+) \oplus (R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{n-2}^-} R_{n-1}^-) \to$$
$$\to (R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1})|_{K_0(p)} \to 0.$$

coming from the inductive hypothesis. $\square$

4. Representations of the Iwahori subgroups

We start here the technical computations which should lead us (in section §5) to the Iwahori-structure of the universal representations $\pi(\underline{r}, 0, 1)$. The aim is to describe the $K_0(p)$-representations $R_{n+1}^{\pm}$ which appeared in the preceeding section §3.

We focus our attention on the representations $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$: the description of $R_{n+1}^{\pm}$ can be obtained with identical techniques (cf. sections §4.1.3 or 4.2). The Iwahori structure of such objects -given by proposition 4.2- may look complicated, but the keypoint is its combinatoric can be controlled by an easy euclidean method which can be outlined as follow.

First of all we detect a “canonical” $\overline{\mathbf{F}}_p$-basis $\mathscr{B}$ for the representation $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$ (definition 4.1). We see that each element $F_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)} \in \mathscr{B}$ is parametrized by a family of f-tuples $\underline{l}_i \in \{0,\dots,p-1\}^f$, family which can be used to define a point (in the naïve sense) $(x_0,\dots,x_{f-1}) \in \mathbf{R}^{f-1}$. In this way, we can associate, bijectively, the elements of the basis $\mathscr{B}$ to the integer points of an f-hypercube of side $p^n - 1$ in $\mathbf{R}^{f-1}$: this is detailed in paragraph 4.1.1.

With this gloss, the $K_0(p)$-socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} 1$ can be simply described by the successive intersections of the f-hypercube with the antidiagonals $X_0 + \cdots + X_{f-1} = \mathit{constant}$, as illustrated in figure IV.3.

This is the content of proposition 4.2 where we verify, by direct computation on Witt vectors, that the behaviour of the canonical elements $F_{\underline{l}_1,\dots,\underline{l}_{f-1}}^{(1,n)}$ fits the previous euclidean picture. It is the technical part of the chapter and rely, as announced in the introduction, on the following three key facts (whose meaning will be clear to the reader of paragraph §4.1.2):

i) the elements of the canonical basis $\mathscr{B}$ are “well behaved” with respect to the action of $g \in K_0(p)$, i.e. one can naturally describe $gF_{\underline{l}_1,\dots,\underline{l}_{f-1}}^{(1,n)}$ as a linear combination of elements of $\mathscr{B}$;

ii) one can compute the homogeneous (pseudo-)degree of the universal Witt polynomials appearing in the developement of $gF_{\underline{l}_1,\dots,\underline{l}_{f-1}}^{(1,n)}$;

iii) the correspondence between the elements of $\mathscr{B}$ and the points in the associated hypercube is well behaved with respect to the homogeneous degree of the universal Witt polynomials.

As annonced the same techniques let us detect the $K_0(p)$-structure for the representations $R_{n+1}^{\pm}$: the involved combinatoric can be handled with the help of a simple euclidean picture (an f-parallelepipoid). The precise statements are propositions 4.9 and 4.10 which deal with R_{n+1}^{-} and R_{n+1}^{+} respectively.

The constructions and computations of this section let us, as an application, determine the Iwahori structure for principal and special series: this is the object of §4.3. Again, in terms of euclidean space, we see that the successive layers for the $K_0(p)$-socle filtration are detected

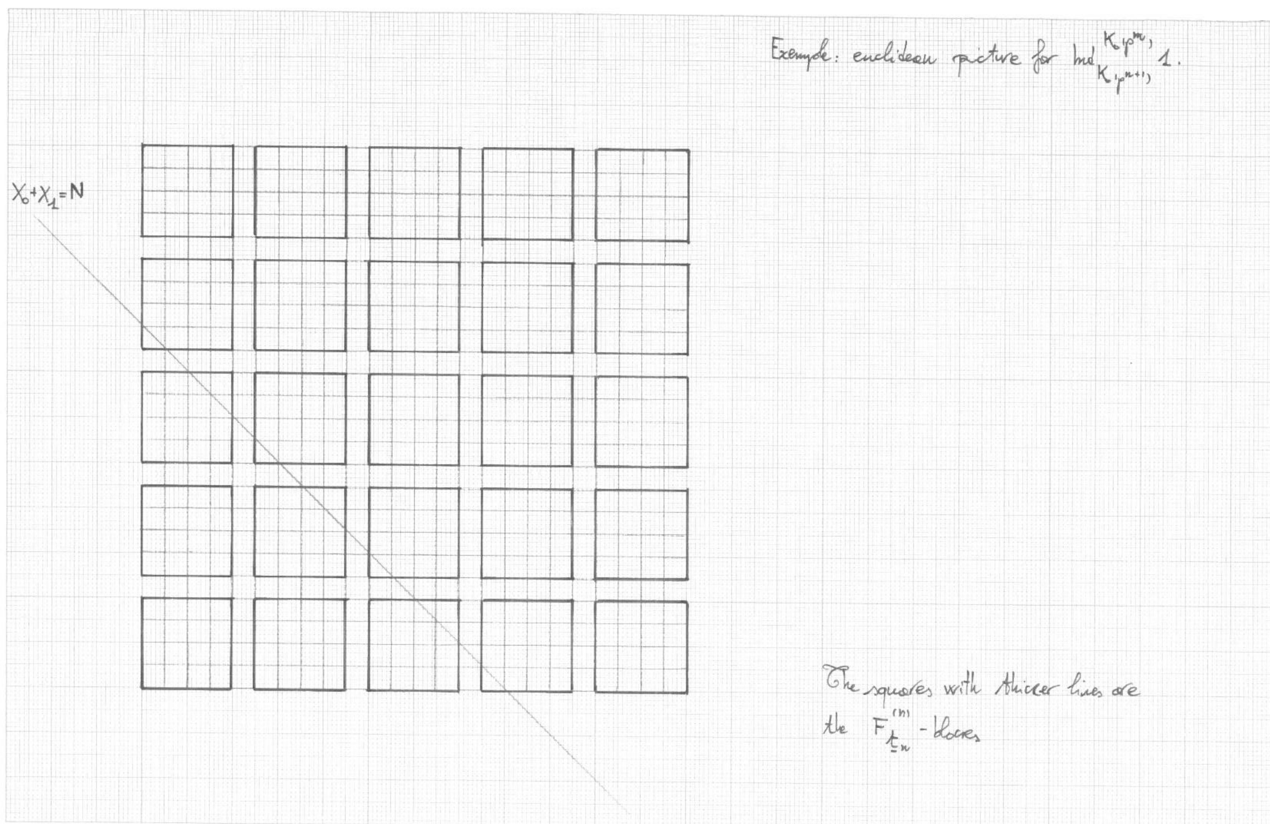


FIGURE IV.3. The structure of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$.

by the intersections of $\mathbf{N}^f$ (the "hypercube" associated to such series) with the hyperplans $X_0 + \cdots + X_{f-1} = constant$.

4.1 The negative case.

Let $1 \leqslant m \leqslant n$ be integers. In this section we examine the $K_0(p)$-socle filtration (and the extensions between two consecutive graded pieces) for the representations $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi$ where $\chi : K_0(p^{n+1}) \to \overline{\mathbf{F}}_p^{\times}$ is a smooth character of $K_0(p^{n+1})$ (i.e. the inflation of a character of the finite Borel $B(\mathbf{F}_q)$ by the morphism $K_0(p^{n+1}) \twoheadrightarrow B(\mathbf{F}_q)$). Thanks to the canonical isomorphism :

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi \cong (\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1) \otimes \chi$$

we can assume that $\chi = 1$ is the trivial character. Finally, let $\{e\}$ be an $\overline{\mathbf{F}}_p$-basis for the underlying vector space associated to the character χ.

We introduce now the canonical base of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ and its interpretation in terms of lattices of $\mathbf{R}^f$.

Definition 4.1. *For $j \in \{m, \ldots, n\}$ let $\underline{l}_j = (l_j^{(0)}, \ldots, l_j^{(f-1)}) \in \{0, \ldots, p-1\}^f$ be a f-tuple. We define the element $F_{\underline{l}_m,\ldots,\underline{l}_n}^{(m,n)} \in \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ as*

$$F_{\underline{l}_m,\ldots,\underline{l}_n}^{(m,n)} \stackrel{\text{def}}{=} \sum_{j=m}^{n} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, e].$$

For a notational convenience, we define $F_{\underline{l}_{n+1},\ldots,\underline{l}_n}^{(n+1,n)} \stackrel{\text{def}}{=} [1, e]$ and $\underline{l}_{n+1} \stackrel{\text{def}}{=} \underline{0}$.

The set

$$\mathscr{B} \stackrel{\text{def}}{=} \left\{ F_{\underline{l}_m,\ldots,\underline{l}_n}^{(m,n)} \in \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1, \text{ for } (\underline{l}_m, \ldots, \underline{l}_n) \in \left\{\{0, \ldots, p-1\}^f\right\}^{n+1-m} \right\}$$

is an $\overline{\mathbf{F}}_p$-basis for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$.

The fact that $\mathscr{B}$ is an $\overline{\mathbf{F}}_p$ basis for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ is again an induction together with a Vandermonde argument as for lemma 2.6.

4.1.1 ***Interpretation in terms of lattices.*** As anticipated in the introduction, each element of $\mathscr{B}$ can be seen as a "point" of a $\mathbf{Z}$-lattice in the standard euclidean f-dimensional space $\mathbf{R}^f$: such correspondence is given by the injective map

$$\begin{aligned} \mathscr{B} &\stackrel{P}{\hookrightarrow} \mathbf{R}^f \\ F_{\underline{l}_m,\ldots,\underline{l}_n}^{m,n} &\mapsto \Big(\sum_{j=m}^{n} p^{j-m} l_j^{(\lfloor j-m \rfloor)}, \ldots, \sum_{j=m}^{n} p^{j-m} l_j^{(\lfloor f-1+j-m \rfloor)} \Big) \end{aligned} \tag{33}$$

whose image will be denoted by $\mathscr{R}$. We notice that $\mathscr{R}$ is a f-hypercube of side $p^{n-m+1} - 1$. It has a natural recurrent structure: for a fixed f-tuple $\underline{t}_n \in \{0, \ldots, p-1\}^f$ the subset

$$\big\{ F_{\underline{l}_m,\ldots,\underline{l}_{n-1},\underline{t}_n}^{(m,n)} \in \mathscr{B} \quad \underline{l}_j \in \{0, \ldots, p-1\}^f, \text{ for } m \leqslant j \leqslant n-1 \big\}$$

is mapped onto an f-hypercube of side $p^{n-m}-1$, which will be referred as the $F^{(n)}_{\underline{t}_n}$-*block*. The hypercube $\mathscr{R}$ is then obtained as the juxtaposition of the $F^{(n)}_{\underline{t}_n}$-blocks for varying $\underline{t}_n \in \{0,\dots,p-1\}^f$.

We are therefore allowed to apply the terminology of real euclidean spaces to the elements of $\mathscr{B}$, meaning their image through the map P. In particular if $e_i \stackrel{\text{def}}{=} (\delta_{0,i},\dots,\delta_{f-1,i}) \in \{0,1\}^f$ we define $F^{m,n}_{(\underline{l}_m,\dots,\underline{l}_n)-e_i}$ by

$$F^{m,n}_{(\underline{l}_m,\dots,\underline{l}_n)-e_i} = \begin{cases} 0 \quad \text{if } P^{\leftarrow}(P(F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}) - e_i) = \emptyset \\ \text{the only element of } P^{\leftarrow}(P(F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}) - e_i) \text{ otherwise.} \end{cases}$$

In order to give the statement concerning the $K_0(p^m)$-structure of $\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}\chi$ we still need some notation. If $(\underline{l}_m,\dots,\underline{l}_n)$ is a $(n+1-m)f$-tuple, we define

$$N_{m,n}(\underline{l}_m,\dots,\underline{l}_n) \stackrel{\text{def}}{=} \sum_{s=0}^{f-1} l^{(s)}_m + p(\sum_{s=0}^{f-1} l^{(s)}_{m+1}) + \dots + p^{n-m}(\sum_{s=0}^{f-1} l^{(s)}_n)$$
$$e(\underline{l}_m,\dots,\underline{l}_n) \stackrel{\text{def}}{=} (\sum_{s=0}^{f-1} p^s l^{(s)}_m) + \dots + (\sum_{s=0}^{f-1} p^s l^{(s)}_n);$$

in particular any $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ lies on the antidiagonal $X_0+\dots+X_{f-1} = N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$.

Let $N \in \mathbf{N}$. We define the $\overline{\mathbf{F}}_p$-linear subspace

$$(\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)_N \stackrel{\text{def}}{=} \langle F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B} \quad \text{s.t.} \quad N_{m,n}(\underline{l}_m,\dots,\underline{l}_n) < N\rangle_{\overline{\mathbf{F}}_p};$$

it is the subspace generated by the functions lying strictly below the antidiagonal $X_0+\dots X_{f-1} = N$.

We refer the reader to figure IV.3 to have the euclidean interpretation in the case $f=2$.

Let $(\underline{l}_m,\dots,\underline{l}_n)$ a fixed -tuple. For $s \in \{0,\dots,f-1\}$, we define

$$\Xi_s \stackrel{\text{def}}{=} \big\{a \in \{m,\dots,n\}, \quad \text{s.t. } l_a^{\lfloor s+a-m\rfloor} \neq 0\big\}$$

and we set

$$a_0(s) \stackrel{\text{def}}{=} \begin{cases} \min(\Xi_s) & \text{if } \Xi_s \neq \emptyset \\ n+1 & \text{otherwise.} \end{cases}$$

The euclidean meaning of $a_0(s)$ is clear: if we consider the $F^{(a_0(s),n)}_{\underline{l}_{a_0(s)},\dots,\underline{l}_n}$-block then the function $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ lies on its s-th face (which is a $(f-1)$-hypercube of side $p^{a_0(s)-m}-1$).

The $K_0(p^m)$-structure of $\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}\chi$ is then given by the following

Proposition 4.2. *Let* $\underline{r} \stackrel{\text{def}}{=} (r_0,\dots,r_{f-1}) \in \{0,\dots,p-1\}^{f-1}$ *be a* $f-tuple$*,* m,n *be integers such that* $1 \leqslant m \leqslant n$ *and let* $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}\chi^s_{\underline{r}}$ *be as in definition 4.1. If* $a,b,c,d \in \mathscr{O}_F$ *are* *integers such that* $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ p^m c & d \end{bmatrix} \in K_0(p^m)$ *we have*

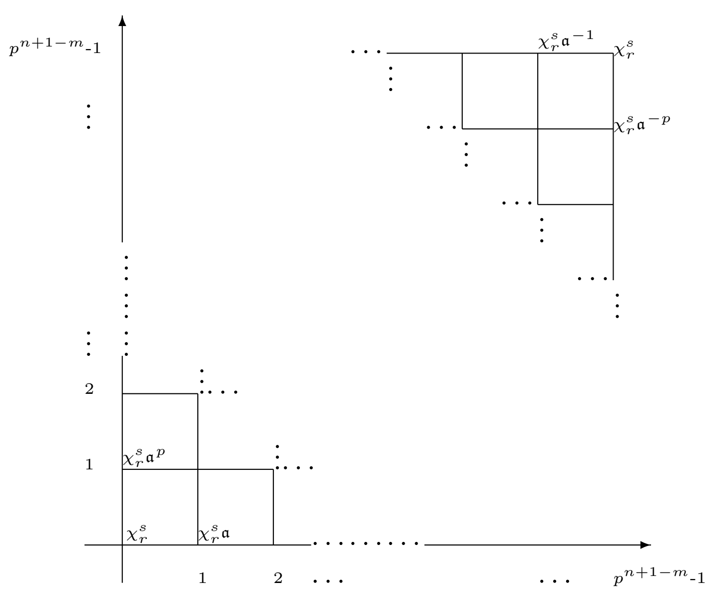
$$gF^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} = \mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)}\chi^s_{\underline{r}}(g)(F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} - \sum_{s=0}^{f-1}(\overline{ca}^{-1})^{p^s} l^{\lfloor s+a_0(s)-m\rfloor}_{a_0(s)} F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n - e_s} + y)$$

where, putting $N \stackrel{\text{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$, *we have* $y \in (\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}\chi^s_{\underline{r}})_{N-1}$.

In particular, the $K_0(p)$*-socle filtration, as well as the extensions between two consecutive graded pieces, of* $\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}\chi^s_{\underline{r}}$ *is described by the associated lattice* $\mathscr{R}$*.*

We emphatise again the meaning of proposition 4.2 in terms of lattices in $\mathbf{R}^f$: the socle filtration of $\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}\chi$ is given by cutting up the hypercube $\mathscr{R}$ by the antidiagonals $X_0 + \dots + X_{f-1} = N$ (precisely, soc_N is obtained by cutting the antidiagonal $X_0 + \dots + X_{f-1} = N$); the extensions between two consecutive graded pieces are visualized by the segments of length 1 obtained by cutting $\mathscr{R}$ by two consecutive antidiagonals $X_0+\dots+X_{f-1} = N$, $X_0+\dots+X_{f-1} = N-1$.

Here below an exemple for $f = 2$.



Here, each "point" in the lattice corresponds to a function $F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$ according to the map P described in (33). The N-th composition factor $\mathrm{soc}_N(\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)$ of the socle filtration can be read as the intersection of R with the semispace $X_0 + \dots + X_{f-1} \leqslant N$, and the N-th graded piece $\mathrm{soc}_N(\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)/\mathrm{soc}_{N-1}(\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)$ as the intersection with the antidiagonal $X_0 + \dots + X_{f-1} = N$. Finally, a "point" of coordinates $(\sum_{j=m}^n p^{j-m} l_j^{(\lfloor j-m\rfloor)}, \sum_{j=m}^n p^{j-m} l_j^{(\lfloor 1+j-m\rfloor)})$ should be understood as the character $\chi^s_{\underline{r}}\mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)}$.

4.1.2 ***Proof of proposition 4.2.*** The section is devoted to the proof of proposition 4.2. Thanks to the decomposition

$$K_0(p^m) = H \cdot \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix} \qquad (34)$$

for $m \geqslant 1$ we are led to study separately the actions of lower unipotent, diagonal and upper unipotent matrices on the elements of the canonical basis $\mathscr{B}$: this will be the object of the next three paragraphs.

The action of lower unipotents matrices. We study here the action of the closed subgroup $\begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix}$ of $K_0(p^m)$ on $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$; we first need to introduce a family of $\overline{\mathbf{F}}_p$-subspaces of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$.

Let $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$ and set $(x_0,\dots,x_{f-1}) \overset{\text{def}}{=} P(F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}) \in \mathscr{R}$. We define the $\overline{\mathbf{F}}_p$-subspace $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ via

$$P(\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}) \overset{\text{def}}{=} \{(x'_0,\dots,x'_{f-1}) \in \mathscr{R} \quad \text{s.t. it exists}\, n \geqslant 0\, \text{for which}$$
$$n(p-1) \leqslant \sum_{s=0}^{f-1}(x_s - x'_s) < (n+1)(p-1)\, \text{and}\, x'_j \leqslant x_j + n\, \text{for all}\, j = 0,\dots,f-1\}.$$

The image $P(\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}) \subseteq \mathbf{R}^f$ looks as a snowflake: in figure IV.4 an exemple for $f = 2$ (and $p = 5$).

It is immediate to check that if $F^{(m,n)}_{\underline{l}'_m,\dots,\underline{l}'_n} \in \mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ then $\mathfrak{W}_{(\underline{l}'_m,\dots,\underline{l}'_n)} \subseteq \mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$. The action of $\begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix}$ is then described in the following

Proposition 4.3. *Let $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$, and write $N \overset{\text{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$. Let $g = \begin{bmatrix} 1 & 0 \\ p^m c & 1 \end{bmatrix} \in \begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F & 1 \end{bmatrix}$ for $c \in \mathscr{O}_F$. Then we have*

$$g \cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} = F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} - \sum_{s=0}^{f-1} \overline{c}^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)-m \rfloor} F^{(m,n)}_{(\underline{l}_m,\dots,\underline{l}_n)-e_s} + y$$

for a suitable $y \in (\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)_{N-1}$. More precisely, via the projection

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1 \overset{pr}{\twoheadrightarrow} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1/(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1)_{N-(p^f+2)},$$

the image of the element y is contained in the image of the subspace $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$.

Proof. As the action of $\begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$ is continuous, we can assume that c belongs to a set of topological generators (for the additive structure) of $\mathscr{O}_F$; in particular, we can assume $c = [\mu^{\frac{1}{p^m}}]$ for $\mu \in \mathbf{F}_q$.

Using the notations of §6.2, we can write the following equality in $p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$:

$$p^m[\mu] + \sum_{j=m}^{n} p^j[\lambda_j^{\frac{1}{p^j}}] = \sum_{j=m}^{n} p^j[\lambda_j^{\frac{1}{p^j}} + (\widetilde{S}_{j-m}^{\frac{1}{p^j}})] \tag{35}$$

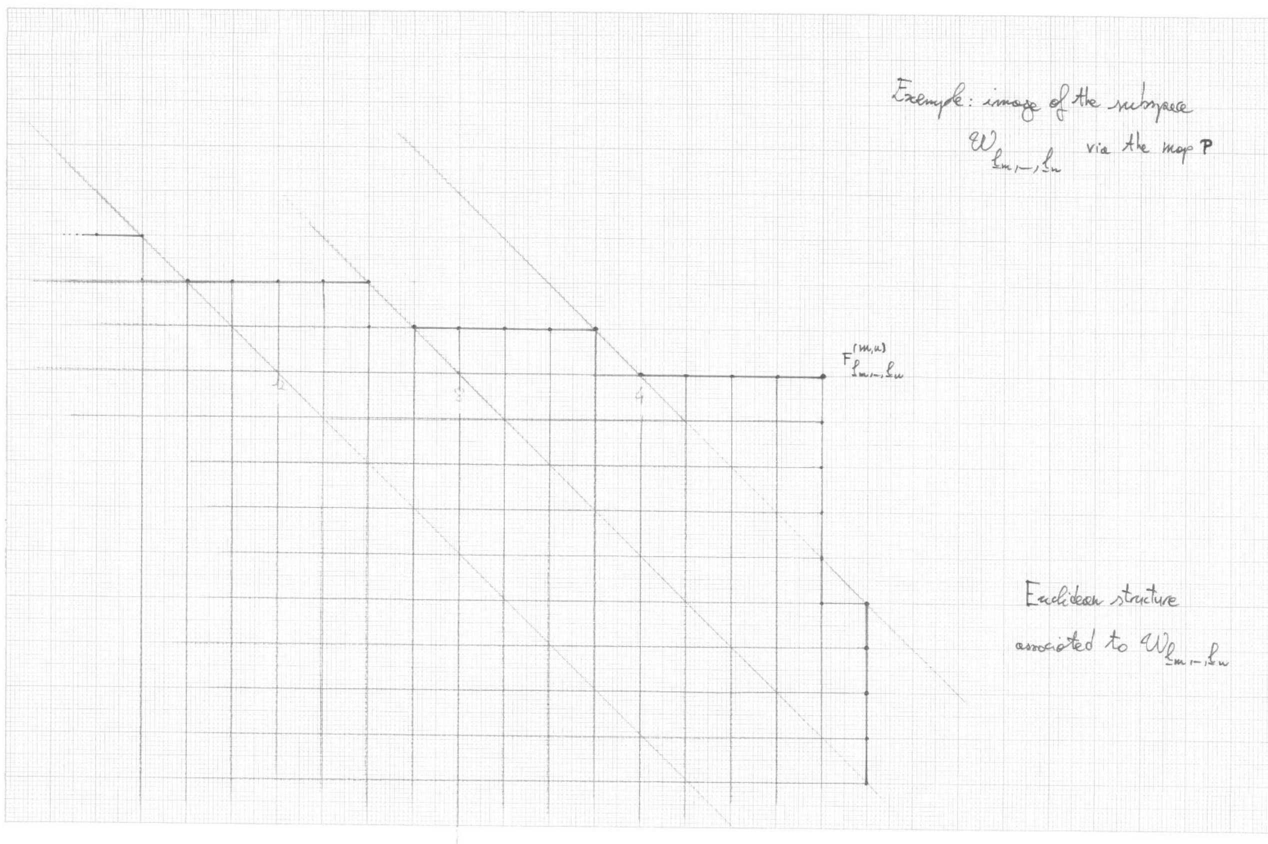


FIGURE IV.4. Euclidean interpretation of $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$.

A direct computation describes the action of g on the function $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$:

$$\begin{bmatrix} 1 & 0 \\ p^m[\mu] & 1 \end{bmatrix} F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} =$$
$$= \sum_{j=m}^{n-1} \sum_{\underline{i}_j \leqslant \underline{l}_j} \binom{\underline{l}_j}{\underline{i}_j} (-s_0(\widetilde{S}_0)^{\underline{i}_m}) \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (-s_j(\widetilde{S}_{j-m+1})^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, F^n_{\underline{l}_n - \underline{i}_n}].$$

As $deg(s_{j-1}(\widetilde{S}_j)) \leqslant p^j$ for each $j \in \{1, \dots, n-m\}$ we can apply proposition 7.3 (with $T_{m+j} = s_{j-1}(\widetilde{S}_j)$) to conclude that

$$g \cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} = F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} + \sum_{s=0}^{f-1} \beta_s F^{(m,n)}_{(\underline{l}_m,\dots,\underline{l}_n)-e_s} + y$$

where $y \in \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ is the element described in the statement, for suitable elements $\beta_s \in \mathbf{F}_q$. We are now left to prove that $\beta_s = -(\mu^{\frac{1}{p^m}})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)-m \rfloor}$.

We use the notations of proposition 7.3 and we recall that, for $b = m+1, \dots, n$, a polynomial $-s_{b-m-1}(\widetilde{S}_{b-m}(\underline{X}, Y))$ is homogeneous of degree p^{b-m} if X_a has degree p^a, Y degree p^0 (and $\widetilde{S}_0 = Y$). In particular if we pick an element

$$x \stackrel{\text{def}}{=} \sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m} \begin{bmatrix} 1 & 0 \\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e]$$

appearing in the development of $gF^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$ we have, for $b \in \{m+1, \dots, n\}$,

$$\sum_{a=m}^{b-1} p^{a-m} \kappa_a^{(b),s} = i_b^{(s)} p^{b-m} - \alpha_b^{(s)}$$

where $i_b^{(s)}(p^{b-m}-1) \geqslant \alpha_b^{(s)} \geqslant i_b^{(s)}$ is the exponent of Y in the fixed monomial of $-s_{b-1-m}(\widetilde{S}_{b-m})^{i_b^{(s)}}$ (recall that any monomial $Y^c \prod_{i=0}^{b-1-m} X_i^{a_i}$ with $c = 0$ appears in the development of $-s_{b-1-m}(\widetilde{S}_{b-m})$

with coefficient zero). Considering that $p \geqslant 3$ the inequalities

$$\mathfrak{s}(\kappa_m) + p\,\mathfrak{s}(\kappa_{m+1}) + \cdots + p^{n-m}\,\mathfrak{s}(\kappa_n) \leqslant$$

$$\begin{aligned}\leqslant&(\mathfrak{s}(\underline{l}_m - \underline{i}_m) + \mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_m^{(m+1)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m)\rfloor}\kappa_m^{(n)})) + \\ &+p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1}) + \mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_{m+1}^{(m+2)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m-1)\rfloor}\kappa_{m+1}^{(n)})) + \ldots \\ &\cdots + p^{n-m-1}(\mathfrak{s}(\underline{l}_{n-1} - \underline{i}_{n-1}) + \mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_{n-1}^{(n)})) + p^{n-m}(\mathfrak{s}(\underline{l}_n - \underline{i}_n)) \leqslant\end{aligned}$$

$$\begin{aligned}&\leqslant \mathfrak{s}(\underline{l}_m - \underline{i}_m) + \sum_{s=0}^{f-1}\mathfrak{s}(\kappa_m^{(m+1),s}) + \\ &p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m-1})) + (\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(m+2),s} + p\,\mathfrak{s}(\kappa_{m+1}^{(m+2),s})))) + \ldots \\ &\cdots + (\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(n),s}) + p\,\mathfrak{s}(\kappa_{m+1}^{(n),s}) + \cdots + p^{n-m-1}\,\mathfrak{s}(\kappa_{n-1}^{(n),s}))) + p^{n-m}\,\mathfrak{s}(\underline{l}_n - \underline{i}_n) \leqslant\end{aligned}$$

$$\leqslant \sum_{a=m}^{n} p^{a-m}(\mathfrak{s}(\underline{l}_a - \underline{i}_a)) + \sum_{b=m+1}^{n}(p^{b-m}(\mathfrak{s}(\underline{i}_b)) - \sum_{s=0}^{f-1}\alpha_b^{(s)})$$

have to be equalities if we furthermore require our element to lie on the hyperplane $X_0 + \cdots + X_{f-1} = N-1$; in particular we must have $i_b^{(s)} = 0$ for all couples $(b, s) \in \{m, \ldots, n\} \times \{0, \ldots, f-1\}$ except one and only one, say (b_0, s_0), for which we must have $i_{b_0}^{(s_0)} = 1$.

We notice that for $b_0 \neq m$ we require furthermore that $\alpha_{b_0} = 1$ i.e. the exponent of Y appearing in the fixed monomial of $-s_{b_0-m-1}(\widetilde{S}_{b_0-m})$ is 1. Thanks to lemmas 6.3 and 6.4 we check that

$$x = -(\mu^{\frac{1}{p^m}})^{p^{s_0}}(l_{a_0(s)}^{\lfloor s+a_0(s)-m\rfloor})F^{(m,n)}_{\underline{l}_m,\ldots,\underline{n}-e_{s_0}}$$

as required. □

The action of diagonal matrices. We are going to study the action of the subgroup

$$\begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix}$$

on the elements of $\mathscr{B}$. If $z \in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$, an elementary computation shows that

$$\begin{bmatrix} 1+pa & 0 \\ 0 & 1+pd \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix}\mathfrak{k}$$

where $\mathfrak{k} \in K_0(p^{n+1})$ is upper unipotent modulo p and $z' \in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ is determined by the condition

$$z' \equiv (1+pa)^{-1}(1+pd)z \bmod p^{n+1}. \tag{36}$$

We can therefore content ourself studying the action of an element of the form $x \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1+p\alpha \end{bmatrix}$ for $\alpha \in \mathscr{O}_F$.

