Invariant elements for p-modular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$

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Abstract

Let p be an odd rational prime and F a p-adic field. We give a realization of the universal p-modular representations of $\mathbf{GL}_2(F)$ in terms of an explicit Iwasawa module. We specialize our constructions to the case $F = \mathbf{Q}_p$, giving a detailed description of the invariants under principal congruence subgroups of irreducible admissible p-modular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$, generalizing previous work of Breuil and Paskunas [BP]. We apply these results to the local-global compatibility of Emerton [Eme10], giving a generalization of the classical multiplicity one results for the Jacobians of modular curves with arbitrary level at p.

Contents

1. Introduction

Let F be a p-adic field, with ring of integers $\mathscr{O}_F$ and residue field k_F. This article is framed in the broad context of the p-modular Langlands correspondence, aimed to match continuous Galois representations of $\mathrm{Gal}(\overline{F}/F)$ over finite dimensional $\overline{\mathbf{F}}_p$-vector spaces with certain $\overline{\mathbf{F}}_p$-valued, smooth representations of the F-points of p-adic reductive groups.

This correspondence has first been defined in the particular case of $F = \mathbf{Q}_p$ and the group $\mathbf{GL}_2$, thanks to the parametrization of *supersingular* representations of $\mathbf{GL}_2(\mathbf{Q}_p)$ (cf. [Bre03a]). It is now completely established in the wide horizon of the p-adic Langlands correspondence for $\mathbf{GL}_2(\mathbf{Q}_p)$ (cf. [Col1], [Kis], [Pas1]) and admits a cohomological realization according to the local-global compatibility of Emerton [Eme10].

For other groups the situation turns out to be extremely more delicate. While p-modular Galois representations are well understood, the theory of p-modular representation of p-adic reductive

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groups is at its beginning, starting with the pionieristic work of Barthel and Livné [BL94], [BL95] and recently achieved in greater generality by Herzig [Her2]. Even for $\mathbf{GL}_2$, recent constructions of Breuil and Paskunas [BP] and Hu [Hu2] show a troubling proliferation of supersingular representations as soon as $F \neq \mathbf{Q}_p$. This phenomenon remains, at present, unexplained.

Nevertheless, the work [BP], [Hu] highlight that the crucial point in order to understand an irreducible admissible p-modular $\mathbf{GL}_2(F)$ representation π relies in a complete control of its *internal structure*, i.e. of the extensions between irreducible representations of certain congruence subgroups appearing as *subquotients* of π. A exhaustive study in this direction has started in [Mo1], where the author realizes the $\mathbf{GL}_2(\mathbf{Z}_p)$-socle filtration for irreducible admissible $\mathbf{GL}_2(\mathbf{Q}_p)$-representations.

In this article we pursue the investigation undertaken in [Mo1], clarifying the internal behavior of *universal* p-modular representations of $\mathbf{GL}_2(F)$ by means of structure theorems, showing the prominent role of an explicit Iwasawa module. This enables us, in the particular case of $F = \mathbf{Q}_p$, to describe exhaustively the space of fixed vectors of supersingular representations under principal congruence subgroups, generalizing previous results of Breuil and Paskunas [BP]. Thanks to the local-global compatibility theorems of Emerton [Eme10], we are able to generalize the classical "multiplicity one" results ([Maz], [Rib], [Edi], [Kha]) in the case of modular curves whose level is highly divisible by p.

We give a more precise account of the main results appearing in this paper.

Let k be a finite extension of k_F (the "field of coefficients"): all representations are on k linear spaces. From the classification of Barthel and Livné [BL94], a supersingular representation π of $\mathbf{GL}_2(F)$ is, up to twist, an irreducible admissible quotient of an explicit universal representation $\pi(\sigma, 0)$. The latter is defined as the cokernel of a canonical Hecke operator on the compact induction $\mathrm{ind}_{\mathbf{GL}_2(\mathscr{O}_F)F^\times}^{\mathbf{GL}_2(F)}\sigma$, where σ is an irreducible smooth representation of $\mathbf{GL}_2(\mathscr{O}_F)F^\times$.

According to the work of Breuil and Paskunas [BP] and Hu [Hu2], the representation π is *completely determined* by its structure as $\mathbf{GL}_2(\mathscr{O}_F)$ and N representation, where N is the normalizer of the Iwahori subgroup I of $\mathbf{GL}_2(\mathscr{O}_F)$ (the crucial point being that $\mathbf{GL}_2(F)$ is canonically identified with the amalgamation of $\mathbf{GL}_2(\mathscr{O}_F)F^\times$ and N along their intersection $IF^\times$).

Our first results give a realization of the $\mathbf{GL}_2(\mathscr{O}_F)$ and the N restriction of $\pi(\sigma, 0)$ in terms of certain $k[I]$-modules $R^-_{\infty,0}$, $R^-_{\infty,-1}$:

Theorem 1.1 (Corollary 3.5). *There is a canonical $\mathbf{GL}_2(\mathscr{O}_F)F^\times$-isomorphism $\pi(\sigma,0)|_{\mathbf{GL}_2(\mathscr{O}_F)F^\times} \cong R_{\infty,0} \oplus R_{\infty,-1}$ where the representations $R_{\infty,0}$, $R_{\infty,-1}$ fit in the following exact sequences of $k[\mathbf{GL}_2(\mathscr{O}_F)]$-modules:*

$$0 \to V_1 \to \mathrm{ind}_I^{\mathbf{GL}_2(\mathscr{O}_F)}\big(R^-_{\infty,0}\big) \to R_{\infty,0} \to 0$$
$$0 \to V_2 \to \mathrm{ind}_I^{\mathbf{GL}_2(\mathscr{O}_F)}\big(R^-_{\infty,-1}\big) \to R_{\infty,-1} \to 0$$

for suitable subquotients V_1, V_2 of an induction from a smooth character of the Iwahori subgroup, depending on σ.

