Invariant elements under some congruence subgroups for irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$ representations over $\overline{\mathbf{F}}_p$

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Abstract

Let p be an odd prime number. Using the explicit description for irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$ made in [Mo1], we determine all invariant elements of such representations under the actions of the congruence subgroups K_t, I_t, for any integer $t \geqslant 1$. In particular, we have the dimension of the K_t-invariants for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, for any $t \geqslant 1$.

1. Introduction

Let p be a prime number. The efforts to describe a "p-adic analogue" of the classical local Langlands correspondence met great progresses in the last few years. After a first, conjectural approach studied by Breuil in [Bre04] and [Bre03b], the works of Berger-Breuil [BB] and Colmez [Col] establish a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. Moreover, such a correspondence is compatible with respect to the reduction of coefficients modulo p: we get a semisimple mod p Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ (again, conjectured by Breuil in [Bre03b] and proved by Berger in [Ber]).

But, if the local field is different from $\mathbf{Q}_p$ the situation is far from being defined. In the direction of a semisimple Langland correspondence for $\mathrm{GL}_2(F)$ for F a non-Archimedean local field, we find the works of Barthel and Livné [BL94] and [BL95]. In those papers the authors classify the smooth irreducible admissible $\mathrm{GL}_2(F)$-representations into four classes: besides characters, principal series and special series, they find a new family of irreducible objects, referred to as "supersingular" whose nature is still very mysterious. Supersingular representations are actually characterised as the subquotients of the cokernel of some "canonical Hecke operator" T, but for $F \neq \mathbf{Q}_p$ such cokernels are not even admissible (cf. [Bre03a], Remarque 4.2.6); moreover the works of Paskunas [Pa04], Breuil-Paskunas [BP] and Hu [Hu] show that for $F \neq \mathbf{Q}_p$ there exists a huge number of supersingular representations with respect to Galois representations (whose classification is indeed well known).

We focus here on the case $F = \mathbf{Q}_p$ where $p$is an odd prime. In this situation the work of Breuil [Bre03a] (followed later by other proofs by Ollivier in [Oll], Emerton in [Eme08]) show that the cokernels of the aforementioned Hecke operators T are actually irreducible, completing the classification for smooth irreducible admissible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$. In the work [Mo1] we develop an explicit approach to the description of irreducible representations for $\mathrm{GL}_2(\mathbf{Q}_p)$: studying the action of T on some privileged elements we are able to describe in great detail supersingular representations (and principal and special series as well), in particular detecting the socle filtrations for their $K\mathbf{Q}_p^{\times}$-restriction.

In the present work, we pursue the study of such explicit elements of irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-

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representations in order to describe their invariants under some congruence subgroups of K.

If we first focus on K_t, i.e. the kernel of the mod p^t-reduction map on K (where $t \geqslant 1$) we see in §3 that taking K_t-invariants of a supersingular representation π comes down, roughly speaking, to "cut" its socle filtration.

The main result (corollary 3.9) is that we can detect *precisely* where such a cutting occurs: if we refer to the socle filtration of a supersingular representation $\pi(r,0,1)$ as "two lines of weights" we get

THEOREM 1.1. *Let $t \geqslant 1$ be an integer. The socle filtration of $\pi(r,0,1)^{K_t}$ is described by*

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^{r+1} \text{—} \dots \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^r \text{—} \mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2 \otimes \det{}^{r+1}$$
$$\bigoplus$$
$$\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2 \otimes \det{}^r \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a} \text{—} \dots \text{—} \mathrm{Ind}_I^K \chi_r^s \text{—} \mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2 \otimes \det$$

where we have $p^{t-1}-1$ parabolic induction in each line and the weight $\mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2 \otimes \det^{r+1}$ in the fist line (resp. $\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2 \otimes \det$ in the second line) appears only of $p-3-r \geqslant 0$ (resp. $r-2 \geqslant 0$).

In particular, we have the dimension of the spaces of K_t-invariants (corollary 3.8):

COROLLARY 1.2. *Let $t \geqslant 1$ be an integer and $r \in \{0,\dots,p-1\}$. The dimension of K_t invariant for a supersingular representation is*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{K_t}) = (p+1)(2p^{t-1}-1) + \begin{cases} p-3 & \text{if } r \notin \{0,p-1\} \\ p-2 & \text{if } r \in \{0,p-1\} \end{cases}$$

Moreover, if we write I_t for the subgroup of K_{t-1} whose elements are upper unipotent mod p^t, we are able, by similar techniques, to describe in greatest detail the space of I_t-invariant of any supersingular representations π of $\mathrm{GL}_2(\mathbf{Q}_p)$. Again, we can roughly say that taking I_t-invariants comes down to "cut" the socle filtration of π, but this time some "reminders elements" appear.

The results of section 4 tells us *exactly* where such cutting occurs and who the reminders elements are. As the combinatoric of such result is a bit heavy, we prefere to omit the statements here, refering the interested reader directly to propositions 4.4, 4.8, 4.11, 4.14 in §4.

We can anyway remark that an immediate corollary is then the $\overline{\mathbf{F}}_p$ dimension of such invariant spaces:

COROLLARY 1.3. *Let $r \in \{0,\dots,p-1\}$ and $t \in \mathbf{N}_>$ be integers. Then:*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{I_t}) = 2(2p^{t-1}-1).$$

The proof of such results relies on the explicit description made in [Mo1] and can be sketched as follow.

We reduce of course to the direct sum decomposition of $\pi|_{K\mathbf{Q}_p^\times}$ (for π a supersingular representation) in terms of the inductive limits of the amalgamed sums $R_i/R_{i-1} \oplus_{R_{i+1}} \cdots \oplus_{R_n} R_{n+1}$ ($i \in \{0,1\}$), treating each summand separately.

We are then able (lemmas 3.2 and 3.3) to give a first estimate of the behaviour of K_t-invariants in terms of the filtrations $\{R_i/R_{i-1} \oplus_{R_{i+1}} \cdots \oplus_{R_n} R_{n+1}\}_{n\in\mathbf{N}}$; for instance for t and n odd we get the following exact sequence:

$$0 \to R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} \to (\varinjlim_{n \text{ odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_t} \to (R_{t+1}/R_t)^{K_t}.$$

Finally (proposition 3.7) in order to extract the K_t-invariants from the previous exact sequence, we exploit the description of the generators of the socle filtration for R_{t+1}/R_t: we get some explicit nullity conditions of certains elements of the amalgamed sums, conditions which can easily be translated into a condition inside $\mathrm{soc}(\pi)$ (where we are able to do direct computations) via an inductive process by means of the operator T (cf. lemma 2.12).

The proof concerning the I_t invariants is similar. For instance, for n odd, we get a first estimate by an exact sequence of the form

$$0 \to \mathbf{V}_0 \to (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{I_t} \to \mathbf{V}^{I_t}$$

where $\mathbf{V}_0$ is a suitable subobject of $(\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_{t-1}}$ and

$$\mathbf{V} = (\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_t}/\mathbf{V}_0$$

(cf. §4.1 and §4.2). We then describe the I_t-invariants of the spaces of the form $\mathbf{V}$, via a decomposition into stable subspaces (cf. for instance propositions 4.3 and 4.7), from which we deduce the I_t-invariants of the inductive limits through some nullity conditions completely analogous to those of proposition 3.7 (cf. propositions 4.4, 4.8, 4.11, 4.14).

We outline here that by similar techniques we are able to describe the space of $\Gamma_0(p^k)$ and $\Gamma_1(p^k)$ invariants for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$ (cf. [Mo2]). Such spaces appear naturally in the study of torsion points in the cohomology of certain modular curves.

The plan of the paper is the following.

Section 2 is devoted to a brief summary of the results in [Mo1], [Bre] and [BP], in order to handle the computational techniques for the rest of the paper. More precisely, in 2.1 we re-interpret the $K\mathbf{Q}_p^{\times}$-restriction of a supersingular representations π in terms of certains induced representations R_n endowed with Hecke operators $T_n^{\pm}$; we give then a precise description of the socle filtration of π in 2.2 (cf. proposition 2.7) using "certains explicit elements" $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$. We recall also (§2.2.1) some classical results concerning $\mathrm{GL}_2(\mathbf{F}_p)$-parabolic induction for $B(\mathbf{F}_p)$-representation. Finally, we deal with some explicit computations on Witt vectors (lemmas 2.10 and 2.11) and study a nullity condition for some elements of the amalgamed sums introduced in 2.1 (cf. lemma 2.12).

Section 3 is devoted to an exhaustive description of K_t-invariants for supersingular representations. After a first estimate (cf. lemmas 3.2 and 3.3) we introduce in definition 3.4 the elements $x^{(')}_{l_1,\dots,l_{t-1}}$, $y^{(')}_{l_1,\dots,l_{t-1}}$ $z^{(')}_{l_1,\dots,l_{t-1}}$. Their behaviour let us refine the previous estimates. They indeed lead us to introduce the subobjects $\sigma(p-2)$, $\sigma(p-3)$, etc.. of definition 3.6, which let us complete the analysis of K_t-invariants stated in proposition 3.7. As a byproduct, we compute the $\overline{\mathbf{F}}_p$-dimension of such spaces.

Section 4 is concerned on the I_t-invariants and is divided into four numbers (completely analogous to each other) §4.1.1, §4.1.2, §4.2.1 and §4.2.2 (according to the parity of t and the direct summand in the decomposition of $\pi|_{K\mathbf{Q}_p^{\times}}$). In each number we start from a first estimate of such invariants by means of an exact sequence issued from the results in 3; we then introduce some explicit elements (cf. definitions 4.1, 4.5, 4.9, 4.12) the study of which let us describe precisely the space of I_t-invariants in each term of the aforementioned exact sequences (cf. propositions 4.3, 4.4, 4.7, 4.8, etc..).

Finally, in section §5 we describe precisely the spaces of K_t and I_t-invariants for principal and special series (where the computations are much simpler than in the supersingular case!).

We introduce now the main notations, convention and structure of the paper.

We fix a prime number p, which will always be assumed to be odd. We write $\mathbf{Q}_p$ (resp. $\mathbf{Z}_p$) for the p-adic completion of $\mathbf{Q}$ (resp. $\mathbf{Z}$) and $\mathbf{F}_p$ the field with p elements; $\overline{\mathbf{F}}_p$ is then a fixed algebraic closure of $\mathbf{F}_p$. For any $\lambda \in \mathbf{F}_p$ (resp. $x \in \mathbf{Z}_p$) we write $[\lambda]$ (resp. $\overline{x}$) for the Teichmüller lift (resp. for the reduction modulo p), defining $[0] \overset{\text{def}}{=} 0$.

We write $G \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Q}_p)$, $K \overset{\text{def}}{=} \mathrm{GL}_2(\mathbf{Z}_p)$ the maximal compact subgroup, I the Iwahori subreoup of K (i.e. the elements of K whose reduction modulo p is upper triangular) and I_1 for the pro-p-iwahori (i.e. the elements of I whose reduction is unipotent). For an integer $t \geqslant 1$ we define $K_t \overset{\text{def}}{=} \ker K \twoheadrightarrow \mathrm{GL}_2(\mathbf{Z}_p/p^t\mathbf{Z}_p)$ and

$$I_t \overset{\text{def}}{=} \{\begin{bmatrix} 1+p^t a & p^{t-1}b \\ p^t c & 1+p^t d \end{bmatrix} \in K \quad a,b,c,d \in \mathbf{Z}_p\}, U_t \overset{\text{def}}{=} \{\begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix} \in K \quad b \in \mathbf{Z}_p\}.$$

Moreover, let $Z \overset{\text{def}}{=} Z(G) \cong \mathbf{Q}_p^\times$ be te center of G and $B(\mathbf{Q}_p)$ (resp. $B(\mathbf{F}_p)$) the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ (resp. $\mathrm{GL}_2(\mathbf{F}_p)$).

For $r \in \{0, \dots, p-1\}$ we denote by σ_r the algebraic representation $\mathrm{Sym}^r \overline{\mathbf{F}}_p^2$ (endowed with the natural action of $\mathrm{GL}_2(\mathbf{F}_p)$). Explicitely, if we consider the identification $\mathrm{Sym}^r \overline{\mathbf{F}}_p^2 \cong \overline{\mathbf{F}}_p[X,Y]_r^h$ (where $\overline{\mathbf{F}}_p[X,Y]_r^h$ means the graded component of degree r for the natural grading on $\overline{\mathbf{F}}_p[X,Y]$) then

$$\sigma_r(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) X^{r-i}Y^i \overset{\text{def}}{=} (aX+cY)^{r-i}(bX+dY)^i$$

for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{F}_p)$, $i \in \{0, \dots, r\}$. We then endow σ_r with the action of K obtained by inflation $K \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_p)$ and, by imposing a trivial action of $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$, we get a smooth KZ-representation. Such a representation is still noted as σ_r, not to overload the notations.

If $r \in \{0, \dots, p-1\}$ it follows from the results in [BL95] that we have an isomorphism of $\overline{\mathbf{F}}_p$-algebras

$$\mathrm{End}_G(\mathrm{Ind}_{KZ}^G \sigma_r) \xrightarrow{\sim} \overline{\mathbf{F}}_p[T]$$

for a suitable endomorphism T, which depends on r, and where $\mathrm{Ind}_{KZ}^G \sigma_r$ is the usual compact induction (cf. [Bre], §3.2 for a detailed description of compact inductions). We then write $\pi(r,0,1)$ to mean the cokernel $\mathrm{coker}(\mathrm{Ind}_{KZ}^G \sigma_r \xrightarrow{T} \mathrm{Ind}_{KZ}^G \sigma_r$; such representations exhaust all supersingular representations for $\mathrm{GL}_2(\mathbf{Q}_p)$ (cf. Breuil's [Bre03a], Corollaire 4.1.1 et 4.1.4).

If H stands for the maximal torus of $\mathrm{GL}_2(\mathbf{F}_p)$ and $\chi : H \to \overline{\mathbf{F}}_p^\times$ is a multiplicative character we will write χ^s for the conjugate character defined by $\chi^s(h) \overset{\text{def}}{=} \chi(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} h \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ for $h \in H$. Characters of H will be seen as characters of $B(\mathbf{F}_p)$ or (by inflation) of (a filter of neighborhood of 1 in) I withouth any commentary.

With "representation" we always mean a smooth representations with central character with coefficients in $\overline{\mathbf{F}}_p^\times$. If V is a $\widetilde{K}$-representation, for $\widetilde{K}$ a subgroup of K, and $v \in V$, we write $\langle \widetilde{K} \cdot v \rangle$ to denote the sub-$\widetilde{K}$ representation of V generated by v. For a $\widetilde{K}$-representation V we write $\mathrm{soc}_{\widetilde{K}}(V)$ (or $\mathrm{soc}(V)$, or $\mathrm{soc}^1(V)$ if $\widetilde{K}$ is clear from the context) to denote the maximal semisimple sub-representation of V. Inductively, the subrepresentation $\mathrm{soc}^i(V)$ of V being defined, we define $\mathrm{soc}^{i+1}(V)$ as the inverse image of $\mathrm{soc}^1(V/\mathrm{soc}^i(V))$ via the projection $V \twoheadrightarrow V/\mathrm{soc}^i(V)$. We therefore obtain an increasing filtration $\{\mathrm{soc}^n(V)\}_{n \in \mathbf{N}_>}$ which will be referred to as the socle filtration for V; we will say that a subrepresentation W of V "comes from the socle filtration" if we have $W =$

$\mathrm{soc}^n(V)$ for some $n \in \mathbf{N}_>$ (with the convention that $\mathrm{soc}^0(V) \overset{\mathrm{def}}{=} 0$). The sequence of the graded pieces of the socle filtration for V will be shortly denoted by

$$\mathrm{SocFil}(V) \overset{\mathrm{def}}{=} \mathrm{soc}^1(V) \text{—} \mathrm{soc}^1(V)/\mathrm{soc}^0(V) \text{—} \dots \text{—} \mathrm{soc}^{n+1}(V)/\mathrm{soc}^n(V) \text{—} \dots$$

We recall the Kroneker delta: if S is any set, and $s_1, s_2 \in S$ we define

$$\delta_{s_1,s_2} \overset{\mathrm{def}}{=} \begin{cases} 0 & \text{if} \quad s_1 \neq s_2 \\ 1 & \text{if} \quad s_1 = s_2. \end{cases}$$

Moreover, for $x \in \mathbf{Z}$, we define $\lfloor x \rfloor \in \{0, \dots, p-2\}$ by the condition $\lfloor x \rfloor \equiv x \bmod p-1$.

2. Preliminaries and definitions

The aim of this section is to give the necessary tools to deal with the explicit computations needed for the description of K_t and I_t-invariants of supersingular representations $\pi(r, 0, 1)$. In §2.1 and §2.2 we recall the socle filtration of the KZ-representations $\pi(r, 0, 1)|_{KZ}$ made in [Mo1], together with the generators for the irreducible factors of the graded pieces of such filtration. Some classical results concerning $\mathrm{GL}_2(\mathbf{F}_p)$-parabolic induction for $B(\mathbf{F}_p)$-representations will be recalled in §2.2 as well, while §2.3 is devoted to some explicit computations on Witt vectors and elements of the amalgamed sums $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$. These computations will be a key tool in §3 and §4

2.1 On the KZ restriction of supersingular representations

We fix $r \in \{0, \dots, p-1\}$ and consider the supersingular representation $\pi(r, 0, 1)$; our goal is to give an exhaustive description of the objects involved in proposition 2.3. For this purpose, we recall the definition of the K-representations R_n, where $n \in \mathbf{N}$ as well as the "Hecke" operators $T_n^{\pm} : R_n \to R_{n\pm1}$, leading us to the decomposition of proposition 2.3. The reader is invited to refer to [Mo1] for the omitted details.