PROPOSITION 4.4. *Let* $g \in \begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix}$ *and fix* $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$*; write* $N \overset{\mathrm{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$*.*

We then have the equality

$$g \cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} = F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} + y$$

where $y \in \mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)_{N-1}$*.*

More precisely, via the projection

$$\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1 \overset{pr}{\twoheadrightarrow} \mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1/(\mathrm{Ind}^{K_0(p^m)}_{K_0(p^{n+1})}1)_{N-(p^f+2)},$$

the image of y *is contained in the image of the subspace* $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ *and writing*

$$y = \sum_{i\in I}\beta_i F^{(m,n)}_{\underline{l}_m(i),\dots,\underline{l}_n(i)}$$

(for a suitable set of indexes I *and scalars* $\beta_i \in \overline{\mathbf{F}}_p^\times$*) we have that each function* $F^{(m,n)}_{\underline{l}_m(i),\dots,\underline{l}_n(i)}$ *which is not in the kernel* $\ker(pr)$ *lies on an hyperplane*

$$X_0 + \dots + X_{f-1} = N - t(p-1)$$

for some $t \in \mathbf{N}_>$*.*

Proof. The proof is completely analogous to the proof of proposition 4.3. As remarked above, it is enough to consider the case $x = \begin{bmatrix} 1 & 0 \\ 0 & 1+p\alpha \end{bmatrix}$ where $\alpha = \sum_{j=0}^\infty p^j[\alpha_j^{\frac{1}{p^j}}]$. Using the notations of §6.3 we see that

$$(1+p\alpha)(\sum_{j=m}^{n} p^j[\lambda_m^{\frac{1}{p^j}}]) \equiv \sum_{j=m}^{n} p^j[\lambda_j^{\frac{1}{p^j}} + \widetilde{Q}_j^{\frac{1}{p^j}}] \bmod p^{n+1}$$

and we deduce

$$\begin{aligned}&\begin{bmatrix} 1 & 0 \\ 0 & 1+p\alpha \end{bmatrix} F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} = \\ &= \sum_{j=m}^{n-1}\sum_{\underline{i}_j\leqslant\underline{l}_j}\binom{\underline{l}_j}{\underline{i}_j}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(-q_{j-m}(\widetilde{Q}_{j+1-m}))^{\underline{i}_{j+1}}\begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix}[1, F^{(n)}_{\underline{l}_n-\underline{i_n}}]\end{aligned} \tag{37}$$

(where we convene that $\underline{i}_m = \underline{0}$ and with the obvious conventions if $n \in \{m, m+1\}$). As each polynomial $(-q_{j-1}(\widetilde{Q}_j)) \in \mathbf{F}_p[\lambda_m,\dots,\lambda_{j-1-m}]$, for $1 \leqslant j \leqslant n-m$ is homogeneous of degree p^j (for the shifted grading for which λ_{m+h} is homogeneous of degree p^h for $h \geqslant 0$) we can apply proposition 7.3 with $T_{m+j} = (-q_{j-1}(\widetilde{Q}_j))$ to get the first part of the statement.

We are left to prove 2). Consider an integer $t \in \mathbf{N}$ and an hyperplane $\mathfrak{H} : X_0 + \dots X_{f-1} = N - t$. Following the proof of proposition 7.3, a necessary condition for an element

$$\sum_{\lambda_m\in\mathbf{F}_q}(\lambda_m^{\frac{1}{p^m}})^{\kappa_m}\begin{bmatrix} 1 & 0 \\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix}\dots\sum_{\lambda_n\in\mathbf{F}_q}(\lambda_n^{\frac{1}{p^n}})^{\kappa_n}\begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix}[1, e]$$

appearing in the developement of (37) to lie in $\mathfrak{H}$ is then

$$\sum_{j=m}^{n} p^{j-m}\,\mathfrak{s}(\kappa_j) \equiv N - t \bmod p-1.$$

Again, as each polynomial $(-q_{j-1}(\widetilde{Q}_j))$, for $1 \leqslant j \leqslant n-m$ is homogeneous of degree p^j, and $\mathfrak{s}(h) \equiv h \bmod p-1$ we deduce that inequalities 50, 51, 52 and 53 appearing in the proof of proposition 7.3 are actually *equalities* in $\mathbf{Z}/(p-1)$ so that we get

$$\sum_{j=m}^{n} p^{j-m}\,\mathfrak{s}(\kappa_j) \equiv N - \mathfrak{s}(\underline{i_m}) \bmod p-1 = N.$$

The conclusion follows. □

The action of upper unipotent matrices. We are left to study the action of the closed subgroup $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ on the elements of $\mathscr{B}$. We recall that the action of $K_0(p^m)$ is continuous on $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ and the natural topology on $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ coincides with the topology induced (via the natural immersion) by $K_0(p^m)$. Thanks to the isomorphisms of abelian topological groups

$$\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix} \cong \mathscr{O}_F \cong (\mathbf{Z}_p)^f$$

where the latter isomorphism is determined by the choice of a primitive element $\alpha \in \mathbf{F}_q$ of $\mathbf{F}_q$ over $\mathbf{F}_p$ (cf. Serre [Ser63], proposition 16 Ch.I) it is enough to study the action of elements $g \in \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ of the form $g = \begin{bmatrix} 1 & [x] \\ 0 & 1 \end{bmatrix}$ for $x \in \mathbf{F}_q$.

We start with an elementary computation:

LEMMA 4.5. *Let $z \in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ and $x \in \mathbf{F}_q$. We have the following equality:*

$$\begin{bmatrix} 1 & [x] \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix}\mathfrak{k}$$

where $\mathfrak{k} \in K_0(p^{n+1})$ is upper unipotent modulo p and $z' \in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ is uniquely determined by the condition

$$z' \equiv z(1+z[x])^{-1} \bmod p^{n+1} \equiv \sum_{j=0}^{N}(z^{j+1}[x^j]) \bmod p^{n+1}$$

for $N \overset{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$.

Proof. Omissis. □

We are now left to use lemma 4.5 and the results of §6.4 in order to describe the required action of $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$:

PROPOSITION 4.6. *Let $g \in \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ and fix $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$. Write* [5] $N \overset{\text{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$. *In the quotient space*

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1/(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1)_{N-(p^m-2)+1}$$

[5] of course, this N does not have anything to do with $N \overset{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$. We believe this conflict of notations will not give rise to any confusion, as the meaning of N will be clear from the context.

we have the equality

$$g \cdot F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} = F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}.$$

Proof. As remarked at the begining of this paragraph, we can assume $g = \begin{bmatrix} 1 & [x] \\ 0 & 1 \end{bmatrix}$ where $x \in \mathbf{F}_q$.

Using lemma 4.5 and the results (and notations) of §6.4.1 we get the following equality in $\mathscr{O}_F/(p^{n+1})$:

$$\sum_{j=0}^{N} z^{j+1}[x^j] \equiv \sum_{j=m}^{n} p^j[\lambda_j^{\frac{1}{p^j}} + \widetilde{U}_j^{\frac{1}{p^j}}] \operatorname{mod} p^{n+1}$$

so that, inside $\operatorname{Ind}^{K_0(p^m)}_{K_0(p^{n+1})} 1$, we have:

$$gF^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} = \sum_{j=m}^{n-1} \sum_{\underline{i}_j \leqslant \underline{l}_j} \binom{\underline{l}_j}{\underline{i}_j} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (-u_j(\widetilde{U}_{j+1}^{\frac{1}{p^{j+1}}}))^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, F^{(n)}_{\underline{l}_n - \underline{i}_n}]$$

where we convene that $\underline{i}_m = \underline{0}$ and we recall that $\widetilde{U}_j = 0$ for $m \leqslant j \leqslant 2m-1$ As for each $2m \leqslant j \leqslant n$ the polynomial $-u_{j-1}(\widetilde{U}_j)$ is pseudo-homogeneous of degree $p^j - p^m(p^m-2)$ the conclusion follows from proposition 7.4, with $V_j = -u_{j-1}(\widetilde{U}_j)$. □

Proof of proposition 4.2. The last step in order to complete the proof of proposition 4.2 is immediate:

Proposition 4.7. *Let $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$ and let $a, d \in \mathbf{F}_q$. We then have the following equality in $\operatorname{Ind}^{K_0(p^m)}_{K_0(p^{n+1})} 1$:*

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n} = \mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)}(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}) F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}.$$

In particular

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n - e_s} = \mathfrak{a}^{e(\underline{l}_m,\dots,\underline{l}_n)-p^s}(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}) F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n - e_s}.$$

Proof. We just remark that for $z = \sum_{j=m}^{n} p^j[\lambda_j] \in p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ we have

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z[a^{-1}d] & 1 \end{bmatrix} \begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}$$

and that

$$z[a^{-1}d] = \sum_{j=m}^{n} p^j[\lambda_j(a^{-1}d)].$$

□

Finally, for $a, b, c, d \in \mathscr{O}_F$ as in the statement of proposition 4.2, we recall the matrix equality

$$\begin{bmatrix} a & b \\ p^m c & d \end{bmatrix} = \begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p^m z & 1 \end{bmatrix} \begin{bmatrix} 1+px & 0 \\ 0 & 1+pw \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$$

where $x, y, z, w \in \mathscr{O}_F$ are suitable integers verifying $\overline{z} = \overline{cd^{-1}}$. The result follows now from propositions 4.3, 4.4, 4.6 and lemma 4.7. □

Remark 4.8. *We note that the bijection (33)* depends *on the immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ fixed in the introduction and should be noted as P_τ. As another immersion $\tau' : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ is obtained by composing τ with a power ϕ^a of the frobenius on $\mathbf{F}_q$ we see that the map $P_{\tau'}$ is obtained by composing P_τ with a power Φ^a, where $\Phi \in \mathrm{End}(\mathbf{R}^f)$ is defined by $\Phi(e_s) = e_{\lfloor s+1 \rfloor}$. Hence, as the antidiagonal is fixed under Φ, proposition 4.2 does not depend on τ.*

Another approach to the proof of proposition 4.2? As remarked by Paskunas, it is likely that proposition 4.2 admits an alternative proof, passing through Pontryagin duality. More precisely, for $k \in \mathbf{N}$ define

$$H_0(p^k) \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ p^k\mathscr{O}_F & 1 \end{bmatrix}, \qquad K_0'(p^k) \overset{\text{def}}{=} \begin{bmatrix} 1+p\mathscr{O}_F & \mathscr{O}_F \\ p^k\mathscr{O}_F & 1+p\mathscr{O}_F \end{bmatrix}.$$

Then, by Mackey's decomposition, we have

$$\mathrm{Ind}_{K_0(p^{p^{n+1}})}^{K_0(p)} 1|_{H_0(p)} \cong \mathrm{Ind}_{H_0(p^{n+1})}^{H_0(p)} 1$$

and, passing to the Pontryagin dual, we get the follwing profinite $\overline{\mathbf{F}}_p[[H_0(p)]]$-module:

$$(\mathrm{Ind}_{H_0(p^{n+1})}^{H_0(p)} 1)^\vee \cong \overline{\mathbf{F}}_p[[H_0(p)]] \widehat{\otimes}_{\overline{\mathbf{F}}_p[[H_0(p^{n+1})]]} 1.$$

As $H_0(p)/H_0(p^{n+1}) \cong \mathscr{O}_F/(p^n) \cong (\mathbf{Z}_p/(p^n))^f$ (isomorphism of additives abelian groups), the usual properties of completed tensor products give us

$$\begin{aligned} N_{n+1} \overset{\text{def}}{=} \overline{\mathbf{F}}_p[[H_0(p)]] \widehat{\otimes}_{\overline{\mathbf{F}}_p[[H_0(p^{n+1})]]} 1 \cong \overline{\mathbf{F}}_p[[H_0(p)/H_0(p^{n+1})]] &\cong (\overline{\mathbf{F}}_p[\mathbf{Z}_p/(p^n)])^{\otimes f} \\ &\cong \overline{\mathbf{F}}_p[X_0, \dots, X_{f-1}]/(X_0^{p^n}, \dots, X_{f-1}^{p^n}). \end{aligned}$$

The latter is an Artinian local ring with maximal ideal $\mathfrak{n} = (X_0, \dots, X_{f-1})$. We therefore see that the combinatoric of the radical filtration for N_{n+1} is described by an f-hypercube of side $p^n - 1$.

Consider now the $\overline{\mathbf{F}}_p[[K_0'(p)]]$-profinite module

$$M_{n+1} \overset{\text{def}}{=} \overline{\mathbf{F}}_p[[K_0'(p)]] \widehat{\otimes}_{\overline{\mathbf{F}}_p[[K_0'(p^{n+1})]]} 1.$$

We are tempted to show that the $\overline{\mathbf{F}}_p[[H_0]]$-restriction commutes with respect to formation of the radical filtration, i.e. the $\overline{\mathbf{F}}_p[[H_0]]$-radical filtration of the restriction $M_{n+1}|_{\overline{\mathbf{F}}_p[[H_0]]}$ coincide with the $\overline{\mathbf{F}}_p[[H_0]]$-restriction of the $\overline{\mathbf{F}}_p[[K_0'(p)]]$-radical filtration for M_{n+1} (indeed, $M_{n+1}|_{\overline{\mathbf{F}}_p[[H_0]]} = N_{n+1}$). As the maximal ideal $\mathfrak{m}$ of $\overline{\mathbf{F}}_p[[K_0'(p)]]$ (which is a Noetherian local ring) is topologically generated by the elements $(g-1)$ for $g \in K_0'(p)$, it would be sufficient to show that for any $g \in K_0'(p)$, $h \in H_0(p)$ we have

$$(g-1)(h-1) - (h-1)(g-1) \in \mathfrak{m}^3. \tag{38}$$

Indeed, an immediate induction would yield the stability of the radical filtration with respect to the $\overline{\mathbf{F}}_p[[H_0(p)]]$-restriction functor. The computatios of Schneider-Venjakob ([SV],lemma 4.3) or the classical result of Wilson ([Wil], theorem 8.7.7) show that condition (38) is verified if

$$[H_0(p), K_0'(p)] \leqslant (K_0'(p))^p \tag{39}$$

An argument by successive approximations (similar to the proof of [DDSMS], theorem 5.2) shows that $(K_0'(p))^p = I_2 \overset{\text{def}}{=} \begin{bmatrix} 1+p^2\mathscr{O}_F & p\mathscr{O}_F \\ p^2\mathscr{O}_F & 1+p2\mathscr{O}_F \end{bmatrix}$ and one can check that the inclusion (39) does *not* hold.

The stability of the radical filtration with respect to $\overline{\mathbf{F}}_p[[H_0(p)]]$ looks more complicated that expected.

4.1.3 ***The structure of the representations*** R_n^-**.** Fix an integer $n \in \mathbf{N}$. We describe here the socle filtration (and the extensions between two consecutive graded pieces) for the $K_0(p)$-representations R_{n+1}^-. Again, we can identify the negative elements of R_{n+1}^- with the points of a lattice of $\mathbf{R}^f$ according to the following injective map

$$\begin{aligned} \mathscr{B}_{n+1}^- &\hookrightarrow \mathbf{R}^f \\ F_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)}(\underline{l}_{n+1}) &\mapsto \Big(\sum_{a=1}^{n+1} p^{a-1} l_a^{\lfloor s+a-1 \rfloor}\Big)_{s\in\{0,\dots,f-1\}} \end{aligned}$$

whose image will be denoted by $\mathscr{R}_{n+1}^-$; we define in the evident way the subspaces $(R_{n+1}^-)_N$ for $N \in \mathbf{N}$.

The structure of R_{n+1}^- is then sumarized in the following

Proposition 4.9. *Let $n \in \mathbf{N}$, $F_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)}(\underline{l}_{n+1}) \in \mathscr{B}_{n+1}^-$ and let $a,b,c,d \in \mathscr{O}_F$ be such that $g \overset{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Define finally the integer $N \overset{\text{def}}{=} N_{1,n+1}(\underline{l}_1,\dots,\underline{l}_{n+1})$.*

We have the equality

$$gF_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)}(\underline{l}_{n+1}) = \mathfrak{a}^{e(\underline{l}_1,\dots,\underline{l}_{n+1})}\chi_{\underline{r}}^s(g)\Big(F_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)}(\underline{l}_{n+1}) - \sum_{s=0}^{f-1}(\overline{ca}^{-1})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)-1\rfloor}(-1)^{\delta_{a_0(s),n+1}} F_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)}(\underline{l}_{n+1}) + y\Big)$$

where $y \in (R_{n+1}^-)_{N-1}$.

In particular, the $K_0(p)$-socle filtration of R_{n+1}^-, as well as the extensions between two consecutive graded pieces, are described by the associated lattice $\mathscr{R}_{n+1}^-$.

Proof. We notice that we have a $K_0(p^{n+1})$-equivariant monomorphism

$$\begin{aligned} \sigma_{\underline{r}}^{(n+1)} &\hookrightarrow \mathrm{Ind}_{K_0(p^{n+2})}^{K_0(p^{n+1})}\chi_{\underline{r}}^s \\ X^{\underline{r}-\underline{l}_{n+1}}Y^{\underline{l}_{n+1}} &\mapsto (-1)^{\underline{l}_{n+1}} \sum_{\lambda_{n+1}\in\mathbf{F}_q} (\lambda_{n+1}^{\frac{1}{p^{n+1}}})^{\underline{l}_{n+1}} \begin{bmatrix} 1 & 0 \\ p^{n+1}[\lambda_{n+1}^{\frac{1}{p^{n+1}}}] & 1 \end{bmatrix}[1,e]. \end{aligned}$$

By transitivity and exactness of the induction functor $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}(\bullet)$ we get a $K_0(p)$-equivariant monomorphism

$$\begin{aligned} R_{n+1}^- &\hookrightarrow \mathrm{Ind}_{K_0(p^{n+2})}^{K_0(p)}\chi_{\underline{r}}^s \\ F_{\underline{l}_1,\dots,\underline{l}_n}^{(1,n)}(\underline{l}_{n+1}) &\mapsto (-1)^{\underline{l}_{n+1}} F_{\underline{l}_1,\dots,\underline{l}_n,\underline{l}_{n+1}}^{(1,n+1)}. \end{aligned}$$

The conclusion is now immediate from proposition 4.2. $\square$

4.2 The positive case

This section is again divided into two parts. We begin with the study of the $K_0(p)$-representations R_{n+1}^+, for $n \in \mathbf{N}$: they are described in proposition 4.10. We subsequently switch our attention introducing other $K_0(p)$ representations (the $(\mathrm{Ind}_{K_0(p^{n+1})}^K \chi^s)^+$, defined in §4.3) which will let us describe the $K_0(p)$-restriction of principal and special series (see §4.3).

The philosophy is completely analogous to the one of the previous paragraph: we verify by a direct computation on the ring of Witt vectors that the $K_0(p)$-structure of such objects can be described in terms of f-parallelepipoids in the euclidean space $\mathbf{R}^f$.

Fix $n \in \mathbf{N}$. We introduce the injective map

$$\mathscr{B}^+_{n+1} \hookrightarrow \mathbf{R}^f$$
$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \mapsto (\sum_{i=0}^{n+1} p^i l_i^{(\lfloor s+i \rfloor)})_{s\in\{0,\dots,f-1\}}$$

which let us interpret the positive elements of R^+_{n+1} as points in a convenient lattice of $\mathbf{R}^f$. The image of such map (which is a parallelepipoid of side $p^{n+1}(r_s+1)-1$) will be denoted as $\mathscr{R}^+_{n+1}$. We still need the following notations (see also §4.1.1):

i) for a $(n+2)f$-tuple $(\underline{l}_0,\dots,\underline{l}_{n+1}) \in \big\{\{0,\dots,p-1\}^f\big\}^{n+2}$ define the integers

$$N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1}) \stackrel{\text{def}}{=} \sum_{a=0}^{n+1} p^a \mathfrak{s}(\underline{l}_a)$$
$$e(\underline{l}_0,\dots,\underline{l}_{n+1}) \stackrel{\text{def}}{=} (\sum_{s=0}^{f-1} p^s l_0^{(s)}) + \cdots + (\sum_{s=0}^{f-1} p^s l_{n+1}^{(s)});$$

ii) for $N \in \mathbf{N}$ we define the $\overline{\mathbf{F}}_p$-linear subspace

$$(R^+_{n+1})_N \stackrel{\text{def}}{=} \left\langle F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1} \quad \text{s.t.} \quad N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1}) < N \right\rangle_{\overline{\mathbf{F}}_p};$$

iii) for $s \in \{0,\dots,f-1\}$, we define

$$\Xi_s \stackrel{\text{def}}{=} \big\{a \in \{0,\dots,n+1\}, \quad \text{s.t. } l_a^{\lfloor s+a \rfloor} \neq 0\big\}$$

and we set

$$a_0(s) \stackrel{\text{def}}{=} \begin{cases} \min(\Xi_s) & \text{if } \Xi_s \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

For a given positive element $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$ we define the subspace $\mathfrak{W}_{(\underline{l}_0,\dots,\underline{l}_{n+1})}$ in the evident, similar way.

The structure of R^+_{n+1} is then given by

Proposition 4.10. *Let $n \in \mathbf{N}$, $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1}$ and let $a,b,c,d \in \mathscr{O}_F$ be such that $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Define finally the integer $N \stackrel{\text{def}}{=} N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1})$. We then have*

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = (\mathfrak{a}^{-1})^{e(\underline{l}_0,\dots,\underline{l}_{n+1})}\chi_{\underline{r}}(g)(F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) - \sum_{s=0}^{f-1}(\overline{bd}^{-1})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s) \rfloor}(-1)^{\delta_{a_0(s),n+1}} F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) + y)$$

where $y \in (R^+_{n+1})_{N-1}$.

In particular, the $K_0(p)$-filtration, as well as the extensions between two consecutive pieces, is described by the associated lattice $\mathscr{R}^+_{n+1}$.

The rest of this section is devoted to the proof of proposition 4.10. Thanks to decomposition (34) (specialized in $m = 1$) we can study separately the actions of lower unipotent, diagonal and upper unipotent matrices on the elements of R^+_{n+1}: this will be the object of the next three paragraphs.

As said in remark 4.8, the choice of the immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$ is irrelevant.

The action of upper unipotent matrices. We start from the action of $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$. We have the

LEMMA 4.11. *Let $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1}$, $g \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$; write $N \overset{\text{def}}{=} N_{0,n+1}(\underline{l}_0, \dots, \underline{l}_{n+1})$.*

Then we have

$$g \cdot F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) - \sum_{s=0}^{f-1} \overline{b}^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s)\rfloor} (-1)^{\delta_{a_0(s),n+1}} F^{0,n}_{(\underline{l}_0,\dots,\underline{l}_n)-e_i}(\underline{l}_{n+1}) + y$$

where for a suitable $y \in (R^+_{n+1})_{N-1}$. More precisely, via the projection

$$R^+_{n+1} \overset{pr}{\twoheadrightarrow} R^+_{n+1}/(R^+_{n+1})_{N-(p^f+2)},$$

the image of the element y is contained in the image of the subspace $\mathfrak{W}_{(\underline{l}_0,\dots,\underline{l}_n)}$.

Proof. Recalling the twisted action of $K_0(p^{n+1})$ on $\sigma^{(n+1)}_{\underline{r}}$, the proof follows closely the proof of proposition 4.3, with some obvious notational modifications. Assuming $b = [\mu]$ for some $\mu \in \mathbf{F}_q$, a computation in the ring of truncated Witt polynomial gives

$$[\mu] + \sum_{i=0}^{n} p^i [\lambda_i^{\frac{1}{p^i}}] \equiv \sum_{i=0}^{n} p^i [\lambda_i^{\frac{1}{p^i}} + \widetilde{S}_i^{\frac{1}{p^i}}] + p^{n+1}[\widetilde{S}_{n+1}^{\frac{1}{p^{n+1}}}] \mod p^{n+2}.$$

We deduce

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) =$$
$$= \sum_{j=0}^{n} \sum_{\underline{i}_j \leqslant \underline{l}_j} \binom{\underline{l}_j}{\underline{i}_j} \sum_{\underline{i}_{n+1} \leqslant \underline{j}_{n+1}} \binom{\underline{j}_{n+1}}{\underline{i}_{n+1}} (T_0)^{\underline{i}_0} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (T_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, f_{\underline{l}_{n+1}-\underline{i}_{n+1}}]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix} 1 & 0 \\ p^0[\lambda_0] & 1 \end{bmatrix}$ instead of $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ and where we have set

$$f_{\underline{l}_{n+1}-\underline{i}_{n+1}} \overset{\text{def}}{=} (-1)^{\underline{i}_{n+1}} X^{\underline{r}-(\underline{l}_{n+1}-\underline{i}_{n+1})} Y^{\underline{l}_{n+1}-\underline{i}_{n+1}},$$

$T_0 \overset{\text{def}}{=} -s_0(\widetilde{S}_0)$, $T_{j+1} \overset{\text{def}}{=} -s_j(\widetilde{S}_{j+1})$ for $j \in \{0, \dots, n\}$. We check again that proposition 7.3 applies, giving the first part of the statement. Again, the result concerning the linear coefficients of the functions $F^{0,n}_{(\underline{l}_0,\dots,\underline{l}_n)-e_i}(\underline{l}_{n+1})$ is deduced exactly as in the proof of proposition 4.3: the details are left to the reader. $\square$

The action of diagonal matrices. We study here the action of the subgoup

$$\begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix}$$

on the elements of $\mathscr{B}^+_{n+1}$. The result is the following:

Lemma 4.12. *Let* $g \in \begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix}$ *and fix* $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1}$.

We then have the equality

$$g \cdot F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) + y$$

where $y \in (R^+_{n+1})_{N-1}$.

More precisely, via the projection

$$R^+_{n+1} \overset{pr}{\twoheadrightarrow} R^+_{n+1}/(R^+_{n+1})_{N-(p^f+2)},$$

the image of y *is contained in the image of the subspace* $\mathfrak{W}_{(\underline{l}_0,\dots,\underline{l}_{n+1})}$ *and writing*

$$y = \sum_{i\in I} \beta_i F^{(0,n)}_{\underline{l}_0(i),\dots,\underline{l}_n(i)}(\underline{l}_{n+1}(i))$$

(for a suitable set of indexes I *and scalars* $\beta_i \in \overline{\mathbf{F}}_p^\times$*) we have that each function* $F^{(0,n)}_{\underline{l}_0(i),\dots,\underline{l}_n(i)}(\underline{l}_{n+1}(i))$ *which is not in the kernel* $\ker(pr)$ *lies on an hyperplane*

$$X_0 + \dots + X_{f-1} = N - t(p-1)$$

for some $t \in \mathbf{N}_>$.

Proof. A direct computation shows that If $z \in \mathscr{O}_F/p^{n+1}\mathscr{O}_F$ then

$$\begin{bmatrix} 1+pa & 0 \\ 0 & 1+pd \end{bmatrix} \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z' & 1 \\ 1 & 0 \end{bmatrix} \mathfrak{k}$$

where $\mathfrak{k} \in K_0(p^{n+2})$ is upper unipotent modulo p and $z' \in \mathscr{O}_F/p^{n+2}\mathscr{O}_F$ is determined by the condition

$$z' \equiv (1+pa)(1+pd)^{-1} z \bmod p^{n+2}.$$

We can therefore content ourself studying the action of an element of the form $g \overset{\text{def}}{=} \begin{bmatrix} 1+p\alpha & 0 \\ 0 & 1 \end{bmatrix}$ for $\alpha \in \mathscr{O}_F$. We have

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = \sum_{j=0}^{n} \sum_{\underline{i}_{j+1} \leqslant \underline{l}_{j+1}} \binom{\underline{l}_{j+1}}{\underline{i}_{j+1}} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (T_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j] & 1 \end{bmatrix} [1, f_{\underline{l}_{n+1}-\underline{i}_{n+1}}]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix} 1 & 0 \\ p^0[\lambda_0] & 1 \end{bmatrix}$ instead of $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ and where we have set

$$f_{\underline{l}_{n+1}-\underline{i}_{n+1}} \overset{\text{def}}{=} (-1)^{\underline{i}_{n+1}} X^{\underline{r}-(\underline{l}_{n+1}-\underline{i}_{n+1})} Y^{\underline{l}_{n+1}-\underline{i}_{n+1}}$$

and $T_{j+1} \overset{\text{def}}{=} -q_j(\widetilde{Q}_{j+1})$ for $j \in \{0,\dots,n\}$. We deduce the statement using the very same arguments of the proof of proposition 4.4: the details are left to the reader. $\square$

The action of lower unipotent. We finally deal with the action of the subgroup $\begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$. We have

LEMMA 4.13. *Let* $g \in \begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$ *and fix* $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1}$. *Write*[6] $N \overset{\text{def}}{=} N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1})$. *In the quotient space*

$$R^+_{n+1}/(R^+_{n+1})_{N-(p-3)}$$

we have the equality

$$g\cdot F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}).$$

Proof. The proof is analogous to the proof of proposition 4.6 using this time the results and notations of §6.4.2. Again, we assume $g = \begin{bmatrix} 1 & 0 \\ p[x] & 1 \end{bmatrix}$; a computation shows that for $z \in \mathscr{O}_F/p^{n+1}\mathscr{O}_F$ we have

$$\begin{bmatrix} 1 & 0 \\ p[x] & 1 \end{bmatrix}\begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z' & 1 \\ 1 & 0 \end{bmatrix}\mathfrak{k}$$

where $\mathfrak{k} \in K_0(p^{n+2})$ is upper unipotent modulo p and $z' \in \mathscr{O}_F/p^{n+2}$ is determined by the condition

$$z' \equiv \sum_{j=0}^{n+1} p^j[x^j]z^{j+1} \mod p^{n+2}.$$

We then deduce from the results of §6.4.2 that

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = \sum_{j=0}^{n}\sum_{\underline{i}_{j+1}\leqslant\underline{l}_{j+1}}\binom{\underline{l}_{j+1}}{\underline{i}_{j+1}}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}\begin{bmatrix} 1 & 0 \\ p^j[\lambda_j] & 1 \end{bmatrix}[1, f_{\underline{l}_{n+1}-\underline{i}_{n+1}}]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix} 1 & 0 \\ p^0[\lambda_0] & 1 \end{bmatrix}$ instead of $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ and where we have set $\underline{i}_0 \overset{\text{def}}{=} \underline{0}$, $f_{\underline{l}_{n+1}-\underline{i}_{n+1}} \overset{\text{def}}{=} (-1)^{\underline{i}_{n+1}}X^{\underline{r}-(\underline{l}_{n+1}-\underline{i}_{n+1})}Y^{\underline{j}_{n+1}-\underline{i}_{n+1}}$ and $V_{j+1} \overset{\text{def}}{=} -u_j(\widetilde{U}_{j+1})$ for $j \in \{0,\dots,n\}$. Thanks to lemma 6.18 we can apply proposition 7.5 to get the desired result □

End of the proof of proposition 4.10. If $z \in \mathscr{O}_F$ and $a, d \in \mathbf{F}_q^\times$ we have the matrix equality

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}\begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z[ad^{-1}] & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} [d] & 0 \\ 0 & [a] \end{bmatrix}$$

so that

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = (\mathfrak{a}^{-1})^{e(\underline{l}_0,\dots,\underline{l}_{n+1})}\chi_{\underline{r}}(\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix})F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}).$$

The result follows from lemmas 4.11, 4.12 and 4.13, noticing that

$$\begin{bmatrix} a & b \\ p^m c & d \end{bmatrix} = \begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix}\begin{bmatrix} 1 & 0 \\ p^m z & 1 \end{bmatrix}\begin{bmatrix} 1+px & 0 \\ 0 & 1+pw \end{bmatrix}\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$$

for suitable $x, y, z, w \in \mathscr{O}_F$ verifying $\overline{y} = \overline{ba^{-1}}$. □

[6] again this N does not have anything to do with $N \overset{\text{def}}{=} \lfloor\frac{n+1}{m}\rfloor$. We believe this conflict of notation will not give rise to any confusion, as the meaning of N will be clear from the context.