The second structure theorem clarifies a result already appearing in [Mo5] (Proposition 3.5) and is concerned with the N-restriction of the universal representation $\pi(\sigma, 0)$. We remark that if $F = \mathbf{Q}_p$ this is a result of Paskunas ([Pas2], Theorem 6.3 and Corollary 6.5).

Theorem 1.2 (Propositions 3.7 and 3.8). *In the notations of Theorem 1.1, we have the following I-equivariant exact sequences*

$$0 \to W_1 \to \big(R^-_{\infty,-1}\big)^s \oplus R^-_{\infty,0} \to R_{\infty,0}|_I \to 0$$
$$0 \to W_2 \to \big(R^-_{\infty,0}\big)^s \oplus R^-_{\infty,-1} \to R_{\infty,-1}|_I \to 0$$

where W_1, W_2 *are convenient* 1*-dimensional* $k[I]$*-modules. Moreover, the action of the element* $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ *on the universal representation* $\pi(\sigma,0)$ *induces the* $k[I]$*-equivariant involution*

$$\begin{aligned}(R^-_{\infty,-1})^s \oplus R^-_{\infty,0} &\overset{\sim}{\longrightarrow} \Big((R^-_{\infty,0})^s \oplus R^-_{\infty,-1}\Big)^s \\ (v_1,v_2) &\longmapsto (v_2,v_1)\end{aligned}$$

which restricts to an isomorphism $W_1 \overset{\sim}{\to} W_2^s$.

Here, the notation $(*)^s$ means that we are considering the action of I on $*$ obtained by conjugation by the element $\begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ (which is a representative for the only nontrivial coset of $N/IF^\times$).

An exhaustive control of the subquotients for the $k[I]$-modules $R^-_{\infty,0}$, $R^-_{\infty,-1}$ is crucial in order to extract the most subtle properties of supersingular representations for $\mathbf{GL}_2(F)$. For instance, in [Hu2] Hu gives a method to detect a subquotient of $R^-_{\infty,0} \oplus R^-_{\infty,-1}$ which essentially *characterizes* a supersingular quotient of $\pi(\sigma,0)$ and the studies of Schraen on the homological properties of $R^-_{\infty,0}$, $R^-_{\infty,-1}$ show that supersingular representations are *not* of finite presentation when $[F:\mathbf{Q}_p]=2$ ([Sch]).

The $k[I]$-modules $R^-_{\infty,0}$, $R^-_{\infty,-1}$ admit an explicit construction, which is recalled in §3.1. Their Pontryagin duals are obtained as limits (over $\mathbf{N}$) of finitely presented modules whose first syzygy requires a strictly increasing number of generators as soon as $F \neq \mathbf{Q}_p$; in particular, $R^-_{\infty,0}$, $R^-_{\infty,-1}$ are not admissible unless $F=\mathbf{Q}_p$. A first study of $R^-_{\infty,0}$, $R^-_{\infty,-1}$ has been pursued by the author in [Mo5] (by representation theoretic methods) and in [Mo6], [Mo7] (using methods from Iwasawa theory).

In the case $F=\mathbf{Q}_p$ the behavior of $R^-_{\infty,0}$, $R^-_{\infty,-1}$ is particularly simple:

Theorem 1.3 ([Mo1], Proposition 5.10). *Let* $F=\mathbf{Q}_p$. *For* $\bullet \in \{0,-1\}$ *the* $k[I]$*-module* $R^-_{\infty,\bullet}$ *is uniserial.*

This phenomenon, which can equally be deduced from the results of [Pas2] (Propositions 4.7 and 5.9), is at the heart of the studies carried out in [Mo1], [Mo4], [AM] and lets us detect in greatest detail the space of invariant vectors of supersingular representations $\pi(\sigma,0)$ of $\mathbf{GL}_2(\mathbf{Q}_p)$ under certain congruence subgroups.

The following theorem is a sharpening of the main result of [Mo1] and of [BP], Proposition 20.1:

Theorem 1.4 (Corollary 4.14). *Let* $t \geqslant 1$ *and let* K_t *be the principal congruence subgroup of* $\mathbf{GL}_2(\mathscr{O}_F)$ *of level* p^t. *Assume* $\sigma = \mathrm{Sym}^r k^2$ *where* $r \in \{0,\dots,p-1\}$.

The space of K_t *fixed vectors for the supersingular representation* $\pi(\sigma,0)$ *decomposes into the direct sum of two* $k[K]$*-modules* $(\pi(\sigma,0))^{K_t} = (R_{\infty,0})^{K_t} \oplus (R_{\infty,-1})^{K_t}$. *Each direct summand admits a* K*-equivariant filtration whose graded pieces are described by:*

$$(R_{\infty,0})^{K_t}: \quad \mathrm{Sym}^r k^2 \text{---} \mathrm{ind}_I^K \chi_{r+2}\det{}^{-1} \text{---} \mathrm{ind}_I^K \chi_{r+4}\det{}^{-2} \text{---} \dots \text{---} \mathrm{ind}_I^K \chi_r \text{---} \mathrm{Sym}^{p-3-r}k^2 \otimes \det{}^{r+1}$$