For any $n \in \mathbf{N}$ we define the following subgroup of K:

$$K_0(p^n) \overset{\mathrm{def}}{=} \{ \begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K, \text{ where } c \in \mathbf{Z}_p \}$$

(in particular, $K_0(p^0) = K$ and $K_0(p)$ is the Iwahori subgroup). For $0 \leqslant r \leqslant p-1$ and $n \in \mathbf{N}$ we define the $K_0(p^n)$-representation σ_r^n over $\overline{\mathbf{F}}_p$ as follow. The associated $\overline{\mathbf{F}}_p$-vector space of σ_r^n is $\mathrm{Sym}^r \overline{\mathbf{F}}_p^2$, while the left action of $K_0(p^n)$ is given by

$$\sigma_r^n(\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix}) \cdot X^{r-j} Y^j \overset{\mathrm{def}}{=} \sigma_r(\begin{bmatrix} d & c \\ p^n b & a \end{bmatrix}) \cdot X^{r-j} Y^j$$

for any $\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K_0(p^n)$, $0 \leqslant j \leqslant r$; in particular, the σ_r^n's are smooth and σ_r^0 is isomorphic to σ_r. Finally, we define

$$R_r^n \overset{\mathrm{def}}{=} \mathrm{Ind}_{K_0(p^n)}^K \sigma_r^n.$$

If r is clear from the context, we will write simply R_n instead of R_r^n.

We recall that an $\overline{\mathbf{F}}_p$-basis for R_n is then described by

$$\mathscr{B}_n \overset{\mathrm{def}}{=} \{ [\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j} Y^j], [\begin{bmatrix} 1 & 0 \\ p\mu & 1 \end{bmatrix}, X^{r-j} Y^j] \text{ for } \mu \in I_n,\ \mu' \in I_{n-1}, 0 \leqslant j \leqslant r \}$$

Each of the K-representations R_n is endowed with "natural" Hecke operators $T_n^{\pm}$. Their definitions and properties are summed up in the next

Proposition 2.1. *For all $n \in \mathbf{N}$ we have a K-equivariant monomorphism $T_n^+ : R_n \hookrightarrow R_{n+1}$ characterised by:*

$$T_n^+([1_K, X^{r-j}Y^j]) = \sum_{\mu_n \in \mathbf{F}_p} (-\mu_n)^j \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1_K, X^r] \text{ if } n > 0$$

$$T_0^+([1_K, X^{r-j}Y^j]) = \sum_{\mu_0 \in \mathbf{F}_p} (-\mu_0)^{r-j} [\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix}, X^r] + [1_K, \delta_{j,0}X^r] \text{ if } n = 0.$$

For all $n \in \mathbf{N}_>$ we have a K-equivariant epimorphism $T_n^- : R_n \twoheadrightarrow R_{n-1}$; such a morphism is characterised by the conditions:

$$T_n^-([\begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1}] & 1 \end{bmatrix}, X^{r-j}Y^j]) = [1_K, \delta_{r,j}(\mu_{n-1}X + Y)^r] \text{ if } n \geqslant 2$$

$$T_1^-([1_K, X^{r-j}Y^j]) = \delta_{r,j}Y^r \text{ if } n = 1$$

for $\mu_{n-1} \in \mathbf{F}_p$.

Proof: Omissis. Cf. [Mo1] §3.2.♯

We identify R_n as a K-subrepresentation of R_{n+1} via the monomorphism T_n^+ without any further commentary. For any odd integer $n \geqslant 1$ we use the hecke operators $T_n^{\pm}$ to define (inductively) the amalgamed sum $R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$ via the following co-cartesian diagram

$$\begin{tikzcd}
R_n \arrow[r, hook, "T_n^+"] \arrow[d, twoheadrightarrow, "-pr_{n-1}\circ T_n^-"'] & R_{n+1} \arrow[d, dotted, twoheadrightarrow, "pr_{n+1}"] \\
R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, dotted] & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}.
\end{tikzcd}$$

Similarly we define the amalgamed sums $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ for any positive even integer $n \in \mathbf{N}_>$. Then

Proposition 2.2. *For any odd integer $n \in \mathbf{N}$, $n \geqslant 1$ we have a natural commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d, twoheadrightarrow, "-pr_{n-1}\circ T_n^-"] & R_{n+1} \arrow[r] \arrow[d, twoheadrightarrow, "pr_{n+1}"] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r, "\pi"] & R_{n+1}/R_n \arrow[r] & 0
\end{tikzcd}$$

with exact lines.

We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N}\setminus\{0\}}.$$

Proof: Omissis. Cf. [Mo1], proposition 4.1 ♯

As claimed at the beginning of the paragraph, we can translate the KZ-restriction of $\pi(r,0,1)|_{KZ}$ in terms of the R_n's and $T_n^{\pm}$:

Proposition 2.3. *We have a KZ-equivariant isomorphism:*

$$\pi(r,0,1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{m \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1})$$

where we define an action of Z on the left hand side by making $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ act trivially.

Proof: Omissis. Cf. [Mo1], proposition 3.9.♯

2.2 Socle filtration for $\pi(r,0,1)|_{KZ}$ and parabolic inductions

Let us fix an integer $r\in\{0,\dots,p-1\}$. In this paragraph we are going to define a filtration (definition 2.4) on the inductive limits of proposition 2.3. Such filtration is rather finer than the one which can be deduced from proposition 2.2 and will let us describe the socle filtration for $\pi(r,0,1)|_{KZ}$. In what follows, we will assume the obvious conventions $R_0\oplus_{R_{-1}}R_0\overset{\mathrm{def}}{=}R_0$ and $R_1/R_0\oplus_{R_0}R_1\overset{\mathrm{def}}{=}R_1/R_0$.

Definition 2.4. *Let $n\in\mathbf{N}$, $0\leqslant h\leqslant r$. We define $\mathrm{Fil}^h(R_{n+1})$ as the K-subrepresentation of R_{n+1} generated by $[1_K,X^{r-h}Y^h]$. For $h=-1$, we define $\mathrm{Fil}^{-1}(R_{n+1})\overset{\mathrm{def}}{=}0$.*

The family $\{\mathrm{Fil}^h(R_{n+1})\}_{h=-1}^r$ defines a separated and exhaustive filtration on R_{n+1}, and for each $h\in\{0,\dots,r\}$ the family

$$\mathscr{B}_{n+1,t}\overset{\mathrm{def}}{=}\left\{[\begin{bmatrix}\mu & 1\\ 1 & 0\end{bmatrix},X^{r-i}Y^i],\ [\begin{bmatrix}1 & 0\\ p\mu' & 1\end{bmatrix},X^{r-i}Y^i]\,\mu\in I_{n+1},\ \mu'\in I_n,\ 0\leqslant i\leqslant h\right\}$$

is an $\overline{\mathbf{F}}_p$ basis for $\mathrm{Fil}^h(R_{n+1})$. By Frobenius reciprocity we get a K-isomorphism

$$\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^h\overset{\sim}{\longrightarrow}\mathrm{Fil}^h(R_{n+1})/\mathrm{Fil}^{h-1}(R_{n+1})$$

(cf. [Mo1], lemma 4.4).

To give explicit description for the socle filtration of the induced representation $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^h$ needs the introduction of the following elements.

Definition 2.5. *Fix $n\in\mathbf{N}$ and let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple. We define then*

$$F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\overset{\mathrm{def}}{=}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\cdots\sum_{\mu_n\in\mathbf{F}_p}\mu_n^{l_n}\begin{bmatrix}1 & 0\\ p^n[\mu_n] & 1\end{bmatrix}[1,e]$$

where e is an $\overline{\mathbf{F}}_p$-basis for the underlying vector space associated to the $K_0(p^{n+1})$-representation χ_r^s.

For a fixed n-tuple $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ we set $h'\overset{\mathrm{def}}{=}\sum_{j=1}^n l_j$. Then

$$F_0^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\overset{\mathrm{def}}{=}\begin{cases}\displaystyle\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\not\equiv 0\,[p-1]; \\ \\ \displaystyle\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}]+(-1)^{h+h'}[1_K,F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\equiv 0\,[p-1]\end{cases}$$

$$F_1^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\overset{\mathrm{def}}{=}\begin{cases}[1_K,F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\not\equiv 0\,[p-1]; \\ \\ \displaystyle\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}] \\ \qquad\text{if } r-2(h+h')\equiv 0\,[p-1].\end{cases}$$

Such definitions look a bit awkward, but they come essentially from the description of the socle filtration for $\mathrm{GL}_2(\mathbf{F}_p)$-parabolic inductions (proposition 2.9)

We provide the set $\{0,1\}\times\{0,\ldots,p-1\}^n$ with the antilexicographic ordering, writing $(i+1,l_1,\ldots,l_n)$ for the $n+1$-tuple immediately succeeding $(i,l_1,\ldots,l_n)$. We introduce then the quotients

$$Q(h)^{(0,n+1)}_{i,l_1,\ldots,l_n} \overset{\text{def}}{=} \mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a}^h/(\langle K\cdot F_j^{(0)} * \ldots F_{j_n}^{(n)}\rangle, \text{ for } (j,j_0,\ldots,j_n)\prec(i,l_1,\ldots,l_n)).$$

We remark that such notations do not keep track of the integer r; moreover if there will not be any ambiguities on h, we will simply write $Q^{(0,n+1)}_{i,l_1,\ldots,l_n}$ instead of $Q(h)^{(0,n+1)}_{i,l_1,\ldots,l_n}$. We believe such notations will not arise any confusion: the meaning will be clear from the context (cf. §3, §4).
We are now able to give a complete description for the socle filtration of $\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a}^h$:

Proposition 2.6. *Let $(l_1,\ldots,l_n)\in\{0,\ldots,p-1\}^n$ be an n-tuple, and let $h'\overset{\text{def}}{=}\sum_{i=1}^n l_i$. Then*

i) the socle of $Q^{(0,n+1)}_{1,l_1,\ldots,l_n}$ is described by

$$\mathrm{soc}(Q^{(0,n+1)}_{1,l_1,\ldots,l_n}) = \langle KF_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \cong \mathrm{Sym}^{p-1-\lfloor r-2(h+h')\rfloor}\overline{\mathbf{F}}_p^2\otimes \det{}^{r-(h+h')}$$

moreover, $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ is an H-eigenvector whose associated eigencharacter is

$$\chi_{2(h+h')-r}\det{}^{r-(h+h')}.$$

ii) the socle of $Q^{(0,n+1)}_{0,l_1,\ldots,l_n}$ is described by

$$\mathrm{soc}(Q^{(0,n+1)}_{0,l_1,\ldots,l_n}) = \begin{cases} \langle KF_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \cong \mathrm{Sym}^{\lfloor r-2(h+h')\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{h+h'} \\ \text{if } r-2(h+h')\not\equiv 0[p-1]; \\ \\ \langle KF_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \oplus \langle KF_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \cong \\ \cong \det{}^{h+h'}\oplus\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2\otimes\det{}^{h+h'} \\ \text{if } r-2(h+h')\equiv 0[p-1]. \end{cases}$$

Further, $F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ (and moreover $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ if $r-2(h+h')\equiv 0\,[p-1]$) is an H-eigenvector whose associated eigencharacter is $\chi_{r-2(h+h')}\det^{h+h'}$.

Proof: Omissis. Cf. [Mo1], proposition 6.6.♯

The filtration $\{\mathrm{Fil}^h(R_{n+1})\}_{h=-1}^r$ induces a natural filtration on the quotient R_{n+1}/R_n such that $\mathrm{Fil}^h(R_{n+1}/R_n)/\mathrm{Fil}^{h-1}(R_{n+1}/R_n)\cong\mathrm{Fil}^h(R_{n+1})/\mathrm{Fil}^{h-1}(R_{n+1})$ for all $h>0$; concerning $h=0$ we see (cf. [Mo1] lemma 8.3) that $\mathrm{Fil}^0(R_{n+1}/R_n)\cong Q^{(0,n+1)}_{0,\ldots,0,r+1}$. The main result of [Mo1] (cf. proposition 9.1) is that we can describe the socle filtration of $\pi(r,0,1)|_{KZ}$ in terms of the socle filtration of the quotients R_{n+1}/R_n. Precisely:

Proposition 2.7. *Let $r\in\{0,\ldots,p-1\}$, $n\in\mathbf{N}$. Then*

i) The socle filtration for R_{n+1}/R_n is described by

$$\begin{aligned}&\mathrm{SocFil}(R_{n+1}/R_n) = \\ &= \mathrm{SocFil}(Q^{(0,n+1)}_{0,\ldots,r+1}) \text{—} \mathrm{SocFil}(\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a}) \text{—} \ldots \text{—} \mathrm{SocFil}(\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a}^r)\end{aligned}$$

(where, if $r=p-1$, we forget about $\mathrm{SocFil}(Q^{(0,n+1)}_{0,\ldots,r+1})$ and the socle filtration starts from $\mathrm{SocFil}(\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\mathfrak{a})$)

ii) If n is odd, the socle filtration for $R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}$ is described by

$$\begin{aligned}&\mathrm{SocFil}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}) = \\ &= R_0\text{—}\mathrm{SocFil}(R_2/R_1)\text{—}\mathrm{SocFil}(R_4/R_3)\text{—}\ldots\text{—}\mathrm{SocFil}(R_{n+1}/R_n).\end{aligned}$$

iii) *If n is even, the socle filtration for $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ is described by*

$$\begin{aligned}&\mathrm{SocFil}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}) = \\ &= \mathrm{SocFil}(R_1/R_0) \text{—} \mathrm{SocFil}(R_3/R_2) \text{—} \mathrm{SocFil}(R_5/R_4) \text{—} \ldots \text{—} \mathrm{SocFil}(R_{n+1}/R_n).\end{aligned}$$

Proof: Omissis. Cf. [Mo1], proposition 7.1 and 9.1.♯

In particular, we are able to compute the dimension of some subquotients of $\pi(r, 0, 1)|_{KZ}$.

LEMMA 2.8. *Let $r \in \{0, \ldots, p-1\}$.*

i) *Let $t \in \mathbf{N}_>$; then*

$$\dim_{\overline{\mathbf{F}}_p}(\mathrm{Fil}^0(R_t/R_{t-1})) = \begin{cases} (p-1-r)(p+1)p^{t-2} & \text{if } t \geqslant 2 \\ p-r \quad \text{if } t=1. \end{cases}$$

ii) *Let $t \in \mathbf{N}_>$; then*

$$\dim_{\overline{\mathbf{F}}_p}(R_t/R_{t-1}) = \begin{cases} (r+1)(p^2-1)p^{t-2} & \text{if } t \geqslant 2 \\ p(r+1) \quad \text{if } t=1. \end{cases}$$

iii) *If n is odd then*

$$\dim_{\overline{\mathbf{F}}_p}(R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) = (r+1)p^{n+1}.$$

iv) *If n is even, then*

$$\dim_{\overline{\mathbf{F}}_p}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}) = (r+1)p^{n+1}.$$

Proof: It is an elementary computation, using [Mo1], Corollary 6.5-*iii)* and the decompositions of proposition 2.7.♯

2.2.1 *Induced representations for $B(\mathbf{F}_p)$.* Let us consider the $B(\mathbf{F}_p)$-character $\chi_i^s\mathfrak{a}^j$. If e is a fixed $\overline{\mathbf{F}}_p$-basis for the underlying vector space associated to $\chi_i^s\mathfrak{a}^j$, we define the following elements of the induced representations $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$:

$$f_k \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^k \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, e]$$

where $k \in \{0, \ldots, p-1\}$. We can give an explicit description of the socle filtration for $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ in terms of the functions f_k:

PROPOSITION 2.9. *Let $i, j \in \{0, \ldots, p-1\}$. Then*

i) *for $k \in \{0, \ldots, p-1\}$, f_k is an H-eigenvector, whose associated eigencharacter is $\chi_{i-2j}\det^j\mathfrak{a}^{-k}$, and the family*

$$\mathscr{B} \stackrel{\mathrm{def}}{=} \{f_k,\ 0 \leqslant k \leqslant p-1,\ [1, e]\}$$

is an $\overline{\mathbf{F}}_p$-basis for $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$.

ii) *If $i - 2j \not\equiv 0\,[p-1]$ then we have a nontrivial extention*

$$0 \to \mathrm{Sym}^{\lfloor i-2j \rfloor}\overline{\mathbf{F}}_p^2 \otimes \det^j \to \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j \to \mathrm{Sym}^{p-1-\lfloor i-2j \rfloor}\overline{\mathbf{F}}_p^2 \otimes \det^{i-j} \to 0.$$

The families

$$\{f_0, \ldots, f_{\lfloor i-2j \rfloor - 1}, f_{\lfloor i-2j \rfloor} + (-1)^{i-j}[1, e]\}, \{f_{i-2j}, \ldots, f_{p-1}\}$$

induce a basis for the socle and the cosocle of $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ respectively.