4.2.1 ***On some other*** $K_0(p)$***-representations.*** As annonced in the introduction, we define and study some $K_0(p)$-representations (denoted as $\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi^{+}$) which naturally appear dealing with the Iwahori structure of principal and special series. The reader will realize soon that the behaviour of the representations $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^{+}$ can be treated with the same methods of §4.2 and 4.1; the proofs will be therefore omitted.

Fix an integer $n \in \mathbf{N}$, a smooth character $\chi : K_0(p^{n+1}) :\to \overline{\mathbf{F}}_p^{\times}$ and an $\overline{\mathbf{F}}_p$-basis $\{e\}$ for the underlying vector space of χ. The $K_0(p)$-representation $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^{+}$ is defined as the $K_0(p)$-subrepresentation induced by $\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi$ on the $\overline{\mathbf{F}}_p$-subspace

$$\langle [\begin{bmatrix} [z] & 1 \\ 1 & 0 \end{bmatrix}, e] \in \mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi, \quad z \in I_{n+1}\rangle_{\overline{\mathbf{F}}_p}$$

(the $K_0(p)$-stability of such $\overline{\mathbf{F}}_p$-linear space is immediately verified). Again, we have the

Definition 4.14. *Let $j \in \{0, \dots, n\}$ and let $\underline{l}_j \in \{0, \dots, p-1\}^f$ be a f-tuple. We define the following element of $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^{+}$:*

$$F_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)} \stackrel{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \sum_{j=1}^{n} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} [1, e].$$

The family

$$\mathscr{B}^{+} \stackrel{\text{def}}{=} \{F_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)} \in (\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^{+}, \quad \underline{l}_j \in \{0, \dots, p-1\}^f \quad \text{for all } j \in \{0, \dots, n\}\}$$

is an $\overline{\mathbf{F}}_p$-basis for $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^{+}$.

Exactly as we did for R_{n+1}^{+}, each given element $F_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)}$ of $\mathscr{B}^{+}$ will be read as a point in a convenient lattice $\mathscr{R}$ of $\mathbf{R}^f$ and the integers $a_0(s)$ (for $s \in \{0, \dots, f-1\}$) can be assigned. Moreover, if $N \in \mathbf{N}$, the subspaces $((\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^{+})_N$ are defined in the similar, evident way (see the introduction of §4.2 for details).

The structure of the representations $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi)^{+}$ is then described in the next

Proposition 4.15. *Let $\underline{r} \in \{0, \dots, p-1\}^f$ be an f-tuple, $n \in \mathbf{N}$ an integer and let $a, b, c, d \in \mathscr{O}_F$ be such that $g \stackrel{\text{def}}{=} \begin{bmatrix} a & b \\ pc & d \end{bmatrix} \in K_0(p)$. Fix an element $F_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)} \in \mathscr{B}^{+}$ and set $N \stackrel{\text{def}}{=} N_{0,n}(\underline{l}_0, \dots, \underline{l}_n)$.*

Then

$$g \cdot F_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)} = (\mathfrak{a}^{-1})^{e(\underline{l}_0,\dots,\underline{l}_n)} \chi_{\underline{r}}(g) F_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)} - \sum_{s=0}^{f-1} (\overline{bd}^{-1})^{p^s} l_{a_0(s)}^{\lfloor s+a_0(s) \rfloor} F_{(\underline{l}_0,\dots,\underline{l}_n)-e_s}^{0,n} + y$$

where $y \in (\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s^{+}})_{N-1}$.

In particular the $K_0(p)$-socle filtration of $(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_{\underline{r}}^{s})^{+}$, as well as the extensions of two consecutive graded pieces, are described by the associated lattice $\mathscr{R}$.

Proof. Omissis. □

4.3 The Iwahori structure of Principal and Special Series

We are now able to describe easely the Iwahori-structure of principal and special series for $\mathrm{GL}_2(F)$. Such result is essentially a formal consequence of the previous sections §4.1 and §4.2.1.

For $\lambda \in \overline{\mathbf{F}}_p^{\times}$ and $\underline{r} \in \{0, \dots, p-1\}^f$ we consider the smooth parabolic induction

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$$

where ω denotes the mod p cyclotomic character and μ_λ the unramified character verifying $\mu_\lambda(p) = \lambda$. It is well known that for $(\underline{r}, \lambda) \notin \{(\underline{0}, \pm 1), (\underline{p-1}, \pm 1)\}$ such inductions are irreducible, while, if $(\underline{r}, \lambda) \in \{(\underline{0}, \pm 1), (\underline{p-1}, \pm 1)\}$ they have length 2 and a unique infinite dimensional factor, the Steinberg representation (see also [BL94]). Thanks to the Iwahori decomposition and Mackey theorem we have

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_K \xrightarrow{\sim} \mathrm{Ind}_{B(\mathscr{O}_F)}^{\mathrm{GL}_2(\mathscr{O}_F)} \chi_{\underline{r}}^s$$

and, since the elements $f \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$ are locally constant functions and $B(\mathscr{O}_F)\backslash \mathrm{GL}_2(\mathscr{O}_F)$ is compact we have a natural isomorphism

$$\mathrm{Ind}_{B(\mathscr{O}_F)}^{\mathrm{GL}_2(\mathscr{O}_F)} \chi_{\underline{r}}^s \xrightarrow{\sim} \varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s.$$

Again, we can use Mackey decomposition to deduce

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s|_{K_0(p)} \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s \oplus (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+$$

so that, by exactness property of filtrant inductive limit, we get

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_{K_0(p)} \xrightarrow{\sim} (\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s) \oplus (\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+). \tag{40}$$

The isomorphism (40) let us reduce to the case of the finite inductions $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$, $\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^{s\,+}$, whose structure is completely described in propositions 4.2 and 4.15. Therefore

THEOREM 4.16. *Let $\lambda \in \overline{\mathbf{F}}_p^{\times}$ and $\underline{r} \in \{0, \dots, p-1\}^f$ an f-tuple. For any $m \in \mathbf{N}_>$ we write*

$$F_{\underline{0}, \dots, \underline{0}, \dots}^{(m, \infty)} \in \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}$$

to denote the characteristic function of $K_0(p^m)$.

The $K_0(p)$-restriction of the parabolic induction admits a natural splitting

$$\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \mu_\lambda \otimes \omega^{\underline{r}} \mu_{\lambda^{-1}}|_{K_0(p)} \xrightarrow{\sim} (\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s) \oplus (\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+).$$

Moreover an $\overline{\mathbf{F}}_p$-basis $\mathscr{B}^-$ for $\varinjlim_{n \in \mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$ (risp. $\mathscr{B}^+$ for $\varinjlim_{n \in \mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+$) is described by the elements

$$F_{\underline{l}_1, \dots, \underline{l}_n, \dots}^{(1, \infty)} \stackrel{\mathrm{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_q} (\lambda_1^{\frac{1}{p}})^{\underline{l}_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1^{\frac{1}{p}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \dots$$

(risp. the elements

$$F_{\underline{l}_0, \dots, \underline{l}_n, \dots}^{(0, \infty)} \stackrel{\mathrm{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \begin{bmatrix} [\lambda_1] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_1} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} \dots)$$

for a varying sequence $(\underline{l}_n)_{n \in \mathbf{N}_>} \in \{0, \dots, p-1\}^{(\mathbf{N}_>)}$ (resp. $(\underline{l}_n)_{n \in \mathbf{N}} \in \{0, \dots, p-1\}^{(\mathbf{N})}$).

If we associate the elements of such basis to points in $\mathbf{R}^f$ *according to the law*

$$F^{(1,\infty)}_{\underline{l}_1,\ldots,\underline{l}_n,\ldots,} \mapsto \Big(\sum_{i=1}^{\infty} p^{i-1} l_i^{\lfloor s+i-1 \rfloor}\Big)_{s\in\{0,\ldots,f-1\}}$$

$$F^{(,\infty)}_{\underline{l},\ldots,\underline{l}_n,\ldots,} \mapsto \Big(\sum_{i=0}^{\infty} p^{i} l_i^{\lfloor s+i \rfloor}\Big)_{s\in\{0,\ldots,f-1\}}$$

and write $\mathscr{R}^-$ *(resp* $\mathscr{R}^+$*) to denote the image of* $\mathscr{B}^-$ *(resp.* $\mathscr{B}^+$*) by this map, then the* $K_0(p)$*-socle filtration for* $\varinjlim_{n\in\mathbf{N}} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^s$ *(resp. for* $\varinjlim_{n\in\mathbf{N}} (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^s)^+$*), as well as the extentions between two graded pieces, is described by the associated lattice* $\mathscr{R}^-$ *(risp.* $\mathscr{R}^+$*).*

The Iwahori structure of irreducible principal series follows.

As far as the Steinberg representation is concerned, we just need to notice the following fact:

Lemma 4.17. *Assume* $\underline{r} \in \{\underline{0}, \underline{p-1}\}$ *and let* $n \in \mathbf{N}$*. We have a* $K_0(p)$*-equivariant exact sequence*

$$0 \to \langle (F^{(0)}_{\underline{0}}, F^{(1,0)}_{\emptyset}) \rangle \to \mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^{s^+} \oplus \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^{s} \to (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^{s}/\langle 1 \rangle)|_{K_0(p)} \to 0.$$

Proof. The proof is an induction on n, the case $n = 0$ being well known (cf. [BP], lemma 2.6).

For the general case, we leave to the reader the easy task to check that we have a natural commutative diagram with exact lines

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ind}_{K_0(p^n)}^{K_0(p)} \chi_{\underline{r}}^{s} \oplus \mathrm{Ind}_{K_0(p^n)}^{K} \chi_{\underline{r}}^{s^+} \arrow[r, two heads] \arrow[d] & (\mathrm{Ind}_{K_0(p^n)}^{K} \chi_{\underline{r}}^{s}/\langle 1 \rangle) \arrow[d] \\
\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^{s} \oplus \mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^{s^+} \arrow[r, two heads] \arrow[d] & (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^{s}/\langle 1 \rangle) \arrow[d] \\
(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_{\underline{r}}^{s}/\mathrm{Ind}_{K_0(p^n)}^{K_0(p)} \chi_{\underline{r}}^{s}) \oplus (\mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^{s^+}/\mathrm{Ind}_{K_0(p^n)}^{K} \chi_{\underline{r}}^{s^+}) \arrow[r, two heads, "\cong"] \arrow[d] & \mathrm{Ind}_{K_0(p^{n+1})}^{K} \chi_{\underline{r}}^{s}/\mathrm{Ind}_{K_0(p^n)}^{K} \chi_{\underline{r}}^{s} \arrow[d] \\
0 & 0
\end{tikzcd}$$

so that the snake lemma and the inductive hypothesys, giving an exact sequence

$$0 \to \langle (F^{(0)}_{\underline{0}}, F^{(1,0)}_{\emptyset}) \rangle \to \mathrm{Ind}_{K_0(p^n)}^{K} \chi_{\underline{r}}^{s^+} \oplus \mathrm{Ind}_{K_0(p^n)}^{K_0(p)} \chi_{\underline{r}}^{s} \to (\mathrm{Ind}_{K_0(p^n)}^{K} \chi_{\underline{r}}^{s}/\langle 1 \rangle)|_{K_0(p)} \to 0,$$

let us conclude. □

5. The structure of the universal representation

In this section we show how the techical results of §4 concerning the representations $R^{\pm}_{n+1}$ and the formalism of §3 let us describe the Iwahori structure for the universal representation $\pi(\underline{r}, 0, 1)$. Again we develop an euclidean dictionary which enable us to handle the involved combinatoric

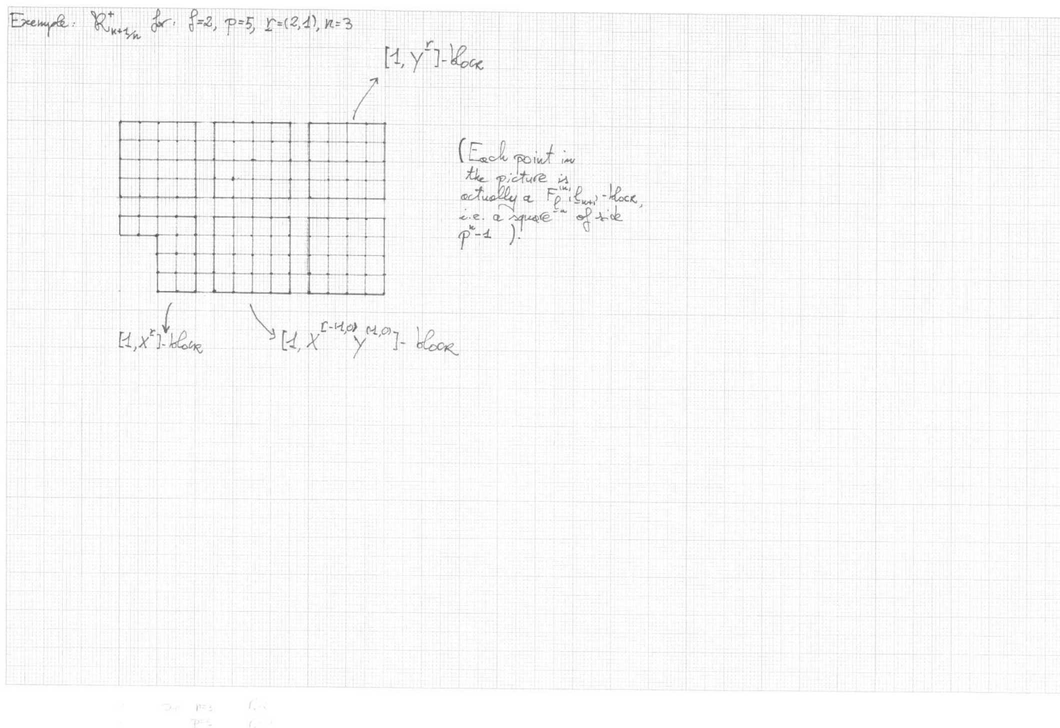


FIGURE IV.5. Euclidean structure for $R^{\pm}_{n+1}/R^{\pm}_{n}$.

of $\pi(\underline{r}, 0, 1)|_{K_0(p)}$: the conclusion is then proposition 5.18, which loosely speaking shows that the required structure is obtained by a juxtaposition of the blocks $R^{\pm}_{n+1}$ in a fractal way. As a byproduct, we will exhibit a natural injective map

$$c\text{–}\mathrm{Ind}_{KZ}^{G} V \hookrightarrow \pi(\underline{r}, 0, 1)$$

where $V \leqslant \pi(\underline{r}, 0, 1)|_{KZ}$ is a convenient KZ-subrepresentation of $\pi(\underline{r}, 0, 1)|_{KZ}$. We remark that a similar injective map has been detected independently by Paskunas in an unpublished draft.

We give here a more precise description of this section. Thanks to proposition 3.7 we can content ourselves to the study of the representations $\varinjlim_{n\,\mathrm{odd}} R_0^+ \oplus_{R_1^+} \cdots \oplus_{R_n^+} R^+_{n+1}$ and $\varinjlim_{n\,\mathrm{odd}} R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R^-_{n+1}$. As seen in proposition 3.5, such $K_0(p)$-representations have a natural filtration whose graded pieces are isomorphic to the quotients R^+_{n+1}/R^+_n, R^-_{n+1}/R^-_n respectively.

Such quotients are studied in §5.1. As we did in sections §4.1.3 and §4.2 -concerning the $K_0(p)$-structure of R^+_{n+1} and R^-_{n+1}- we introduce a natural correspondence between a "canonical" $\overline{\mathbf{F}}_p$-base $\mathscr{B}^{\pm}_{n+1/n}$ for $R^{\pm}_{n+1}/R^{\pm}_n$ and a convenient lattice (denoted as $\mathscr{R}^{\pm}_{n+1/n}$) in $\mathbf{R}^f$. Thanks to the behaviour of the canonical Hecke operator $(T^+_n)^{\mathrm{pos,neg}}$ with respect to the elements of $\mathscr{B}^{\pm}_{n+1/n}$ we see that such a lattice is in fact the set-theoretic difference of the lattices $\mathscr{R}^{\pm}_{n+1}$ and $\mathscr{R}^{\pm}_n$ (cf. lemma 5.1): figure IV.5 shows this phenomenon for $f = 2$.

Unfortunately, we can not use directly the results of sections 4 to concude that the $K_0(p)$-structure of $R^{\pm}_{n+1}/R^{\pm}_n$ is predicted by the lattice $\mathscr{R}^{\pm}_{n+1/n}$: in fact propositions 4.9 and 4.10 describe the extensions detected by functions $f_1, f_2 \in \mathscr{B}^{\pm}_{n+1}$ *lying on adjacent antidiagonals.*

It is therefore necessary to perfect the estimates made in the proofs of propositions 4.9, 4.10: this is the object of §5.1.1. We remark that the behaviour of $(R_1/R_0)^+$ (resp. $R_0^- \oplus_{R_1^-} R_2^-$) is slighty different from that of R^+_{n+1}/R^+_n for $n \geqslant 1$ (resp. R^-_{n+1}/R^-_n for $n \geqslant 2$) (and treated in §5.1.2).

In section §5.2 we determine the structure of the amalgamed sums $\cdots \oplus_{R^{\pm}_n} R^{\pm}_{n+1}$: their structure can be *easily* determined from the results concerning of $R^{\pm}_{n+1}/R^{\pm}_n$. Indeed, thanks to the behaviour of the canonical basis of $R^{\pm}_n$ with respect to the Hecke operators $(T^-_n)^{\mathrm{pos,neg}}$ we see that the convenient euclidean pictured is obtained by gluenig the lattice $\mathscr{R}^{\pm}_{n+1/n}$ with (a suitable translation of) the lattice associated to $\dots_{R^{\pm}_{n-2}} R^{\pm}_{n-1}$ (which we assume inductively to have been described). Again, the $K_0(p)$-socle filtration is expected to be obtained by successive intersections of such lattice with parallels antidiagonals, as it was for $R^{\pm}_{n+1}/R^{\pm}_n$, but a simple computation shows that the hyperplanes giving the J-th layer of the socle filtation of $\mathscr{R}^{\pm}_{n+1/n}$ *lie always below* the hyperplanes giving the J-th layer of the socle filtration for $\dots_{R^{\pm}_{n-2}} R^{\pm}_{n-1}$. As $\dots_{R^{\pm}_{n-2}} R^{\pm}_{n-1}$ is a $K_0(p)$-*subrepresentation* of $\dots_{R^{\pm}_n} R^{\pm}_{n+1}$ we are able to deduce the desired result of proposition 5.18.

In figure IV.6 an exemple of the glueing of blocks [7] and their fractal stucture.

As annonced, we can combine lemma 5.1 and proposition 3.5 to exhibit a natural *injective* morphism -whose existence was known informally by an unpublished work of Paskunas-

$$c\text{--}\mathrm{Ind}_{KZ}^G V \hookrightarrow \pi(\underline{r}, 0, 1)|_{KZ}$$

where $V \leqslant \pi(\underline{r}, 0, 1)|_{KZ}$ is a convenient KZ-subrepresentation of $\pi(\underline{r}, 0, 1)|_{KZ}$: this is the object of proposition 5.12.

As the cutting hyperplanes are fixed by the linear transformation $e_s \mapsto e_{\lfloor s+1 \rfloor}$ of $\mathbf{R}^f$ the results of §5.1 and §5.2 do not depend on the immersion $\tau : \mathbf{F}_q \hookrightarrow \overline{\mathbf{F}}_p$, see remark 4.8.

5.1 The structure of the quotients $R^{\bullet}_{n+1}/R^{\bullet}_n$

In the flavour of §4.1.3 and §4.2 we start by describing a suitable $\overline{\mathbf{F}}_p$-basis for the quotients $R^{\bullet}_{n+1}/R^{\bullet}_n$.

LEMMA 5.1. *Let $n \in \mathbf{N}_{\geqslant 1}$.*

1) *An $\overline{\mathbf{F}}_p$-basis $\mathscr{B}^+_{n+1/n}$ for R^+_{n+1}/R^+_n is described as the homomorphic image (via the natural projection $R^+_{n+1} \twoheadrightarrow R^+_{n+1}/R^+_n$) of the elements*

$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(l_{n+1}) \in \mathscr{B}^+_{n+1}$$

such that $\underline{l}_n \not\leqslant \underline{r}$ if $\underline{l}_{n+1} = \underline{0}$.

2) *An $\overline{\mathbf{F}}_p$-basis $\mathscr{B}^-_{n+1/n}$ for R^-_{n+1}/R^-_n is described as the homomorphic image (via the natural projection $R^-_{n+1} \twoheadrightarrow R^-_{n+1}/R^-_n$) of the elements*

$$F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(l_{n+1}) \in \mathscr{B}^-_{n+1}$$

[7] strictly speaking, the figure gives the glueing of blocks R^+_{n-1}/R^+_{n-2} and R^+_{n+1}/R^+_n, i.e. the structure of $R^+_{n-1}/R^+_{n-2} \oplus_{R^+_n} R^+_{n+1}$. If we want to get the picture of the *whole* amalgamed sum $\cdots \oplus_{R^+_n} R^+_{n+1}$ we should insert a "even smaller" structure inside the point $(1,2)$ of the rectangle drawed on the left in figure IV.6.

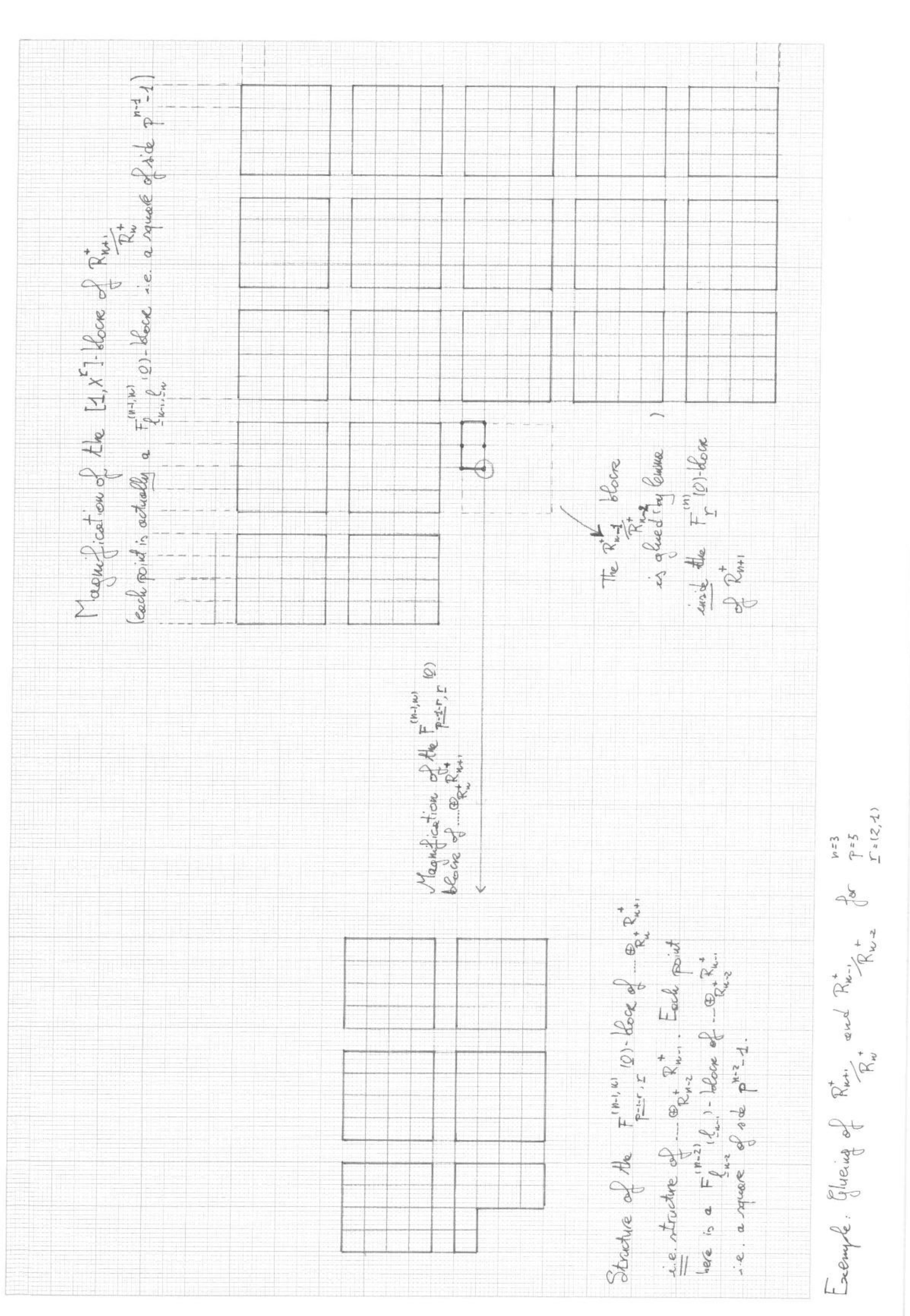


FIGURE IV.6. The glueing and the fractal structure.

such that $\underline{l}_n \not\leqslant \underline{r}$ *if* $\underline{l}_{n+1} = \underline{0}$.

If $n = 0$ *then an* $\overline{\mathbf{F}}_p$*-basis for* $(R_1/R_0)^+$ *is described as the homomorphic image (via the natural projection* $R_1^+ \twoheadrightarrow (R_1/R_0)^+$*) of the elements*

$$F^{(0)}_{\underline{l}_0}(\underline{l}_1)$$

such that $\underline{l}_1 \not\leqslant \underline{r}$ *if* $\underline{l}_1 = \underline{0}$ *and of the element* $F^{(0)}_{\underline{r}}(\underline{0})$.

Proof. The result follows immediately from the definition of the operators $(T_n^+)^{\mathrm{pos,neg}}$. Indeed, for $n \geqslant 1$ we have (with the obvious conventions if $n = 1$):

$$(T_n^+)^{\mathrm{pos}}(F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n)) = (-1)^{\underline{l}_n} F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{0});$$
$$(T_n^+)^{\mathrm{neg}}(F^{(1,n-1)}_{\underline{l}_1,\ldots,\underline{l}_{n-1}}(\underline{l}_n)) = (-1)^{\underline{l}_n} F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{0})$$

while, for $n = 0$ we have

$$T_0(F^{(0,-1)}_{\emptyset}(\underline{l}_0)) = F^{(0)}_{\underline{l}_0}(\underline{0}) + (-1)^{\underline{r}}\delta_{\underline{l}_0,\underline{0}} F^{(1,0)}_{\emptyset}(\underline{0}).$$

□

As usual the elements of the basis $\mathscr{B}^{\pm}_{n+1/n}$ will be read as the elements of a convenient lattice $\mathscr{R}^{\pm}_{n+1/n}$ of $\mathbf{R}^f$.

Interpretation in terms of euclidean data. Exactely as we did in sections §4.1.3 and §4.2 we have natural injections $\mathscr{B}^{\pm}_{n+1/n} \hookrightarrow \mathbf{R}^f$ which let us interpret the elements of $\mathscr{B}^{\pm}_{n+1/n}$ as points in a convenient lattice $\mathscr{R}^{\pm}_{n+1/n}$ of $\mathbf{R}^f$: the details can safely be left to the reader.

The euclidean interpretation of lemma 5.1 is therefore clear: for $n \geqslant 1$ the lattice $\mathscr{R}^{+}_{n+1/n}$ (resp. $\mathscr{R}^{-}_{n+1/n}$) of $\mathbf{R}^f$, which is expected to describe the $K_0(p)$-structure of R^+_{n+1}/R^+_n (resp. R^-_{n+1}/R^-_n), is obtained from the lattice of R^+_{n+1} (resp. R^-_{n+1}) by removing the simplex

$$\{(x_0,\ldots,x_{f-1}) \in \mathscr{R}^+_{n+1} \quad \text{s.t.}\, x_s < p^n(r_{\lfloor n+s\rfloor}+1)\,\text{for all}\, s = 0,\ldots,f-1\}$$

(resp.

$$\{(x_0,\ldots,x_{f-1}) \in \mathscr{R}^-_{n+1} \quad \text{s.t.}\, x_s < p^{n-1}(r_{\lfloor n+s-1\rfloor}+1)\,\text{for all}\, s = 0,\ldots,f-1\})$$

(equivalently, $\mathscr{R}^{\pm}_{n+1/n}$ is obtained as the set-theoretical difference of $\mathscr{R}^{\pm}_{n+1} \setminus \mathscr{R}^{\pm}_{n}$).

We refer the reader to figure IV.5 for an exemple in residual degree $f = 2$.

The lattice $\mathscr{R}^+_{1/0}$ associated to $(R_1/R_0)^+$ similarly obtained from the lattice associated to R_1^+, by removing the subset

$$\big\{(x_0,\ldots,x_{f-1}) \in \mathscr{R}^+_{n+1} \quad \text{s.t.}\, x_s < (r_{\lfloor n+s\rfloor}+1)\,\text{for all} s = 0,\ldots,f-1\big\} \setminus \big\{(r_0,\ldots,r_{f-1})\big\}.$$

To be precise, the lattice $\mathscr{R}^+_{1/0}$ (resp. the lattice naturally associated to $R_0^- \oplus_{R_1^-} R_2^-$) does *not* describe the $K_0(p)$-structure of $(R_1/R_0)^+$ (resp. $R_0^- \oplus_{R_1^-} R_2^-$) *sic et simpliciter*. But a harmless modification of the formalism used for $\mathscr{R}^+_{n+1/n}$ if $n \geqslant 1$ (resp. $\mathscr{R}^-_{n+1/n}$ if $n \geqslant 2$) let us detect their $K_0(p)$-socle filtration: see section §5.1.2 and propositions 5.8, 5.9 and 5.10 for details.