$$(R_{\infty,-1})^{K_t}: \quad \mathrm{Sym}^{p-1-r} k^2 \otimes \det{}^r \text{---} \mathrm{ind}_I^K \chi_{-r+2}\det{}^{r-1} \text{---} \mathrm{ind}_I^K \chi_{-r+4}\det{}^{r-2} \text{---} \dots \text{---} \mathrm{ind}_I^K \chi_{-r}\det{}^r \text{---} \mathrm{Sym}^{r-2}k^2 \otimes \det$$

where we have $p^{t-1}-1$ *parabolic inductions in each line and the algebraic representation* $\mathrm{Sym}^{p-3-r}k^2 \otimes \det^{r+1}$ *in the first line (resp.* $\mathrm{Sym}^{r-2}k^2 \otimes \det$ *in the second line) appears only if* $p-3-r \geqslant 0$ *(resp.* $r-2 \geqslant 0$*).*

We recall that for any $n \in \mathbf{N}$ the natural $\mathbf{GL}_2(\mathbf{F}_p)$-representation $\mathrm{Sym}^n k^2$ is viewed as a $\mathbf{GL}_2(\mathbf{Z}_p)$-representation by inflation and that the smooth character χ_n of the Iwahori I is defined by $\begin{bmatrix} a & b \\ pc & d \end{bmatrix} \mapsto a^n \bmod p$. One can indeed prove that the $\mathbf{GL}_2(\mathbf{Z}_p)$ socle filtration for $(R_{\infty,0})^{K_t}$, $(R_{\infty,-1})^{K_t}$ is obtained by the evident refinement of the filtration described by Theorem 1.4 (cf. Corollary 4.14).

The statement of Theorem 1.4 is deduced from Theorem 1.1, through a careful study of the K_t fixed vectors of the Iwasawa modules $R^-_{\infty,0}$, $R^-_{\infty,-1}$. As the latter are unserial, their K_t fixed elements can be easily recovered with a direct argument on Witt vectors (cf. Proposition 4.4).

In a similar fashion, we detect the fixed vectors for the congruence subgroup I_t, which is defined as the subgroup of K_{t-1} whose elements are upper unipotent modulo p^t:

Theorem 1.5 (Corollary 3.11, Propositions 4.6, 4.9). *Let $t \geqslant 1$ and assume $\sigma = \mathrm{Sym}^r k^2$ for $r \in \{0, \dots, p-1\}$.*

The space of I_t fixed vectors for the supersingular representation $\pi(\sigma, 0)$ decomposes as $(\pi(\sigma,0))^{I_t} = (R_{\infty,0})^{I_t} \oplus (R_{\infty,-1})^{I_t}$. Each direct summand is a $k[I]$-module admitting an equivariant filtration whose graded pieces are described by:

$$
(R_{\infty,0})^{I_t}: \quad
\begin{array}{cccccccc}
 & & \chi_{r+2}\det^{-1} & \text{---} & \chi_{r+4}\det^{-2} & \text{---} & \cdots & \text{---}\ \chi_r \\
 & \diagup & & & & & & \\
\chi_r & & \oplus & & \oplus & & \oplus & \\
 & \diagdown & & & & & & \\
 & & \chi_{r-2}\det & \text{---} & \chi_{r-4}\det^{2} & \text{---} & \cdots & \text{---}\ \chi_r
\end{array}
$$

$$
(R_{\infty,-1})^{I_t}: \quad
\begin{array}{cccccccc}
 & & \chi_{-r+2}\det^{r-1} & \text{---} & \chi_{-r+4}\det^{r-2} & \text{---} & \cdots & \text{---}\ \chi_{-r}\det^{r} \\
 & \diagup & & & & & & \\
\chi_{-r}\det^{r} & & \oplus & & \oplus & & \oplus & \\
 & \diagdown & & & & & & \\
 & & \chi_{-r-2}\det^{r+1} & \text{---} & \chi_{-r-4}\det^{r+2} & \text{---} & \cdots & \text{---}\ \chi_{-r}\det^{r}
\end{array}
$$

and we have $p^{t-1} - 1$ characters on each horizontal line.

We point out that Theorem 1.4 and 1.5 had first been proved by the author in [Mo2], essentially with the same technical tools, but the lack of the structure Theorems 1.1 and 1.2 required a considerable amount of delicate estimates on Witt vectors. Moreover, our techniques could be applied to detect the fixed vectors for irreducible admissible $\mathbf{GL}_2(\mathbf{Q}_p)$-representations under other congruence subgroups (see for instance [Mo3]).

As we remarked above, a precise control of K_t, I_t invariants has global applications, thanks to the geometric realization of the p-adic Langlands correspondence by Emerton [Eme10]. Let $\overline{\rho}: \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}_2(k)$ be a continuous, irreducible odd Galois representation which we assume to be absolutely irreducible at p. Let N be its Artin conductor and κ its minimal weight (cf. [Ser87]); up to twist, we may assume $2 \leqslant \kappa \leqslant p$. Let $Y(Np^t)$ be the modular curve (defined over $\mathbf{Q}$) of level Np^t and $\mathfrak{m}_{\overline{\rho}}$ the maximal ideal in the spherical Hecke algebra of $H^1_{\acute{e}t}(Y(Np^t) \times_{\mathbf{Q}} \overline{\mathbf{Q}}, k)$ corresponding to $\overline{\rho}$.