iii) If $i - 2j \equiv 0\,[p-1]$ *then* $\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ *is semisimple and*

$$\mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)\chi_i^s\mathfrak{a}^j} \xrightarrow{\sim} \det{}^j \oplus \mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2 \otimes \det{}^j;$$

The families

$$\{f_0 + (-1)^j[1,e]\},\{f_0, f_1, \dots, f_{p-2}, f_{p-1} + (-1)^j[1,e]\}$$

induce an $\overline{\mathbf{F}}_p$*-basis for* $\det^j$ *and* $\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2 \otimes \det^j$ *respectively.*

Proof: Omissis. Cf. [BP], lemmas 2.5, 2.6, 2.7.♯

2.3 Computations on Witt vectors.

In this paragraph we collect all the technical computations needed for the study of K_t and I_t-invariants. For $\mu, \lambda \in \mathbf{F}_p$ we define

$$P_\lambda(\mu) \overset{\text{def}}{=} -\sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p}\lambda^{p-j}\mu^j \in \mathbf{F}_p.$$

Then, we have the following results concerning the sum of Witt vectors in $\mathbf{Z}_p$:

Lemma 2.10. *Let* $\lambda \in \mathbf{F}_p$, $\sum_{j=0}^n p^j[\mu_j] \in I_{n+1}$. *Then the following equality holds in* $\mathbf{Z}_p/(p^{n+1})$*:*

$$[\lambda] + \sum_{j=0}^n p^j[\mu_j] \equiv [\lambda + \mu_0] + p[\mu_1 + P_\lambda(\mu_0)] + \dots + p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1})]$$

where, for all $j = 1, \dots, n-2$, *the* $P_{\lambda,\dots,\mu_j}(X)$*'s (resp.* $P_{\lambda,\mu_0}(X)$*, resp.* $P_\lambda(X)$*) are suitable polynomials in* $\mathbf{F}_p[X]$*, of degree* $p-1$*, depending only on* $\lambda, \dots, \mu_j$ *(resp. on* λ, μ*, resp. on* λ*), and whose dominant coefficient is* $-P_{\lambda,\dots,\mu_{j-1}}(\mu_j)$ *(resp.* $-P_\lambda(\mu_0)$*, resp.* $-\lambda$*).*

Proof: Immediate exercise on Witt vectors in $\mathbf{Z}_p$. ♯

Lemma 2.11. *Let* $\lambda \in \mathbf{F}_p$, $z \overset{\text{def}}{=} \sum_{j=1}^n p^j[\mu_j]$ *and* $k \geqslant 0$. *Then it exists a* p*-adic integer* $z' = \sum_{j=1}^n p^j[\mu'_j] \in \mathbf{Z}_p$ *such that*

$$z \equiv z'(1 + zp^k[\lambda]) \bmod p^{n+1}.$$

Furthermore, for $j = k+3, \dots, n$ *(resp.* $j = k+2$*, resp.* $j \leqslant k+1$*) we have the following equality in* $\mathbf{F}_p$*:*

$$\mu_j = \mu'_j + \mu_{j-k-1}\mu'_1\lambda + \dots + \mu_1\mu_{j-k-1}\lambda + S_{j-2}(\mu_{j-1})$$

(resp. $\mu_{k+2} = \mu'_{k+2} + \mu'_1\mu_1\lambda$ *for* $j = k-2$*, resp.* $\mu_j = \mu'_j$ *if* $j \leqslant k+1$*) where* $S_{j-2}(X) \in \mathbf{F}_p[X]$ *is a polynomial of degree* $p-1$*, depending only on* $\lambda, \dots, \mu_{j-2}$ *and leading coefficient* $-s_{\lambda,\dots,\mu_{j-2}} \overset{\text{def}}{=} \mu'_{j-1} - \mu_{j-1}$.

Proof: Exercise on Witt vectors.♯

To conclude, we give a criterion to detect wether a certain element (which naturally appears in the study of K_t and I_t-invariants) is zero in the amalgamed sums $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ (n odd) and $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ (n even).

Lemma 2.12. *Let $k \geqslant 2$ and fix an $(k-1)$-tuple $(l_1, \dots, l_{k-1})$. If we set*

$$x_{l_1,\dots,l_{k-1}} \stackrel{\text{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_{k-1} \in \mathbf{F}_p} \mu_{k-2}^{l_{k-2}} \begin{bmatrix} 1 & 0 \\ p^{k-2}[\mu_{k-2}] & 1 \end{bmatrix} \sum_{\mu_{k-1} \in \mathbf{F}_p} \mu_{k-1}^{l_{k-1}} [1, (\mu_{k-1}X + Y)^r] \in R_{k-1}.$$

i) Assume k odd. We describe the image of $x_{l_1,\dots,l_{k-1}}$ in the amalgamed sum $R_0 \oplus_{R_1} \cdots \oplus_{R_{k-2}} R_{k-1}$ as follow:

- *a) $x_{l_1,\dots,l_{k-1}} \equiv 0$ if $(l_1, \dots, l_{k-1}) \prec (r, p-1-r, \dots, r, p-1-r)$;*
- *b) if $(r, p-1-r, \dots, r, p-1-r) \prec (l_1, \dots, l_{k-1})$, then the image of $x_{l_1,\dots,l_k-1}$ induces a I_1-invariant generator in a subquotient of $R_0 \oplus_{R_1} \cdots \oplus_{R_{k-2}} R_{k-1}$ of the form $\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t'}$ for some suitable $t' \in \mathbf{N}$;*
- *c) equal to (the image of) $(-1)^{(r+2)(\frac{k-1}{2})} Y^r \in R_0$ if $(l_1, \dots, l_{k-1}) = (r, p-1-r, \dots, r, p-1-r)$.*

ii) Assume k even. We describe the image of $x_{l_1,\dots,l_{k-1}}$ in the amalgamed sum $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{k-2}} R_{k-1}$ as follow:

- *a′) $x_{l_1,\dots,l_k-1} \equiv 0$ if $(l_1, \dots, l_{k-1}) \prec (p-1-r, r, \dots, r, p-1-r)$;*
- *b′) if $(p-1-r, r, \dots, r, p-1-r) \prec (l_1, \dots, l_{k-1})$, then the image of $x_{l_1,\dots,l_k-1}$ induces a $I_!$-invariant generator of a subquotient of $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{k-2}} R_{k-1}$ of the form $\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t'}$ for some suitable $t' \in \mathbf{N}$;*
- *c′) equal to (the image of) $(-1)^{(r+2)(\frac{k-2}{2})}(-1)[1, X^r] \in \mathrm{Fil}^0(R_1/R_0)$ if $(l_1, \dots, l_{k-1}) = (p-1-r, r, \dots, r, p-1-r)$.*

Proof: It is an induction on k; we treat the case k even, the other being completely similar. The result is clearly true for $k=2$. For the general case, we consider the image of the element

$$u \stackrel{\text{def}}{=} \sum_{j=0}^{r} \binom{r}{j} \sum_{\mu_{k-2} \in \mathbf{F}_p} \mu_{k-2}^{l_{k-2}} \begin{bmatrix} 1 & 0 \\ p^{k-2}[\mu_{k-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j] \sum_{\mu_{k-1} \in \mathbf{F}_p} \mu_{k-1}^{l_{k-1}+r-j} \in R_{k-1}$$

in R_{k-1}/R_{k-2} via the natural projection $R_{k-1} \twoheadrightarrow R_{k-1}/R_{k-2}$. We see then if $(r+1, p-1-r) \preceq (l_{k-2}, l_{k-1})$ such an image is nonzero in R_{k-1}/R_{k-2}; we deduce that the image of $x_{l_1,\dots,l_{k-1}}$ in R_{k-1}/R_{k-2} is a K-generator of a subquotient of the form $\mathrm{Ind}_{K_0(p^{k-1})}^K \chi_r^s \mathfrak{a}^{t'}$, for a suitable $t' \in \mathbf{N}$. If $l_{k-1} = p-1-r$ and $l_{k-2} \leqslant r$ we see that u is in the image of T_{k-2}^+:

$$u = T_{k-2}^+((-1)^{l_{k-2}+1}[1, X^{r-l_{k-2}}Y^{l_{k-2}}]).$$

If $l_{k-2} < r$, then $T_{k-2}^-([1, X^{r-l_{k-2}}Y^{l_{k-2}}]) = 0 \in R_{k-3}$, while for $l_{k-2} = r$ we get

$$-T_{k-2}^-((-1)^{r+1}[1, X^{r-l_{k-2}}Y^{l_{k-2}}] = (-1)^{r+2}[1, Y^r] \in R_{k-3}.$$

This let us establish the inductive step and the proof is complete.♯

3. Study of K_t-invariants

Fix an integer $r \in \{0, \dots, p-2\}$; in this section we use the explicit description of the socle filtration of $\pi(r, 0, 1)|_{KZ}$ to deduce the space of K_t-invariants $\pi(r, 0, 1)^{K_t}$.

We start from rough estimates of such spaces in terms of the filtrations $R_i/(R_{i-1}) \oplus_{R_{i+1}} \cdots \oplus_{R_n} R_{n+1}$ in lemmas 3.2 and 3.3: they let us rule out a wide range of possibilities for the K_t invariants. For those cases which are not covered by the previous estimates, we pursue a detailed (and, unfortunately, rather technical) analysis, by means of the elements introduced in definition 3.4. Such analysis lead us to refine the results of lemmas 3.2 and 3.3 in proposition 3.5 from which we deduce the exact description of the K_t-invariants given in proposition 3.7.

First of all, we have

$$K_t = \begin{bmatrix} 1 & 0 \\ p^t\mathbf{Z}_p & 1 \end{bmatrix} \begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix} \begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}.$$

Furthermore

Lemma 3.1. *If σ is a smooth K-representation over $\overline{\mathbf{F}}_p$ and $t \in \mathbf{N}$, then*

$$\mathrm{soc}_K(\sigma) = \mathrm{soc}_{K/K_t}(\sigma^{K_t}).$$

Proof: It is enough to recall that for any irreducible smooth K representation τ we have $\tau^{K_1} = \tau$. $\sharp$

Lemma 3.2. *Let $t \geqslant 1$. Then*

i) If t is odd then

$$(\varinjlim_{n,\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_t} = (R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1})^{K_t}$$

$$(\varinjlim_{n,\,\mathrm{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})^{K_t} = \begin{cases} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-1}} R_t)^{K_t} & \text{if} \quad r \neq 0 \\ (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t+1}} R_{t+2})^{K_t} & \text{if} \quad r = 0. \end{cases}$$

ii) If t is even, then

$$(\varinjlim_{n,\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{K_t} = \begin{cases} (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-1}} R_t)^{K_t} & \text{if} \quad r \neq 0 \\ (R_0 \oplus_{R_1} \cdots \oplus_{R_{t+1}} R_{t+2})^{K_t} & \text{if} \quad r = 0. \end{cases}$$

$$(\varinjlim_{n,\,\mathrm{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})^{K_t} = (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_t} R_{t+1})^{K_t}$$

Proof: We first prove the statement for $r \neq 0$. Let $n > t$ and assume we have $z \in (\cdots \oplus_{R_n} R_{n+1})^{K_t}$ such that $\pi(z) \neq 0$ in R_{n+1}/R_n (where π denotes the natural projection of proposition 2.2). As K_t is normal in K, we conclude that $\pi(\langle K \cdot z \rangle) \leqslant (R_{n+1}/R_n)^{K_t}$ and, by lemma 3.1, $\mathrm{soc}_K(\pi(\langle K \cdot z\rangle)) \cap \mathrm{soc}_K(R_{n+1}/R_n) \neq 0$. By the explicit description of $\mathrm{soc}_K(R_{n+1}/R_n)$ we deduce that it exists $y \in (\cdots \oplus_{R_{n-2}} R_{R-1})$ such that we are in one of the following situations:

i) the element

$$\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x' + y$$

is K_t-invariant (in the amalgamed sum);

ii) we have $p-3-r=0$ and the element

$$\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x' + (-1)^{r+1}x' + y$$

is K_t-invariant (in the amalgamed sum);

where we put

$$x' \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{n-1} \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1}] & 1 \end{bmatrix} \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{r+1} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1, X^r].$$

Consider now the projection

$$(\cdots \oplus_{R_n} R_{n+1}) \twoheadrightarrow R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}) \oplus_{R_n} R_{n+1}.$$

As the space $(R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}))$ is fixed under the action of $\begin{bmatrix} 1 & p^n\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$, it follows that the elements $\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x'$ (resp. $\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x' + (-1)^{r+1}x'$) should be fixed under the action of $\begin{bmatrix} 1 & p^n\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$ inside $R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}) \oplus_{R_n} R_{n+1}$, which is absurde. Indeed, a computation using lemma 2.10 shows

$$\begin{bmatrix} 1 & p^{n-1}[\lambda] \\ 0 & 1 \end{bmatrix} x - x = \sum_{j=1}^{r+1} \binom{r+1}{j} (-1)^{r+1-j} T_n^+(v_j)$$

where

$$v_j \stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{n-1}\in\mathbf{F}_p} (-P_\lambda(\mu_{n-1}))^j \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1}+\lambda] & 1 \end{bmatrix} [1, X^{j-1}Y^{r-(j-1)}].$$

Using the operator $-T_n^-$ and the natural projection $R_n \twoheadrightarrow R_n/\mathrm{Fil}^{r-1}(R_n)$ we get

$$-T_n^-(\sum_{j=1}^{r+1} \binom{r+1}{j} (-1)^{r+1-j} v_j) \equiv$$

$$\equiv (r+1)(-1)^{r+2}\lambda \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{n-2}\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{n-2}[\mu_{n-2}] & 1 \end{bmatrix} [1, Y^r] \mod \mathrm{Fil}^{r-1}(R_{n-1})$$

$$(\text{resp. } \equiv (r+1)(-1)^{r+2}\lambda \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] \quad \text{if } n = 2)$$

and such an element is nonzero in $R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1})$ if $r \neq 0$. As x' is anyway $\begin{bmatrix} 1 & p^{n-1}[\lambda] \\ 0 & 1 \end{bmatrix}$-invariant in R_{n+1} we deduce that the elements in $i)$, $ii)$ can be K_t-invariant only if $n-1 < k$: this let us conclude the case $r \neq 0$.

We pass to the the case $r = 0$ and and let $n > t+1$. Using the same arguments for the case $r \neq 0$ we see that if we have $z \in (\cdots \oplus_{R_n} R_{n+1})^{K_t}$ such that $\pi(z) \neq 0$ in R_{n+1}/R_n it would exists an element $\overline{y} \in R_{n-1}/R_{n-2}$ such that $w \stackrel{\mathrm{def}}{=} \overline{y} + \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x'$ (resp. $w \stackrel{\mathrm{def}}{=} \overline{y} + \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} x' + (-1)^{r+1}x'$ if $r = p-3$) is $\begin{bmatrix} 1 & p^{n-2}\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$ invariant inside $R_{n-1}/R_{n-2} + \langle w \rangle$. On the other hand, we have a decomposition of R_{n-1}/R_{n-2} in $\begin{bmatrix} 1 & p^{n-2} \\ 0 & 1 \end{bmatrix}$-stable subspaces. Indeed R_{n-1}/R_{n-2} is a quotient of $\mathrm{Ind}_{K_0(p^{n-1})}^{K} 1$ and if we put

$$w'_{l_1,\dots,l_{n-2}}(0) \stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{n-1}\in\mathbf{F}_p} \mu_{n-1}^{l_{n-1}} \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1}] & 1 \end{bmatrix} [1, e]$$

the latter admits the following decomposition in $\begin{bmatrix} 1 & p^{n-2}\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-stable subspaces:

a) for a fixed -tuple $(l_0,\dots,l_{n-3})\in\{0,\dots,p-1\}^{n-2}$ the $\overline{\mathbf{F}}_p$-subspace generated by the elements

$$\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}w_{l_1,\dots,l_{n-3},j}(0)$$

where $j\in\{0,\dots,p-1\}$;

b) the subspace generated by the elements $w'_{l_1,\dots,l_{n-2}}(0)$ for $(l_1,\dots,l_{n-3})\in\{0,\dots,p-1\}^{n-3}$.

For $r\neq p-3$ (resp. $r=p-3$) we study, analogously to the case $r\neq 0$, the action of $\begin{bmatrix}1 & p^{n-2}\mathbf{Z}_p\\ 0 & 1\end{bmatrix}$ on the element $\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}x'+(-1)^{r+1}x'$ (resp. $\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}x'+(-1)^{r+1}x'+(-1)^{r+1}x'$); using the previous decomposition in stable subspaces for R_{n-1}/R_{n-2} we deduce again a contraddiction with the assumption $\pi(z)\neq 0$ (the computational details are left to the reader). $\sharp$

On the other hand, we have

Lemma 3.3. *Let $t\geqslant 1$. Then:*

i) *for t odd we have*

$$\begin{aligned}(R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1})^{K_t}&=R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}\\ (R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2})^{K_t}&=R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}\quad\text{if } r\neq 0\\ (R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-1}}R_{t})^{K_t}&=R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-1}}R_{t}\quad\text{if } r=0\end{aligned}$$

ii) *for t even we have*

$$\begin{aligned}(R_0\oplus_{R_1}\cdots\oplus_{R_{t-3}}R_{t-2})^{K_t}&=R_0\oplus_{R_1}\cdots\oplus_{R_{t-3}}R_{t-2}\quad\text{if } r\neq 0\\ (R_0\oplus_{R_1}\cdots\oplus_{R_{t-1}}R_{t})^{K_t}&=R_0\oplus_{R_1}\cdots\oplus_{R_{t-1}}R_{t}\quad\text{if } r=0\\ (R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-2}}R_{t-1})^{K_t}&=R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-2}}R_{t-1}\end{aligned}$$

where we convene that $R_1/R_0\oplus_{R_{-2}}R_{-1}\overset{def}{=}0$.