We will describe in detail the $K_0(p)$-structure of R^+_{n+1}/R^+_n for $n \geqslant 1$; as annonced, the negative case (for $n \geqslant 2$) will be left to the reader.

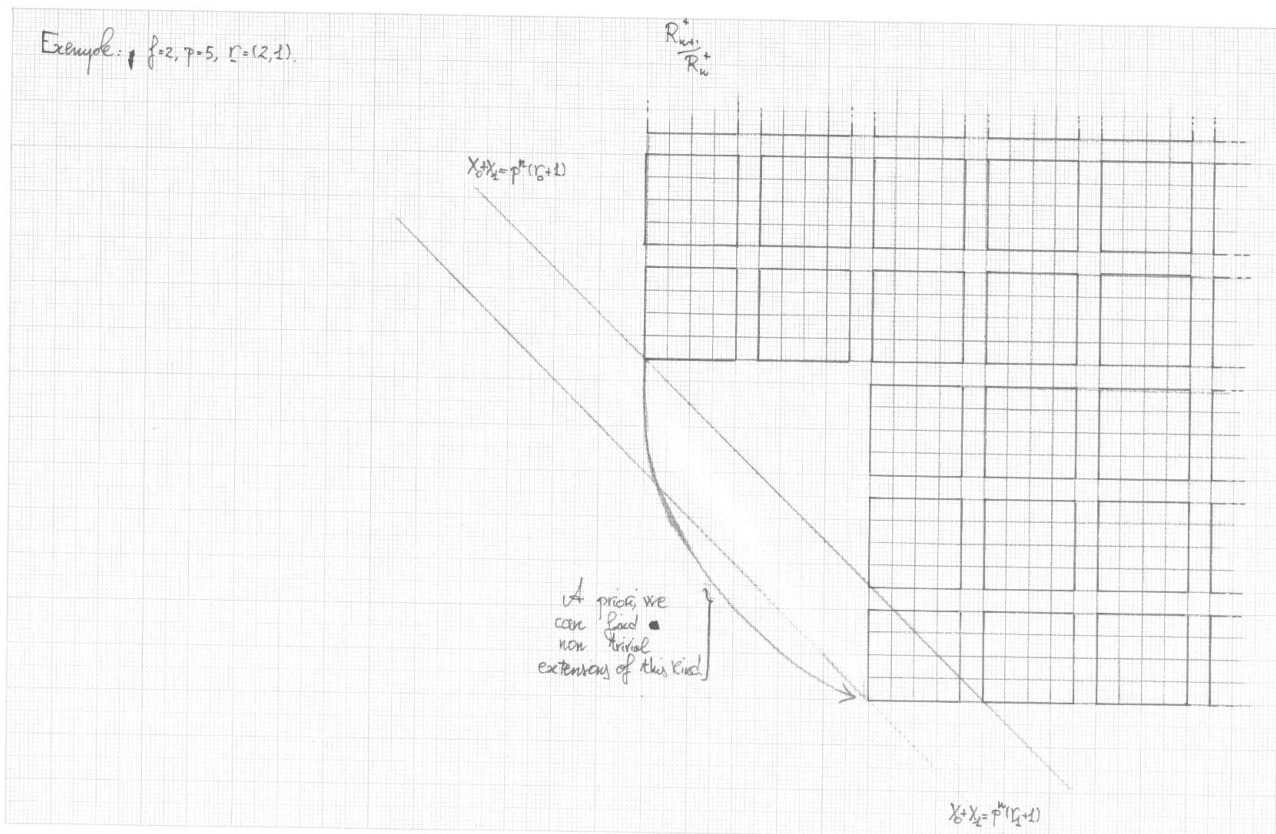


FIGURE IV.7. A priori, we can have disagreeable glueing phenomena.

Preliminaries: partitioning the lattice. As annonced in the introduction to §5, the mere knowledge of the $K_0(p)$-socle filtration for R^+_{n+1} does *not* allow us determine the structure of the quotient R^+_{n+1}/R^+_n. Indeed proposition 4.9 let us determine the extensions detected by functions $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_n), F^{(0,n)}_{\underline{l}'_0,\dots,\underline{l}'_n}(\underline{l}'_n) \in \mathscr{B}^+_{n+1}$ *lying on adjacent antidiagonals*. We could therefore get, *a priori*, a nontrivial extension between them if $\underline{l}_j = \underline{l}'_j = \underline{0}$ for all $j \neq n$ and $\underline{l}_n = (0,\dots,0,r_s,0,\dots,0)$ $\underline{l}_n = (0,\dots,0,r_{s'},0,\dots,0)$ for $s \neq s'$ as illustred in the figure IV.7.

Notice that this phenomena happens only if $F \neq \mathbf{Q}_p$: if $F = \mathbf{Q}_p$ the structure of the quotients is immediate from the structure of R^+_{n+1}.

We modify the strategy of section 4.2. We show that the $K_0(p)$-strucure of R^+_{n+1} is again predicted by $\mathscr{R}^+_{n+1}$ but each cutting antidiagonal $X_0 + \cdots + X_{f-1} = constant$ of section §4.2 is now replaced by f-antidiagonals of the form $X_0 + \cdots + X_{f-1} = p^n(r_{\lfloor n+s \rfloor}+1) + constant$: we will say that $X_0 + \cdots + X_{f-1} = p^n(r_{\lfloor n+s \rfloor}+1) + constant$ is the *s-th cutting hyperplane* of R^+_{n+1}/R^+_n. This means that we naturally divide the lattice $\mathscr{R}^+_{n+1/n}$ into sub-blocks $\mathfrak{V}_{s_{m+k}}$ of increasing size for $k \in \{0,\dots,f-1\}$ (cf. definition 5.2); the J-th composition factor for the $K_0(p)$-socle filtration of R^+_{n+1}/R^+_n is then obtained as the sum of the subspaces determined by the intersection of the block $\mathfrak{V}_{s_{m+k}}$ with the antidiagonal $X_0 + \cdots + X_{f-1} = p^n(r_{s_{m+k}} + 1) + constant$, for varying $k \in \{0, \ldots, f-1\}$. This is the content of proposition 5.3. In figure IV.8, an exemple of how the inreasing block (and successive cuttings) look like.

We determine the decomposition of $\mathscr{R}^+_{n+1/n}$ into increasing blocks. Fix $n \geqslant 0$ and define $s_m \in \{0, \ldots, f-1\}$ by the condition

$$r_{\lfloor s_m+n \rfloor} = \max\{r_{\lfloor s+n \rfloor}\}.$$

We fix an ordering

$$p-1 \geqslant r_{\lfloor s_m+n \rfloor} \geqslant r_{\lfloor s_{m+1}+n \rfloor} \geqslant \cdots \geqslant r_{\lfloor s_{m+f-1}+n \rfloor} \geqslant 0$$

and define the following $\overline{\mathbf{F}}_p$-subspaces of R^+_{n+1}/R^+_n:

Definition 5.2. *For $k \in \{0, \ldots, f-1\}$ define $\mathfrak{V}_{s_{m+k}}$ as the $\overline{\mathbf{F}}_p$-subspace of R^+_{n+1}/R^+_n generated by the elements $F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1/n}$ verifying the properties:*

i) for $s \notin \{s_m, \ldots, s_{m+k}\}$ we have

$$l_n^{\lfloor s+n \rfloor} \leqslant r_{\lfloor s+n \rfloor};$$

ii) for $s \notin \{s_m, \ldots, s_{m+k}\}$ we have

$$l_{n+1}^{\lfloor s+n+1 \rfloor} = 0.$$

By abuse of notation, we will also write $\mathfrak{V}_{s_{m+k}}$ to denote the image of the canonical basis (in the obvious sense) of $\mathfrak{V}_{s_{m+k}}$ in the lattice $\mathscr{R}^+_{n+1/n}$. The geometric meaning of the previous definition is the following: the block $\mathfrak{V}_{s_{m+k}}$ is described as the intersection of the subset

$$\{X_{s_{m+k+1}} < p^n(r_{\lfloor s_{m+k+1}+n \rfloor} + 1)\} \cap \cdots \cap \{X_{s_{m+f-1}} < p^n(r_{\lfloor s_{m+f-1}+n \rfloor} + 1)\}$$

with the lattice $\mathscr{R}^+_{n+1/n}$: in other words, we give restrictions on the coordinates $x_{s_{m+k+1}}, \ldots, x_{s_{m+f-1}}$ of a point $(x_0, \ldots, x_{f-1}) \in \mathscr{R}^+_{n+1/n}$ to lie in the block $\mathfrak{V}_{s_{m+k}}$.

Notice that in order to detect if a function $F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathscr{B}^+_{n+1/n}$ belongs to the subspace $\mathfrak{V}_{s_{m+k}}$ we *only need to study the last two f-tuples $\underline{l}_n$, $\underline{l}_{n+1}$*.

Obviously, the subspaces $\mathfrak{V}_{s_{m+k}}$ describe (for $n \geqslant 1$) an exhaustive increasing filtration on R^+_{n+1}/R^+_n as a $\overline{\mathbf{F}}_p$*-vector space.*

The following crucial result shows that the lattice $\mathscr{R}^+_{n+1/n}$ let us detect the required $K_0(p)$-structure for $n \geqslant 1$.

Proposition 5.3. *Assume $n \in \mathbf{N}_{\geqslant 1}$. Let $a, b, c, d \in \mathscr{O}_F$, $g \overset{\text{def}}{=} \begin{bmatrix} 1+pa & b \\ pc & 1+pd \end{bmatrix} \in K_0(p)$, fix an element $F^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathfrak{V}_{s_{m+k}}$ for some $k \in \{0, \ldots, f-1\}$ and write $N_{0,n+1}(\underline{l}_0, \ldots, \underline{l}_{n+1}) = p^n(r_{\lfloor s_{m+k}+n \rfloor} + 1) + J$ for some $J \in \mathbf{N}$. Finally, consider the linear development*

$$gF^{(0,n)}_{\underline{l}_0,\ldots,\underline{l}_n}(\underline{l}_{n+1}) = \sum_{i \in I} \beta(i) F^{(0,n)}_{\underline{l}_0(i),\ldots,\underline{l}_n(i)}(\underline{l}_{n+1}(i))$$

(where I is a suitable set of indices $\beta(i) \in \overline{\mathbf{F}}_p^\times$ are scalars).

Fix an index $i_0 \in I$ and assume there exists $k' \in \{k+1, \ldots, f-1\}$, minimal with respect to

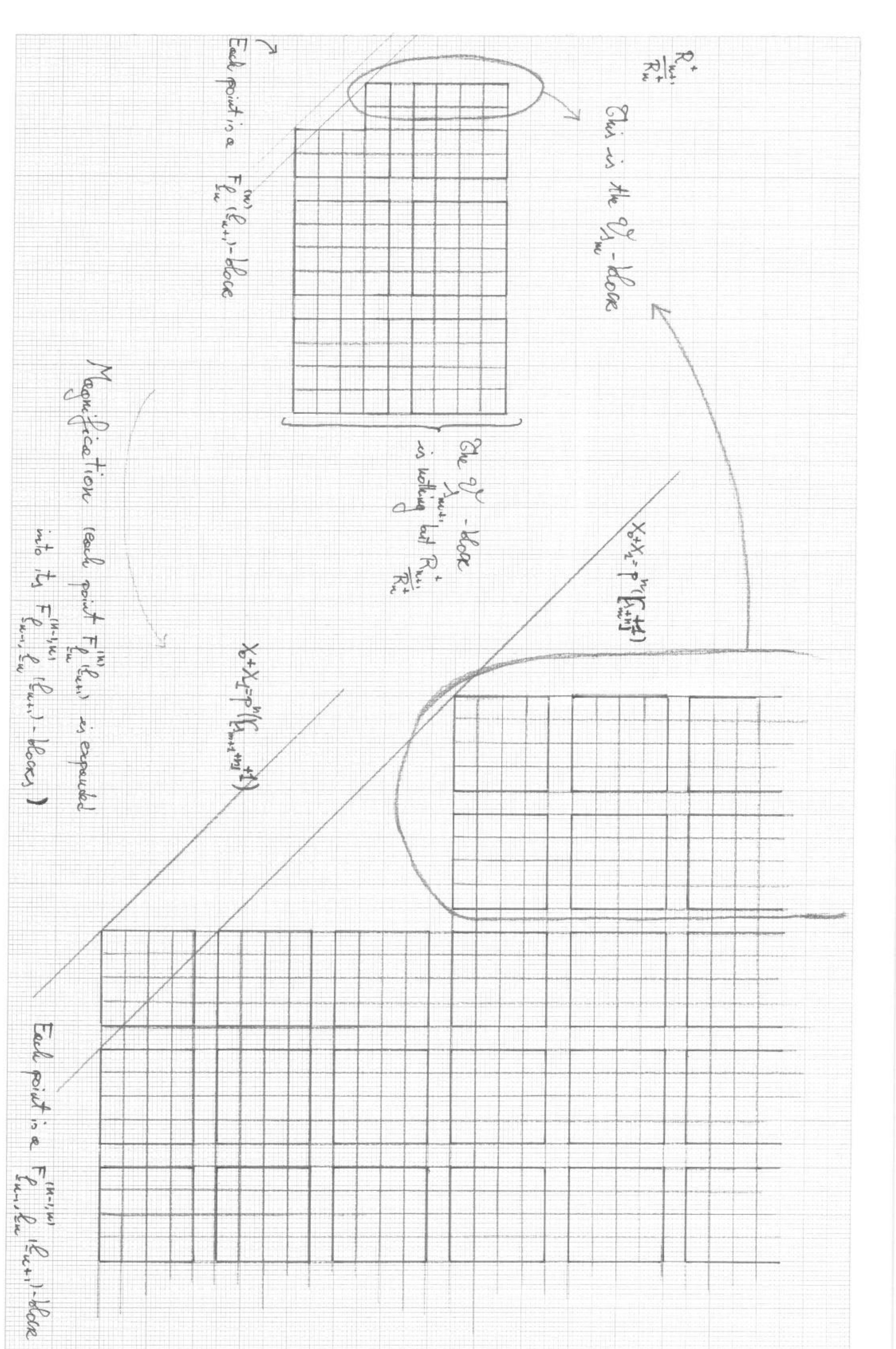


FIGURE IV.8. Exemple of bloks subdivision and cutting.

the property $F^{(0,n)}_{\underline{l}_0(i_0),\dots,\underline{l}_n(i_0)}(\underline{l}_{n+1}(i_0)) \in \mathfrak{V}_{s_{m+k'}} \setminus \mathfrak{V}_{s_{m+k}}$.

Then we have

$$N_{0,n+1}(\underline{l}_0(i_0),\dots,\underline{l}_{n+1}(i_0)) \leqslant p^n(r_{\lfloor s_{m+k'}+n\rfloor}+1)+J-2. \tag{41}$$

In particular, the lattice $\mathscr{R}^+_{n+1/n}$ *describes the* $K_0(p)$*-socle filtration, as well as the extensions between two consecutive graded pieces, of* R^+_{n+1}/R^+_n.

We insist on the geometric meaning of proposition 5.3: we pick a function in the k-th block $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in \mathfrak{V}_{s_{m+k}}$, liyng on the antidiagnal $X_0+\cdots+X_{f-1}=p^n(r_{s_{m+k}}+1)+J$ and $F^{(0,n)}_{\underline{l}_0(i_0),\dots,\underline{l}_n(i_0)}(\underline{l}_{n+1}(i_0))$ a function appearing (with nonzero linear coefficient) in the linear development of $gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$. If $F^{(0,n)}_{\underline{l}_0(i_0),\dots,\underline{l}_n(i_0)}(\underline{l}_{n+1}(i_0))$ happens to belong to a strictly bigger block, say $\mathfrak{V}_{s_{m+k'}}$ with $k'>k$ and minimal with respect to this property, then it lies *strictly below* the antidiagonal $X_0+\cdots+X_{f-1}=p^n(r_{s_{m+k'}}+1)+J-1$.

Thanks to this phenomenon, we can invoke proposition 4.10 to deduce the $K_0(p)$-structure for R^+_{n+1}/R^+_n from the associated lattice $\mathscr{R}^+_{n+1/n}$: the J-composition factor for the socle filtration of R^+_{n+1}/R^+_n is determined as the sum of the f subspaces obtained by intersecting each block $\mathfrak{V}_{s_{m+k}}$ with the corresponding antidiagonal $X_0+\cdots+X_{f-1}=p^n(r_{s_{m+k}}+1)+J$ (as in figure IV.8).

Notice moreover that the statement of proposition 5.3 is empty if $f=1$: in the rest of §5.1 we will assume $f\geqslant 2$.

5.1.1 ***Proof of proposition 5.3.*** The rest of this section is devoted to the proof of proposition 5.3. Thanks to decomposition (34) we can study separately the actions of lower unipotent, diagonal and upper unipotent matrices on the elements of R^+_{n+1}: this will be the object of the next three paragraphs. The proofs are similar to the proofs of propositions 4.11, 4.12 and 4.13, but need a delicate extra argument due to the irregular sttructure of the lattice $\mathscr{R}^+_{n+1/n}$.

The action of upper unipotent matrices. We study here the case where $g\in\begin{bmatrix}1 & \mathscr{O}_F\\ 0 & 1\end{bmatrix}$, and again we assume $g=\begin{bmatrix}1 & [\mu]\\ 0 & 1\end{bmatrix}$ for $\mu\in\mathbf{F}_q$. Exactly as in lemma 4.11 we write

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) =$$
$$=\sum_{j=1}^{n+1}\sum_{\underline{i}_j\leqslant\underline{l}_j}\binom{\underline{l}_j}{\underline{i}_j}\sum_{\underline{i}_0\leqslant\underline{l}_0}\binom{\underline{l}_0}{\underline{i}_0}(T_0)^{\underline{i}_0}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(T_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}\begin{bmatrix}1 & 0\\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}[1,f_{\underline{l}_{n+1}-\underline{i}_{n+1}}]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix}1 & 0\\ p^0[\lambda_0] & 1\end{bmatrix}$ instead of $\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}$ and where we have set

$$f_{\underline{l}_{n+1}-\underline{i}_{n+1}} \stackrel{\text{def}}{=} (-1)^{\underline{i}_{n+1}}X^{\underline{r}-(\underline{l}_{n+1}-\underline{i}_{n+1})}Y^{\underline{l}_{n+1}-\underline{i}_{n+1}},$$

$T_0 \overset{\text{def}}{=} -s_0(\widetilde{S}_0)$, $T_{j+1} \overset{\text{def}}{=} -s_j(\widetilde{S}_{j+1})$ for $j \in \{0,\dots,n\}$.

Developing the polynomials T_{j+1}'s we write

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = \sum_{i\in I}\beta(i)F^{(0,n)}_{\underline{l}_0(i),\dots,\underline{l}_n(i)}(\underline{l}_{n+1}(i))$$

(for a suitable set of indices I) and we pick a vector v appearing in the linear development of $gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})$:

$$v \overset{\text{def}}{=} F^{(0,n)}_{\lceil\underline{\kappa}_0\rceil,\dots,\lceil\underline{\kappa}_n\rceil}(\lceil\underline{\kappa}_{n+1}\rceil);$$

where, as in proposition 7.3, we write for $0 \leqslant a \leqslant n+1$

$$\underline{\kappa}_a = \underline{l}_a - \underline{i}_a + \sum_{b=a+1}^{n+1} p^{\lfloor a-b\rfloor}\kappa_a^{(b)}$$

and, for $a+1 \leqslant b \leqslant n+1$ we have

$$\kappa_a^{(b)} = \sum_{s=0}^{f-1} p^s \kappa_a^{(b),s}$$

where $\kappa_a^{(b),s}$ is the exponent of λ_a in $T_b^{i_b^{(s)}}$. By the definition of the subspace $\mathfrak{V}_{s_{m+k}}$ we see that

$\underline{\kappa}_n = \underline{l}_n - \underline{i}_n + p^{\lfloor -1\rfloor}\kappa_n^{(n+1)} =$

$$= \sum_{h=0}^{k} p^{\lfloor s_{m+h}+n\rfloor}(l_n^{(\lfloor s_{m+h}+n\rfloor)} - i_n^{(\lfloor s_{m+h}+n\rfloor)} + \kappa_n^{(n+1),\lfloor s_{m+h}+n+1\rfloor}) + \sum_{h=k+1}^{f-1} p^{\lfloor s_{m+h}+n\rfloor}(l_n^{(\lfloor s_{m+h}+n\rfloor)} - i_n^{(\lfloor s_{m+h}+n\rfloor)})$$

If $v \notin \mathfrak{V}_{s_{m+k}}$ then

$$k' \overset{\text{def}}{=} \min\big\{c \in \{k+1,\dots f-1\},\ \text{s.t.}\ \lceil\kappa_n^{(\lfloor s_{m+c}+n\rfloor)}\rceil > r_{(\lfloor s_{m+c}+n\rfloor)}\big\}(\ > k\ )$$

and we *necessarly* have $\underline{\kappa}_n \neq 0$ and the equality

$$\mathfrak{s}(\underline{l}_n - \underline{i}_n + p^{\lfloor -1\rfloor}\kappa_n^{(n+1)}) = \sum_{s=0}^{f-1} l_n^{(s)} - i_n^{(s)} + \kappa_n^{(n+1),\lfloor s+1\rfloor} - \widetilde{j}(p-1)$$

for a suitable $\widetilde{j} \geqslant 1$. Following the inequalities (51), (52), (53) of proposition 7.3 (i.e. using the subadditivity of the function $\mathfrak{s}$ and the fact that the polynomials T_j are homogeneous of degree p^j if λ_i is defined to have degree p^i) we get

$$\mathfrak{s}(\underline{\kappa}_0) + \dots p^{n+1}\mathfrak{s}(\underline{\kappa}_{n+1}) \leqslant p^n(r_{\lfloor s_{m+k}+n\rfloor}+1) + J - \mathfrak{s}(\underline{i}_0) + p^n(p-1)\widetilde{j}.$$

As $n \geqslant 1$ the inequality

$$p^n(r_{\lfloor s_{m+k}+n\rfloor} - r_{\lfloor s_{m+k'}+n\rfloor}) \leqslant \widetilde{j}p^n(p-1) + \mathfrak{s}(\underline{i}_0) - 2$$

is then obvious if either $\widetilde{j} \geqslant 2$ or $r_{\lfloor s_{m+k'}+n\rfloor} > 0$.

Assume finally $\widetilde{j} = 1$ and $r_{\lfloor s_{m+k'}+n\rfloor} = 0$. Therefore the p-adic development of $\lceil\kappa_n\rceil$ has the form

$$(l_n^{(0)} - i_n^{(0)} + \kappa_n^{(n+1),1},\dots,l_n^{(s)} - i_n^{(s)} + \kappa_n^{(n+1),s+1} - p, l_n^{(s+1)} - i_n^{(s+1)} + \kappa_n^{(n+1),s+2} + 1,\dots)$$

for a unique $s \in \{s_m,\dots,s_{m+k}\}$. The condition $x \notin \mathfrak{V}_{s_{m+k}}$ imposes $\lfloor s+1\rfloor \notin \{s_m,\dots,s_{m+k}\}$ and the minimality condition on k' imposes $\lfloor s_{m+k'}+n\rfloor = \lfloor s+1\rfloor$, in particular $r_{\lfloor s+1\rfloor} = 0$. As

$\kappa_n^{(n+1),s+1}$ is the coefficient of $\lambda_n^{\frac{1}{p^n}}$ in the fixed monomial of $s(\widetilde{S}_{n+1})^{i_{n+1}^{\lfloor s+1 \rfloor}}$ and $i_n^{\lfloor s+1 \rfloor} \leqslant r_{\lfloor s+1 \rfloor}$ we get an absurde.

The action of diagonal matrices. The next step is to study the action of an element $g \in \begin{bmatrix} 1+p\mathscr{O}_F & 0 \\ 0 & 1+p\mathscr{O}_F \end{bmatrix}$; again we can assume $g = \begin{bmatrix} 1+p\alpha & 0 \\ 0 & 1 \end{bmatrix}$. The arguments are completely analogous to those of the previous paragraph, in this case using the fact that the polynomials $q_{j-1}(\widetilde{Q}_j)$ of §6.3 are homogeneous of degree p^j. The details are left to the reader.

The action of lower unipotent matrices. In this section we deal with the action of an element $g \in \begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$; again, we assume $g = \begin{bmatrix} 1 & 0 \\ p[\mu] & 1 \end{bmatrix}$. This case is more delicate than the previous and we need to recall and carry on the accurate estimates seen in the appendix A §6.4.2.

Still others remarks on some universal Witt polynomials. For notation and convention we invite the reader to see Appendix A-§6.4.2. Let $z = (\lambda_0, \dots, \lambda_n, 0) \in \mathbf{W}_{n+1}(\mathbf{F}_q)$ and write

$$\sum_{j=0}^{n+1} p^j [\mu] z^{j+1} = (U_0, \dots, U_{n+1}).$$

for $U_j \in \mathbf{F}_p[\lambda_0, \dots, \lambda_j, \mu]$. We recall that U_h is obtained by specializing the universal polynomial $S_h^{n+2}(\underline{X}(1), \dots, \underline{X}(n+2))$ at

$$\underline{X}(j+1) = (0, \dots, 0, \underbrace{(Pot_0^{j+1}(\underline{\lambda}))^{p^j}(x^j)^{p^j}}_{\text{position } j}, \dots, \underbrace{(Pot_l^{j+1}(\underline{\lambda}))^{p^j}(x^j)^{p^{j+l}}}_{\text{position } j+l}, \dots).$$

We recall moreover that a monomial $\mathfrak{X}$ of $S_h^{n+2}(\underline{X}(1), \dots, \underline{X}(n+2))$ has the form

$$\mathfrak{X} = \prod_{l_0=0}^{h} X_{l_0}(1)^{a_{l_0}(0)} \cdots \prod_{l_{n+0}=0}^{h} X_{l_{n+1}}(n+2)^{a_{l_{n+1}}(n+1)}$$

where the integers $a_{l_i}(i)$ verify

$$\sum_{l_0=0}^{h} p^{l_0} a_{l_0}(0) + \cdots + \sum_{l_{n+1}}^{h} p^{l_{n+1}} a_{l_{n+1}}(n+1) = p^h;$$

Therefore a monomial $\lambda_0^{\alpha_0} \cdot \dots \cdot \lambda_h^{\alpha_h}$ issued from U_h verifies

$$\sum_{j=0}^{h} p^j \mathfrak{s}(\alpha_j) \leqslant \sum_{j=0}^{n+1} (j+1)(\sum_{i=j}^{h} p^{i-j} a_i(j)) = p^h - \sum_{j=1}^{h} (p^j - (j+1)) x_j$$

where we have set

$$x_j \stackrel{\text{def}}{=} \sum_{i=j}^{h} p^{i-j} a_i(j).$$

We focus our attention for the case $h = n+1$, obtaining thus the following

LEMMA 5.4. *A monomial of $\widetilde{U}_{n+1}$ has the following form*

$$\lambda_n^{a_n(0)+pa_{n+1}(1)} \cdot \lambda_{n-1}^{\alpha_{n-1}} \cdot \dots \cdot \lambda_0^{\alpha_0}$$

whose the exponents verify the following properties:

1) *we have* $a_n(0) \in \{0, \ldots, p-1\}$ *and* $a_{n+1}(1) \in \{0, 1\}$,
2) *letting* $x_j \overset{\text{def}}{=} \sum_{i=j}^{n+1} p^{i-j} a_i(j)$ *we have*

$$\sum_{j=0}^{n} p^j \mathfrak{s}(\alpha_j) + (a_n(0) + a_{n+1}(1)) \leqslant p^h - \sum_{j=1}^{h} (p^j - (j+1)) x_j$$

3) *if* $a_{n+1}(1) = 1$ *then the monomial has the form*

$$\lambda_0^{p^{n+1}} \lambda_n^p.$$

Proof. The fact that $\alpha_n(0) \neq p$ follows from the fact that in the polynomial S_{n+1}^{n+2} the coefficient of $X_n(1)^p$ is zero (the proof is the usual one: see lemma 6.15). Assertion 2) is deduced from 1) (and the fact that $f \geqslant 2$). Assertion 3) follows noticing that $(Pot_n^2(z))^p = 2\lambda_0^{p^{n+1}} \lambda_n^p + x$ where $x \in \mathbf{F}_p[\lambda_0, \ldots, \lambda_{n-1}]$. $\square$

We recall the ring morphism $u_n : \mathbf{F}_p[\lambda_0, \ldots, \lambda_n \mu] \to \mathbf{F}_p[\lambda_0, \ldots, \lambda_n \mu]$ (cf. 6.4.2). If $i_{n+1}^{(s)} \in \mathbf{N}$ deduce the following

Lemma 5.5. *In the preceeding notations, a monomial issued from* $u_n(\widetilde{U}_{n+1})^{i_{n+1}^{(s)}}$ *has the following form*

$$(\lambda_0^{p^{n+1}} \lambda_n^p)^{B_{n+1}^{(s)}(1)} \lambda_n^{B_n^{(s)}(0)} \lambda_{n-1}^{\beta_{n-1}} \cdot \ldots \lambda_0^{\beta_0}$$

where the exponents verify the following properties:

1) *we have*

$$\mathfrak{s}(\beta_0 + p^{n+1} B_{n+1}^{(s)}(1)) + \sum_{j=1}^{n} p^j \mathfrak{s}(\beta_j) + (B_n^{(s)}(0) + B_{n+1}^{(s)}(1)) \leqslant p^{n+1} i_{n+1}^{(s)} - \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (p^{j-i} - (j+1)) A_i(j)$$

for suitable integers $A_i(j) \in \mathbf{N}$*;*

2) *we have* $A_i(j) = 0$ *for all couples* (i, j) *if and only if* $i_{n+1}^{(s)} = 0$*;*
3) *we have* $0 \leqslant B_{n+1}^{(s)}(1) \leqslant A_{n+1}^{(s)}(1)$.