The result is the following:

THEOREM 1.6 (Proposition 6.1). *Let $K(N)$ be the kernel of the map*

$$\prod_{\ell|N}\mathbf{GL}_2(\mathbf{Z}_\ell)\to\prod_{\ell|N}\mathbf{GL}_2(\mathbf{Z}_\ell/N)$$

and define

$$d\stackrel{\mathrm{def}}{=}\dim_k\Big(\bigotimes_{\ell|N}\pi(\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)})\Big)^{K(N)}$$

where $\pi(\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)})$ is the smooth $\mathbf{GL}_2(\mathbf{Q}_\ell)$-representation attached to $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)}$ by the Emerton-Helm p-modular Langlands correspondence ([EH]).

If $t\geqslant 1$ and $Np^t>4$ we have

$$\dim_k\big(H^1_{\acute{e}t}(Y(Np^t)\times_{\mathbf{Q}}\overline{\mathbf{Q}},k)[\mathfrak{m}_{\overline{\rho}}]\big)=2d\big(2p^{t-1}(p+1)-3\big)\qquad \text{if } \kappa-2=0$$
$$\dim_k\big(H^1_{\acute{e}t}(Y(Np^t)\times_{\mathbf{Q}}\overline{\mathbf{Q}},k)[\mathfrak{m}_{\overline{\rho}}]\big)=2d\big(2p^{t-1}(p+1)-4\big)\qquad \text{if } \kappa-2\neq 0.$$

Thanks to the relation between the étale cohomology of the modular curve $Y(Np^t)$ and the Tate module of its Jacobian, Theorem 1.6 generalizes the classical multiplicity one theorems ([Rib], [Maz]) to modular curves of arbitrary level at p. It is consistent with the results of Khare [Kha], where it is shown that the dimension of the $\overline{\rho}$-isotypical component of the Jacobian of $X_1(Np^t)$ tends to infinity as the level at p increases.

The organization of the paper is the following.

We start (§2) by recalling the construction of the universal representation $\pi(\sigma,0)$ for a p-adic field F, as well as some properties of finite parabolic induction for $\mathbf{GL}_2(\mathbf{F}_p)$ which will be used later on to describe the K_t-invariants for irreducible admissible representations of $\mathbf{GL}_2(\mathbf{Q}_p)$.

Section §3 is devoted to the realization of the structure theorems for universal representations. We first refine the constructions of §2 in order to define the Iwasawa modules $R^-_{\infty,0}$, $R^-_{\infty,-1}$ (§3.1); we subsequently specialize to the case $F=\mathbf{Q}_p$ (§3.2).

The space of invariant vectors for irreducible admissible representations is worked out in section 4. We first detect the invariants for the Iwasawa modules $R^-_{\infty,0}$, $R^-_{\infty,-1}$ (§4.1), relying crucially on the fact that such objects are unimodular (Proposition 4.4). We then use the structure theorems of section 3 to deduce the space of K_t and I_t fixed vectors for supersingular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$.

Section 5 is devoted to the case of principal and special series representations for $\mathbf{GL}_2(\mathbf{Q}_p)$. The results are somehow similar, but can be detected with much less efforts.

Finally, we give in §6 a precise description of the global application of Theorems 1.4, 1.5 for the multiplicity spaces of mod p cohomology of modular curves.

1.1 Notation

Let p be an odd prime. We consider a p-adic field F, with ring of integers $\mathscr{O}_F$, uniformizer ϖ and (finite) residue field k_F. Let $q\stackrel{\mathrm{def}}{=}\mathrm{Card}(k_F)$ be its cardinality and $f\stackrel{\mathrm{def}}{=}[k_F:\mathbf{F}_p]$ the residual degree. We write $x\mapsto\overline{x}$ for the reduction morphism $\mathscr{O}_F\to k_F$ and $\overline{x}\mapsto[\overline{x}]$ for the Teichmüller lift $k_F^\times\to\mathscr{O}_F^\times$ (we set $[0]\stackrel{\mathrm{def}}{=}0$).

Consider the general linear group $\mathbf{GL}_2$, whose F-points will be denoted by $G\stackrel{\mathrm{def}}{=}\mathbf{GL}_2(F)$. We fix the maximal torus $\mathbf{T}$ of diagonal matrices and the unipotent radical $\mathbf{U}$ of upper unipotent matrices, so that $\mathbf{B}\stackrel{\mathrm{def}}{=}\mathbf{T}\ltimes\mathbf{U}$ is the Borel subgroup of upper triangular matrices. We write $\overline{\mathbf{B}}=\mathbf{T}\ltimes\overline{\mathbf{U}}$ for the opposite Borel, and $Z\stackrel{\mathrm{def}}{=}Z(G)$ for the center of the F-points of $\mathbf{GL}_2$. Let $\mathscr{T}$ be the Bruhat-Tits tree associated to $\mathbf{GL}_2(F)$ (cf. [Ser77]) and consider the hyperspecial maximal compact subgroup $K \overset{\text{def}}{=} \mathbf{GL}_2(\mathscr{O}_F)$.

The object of study of this article are the following congruence subgroups of K:

$$K_t \overset{\text{def}}{=} \ker\big(K \overset{red_t}{\longrightarrow} \mathbf{GL}_2(\mathscr{O}_F/(\varpi^t))\big), \qquad I_t \overset{\text{def}}{=} \left(red_t^{-1}\big(\mathbf{U}(\mathscr{O}_F/(\varpi^t))\big)\right) \cap K_{t-1}$$

where $t \in \mathbf{N}$ and red_t denotes the mod ϖ^t reduction map. For notational convenience, we introduce the following objects

$$s \overset{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbf{GL}_2(F), \qquad \alpha \overset{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix} \in \mathbf{GL}_2(F), \qquad K_0(\varpi) \overset{\text{def}}{=} red_1^{-1}\big(\mathbf{B}(k_F)\big).$$

Let E be a p-adic field, with ring of integers $\mathscr{O}$ and finite residue field k (the "coefficient field"). Up to enlarging E, we can assume that $\mathrm{Card}\big(\mathrm{Hom}_{\mathbf{F}_p}(k_F, k)\big) = [k_F : \mathbf{F}_p]$.