Proof: If $\kappa\in K$ and $z\in I_t$ then

$$\kappa\begin{bmatrix}z & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}z & 1\\ 1 & 0\end{bmatrix}\kappa_1;\kappa\begin{bmatrix}1 & 0\\ z & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ z & 1\end{bmatrix}\kappa_2$$

for suitable $\kappa_1,\kappa_2\in K_t$. As the action of K_t is trivial on σ_r^j for $j<t$ (resp for $j\leqslant t$ if $r=0$) we get the desired result. $\sharp$

We are thus able to insert the K_t invariants into a natural exact sequence coming from the filtrations of proposition 2.2. For instance, for t odd we have

$$\begin{tikzcd}0 \arrow[r] & (R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1})^{K_t} \arrow[r] \arrow[d, equal] & (R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1})^{K_t} \arrow[r] \arrow[d, hook] & (R_{t+1}/R_t)^{K_t} \arrow[d, hook] & \\ 0 \arrow[r] & R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1} \arrow[r] & R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1} \arrow[r] & R_{t+1}/R_t \arrow[r] & 0\end{tikzcd}$$

and the reader can deduce similar diagrams, according to lemmas 3.2, 3.3. In particular, we are lead to the study of the K_t invariants of the quotients R_{n+1}/R_n, which is the object of the next proposition. We introduce the following notations:

Definition 3.4. *Let $t\geqslant 2$ be an integer, $(l_1,\dots,l_{t-1})\in\{0,\dots,p-1\}^{k-1}$ be an $(t-1)$-tuple. We define:*

i)

$$x_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t\in\mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r];$$
$$x'_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t\in\mathbf{F}_p} \mu_t^{r+1} [1, X^r]$$

which will be seen as elements of R_{t+1}, R_{t+1}/R_t or in the amalgamed sum, accordingly to the context.

ii) *if $r \neq 0$ we define*

$$y_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{r-1}Y];$$
$$y'_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{r-1}Y]$$

which will be seen as elements of R_t, R_t/R_{t-1} or in the amalgamed sum, accordingly to the context.

iii) *if $r = 0$ and X^r is a fixed $\overline{\mathbf{F}}_p$ basis of $\mathrm{Sym}^0\overline{\mathbf{F}}_p^2$, we define*

$$z_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} \sum_{\mu_{t+1}\in\mathbf{F}_p} \mu_{t+1} \begin{bmatrix} 1 & 0 \\ p^{t+1}[\mu_{t+1}] & 1 \end{bmatrix} [1, X^r];$$
$$z'_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots$$
$$\dots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} \sum_{\mu_{t+1}\in\mathbf{F}_p} \mu_{t+1} \begin{bmatrix} 1 & 0 \\ p^{t+1}[\mu_{t+1}] & 1 \end{bmatrix} [1, X^r]$$

which will be seen as elements of R_{t+2}, R_{t+2}/R_{t+1} or in the amalgamed sum, accordingly to the context.

The result concerning the K_t-invariants of R_{t+1}/R_t is the following

Proposition 3.5. *Let $t \geqslant 1$, $r \in \{0, \dots, p-2\}$. Then*

i) *the K_t invariants of R_{t+1}/R_t are described by*

$$(R_{t+1}/R_t)^{K_t} = \mathrm{Ind}_{K_0(p^t)}^{K} F_{r+1}^{(t)} \hookrightarrow \mathrm{Fil}^{(0)}(R_{t+1}/R_t);$$

ii) *if $r \neq 0$ and $t \geqslant 2$ the K_t-invariants of R_t/R_{t-1} are described by*

$$(R_t/R_{t-1})^{K_t} = (\mathrm{Fil}^0(R_t/R_{t-1}) + \tau_t) \hookrightarrow \mathrm{Fil}^1(R_t/R_{t-1})$$

where τ_t is the K-subrepresentation of $\mathrm{Fil}^1(R_t/R_{t-1})$ generated by (the image of) the elements $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ with $(l_1,\dots,l_{t-1}) \prec (0,\dots,0,r+1)$. If $t=1$ then the K_1-invariants of R_1/R_0 are described by

$$(R_1/R_0)^{K_1} = (\mathrm{Fil}^0(R_1/R_0) + \tau_1) \hookrightarrow \mathrm{Fil}^1(R_1/R_0)$$

where $\tau_1 = 0$ if $r \in \{0,1\}$ and τ_1 is the K-subrepresentation of $\mathrm{Fil}^1(R_1/R_0)$ generated by $\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-1}Y]$ if $r \geqslant 3$

(resp. by $\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-1}Y] - [1, X^{r-1}Y]$ if $r=2$).

iii) *If $r=0$ then the K_t-invariants of R_{t+2}/R_{t+1} are described by*

$$(R_{t+2}/R_{t+1})^{K_t} = \mathrm{Ind}_{K_0(p^k)}^K F_0^{(t)} * F_1^{(t+1)} \hookrightarrow Q_{0,\dots,0,1}^{(0,t+2)}$$

Proof: First, let $z \in (R_{t+1}/R_t)^{K_t}$, say $z \in \mathrm{Fil}^t(R_{t+1}/R_t) \setminus \mathrm{Fil}^{t-1}(R_{t+1}/R_t)$. We deduce, as in the proof of lemma 3.2 that one of the following condition must hold:

a) the element

$$\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} v'_t$$

is K_t-invariant;

b) we have $r - 2t \equiv 0\,[p-1]$ and the element

$$\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} v'_t + (-1)^t v'_t$$

is K_t-invariant,

where we put

$$v'_t \stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_t\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^{r-t}Y^t].$$

For $t \geqslant 1$ we study the action of $\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix}$ on the elements in *a)*, *b)* modulo Fil^{t-2} to deduce that such elements can not be K_t-invariant: we conclude that

$$(R_{t+1}/R_t)^{K_t} = \mathrm{Fil}^0(R_{t+1}/R_t).$$

A similar argument, using the exact sequence

$$0 \to \mathrm{Ind}_{K_0(p^t)}^K F_{r+1}^{(t)} \to Q_{0,\dots,0,r+1}^{(0,t+1)} \to Q_{0,\dots,0,r+2}^{(0,t+1)} \to 0$$

shows that

$$(Q_{0,\dots,0,r+1}^{(0,t+1)})^{K_t} = (\mathrm{Ind}_{K_0(p^t)}^K F_{r+1}^{(t)})^{K_t}.$$

As the latter is K_t-invariant we get the desired result.

ii) Assume $t \geqslant 2$. With the same arguments of *i)* we can check that

$$(R_t/R_{t-1})^{K_t} = (\mathrm{Fil}^1(R_t/R_{t-1}))^{K_t}$$

and, using the definition of $Q(1)^{(0,t)}_{0,\dots,0,r+1}$ and the fact that $\mathrm{Fil}^0(R_t/R_{t-1})$ is a quotient of $\mathrm{Ind}^K_{K_0(p^t)}\chi_r^s$, we have

$$(\mathrm{Fil}^1(R_t/R_{t-1}))^{K_t}=(\mathrm{Fil}^0(R_t/R_{t-1})+\tau_t)^{K_t}.$$

We can now check directly that the action of

$$\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0\\ 0 & 1+p^t\mathbf{Z}_p\end{bmatrix},\begin{bmatrix} 1 & 0\\ p^t\mathbf{Z}_p & 1\end{bmatrix}$$

on $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}\in R_t/R_{t-1}$ is trivial, provided that $l_{t-1}\leqslant r$, and we conclude.

The case of $t=1$ is a similar computation, and it is left to the reader.

iii) It is similar to the previous one and left to the reader. $\sharp$

Thanks to proposition 3.5 it suffices to study the behaviour of the elements of the form $x^{(\prime)}_{l_1,\dots,l_{k-1}}$, $y^{(\prime)}_{l_1,\dots,l_{k-1}}$, $z^{(\prime)}_{l_1,\dots,l_{k-1}}$ in order to describe completely the K_t-invariants of supersingular representations. First of all we introduce the objects:

Definition 3.6. *Let $t\geqslant 1$. We define the following subrepresentations of R_{t+1}, R_t:*

i) for $t\geqslant 2$, t odd, let

- *a_1) $\sigma(p-2)$ as the K-subrepresentation of R_{t+1} generated by $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$ with $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,r,p-1-r)$;*
- *a_2) $\sigma(p-3)$ as the K-subrepresentaiton of R_{t+1} generated by $\sigma(p-2)$ and the element $x_{r,p-1-r,\dots,r,p-1-r}+(-1)^{r+1}x'_{r,p-1-r,\dots,r,p-1-r}$;*
- *a_3) $\sigma(<p-3)$ as the K-subrepresentation of R_{t+1} generated by $\sigma(p-2)$ and the element $x_{r,p-1-r,\dots,r,p-1-r}$.*
- *b_1) if $r\neq 0$, $\sigma^s_y(1)$ (resp. $\sigma^s_z(0)$ if $r=0$) as the K-subrepresentation of R_t generated by $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ (resp. $z_{l_1,\dots,l_{t-1}}$, $z'_{l_1,\dots,l_{t-1}}$) with $(l_1,\dots,l_{t-1})\prec(p-1-r,r,\dots,p-1-r,r)$ if $r\neq 0$ (resp. if $r=0$).*
- *b_2) if $r\neq 0$, $\sigma^s_y(2)$ as the K-subrepresentaiton of R_t generated by $\sigma^s_y(1)$ and the element $y_{p-1-r,r,\dots,p-1-r,r}+(-1)^{(r-2)+1}y'_{p-1-r,r,\dots,p-1-r,r}$;*
- *b_3) if $r\neq 0$, $\sigma^s_y(>2)$ as the K-subrepresentation of R_t generated by $\sigma^s_y(1)$ and the element $y_{p-1-r,r,\dots,r,p-1-r,r}$.*

ii) For $t\geqslant 2$, t even, let

- *a'_1) $\sigma_y(p-2)$ (resp. $\sigma_z(p-2)$) the K subrepresentation of R_t generated by $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ (resp. $z_{l_1,\dots,l_{t-1}}$, $z'_{l_1,\dots,l_{t-1}}$) with $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,p-1-r,r)$ if $r\neq 0$ (resp. if $r=0$);*
- *a'_2) $\sigma_y(p-3)$ (resp. $\sigma_z(p-3)$) as the K-subrepresentaiton of R_t generated by $\sigma_y(p-2)$ (resp. $\sigma_z(p-2)$) and the element $y_{r,p-1-r,\dots,r,p-1-r}+(-1)^{r+1}y'_{r,p-1-r,\dots,p-1-r,r}$ if $r\neq 0$ (resp. $z_{r,p-1-r,\dots,r,p-1-r}+(-1)^{r+1}z'_{r,p-1-r,\dots,p-1-r,r}$ if $r=0$);*
- *a'_3) $\sigma_y(<p-3)$ (resp. $\sigma_z(<p-3)$) as the K-subrepresentation of R_t generated by $\sigma_y(p-2)$ (resp. $\sigma_z(p-2)$) and the element $y_{r,p-1-r,\dots,p-1-r,r}$ (resp. $z_{r,p-1-r,\dots,p-1-r,r}$).*
- *b'_1) $\sigma^s(0)$ and $\sigma^s(1)$ as the K-subrepresentation of R_{t+1} generated by $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$ with $(l_1,\dots,l_{t-1})\prec(p-1-r,r,\dots,r,p-1-r)$;*
- *b'_2) $\sigma^s(2)$ as the K-subrepresentaiton of R_{t+1} generated by $\sigma^s(1)$ and the element*

$$x_{p-1-r,r,\dots,r,p-1-r}+(-1)^{(r-2)+1}x'_{r,p-1-r,\dots,r,p-1-r};$$

- *b'_3) $\sigma^s(>2)$ as the K-subrepresentation of R_{t+1} generated by $\sigma^s(1)$ and the element*

$$x_{p-1-r,r,\dots,r,p-1-r}.$$

iii) Assume $t=1$. We define:

$a_1'')$ $\sigma(p-2)=0$;

$a_2'')$ $\sigma(p-3)$ as the K-subrepresentaiton of R_2 generated by

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{r+1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}[1,X^r]+(-1)^{r+1}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{r+1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}[1,X^r]$$

$a_3'')$ $\sigma(<p-3)$ as the K-subrepresentation of R_2 generated by the element

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{r+1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}[1,X^r].$$

$b_1'')$ $\sigma_y^s(1)=\sigma_z^s(0)=0$;

$b_2'')$ $\sigma_y^s(2)$ as the K-subrepresentation of R_1 generated by:

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-1}Y]+(-1)^{r+1}[1,X^{r-1}Y] \text{ for } r\geqslant 1;$$

$b_3'')$ $\sigma_y^2(>2)$ as the K-subrepresentation of R_1 generated by:

$$\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-1}Y] \text{ for } r\geqslant 1.$$

With the above formalism, we are ready to describe completely the K_t-invariants of supersingular representations $\pi(r,0,1)$ with $r\in\{0,\dots,p-2\}$ (and therefore also for $r=p-1$ since $\pi(0,0,1)\cong\pi(p-1,0,1)$.

Proposition 3.7. *Let $t\geqslant 1$ and $r\in\{0,\dots,p-2\}$; then*

i) Assume t odd. Then

$a_1)$ the K_t-invariants of $\varinjlim_{n,\,\mathrm{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})$ are precisely:

$$R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\begin{cases}\sigma(p-2) & \text{if } r=p-2\\ \sigma(p-3) & \text{if } r=p-3\\ \sigma(<p-3) & \text{if } r<p-3.\end{cases}$$

$b_1)$ the K_t-invariants of $\varinjlim_{n,\,\mathrm{even}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1})$ for $r\neq 0$ are precisely:

$$R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}+pr_t(\mathrm{Fil}^0(R_t))+\begin{cases}\sigma_y^s(1) & \text{if } r=1\\ \sigma_y^s(2) & \text{if } r=2\\ \sigma_y^s(>2) & \text{if } r>2.\end{cases}$$

while, if $r=0$

$$R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-1}}R_t+\sigma_z^s(0).$$

ii) Assume t even. Then

$a_2)$ the K_t-invariants of $\varinjlim_{n,\,\mathrm{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})$ for $r\neq 0$ are precisely:

$$R_0\oplus_{R_1}\cdots\oplus_{R_{t-3}}R_{t-2}+pr_t(\mathrm{Fil}^0(R_t))+\begin{cases}\sigma_y(p-2) & \text{if } r=p-2\\ \sigma_y(p-3) & \text{if } r=p-3\\ \sigma_y(<p-3) & \text{if } r<p-3.\end{cases}$$

while, for $r=0$

$$R_0\oplus_{R_1}\cdots\oplus_{R_{t-1}}R_t+\begin{cases}\sigma_z(p-3) & \text{if } p=3\\ \sigma_z(<p-3) & \text{if } p\neq 3.\end{cases}$$

b_2) *the K_t-invariants of* $\varinjlim_{n,\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ *are precisely:*

$$R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1} + \begin{cases} \sigma^s(0) & \text{if } r=0 \\ \sigma^s(1) & \text{if } r=1 \\ \sigma^s(2) & \text{if } r=2 \\ \sigma^s(>2) & \text{if } r>2. \end{cases}$$

Note that $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) = pr_t(\mathrm{Fil}^0(R_t))$ and $R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) = pr_t(\mathrm{Fil}^0(R_t))$; we believe that the redoundant notation of proposition 3.7 is more expressive.

We warn the reader that the proof of proposition 3.7 is rather technical and lenghty, relying on a detailed computations in the amalgamed sums by means of the Hecke operators $T_n^{\pm}$; we inserted it for sake of completeness. We apologize with the reader for its length and technicity.