Proof. From the proof of lemma 6.18 we see that a fixed monomial $\lambda_0^{\alpha_0} \cdot \cdots \cdot \lambda_n^{\alpha_n}$ issued from $u_n(\widetilde{U}_{n+1})^{i_{n+1}^{(s)}}$ is pseudohomogeneous of degree $d \overset{\text{def}}{=} p^{n+1} i_{n+1}^{(s)} - \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (p^{j-i} - (j+1)) a_i(j)$, where the integers $a_i(j) \in \mathbf{N}$ are not all equal to zero, except if $i_{n+1}^{(s)} = 0$. Since $u_j(\lambda_j)$ is pseudohomogeneous of degree p^j (lemma 6.18), it follows from lemma 6.12 that a monomial $\lambda_0^{\beta_0'} \cdot \lambda_n^{\beta_n'}$ issued from $u_n(\lambda_0^{\alpha_0} \cdot \cdots \cdot \lambda_n^{\alpha_n})$ is pseudohomogeneous of degree d so that, in particular, it verifies

$$\sum_{j=0}^{n} p^j \mathfrak{s}(\beta_j') \leqslant p^{n+1} i_{n+1}^{(s)} - \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (p^{j-i} - (j+1)) a_i(j).$$

Moreover, from lemma 5.4, we deduce that $\beta_n' = b_n(0) + p b_{n+1}(1)$ where $0 \leqslant b_n(0) \leqslant a_n(0)$ and $0 \leqslant b_{n+1}(1) \leqslant a_{n+1}(1)$. Using lemma 5.4 the conclusion follows easily. $\square$

We go back to the study of the action of g. As for lemma 4.13, we write

$$gF^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) = \sum_{j=0}^{n}\sum_{\underline{i}_{j+1}\leqslant\underline{l}_{j+1}}\binom{\underline{l}_{j+1}}{\underline{i}_{j+1}}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}\begin{bmatrix}1 & 0\\ p^j[\lambda_j] & 1\end{bmatrix}[1, f_{\underline{l}_{n+1}-\underline{i}_{n+1}}]$$

where for notational convenience, we commit the abuse of writing $\begin{bmatrix}1 & 0\\ p^0[\lambda_0] & 1\end{bmatrix}$ instead of $\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}$ and where we have set $\underline{i}_0 \overset{\text{def}}{=} \underline{0}$,

$$f_{\underline{l}_{n+1}-\underline{i}_{n+1}} \overset{\text{def}}{=} (-1)^{\underline{i}_{n+1}}X^{\underline{r}-(\underline{l}_{n+1}+\underline{i}_{n+1})}Y^{\underline{l}_{n+1}-\underline{i}_{n+1}}$$

and $V_{j+1} \overset{\text{def}}{=} -u_j(\widetilde{U}_{j+1})$ for $j\in\{0,\dots,n\}$. We develop the polynomials $V_{j+1}^{\underline{i}_{j+1}}$, recognizing again a sum of elements of the basis $\mathscr{B}^+_{n+1/n}$: we pick a vector

$$v \overset{\text{def}}{=} F^{(0,n)}_{\lceil\kappa_0\rceil,\dots,\lceil\kappa_n\rceil}(\lceil\kappa_{n+1}\rceil);$$

as in the previous paragraph we write for $0\leqslant a\leqslant n+1$

$$\underline{\kappa}_a = \underline{l}_a - \underline{i}_a + \sum_{b=a+1}^{n+1}p^{\lfloor a-b\rfloor}\kappa_a^{(b)}$$

and, for $a+1\leqslant b\leqslant n+1$ we have

$$\kappa_a^{(b)} = \sum_{s=0}^{f-1}p^s\kappa_a^{(b),s}$$

where $\kappa_a^{(b),s}$ is the exponent of λ_a in $V_b^{i_b^{(s)}}$. Again, using the notations of lemmas 5.4 and 5.5, we focus our attention on

$$\begin{aligned}\underline{\kappa}_n &= \underline{l}_n - \underline{i}_n + p^{\lfloor -1\rfloor}\kappa_n^{(n+1)} =\\ &= \sum_{h=0}^{k}p^{\lfloor s_{m+h}+n\rfloor}(l_n^{(\lfloor s_{m+h}+n\rfloor)} - i_n^{(\lfloor s_{m+h}+n\rfloor)} + B_n^{\lfloor s_{m+h}+1+n\rfloor}(0) + pB_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1)) +\\ &\quad + \sum_{h=k+1}^{f-1}p^{\lfloor s_{m+h}+n\rfloor}(l_n^{(\lfloor s_{m+h}+n\rfloor)} - i_n^{(\lfloor s_{m+h}+n\rfloor)})\end{aligned}$$

(where we can again assume $\underline{\kappa}_n\neq 0$) and we distinguish the following four possibilities.

I). Assume $\sum_{h=0}^{k}B_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1)=0$. The condition $v\notin\mathfrak{V}_{s_{m+k}}$ imposes that

$$\mathfrak{s}(\kappa_n) = \sum_{s=0}^{f-1}l_n^{(s)} - i_n^{(s)} + B_n^{\lfloor s+1\rfloor} - \widetilde{j}(p-1)$$

for $\widetilde{j}\in\mathbf{N}$, $\widetilde{j}\geqslant 1$. We recall that for each $j\in\{0,\dots,n-1\}$ the polynomial V_j is pseudohomogeeous of degree $p^j-(p-2)$ so that the subadditivity of $\mathfrak{s}$ and lemma 5.5 give

$$\sum_{j=0}^{n+1}p^j\mathfrak{s}(\kappa_j) \leqslant \sum_{j=0}^{n+1}p^j\mathfrak{s}(\underline{l}_j) - (p-2)(\sum_{j=0}^{n+1}\mathfrak{s}(\underline{i}_j)) - p^n\widetilde{j}(p-1)$$

and the conclusion follows.

II). Assume $\sum_{h=0}^{k} B_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1)\geqslant 2$. Then we have

$$\sum_{s=0}^{f-1}\sum_{j=0}^{n}p^j\mathfrak{s}(\kappa_j^{(n+1),s})\leqslant p^{n+1}\mathfrak{s}(\underline{i}_{n+1})-2p^n(p-2).$$

The conclusion is now easy and left to the reader.

III). Assume $1=\sum_{h=0}^{k} A_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1)=\sum_{h=0}^{k} B_{n+1}^{\lfloor s_{m+h}+1+n\rfloor}(1)=1$. Let $h_1\in\{0,\dots,k\}$ the unique integer such that $B_{n+1}^{\lfloor s_{m+h_1}+1+n\rfloor}(1)=1$. We can again distinguish the following two subcases:

III)$_{\mathrm{A}}$ Assume

$$\mathfrak{s}(\kappa_n)=\sum_{s=0}^{f-1}(l_n^{(s)}-i_n^{(s)}+B_n^{(s+1)}(0)+B_{n+1}^{(s)}(1))-\widetilde{j}(p-1)$$

for $\widetilde{j}\in\mathbf{N}$, $\widetilde{j}\geqslant 1$. In this case the reader can check that

$$\sum_{j=0}^{n+1}p^j\mathfrak{s}(\kappa_j)\leqslant\sum_{j=0}^{n+1}p^j\mathfrak{s}(\underline{l}_j)-(p-2)(\sum_{j=0}^{n}\mathfrak{s}(\underline{i}_j))-p^n\widetilde{j}(p-1)-(p-2)p^n$$

and the conclusion follows.

III)$_{\mathrm{B}}$ Assume finally

$$\mathfrak{s}(\kappa_n)=\sum_{s=0}^{f-1}(l_n^{(s)}-i_n^{(s)}+B_n^{(s+1)}(0)+B_{n+1}^{(s)}(1)).$$

Such condition, together with $v\notin\mathfrak{V}_{s_{m+k}}$ imposes that $\lfloor s_{m+h_1}+1\rfloor\notin\{s_m,\dots,s_{m+k}\}$; by minimality of k' we conclude that $\lfloor s_{m+h_1}+1\rfloor=s_{m+k'}$; in particular $r_{s_{m+k'}}>0$. We deduce that the choosen monomial of $u_n(\widetilde{U}_{n+1}^{\frac{1}{p^{n+1}}})^{\underline{i}_{n+1}}$ is of the form

$$\lambda_0^{\alpha'_0}\cdot\dots\cdot\lambda_n^{\alpha'_n}(\lambda_0\lambda_n^{\frac{1}{p^n}})^{p^{\lfloor s_{m+h_1}+1+n\rfloor}}$$

where the integers α'_j verify

$$\sum_{j=0}^{n}p^j\mathfrak{s}(\alpha'_j)\leqslant(p^{n+1}-(p-2))(\mathfrak{s}(\underline{i}_{n+1}-1)).$$

By subadditivity of the function $\mathfrak{s}$ we find finally

$$\sum_{j=0}^{n+1}p^j\mathfrak{s}(\kappa_j)\leqslant\sum_{j=0}^{n+1}p^j\mathfrak{s}(\underline{l}_j)-(p-2)(\sum_{j=0}^{n}\mathfrak{s}(\underline{i}_j))+(p^{n+1}-(p-2))(\mathfrak{s}(\underline{i}_{n+1})-1)+$$
$$+(1+p^n)-p^{n+1}\mathfrak{s}(\underline{i}_{n+1})$$

(where the integer $1+p^n$ is deduced from the monomial $\lambda_0\lambda_n^{\frac{1}{p^n}}$) and the conclusion follows easily (notice that $\sum_{j=0}^{n+1}\mathfrak{s}(\underline{i}_j)\geqslant 1$).

The proof of proposition 5.3 is therefore complete.

Remark 5.6. *The reader has noticed that if we assume $r_s\leqslant p-2$ for all $s\in\{0,\dots,f-1\}$ then the inequality (41) in the statement can be replaced by the following, stronger, inequality*

$$N_{0,n+1}(\underline{l}_0(i_0),\dots,\underline{l}_{n+1}(i_0))\leqslant p^n+J-2.$$

5.1.2 ***The case*** $n = 0$**.** In this section we show that the $K_0(p)$-structure of $(R_1/R_0)^+$ is actually slightly more complicated than expected, at least under some particular conditions on the f-tuple $\underline{r}$. The negative counterpart will be the $K_0(p)$-structure of $R_0^- \oplus_{R_1^-} R_2^-$ which is left to the reader. The aim is to give an analogue of proposition 5.3 in the case $n = 0$: in the next three paragraphs we will analyse where and how a statement of such a kind fails to hold true, detecting some condition on the f-tuple $\underline{r}$. The main statements are propositions 5.8, 5.9 and 5.10, where we see that the $K_0(p)$-socle filtration for $(R_1/R_0)^+$ can be obtained from the associated lattice $\mathscr{R}^+_{1/0}$, with some harmless adjustment in few special cases (according to the combinatoric of $\underline{r}$).

In what follows, we fix $k \in \{0, \dots, f-1\}$ and an element $F^{(0)}_{\underline{l}_0}(\underline{l}_1) \in \mathfrak{V}_{s_{m+k}} \setminus \langle F^{(0)}_{\underline{r}}(\underline{0})\rangle_{\overline{\mathbf{F}}_p}$. Let $g \in K_0(p)$. We fix an element $v = F^{(0)}_{\lceil \underline{\kappa}_0 \rceil}(\lceil \underline{\kappa}_1 \rceil)$ appearing (with a nonzero linear coefficient) in the $\overline{\mathbf{F}}_p$-linear development of $gF^{(0)}_{\underline{l}_0}(\underline{l}_1)$, for suitable integers $\underline{\kappa}_0, \underline{\kappa}_1 \in \mathbf{N}$.

We assume there exists an integer $k' \in \{k+1, \dots, f-1\}$ such that $v \notin \mathfrak{V}_{s_{m+k'}} \setminus \mathfrak{V}_{s_{m+k}}$ and k' is minimal with respect to this property.

The next lemma can be verified by an easy computation on the ring $\mathbf{W}_1(\mathbf{F}_q)$:

Lemma 5.7. *In the previous hypothesis we have*

$$N_{0,1}(\underline{\kappa}_0, \underline{\kappa}_1) = N_{0,1}(\underline{l}_0, \underline{l}_1) - \epsilon$$

where

1) *if* $g \in \begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ *then* $\epsilon = \mathfrak{s}(\underline{i}_0) + \mathfrak{s}(\underline{i}_1) + \widetilde{j}(p-1)$ *where* $\widetilde{j} \geqslant 1$ *and* $\mathfrak{s}(\underline{i}_0) + \mathfrak{s}(\underline{i}_1) \geqslant 1$*;*
2) *if* $g \in \begin{bmatrix} 1 + p\mathscr{O}_F & 0 \\ 0 & 1 + p\mathscr{O}_F \end{bmatrix}$ *then* $\epsilon = \mathfrak{s}(\underline{i}_1)(p-1) + \widetilde{j}(p-1)$ *where* $\mathfrak{s}(\underline{i}_1) \geqslant 1$ *and* $\widetilde{j} \in \mathbf{N}$*;*
3) *if* $g \in \begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$ *then* $\epsilon = \mathfrak{s}(\underline{i}_1)(p-2) + \widetilde{j}(p-1)$ *where* $\mathfrak{s}(\underline{i}_1) \geqslant 1$ *and* $\widetilde{j} \in \mathbf{N}$.

Moreover:

1_A) *if in case* 1) *we have* $\widetilde{j} = 1$ *then we necessarily have* $s_{m+k'} = \lfloor s+1 \rfloor$ *for an index* s *verifying* $s \in \{s_m, \dots, s_{m+k}\}$ *and* $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$*; moreover* $r_{s_{m+k'}} > 0$*;*

2_B) *if in case* 2) *we have* $\widetilde{j} = 0$ *and* $\mathfrak{s}(\underline{i}_1) = 1$ *then we have*

$$\lceil \kappa_0 \rceil = (l_0^{(0)}, \dots, l_0^{(s)}, l_0^{\lfloor s+1 \rfloor} + 1, l_0^{\lfloor s+2 \rfloor}, \dots, l_0^{(f-1)})$$

where the index s *verify* $s \in \{s_m, \dots, s_{m+k}\}$ *and* $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$*. Furthermore* $r_{\lfloor s+1 \rfloor} = r_{s_{m+k'}} > 0$*.*

3_B) *if in case* 3) *we have* $\widetilde{j} = 0$ *and* $\mathfrak{s}(\underline{i}_1) = 1$ *then we have*

$$\lceil \kappa_0 \rceil = (l_0^{(0)}, \dots, l_0^{(s)}, l_0^{\lfloor s+1 \rfloor} + 2, l_0^{\lfloor s+2 \rfloor}, \dots, l_0^{(f-1)})$$

where the index s *verify* $s \in \{s_m, \dots, s_{m+k}\}$ *and* $\lfloor s+1 \rfloor \notin \{s_m, \dots, s_{m+k}\}$*. Furthermore* $r_{\lfloor s+1 \rfloor} = r_{s_{m+k'}} > 0$*.*

Proof. The proof, a direct computation, is left to the reader. □

The description of the socle filtration for $(R_1/R_0)^+$ is can be easily deduced from lemma 5.7. Nevertheless we still have to distinguish three cases, according to the combinatoric of the f-tuple $\underline{r}$.

PROPOSITION 5.8. *Assume that the f-tuple verifies one of the following hypothesis:*

I_A). For each $s \in \{0, \dots, f-1\}$ the condition

$$\begin{cases} r_s \geqslant r_{\lfloor s+1 \rfloor} \geqslant 1 \\ r_s - r_{\lfloor s+1 \rfloor} \in \{p-2, p-3\} \end{cases}$$

is false.

I_B). The f-tuple is of the form $(0, \dots, 0, r_{s_m}, 0, \dots, 0)$.

Then the socle filtration, together with the extensions between two consecutive graded pieces, of $(R_1/R_0)^+$ is described by the associated lattice $\mathscr{R}^+_{1/0}$.

Proof. Assume first that $\sum_{s=0}^{f-1}(r_s) \geqslant r_s + 1$ for all $s \in \{0, \dots, f-1\}$. Then it sufficies to show that the socle filtration (and the extensions between two consecutive graded pieces) of $(R_1/R_0)^+/\langle F_{\underline{r}}^{(0)}(\underline{0})\rangle$ is described by the lattice $\mathscr{R}^+_{1/0} \setminus \{F_{\underline{r}}^{(0)}(\underline{0})\}$.

To this aim, it is enough to show that if $N_{(0,1)}(\underline{l}_0, \underline{l}_1) = (r_{s_{m+k}} + 1) + J$ then

$$N_{(0,1)}(\underline{\kappa}_0, \underline{\kappa}_1) \leqslant (r_{s_{m+k'}} + 1) + J - 2.$$

The latter inequality can be checked to hold true using lemma 5.7.

Assume now hypothesis I_B) to hold true. In this case it sufficies to show that if $F_{\underline{l}_0}^{(0)}(\underline{l}_1) \in \mathfrak{V}_{s_m}$ verify $N_{(0,1)}(\underline{l}_0, \underline{l}_1) = (r_{s_{m+k}}) + J$ then

$$N_{(0,1)}(\underline{\kappa}_0, \underline{\kappa}_1) \leqslant (r_{s_{m+k'}} + 1) + J - 2.$$

As $r_{s_{m+k'}} = 0$, this certainly hold true if $r_{s_m} < p-1$. But if $r_{s_m} = p-1$ then $\mathfrak{V}_{s_m} = \{0\}$ and the conclusion follows. □

PROPOSITION 5.9. *Assume that for all $s \in \{0, \dots, f-1\}$ we have $\sum_{s=0}^{f-1}(r_s) \geqslant r_s + 1$ and that the condition*

$$\begin{cases} r_s \geqslant r_{\lfloor s+1 \rfloor} \geqslant 1 \\ r_s - r_{\lfloor s+1 \rfloor} = p-2 \end{cases}$$

is false.

Then the socle filtration for $(R_1/R_0)^+$ is described by the lattice $\mathscr{R}^+_{1/0}$.

Proof. As $\sum_{s=0}^{f-1}(r_s + 1) \geqslant r_s + 1$ for all $s \in \{0, \dots, f-1\}$ it sufficies to show that the socle filtration of $(R_1/R_0)^+/\langle F_{\underline{r}}^{(0)}(\underline{0})\rangle$ is described by the lattice $\mathscr{R}^+_{1/0} \setminus \{F_{\underline{r}}^{(0)}(\underline{0})\}$. In euclidean termes, this means that we have to prove the inequality

$$N_{(0,1)}(\underline{\kappa}_0, \underline{\kappa}_1) \leqslant (r_{s_{m+k'}} + 1) + J - 1$$

if $N_{(0,1)}(\underline{l}_0, \underline{l}_1) = (r_{s_{m+k}} + 1) + J$. This can be checked by lemma 5.7. □

We finally deal with the remaining case -the socle filtration is here slightly more complicated: in euclidean terms, the blocks $\mathfrak{V}_{s_{m+k}}$ for $r_{s_{m+k}} = p-1$ should be cutted by the hyperplanes $X_0 + \dots + X_{f-1} = (r_{s_{m+k}} + 1) + J$ or $X_0 + \dots + X_{f-1} = (r_{s_{m+k}} + 1) + J - 1$ according to a condition on $r_{s_{m+k}+1}$.

PROPOSITION 5.10. *Assume there exists an index $s \in \{0, \dots, f-1\}$ such that $r_s = p-1$ and $r_{\lfloor s+1 \rfloor} = 1$. Up to reordering, we assume there exists integers $0 \leqslant k_1 \leqslant k_0$ such that $r_{s_{m+j}} = p-1$ for all $j \in \{0, \dots, k_0\}$ and*

$$\begin{cases} r_{\lfloor s_{m+j}+1 \rfloor} \neq 1 & \text{if} \quad 0 \leqslant j \leqslant k_1 - 1, \\ r_{\lfloor s_{m+j}+1 \rfloor} = 1 & \text{if} \quad k_1 \leqslant j \leqslant k_0. \end{cases}$$

Then the J-th factor for the socle filtration of $(R_1/R_0)^+$ is described by the subspace

$$\mathscr{V}_J \stackrel{\text{def}}{=} \langle F_{\underline{r}}^{(0)}(\underline{0})\rangle_{\overline{\mathbf{F}}_p} + \sum_{k=0}^{f-1} \langle F_{\underline{l}_0}(\underline{l}_1) \in \mathfrak{V}_{s_{m+k}}, \quad N_{(0,1)}(\underline{l}_0, \underline{l}_1) \leqslant (r_{s_{m+k}}+1) + J - \delta_{k_1\leqslant k\leqslant k_0}\rangle_{\overline{\mathbf{F}}_p}.$$

In particular, the socle filtration is deduced from the lattice $\mathscr{R}^+_{1/0}$ by cutting the k-th block by the hyperplane $X_0+\cdots+X_{f-1} = (r_{s_{m+k}}+1)+J-\delta_{k_1\leqslant k\leqslant k_0}$.

Proof. As $\sum_{s=0}^{f-1}(r_s+1) \geqslant r_s+1$ for all $s\in\{0,\dots,f-1\}$ it sufficies to show that the socle filtration of $(R_1/R_0)^+/\langle F_{\underline{r}}^{(0)}(\underline{0})\rangle$ is described by the lattice $\mathscr{R}^+_{1/0}\setminus\{F_{\underline{r}}^{(0)}(\underline{0})\}$ by by cutting the k-th block by the hyperplane $X_0+\cdots+X_{f-1} = (r_{s_{m+k}}+1)+J-\delta_{k_1\leqslant k\leqslant k_0}$. To this aim, we have to prove the inequality

$$N_{(0,1)}(\underline{\kappa}_0,\underline{\kappa}_1) \leqslant ((r_{s_{m+k'}}+1)+J-\delta_{k_1\leqslant k'\leqslant k_0})-1$$

if $N_{(0,1)}(\underline{l}_0,\underline{l}_1) = (r_{s_{m+k}}+1)+J-\delta_{k_1\leqslant k\leqslant k_0}$. As, by lemma 5.7, we have $N_{(0,1)}(\underline{\kappa}_0,\underline{\kappa}_1) \leqslant N_{(0,1)}(\underline{l}_0,\underline{l}_1)-(p-2)$ the result follows. □

5.1.3 ***Application: the universal representation contains infinitely many compact inductions.*** As annonced in the introduction of §5 we are able to describe a G-equivariant natural injection

$$c\text{–}\mathrm{Ind}_{KZ}^G V \hookrightarrow \pi(\underline{r},0,1)$$

for $\underline{r}\notin\{\underline{0},\underline{p-1}\}$ where V is a convenient KZ-subrepresentation of $\pi(\underline{r},0,1)|_{KZ}$. An analogous result has been discovered by Paskunas in an unpublished draft.

The proof can be outlined as follow. Via the isomorphism of proposition 2.9 we define the representation V as a suitable subrepresentation of R_1/R_0: by Frobenius reciprocity we get a morphism $\phi: c\text{–}\mathrm{Ind}_{KZ}^G V \to \pi(\underline{r},0,1)$. From a basis of V we construct a convenient $\overline{\mathbf{F}}_p$-basis for the compact induction $c\text{–}\mathrm{Ind}_{KZ}^G V$ and therefore we only have to check that ϕ maps such basis into a linearly independent family of $\pi(\underline{r},0,1)$.

This can be easily verified combinig proposition 3.5, lemma 5.1 and proposition 3.7.

We start from the following elementary fact:

Lemma 5.11. *The K subrepresentation $\mathrm{Fil}^{\underline{0}}(R_1)$ of R_1 generated by $[1,X^{\underline{r}}]$ is naturally isomorphic to the finite principal series $\mathrm{Ind}_{K_0(p)}^K \chi_{\underline{r}}^s$ and $\mathrm{soc}(\mathrm{Fil}^{\underline{0}}(R_1))\cong R_0$ via the monomorphism $R_0\hookrightarrow R_1$.*

Proof. Obvious. □

Let $\widetilde{V}$ denote the kernel of the natural map

$$\mathrm{Fil}^{\underline{0}}(R_1)/R_0 \twoheadrightarrow \mathrm{cosoc}(\mathrm{Fil}^{\underline{0}}(R_1));$$

we define $V\leqslant \pi(\underline{r},0,1)|_{KZ}$ as the homomorphic image of $\widetilde{V}$ via the isomrphism given in 2.9. Therefore, by Frobenius reciprocity, we get a morphism

$$\phi: c\text{–}\mathrm{Ind}_{KZ}^G V \to \pi(\underline{r},0,1).$$

We claim that

Theorem 5.12. *Assume $\underline{r}\notin\{\underline{0},\underline{p-1}\}$. Then ϕ is a monomorphism.*

Proof. We show that the composite morphism of ϕ with the isomorphism (31)

$$c\text{−}\mathrm{Ind}_{KZ}^{G}V \xrightarrow{\phi} \pi(\underline{r},0,1) \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{n\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$$

maps an $\overline{\mathbf{F}}_p$-basis of $c\text{−}\mathrm{Ind}_{KZ}^{G}V$ onto a linearly independent family of the amalgamed sums on the right hand side.

By the well known results concerning the structure of finite principal series for $\mathrm{GL}_2(\mathbf{F}_q)$ we have

Lemma 5.13. *Assume $\underline{r} \notin \{\underline{0}, \underline{p-1}\}$. For an f-tuple $\underline{t} \in \{0,\dots,p-1\}^f$ such that $\underline{t} \nleq \underline{r}$ and $\underline{r} \nleq \underline{t}$ the element $v_{\underline{t}} \in V$ is defined as*

$$v_{\underline{t}} \overset{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_q} \mu_0^{\underline{t}} \begin{bmatrix} p & [\mu_0] \\ 0 & 1 \end{bmatrix} [1, X^{\underline{r}}].$$

An $\overline{\mathbf{F}}_p$-basis $\mathcal{V}$ for the compact induction is described by the elements

$$\begin{aligned}
&G_{\emptyset}^{(0,-1)}(\underline{t}) \overset{\mathrm{def}}{=} [1, v_{\underline{t}}]\\
&G_{\underline{l}_0,\dots,\underline{l}_n}^{(1,n)}(\underline{t}) \overset{\mathrm{def}}{=} \sum_{\lambda_1 \in \mathbf{F}_q} (\lambda_1^{\frac{1}{p}})^{\underline{l}_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1^{\frac{1}{p}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n} \begin{bmatrix} 0 & 1 \\ p^{n+1} & 0 \end{bmatrix} [1, v_{\underline{t}}]\\
&G_{\underline{l}_0,\dots,\underline{l}_n}^{(0,n)}(\underline{t}) \overset{\mathrm{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_q} \lambda_0^{\underline{l}_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, G_{\underline{l}_0,\dots,\underline{l}_n}^{(1,n)}(\underline{t})]
\end{aligned}$$

where $n \in \mathbf{N}$, $\underline{l}_j \in \{0,\dots,p-1\}^f$ for all $j \in \{0,\dots,n\}$, and $\underline{t} \in \{0,\dots,p-1\}^f$ verify the conditions $\underline{t} \nleq \underline{r}$ and $\underline{r} \nleq \underline{t}$.

Proof. It is well known that the family

$$\left\{v_{\underline{t}}, \quad \text{such that}\, \underline{t} \nleq \underline{r}\, \text{and}\, \underline{r} \nleq \underline{t}\right\}$$

describe an $\overline{\mathbf{F}}_p$-basis for V (see for istance [BP], lemma 2.7). By coset decomposition (30) we deduce that an $\overline{\mathbf{F}}_p$-basis for the compact induction is given by the family

$$\begin{aligned}
&[1, v_{\underline{t}}]\\
&[\begin{bmatrix} p^{n+1} & \sum_{j=0}^{n} p^j [\lambda_j^{\frac{1}{p^j}}] \\ 0 & 1 \end{bmatrix}, v_{\underline{t}}]\\
&[\begin{bmatrix} 1 & 0 \\ \sum_{j=1}^{n} p^j [\lambda_j^{\frac{1}{p^j}}] & p^{n+1} \end{bmatrix}, v_{\underline{t}}]
\end{aligned}$$

for $n \in \mathbf{N}$, $\lambda_j \in \mathbf{F}_q$ and $\underline{t}$ is as in the statement.

An immediate induction using Vandermonde matrices yields the results. □

We recall that the morphism ϕ is G-equivariant and the isomorphism (31) is KZ-equivariant. Therefore, thanks to the matrix equality

$$\begin{bmatrix} 0 & 1 \\ p^{n+1} & 0 \end{bmatrix} \begin{bmatrix} p & [\mu_0] \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p^{n+1}[\mu_0] & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ p^{n+1} & 0 \end{bmatrix}$$

we see that

$$\phi(G^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{t})) = pr(F^{(0,n+1)}_{\underline{l}_0,\dots,\underline{l}_n,\underline{t}}(\underline{0}))$$
$$\phi(G^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{t})) = pr(F^{(1,n+1)}_{\underline{l}_1,\dots,\underline{l}_n,\underline{t}}(\underline{0}))$$
$$\phi(G^{(0,-1)}_{\emptyset}(\underline{t})) = pr(F^{(0)}_{\underline{t}}(\underline{0}))$$

where we wrote pr to denote the natural epimorphisms of proposition 3.7.

But the kernel of the epimorphism pr is known and we dispose of a suitable $\overline{\mathbf{F}}_p$-basis of the inductive limits $\varinjlim_{n\,\mathrm{odd}} R_0^{\pm} \oplus_{R_1^{\pm}} \cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$, $\varinjlim_{n\,\mathrm{even}} (R_1/R_0)^{\pm} \oplus_{R_2^{\pm}} \cdots \oplus_{R_n^{\pm}} R_{n+1}^{\pm}$ by the evident induction using proposition 3.5 and lemma 5.1. An immediate check let us conclude that the elements $pr(F^{(0,n+1)}_{\underline{l}_0,\dots,\underline{l}_n,\underline{t}}(\underline{0}))$, $pr(F^{(1,n+1)}_{\underline{l}_1,\dots,\underline{l}_n,\underline{t}}(\underline{0}))$ and $pr(F^{(0)}_{\underline{t}}(\underline{0}))$ of the inductive limits $\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$, $\varinjlim_{n\,\mathrm{even}} (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ are linearly independent, as required. □

Remark 5.14. *Let $\mathfrak{V}$ the image of the composite map obtained by ϕ and the isomorphism (31). By the proof of proposition 5.12 the reader can easily describe, in terms of the lattices $\cdots \oplus_{\mathscr{R}_n^{\pm}} \mathscr{R}_{n+1}^{\pm}$, the inverse image of $\mathfrak{V}$ by the natural epimorphism pr of proposition 3.7.*

5.2 The structure of the amalgamed sums

We are now ready to describe two blocks $R^{\bullet}_{n+1}/R^{\bullet}_n$ and $R^{\bullet}_{n-1}/R^{\bullet}_{n-2}$ should be glued together. We will see that such glueing is more or less a formal consequence of the geometric interpretation of the amalgamed sums, as annonced in the introduction of §5.

Like in section 5.1 we will give the detailed proofs for the positive case: the negative part is deduced analogously.

First, we want to understand the image of an element $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in R^+_{n+1}$ (resp. $F^{(1,n)}_{\underline{l}_1,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in R^-_{n+1}$) via the projection $(pr_{n+1})^{\mathrm{pos}}$ (resp. $(pr_{n+1})^{\mathrm{neg}}$) of lemma 3.5.