A representation σ of a subgroup H of $\mathbf{GL}_2(\mathbf{Q}_p)$ is always understood to be smooth with coefficients in k. If $h \in H$ we will sometimes write $\sigma(h)$ to denote the k-linear automorphism induced by the action of h on the underlying vector space of σ. We denote by $(\sigma)^H$ the linear space of H fixed vectors of σ.

Let $H_2 \leqslant H_1$ be compact open subgroups of K. For a smooth representation σ of H_2 we write $\mathrm{ind}_{H_2}^{H_1}\sigma$ to denote the (compact) induction of σ from H_2 to H_1. If $v \in \sigma$ and $h \in H_1$ we write $[h, v]$ for the unique element of $\mathrm{ind}_{H_2}^{H_1}\sigma$ supported in H_2h^{-1} and sending h^{-1} to v. We deduce in particular the following equalities:

$$h' \cdot [h, v] = [h'h, v], \qquad [hk, v] = [h, \sigma(k)v] \tag{1}$$

for any $h' \in H_1$, $k \in H_2$.

The previous construction will mainly be used when $H_1 = K$, $H_2 = K_0(\varpi)$. In this situation we define, for any $v \in \sigma$ and $l \in \mathbf{N}$, the element

$$f_l(v) \overset{\text{def}}{=} \sum_{\lambda \in k_F} \lambda^l \begin{bmatrix} [\lambda] & 1 \\ 1 & 0 \end{bmatrix} [1, v] \in \mathrm{ind}_{K_0(\varpi)}^{K}\sigma.$$

If $Z \cong F^\times$ is the center of $\mathbf{GL}_2(F)$ and σ is a representation of KZ we will similarly write $\mathrm{ind}_{KZ}^{\mathbf{GL}_2(F)}\sigma$ for the subspace of the full induction $\mathrm{Ind}_{KZ}^{\mathbf{GL}_2(F)}\sigma$ consisting of functions which are compactly supported modulo the center Z (cf. [Bre03a], §2.3). For $g \in \mathbf{GL}_2(F)$, $v \in \sigma$ we use the same notation $[g, v]$ for the element of $\mathrm{ind}_{KZ}^{\mathbf{GL}_2(F)}\sigma$ having support in KZg^{-1} and sending g^{-1} to v; the element $[g, v]$ verifies similar compatibility relations as in (1).

A *Serre weight* is an absolutely irreducible representation of K. Up to isomorphism they are of the form

$$\bigotimes_{\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)} \big(\det{}^{t_\tau} \otimes_{k_F} \mathrm{Sym}^{r_\tau} k_F^2\big) \otimes_{k_F, \tau} k \tag{2}$$

where $r_\tau, t_\tau \in \{0, \dots, p-1\}$ for all $\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)$ and $t_\tau < p-1$ for at least one τ. This gives a bijective parametrization of isomorphism classes of Serre weights by $2f$-tuples of integers $r_\tau, t_\tau \in \{0, \dots, p-1\}$ such that $t_\tau < p-1$ for some τ. The Serre weight characterized by $t_\tau = 0$, $r_\tau = p-1$ for all $\tau \in \mathrm{Gal}(k_F/\mathbf{F}_p)$ will be referred as the *Steinberg* weight and denoted by $\overline{St}$.

Recall that the K representations $\mathrm{Sym}^{r_\tau} k_F^2$ can be identified with $k_F[X, Y]^h_{r_\tau}$, the linear subspace of $k_F[X, Y]$ described by the homogeneous polynomials of degree r_τ.

In this case, the action of K is described by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot X^{r_\tau - i} Y^i \overset{\text{def}}{=} (\overline{a}X + \overline{c}Y)^{r_\tau - i}(\overline{b}X + \overline{d}Y)^i$$

for any $0 \leqslant i \leqslant r_\tau$.

We fix once and for all a field homomorphism $k_F \hookrightarrow k$. The results of this paper do not depend on this choice.

Up to twist by a power of det, a Serre weight has now the more concrete expression

$$\sigma_{\underline{r}} \cong \bigotimes_{i=0}^{f-1} \left(\mathrm{Sym}^{r_i} k^2\right)^{\mathrm{Frob}^i} \tag{3}$$

where $\underline{r} = (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$ and $\left(\mathrm{Sym}^{r_i} k^2\right)^{\mathrm{Frob}^i}$ is the representation of K obtained from $\mathrm{Sym}^{r_i} k^2$ via the homomorphism $\mathbf{GL}_2(k_F) \to \mathbf{GL}_2(k_F)$ induced by the i-th Frobenius $x \mapsto x^{p^i}$ on k_F.

We will usually extend the action of K on a Serre weight to the group KZ, by imposing the scalar matrix $\varpi \in Z$ to act trivially.

A k-valued character χ of the torus $\mathbf{T}(k_F)$ will be considered, by inflation, as a smooth character of any subgroup of $K_0(\varpi)$. We will write χ^s to denote the conjugate character of χ, defined by

$$\chi^s(t) \overset{\text{def}}{=} \chi(sts)$$

for any $t \in \mathbf{T}(k_F)$.