Proof: i), $\mathbf{a_1}$). For $t \geqslant 2$ we fix an $(t-1)$-tuple $(l_1, \ldots, l_{t-1}) \in \{0, \ldots, p-1\}^{t-1}$ and we consider the action of $\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix}$ on the element $x_{l_1,\ldots,l_{t-1}}$ (the case $t=1$ is similar and left to the reader). We have the following equality in R_{t+1}:

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} x_{l_1,\ldots,l_{t-1}} = x_{l_1,\ldots,l_{t-1}} + \sum_{j=0}^{r+1} \binom{r+1}{j} (-\lambda)^j (-1)^{r+1-j} T_n^+(v_j)$$

where we put

$$v_j \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{j-1} Y^{r-(j-1)}] \in R_t.$$

Since

$$-T_n^-(\sum_{j=1}^{r+1} \binom{r+1}{j} (-\lambda)^j (-1)^{r+1-j} v_j =$$
$$= (r+1)(-1)^{r+2}(\lambda) \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} [1, (\mu_{t-1} X + Y)^r]$$

we conclude by lemma 2.12 (using of course the definition of T_1^- and $r \leqslant p-2$ in the case $(l_1, \ldots, l_{t-1}) = (r, p-1-r, \ldots, p-1-r)$) that

-) the element $x_{l_1,\ldots,l_{t-1}}$ is $\begin{bmatrix} 1 & p^t \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant in the amalgamed sum if $(l_1, \ldots, l_{t-1}) \preceq (r, p-1-r, \ldots, p-1-r)$;

-) the element $x_{l_1,\ldots,l_{t-1}}$ is not $\begin{bmatrix} 1 & p^t \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant in the amalgamed sum if $(r, p-1-r, \ldots, p-1-r) \prec (l_1, \ldots, l_{t-1})$.

Moreover, as

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} \begin{bmatrix} 1 + p^t * & p^t[\lambda] \\ p^{t+1} * & 1 + p^t * \end{bmatrix}$$

for $z' = \sum_{j=1}^{t} p^j [\mu_j]$, we see that $x'_{l_1,\ldots,l_{t-1}}$ is $\begin{bmatrix} 1 & p^t \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant (already inside R_{t+1}).

If we define σ as the K-subrepresentation of R_{k+1} generated by the elements $x_{l_1,\ldots,l_{t-1}}$, $x'_{l_1,\ldots,l_{t-1}}$

with $(l_1,\dots,l_{t-1})\preceq(r+1,p-1-r,r,\dots,p-1-r)$ we deduce from the diagram:

$$\begin{tikzcd}
0 \arrow[r] & (R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\sigma)^{K_t} \arrow[r] \arrow[d, hook] & (R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1})^{K_t} \arrow[r] \arrow[d, hook] & (R_{t+1}/(R_t+\sigma))^{K_t} \arrow[d, hook] & \\
0 \arrow[r] & R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\sigma \arrow[r] & R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1} \arrow[r] & R_{t+1}/(R_t+\sigma) \arrow[r] & 0
\end{tikzcd}$$

that

$$(R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1})^{K_t}=(R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\sigma)^{K_t},$$

as the K-socle of $R_{t+1}/(R_t+\sigma)$ is generated by $x_{r+1,p-1-r,r,\dots,p-1-r}$ (and $x_{r+1,p-1-r,r,\dots,p-1-r}+(-1)^{r+2}$ if the K-socle is semisimple).

Similarly, we study the action of $\begin{bmatrix}1+p^ta & 0\\ 0 & 1+p^td\end{bmatrix}$ on $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$, for $(l_1,\dots,l_{t-1})\preceq(r+1,p-1-r,r,\dots,p-1-r)$ and $a,d\in\mathbf{Z}_p^\times$. From the equality

$$\begin{bmatrix}1+p^ta & 0\\ 0 & 1+p^td\end{bmatrix}\begin{bmatrix}z & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}z(1+p^td)^{-1}(1+p^ta) & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1+p^td & 0\\ 0 & 1+p^ta\end{bmatrix}$$

we deduce the following equality in R_{t+1}:

$$\begin{bmatrix}1+p^ta & 0\\ 0 & 1+p^td\end{bmatrix}x_{l_1,\dots,l_{t-1}}=x_{l_1,\dots,l_{t-1}}+\sum_{j=1}^{r+1}\binom{r+1}{j}(\overline{d-a})^j(-1)^{r+1-j}T_t^+(v'_j)$$

where

$$v'_j\overset{\text{def}}{=}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^j\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\cdots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,X^{j-1}Y^{r-(j-1)}]\in R_t.$$

We deduce from lemma 2.12 (using again the definition of T_1^- and $r\leqslant p-2$ in the case $(l_1,\dots,l_{t-1})=(r,p-1-r,\dots,p-1-r)$) that

- -) the element $x_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix}1+p^t\mathbf{Z}_p & 0\\ 0 & 1+p^t\mathbf{Z}_p\end{bmatrix}$-invariant in the amalgamed sum if $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,p-1-r)$ or if $(l_1,\dots,l_{t-1})=(r,p-1-r,\dots,p-1-r)$ and $r\leqslant p-3$;
- -) the element $x_{l_1,\dots,l_{t-1}}$ is not $\begin{bmatrix}1+p^t\mathbf{Z}_p & 0\\ 0 & 1+p^t\mathbf{Z}_p\end{bmatrix}$-invariant in the amalgamed sum if $(r,p-1-r,\dots,p-1-r)=(l_1,\dots,l_{t-1})$ and $r=p-2$.

Moreover, the equality

$$\begin{bmatrix}1+p^ta & 0\\ 0 & 1+p^td\end{bmatrix}\begin{bmatrix}1 & 0\\ z' & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ z' & 1\end{bmatrix}\begin{bmatrix}1+p^t* & 0\\ p^{t+1}* & 1+p^t*\end{bmatrix}$$

for $z'=\sum_{j=1}^t p^j[\mu_j]$ shows that the action of $\begin{bmatrix}1+p^t\mathbf{Z}_p & 0\\ 0 & 1+p^t\mathbf{Z}_p\end{bmatrix}$ on $x'_{l_1,\dots,l_{t-1}}$ is trivial (already in R_{t+1}). As above, we conclude that

- -) if $r=p-2$ then
$$(R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1})^{K_t}=(R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\sigma(p-2))^{K_t};$$
- -) if $r\leqslant p-3$ then
$$(R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1})^{K_t}=(R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\sigma)^{K_t}.$$

We are now left to study the action of $\begin{bmatrix}1 & 0\\ p^t\mathbf{Z}_p & 1\end{bmatrix}$ on $x_{l_1,\dots,l_{t-1}}$, $x'_{l_1,\dots,l_{t-1}}$, for $(l_1,\dots,l_{t-1})\preceq$

$(r+1,p-1-r,r,\ldots,p-1-r)$. For $z=\sum_{j=0}^{t}p^j[\mu_j]$ we have the equality

$$\left[\begin{array}{cc}1 & 0\\ p^t[\lambda] & 1\end{array}\right]\left[\begin{array}{cc}z & 1\\ 1 & 0\end{array}\right]=\left[\begin{array}{cc}z_1 & 1\\ 1 & 0\end{array}\right]\left[\begin{array}{cc}1+p^t* & p^t[\lambda]\\ p^{t+1}* & 1+p^t*\end{array}\right]$$

where $z_1=\sum_{j=0}^{t-1}p^j[\mu_j]+p^t[\mu_t+\mu_0^2\lambda]$. We can use lemma 2.12 (and the definition of T_1^- in the case $(r,p-1-r,\ldots,p-1-r)=(l_1,\ldots,l_{t-1})$) to deduce the following equality in $R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1}$:

$$\left[\begin{array}{cc}1 & 0\\ p^t[\lambda] & 1\end{array}\right]x_{l_1,\ldots,l_{t-1}}-x_{l_1,\ldots,l_{t-1}}=$$
$$=\begin{cases}0 \quad \text{if either}\,(l_1,\ldots,l_{t-1})\prec(r,p-1-r,\ldots,p-1-r)\,\text{or}\\ \qquad (l_1,\ldots,l_{t-1})=(r,p-1-r,\ldots,p-1-r)\,\text{and}\,r<p-3\\ \\ (r+1)\lambda(-1)^{(r+2)\frac{k-1}{2}}Y^r \quad \text{if}\,(l_1,\ldots,l_{t-1})=(r,p-1-r,\ldots,p-1-r)\,\text{and}\,r=p-3.\end{cases}$$

On the other hand we have

$$\left[\begin{array}{cc}1 & 0\\ p^t[\lambda] & 1\end{array}\right]x'_{l_1,\ldots,l_{t-1}}-x'_{l_1,\ldots,l_{t-1}}=$$
$$=\begin{cases}0 \quad \text{if either}\,(l_1,\ldots,l_{t-1})\prec(r,p-1-r,\ldots,p-1-r)\\ (r+1)\lambda(-1)^{(r+2)\frac{k-1}{2}}Y^r \quad \text{if}\,(l_1,\ldots,l_{t-1})=(r,p-1-r,\ldots,p-1-r).\end{cases}$$

As

$$\mathrm{soc}(R_{t+1}/R_t+\sigma(\bullet))=\mathrm{soc}(Q^{(0,t+1)}_{1,r,p-1-r,\ldots,p-1-r,r+1})$$

(where $\sigma(\bullet)\in\{\sigma(p-3),\sigma(<p-3)\}$ according to r as in the statement of i)-a_1)) we conclude, as above, that

$$(R_0\oplus_{R_1}\cdots\oplus_{R_t}R_{t+1})^{K_t}=(R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\sigma(\bullet))^{K_t}$$

and the result follows, as $\sigma(\bullet)$ are generated by K_t-invariant elements.

i), b₁) The proof is similar to the previous. First of all, notice that $\mathrm{Fil}^0(R_t)\cong\mathrm{Ind}_{K_0(p^t)}^{K}\chi_r^s$ is K_t-invariant. Therefore, we focus on the action of K_t on the elements $y_{l_1,\ldots,l_{t-1}}$, $y'_{l_1,\ldots,l_{t-1}}$ if $r\neq0$ (resp. $z_{l_1,\ldots,l_{t-1}}$, $z'_{l_1,\ldots,l_{t-1}}$ if $r=0$). We notice moreover that, if $r\neq0$ and $t=1$, the K_1 invariants of $\varinjlim_{n\,\text{even}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1})$ are described by lemma 3.2 and 3.5: we can exclude this situation in the reminder of the proof of $i),b_1)$.

Assume now $r\neq0$. As above, we have the following equality in R_t for $l_{t-1}\leqslant r$:

$$\left[\begin{array}{cc}1 & p^t[\lambda]\\ 0 & 1\end{array}\right]y_{l_1,\ldots,l_{t-1}}-y_{l_1,\ldots,l_{t-1}}=\mu(-1)^{l_{t-1}}T^+_{t-1}(w)$$

where

$$w\stackrel{\text{def}}{=}\sum_{\mu_0\in\mathbf{F}_p}\left[\begin{array}{cc}[\mu_0] & 1\\ 1 & 0\end{array}\right]\cdots\sum_{\mu_{t-2}\in\mathbf{F}_p}\mu_{t-2}^{l_{t-2}}\left[\begin{array}{cc}1 & 0\\ p^{t-2}[\mu_{t-2}] & 1\end{array}\right][1,X^{r-l_{t-1}}Y^{l_{t-1}}]\in R_{t-1}$$

Using lemma 2.12 see that

-) the element $y_{l_1,\ldots,l_{t-1}}$ is $\left[\begin{array}{cc}1 & p^t\mathbf{Z}_p\\ 0 & 1\end{array}\right]$-invariant in the amalgamed sum if $(l_1,\ldots,l_{t-1})\prec(p-r,r,\ldots,p-1-r,r)$;

-) the element $y_{l_1,\ldots,l_{t-1}}$ is not $\left[\begin{array}{cc}1 & p^t\mathbf{Z}_p\\ 0 & 1\end{array}\right]$-invariant in the amalgamed sum if $(p-r,r,\ldots,p-1-r,r)\preceq(l_1,\ldots,l_{t-1})$.

We see again that $y'_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariant (already in R_t), and we conclude by the usual argument that

$$(\varinjlim_{n\,\text{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}))^{K_t} = (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) + \sigma)^{K_t}$$

where σ is the (image of the) K-subrepresentation of R_t generated by the elements $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ where $(l_1,\dots,l_{t-1}) \prec (p-r,r,p-1-r,\dots,p-1-r,r)$.

We pass to the action of $\begin{bmatrix} 1+p^ta & 0 \\ 0 & 1+p^td \end{bmatrix}$, with $a,d \in \mathbf{Z}_p^\times$. Exactly as in the proof of $i)$-$a_1)$ we use the matrix relation

$$\begin{bmatrix} 1+p^ta & 0 \\ 0 & 1+p^td \end{bmatrix}\begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z(1+p^td)^{-1}(1+p^ta) & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+p^td & 0 \\ 0 & 1+p^ta \end{bmatrix}$$

and lemma 2.12 to see that

-) the element $y_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$-invariant in the amalgamed sum if $(l_1,\dots,l_{t-1}) \prec (p-1-r,r,\dots,p-1-r,r)$ or if $(l_1,\dots,l_{t-1}) = (p-1-r,r,\dots,p-1-r,r)$ and $r \geqslant 2$;

-) the element $x_{l_1,\dots,l_{t-1}}$ is not $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$-invariant in the amalgamed sum if $(p-1-r,r,\dots,p-1-r,r) = (l_1,\dots,l_{t-1})$ and $r = p-2$.

Moreover, as $y'_{l_1,\dots,l_{t-1}}$ is $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$-invariant (already in R_t), we deduce

$$(\varinjlim_{n\,\text{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}))^{K_t} = \begin{cases} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) + \sigma_y^s(1))^{K_t} \\ \quad \text{if } r=1 \\ \\ (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) + \sigma)^{K_t} \\ \quad \text{if } r \geqslant 2. \end{cases}$$

We are left to study the action of $\begin{bmatrix} 1 & 0 \\ p^t\mathbf{Z}_p & 1 \end{bmatrix}$ on $y_{l_1,\dots,l_{t-1}}$, $y'_{l_1,\dots,l_{t-1}}$ in the situation $(l_1,\dots,l_{t-1}) \preceq (p-r,r,p-1-r,\dots,p-1-r,r)$. For $z \in I_{t-1}$ we have the equality

$$\begin{bmatrix} 1 & 0 \\ p^t[\lambda] & 1 \end{bmatrix}\begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+p^t* & p^t[\lambda] \\ p^t[-\lambda\mu_0^2] & 1+p^t* \end{bmatrix}.$$

We can use lemma 2.12 to deduce the following equality in $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-1}} R_t$:

$$\begin{bmatrix} 1 & 0 \\ p^t[\lambda] & 1 \end{bmatrix} y_{l_1,\dots,l_{t-1}} - y_{l_1,\dots,l_{t-1}} = \\ = \begin{cases} 0 \quad \text{if } (l_1,\dots,l_{t-1}) \prec (p-1-r,r\dots,p-1-r,r) \text{ or} \\ \qquad (l_1,\dots,l_{t-1}) = (p-1-r,r,\dots,p-1-r,r) \text{ and } r \geqslant 3 \\ \\ -\lambda(-1)^{(r+2)\frac{t-1}{2}} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^2 \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^r] \quad \text{if} \\ \quad (l_1,\dots,l_{t-1}) = (p-1-r,r,\dots,p-1-r,r) \text{ and } r=2. \end{cases}$$

We compute now

$$\begin{bmatrix} 1 & 0 \\ p^t[\lambda] & 1 \end{bmatrix} x'_{l_1,\dots,l_{t-1}} - x'_{l_1,\dots,l_{t-1}} =$$
$$= \begin{cases} 0 \quad \text{if either } (l_1,\dots,l_{t-1}) \prec (p-1-r,r,\dots,p-1-r,r) \\ \lambda(-1)^{(r+2)\frac{t-1}{2}}[1,X^r] \quad \text{if } (l_1,\dots,l_{t-1}) = (p-1-r,r,\dots,p-1-r,r). \end{cases}$$

As

$$\mathrm{soc}(R_t/(\mathrm{Fil}^0(R_t)+\sigma^s_y(\bullet))) = \mathrm{soc}(Q^{(0,t)}(1)_{1,p-1-r,r\dots,p-1-r,r})$$

(where $\sigma(\bullet) \in \{\sigma^s_y(2), \sigma^y_s(>2)\}$ according to r) we conclude, as above, that

$$(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-1}} R_t)^{K_t} = (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2} + pr_t(\mathrm{Fil}^0(R_t)) + \sigma^s_y(\bullet))^{K_t}$$

and the result follows, as $\sigma^s_y(\bullet)$ are generated by K_t-invariant elements.

We consider the case $r=0$. We see that the following equalities hold in the amalgamed sum (with the obvious conventions if $t=1$):

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} z_{l_1,\dots,l_{t-1}} - z_{l_1,\dots,l_{t-1}} =$$
$$= \lambda \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}\in\mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1,e];$$
$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} z'_{l_1,\dots,l_{t-1}} - z'_{l_1,\dots,l_{t-1}} = 0$$

and therefore, the study of $\begin{bmatrix} 1 & p^t\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariance can be recovered from the formalism of the case $r \neq 0$, $t \geqslant 3$. Notice that, if $t=1$ and

$$z \overset{\text{def}}{=} \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \sum_{\mu_1} \mu_1 \begin{bmatrix} 1 & 0 \\ p^2[\mu_2] & 1 \end{bmatrix} [1,e]$$

we get

$$\begin{bmatrix} 1 & p^t[\lambda] \\ 0 & 1 \end{bmatrix} z - z = \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1,e];$$

as the latter is nonzero in R_1/R_0 we can conclude that the K_1-invariants of the inductive limit are simply the elements of R_1/R_0.