Lemma 5.15. *Let $n \in \mathbf{N}_{\geqslant 1}$. The image of the element $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}) \in R^+_{n+1}$ via the projection pr^{pos}_{n+1} is described as follow:*

1) *If either $\underline{l}_{n+1} \neq \underline{0}$ or $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \nleqslant \underline{r}$ then*
$$\pi_{n+1}(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})) = \pi_{n+1}(F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1}));$$
2) *If $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ and $\underline{l}_{n-1} \geqslant \underline{p-1-r}$ then*
$$(-1)^{\underline{r}}(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})) = \iota^{\mathrm{pos}}_{n-1}(F^{(0,n-2)}_{\underline{l}_0,\dots,\underline{l}_{n-2}}(\underline{l}_{n-1} - \underline{p-1-r})) + \delta_{\underline{r},\underline{p-1}}\delta_{\underline{l}_{n-1},\underline{p-1}}\iota^{\mathrm{pos}}_{n-1}(F^{(0,n-2)}_{\underline{l}_0,\dots,\underline{l}_{n-2}}(\underline{0}));$$
3) *If either $\underline{l}_{n+1} = \underline{0}$, $\underline{l}_n = \underline{r}$ and $\underline{l}_{n-1} \ngeqslant \underline{p-1-r}$ or $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \lneqq \underline{r}$ then*
$$(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{l}_{n+1})) = 0.$$

Proof. Assertion 1) is clear by lemma 5.1. We assume now that $\underline{l}_{n+1} = \underline{0}$ and $\underline{l}_n \leqslant \underline{r}$. Thus,
$$F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{0})) = (-1)^{\underline{l}_n}(T^+_n)^{\mathrm{pos}}(F^{(0,n-1)}_{\underline{l}_0,\dots,\underline{l}_{n-1}}(\underline{l}_n))$$
so that we get the following equality in the amalgamed sum $\cdots \oplus_{R^+_n} R^+_{n+1}$:
$$(pr_{n+1})^{\mathrm{pos}}(F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_n}(\underline{0})) = \iota^+_{n-1} \circ pr^+_{n-1} \circ (-T^-_n)^{\mathrm{pos}}((-1)^{\underline{l}_n}(F^{(0,n-1)}_{\underline{l}_0,\dots,\underline{l}_{n-1}}(\underline{l}_n))).$$

In order to get the statement, we are now left to describe

$$(T_n^-)^{\mathrm{pos}}((F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n))).$$

Let assume $n \geqslant 2$ (the case $n=1$ is treated in an analogous way and is left to the reader). By the characterisation of the operator T_n^- we have

$$(T_n^-)^{\mathrm{pos}}((F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n))) = 0$$

if $\underline{l}_n \neq \underline{r}$, while, for $\underline{l}_n = \underline{r}$, we have

$$\begin{aligned}
&(T_n^-)^{\mathrm{pos}}((F^{(0,n-1)}_{\underline{l}_0,\ldots,\underline{l}_{n-1}}(\underline{l}_n)))) = \\
&= \sum_{j=0}^{n-2}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j}\begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}[1,\sum_{\lambda_{n-1}\in\mathbf{F}_q}(\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{\underline{l}_{n-1}}(\lambda_{n-1}^{\frac{1}{p^{n-1}}}X+Y)\underline{r}] = \\
&= \sum_{\underline{i}\leqslant\underline{r}}\binom{\underline{r}}{\underline{i}}\sum_{j=0}^{n-2}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j}\begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1\end{bmatrix}[1, X^{\underline{r}-\underline{i}}Y^{\underline{i}}\sum_{\lambda_{n-1}\in\mathbf{F}_q}(\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{\underline{l}_{n-1}+\underline{r}-\underline{i}}].
\end{aligned}$$

By lemma 3.1, the quantity

$$\sum_{\lambda_{n-1}\in\mathbf{F}_q}(\lambda_{n-1}^{\frac{1}{p^{n-1}}})^{\underline{l}_{n-1}+\underline{r}-\underline{i}}$$

is non zero (indeed assuming the value -1) if and only if $\underline{l}_{n+1}+\underline{r}-\underline{i}\equiv 0 \bmod q-1$ and $\underline{l}_{n+1}+\underline{r}-\underline{i}\neq 0$. The result follows. $\square$

The result concerning the negative part is similar

LEMMA 5.16. *Let $n\in\mathbf{N}_{\geqslant 1}$. The image of the element $F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{l}_{n+1})\in R^-_{n+1}$ via the projection pr^{neg}_{n+1} is described as follow:*

1) *If either $\underline{l}_{n+1}\neq\underline{0}$ or $\underline{l}_{n+1}=\underline{0}$ and $\underline{l}_n\not\leqslant\underline{r}$ then*
$$\pi_{n+1}(pr_{n+1})^{\mathrm{neg}}(F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{l}_{n+1})) = \pi_{n+1}(F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{l}_{n+1}));$$
2) *If $\underline{l}_{n+1}=\underline{0}$, $\underline{l}_n=\underline{r}$ and $\underline{l}_{n-1}\geqslant\underline{p-1-r}$ (the latter condition being empty if $n=1$) then*
$$(-1)^{\underline{r}}(pr_{n+1})^{\mathrm{neg}}(F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{l}_{n+1})) = \iota^{\mathrm{neg}}_{n-1}(F^{(1,n-2)}_{\underline{l}_1,\ldots,\underline{l}_{n-2}}(\underline{l}_{n-1}-\underline{p-1-r}))+\delta_{\underline{r},\underline{p-1}}\delta_{\underline{l}_{n-1},\underline{p-1}}\iota^{\mathrm{pos}}_{n-1}(F^{(1,n-2)}_{\underline{l}_1,\ldots,\underline{l}_{n-2}}(\underline{0}));$$
3) *If either $\underline{l}_{n+1}=\underline{0}$, $\underline{l}_n=\underline{r}$ and $\underline{l}_{n-1}\not\geqslant\underline{p-1-r}$ (the latter condition being empty if $n=1$) or $\underline{l}_{n+1}=\underline{0}$ and $\underline{l}_n\lneq\underline{r}$ then*
$$(pr_{n+1})^{\mathrm{pos}}(F^{(1,n)}_{\underline{l}_1,\ldots,\underline{l}_n}(\underline{l}_{n+1})) = 0.$$

Proof. It is analogous to the proof of proposition 5.15 and it is left to the reader. $\square$

Interpretation in terms of euclidean data. We dispose of a canonical $\overline{\mathbf{F}}_p$-basis for the representation $\cdots\oplus_{R_n^\pm}R^\pm_{n+1}$, which is obtained in the obvious way by an induction from proposition 3.5 and lemma 5.1.

Exactly as we did in §5.1 we have a natural way to associate an element of such canonical basis to a point in $\mathbf{R}^f$: again, we obtain a lattice, which we will denote by $\cdots\oplus_{\mathscr{R}_n^\pm}\mathscr{R}^\pm_{n+1}$.

In such euclidean setting proposition 5.15 is clear: it tells that lattice $\cdots\oplus_{\mathscr{R}_n^+}\mathscr{R}^+_{n+1}$ is obtained as the union of the lattice $\mathscr{R}^+_{n+1/n}$ associated to R^+_{n+1}/R^+_n and the image of the lattice $\cdots\oplus_{\mathscr{R}^+_{n-2}}$

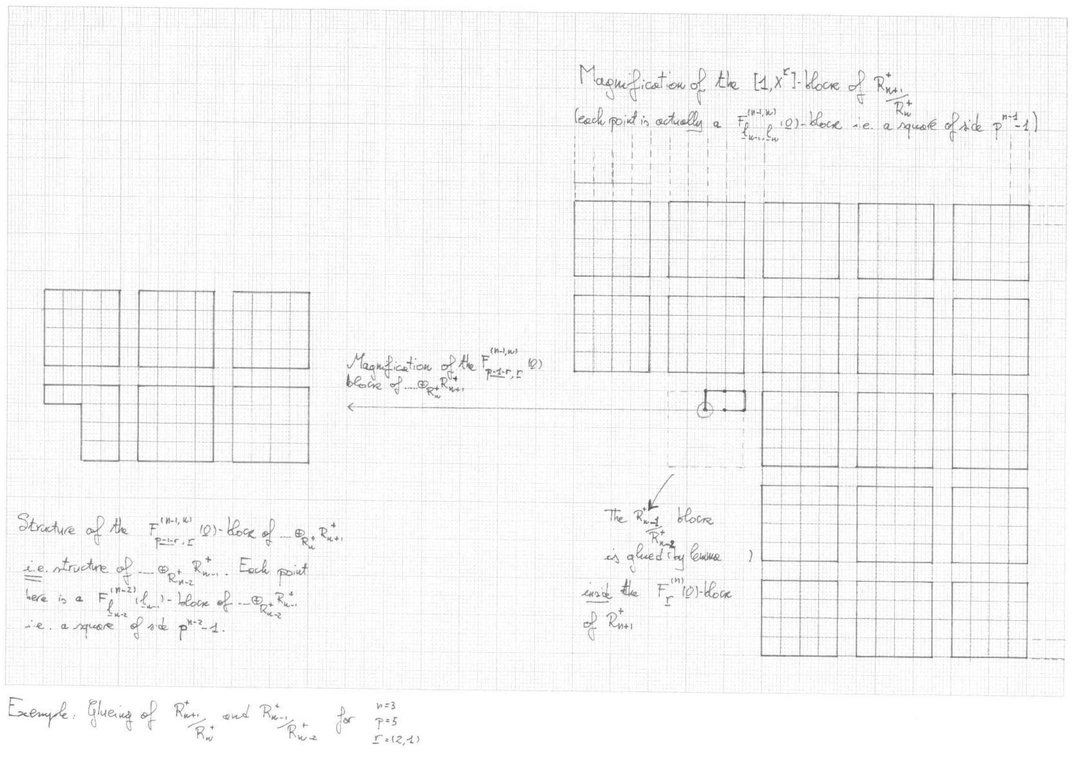


FIGURE IV.9. Again, the glueing and the fractal structure.

$\mathscr{R}^+_{n-1}$ associated to the amalgamed sum $\cdots \oplus_{R^+_{n-2}} R^+_{n-1}$ (which, inductively, can be assumed to be known) by the traslation

$$\begin{aligned} \mathbf{R}^f &\to \mathbf{R}^f \\ (x_i)_i &\mapsto (x_i + p^{n-1}(p-1-r_{\lfloor i+n-1 \rfloor}) + p^n r_{\lfloor i+n \rfloor}). \end{aligned} \tag{42}$$

Notice that in particular the lattice $\cdots \oplus_{\mathscr{R}^+_{n-2}} \mathscr{R}^+_{n-1}$ is glued inside the $F^n_{\underline{r}}(\underline{0})$-block of $\mathscr{R}^+_{n+1}$.

We stress again in figure IV.9 the glueing and the fractal structure for $f = 2$ (noticing the glueing of $\cdots \oplus_{\mathscr{R}^+_{n-2}} \mathscr{R}^+_{n-1}$ inside the $F^{(n)}_{\underline{r}}(\underline{0})$-block of $\mathscr{R}^+_{n+1/n}$).

The evident analogous considerations for the negative part $\cdots \oplus_{\mathscr{R}^-_n} \mathscr{R}^-_{n+1}$ are left to the reader.

REMARK 5.17. *Notice that if $f = 1$ then it follows directly from propositions 5.15 and 5.16 that the $K_0(p)$-structure (and the extensions between two consecutive graded pieces) of the representations $\ldots_{R^\bullet_n} R^\bullet_{n+1}$ are given by the associated lattices $\cdots \oplus_{\mathscr{R}^\bullet_n} \mathscr{R}^\bullet_{n+1}$. In particular, each of these representations has a space of I_1 invariants of dimension 1.*

By remark 5.17 we can assume $f \geqslant 2$. In the next proposition we describe the socle filtration (and the extension between two consecutive graded pieces) of the $K_0(p)$-representations $\cdots \oplus_{R^+_n} R^+_{n+1}$ for $n \geqslant 1$; the corresponding result for $\cdots \oplus_{R^-_n} R^-_{n+1}$ is similar and left to the reader.

The euclidean leitfaden which we are going to follow in order to prove the main result given in proposition 5.18 is the following. As $\cdots \oplus_{R^+_{n-2}} R^+_{n-1}$ is a $K_0(p)$-*subrepresentation* of $\cdots \oplus_{R^+_n} R^+_{n+1}$

the *only* thing we have to check is the following:

> each of the J-th cutting hyperplanes $X_0 + \cdots + X_{f-1} = p^n(r_{\lfloor n+s\rfloor} + 1) + J$ of the lattice $\mathscr{R}^+_{n+1/n}$ lies strictly below [8] any of the $J-1$-cutting hyperplanes of the lattice $\cdots \oplus_{\mathscr{R}^+_{n-2}} \mathscr{R}^+_{n-1}$.

Note that, as the cutting hyperplanes are parallel, we can assume $J = 0$.

Fix $n \geqslant 1$ and define

$$M_n \stackrel{\mathrm{def}}{=} \sum_{s=0}^{f-1} p^{n-1}(p-1-r_{\lfloor s+n-1\rfloor}) + p^n r_{\lfloor s+n\rfloor}$$

(so that the hyperplane $X_0+\cdots+X_{f-1} = M_n$ contains the image of the point $\underline{0}$ via the translation (42)).

Theorem 5.18. *Let $n \geqslant 1$ and consider the $K_0(p)$-representation $\cdots \oplus_{R^+_n} R^+_{n+1}$.*

The socle filtration and the extensions between two consecutive graded pieces are described by the associated lattice $\cdots \oplus_{\mathscr{R}^+_n} \mathscr{R}^+_{n+1}$, with the conventions of section §5.1.2 and propositions 5.8, 5.9 and 5.10 concerning the lattice associated to the $K_0(p)$-structure of $(R_1/R_0)^+$.

Proof. By the eucildean interpretation of the $K_0(p)$-structure of $\cdots \oplus_{R^+_n} R^+_{n+1}$ and an immediate induction we see that it is enough to prove the inequalities

1) for $n \geqslant 3$

$$p^n(r_{s_0}+1) < M_n + p^{n-2}(r_{s_1}+1)$$

for any all indexes $s_0, s_1 \in \{0, \ldots, f-1\}$;

2 for $n = 2$ and $s_0, s_1 \in \{0, \ldots, f-1\}$

$$p^2(r_{s_0}+1) < M_2 + (r_{s_1}+1) - \delta$$

where $\delta \in \{0, 1\}$ is nonzero if and only if either the f-tuple $\underline{r}$ verifies the hypothesis I_B) of proposition 5.8 and $s_1 = s_m$ or the the f-tuple $\underline{r}$ verifies the hypothesis of proposition 5.10 $s_1 \in \{s_{m+k_1}, \ldots, s_{m+k_0}\}$.

3) if $n = 1$

$$p(r_{s_0}+1) \leqslant M_1.$$

Inequality 1) is immediately verified, and 2), 3) are trivial if $f \geqslant 3$ or $f = 2$ and $(r_0, r_1) \notin \{(p-1,0),(0,p-1),(p-2,0),(0,p-2)\}$. Notice that if $f = 2$ and $(r_0, r_1) \in \{(p-1,0),(0,p-1)\}$ then $\mathfrak{V}_{s_m} = \{0\}$ so that it sufficies to prove inequalities 2) and 3) only for $s_0 = s_{m+1}$, i.e. $r_{s_0} = 0$, which is true. The remaining case $f = 2$ and $(r_0, r_1) \in \{(p-2,0),(0,p-2)\}$ is trivially checked and the proof is complete. $\square$

6. Appendix A: Some remarks on Witt polynomials

The aim of this appendix is to collect some technical results concerning Witt polynomials. After a section of general reminders (§6.1), we will treat in detail the case of the universal polynomials

[8] if $f = 2$ and $n = 1$ we will see that, in few cases depending on the f-tuple $\underline{r}$, the J-th cutting hyperplane $X_0 + \cdots + X_{f-1} = p(r_{\lfloor n+s\rfloor} + 1) + J$ of R^+_2/R^+_1 coincide with a J-th cutting hyperplane for R^+_0. A direct check shows that the $K_0(p)$-structure is the desired one.

for the sum and the product (§6.2 and §6.3). In section §6.4 we study the Witt polynomials of a certain power series in the ring $W(\mathbf{F}_q)$: in this situation it is more complicate to keep track of the exponents of such polynomials. We are therefore led to introduce the notion of "pseudo homogeneity" (definition 6.11), a weak condition which nevetheless gives us a small control, sufficient for our aim (see also proposition 7.4 and 7.5).

6.1 Reminder on Witt polynomials

The description of the socle filtration for the aforementioned representations of $\mathrm{GL}_2(F)$ relies crucially on the behaviour of the universal Witt polynomials. After some generalities, we focus on specific situations related to the study of the action of lower unipotent, diagonal and upper unipotent matrices in $\mathrm{GL}_2(\mathscr{O}_F)$.

For $n \in \mathbf{N}$ the n-th Witt polynomial $W_n(\underline{X}) \in \mathbf{Z}[X_0, \dots, X_n]$ is defined by

$$W_n(\underline{X}) \stackrel{\text{def}}{=} \sum_{i=0}^{n} X_i^{p^{n-i}} p^i.$$

As the ring endomorphism

$$\begin{aligned} \mathbf{Z}[\frac{1}{p}][X_0, \dots, X_n] &\xrightarrow{\omega_n} \mathbf{Z}[\frac{1}{p}][X_0, \dots, X_n] \\ X_j &\longmapsto W_j(X_0, \dots, X_j) \end{aligned}$$

is bijective, we get a family of polynomials $M_0(X_0), \dots, M_n(X_0, \dots, X_n) \in \mathbf{Z}[\frac{1}{p}][X_0, \dots, X_n]$ which are uniquely determined by the condition:

$$M_j(W_0(\underline{X}), \dots, W_n(\underline{X})) = X_j.$$

They are of course described inductively by

$$M_n = \frac{1}{p^n}(X_n - p^{n-1}M_{n-1}(\underline{X})^p - \cdots - pM_1(X_0, X_1)^{p^{n-1}} - M_0(X_0)^{p^n}).$$

The following lemma let us deduce the universal Witt polynomials describing the ring structure of $W(\mathbf{F}_q)$:

Proposition 6.1. *Let $\Phi \in \mathbf{Z}[\zeta, \xi]$ be a polynomial in the variables ζ, ξ. For all $n \in \mathbf{N}$ there exist polynomials $\phi_n \in \mathbf{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$, uniquely determied by the conditions*

$$W_n(\phi_0, \dots, \phi_n) = \Phi(W_n(X_0, \dots, X_n), W_n(Y_0, \dots, Y_n)).$$

sketch. The proof is constructive: we considering the commutative diagramm

$$\begin{tikzcd} \mathbf{Z}[\frac{1}{p}][X_0, \dots, X_n] \arrow[r, "\omega_n", "\sim"'] \arrow[d, "f"] & \mathbf{Z}[\frac{1}{p}][X_0, \dots, X_n] \arrow[d, dotted] \\ \mathbf{Z}[\frac{1}{p}][X_0, \dots, X_n, Y_0, \dots, Y_n] \arrow[r, "\omega_n \otimes \omega_n", "\sim"'] & \mathbf{Z}[\frac{1}{p}][X_0, \dots, X_n, Y_0, \dots, Y_n] \end{tikzcd}$$

where $f : \mathbf{Z}[\frac{1}{p}][\underline{X}] \to \mathbf{Z}[\frac{1}{p}][\underline{X}, \underline{Y}]$ is defined by $f(X_j) \stackrel{\text{def}}{=} \Phi(X_j, Y_j)$ for any $j \in \{0, \dots, n\}$; the polynomial ϕ_n is then given by

$$\phi_n(\underline{X}, \underline{Y}) \stackrel{\text{def}}{=} (\omega_n \otimes \omega_n) \circ f \circ \omega_n^{-1}(X_n).$$

The fact that such ϕ_n's have integer coefficients is then an induction on n. $\square$

We apply then proposition 6.1 to the polynomials

$$\Phi(\zeta,\xi)=\zeta+\xi, \Phi(\zeta,\xi)=\zeta\xi$$

to get the universal polynomials for the sum and the product respectively. They will be denoted as $S_n, Prod_n \in \mathbf{Z}[X_0,\dots,X_n,Y_0,\dots,Y_n]$ and are described inductively by

$$S_n(\underline{X},\underline{Y})=\frac{1}{p^n}(W_n(\underline{X})+W_n(\underline{Y})-p^{n-1}S_{n-1}(\underline{X},\underline{Y})^p-\dots-pS_1(\underline{X},\underline{Y})^{p^{n-1}}-S_0(\underline{X},\underline{Y})^{p^n})$$

$$Prod_n(\underline{X},\underline{Y})=\frac{1}{p^n}(W_n(\underline{X})W_n(\underline{Y})-p^{n-1}Prod_{n-1}(\underline{X},\underline{Y})^p-\dots-pProd_1(\underline{X},\underline{Y})^{p^{n-1}}-Prod_0(\underline{X},\underline{Y})^{p^n}).$$

In section 4 we are interested in such operations as either rise to the N-power or the sum of N elements. We can of course adapt the arguments of proposition 6.1 (or, use an induction on N) to determine the universal Witt polynomials associated to such operations. We will write $Pot_n^N(\underline{X})\in\mathbf{Z}[X_0,\dots,X_n]$, $S_n^N(\underline{X(1)},\dots,\underline{X(N)})\in\mathbf{Z}[X(1)_0,\dots,X(1)_n,\dots,X(N)_0,\dots,X(N)_n]$ for the n-th Witt polynomial associated to the rise to the N-power and the sum of N elements respectively. We have then the recursive relations:

$$Pot_n^N(\underline{X})=\frac{1}{p^n}(W_n(\underline{X})^N-p^{n-1}Pot_{n-1}^N(\underline{X})^p-$$
$$\dots-pPot_1^N(\underline{X})^{p^{n-1}}-Pot_0^N(\underline{X})^{p^n})$$
$$S_n^N(\underline{X(1)},\dots,\underline{X(N)})=\frac{1}{p^n}(\sum_{j=1}^N W_n(\underline{X(j)})-p^{n-1}S_{n-1}^N(\underline{X(1)},\dots,\underline{X(N)})^p-$$
$$\dots-pS_1^N(\underline{X(1)},\dots,\underline{X(N)})^{p^{n-1}}-S_0^N(\underline{X(1)},\dots,\underline{X(N)})^{p^n}).$$

6.2 Some special polynomials-I

In this paragraph we collect some thechnical results concerning some Witt polynomials which appear naturally in the study of the action of $\begin{bmatrix}1&0\\ \mathscr{O}_F&1\end{bmatrix}$ (resp. $\begin{bmatrix}1&p\mathscr{O}_F\\0&1\end{bmatrix}$) for the representations of §4.2 (resp. of §4.1).

For $n\in\mathbf{N}$ we define $S_n(\underline{X},Y_0)\in\mathbf{Z}[X_0,\dots,X_n,Y_0]$ as the specialisation of $S_n(\underline{X},\underline{Y})$ at $\underline{Y}=(Y_0,0,\dots,0,\dots)$. We recall

Lemma 6.2. *For $n\in\mathbf{N}$ the polynomial $S_n(\underline{X},\underline{Y})$ is an homogeneous polynomial in $\underline{X},\underline{Y}$, of degree p^n if we define the elemets X_j,Y_j to be homogeneous of degree p^j.*

Proof. Elementary. □

Thus, if we set

$$\widetilde{S}_n(\underline{X},Y_0)\stackrel{\text{def}}{=}S_n(\underline{X},Y_0)-X_n$$

we see that $\widetilde{S}_j(\underline{X},Y_0)$ is a polynomial in $\mathbf{Z}[X_0,\dots,X_{n-1},Y_0]$, homogeneous of degree p^n. Moreover, as $\widetilde{S}_n(\underline{X},0)=0$ we see that $\widetilde{S}_n(\underline{X},Y_0)$ belongs to the ideal generated by Y_0.

We define inductively the following family of automorphisms: we put

$$\begin{aligned}s_0:\mathbf{Z}[X_0,Y_0]&\to\mathbf{Z}[X_0,Y_0]\\ X_0&\mapsto X_0-Y_0\\ Y_0&\mapsto Y_0\end{aligned}$$

and, assuming $s_{j-1} : \mathbf{Z}[X_0, \ldots, X_{j-1}, Y_0] \to \mathbf{Z}[X_0, \ldots, X_{j-1}, Y_0]$ being constructed, we define

$$\begin{aligned} s_j : \mathbf{Z}[X_0, \ldots, X_j, Y_0] &\to \mathbf{Z}[X_0, \ldots, X_j, Y_0] \\ X_j &\mapsto X_j - s_{j-1}(\widetilde{S}_j) \end{aligned}$$

By their very construction, the s_j's are graded homomorphisms; in particular $s_j(\widetilde{S}_j)$ is homogeneous of degree p^j, and belongs to the ideal (Y_0) inside $\mathbf{Z}[X_0, \ldots, X_j, Y_0]$. We can actually prove the following result

Lemma 6.3. *For any $n \geqslant 1$ we have*

$$s_{n-1}(S_n(\underline{X}, Y_0) - X_n) = -(S_n(\underline{X}, -Y_0) - X_n).$$

Proof. The case $n = 1$ is elementary:

$$s_0(S_1(X_0, X_1, Y_0) - X_1) = s_0(\frac{1}{p}(X_0^p + Y_0^p - (X_0 + Y_0)^p)) = \frac{1}{p}((X_0 - Y_0)^p + Y_0^p - X_0^p) = -(S_1(X_0, X_1, Y_0) - X_1).$$

Concerning the general case, we write

$$S_n(X_0, \ldots, X_n, Y_0) - X_n = \frac{1}{p^n}\big[X_0^{p^n} + Y_0^{p^n} - p^{n-1}(S_{n-1}(\underline{X}, Y_0)^p - X_{n-1}^p) - \ldots \tag{43}$$

$$\cdots - p(S_1(X_0, X_1, Y_0)^{p^{n-1}} - X_1^{p^{n-1}}) - (X_0 + Y_0)^{p^n}\big]. \tag{44}$$

For $j \in \{1, \ldots, n-1\}$ we have

$$\begin{aligned} s_j(S_j(X_0, \ldots, X_j, Y_0)^{p^{n-j}} - X_j^{p^{n-1}}) &= (s_{j-1}(S_j(X_0, \ldots, X_j, Y_0) - X_j) + s_j(X_j))^{p^{n-j}} - (s_j(X_j))^{p^{n-j}} \\ &= X_j^{p^{n-j}} - (X_j - s_{j-1}(S_j(X_0, \ldots, X_j, Y_0) - X_j))^{p^{n-j}} \\ &= X_j^{p^{n-j}} - (X_j + S_j(X_0, \ldots, X_j, -Y_0) - X_j))^{p^{n-j}} \\ &= -(S_j(X_0, \ldots, X_j, -Y_0)^{p^{n-j}} - X_j^{p^{n-j}}). \end{aligned}$$

As $s_{n-1}(S_n(X_0, \ldots, X_n, Y_0) - X_n) = s_n(S_n(X_0, \ldots, X_n, Y_0) - X_n)$ we are left compute

$$\begin{aligned} & s_n(\frac{1}{p^n}\big[X_0^{p^n} + Y_0^{p^n} - p^{n-1}(S_{n-1}(\underline{X}, Y_0)^p - X_{n-1}^p) - \ldots \\ & \qquad \cdots - p(S_1(X_0, X_1, Y_0)^{p^{n-1}} - X_1^{p^{n-1}}) - (X_0 + Y_0)^{p^n}\big]) = \\ & \frac{1}{p^n}\big[(X_0 - Y_0)^{p^n} + Y_0^{p^n} - p^{n-1}s_{n-1}(S_{n-1}(\underline{X}, Y_0)^p - X_{n-1}^p) - \ldots \\ & \qquad \cdots - ps_1(S_1(X_0, X_1, Y_0)^{p^{n-1}} - X_1^{p^{n-1}}) - (X_0)^{p^n}\big] \end{aligned}$$

and the result follows as $s_j(S_j(X_0, \ldots, X_j, Y_0)^{p^{n-j}} - X_j^{p^{n-1}}) = -(S_j(X_0, \ldots, X_j, -Y_0)^{p^{n-j}} - X_j^{p^{n-j}})$ for all $j \in \{1, \ldots n-1\}$. □

We will also need a cleaner statement concerning the monomials of $S_n(X_0, \ldots, X_n, Y_0)$:

Lemma 6.4. *For all $n \geqslant 1$ the coefficient of the monomial $X_0^{p-1} \ldots X_{n-1}^{p-1} Y_0$ appearing in the development of the universal Witt polynomial $S_n(X_0, \ldots, X_n, Y_0)$ is 1.*

Proof. The proof is again an induction on n: the case $n = 1$ is evident.

For the general case, consider

$$S_n(\underline{X}, Y_0) = \frac{1}{p^n}(W_n(\underline{X}) + Y_0^{p^n} - p^{n-1}S_{n-1}(\underline{X}, Y_0)^p - \cdots - pS_1(\underline{X}, Y_0)^{p^{n-1}} - S_0(\underline{X}, Y_0)^{p^n}).$$

A monomial of the form $X_0^{p-1}\dots X_{n-1}^{p-1}Y_0$ lies therefore inside

$$-\frac{1}{p}(S_{n-1}(X_0,\dots,X_{n-1},Y_0)^p - X_{n-1}^{p-1})$$

and the inductive hypothesis yields

$$S_{n-1}(X_0,\dots,X_{n-1},Y_0) = X_{n-1} + X_0^{p-1}\dots X_{n-2}^{p-1}Y_0 + x(X_0,\dots,X_{n-2},Y_0)$$

where $x(X_0,\dots,X_{n-2},Y_0)\in\mathbf{Z}[X_0,\dots,X_{n-2},Y_0]$ doesn't contains the monomial $X_0^{p-1}\dots X_{n-2}^{p-1}Y_0$. Finally, we have

$$(S_{n-1}(X_0,\dots,X_{n-1},Y_0))^p = \sum_{\substack{i+j+k=p\\ 0\leqslant i,j,k}} \frac{p!}{i!j!k!}X_{n-1}^i(X_0^{p-1}\dots X_{n-2}^{p-1}Y_0)^j(x(X_0,\dots,X_{n-2},Y_0))^k$$

and the conclusion follows. □

6.3 Some special polynomials -II

In this section we deal with some Witt polynomials which appear naturally when we study the action of the diagonal matrices $\begin{bmatrix} 1+p\mathscr{O}_F & 0\\ 0 & 1+\mathscr{O}_F\end{bmatrix}$. Recall that

Lemma 6.5. *Let $n\in\mathbf{N}$. The n-th universal Witt polynomial of the product $Prod_n(\underline{X},\underline{Y})$ is an homogeneous element of $(Z[\underline{Y}])[\underline{X}]$ (resp. $(Z[\underline{X}])[\underline{Y}]$) provided that X_j (resp. Y_j) is homogeneous of degree p^j for any $0\leqslant j\leqslant n$.*

Proof. Elementary. □

Remark 6.6. *In the present paragraph, we will be concerned with the image in $\mathbf{F}_p[\underline{X},\underline{Y}]$ of the universal Witt polynomials $S_n(\underline{X},\underline{Y}), Prod_n(\underline{X},\underline{Y})$. Such images will be denoted again by $S_n(\underline{X},\underline{Y}), Prod_n(\underline{X},\underline{Y})$, in order not to overload notations. As $p\cdot 1=0$ multiplication by p is the composite of Frobenius and Verschiebung.*

For $N\in\mathbf{N}$, let $z'=(\lambda'_0,\dots,\lambda'_N,0\dots,0,\dots)\in W(\mathbf{F}_q)$ and let $\alpha=(\alpha_0,\alpha_1,\dots)\in W(\mathbf{F}_q)$; we need to describe

$$z'+p\alpha\cdot z' \mod p^{N+1} \tag{45}$$

in terms of the universal Witt polynomials.