Similarly, if τ is any representation of $K_0(\varpi)$, we will write τ^s to denote the conjugate representation, defined by

$$\tau^s(h) = \tau(\alpha h \alpha)$$

for any $h \in K_0(\varpi)$.

Finally, if σ is a Serre weight, we write $\sigma^{[s]}$ for the unique Serre weight non isomorphic to σ and whose highest weight space affords the character $\left((\sigma)^{K_0(\varpi)}\right)^s$. Concretely, if σ appears in the K socle of an induction $\mathrm{ind}_{K_0(\varpi)}^K \chi$, then $\sigma^{[s]}$ appears in the socle of $\mathrm{ind}_{K_0(\varpi)}^K \chi^s$.

If $\underline{r} = (r_0, \dots, r_{f-1}) \in \{0, \dots, p-1\}^f$ is an f-tuple we define the characters of $\mathbf{T}(k_F)$:

$$\chi_{\underline{r}}\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \overset{\text{def}}{=} a^{\sum_{i=0}^{f-1} p^i r_i}, \qquad \mathfrak{a}\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \overset{\text{def}}{=} ad^{-1}.$$

If $H \leqslant K$ is an open subgroup and τ is a representation of H we write $\left\{\mathrm{soc}^i(\tau)\right\}_{i \in \mathbf{N}}$ to denote its socle filtration (we set $\mathrm{soc}^0(\tau) \overset{\text{def}}{=} \mathrm{soc}(\tau)$). We will use the notation

$$\mathrm{soc}^0(\tau) — \mathrm{soc}^1(\tau)/\mathrm{soc}^0(\tau) — \dots — \mathrm{soc}^{n+1}(\tau)/\mathrm{soc}^n(\tau) — \dots$$

to denote the sequence of consecutive graded pieces of the socle filtration for τ (in particular, each $\mathrm{soc}^{i+1}(\tau)/\mathrm{soc}^i(\tau) — \mathrm{soc}^{i+2}(\tau)/\mathrm{soc}^{i+1}(\tau)$ is a non-split extension).

More generally, if τ is an H-representation endowed with an increasing filtration $\left\{\tau_i\right\}_{i \in \mathbf{N}}$ we will write

$$\mathrm{socfil}(\tau_0) — \mathrm{socfil}(\tau_1/\tau_0) — \dots — \mathrm{socfil}(\tau_{i+1}/\tau_i) — \dots$$

to mean that

i) the socle filtration for τ is obtained, by refinement, from the filtration induced on τ by the socle filtration of each graded piece τ_{i+1}/τ_i;

ii) the sequence of consecutive graded pieces of the socle filtration for τ is obtained as the juxtaposition of the sequences of the graded pieces associated to the socle filtration of each τ_{i+1}/τ_i.

If S is any set, and $s_1, s_2 \in S$ we define the Kronecker delta

$$\delta_{s_1,s_2} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad s_1 \neq s_2 \\ 1 & \text{if} \quad s_1 = s_2. \end{cases}$$

Moreover, for $x \in \mathbf{Z}$, we define $\lfloor x \rfloor \in \{0, \dots, p-2\}$ (resp. $\lceil x \rceil \in \{1, \dots, p-1\}$) by the condition $\lfloor x \rfloor \equiv x \equiv \lceil x \rceil \bmod p-1$.

2. Reminders on the universal representations for $\mathbf{GL}_2$

We recall here the precise definition of the universal representation of $\mathbf{GL}_2$. We provide an explicit construction in terms of Hecke operators and Mackey decomposition, which turns out to be useful to realize the structure theorems of §3. We end the section collecting some results on finite inductions for smooth characters of the Iwahori subgroup.

The main references are the work of Breuil [Bre03a], §2 and [Mo1], §2 and §3.

2.1 Construction of the universal representation

We fix an f-tuple $\underline{r} \in \{0, \dots, p-1\}^f$ and write $\sigma = \sigma_{\underline{r}}$ for the associated Serre weight described in (2). In particular, the highest weight space of σ affords the character $\chi_{\underline{r}}$. We recall ([BL95], [Her1]) that the Hecke algebra $\mathscr{H}_{KZ}(\sigma) \stackrel{\text{def}}{=} \mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is commutative and isomorphic to the algebra of polynomials in one variable over k:

$$\mathscr{H}_{KZ}(\sigma) \xrightarrow{\sim} k[T].$$

The Hecke operator T is supported on the double coset $K\alpha KZ$ and completely determined as a suitable linear projection on σ (cf. [Her1], Theorem 1.2); it admits an explicit description in terms of the Bruhat-Tits tree of $\mathbf{GL}_2(F)$ (cf. [Bre03a], §2.5).

The universal representation $\pi(\sigma, 0, 1)$ for $\mathbf{GL}_2$ is then defined by the exact sequence

$$0 \to \mathrm{ind}_{KZ}^G \sigma \xrightarrow{T} \mathrm{ind}_{KZ}^G \sigma \to \pi(\sigma, 0, 1) \to 0.$$

In the rest of this section we study the KZ-restriction of $\pi(\sigma, 0, 1)$ in terms of its Mackey decomposition, giving a precise construction by means of a family of suitable Hecke operators.