We can now assume $t \geqslant 2$; from the the action of $\begin{bmatrix} 1+p^t\mathbf{Z}_p & 0 \\ 0 & 1+p^t\mathbf{Z}_p \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ p^t\mathbf{Z}_p & 1 \end{bmatrix}$ on $z_{l_1,\dots,l_{t-1}}$, $z'_{l_1,\dots,l_{t-1}}$ we see as above that their K_t-invariance can be reduced to the formalism for the case $r \neq 0$. The conclusion follows.

ii) The proof is completely analogous to the case *i*), without any new ideas. It is therefore left to the reader. $\sharp$

The formalism of proposition 3.7 may look a bit heavy, but we can use the description of the socle filtration for $\pi(r,0,1)|_{KZ}$ to have an immediate idea of what is going on. Roughly speaking, when we extract the K_t-invariants from $\pi(r,0,1)$ we are "cutting" the socle filtration, and proposition 3.7 tells us precisely where such a "cutting" occurs. For instance, the description of ($\varinjlim_{n,\,\mathrm{odd}}$ $(R_0 \oplus_{R_1}$

$\cdots \oplus_{R_n} R_{n+1}))^{K_t}$, for t odd, in terms of the socle filtration is the following:

$$\mathrm{SocFil}(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1}) \text{—} \begin{cases} \mathrm{SocFil}Q^{(0,t+1)}_{0,\dots,0,r+1} \setminus \mathrm{SocFil}Q^{(0,t+1)}_{0,r,p-1-r,\dots,p-1-r,r+1} \\ \text{if } r = p-2 \\ \\ \mathrm{SocFil}Q^{(0,t+1)}_{0,\dots,0,r+1} \setminus \mathrm{SocFil}Q^{(0,t+1)}_{1,r,p-1-r,\dots,p-1-r,r+1} \\ \text{if } r \leqslant p-3 \end{cases}$$

where we used the notation "$\mathbf{V}_1 \setminus \mathbf{V}_2$" to mean that we have to rule out the factors of the socle filtration of $\mathbf{V}_2$ from the socle filtration of $\mathbf{V}_1$ (or, more scientifically but less immaginative, to mean the socle filtration of the kernel $\ker(\mathbf{V}_1 \twoheadrightarrow \mathbf{V}_2)$ of the natural projection).

Corollary 3.8. *Let $r \in \{0, \dots, p-1\}$, $t \geqslant 1$. The $\overline{\mathbf{F}}_p$-dimension of $(\pi(r,0,1))^{K_t}$ is then:*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{K_t}) = (p+1)(2p^{t-1}-1) + \begin{cases} p-3 & \text{if } r \notin \{0, p-1\} \\ p-2 & \text{if } r \in \{0, p-1\} \end{cases}$$

Proof: Thanks to the isomorphism $\pi(0,0,1) \cong \pi(p-1,0,1)$ we can assume $r \leqslant p-2$. Let us assume t odd (the case t even is analogous). Using [Mo1], corollary 6.5 we get

$$\dim_{\overline{\mathbf{F}}_p}(\sigma(\bullet)/R_t) = (p+1)p^t - (p+1)p^{t-1}(r+1) - ((p+1)p^t - (p+1)\sum_{j=1}^{t} p^{j-1} l_j - (p-2-r))$$

where $(l_1, \dots, l_t - 1) = (r, p-1-r, \dots, r, p-1-r, r+1)$ if $t \geqslant 2$; thus

$$\dim_{\overline{\mathbf{F}}_p}(\sigma(\bullet)/R_t) = (p-r)(p^{t-1}-1) + (p-2-r).$$

Similarly we find, for $r \neq 0$,

$$\begin{aligned} \dim_{\overline{\mathbf{F}}_p}(\sigma^s_y(\bullet)/\mathrm{Fil}^0(R_t)) &= p^{t-1}(p+1) - (p^{t-1}(p+1) - (p+1)\sum_{j=1}^{t-1} p^{j-1} l_j - (r-1)) \\ &= (r+1)(p^t-1) + (r-1) \end{aligned}$$

where $(l_1, \dots, l_{t-1}) = (p-1-r, r, \dots, p-1-r, r)$ if $t \geqslant 2$; if $r = 0$ we similarly get

$$\dim_{\overline{\mathbf{F}}_p}(\sigma^s_z(\bullet)/\mathrm{Fil}^0(R_t)) = (r+1)(p^t-1).$$

As

$$\begin{aligned} &\dim_{\overline{\mathbf{F}}_p}(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1}) + \dim_{\overline{\mathbf{F}}_p}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2}) + \\ &\quad + \dim_{\overline{\mathbf{F}}_p}(\mathrm{Fil}^0(R_t/R_{t-1})) = (p+1)p^{t-1} \end{aligned}$$

the result follows. ♯

With respect to the description of the socle filtration of $\pi(r,0,1)|_K$ as "two lines of weights", proposition 3.7 let us deduce the following result:

Corollary 3.9. *Let $t \geqslant 1$ be an integer, $r \in \{0, \dots, p-1\}$.*

1) *The socle filtration for $(\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}))^{K_t}$ is described by:*

$$\mathrm{Sym}^r \overline{\mathbf{F}}_p^2 \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^{r+1} \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^{r+2} \text{—} \dots \text{—} \mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^r \text{—} \mathrm{Sym}^{p-3-r} \overline{\mathbf{F}}_p^2 \otimes \det{}^{r+1}$$

where the number of parabolic inductions $\mathrm{Ind}_I^K \chi_r^s \mathfrak{a}^j$ is $p^{t-1}-1$ and last weight $\mathrm{Sym}^{p-3-r} \overline{\mathbf{F}}_p^2 \otimes \det^{r+1}$ appears only if $p-3-r \geqslant 0$.

2) *The socle filtration for* $(\varinjlim_{n,\text{even}} ((R_1/R_0)\oplus_{R_2}\cdots\oplus_{R_n} R_{n+1}))^{K_t}$ *is described by:*

$$\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2\otimes\det{}^r\text{---}\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}\text{---}\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^2\text{---}\ldots\text{---}\mathrm{Ind}_I^K\chi_r^s\text{---}\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det$$

where the number of parabolic inductions $\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^j$ *is* $p^{t-1}-1$ *and last weight* $\mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2\otimes\det$ *appears only if* $r-2\geqslant 0$.

Proof: We sketch here the proof for t odd, $r\in\{0,\ldots,p-2\}$. Using the computations in the proof of corollary 3.8 and the result in lemma 2.8 we see that

$$\dim_{\overline{\mathbf{F}}_p}(\varinjlim_{n,\text{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}))^{K_t}=(r+1)+(p+1)(p^{t-1}-1)+(p-2-r)$$

and

$$\dim_{\overline{\mathbf{F}}_p}(\varinjlim_{n,\text{even}}((R_1/R_0)\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}))^{K_t}=(p-r)+(p+1)(p^{t-1}-1)+(r-1)+\delta_{r,0}$$

As the dimension of the parabolic inductions $\mathrm{Ind}_I^K\chi_r^s\mathfrak{a}^j$ is $p+1$ we conclude from proposition 3.7. $\sharp$

4. Study of I_t-invariants.

Let $t\geqslant 1$ be an integer and $r\in\{0,\ldots,p-1\}$. The aim of this section is to study in detail the space of I_t-invariant of supersingular representations $\pi(r,0,1)$; thanks to the isomorphism $\pi(0,0,1)\cong\pi(p-1,0,1)$ we will assume $r\leqslant p-2$, unless otherwise specified. Moreover the relations

$$K_{t-1}\leqslant I_t\leqslant K_t, I_t=\begin{bmatrix}1&p^{t-1}\mathbf{Z}_p\\0&1\end{bmatrix}\begin{bmatrix}1+p^t\mathbf{Z}_p&0\\0&1+p^t\mathbf{Z}_p\end{bmatrix}\begin{bmatrix}1&0\\p^t\mathbf{Z}_p&1\end{bmatrix}$$

show that the hard task consist in studying the $\begin{bmatrix}1&p^{t-1}[\lambda]\\0&1\end{bmatrix}$-invariants (for $\lambda\in\mathbf{F}_p$) of $\pi(r,0,1)^{K_t}$, the latter being completely described in proposition 3.7. We distingush two cases, accordingly to the parity (?) of t.

4.1 The case t odd.

In the present section, we assume $t\geqslant 1$, t odd. We then can write, accordingly to the value of r,

$$(\varinjlim_{n,\text{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}))^{I_t}\leqslant R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}+\begin{cases}\sigma(p-2)&\text{if } r=p-2\\\sigma(p-3)&\text{if } r=p-3\\\sigma(<p-3)&\text{if } r<p-3.\end{cases}$$

$$(\varinjlim_{n,\text{odd}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}))^{I_t}\geqslant pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$$

with the obvious convention that $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))=R_0$ if $t=1$. Notice that all vectors in the spaces $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ are $\begin{bmatrix}1&p^{t-1}[\lambda]\\0&1\end{bmatrix}$-invariants. Similarly, we get

$$(\varinjlim_{n,\text{even}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}))^{I_t}\leqslant pr_t(\mathrm{Fil}^0(R_t))+\begin{cases}\sigma_z^s(0)&\text{if } r=0\\\sigma_y^s(1)&\text{if } r=1\\\sigma_y^s(2)&\text{if } r=2\\\sigma_y^s(>2)&\text{if } r>2.\end{cases}$$

$$(\varinjlim_{n,\text{even}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}))^{I_t}\geqslant R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}$$

with the obvious convention that $pr_1(\mathrm{Fil}^0(R_1/R_0)) = \mathrm{Fil}^0(R_1/R_0)$. Notice that all vectors in the spaces $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2}$ are $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$-invariants.

From now onwards we assume $t > 1$: the case $t = 1$ it is well known (cf. [Bre03a], Théoréme 3.2.4) and can anyway be treated with analogous techniques.

4.1.1 ***Concerning*** $R_0 \oplus_1 \cdots \oplus_{R_n} R_{n+1}$, n ***odd.*** We conside the K-equivariant exact sequence

$$0 \to pr_{t-1}(\mathrm{Fil}^0(R_{t-1})) \to R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1} + \sigma(\bullet) \to \mathbf{V} \to 0$$

where $\bullet$ depends on r accordingly to proposition 3.7-a_1). We introduce the following elements:

Definition 4.1. *Let $t > 1$ be odd, $(l_0, \dots, l_{t-2}) \in \{0, \dots, p-1\}^{t-1}$ a $(t-1)$-tuple.*

i) For $j \in \{0, \dots, r\}$ we define the following elements of R_{t-1}

$$\mathfrak{x}_{l_0,\dots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

$$\mathfrak{x}'_{l_1,\dots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

ii) For $l_{t-1} \in \{0, \dots, p-1\}$, we define the following elements of R_{t+1}

$$\mathfrak{y}_{l_0,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r]$$

$$\mathfrak{y}'_{l_1,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} [\mu_1] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1}} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

The rôle of such elements is explained by the next

Lemma 4.2. *An $\overline{\mathbf{F}}_p$-basis for $\mathbf{V}$ is described as follow:*

*i) the elements $\mathfrak{y}_{l_0,\dots,l_{t-1}}$, $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$
where $(l_1, \dots, l_{t-1}) \prec (r, p-1-r, \dots, r, p-1-r)$, $l_0 \in \{0, \dots, p-1\}$;*

*ii) if $r \leqslant p-3$, the elements $\mathfrak{y}_{j,r,p-1-r,\dots,r,p-1-r}$, $j \in \{0, \dots, p-3-r-1\}$ and
$\mathfrak{y}_{p-3-r,r,p-1-r,\dots,r,p-1-r} + (-1)^{(r+1)+p-3-r}\mathfrak{y}'_{p-3-r,r,p-1-r,\dots,r,p-1-r}$*

iii) if $r \neq 0$, the elements

$$\mathfrak{x}_{l_0,\dots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

$$\mathfrak{x}'_{l_1,\dots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-2}} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

where $j \in \{1, \dots, r\}$.

Proof: It is a formal fact to verify that we have a K-equivariant exact sequence

$$0 \to R_{t-1}/\mathrm{Fil}^1(R_{t-1}) \to \mathbf{V} \to \sigma(\bullet)/R_t \to 0;$$

the the assertion is then an immediate consequence. $\sharp$

Thanks to lemma 4.2 we can describe the structure of $\mathbf{V}^{I_t}$.

Proposition 4.3. *An $\overline{\mathbf{F}}_p$-basis for $\mathbf{V}^{U_t}$ is given by*

a) the elements $\mathfrak{x}'_{l_1,\dots,l_{t-2}}(j)$ for $(l_1,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-2}$, $j\in\{1,\dots,r\}$ if $r\geqslant 1$;

b) the elements $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$ where $(l_1,\dots,l_{t-1})\prec(r,p-1-r,\dots,r,p-1-r)$

c) if $r\neq 0$ the elements

$$\mathfrak{x}_{l_0,\dots,l_{t-2}}(1)$$

where $(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}$, while, if $r=0$, the elements

$$\mathfrak{y}_{l_0,\dots,l_{t-2},0}$$

where $(l_1,\dots,l_{t-2})\preceq(r,\dots,p-1-r,r)$ and $l_0\in\{0,\dots,p-1\}$.

Proof: Assume $r\neq 0$ (the case $r=0$ is strictly analogous). First of all, we look for a decomposition of $\mathbf{V}$ into $\begin{bmatrix}1 & p^{t-1}\mathbf{Z}_p\\ 0 & 1\end{bmatrix}$-stable subspaces. We deduce immediately the following equalities (in R_{t-1}):

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{x}_{l_0,\dots,l_{t-2}}(j)=\sum_{i=0}^{j}\binom{j}{i}\lambda^{j-i}\mathfrak{x}_{l_0,\dots,l_{t-2}}(j-i);\tag{1}$$

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{x}'_{l_1,\dots,l_{t-2}}=\mathfrak{x}'_{l_1,\dots,l_{t-2}};$$

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{y}'_{l_1,\dots,l_{t-2}}=\mathfrak{y}'_{l_1,\dots,l_{t-2}}.$$

Using the operators $T_t^{\pm}$, we get the following equality inside $R_0\oplus_{R_1}\cdots\oplus_{R_{t-2}}R_{t-1}$:

$$\begin{aligned}\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}\mathfrak{y}_{l_0,\dots,l_{t-1}}=&\sum_{i=0}^{l_{t-1}}\binom{l_{t-1}}{i}(-\lambda)^i\mathfrak{y}_{l_0,\dots,l_{t-2},(l_{t-1}-i)}+\\ &+\sum_{i=0}^{l_{t-1}}\binom{l_{t-1}}{i}(-\lambda)^i(r+1)(-1)^{r+1}v_i\end{aligned}\tag{2}$$

where

$$\begin{aligned}v_i\stackrel{\text{def}}{=}&\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots\\ &\dots\sum_{\mu_{t-2}\in\mathbf{F}_p}\mu_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\mu_{t-2}] & 1\end{bmatrix}\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}-i}P_{-\lambda}(\mu_{t-1})[1,(\lambda_{t-1}X+Y)^r].\end{aligned}$$

In particular, each v_i belongs to the linear space generated by the elements $\mathfrak{x}_{l_0\dots,l_{t-2}}(j)$ and, for $l_{t-1}=0$, we see that the coefficient of $[1,Y^r]$ in $\sum\limits_{\mu_{t-1}\in\mathbf{F}_p}P_{-\lambda}(\mu_{t-1})[1,(\lambda X+Y)^r]$ is $-\lambda$.

We deduce that the following spaces are $\begin{bmatrix}1 & p^{t-1}\mathbf{Z}_p\\ 0 & 1\end{bmatrix}$-stable: the spaces

$$\langle\mathfrak{x}'_{l_1,\dots,l_{t-2}}(j)\rangle_{\overline{\mathbf{F}}_p};\langle\mathfrak{y}'_{l_1,\dots,l_{t-1}}\rangle_{\overline{\mathbf{F}}_p}$$

and, for a given $(t-1)$-tuple $(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}$, the space $\mathbf{V}_{l_0,\dots,l_{t-2}}$ which is defined as the $\overline{\mathbf{F}}_p$-vector subspace of $\mathbf{V}$ generated by $\mathfrak{x}_{l_0,\dots,l_{t-2}}(j)$, for $j\in\{1,\dots,r\}$ and

-) the elements $\mathfrak{y}_{l_0,\dots,l_{t-2},i}$ with $i\in\{0,\dots,p-1-r\}$ if either $(l_1,\dots,l_{t-2})\prec(r,p-1-r,\dots,p-1-r,r)$ or $(l_1,\dots,l_{t-2})=(r,p-1-r,\dots,p-1-r,r)$ and $l_0<p-3-r$;

-) the elements $\mathfrak{y}_{l_0,\dots,l_{t-2},i}$ with $i \in \{0,\dots,p-2-r\}$ if $(l_1,\dots,l_{t-2}) = (r, p-1-r,\dots,p-1-r,r)$ and $l_0 > p-3-r$;

-) the elements $\mathfrak{y}_{l_0,\dots,l_{t-2},i}$ with $i \in \{0,\dots,p-2-r\}$ and the element

$$\mathfrak{y}_{l_0,\dots,l_{t-2},p-1-r} + (-1)^{p-3-r+(r+1)}\mathfrak{y}'_{l_1,\dots,l_{t-2},p-1-r}$$

if $(l_1,\dots,l_{t-2}) = (r, p-1-r,\dots,p-1-r,r)$ and $l_0 = p-3-r$.