Lemma 6.7. *For $0\leqslant j\leqslant$ he j-th Witt polynomial of the development of (45) is an homogeneous element $Q_j(\underline{\lambda'},\underline{\alpha})$ of degree p^j in $(\mathbf{F}_p[\alpha_0,\dots,\alpha_{j-1}])[\lambda'_0,\dots,\lambda'_j]$ if we define, for $0\leqslant s\leqslant j$, λ'_s to be homogeneous of degree p^s.*

Proof. It is a strightforward consequence of lemmas 6.2 and 6.5. More precisely, from 6.5 we see that

$$p\cdot z'\cdot\alpha = (0, Prod_0(\lambda'^p_0,\alpha_0^p),\dots,\underbrace{Prod_{j-1}(\lambda'^p_0,\dots,\lambda'^p_{j-1},\alpha_0^p,\dots,\alpha_{j-1}^p)}_{j\text{ th entry}}\dots)$$

where each $Prod_{j-1}(\underline{\lambda'},\underline{\alpha})^p$ is homogeneous of degree p^j (provided that λ'_s is homogeneous of degree p^s for $0\leqslant s\leqslant j-1$). Furthermore, $Q_j(\underline{\lambda'},\underline{\alpha})$ is the specialisation of $S_j(\underline{X},\underline{Y})$ at $\underline{X}=z',\underline{Y}=p\cdot z'\cdot\alpha$ and we use lemma 6.2 to get the desired result. □

As we did in §6.2 we define (for $0 \leqslant j \leqslant N$)

$$\widetilde{Q}_j \stackrel{\text{def}}{=} Q_j(\underline{\lambda'}, \underline{\alpha}) - \lambda'_j.$$

For $j \neq 0$ it is a polynomial in $(\mathbf{F}_p[\alpha_0, \ldots, \alpha_{j-1}])[\lambda'_0, \ldots, \lambda'_{j-1}]$, homogeneous of degree p^j

We can finally define, inductively, a family of ring homomorphisms: we let

$$q_0 : \mathbf{F}_p[\lambda'_0] \to \mathbf{F}_p[\lambda'_0]$$

be the identity map, and, assuming q_{j-1} being constructed for $j \geqslant 1$, we define

$$q_j : \mathbf{F}_p[\lambda'_0, \ldots, \lambda_j, \alpha_0, \ldots, \alpha_{j-1}] \to \mathbf{F}_p[\lambda'_0, \ldots, \lambda'_j, \alpha_0, \ldots, \alpha_{j-1}]$$

by the condition

$$\begin{aligned} \lambda'_j &\mapsto \lambda'_j - q_{j-1}(\widetilde{Q}_j) \\ \alpha_{j-1} &\mapsto \alpha_{j-1} \\ q_j|_{\mathbf{F}_p[\lambda'_0, \ldots, \lambda_{j-1}, \alpha_0, \ldots, \alpha_{j-2}]} &= q_{j-1} \end{aligned}$$

(and the obvious formalism: if $j = 1$ we just forget α_{j-2} from the formulas).

We deduce:

Lemma 6.8. *For $0 \leqslant j \leqslant N$, the polynomial $q_{j-1}(\widetilde{Q}_j)$ is homogeneous of degree p^j in $\lambda'_0, \ldots, \lambda'_{j-1}$.*

Proof. The morphism q_{j-1} is a graded ring homomorphism. □

6.4 Some special Witt polynomials -III

In this paragraph we study some Witt polynomials giving the action of $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$) for the representations of §4.1 (resp. of §4.2). Such study is more delicate than the previous sections (§6.2 and §6.3) and relies crucially on the fact that we deal with Witt vectors $x \in W(\mathbf{F}_q)$ which are NOT invertible.

We start with a general remark

Lemma 6.9. *Let $N, n \in \mathbf{N}$.*

i) The n-th universal Witt polynomial of the rise to the N-th power $Pot_n^N(\underline{X})$ is an homogeneous element of degree Np^n in $\mathbf{Z}[X_0, \ldots, X_n]$ provided that X_j is homogeneous of degree p^j for any $0 \leqslant j \leqslant n$.

ii) The n-th universal Witt polynomial associated to the sum of N elements $S_n^N(\underline{X(1)}, \ldots, \underline{X(N)})$ is an homogeneous element of degree p^n in $\mathbf{Z}[X(1)_0, \ldots, X(1)_n, \ldots, X(N)_0, \ldots, X(N)_n]$ if we define $X(l)_j$ to be homogeneous of degree p^j, for any $l \in \{1, \ldots, N\}$.

Proof. Elementary. □

As in §6.3 we have the following

Remark 6.10. *In the present paragraph, we will be concerned with polynomials with coefficients in $\mathbf{F}_p$ obtained by reducing modulo p the coefficients of the universal Witt polynomials $S_n^N(\underline{X}, \underline{Y})$, $Pot_n^N(\underline{X})$, $S_n(\underline{X}, \underline{Y})$, $Prod_n(\underline{X}, \underline{Y})$. In order not to overload notations, such images will be denoted again by $S_n^N(\underline{X}, \underline{Y}), \ldots$. As $p \cdot 1 = 0$, recall that multiplication by p is the composite of Frobenius and Verschiebung.*

Fix $0 \leqslant m \leqslant n$ and consider the ring $\mathbf{F}_p[\lambda_m, \dots, \lambda_n]$.

Definition 6.11. *Let $M \in \mathbf{N}$. A monomial $\lambda_m^{\alpha_m} \dots \lambda_n^{\alpha_n} \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is said to be* pseudo-homogeneous of degree M *if the following holds:*

there exist an integer $L \in \mathbf{N}$ and integers $\beta_l(j)$ for $j \in \{1, \dots, L\}$, $l \in \{m, \dots, n\}$ such that

i) for all $l \in \{m \dots, n\}$ we have

$$\alpha_l = \sum_{j=1}^{L} p^{j-1} \beta_l(j)$$

ii) we have

$$p^m (\sum_{j=1}^{L} \beta_m(j)) + \dots + p^n (\sum_{j=1}^{L} \beta_n(j)) \leqslant M.$$

A polynomial in $\mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is said to be pseudo-homogeneous of degree M *if it is a sum of monomials each of which is pseudo homogeneous of degree M.*

The following result is imediate

Lemma 6.12. *Fix m, n as above. Then:*

i) If $P_1, P_2 \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ are pseudo-homogeneous of degree M_1, M_2 respectively, then $P_1 P_2$ is pseudo-homogeneous of degree $M_1 + M_2$.

ii) if $P_1 \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is pseudo-homogeneous of degree M_1 then P_1^p is again pseudohomogeneous of degree M_1.

Proof. Omissis. $\square$

Remark 6.13. *If $P \in \mathbf{F}_p[\lambda_m, \dots, \lambda_n]$ is pseudo-homogeneous and we specialise P on an element of $\mathbf{F}_q^{n-m+1}$, we see that the integer L in definition 6.11 can be assumed to verify $L \leqslant f$.*

We are now ready to focus our attention some Witt vectors in $W(\mathbf{F}_q)$.

6.4.1 *The negative case.* For $1 \leqslant m \leqslant n$, let $z \stackrel{\text{def}}{=} (0, \dots, 0, \lambda_m, \dots, \lambda_n, 0, \dots)$ and $[x] \stackrel{\text{def}}{=} (x, 0, \dots)$ be elements of $W(\mathbf{F}_q)$. We are interested in the Witt development of

$$\sum_{j=0}^{N} z^{j+1} [x^j] \mod p^{n+1} \tag{46}$$

where $N \stackrel{\text{def}}{=} \lfloor \frac{n+1}{m} \rfloor$. For $j \in \{m, \dots, n\}$ write finally $U_j(\underline{\lambda}, x) \in \mathbf{F}_p[\lambda_m, \dots, \lambda_j, x]$ for the j-th polynomial of the Witt development of (46) and put

$$\widetilde{U}_j(\underline{\lambda}, x) \stackrel{\text{def}}{=} U_j - \lambda_j.$$

We notice that $\widetilde{U}_j = 0$ if $m \leqslant j \leqslant 2m-1$ and $\widetilde{U}_{2m} = \lambda_m^{2p^m}$.

We have a rough estimate of the degree of the $\widetilde{U}_h$

Lemma 6.14. *Let $h \in \{2m, \dots, n\}$. Then $\widetilde{U}_h \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{h-1}, x]$ and is pseudo homogeneous of degree $p^h - p^m(p^m - 2)$.*

Proof. If $\widetilde{z} \stackrel{\text{def}}{=} \lambda_m^{\frac{1}{p^m}}, \dots, \lambda_n^{\frac{1}{p^m}}, 0, \dots)$ then we recall that $Pot_l^{j+1}(\widetilde{z})$ is homogeneous of degree $(j+1)p^l$ (if λ_s is homogeneous of degree p^s). Thus the Witt development of $z^{j+1}[x]^j$ has the form

$$z^{j+1}[x]^j = (0, \dots, 0, \underbrace{Pot_0^{j+1}(\lambda_m^{p^{mj}})(x^j)^{p^{m(j+1)}}}_{\text{position } m(j+1)}, \dots, \underbrace{Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(x^j)^{p^{m(j+1)+l}}}_{\text{position } m(j+1)+l}, \dots)$$

and $Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(x^j)^{p^{m(j+1)+l}}$ is homogeneous of degree $(j+1)p^{l+m(j+1)}$ and actually is pseudo-homogeneous of degree $(j+1)p^{l+m}$.

Thus, if $a_{(j+1)m}(j), \dots, a_h(j)$ is an $h-(j+1)m+1$-tuple of integers, the polynomial

$$\prod_{l=0}^{h-(j+1)m} (Pot_l^{j+1}(\lambda_m^{p^{mj}}, \dots, \lambda_{m+l}^{p^{mj}})(x^j)^{p^{m(j+1)+l}})^{a_{(j+1)m+l}(j)}$$

is pseudo-homogeneous of degree

$$(j+1)(p^m a_{(j+1)m}(j) + \dots + p^{h-mj} a_h(j)).$$

By lemma 6.9 we see that a monomial of $S_h^{N+1}(\underline{X}(1), \dots, \underline{X}(N+1))$ has the following form:

$$\mathfrak{X} \stackrel{\text{def}}{=} \prod_{l_0=0}^{h} X_{l_0}(1)^{a_{l_0}(0)} \cdots \prod_{l_N=0}^{h} X_{l_N}(N+1)^{a_{l_N}(N)}$$

where

$$\sum_{l_0=0}^{h} p^{l_0} a_{l_0}(0) + \dots + \sum_{l_N=0}^{h} p^{l_N} a_{l_N}(N) = p^h.$$

As U_h is the specialisation of $S_h^{(N+1)}$ at

$$(\underline{X}(j+1))_{j \in \{0,\dots,N\}} = (z^{j+1}[x^j])_{j \in \{0,\dots,N\}}$$

we see in particular that $\widetilde{U_h} \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{h-1}, x]$.

Assume now that

1) for $h \geqslant (j+1)m$ we have $a_{l_j}(j) = 0$ for all $l_j < (j+1)m$;
2) for $h < (j+1)m$ we have $a_{l_j}(j) = 0$.

Then lemma 6.12 shows that the specialisation of $\mathfrak{X}$ is pseudo-homogeneous of degree

$$d \stackrel{\text{def}}{=} \sum_{j=0}^{N} (j+1)(\sum_{i=(j+1)m}^{h} p^{i-jm} a_i(j)).$$

Letting

$$x_{j+1} \stackrel{\text{def}}{=} \sum_{i=(j+1)m}^{h} p^{i-mj} a_i(j)$$

for $j \in \{0, \dots, h\}$ we get

$$d = p^h - \sum_{j=0}^{N} (p^{jm} - (j+1)) x_j$$

and the conclusion follows from lemma 6.15 below. $\square$

LEMMA 6.15. *Let $j \in \{0, \dots, N\}$ and let*

$$\mathfrak{X} \overset{\text{def}}{=} \prod_{l_0=0}^{h} X_{l_0}(1)^{a_{l_0}(0)} \cdots \prod_{l_N=0}^{h} X_{l_N}(N+1)^{a_{l_N}(N)}$$

be a monomial of $S_h^{(N+1)}(\underline{X}(1), \dots, \underline{X}(N+1))$.

If $a_{l_i}(i) = 0$ for all $i \neq j$ and $l_i \in \{0, \dots, h\}$ then

$$\mathfrak{X} = X_h(j).$$

Proof. An immediate induction on h shows that if we specialise $S_h^{(N+1)}$ at

$$(X_0(i), \dots, X_h(i)) = (0, \dots, 0)$$

for $i \neq j$ we get

$$S_h^{(N+1)}(\underline{0}, \dots, \underline{0}, \underline{X}(j), \underline{0}, \dots, \underline{0}) = X_h(j)$$

and the claim follows. □

We finally introduce a family of ring homomorphisms, for $m \leqslant j \leqslant n$,

$$u_j : \mathbf{F}_p[\lambda_m, \dots, \lambda_j, x] \to \mathbf{F}_p[\lambda_m, \dots, \lambda_j, x]$$

defined inductively as follow: u_m is the identity map and, assuming u_{j-1} being constructed, we define u_j as the unique extension of u_{j-1} to $\mathbf{F}_p[\lambda_m, \dots, \lambda_j, x]$ such that

$$\lambda_j \mapsto \lambda_j - u_{j-1}(\widetilde{U}_j).$$

We have the

LEMMA 6.16. *Let $h \in \{2m, \dots, n\}$. Then $u_h(\widetilde{U}_h)$ is pseudo homogeneous of degree $p^h - p^m(p^m - 2)$.*

Proof. Arguing by induction, we can assume that $u_l(\lambda_l)$ is pseudohomogeneous of degree p^l for all $l \in \{m, \dots, h-1\}$. As $\widetilde{U}_h$ is pseudohomogeneous of degree $p^h - p^m(p^m - 2)$ by lemma 6.14, the claim follows from lemma 6.12. □

6.4.2 *The positive case* This section is essentially a re-edition of §6.4.1, where we take $m = 0$. The interest of this case will appear in §4.2, where we give a description of the $K_0(p)$-representations R_{n+1}^{+}.

Let $(\lambda_0, \dots, \lambda_n, 0, \dots) \in \mathbf{W}(\mathbf{F}_q)$.

We are interested in the Witt development $(U_0(\lambda_0, x), U_1(\lambda_0, \lambda_1, x), \dots, U_{n+1}(\lambda_0, \dots, \lambda_{n+1}, x), 0, \dots)$ of

$$z(1 + p[x]z)^{-1} \equiv \sum_{j=0}^{n+1} p^j [x] z^{j+1} \bmod p^{n+2}.$$

We check immediately that $U_0 = \lambda_0$ and $U_1 = \lambda_1 + \lambda_0^{2p} x$.

We define for $h = 0, \dots, n+1$ $\widetilde{U}_h \overset{\text{def}}{=} U_h - \lambda_h$. The following result is the analogous of lemma 6.14

LEMMA 6.17. *Let $h \in \{1, \dots, n+1\}$. Then $\widetilde{U}_h \in \mathbf{F}_p[\lambda_0, \dots, \lambda_{h-1}, x]$ is pseudohomogeneous of degree $p^h - (p-2)$.*

Proof. The proof is completely analogous to the proof of lemma 6.14.

We have

$$z^{j+1} = (Pot_0^{j+1}(\underline{\lambda}), \dots, Pot_{n+1}^{j+1}(\underline{\lambda}), \dots)$$

where $Pot_l^{j+1}(\lambda)$ is homogeneous of degree $(j+1)p^l$ (for the natural grading on $\mathbf{F}_p[\lambda_0, \dots, \lambda_l]$); therefore

$$p^j z^{j+1}[x^j] = (0, \dots, 0, \underbrace{(Pot_0^{j+1}(\underline{\lambda}))^{p^j}(x^j)^{p^j}}_{\text{position } j}, \dots, \underbrace{(Pot_l^{j+1}(\underline{\lambda}))^{p^j}(x^j)^{p^{j+l}}}_{\text{position } j+l}, \dots)$$

and $(Pot_l^{j+1}(\underline{\lambda}))^{p^j}(x^j)^{p^{j+l}}$ is therefore pseudohomogeneous of degree $(j+1)p^l$.

We recall that a monomial of $S_h^{(n+1)}(\underline{X}(1), \dots, \underline{X}(n+2))$ is of the following form:

$$\mathfrak{X} = \prod_{l_0=0}^{h} X_{l_0}(1)^{a_{l_0}(0)} \cdots \prod_{l_{n+0}=0}^{h} X_{l_{n+1}}(n+2)^{a_{l_{n+1}}(n+1)}$$

where the integers $a_{l_i}(i)$ verify

$$\sum_{l_0=0}^{h} p^{l_0} a_{l_0}(0) + \dots + \sum_{l_{n+1}}^{h} p^{l_{n+1}} a_{l_{n+1}}(n+1) = p^h.$$

As U_h is the specialisation of $S_h^{(n+1)}$ at $(z^{j+1}p^j[x^j])_{j \in \{0, \dots, n+1\}}$ the latter equality shows in particular that $\widetilde{U}_h \in \mathbf{F}_p[\lambda_0, \dots, \lambda_{h-1}, x]$ for $h \in \{1, \dots, n+1\}$.

Assuming that

1) for $0 \leqslant j \leqslant h$ we have $a_{l_j}(j) = 0$ for all $l_j < j$
2) for $j > h$ we have $a_{l_j}(j) = 0$

the specialisation of $\mathfrak{X}$ is pseudohomogeneous of degree

$$\sum_{j=0}^{n+1}(j+1)(\sum_{i=j}^{h} p^{i-j} a_i(j)) = p^h - \sum_{j=1}^{h}(p^j - (j+1))x_j$$

where we have set

$$x_j \overset{\text{def}}{=} \sum_{i=j}^{h} p^{i-j} a_i(j).$$

The conclusion then follows once we have shown that it exixts $j \in \{1, \dots, h\}$ such that $x_h \neq 0$. This is then an immediate consequence of lemma 6.15. $\square$

As in section §6.4.1 we define inductively, for $h = 0, \dots, n+1$, the ring morphisms

$$u_h : \mathbf{F}_p[\lambda_0, \dots, \lambda_h, x] \to \mathbf{F}_p[\lambda_0, \dots, \lambda_h, x]$$

by the condition $u_h(\lambda_h) \overset{\text{def}}{=} \lambda_h - u_{h-1}(\widetilde{U}_h)$ for $h \geqslant 1$ and $u_0 \overset{\text{def}}{=} id$. Then

Lemma 6.18. *Let $1 \leqslant h \leqslant n+1$. Then $u_h(\widetilde{U}_h)$ is pseudo homogeneous of degree $p^h - (p-2)$.*

Proof. As for lemma 6.16 it is a consequence of lemma 6.12. $\square$

7. Appendix B: Two rough estimates

In this appendix use the material of appendix A to estimate the behaviour of (the reduction modulo p^f-1 of) some elements which appear naturally in the study of the socle filtration for the representations $R^{\pm}_{n+1}$, $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}1$, etc...

The first tool is discussed in §7.1: it is an elementary description of the function $\mathfrak{s}$ giving the cipher sum of the reduction modulo p^f-1 of a natural number. In §7.2 the properties of the function $\mathfrak{s}$ and the results on Witt polynomials stated in §6 will be used to describe in detail some explicit vectors of the aforementioned representations (propositions 7.3, 7.4 and 7.5).

7.1 Remark on the proof of Stickelberger's theorem

In this section we recall the construction and the properties of a certain function $s:\mathbf{Z}\to\mathbf{N}$ which appears in the proof of Stickelberger's theorem.

If $\mathfrak{p}$ is a prime of $\mathbf{Q}(\zeta_{q-1})$ lying above p, the reduction modulo $\mathfrak{p}$, $\mathbf{Z}[\zeta_{q-1}]\to\mathbf{F}_q$ admits a multiplicative section

$$\omega_{\mathfrak{p}}:\mathbf{F}_p^{\times}\to\mathbf{Z}[\zeta_{q-1}]$$

which induces an isomorphisms on the group μ_{q-1} of $q-1$-th roots of unity. If $\mathfrak{P}$ is the prime of $\mathbf{Q}(\zeta_{q-1},\zeta_p)$ lying above $\mathfrak{p}$, we define a function $s:\mathbf{Z}\to\mathbf{N}$ by

$$s(n)\overset{\text{def}}{=}val_{\mathfrak{P}}(g(\omega_{\mathfrak{p}}^{-n}))$$

where $val_{\mathfrak{P}}$ denotes the $\mathfrak{P}$-adic valuation and $g(\omega_{\mathfrak{p}}^{-n})$ denotes the Gauss sum of the character $\omega_{\mathfrak{p}}^{-n}:\mathbf{F}_q^{\times}\to\mu_{q-1}$.

We need to modify slightly this function as follow:

$$\begin{aligned}\mathfrak{s}:\mathbf{N}&\to\mathbf{N}\\ n&\mapsto\begin{cases}s(n)\,\text{if either}\,n\not\equiv 0\,\mathrm{mod}\,q-1\,\text{or}\,n=0\\ f(p-1)\quad\text{otherwise}\end{cases}\end{aligned}$$

The following lemma is then easily deduced from the well known properties of the function s (cf. [Was], §6.2):

Lemma 7.1. *Let $n,m\in\mathbf{N}$. Then:*

a) $\mathfrak{s}(0)=0$ *and* $\mathfrak{s}(1)=1$*;*

b) $0\leqslant\mathfrak{s}(m+n)\leqslant\mathfrak{s}(n)+\mathfrak{s}(m)$*;*

c) $\mathfrak{s}(pn)=\mathfrak{s}(n)$*;*

d) if $0\leqslant n\leqslant q-1$ and $(a_0,\dots,a_{f-1})$ are the cyphers of the p-adic development of n, we have

$$\mathfrak{s}(n)=a_0+a_1+\dots+a_{f-1}.$$

In particular, $\mathfrak{s}(n)\leqslant n$ for any $n\in\mathbf{N}$, with equality if and only if $n\in\{0,\dots,p-1\}$.

We can improve the statement of b):

Lemma 7.2. *Let $b_0,\dots,b_{f-1}\in\mathbf{N}$ be integers.*

Then there exists integers m_s,n_s, where $s\in\{0,\dots,f-1\}$ such that:

1) for all $s\in\{0,\dots,f-1\}$

$$c_s\overset{\text{def}}{=}b_s-pm_s+n_{\lfloor s-1\rfloor}\in\{0,\dots,p-1\};$$

2) *we have*

$$\widetilde{j} \stackrel{\text{def}}{=} \sum_{s=0}^{f-1} m_s = \sum_{s=0}^{f-1} n_s;$$

3) *we have*

$$\sum_{s=0}^{f-1} p^s b_s \equiv \sum_{s=0}^{f-1} p^s c_s \mod p^f - 1;$$

4) *we have the equality*

$$\mathfrak{s}(\sum_{s=0}^{f-1} p^s b_s) = \sum_{s=0}^{f-1} b_s - \widetilde{j}(p-1).$$

Proof. Assume first that $b_s \in \{0,\dots,p-1\}$ for all $s \geqslant 1$ and $b_0 \geqslant p$. There exist (unique) integers m_s, for $s = 0, \dots, f-1$ such that

i) $b_s + m_{s-1} - pm_s \in \{0,\dots,p-1\}$ for all $s \geqslant 1$ and $b_0 - pm_0 \in \{0,\dots,p-1\}$;

ii) we have the equality

$$\sum_{s=0}^{f-1} b_s p^s = (b_0 - pm_0) + \sum_{s=0}^{f-1} p^s(b_s + m_{s-1} - pms) + p^{f-1} m_{f-1}. \tag{47}$$

As we work modulo $q-1$ the equality (47) reads

$$\sum_{s=0}^{f-1} b_s p^s \equiv (b_0 - pm_0 + m_{f-1}) + \sum_{s=0}^{f-1} p^s(b_s + m_{s-1} - pms) \mod q-1.$$

If $b_0 - pm_0 + m_{f-1} \in \{0,\dots,p-1\}$ we get the result. If not, we only have to check that $0 \leqslant b_0 - pm_0 + m_{f-1} < b_0$ (so that the iteration of the preceeding procedure eventually stops). As $-pm_1 + b_1 + m_0 \geqslant 0$ and $b_1 \leqslant p-1$ we get $m_1 \leqslant \frac{p-1+m_0}{p}$ and, inductively, $m_{s+1} \leqslant \frac{p^{s+1}-1+m_0}{p^{s+1}}$. Thus

$$-pm_0 + m_{f-1} \leqslant -pm_0 + \frac{p^{f-1} - 1 + m_0}{p^{f-1}} < 0$$

if $m_0 \geqslant 1$.

For the general case, we notice that there exists unique integers m'_s such that $b_s + m_{s-1} - pm_s \in \{0,\dots,p-1\}$ for all $s \geqslant 1$ and $b_0 - m_0 \in \{0,\dots,p-1\}$. As we work modulo $q-1$ we get

$$\sum_{s=0}^{f-1} b_s p^s \equiv (b_0 - pm_0 + m_{f-1}) + \sum_{s=0}^{f-1} p^s(b_s + m_{s-1} - pms) \mod q-1.$$

and we are in the previous case. $\square$

7.2 Two rough estimates

In this section we study some elements of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ which appear naturally in the study of the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ (but the results adapt immediately for the representations $R_{n+1}^{\pm}$). In particular, we will be able to have a partial control of the action of $K_0(p^m)$ on $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$ (and not only on the graded pieces of the socle filtration).

The following proposition holds for a fixed couple (m,n) of integers such that $0 \leqslant m \leqslant n$; for the $m=0$ case we just have to replace the matrix $\begin{bmatrix} 1 & 0 \\ p^m[\lambda_m] & 1 \end{bmatrix}$ with $\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}$ in the expressions (48) and (49). Finally we recall the definition of the $\overline{\mathbf{F}}_p$-linear subspace $\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}$ of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$ for a given $(n+1-m)f$-tuple $(\underline{l}_m,\dots,\underline{l}_n) \in \big\{\{0,\dots,p-1\}^f\big\}^{n+1-m}$, given in §4.1.2.

Proposition 7.3. *Let $F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$, and $N \overset{\mathrm{def}}{=} N_{m,n}(\underline{l}_m,\dots,\underline{l}_n)$. For $m \leqslant j \leqslant n$ let $T_j \in \mathbf{F}_p[\lambda_m,\dots,\lambda_{j-1}]$ be a polynomial of degree $\deg(T_j) \leqslant p^{j-m}$ (where, for $j \in \{0,\dots,n-1\}$, we define λ_{j+m} to be homogeneous of degree p^j), and $\underline{i}_j$ be a f-tuple such that $\underline{i}_j \leqslant \underline{l}_j$. Finally, fix $M < p^f - 1$. Then the image inside $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1/N-M$ of the element x defined as*

$$x \overset{\mathrm{def}}{=} \sum_{j=m}^{n-1} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (T_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n - \underline{i}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1,e] \quad (48)$$

is contained in the image inside $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1/N-M$ of the subspace

$$\mathfrak{W}_{(\underline{l}_m,\dots,\underline{l}_n)}.$$

Proof. The technique of the proof is very simple: we fix $0 \leqslant t \leqslant M$ and $n \in \mathbf{N}$ such that $n(p-1) \leqslant t < (n+1)(p-1)$. If we write x as a suitable sum of elements $F^{m,n}_{\underline{l}'_m,\dots,\underline{l}'_n}$, the statement is proved if we check that any such element lying in the antidiagonal $X_0 + \dots + X_{f-1} = N - t$ verifies $x'_j \leqslant x_j + n$ for all $j = 0,\dots,f-1$ (where, as usual, $(x_0,\dots,x_{f-1})$, $(x'_0,\dots,x'_{f-1})$ are the coordinates of $F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n}$, $F^{m,n}_{\underline{l}'_m,\dots,\underline{l}'_n}$ via the map (33)).

This is a long computation. If we expand each of the polynomials $T_{m+1}^{\underline{i}_{m+1}},\dots,T_n^{\underline{i}_n}$, we obtain:

$$\sum_{i\in I} \beta_i \sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m(i)} \begin{bmatrix} 1 & 0 \\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n(i)} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1,e] \qquad (49)$$

where I is a suitable set of indices, $\beta_i \in \overline{\mathbf{F}}_p$, and the exponents $\kappa_j(i)$ (for $j \in \{m,\dots,n\}$) admit the following explicit description: [9]

$$\kappa_a = p^{\lfloor -1 \rfloor}\kappa_a^{(a+1)} + \dots + p^{\lfloor -(n-a) \rfloor}\kappa_a^{(n)} + \underline{l}_a - \underline{i}_a$$

and (for $a+1 \leqslant b \leqslant n$)

$$\kappa_a^{(b)} = \kappa_a^{(b),0} + p\kappa_a^{(b),1} + \dots + p^{f-1}\kappa_a^{(b),f-1}$$

where each $\kappa_a^{(b),s}$ is the exponent of λ_a apperaring in a fixed monomial of $(T_b)^{i_b^{(s)}}$.