Let $n \in \mathbf{N}$. We consider the anti-dominant co-weight $\lambda_n \in X(\mathbf{T})_*$ characterized by

$$\lambda_n(\varpi) = \begin{bmatrix} 1 & 0 \\ 0 & \varpi^n \end{bmatrix}$$

and we introduce the subgroup

$$K_0(\varpi^n) \stackrel{\text{def}}{=} \big(\lambda_n(\varpi) K \lambda_n(\varpi^{-1})\big) \cap K = \left\{ \begin{bmatrix} a & b \\ \varpi^n c & d \end{bmatrix} \in K,\ c \in \mathscr{O}_F \right\}.$$

The element $\begin{bmatrix} 0 & 1 \\ \varpi^n & 0 \end{bmatrix}$ normalizes $K_0(\varpi^n)$ and we define the $K_0(\varpi^n)$-representation $\sigma^{(n)}$ as the $K_0(\varpi^n)$ restriction of σ endowed with the twisted action of $K_0(\varpi^n)$ by the element $\begin{bmatrix} 0 & 1 \\ \varpi^n & 0 \end{bmatrix}$. Explicitly,

$$\sigma^{(n)}\left(\begin{bmatrix} a & b \\ \varpi^n c & d \end{bmatrix} \right) \cdot \underline{X}^{\underline{r}-\underline{j}} \underline{Y}^{\underline{j}} \stackrel{\text{def}}{=} \sigma\left(\begin{bmatrix} d & c \\ \varpi^n b & a \end{bmatrix} \right) \underline{X}^{\underline{r}-\underline{j}} \underline{Y}^{\underline{j}}.$$

Finally, we write

$$R_n(\sigma) \stackrel{\text{def}}{=} \mathrm{ind}_{K_0(\varpi^n)}^K \big(\sigma^{(n)}\big).$$

If the Serre weight σ is clear from the context, we set $R_n = R_n(\sigma)$. For notational convenience we define $R_{-1} \overset{\text{def}}{=} 0$.

We have a K-equivariant isomorphism (deduced from Frobenius reciprocity)

$$\begin{aligned} R_n &\xrightarrow{\sim} k[K\lambda_n(\varpi)KZ] \otimes_{k[KZ]} \sigma \\ [1,v] &\longmapsto \lambda_n(\varpi) \otimes s \cdot v \end{aligned} \tag{4}$$

which realizes the Mackey decomposition for $\mathrm{ind}_{KZ}^G\sigma$:

$$\big(\mathrm{ind}_{KZ}^G\sigma\big)|_{KZ} \xrightarrow{\sim} \bigoplus_{n\in\mathbf{N}} R_n.$$

Here, $k[K\lambda_n(\varpi)KZ]$ is the k-linear space on the double coset $K\lambda_n(\varpi)KZ$, endowed with its natural structure of $(k[K], k[KZ])$-bimodule.

The interpretation in terms of the tree of $\mathbf{GL}_2$ is clear: the $k[K]$-module R_n maps isomorphically onto the space of elements of $\mathrm{ind}_{KZ}^G\sigma$ having support on the double coset $K\lambda_n(\varpi)KZ$. In particular, if σ is the trivial weight, a linear basis for R_n is parametrized by the vertices of $\mathscr{T}$ lying at distance n from the central vertex.

The Hecke endomorphism T induces, by transport of structure, a family of K-equivariant morphisms T_n defined on the $k[K]$-modules R_n: $T_n \overset{\text{def}}{=} T|_{R_n}$. From the explicit description of the Hecke operator T one sees (cf. [Mo5], §2.2.1) that $\mathrm{Im}(T_n)$ is a sub-object of $R_{n+1} \oplus R_{n-1}$ so that we can further consider the composition with the canonical the projections

$$T_n^{\pm} : R_n \xrightarrow{T_n} R_{n+1} \oplus R_{n-1} \longrightarrow R_{n\pm1}.$$

It turns out that, for $n \geqslant 1$, the operators $T_n^{\pm}$ are obtained by compact induction (from $K_0(\varpi^n)$ to K) from the following morphisms $t_n^{\pm}$:

$$\begin{aligned} t_n^+ : \sigma^{(n)} &\hookrightarrow \mathrm{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \sigma^{(n+1)} \\ \underline{X}^{\underline{r}-\underline{j}}\underline{Y}^{\underline{j}} &\mapsto \sum_{\lambda_n \in k_F} (-\lambda_n)^{\underline{j}} \begin{bmatrix} 1 & 0 \\ \varpi^n[\lambda_n] & 1 \end{bmatrix} [1, \underline{X}^{\underline{r}}]; \end{aligned}$$

$$\begin{aligned} t_{n+1}^- : \mathrm{ind}_{K_0(\varpi^{n+1})}^{K_0(\varpi^n)} \sigma^{(n+1)} &\twoheadrightarrow \sigma^{(n)} \\ [1, \underline{X}^{\underline{r}-\underline{j}}\underline{Y}^{\underline{j}}] &\mapsto \delta_{\underline{j},\underline{r}} \underline{Y}^{\underline{r}}. \end{aligned}$$

For $n = 0$ we similarly have

$$\begin{aligned} T_0^+ : \sigma^{(0)} &\hookrightarrow R_1 \\ \underline{X}^{\underline{r}-\underline{j}}\underline{Y}^{\underline{j}} &\mapsto \sum_{\lambda_0 \in k_F} (-\lambda_0)^{\underline{r}-\underline{j}} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, \underline{X}^{\underline{r}}] + \delta_{\underline{j},\underline{0}}[1, \underline{X}^{\underline{r}}] \\ T_1^- : R_1 &\twoheadrightarrow \sigma^{(0)} \\ [1, \underline{X}^{\underline{r}-\underline{j}}\underline{Y}^{\underline{j}}] &\mapsto \delta_{\underline{j},\underline{r}} \underline{Y}^{\underline{r}}. \end{aligned}$$

In particular, T_n^+ (resp. T_n^-) are monomorphisms (resp. epimorphisms).