For a fixed $(t-1)$-tuple $(l_0,\dots,l_{t-2}) \in \{0,\dots,p-1\}^{t-1}$ we deduce from the equalities (1) and (2) that there exists an $\overline{\mathbf{F}}_p$-basis of $\mathbf{V}_{l_0,\dots,l_{t-2}}$ such that the matrix associated to the action of $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ is unipotent, and the elements on the superdiagonal are all nonzero. In other words, the $\mathbf{V}_{l_0,\dots,l_{t-2}}$-restriction of the $\mathbf{V}$-endomorphism associated to $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ has a unique eigenvalue (equal to 1) and the associated eigenspace has dimension 1. We see that a generator of such eigenspace is $\mathfrak{x}_{l_0,\dots,l_{t-2}}(1)$ and the proof is complete. The statement concerning the case $r=0$ can be proved with the same techniques and it is left to the reader. $\sharp$

We remark that the elements in a), b) of proposition 4.3 are already U_t invariant inside the amalgamed sum $R_0 \oplus_{R_1} \cdots \oplus_{R_t} R_{t+1}$. Together with the elements inside $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ they are denoted as the *trivial I_t-invariants* We therefore are left to study the U_t-invariance of the elements of the form c) inside $\varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ to complete the description of I_t-invariants.

Proposition 4.4. *An $\overline{\mathbf{F}}_p$-basis for the space of nontrivial I_t-invariants of $\varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ modulo the trivial invariants is described as follow:*

i) if $r \neq 0$, by the family

$$\mathfrak{x}(1)_{l_0,\dots,l_{t-2}}$$

where $(l_0, l_1,\dots,l_{t-2}) \prec (p-1-r, r,\dots,p-1-r,r)$.

ii) if $r = 0$, by the family

$$\mathfrak{y}_{l_0,\dots,l_{t-2},0}$$

where $(l_0,\dots,l_{t-2}) \prec (p-1-r, r,\dots,p-1-r,r)$

Proof: i) The proof is an induction on t, and follows closely the computations of lemma 2.12. Let $t=3$, and consider a I_t-invariant vector v which we can assume of the following form:

$$v = \sum_{(l_0,l_1)\in\{0,\dots,p-1\}^2} c_{l_0,l_1}\mathfrak{x}(1)_{l_0,l_1}$$

for suitable $c_{l_0,l_1} \in \overline{\mathbf{F}}_p$. We have

$$\begin{bmatrix} 1 & p^2[\lambda] \\ 0 & 1 \end{bmatrix} v - v = \lambda \sum_{(l_0,l_1)\in\{0,\dots,p-1\}^2} c_{l_0,l_1}\mathfrak{x}(0)_{l_0,l_1};$$

it is now clear that $\mathfrak{x}(0)_{l_0,l_1} \equiv 0$ inside R_2/R_1 if $l_1 \leqslant r$ while the (image of the) elements $\mathfrak{x}(0)_{l_0,l_1}$ inside R_2/R_1 induce a free family for $l_1 \geqslant r+1$: the I_3-invariance of v shows that $c_{l_0,l_1} = 0$ if $(0, r+1) \preceq (l_0,l_1)$. Therefore, using the operators $T_1^{\pm}$, we get the following equality in the amalgamed sum

$$\sum_{(l_0,l_1)\prec(0,r+1)} c_{l_0,l_1}\mathfrak{x}(0)_{l_0,l_1} = (-1)^{r+1}\sum_{l_0=0}^{p-1} c_{l_0,r}\sum_{\mu_0\in\mathbf{F}_p} \mu_0^{l_0}(X+\mu_0 Y)^r$$

which shows that $c_{l_0,r}=0$ for $l_0\geqslant p-1-r$. This let us establish the case $t=3$.

Concerning the general case, let v be a I_t-invariant element which we can assume of the form

$$v=\sum_{(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}}c_{l_0,\dots,l_{t-2}}\mathfrak{x}(1)_{l_0,\dots,l_{t-2}}$$

for suitable $c_{l_0,\dots,l_{t-2}}\in\overline{\mathbf{F}}_p$. We have

$$\begin{bmatrix}1 & p^{t-1}[\lambda]\\ 0 & 1\end{bmatrix}v-v=\lambda\sum_{(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}}c_{l_0,\dots,l_{t-2}}\mathfrak{x}(0)_{l_0,\dots,l_{t-2}};$$

since $\mathfrak{x}(0)_{l_0,\dots,l_{t-2}}\equiv 0$ inside R_{t-1}/R_{t-2} if $l_{t-2}\leqslant r$ and the family

$$\{\mathfrak{x}(0)_{l_0,\dots,l_{t-2}}\}_{l_{t-2}\geqslant r+1}$$

is linearly independent in R_{t-1}/R_{t-2} we conclude that $c_{l_0,\dots,l_{t-2}}=0$ as soon as $l_{t-2}\geqslant r+1$. Using the operators $T^{\pm}_{t-2}$ and a similar argument (i.e. studying the image of the sum inside R_{t-3}/R_{t-4}) we deduce that we must have $c_{l_0,\dots,l_{t-3},r}=0$ if $l_{t-3}>p-1-r$, therefore getting the following equality in the amalgamed sum:

$$\begin{aligned}&\sum_{(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}}c_{l_0,\dots,l_{t-2}}\mathfrak{x}(0)_{l_0,\dots,l_{t-2}}=\\ &\qquad=(-1)^{r+2}\sum_{(l_0,\dots,l_{t-4})\in\{0,\dots,p-1\}^{t-3}}c_{l_0,\dots,l_{t-4},p-1-r,r}\mathfrak{x}(0)_{l_0,\dots,l_{t-4}}.\end{aligned}$$

The conclusion follows by induction.

ii) It is completely analogous and left to the reader. ♯

4.1.2 ***Concerning*** $R_1/R_0\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}$, n ***even***. We consider now the K-equivariant exact sequence for $r\neq 0$

$$0\to R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}\to pr_t(\mathrm{Fil}^0(R_t))+\sigma_y^s(\bullet)\to\mathbf{W}\to 0$$

and, for $r=0$,

$$0\to R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{t-3}}R_{t-2}\to pr_t(\mathrm{Fil}^0(R_t))+\sigma_z^s(0)\to\mathbf{W}\to 0$$

where $\sigma_y^s(\bullet)$ depends on r accordingly to proposition 3.7i)-b_1). As in the previous section, we introduce the elements

Definition 4.5. *Let $t>1$ be odd, $(l_0,\dots,l_{t-1})\in\{0,\dots,p-1\}^t$ a t-tuple.*

i) we define the following elements of R_t

$$\mathfrak{w}_{l_0,\dots,l_{t-1}}\stackrel{\text{def}}{=}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\cdots\sum_{\mu_{t-1}}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,X^r];$$

$$\mathfrak{w}'_{l_1,\dots,l_{t-1}}\stackrel{\text{def}}{=}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\cdots\sum_{\mu_{t-1}}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,X^r]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

ii) For $r\neq 0$, we define the following elements of R_t

$$\mathfrak{z}_{l_0,\dots,l_{t-1}}\stackrel{\text{def}}{=}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\cdots\sum_{\mu_{t-1}}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,X^{r-1}Y];$$

$$\mathfrak{z}'_{l_1,\dots,l_{t-1}}\stackrel{\text{def}}{=}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}[\mu_1] & 1\\ 1 & 0\end{bmatrix}\cdots\sum_{\mu_{t-1}}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,X^{r-1}Y]$$

which will be also seen as elements of the amalgamed sums accordingly to the context.

iii) *For $r=0$ we define the following elements of R_{t+2}*

$$\mathfrak{h}_{l_0,\dots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots$$

$$\dots\sum_{\mu_{t-1}}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}\sum_{\mu_t\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^t[\mu_t] & 1\end{bmatrix}\sum_{\mu_{t+1}\in\mathbf{F}_p}\mu_{t+1}\begin{bmatrix}1 & 0\\ p^{t+1}[\mu_{t+1}] & 1\end{bmatrix}[1,X^r];$$

$$\mathfrak{h}'_{l_1,\dots,l_{t-1}} \stackrel{\text{def}}{=} \sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}[\mu_1] & 1\\ 1 & 0\end{bmatrix}\dots$$

$$\sum_{\mu_{t-1}}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}\sum_{\mu_t\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^t[\mu_t] & 1\end{bmatrix}\sum_{\mu_{t+1}\in\mathbf{F}_p}\mu_{t+1}\begin{bmatrix}1 & 0\\ p^{t+1}[\mu_{t+1}] & 1\end{bmatrix}[1,X^r]$$

where X^r is a fixed $\overline{\mathbf{F}}_p$-basis of $\mathrm{Sym}^0\overline{\mathbf{F}}_p^2$; such elements will be also seen as elements of the amalgamed sums accordingly to the context.

As for lemma 4.2, we are able to describe an $\overline{\mathbf{F}}_p$-basis for $\mathbf{W}$ in terms of the elements defined in 4.5

LEMMA 4.6. *An $\overline{\mathbf{F}}_p$-basis for $\mathbf{W}$ is described by:*

a) *the elements $\mathfrak{w}_{l_0,\dots,l_{t-1}}$, $\mathfrak{w}'_{l_1,\dots,l_{t-1}}$ where* $l_{t-1}\geqslant r+1$, $(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}$;

b) *if $r\neq 0$, the elements $\mathfrak{z}_{l_0,\dots,l_{t-1}}$, $\mathfrak{z}'_{l_0,\dots,l_{t-1}}$ with* $(l_1,\dots,l_{t-1})\prec(p-1-r,r,\dots,p-1-r,r)$ *and* $l_0\in\{0,\dots,p-1\}$ *as well as the elements* $\mathfrak{z}_{j,p-1-r,r,\dots,p-1-r,r}$ *for* $j\in\{0,\dots,(r-2)-1\}$ *and* $\mathfrak{z}_{r-2,p-1-r,\dots,p-1-r,r}+(-1)^{(r-2)+1}\mathfrak{z}'_{p-1-r,\dots,p-1-r,r}$;

c) *if $r=0$, the elements $\mathfrak{h}_{l_0,\dots,l_{t-1}}$, $\mathfrak{h}'_{l_0,\dots,l_{t-1}}$ with* $(l_1,\dots,l_{t-1})\prec(p-1-r,r,\dots,p-1-r,r)$ *and* $l_0\in\{0,\dots,p-1\}$.

Proof: As in lemma 4.2, it is a formal verification that $\mathbf{W}$ fits into a K-equivariant exact sequence:

$$0\to\mathrm{Fil}^0(R_t/R_{t-1})\to\mathbf{W}\to\sigma^s_\bullet(\bullet)/\mathrm{Fil}^0(R_t)\to 0$$

where $\sigma^s_\bullet(\bullet)$ is defined according to the value of r as in proposition 3.7i)-b_1). The result follows. $\sharp$.

Again, we are going to describe the I_t-invariants of $\mathbf{W}$:

PROPOSITION 4.7. *An $\overline{\mathbf{F}}_p$-basis for $\mathbf{W}^{U_t}$ is given by*

a) *the elements $\mathfrak{w}'_{l_1,\dots,l_{t-1}}$ for $(l_1,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-2}$, and $l_{t-1}\geqslant r+1$*

b) *if $r\neq 0$, the elements $\mathfrak{z}'_{l_1,\dots,l_{t-1}}$ where* $(l_1,\dots,l_{t-1})\prec(p-1-r,r\dots,p-1-r,r)$ *while, if $r=0$, the elements $\mathfrak{h}'_{l_1,\dots,l_{t-1}}$ where* $(l_1,\dots,l_{t-1})\prec(p-1-r,r\dots,p-1-r,r)$

c) *the elements*

$$\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$$

where $(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}$.

Proof: Assume $r \neq 0$ (the case $r = 0$ is analogous). For $\lambda \in \mathbf{F}_p$ we easily get the following equalities in $\mathbf{W}$:

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{w}_{l_0,\dots,l_{t-1}} = \sum_{j=r+1}^{l_{t-1}} \binom{l_{t-1}}{j} (-\lambda)^j \mathfrak{w}_{l_0,\dots,(l_{t-1}-j)}; \tag{3}$$

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{w}'_{l_0,\dots,l_{t-1}} = \mathfrak{w}'_{l_0,\dots,l_{t-1}};$$

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{z}'_{l_0,\dots,l_{t-1}} = \mathfrak{z}'_{l_0,\dots,l_{t-1}}. \tag{4}$$

Moreover, using lemma 2.10, we get the following equality in $\mathbf{W}$:

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} \mathfrak{z}_{l_0,\dots,l_{t-1}} = \sum_{j=0}^{l_{t-1}} \binom{l_{t-1}}{j} (-\lambda)^j \mathfrak{z}_{l_0,\dots,(l_{t-1}-j)} + \sum_{j=0}^{l_{t-1}} \binom{l_{t-1}}{j} (-\lambda)^j w_j \tag{5}$$

where we set

$$w_j \stackrel{\text{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots$$
$$\cdots \sum_{\mu_{t-2} \in \mathbf{F}_p} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}-j} (-P_{-\lambda}(\mu_{t-1})) \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^r].$$

We notice that w_j belongs to the linear space generated by $\mathfrak{w}_{l_0,\dots,l_{t-2},i}$ for $i \in \{r+1,\dots,p-1\}$ and, for $l_{t-1} = 0$, the coefficient of $\mathfrak{w}_{l_0,\dots,l_{t-2},p-1}$ in w_0 is $-\lambda$.

We deduce that the following subspaces of $\mathbf{W}$ give a decomposition of $\mathbf{W}$ in $\begin{bmatrix} 1 & p^{t-1}\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-stable subspaces:

$$\langle \mathfrak{w}'_{l_1,\dots,l_{t-1}} \rangle_{\overline{\mathbf{F}}_p}; \langle \mathfrak{z}'_{l_1,\dots,l_{t-1}} \rangle_{\overline{\mathbf{F}}_p};$$

and, for any fixed $(t-1)$-tuple $(l_0,\dots,l_{t-2}) \in \{0,\dots,p-1\}^{t-1}$, the subspace $\mathbf{W}_{l_0,\dots,l_{t-2}}$, which is defined as the $\overline{\mathbf{F}}_p$-vector subspace of $\mathbf{W}$ generated by $\mathfrak{w}_{l_0,\dots,l_{t-2},j}$ with $j \in \{r+1,\dots,p-1\}$ and

-) the elements $\mathfrak{z}_{l_0,\dots,l_{t-2},i}$ with $i \in \{0,\dots,r\}$ if either $(l_1,\dots,l_{t-2}) \prec (p-1-r, r,\dots,r,p-1-r)$ or $(l_1,\dots,l_{t-2}) = (p-1-r,r,\dots,r,p-1-r)$ and $l_0 < r-2$;
-) the elements $\mathfrak{z}_{l_0,\dots,l_{t-2},i}$ with $i \in \{0,\dots,r-1\}$ and $\mathfrak{z}_{l_0,\dots,l_{t-2},r} + (-1)^{(r-2)+1} \mathfrak{z}'_{l_1,\dots,l_{t-2},r}$ if $(l_1,\dots,l_{t-2}) = (p-1-r,r,\dots,r,p-1-r)$ and $l_0 = r-2$;
-) the elements $\mathfrak{z}_{l_0,\dots,l_{t-2},i}$ with $i \in \{0,\dots,r-1\}$ if $(l_1,\dots,l_{t-2}) = (p-1-r,r,\dots,r,p-1-r)$ and $l_0 > r-2$.

As in proposition 4.3, we see that we can find a basis of $\mathbf{W}_{l_0,\dots,l_{t-2}}$ such that the matrix associated to the action of $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ on $\mathbf{W}_{l_0,\dots,l_{t-2}}$ (for $\lambda \in \mathbf{F}_p^\times$) is upper unipotent with nonzero scalars in the superdiagonal. In other words, the $\mathbf{W}_{l_0,\dots,l_{t-2}}$-restriction of the $\mathbf{W}$-endomorphism associated to $\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix}$ (for $\lambda \in \mathbf{F}_p^\times$) has a unique eigenvalue (equal to 1) and the associated eigenspace has dimension 1. Since such eigenspace is generated by $\mathfrak{w}_{l_0,\dots,l_{t-2},r}$, the conclusion follows.

The case $r = 0$ is strictly analogous; we point out anyway that the equalities of type (4), (5) for $\mathfrak{h}'_{l_1,\dots,l_{t-1}}$, $\mathfrak{h}_{l_0,\dots,l_{t-1}}$ are now established via the operators $T_t^{\pm}$ $\sharp$

We remark that the elements in a), b) of proposition 4.7 are already U_t invariant inside the amalgamed sum $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t+1}} R_{t+2}$. Together with the elements inside $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}}$

R_{t-2} they will be denoted as the *trivial I_t-invariants* We therefore are left to study the U_t-invariance of the elements of the form $c)$ inside $\varinjlim_{n \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ to complete the description of I_t-invariants.