Recall that, by the hypothesys on the T_b's, we have

$$\kappa_m^{(b),s} + p\kappa_{m+1}^{(b),s} + \dots + p^{b-1-m}\kappa_{b-1}^{(b),s} \leqslant p^{b-m} i_b^{(s)}. \qquad (50)$$

[9]from now on, we fix an index $i \in I$, and we put $\kappa_j \overset{\mathrm{def}}{=} \kappa_j(i)$

Thanks to lemma 7.1, we have the following inequalities:

$$\mathfrak{s}(\kappa_m) + p\,\mathfrak{s}(\kappa_{m+1}) + \cdots + p^{n-m}\,\mathfrak{s}(\kappa_n) \leqslant \tag{51}$$

$$\begin{aligned}\leqslant & (\mathfrak{s}(\underline{l}_m - \underline{i}_m) + \mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_m^{(m+1)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m)\rfloor}\kappa_m^{(n)})) + \\ & +p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1}) + \mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_{m+1}^{(m+2)}) + \cdots + \mathfrak{s}(p^{\lfloor -(n-m-1)\rfloor}\kappa_{m+1}^{(n)})) + \dots \\ & \cdots + p^{n-m-1}(\mathfrak{s}(\underline{l}_{n-1} - \underline{i}_{n-1}) + \mathfrak{s}(p^{\lfloor -1\rfloor}\kappa_{n-1}^{(n)})) + p^{n-m}(\mathfrak{s}(\underline{l}_n - \underline{i}_n)) \leqslant \end{aligned}\tag{52}$$

$$\begin{aligned}& \leqslant \mathfrak{s}(\underline{l}_m - \underline{i}_m) + \sum_{s=0}^{f-1}\mathfrak{s}(\kappa_m^{(m+1),s}) + \\ & p(\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m-1})) + (\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(m+2),s} + p\,\mathfrak{s}(\kappa_{m+1}^{(m+2),s})) + \dots \\ & \cdots + (\sum_{s=0}^{f-1}(\mathfrak{s}(\kappa_m^{(n),s}) + p\,\mathfrak{s}(\kappa_{m+1}^{(n),s}) + \cdots + p^{n-m-1}\,\mathfrak{s}(\kappa_{n-1}^{(n),s}))) + p^{n-m}\,\mathfrak{s}(\underline{l}_n - \underline{i}_n) \leqslant \end{aligned}\tag{53}$$

$$\leqslant \mathfrak{s}(\underline{l}_m - \underline{i}_m) + p\,\mathfrak{s}(\underline{i}_{m+1}) + p\,\mathfrak{s}(\underline{l}_{m+1} - \underline{i}_{m+1}) + \cdots + p^{n-m}\,\mathfrak{s}(\underline{i}_n) + p^{n-m}\,\mathfrak{s}(\underline{l}_n - \underline{i}_n)$$

where the inequality (53) is deduced from (50) and lemma 7.1-*d*).

If we impose our function to lie on the hyperplane $X_0 + \cdots + X_{f-1} = t$ we get a "control" on the exponents $\kappa_a^{(b),s}$. More precisely,

i) the inequality (51) give rise to the conditions:

$$\mathfrak{s}(\kappa_a) = \mathfrak{s}(\underline{l}_a - \underline{i}_a) + \mathfrak{s}(\kappa_a^{(a+1)}) + \cdots + \mathfrak{s}(\kappa_a^{(n)}) - u_a(p-1)$$

for $a \in \{m, \dots, n-1\}$ and some $u_a \in \mathbf{N}$;

ii) the inequality (52) give rise to the conditions:

$$\mathfrak{s}(\kappa_a^{(b)}) = \mathfrak{s}(\kappa_a^{(b),0}) + \cdots + \mathfrak{s}(\kappa_a^{(b),f-1}) - w_a^{(b)}(p-1)$$

where $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$ and some $w_a^{(b)} \in \mathbf{N}$;

iii) the inequality (53) give rise to the conditions

$$\mathfrak{s}(\kappa_a^{(b),s}) = \kappa_a^{(b),s} - v_a^{(b),c}(p-1)$$

where $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$, $s \in \{0, \dots, f-1\}$ and some $v_a^{(b),s} \in \mathbf{N}$;

iv) condition $t < (n+1)(p-1)$ imposes finally

$$\sum_{a=m}^{n-1} p^{a-m}u_a + \sum_{a=m}^{n-1} p^{a-m}(\sum_{b=a+1}^{n} w_a^{(b)}) + \sum_{a=m}^{n-1} p^{a-m}(\sum_{b=a+1}^{n}\sum_{s=0}^{f-1} v_a^{(b),s}) \leqslant n.$$

First, notice that the condition $n(p-1) < p^f - 1$ imply $k_a^{(b),s} \leqslant p^f - 1$ for all possible choices of a, b, s (as $\mathfrak{s}(k_a^{(b),s}) \leqslant \lceil k_a^{(b),s} \rceil$). If $k_a^{(b),s}(i)$, for $i \in \{0, \dots, f-1\}$, are the cyphers of the p-adic development of $\kappa_a^{(b),s}$, we then see that *iii*) gives the necessary condition

$$\sum_{i=1}^{f-1} \kappa_a^{(b),s}(i) \leqslant v_a^{(b),s}$$

(indeed, $v_a^{(b),s}$ can uniquely written as $v_a^{(b),s} = \alpha_{a,b,s}(1) + (p+1)\alpha_{a,b,s}(2) + \dots + \alpha_{a,b,s}(f-1)(1 + p + \dots + p^{f-1})$ for suitable integers $\alpha_{a,b,s}(j)$).

Fix now $a \in \{m, \dots, n-1\}$, $b \in \{a+1, \dots, n\}$. Working in $\mathbf{Z}/(p^f - 1)$, we see that

$$\kappa_a^{(b),0} + \dots + p^{f-1}\kappa_a^{(b),f-1} \equiv \sum_{j=0}^{f-1} p^j(\kappa_a^{(b),0}(j) + \kappa_a^{(b),1}(\lfloor j-1 \rfloor) + \dots + \kappa_a^{(b),f-1}(\lfloor j-(f-1) \rfloor)).$$

Using lemma 7.2 we see that condition *ii*) let us deduce the p-adic expansion of $\kappa_a^{(b)}$:

$$\begin{aligned} \kappa_a^{(b)}(j) &= \kappa_a^{(b),0}(j) + \dots + \kappa_a^{(b),f-1}(\lfloor j-(f-1) \rfloor) - p\alpha_a^{(b)}(j) + \beta_a^{(b)}(j) \\ &= \kappa_a^{(b),j}(0) + \rho_a^{(b)}(j) - p\alpha_a^{(b)}(j) \end{aligned} \tag{54}$$

where the integers $\alpha_a^{(b)}(j)$, $\beta_a^{(b)}(j)$ verify

$$\sum_{j=0}^{f-1} \alpha_a^{(b)}(j) = \sum_{j=0}^{f-1} \beta_a^{(b)}(j) = w_a^{(b)}$$

and

$$\rho_a^{(b)}(j) = \kappa_{s\in\{0\dots,f-1\}\setminus\{j\}_a}^{(b),s}(\lfloor j-s \rfloor) + \beta_a^{(b)}(j) \leqslant \sum_{s=0}^{f-1} v_a^{(b),s} + w_a^{(b)}.$$

Similarly, condition *i*) let us deduce the p-adic development of κ_a:

$$\begin{aligned} \kappa_a(j) &= l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^{n} \kappa_a^b(\lfloor j+b-a \rfloor) - pA_a(j) + B_a(j) \\ &= l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^{n} \kappa_a^{(b),\lfloor j+b-a \rfloor}(0) + \mathfrak{R}_a(j) - p(\sum_{b=a+1}^{n} \alpha_a^{(b)}(\lfloor j+b-a \rfloor) + A_a(j)) \end{aligned}$$

where the integers $A_a(j)$, $B_a(j), \mathfrak{R}_a(j)$ verify

$$\sum_{j=0}^{f-1} A_a(j) = \sum_{j=0}^{f-1} B_a(j) = u_a$$

and

$$\mathfrak{R}_a(j) = \sum_{b=a+1}^{n} \rho_a^{(b)}(\lfloor j+b-a \rfloor) + B_a(j) \leqslant u_a + \sum_{b=a+1}^{n} (\sum_{s=0}^{f-1} v_a^{(b),s} + w_a^{(b)}).$$

We finally have all the ingredients to give the rough estimate of the statement. We fix a "coordinate" j. A strightforward but tedious computation gives

$$\begin{aligned} \sum_{a=m}^{n} p^{a-m}\kappa_a(j) &= \sum_{a=m}^{n} p^{a-m}(l_a^{(j)} - i_a^{(j)} + \sum_{b=a+1}^{n} \kappa_a^{(b),\lfloor j+b-a \rfloor}(0) + \mathfrak{R}_a(j) - p\mathfrak{A}_a(j)) \\ &= x_j - \sum_{a=m}^{n} i_a^{\lfloor j+a-m \rfloor} + \sum_{b=m+1}^{n} \sum_{a=m}^{b-1} p^{a-m}\kappa_a^{(b),\lfloor j+b-m \rfloor} + \sum_{a=m}^{n-1} p^{a-m}\mathfrak{R}_a(j) - p(\sum_{a=m}^{n-1} \mathfrak{A}_a(j)). \end{aligned}$$

The conclusion follows as

$$\sum_{a=m}^{n-1} p^{a-m}\mathfrak{R}_a(j) \leqslant \sum_{a=m}^{n-1} p^{a-m}(u_a + \sum_{b=a+1}^{n} w_a^{(b)}) + \sum_{b=a+1}^{n}\sum_{s=0}^{f-1} v_a^{(b),s} \leqslant n$$

and

$$\sum_{a=m}^{b-1} \kappa^{(b),s}(0) \leqslant p^{b-m} i_b^{(s)}$$

for any $b \in \{m+1, \dots, n\}$ and $s \in \{0, \dots, f-1\}$. $\square$

The following rough estimate will help us to understand the action of $\begin{bmatrix} 1 & \mathscr{O}_F \\ 0 & 1 \end{bmatrix}$ (resp. of $\begin{bmatrix} 1 & 0 \\ p\mathscr{O}_F & 1 \end{bmatrix}$) on the representations in §4.1 (resp. §4.2). Apparently, the result is unsatisfactory if we want to describe the K-socle filtration for the representations $\pi(\underline{r}, \lambda, 1)$, unless we impose some conditions, depending on p, on the residue degree f (we expect a condition of the form $f \leqslant \frac{p+1}{2}$).

Proposition 7.4. *Let $1 \leqslant m \leqslant n$ be integers and consider $F^{m,n}_{\underline{l}_m,\dots,\underline{l}_n} \in \mathscr{B}$; let $N \stackrel{\text{def}}{=} N_{m,n}(\underline{l}_m, \dots, \underline{l}_n)$. For $2m \leqslant j \leqslant n$ let $V_j \in \mathbf{F}_p[\lambda_m, \dots, \lambda_{j-1}]$ be a pseudo-homogeneous polynomial of degree $\deg(V_j) \leqslant p^j - p^m(p^m - 2)$ and $\underline{i}_j$ be a f-tuple such that $\underline{i}_j \leqslant \underline{l}_j$. Finally, fix $M < p^m - 2$ and define $V_j \stackrel{\text{def}}{=} 1$, $\underline{i}_j = \underline{0}$ for $m \leqslant j \leqslant 2m-1$.*

The image inside $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1/N - M$ of the element x defined as

$$x \stackrel{\text{def}}{=} \sum_{j=m}^{n-1} \sum_{\lambda_j \in \mathbf{F}_q} (\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j - \underline{i}_j} (V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}} \begin{bmatrix} 1 & 0 \\ p^j[\lambda_j^{\frac{1}{p^j}}] & 1 \end{bmatrix} \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\underline{l}_n - \underline{i}_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e]$$

coincides with the image of $F^{(m,n)}_{\underline{l}_m,\dots,\underline{l}_n}$.

Proof. The idea of the proof is completely analogous of that of proposition 7.3 the main difference being that here we are *not* able to give an estimate of the coordinates of the points appearing in the development of x.

As in 7.3 we consider an element appearing in the development of x:

$$\sum_{\lambda_m \in \mathbf{F}_q} (\lambda_m^{\frac{1}{p^m}})^{\kappa_m(i)} \begin{bmatrix} 1 & 0 \\ p^m[\lambda_m^{\frac{1}{p^m}}] & 1 \end{bmatrix} \dots \sum_{\lambda_n \in \mathbf{F}_q} (\lambda_n^{\frac{1}{p^n}})^{\kappa_n(i)} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n^{\frac{1}{p^n}}] & 1 \end{bmatrix} [1, e].$$

The exponents κ_a (for $a \in \{m, \dots, n\}$) admit the following explicit description:

$$\kappa_a = p^{\lfloor -1 \rfloor}\kappa_a^{(a+1)} + \dots + p^{\lfloor n-a \rfloor}\kappa_a^{(n)} + \underline{l}_a - \underline{i}_a$$

and (for $a+1 \leqslant b \leqslant n$)

$$\kappa_a^{(b)} = \kappa_a^{(b),0} + p\kappa_a^{(b),1} + \dots + p^{f-1}\kappa_a^{(b),f-1}$$

where each $\kappa_a^{(b),s}$ is the exponent of λ_a apperaring in a fixed monomial of $(V_b)^{i_b^{(s)}}$.

As each V_a is pseudo-homogeneous, for each triple (a, b, s) we have

$$\kappa_a^{(b),s} = \beta_a^{(b),s}(1) + \dots + p^{f-1}\beta_a^{(b),s}(f)$$

where the integers $\beta_a^{(b),s}(j)$ verify

$$\sum_{j=1}^{f}\beta_m^{(b),s}(j)+p(\sum_{j=1}^{f}\beta_{m+1}^{(b),s}(j))+\cdots+p^{b-m-1}\sum_{j=1}^{f}(\beta_{b-1}^{(b),s}(j))\leqslant(p^{b-m}-(p^m-2))i_b^{(s)}.$$

As for the inequalities (51), (52), (53), we use lemma 7.1 to obtain

$$\sum_{a=m}^{n}p^{a-m}\mathfrak{s}(\kappa_a)\leqslant N-(p^m-2)(\sum_{a=2m}^{n}\mathfrak{s}(\underline{i}_a))$$

and the conclusion follows. □

We state an analogous result in the case $m=0$.

Proposition 7.5. *Let $n\geqslant 0$ and $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_{n+1}}\in\mathscr{B}^+_{n+1}$; let $N\stackrel{\text{def}}{=}N_{0,n+1}(\underline{l}_0,\dots,\underline{l}_{n+1})$. For $1\leqslant h\leqslant n+1$ let $V_h\in\mathbf{F}_p[\lambda_0,\dots,\lambda_{h-1}]$ be a pseudo homogeneous polynomial of degree $p^h-(p-2)$ and $\underline{i}_h\leqslant\underline{l}_h$ be an f-tuple. We finally fix $M\in\{0,\dots,p-3\}$ and put $\underline{i}_0\stackrel{\text{def}}{=}\underline{0}$, $V_{n+2}\stackrel{\text{def}}{=}1$.*

The image inside $(\mathrm{Ind}_{K_0(p^{n+2})}^{K}1)^+/N-M$ of the element

$$x\stackrel{\text{def}}{=}\sum_{\lambda_0\in\mathbf{F}_q}\lambda_0^{\underline{l}_0-\underline{i}_0}(V_1^{\frac{1}{p}})^{\underline{i}_1}\begin{bmatrix}[\lambda_0]&1\\1&0\end{bmatrix}\sum_{j=1}^{n+1}\sum_{\lambda_j\in\mathbf{F}_q}(\lambda_j^{\frac{1}{p^j}})^{\underline{l}_j-\underline{i}_j}(V_{j+1}^{\frac{1}{p^{j+1}}})^{\underline{i}_{j+1}}\begin{bmatrix}1&0\\p^j[\lambda_j^{\frac{1}{p^j}}]&1\end{bmatrix}[1,e]$$

coincides with the image of $F^{(0,n)}_{\underline{l}_0,\dots,\underline{l}_{n+1}}$.

Proof. The proof is completely analogous to the proof of proposition 7.4. We consider an element of $\mathscr{B}^+_{n+1}$ appearing in the deveolpment of x:

$$\sum_{\lambda_0\in\mathbf{F}_q}\lambda_0^{\kappa_0}\begin{bmatrix}[\lambda_0]&1\\1&0\end{bmatrix}\sum_{a=1}^{n+1}\sum_{\lambda_a\in\mathbf{F}_q}(\lambda_a^{\frac{1}{p^a}})^{\kappa_a}\begin{bmatrix}1&0\\p^a[\lambda_a^{\frac{1}{p^a}}]&1\end{bmatrix}[1,e]$$

where, for $0\leqslant a\leqslant n+1$

$$\kappa_a=\underline{l}_a-\underline{i}_a+\sum_{b=a+1}^{n+1}p^{\lfloor a-b\rfloor}\kappa_a^{(b)}$$

and, for $a+1\leqslant b\leqslant n+1$

$$\kappa_a^{(b)}=\sum_{s=0}^{f-1}\kappa_a^{(b),s}p^s$$

(and $\kappa_a^{(b),s}$ is the exponent of λ_a appearing in a fixed monomial of $(V_b)^{i_b^{(s)}}$).

By pseudo homogeneity of $(V_b)^{i_b^{(s)}}$ we have

$$\kappa_a^{(b),s}=\sum_{j=1}^{f}p^{j-1}\beta_a^{(b),s}(j)$$

for suitable integers $\beta_a^{(b),s}(j)$ verifying

$$\sum_{a=0}^{b-1}p^a(\sum_{j=1}^{f}\beta_a^{(b),s}(j))\leqslant i_b^{(s)}(p^b-(p-2)).$$

Therefore we have $\mathfrak{s}(\kappa_a^{(b),s}) \leqslant i_b^{(s)}(p^b - (p-2))$ and using estimates exactely analogous to (51), (52), (53) we deduce

$$\sum_{a=0}^{n+1} p^a \mathfrak{s}(\kappa_a) \leqslant N - \mathfrak{s}(\underline{i}_0) - (p-2)(\sum_{a=1}^{n+1} \mathfrak{s}(\underline{i}_a)).$$

The conclusion follows. □

Conclusions et perspectives

On pourra conclure que cette thèse évidence la possibilié de déduire des informations non triviales sur les représentations universelles à partir de certains vecteurs convenablement choisis grâce à des manipulations sur des polynômes universels de Witt.

Lorsque $F_{\mathfrak{p}} = \mathbf{Q}_p$ l'interprétation euclidienne montre que l'on peut se contenter de la connaissance des polynômes universels tronqués à l'ordre p^2. En fait, on a même besoin de beaucoup moins d'information : la première partie de cette thèse (partie I) montre qu'il suffit de connaître le dégré et le coefficient dominant de ces polynômes. Le problème pour des extensions (dans notre cas, non ramifiées) non triviales de $\mathbf{Q}_p$ est que l'on ne dispose pas à ce jour d'une méthode de récurrence qui permette de se limiter à des polynômes tronqués : le comportement des représentations $\pi(\sigma, 0, 1)$ nécessite la connaissance du dégré homogène des polynômes ou, plus précisement, du développement p-adique de la réduction modulo $p^f - 1$ des exposants.

Dans le cadre des séries modérément ramifiées, il suffit de connaître la somme des chiffres de ces développements p-adiques. Cette information est alors donnée par une fonction $\mathfrak{s}$ que l'on retrouve dans la preuve du thèorème de Stickelberger (§IV-7.1). Lorsque l'on travaille avec les représentations universelles, les recollements entre blocs de tailles différentes montrent que l'information donnée par la fonction $\mathfrak{s}$ est largement insuffisante et qu'une connnaisance plus fine des développements p-adiques des exposantes est donc nécessaire (c'est le problème abordé au §IV-5.1).

La question que l'on se pose est donc la suivante :

Question 7.6. *Est-il possible de deviner, à partir de la structure euclidienne $\mathfrak{R}$, des sous-représentations de $\pi(\sigma, 0, 1)$, de telle sorte que l'on puisse obtenir des quotients irréductibles et admissibles de $\pi(\sigma, 0, 1)$?*

La définition de diagramme canonique proposée par Hu ([Hu2]) ne dépend pas de l'admissibilité de la représentation et l'on pourra ainsi considérer $D_1(\pi(\sigma, 0, 1))$. Alors $D_1(\pi(\sigma, 0, 1))$ admet une interprétation simple en termes de données euclidiennes : en utilisant les notations de IV-5.2 on voit que $D_1(\pi(\sigma, 0, 1))$ est engendré (sur $\overline{\mathbf{F}}_p$) par les éléments des $\cdots \oplus_{\mathscr{R}_n^+} \mathscr{R}_{n+1}^+ \dots$ ayant une relation non triviale avec les éléments de $\cdots \oplus_{\mathscr{R}_n^-} \mathscr{R}_{n+1}^- \dots$, c'est-à-dire l'espace engendré par les éléments $[1, \sigma^{U(\mathbf{F}_q)}]$ et $[\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}, \sigma^{\overline{U}\mathbf{F}_q}]$ (où l'on désigne par $U(\mathbf{F}_q)$, $\overline{U}(\mathbf{F}_q)$ les matrices unipotentes supérieures et inférieures de $\mathrm{GL}_2(\mathbf{F}_q)$ respectivement). Plus généralement, l'interprétation euclidienne de l'espace $D_1(\pi_{\mathfrak{p}})$ devrait être celle des relations entre les éléments de $\cdots \oplus_{\mathscr{R}_n^+} \mathscr{R}_{n+1}^+ \dots$ et de $\cdots \oplus_{\mathscr{R}_n^-} \mathscr{R}_{n+1}^- \dots$. D'après la théorie de Hu, ce genre de relations devrait permettre de caractériser la représentation $\pi_{\mathfrak{p}}$.

Des discussions avec M. Schein nous ont convaincu que l'on devrait également obtenir une structure euclidienne dans le cadre totalement ramifié : dans ce cas, ce serait une structure fractale dans $\mathbf{R}^e$ si e est la ramification. Cela permetterait de donner une interprétation simple et efficace des résultats énoncés dans [Sch]: le quotient universel V_{e-1} est obtenu en enlevant des blocs "assez petits" à partir des sommets libres de la structure fractale associée à $\pi(\sigma, 0, 1)$. Dans cette interprétation heuristique, la non admissibilité du quotient universel V_{e-1} est claire : pendant la suppression des petits blocs on enlève un sommet libre pour obtenir e-sommets libres ! La difficulté qui subsiste encore dans le cas totalement ramifié, est la détermination du bon ordre à mettre sur la base "canonique" de $\pi(\sigma, 0, 1)$ (que l'on définirait de manière analogue au cas non ramifié). Autrement dit, on retrouve des extensions non triviales entre des sommets non adjacents de la structure euclidienne associé : la combinatoire de ces extensions n'est pas comprise à l'heure actuelle.

On espère enfin que l'étude détaillée des représentations universelles de $\mathrm{GL}_2(\mathbf{Q}_p)$, effectuée dans la première partie de cette thèse, puisse avoir des applications à des questions ouvertes concernant la correspondance locale pour $\mathrm{GL}_2(\mathbf{Q}_p)$. En particulier un des problèmes ouverts est la détermination de la réduction modulo p d'une représentation galoisienne p-adique cristalline ρ à poids de Hodge-Tate $(0, k-1)$ avec $\mathrm{Tr}(\rho(Frob)) = a_p \in \mathfrak{m}_{\overline{\mathbf{Z}}_p}$. La correspondance p-adique nous donne une certaine représentation Π_{k,a_p} ([Bre03b]) de $\mathrm{GL}_2(\mathbf{Q}_p)$, et l'on dispose d'un monomorphisme $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$-équivariant $\mathrm{Sym}^{k-2}\overline{\mathbf{Q}}_p^2 \to \Pi_{k,a_p}$. La représentation algébrique $\mathrm{Sym}^{k-2}\overline{\mathbf{Q}}_p^2$ est alors dite le *type* associée à Π_{k,a_p}. La question suivante est dûe à Breuil

QUESTION 7.7. *Si l'on considère un $\mathbf{Z}_p$-réseau Θ_{k,a_p} dans Π_{k,a_p}, y a-t-il une relation précise entre la filtration par le $\mathrm{GL}_2(\mathbf{Z}_p)$-socle de la réduction modulo p de $\Theta_{k,a_p} \cap \mathrm{Sym}^{k-2}\overline{\mathbf{Q}}_p^2$ et la filtration par le $\mathrm{GL}_2(\mathbf{Z}_p)$-socle de la réduction modulo p de Θ_{k,a_p}?*

Si oui, cette relation permet-elle de deviner la nature de $\overline{\Pi}_{k,a_p}$?

D'après cette question, il s'agirait d'étudier la $\mathrm{GL}_2(\mathbf{Z}_p)$-structure des atomes automorphes de longueur 2 c'est-à-dire la seule extension non triviale de $\mathrm{Ind}_{B\mathbf{Q}_p}^{\mathrm{GL}_2(\mathbf{Q}_p)}\chi_1\otimes\chi_2\omega^{-1}$ par $\mathrm{Ind}_{B\mathbf{Q}_p}^{\mathrm{GL}_2(\mathbf{Q}_p)}\chi_2\otimes\chi_1\omega^{-1}$ si les caractères lisses χ_1, χ_2 de $\mathbf{Q}_p^\times$ vérifient $\chi_1\chi_2^{-1} \notin \{1, \omega^{\pm 1}\}$. L'étude de leur $\mathrm{GL}_2(\mathbf{Z}_p)$-structure devrait se déduire par des techniques de récurrence descandante à partir des cas finis, d'une manière qui est tout-à-fait similaire à ce que l'on fait dans la partie I.

La description explicite des représentations irréductibles permet, comme l'on a vu, de disposer des informations trés fines sur leur structure, les chapitres II et III étant des illustrations. En fait, on pourrait appliquer les techniques du chapitre II à d'autres sous-groupes de congruence de $\mathrm{GL}_2(\mathbf{Z}_p)$. C'est le cas pour les groupes $\Gamma_0(p^t) \stackrel{\mathrm{def}}{=} \begin{bmatrix} \mathbf{Z}_p^\times & \mathbf{Z}_p \\ p^k\mathbf{Z}_p & \mathbf{Z}_p^\times \end{bmatrix}$ et $\Gamma_1(p^t) \stackrel{\mathrm{def}}{=} \begin{bmatrix} 1+p^k\mathbf{Z}_p & \mathbf{Z}_p \\ p^k\mathbf{Z}_p & 1+p^k\mathbf{Z}_p \end{bmatrix}$: on peut déterminer, par un dévissage descendant, l'espace des invariants des représentations supersingulières, à l'aide de calculs similaires à ceux de §II-4 (c'est le contenu de [Mo3])

De plus, la dimension des espaces des invariants des représentations supersingulières possède une signification modulaire plus vaste dûe à Emerton et qui peut se résumer de la manière suivante. Soit $\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ une représentation galoisienne continue irréductible et impaire (i.e. modulaire d'après la conjecture de Serre). Soit N la partie première à p du conducteur d'Artin associé à $\overline{\rho}$ et soit $K_{r,N} \stackrel{\mathrm{def}}{=} \ker(\mathrm{GL}_2(\widehat{Z}) \to \mathrm{GL}_2(\widehat{Z}/(Np^r)))$. Les théorèmes de compatibilité locale-globale de [Eme10], notamment la proposition 6.1.20 et le lemme 5.3.8 donnent un lien précis entre la dimension de $(\pi_{\mathfrak{p}})^{K_r}$ et la dimension de l'espace de cohomologie $H^1_{ét}(Y(K_{r,N})_{\overline{\mathbf{Q}}}, \overline{\mathbf{F}}_p)$, où $Y(K_{r,N})$ est la courbe modulaire (sur $\mathbf{Q}$) de niveau p^rN et $\pi_{\mathfrak{p}}$ est la représentation de $\mathrm{GL}_2(\mathbf{Q}_p)$ associée è $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ par la correspondance de Langlands modulo p.

Les résultats de la partie III pourraient jouer un rôle dans l'étude des facteurs ϵ p-adiques. En effet, les vecteurs localement algébriques des représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ associés aux représentations galoisiennes par la méthode de Colmez admettent un modèle de Kirillov. Cela permet donc de définir des facteurs epsilon et, en utilisant des stuctures entières des modèles de Kirillov, on pourra introduire la notion de facteur epsilon modulo p. D'après la théorie classique on pourrait se poser la question suivante :

Question 7.8. *Quelle est la relation entre les facteurs epsilon modulo p associés à une représentation supersingulière π et sa restriction aux extensions quadratiques $L^\times$ de $\mathbf{Q}_p$? Si ψ est une représentation lisse irréductible de $L^\times$, y a-t-il une relation entre le signe du facteur epsilon local de $\pi \otimes \psi$ et l'espace des formes linéaires $L^\times$-équivariantes* $\mathrm{Hom}_{L^\times}(\pi \otimes \psi, \overline{\mathbf{F}}_p)$*?*

D'ailleurs il est extrêmement difficile à l'heure actuelle d'extraire les facteurs epsilon locaux des constructions de Colmez. De plus, comme on l'a remarqué dans la partie III, on peut améloirer l'énoncé de la proposition III-2.10 : la restriction $\pi|_{L^\times}$ se décompose en une somme directe de $(p+1)$ représentations uniserielles de $\mathscr{O}_L^\times$, à facteurs de Jordan Hölder isomorphes.

Explicit description of irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$

Abstract

Let p be an odd prime number. The classification of irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$ is known thanks to the works of Barthel-Livné [BL94] and Breuil [Bre03a]. In the first chapter we illustrate an exhaustive description of such irreducible representations, through the study of certain functions on the Bruhat-Tits tree of $\mathrm{GL}_2(\mathbf{Q}_p)$. In particular, we are able to detect the socle filtration for the KZ-restriction of supersingular representations, principal series and special series.

Invariant elements under some congruence subgroups for irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$ representations over $\overline{\mathbf{F}}_p$

Abstract

Let p be an odd prime number. Using the explicit description for irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$ made in [Mo1], we determine all invariant elements of such representations under the actions of the congruence subgroups K_t, I_t, for any integer $t \geqslant 1$. In particular, we have the dimension of the K_t-invariants for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, for any $t \geqslant 1$.

On some restriction of supersingular representations for $\mathrm{GL}_2(\mathbf{Q}_p)$

Abstract

If L/F is a quadratic extension of local fields (of characteristic zero) and π a supercuspidal representation of $\mathrm{GL}_2(F)$ a theorem of Tunnel and Saito relates the epsilon local factor associated to π to the $L^\times$-socle of $\pi|_{L^\times}$. In this chapter we consider the problem of giving a detailed description of the $L^\times$-structure of supersingular mod p-representations for the case $F = \mathbf{Q}_p$, in the spirit of a theory of mod p epsilon factors.

On some representations of the Iwahori subgroup

Abstract

Let $p \geqslant 5$ be a prime number. In [BL94] Barthel and Livné described a classification for irreducible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$, for F a p-adic field, discovering some objects, referred as "supersingular", which appear as subquotients of a universal representations $\pi(\underline{r}, 0, 1)$. In this chapter we study in detail the Iwahori structure of such universal representations for F an unramified extension of $\mathbf{Q}_p$. We determine a fractal structure which shows how and why the thechniques used for $\mathbf{Q}_p$ fail and which let us determine"natural" subrepresentations of the universal object $\pi(\underline{r}, 0, 1)$. As a byproduct, we get the Iwahori structure of tamely ramified principal series.

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