Proposition 4.8. *An $\overline{\mathbf{F}}_p$-basis for the space of nontrivial I_t-invariants of $\varinjlim_{n \text{ even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ modulo the trivial invariants is described by the elements $\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$ where $(l_0,\dots,l_{t-2}) \prec (r, p-1-r, \dots, r, p-1-r)$*

Proof: The proof is an induction on t, analogous to proposition 4.4. Assume $t=3$ and consider a I_t-invariant vector which we can assume of the following form:

$$v = \sum_{(l_0,l_1)\in\{0,\dots,p-1\}^2} c_{l_0,l_1} \mathfrak{w}_{l_0,l_1,r+1}$$

for suitable $c_{l_0,l_1} \in \overline{\mathbf{F}}_p$. Using the operators $T_2^{\pm}$ we deduce the following equality in R_1/R_0:

$$\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} v - v = \sum_{(l_0,l_1)\in\{0,\dots,p-1\}^2} c_{l_0,l_1} \sum_{\mu_0\in\mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \sum_{\mu_1\in\mathbf{F}_p} \mu_1^{l_1} [1, (\mu_1 X + Y)^r].$$

We therefore see (thanks to proposition 2.7-*i*)) that $c_{l_0,l_1} = 0$ as soon as $l_1 > p-1-r$, while we can use proposition 2.9-*ii*) and *iii*) to deduce that $c_{l_0,p-1-r} = 0$ for $l_0 \geqslant r$. This establish the case $t=3$.

Concerning the general case, let v be a I_t-invariant vector, which we may assume of the form

$$v = \sum_{(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}} c_{l_0,\dots,l_{t-2}} \mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$$

for suitable $c_{l_0,\dots,l_{t-2}} \in \overline{\mathbf{F}}_p$. Using the operators $T_{t-1}^{\pm}$ we get the following equality in the amalgamed sum

$$\begin{aligned} &\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} v - v = \\ &\quad = (r+1)(-1)^{r+1}(-\lambda) \sum_{(l_0,\dots,l_{t-2})\in\{0,\dots,p-1\}^{t-1}} c_{l_0,\dots,l_{t-2}} \sum_{\mu_0\in\mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \\ &\quad \cdots \sum_{\mu_{t-3}\in\mathbf{F}_p} \mu_{t-3}^{l_{t-3}} \begin{bmatrix} 1 & 0 \\ p^{t-3}[\mu_{t-3}] & 1 \end{bmatrix} \sum_{\mu_{t-2}\in\mathbf{F}_p} \mu_{t-2}^{l_{t-2}} [1, (\mu_{t-2} X + Y)^r]. \end{aligned}$$

We map the latter in R_{t-2}/R_{t-3} to deduce that $c_{l_0,\dots,l_{t-2}} = 0$ if $(r+1, p-1-r) \preceq (l_{t-3}, l_{t-2})$ and therefore we get the following equality in the amalgamed sum:

$$\begin{aligned} &\begin{bmatrix} 1 & p^{t-1}[\lambda] \\ 0 & 1 \end{bmatrix} v - v = \\ &\quad = (r+1)(-1)^{r+2}(-1)^{r+1} \sum_{(l_0,\dots,l_{t-4})\in\{0,\dots,p-1\}^{t-3}} c_{l_0,\dots,l_{t-4},r,p-1-r} \sum_{\mu_0\in\mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \\ &\quad \cdots \sum_{\mu_{t-4}\in\mathbf{F}_p} \mu_{t-4}^{l_{t-4}} \begin{bmatrix} 1 & 0 \\ p^{t-4}[\mu_{t-4}] & 1 \end{bmatrix} [1, (\lambda_{t-4} X + Y)^r]. \end{aligned}$$

This let us conclude the inductive step and the proof is complete. $\sharp$

4.2 The case t even

We assume now t even. The study of I_t-invariants for the inductive limits $\varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ and $\varinjlim_{n\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ is treated in a completely analogous way as we did in paragraph 4.1. We therefore content ourselves to give the results, leaving the computational efforts to the reader.

4.2.1 ***Concerning*** $R_0 \oplus_1 \cdots \oplus_{R_n} R_{n+1}$, n ***odd.*** We now should consider the K-equivariant short exact sequence

$$0 \to R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2} \to pr_t(\mathrm{Fil}^0(R_t)) + \left\{ \begin{array}{l} \sigma_y(\bullet) \text{ if } r \neq 0 \\ \sigma_z(\bullet) \text{ if } r = 0 \end{array} \right. \to \mathbf{V}' \to 0$$

where $\sigma_y(\bullet)$, $\sigma_z(\bullet)$ are defined accordingly to proposition 3.7-a_2). We then introduce the following elements of R_t, R_{t+2}:

Definition 4.9. *Let $t \geqslant 2$, t even.*

i) For any $(t-1)$-tuple $(l_0, \ldots, l_{t-1}) \in \{0, \ldots, p-1\}^{t-1}$, $l_{t-1} \in \{r+1, \ldots, p-1\}$ define

$$\mathfrak{x}_{l_0,\ldots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} [t-1] & 1 \\ 1 & 0 \end{bmatrix} \mu_{t-1}[1, X^r];$$

$$\mathfrak{x}'_{l_1,\ldots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} [t-1] & 1 \\ 1 & 0 \end{bmatrix} \mu_{t-1}[1, X^r];$$

ii) if $r \neq 0$, define

$$\mathfrak{y}'_{l_1,\ldots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} [1, X^{r-1}Y]$$

where $(l_1, \ldots, l_{t-1}) \prec (r, p-1-r, \ldots, p-1-r, r)$.

iii) if $r = 0$, define

$$\mathfrak{z}'_{l_1,\ldots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix}$$
$$\sum_{\mu_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} \sum_{\mu_{t+1} \in \mathbf{F}_p} \mu_{t+1} \begin{bmatrix} 1 & 0 \\ p^{t+1}[\mu_{t+1}] & 1 \end{bmatrix} [1, X^r]$$

where $(l_1, \ldots, l_{t-1}) \prec (r, p-1-r, \ldots, p-1-r, r)$ and X^r is a fixed $\overline{\mathbf{F}}_p$-basis for $\mathrm{Sym}^0 \overline{\mathbf{F}}_p^2$.

The element defined in 4.9 will be seen also as elements of the amalgamed sums, according to the context. We see as above that

Lemma 4.10. *Let $t \geqslant 2$, t even. The following elements are I_t invariant in the inductive limit $\varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$:*

i) the elements $\mathfrak{x}'_{l_1,\ldots,l_{t-1}}$ for $l_{t-1} \geqslant r+1$, $(l_1, \ldots, l_{t-2}) \in \{0, \ldots, p-1\}^{t-2}$;

ii) if $r \neq 0$, the elements $\mathfrak{y}'_{l_1,\ldots,l_{t-1}}$,
where $(l_1, \ldots, l_{t-1}) \prec (r, p-1-r, \ldots, p-1-r, r)$;

iii) if $r = 0$, the elements $\mathfrak{z}'_{l_1,\ldots,l_{t-1}}$,
where $(l_1, \ldots, l_{t-1}) \prec (r, p-1-r, \ldots, p-1-r, r)$;

Proof: Omissis. $\sharp$

As in §4.1.1, the elements of lemma 4.10, as well as the elements of $R_0 \oplus_{R_1} \cdots \oplus_{R_{t-3}} R_{t-2}$ will be referred to as the *trivial I_t-invariants*. The result is then the following:

PROPOSITION 4.11. *An $\overline{\mathbf{F}}_p$-basis for the space of nontrivial I_t-invariants of $\varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ modulo the trivial invariants is described by the elements $\mathfrak{x}_{l_0,\dots,l_{t-2},r+1}$ where $(l_0,\dots,l_{t-2}) \prec (p-1-r, r, \dots, r, p-1-r)$*

Proof: Omissis.$\sharp$

4.2.2 ***Concerning*** $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$*, n even.* We have now to consider the K-equivariant short exact sequence

$$0 \to pr_{t-1}(\mathrm{Fil}^0(R_{t-1})) \to R_1/R_0 \oplus_{R_2} \cdots \oplus_{t-2} R_{t-1} + \left\{ \begin{array}{l} \sigma_z^s(0) \text{ if } r = 0 \\ \sigma_y^s(\bullet) \text{ if } r \neq 0 \end{array} \right. \to \mathbf{W}' \to 0$$

where $\sigma_y^s(\bullet)$ are defined accordingly to proposition 3.7-b_2). We introduce the following elements:

DEFINITION 4.12. *Let $t \geqslant 2$, t even.*

i) *for $l_0, \dots, l_{t-2} \in \{0, \dots, p-1\}$ and $j \in \{0, \dots, r\}$ define*

$$\mathfrak{w}_{l_0,\dots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-2} \in \mathbf{F}_p} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

$$\mathfrak{w}'_{l_1,\dots,l_{t-2}}(j) \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-2} \in \mathbf{F}_p} \mu_{t-2}^{l_{t-2}} \begin{bmatrix} 1 & 0 \\ p^{t-2}[\mu_{t-2}] & 1 \end{bmatrix} [1, X^{r-j}Y^j]$$

where we set, for $t = 2$,

$$\mathfrak{w}'(j) \stackrel{\mathrm{def}}{=} [1, X^{r-j}Y^j].$$

Note that such elements are in $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ (which is K_{t-1}-invariant) iff $j = 0$.

ii) *for $l_0, \dots, l_{t-1} \in \{0, \dots, p-1\}$ define*

$$\mathfrak{z}_{l_0,\dots,l_{t-1}} \stackrel{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{l_0} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r]$$

$$\mathfrak{z}'_{l_1,\dots,l_{t-1}} 1 \stackrel{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \mu_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \cdots \sum_{\mu_{t-1} \in \mathbf{F}_p} \mu_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\mu_{t-1}] & 1 \end{bmatrix} \sum_{\mu_t \in \mathbf{F}_p} \mu_t^{r+1} \begin{bmatrix} 1 & 0 \\ p^t[\mu_t] & 1 \end{bmatrix} [1, X^r].$$

The element defined in 4.12 will be seen also as elements of the amalgamed sums, according to the context. We see as above that

LEMMA 4.13. *Let $t \geqslant 2$, t even. The following elements are I_t invariant in the inductive limit $\varinjlim_{n\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$:*

i) *the elements $\mathfrak{w}'_{l_1,\dots,l_{t-2}}(j)$ for $j \in \{1, \dots, r\}$, $(l_0, \dots, l_{t-2}) \in \{0, \dots, p-1\}^{t-1}$;*

ii) *the elements $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$, where $(l_1, \dots, l_{t-1}) \prec (p-1-r, r, \dots, r, p-1-r)$.*

Proof: Omissis. $\sharp$

As in §4.1.1, the elements of lemma 4.13, as well as the elements of $pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))$ will be referred to as the *trivial I_t-invariants*. The result is then the following:

Proposition 4.14. *An $\overline{\mathbf{F}}_p$-basis for the space of nontrivial I_t-invariants of* $\varinjlim_{n\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$ *modulo the trivial invariants is described by*

i) the elements $\mathfrak{w}_{l_0,\dots,l_{t-2}}(1)$ *where* $(l_0,\dots,l_{t-2}) \prec (r, p-1-r \dots, p-1-r, r)$ *if* $r \neq 0$;

ii) the elements $\mathfrak{z}_{l_0,\dots,l_{t-2},0}$ *where* $(l_0,\dots,l_{t-2}) \prec (r, p-1-r \dots, p-1-r, r)$ *if* $r = 0$.

Proof: Omissis.♯

We are finally in the position to compute the dimension of I_t-invariants for $\pi(r,0,1)$:

Corollary 4.15. *Let* $r \in \{0,\dots,p-1\}$, $t \in \mathbf{N}_>$. *Then*

$$\dim_{\overline{\mathbf{F}}_p}((\pi(r,0,1))^{I_t}) = 2(2p^{t-1}-1).$$

Proof: We assume $t \geqslant 2$ and we will prove the result for t odd (the case t even is similar and left to the reader). We deduce from propositions 4.4 and 4.3 that

$$\dim_{\overline{\mathbf{F}}_p}((\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})^{I_t}/pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))) = rp^{t-2} + \sum_{j=1}^{t-1} p^{j-1} l_j + \sum_{j=0}^{t-2} p^j l'_j$$

where $(l_1,\dots,l_{t-1}) = (r, p-1-r,\dots,r,p-1-r)$ and $(l'_0,\dots,l'_{t-2}) = (p-1-r, r,\dots,p-1-r,r)$: they correspond to the invariants of type $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{x}_{l_0,\dots,l_{t-2}}(1)$ if $r \neq 0$ (resp. $\mathfrak{y}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{y}_{l_0,\dots,l_{t-2},0}$ if $r = 0$).

Similarly, propositions 4.8 and 4.7 give

$$\dim_{\overline{\mathbf{F}}_p}((\varinjlim_{n\,\mathrm{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})^{I_t}/pr_{t-2}(R_{t-2})) = (p-1-r)p^{t-2} + \sum_{j=1}^{t-1} p^{j-1} l_j + \sum_{j=0}^{t-2} p^j l'_j$$

where $(l_1,\dots,l_{t-1}) = (p-1-r, r \dots, p-1-r, r)$ and $(l'_0,\dots,l'_{t-2}) = (r, p-1-r,\dots,r,p-1-r)$: they correspond to the invariants of type $\mathfrak{z}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$ if $r \neq 0$ (resp. $\mathfrak{h}'_{l_1,\dots,l_{t-1}}$ and $\mathfrak{w}_{l_0,\dots,l_{t-2},r+1}$ if $r = 0$). As

$$\dim_{\overline{\mathbf{F}}_p}(pr_{t-1}(\mathrm{Fil}^0(R_{t-1}))) + \dim_{\overline{\mathbf{F}}_p}(R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-3}} R_{t-2}) = p^{t-2}(p+1)$$

(lemma 2.8) an elementary computation yields the desired result for $t \geqslant 2$, t odd.

Since $\pi(r,0,1)^{I_1}$ is 2 dimensional (cf. [Bre03a] Théoréme 3.2.4) the conclusion follows. ♯.

5. The case of principal series and the Steinberg.

We are going to describe briefly the K_t and I_t-invariants of principal series and Steinberg for $\mathrm{GL}_2(\mathbf{Q}_p)$; by Mackey's theorem and the Iwasawa decomposition for $\mathrm{GL}_2(\mathbf{Q}_p)$ it will be enough to study the inductions $\mathrm{Ind}_{K\cap B}^K \chi_r^s$ for $r \in \{0,\dots,p-2\}$. As the techniques involved are completely similar to what we have seen for the supersingular case, we will content ourselves to state the results, leaving the proofs to the reader.

Concerning the K_t-invariants. Let $t \in \mathbf{N}_>$. From the exact sequences

$$0 \to \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \to \mathrm{Ind}_{K_0(p^{n+2})}^K \chi_r^s \to Q_{0,\dots,0,1}^{(0,n+2)} \to 0$$

we see (as in the proof of lemma 3.2) that all K_t-invariants for $\mathrm{Ind}_{K\cap B}^K \chi_r^s$ must be inside $\mathrm{Ind}_{K_0(p^k)}^K \chi_r^s$. More precisely, we have

Proposition 5.1. *Let* $t \in \mathbf{N}_>$. *Then*

$$(\mathrm{Ind}_{K\cap B}^K \chi_r^s)^{K_t} = \mathrm{Ind}_{K_0(p^t)}^K \chi_r^s.$$

In particular, $\dim_{\overline{\mathbf{F}}_p}(\mathrm{Ind}_{K\cap B}^K\chi_r^s) = p^{t-1}(p+1)$.

Proof: Omissis. $\sharp$

Concerning the I_t-invariants. Fix $t\in\mathbf{N}_>$. As $I_t \geqslant K_t$ we see that $(\mathrm{Ind}_{K\cap B}^K\chi_r^s)^{I_t} = (\mathrm{Ind}_{K_0(p^t)}^K\chi_r^s)^{I_t}$. We can therefore use the $\overline{\mathbf{F}}_p$-basis of $\mathrm{Ind}_{K_0(p^t)}^K\chi_r^s$ given by

$$\mathfrak{x}_{l_0,\dots,l_{t-1}} \overset{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\mu_0^{l_0}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\cdots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,e]$$

$$\mathfrak{x}'_{l_1,\dots,l_{t-1}} \overset{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\cdots\sum_{\mu_{t-1}\in\mathbf{F}_p}\mu_{t-1}^{l_{t-1}}\begin{bmatrix}1 & 0\\ p^{t-1}[\mu_{t-1}] & 1\end{bmatrix}[1,e]$$

(where, if $t=1$, we set $\mathfrak{x}' \overset{\mathrm{def}}{=} [1,e]$) to describe completely the space $(\mathrm{Ind}_{K_0(p^t)}^K\chi_r^s)^{I_t}$. We find that

Proposition 5.2. *Let $t\geqslant 2$. Then an $\overline{\mathbf{F}}_p$-basis for the space $(\mathrm{Ind}_{K_0(p^t)}^K\chi_r^s)^{I_t}/(\mathrm{Ind}_{K_0(p^{t-1})}^K\chi_r^s)$ is given by the elements $\mathfrak{x}'_{l_1,\dots,l_{t-1}}$ with $l_{t-1}\geqslant 1$. In particular we have*

$$\dim_{\overline{\mathbf{F}}_p}((\mathrm{Ind}_{K\cap B}^K\chi_r^s)^{I_t}) = 2p^{t-1}$$

for any $t\in\mathbf{N}_>$.

Proof: It follows the arguments in the proofs of propositions 4.3, 4.7 and it is left to the reader.$\sharp$

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