We deduce the following exact sequence of K-representations

$$0 \to \bigoplus_{n\in\mathbf{N}} R_n \xrightarrow{\bigoplus_n T_n} \bigoplus_{n\in\mathbf{N}} R_n \to \pi(\sigma, 0, 1)|_{KZ} \to 0$$

so that, by the exactness of filtered co-limits and the definition of the Hecke operators T_n we obtain

$$\Big(\varinjlim_{n\,\mathrm{odd}} \mathrm{coker}\big(\bigoplus_{j=0}^{\frac{n-1}{2}} T_{2j+1}\big)\Big) \oplus \Big(\varinjlim_{n\,\mathrm{even}} \mathrm{coker}\big(\bigoplus_{j=0}^{\frac{n}{2}} T_{2j}\big)\Big) \cong \pi(\sigma,0,1)|_{KZ} \qquad (5)$$

The representations $\mathrm{coker}\big(\bigoplus_{j=0}^{\frac{n-1}{2}} T_{2j+1}\big)$ can be described in a more expressive way as a suitable push-out of the partial Hecke operators $T_n^{\pm}$. Indeed one verifies that $\mathrm{coker}(T_1) = R_0 \oplus_{R_1} R_2$, where the push out is defined by the following co-cartesian diagram

$$\begin{array}{ccc} R_1 & \overset{T_1^+}{\hookrightarrow} & R_2 \\ {\scriptstyle -T_1^-}\downarrow & & \downarrow{\scriptstyle pr_2} \\ R_0 & \hookrightarrow & R_0 \oplus_{R_1} R_2. \end{array}$$

If we assume we have inductively constructed $pr_{n-1} : R_{n-1} \twoheadrightarrow R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1}$ (where $n \geqslant 3$ is odd), we define the amalgamated sum $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ by the following co-cartesian diagram:

$$\begin{array}{ccc} R_n & \overset{T_n^+}{\hookrightarrow} & R_{n+1} \\ {\scriptstyle -pr_{n-1}\circ T_n^-}\downarrow & & \downarrow{\scriptstyle pr_{n+1}} \\ R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} & \hookrightarrow & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}. \end{array}$$

Using the universal properties of push-outs and cokernels, one obtains a canonical isomorphism of $k[K]$-modules

$$\mathrm{coker}\big(\bigoplus_{j=0}^{\frac{n-1}{2}} T_{2j+1}\big) \cong R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$$

(cf. [Mo5], Proposition 2.8 or [Mo1], Proposition 3.9) together with a commutative diagram with exact lines

$$\begin{array}{ccccccccc} 0 & \to & R_n & \xrightarrow{T_n^+} & R_{n+1} & \to & R_{n+1}/R_n & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} & \xrightarrow{\iota_n} & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} & \to & R_{n+1}/R_n & \to & 0 \end{array} \qquad (6)$$

We construct in a completely analogous fashion the amalgamated sums $(R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$ for $n \in 2\mathbf{N}$, obtaining an isomorphism

$$\mathrm{coker}\big(\bigoplus_{j=0}^{\frac{n}{2}} T_{2j}\big) \cong (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$$

and a similar commutative diagram as in (6).

In order to lighten notations, we put

$$R_{\infty,0} \overset{\mathrm{def}}{=} \varinjlim_{n,\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$$

(where the inductive system is defined by the natural morphisms ι_n appearing in the diagram (6)) and, similarly,

$$R_{\infty,-1} \overset{\text{def}}{=} \varinjlim_{n,\,\text{even}} (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}.$$

If we need to emphasize their dependence on the Serre weight σ we will write $R_{\infty,0}(\sigma)$, $R_{\infty,-1}(\sigma)$.

2.1.1 *Induced representations for* $\mathbf{B}(\mathbf{F}_p)$. In this section we specialize to $k_F = \mathbf{F}_p$ the results of [BP], §2 (see also [BS00]), which describe the structure of a $\mathbf{GL}_2(k_F)$-representation parabolically induced from a character of a Borel subgroup. The results here will be used to complete the computations for the K_t invariant vectors of supersingular representations for $\mathbf{GL}_2(\mathbf{Q}_p)$.

Let $i,j \in \{0,\ldots,p-1\}$ and let us consider the $\mathbf{B}(\mathbf{F}_p)$-character $\chi_i^s\mathfrak{a}^j$. If e is a fixed linear basis for the underlying vector space associated to $\chi_i^s\mathfrak{a}^j$ and if $l \in \mathbf{Z}$, we recall the elements $f_l \overset{\text{def}}{=} f_l(e) \in \mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ defined in §1.1.

The following result clarifies the relation between the elements f_l and the socle filtration for the finite parabolic induction:

PROPOSITION 2.1. *Let* $i,j \in \{0,\ldots,p-1\}$. *Then*

i) for $l \in \{0,\ldots,p-1\}$, f_l *is an* $\mathbf{T}(k_F)$*-eigenvector, whose associated eigencharacter is* $\chi_{i-2j}\det^j\mathfrak{a}^{-l}$, *and the set*

$$\mathscr{B} \overset{\text{def}}{=} \{f_l,\ [1,e]\ 0 \leqslant l \leqslant p-1,\}$$

is an linear basis for $\mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$.

ii) If $i-2j \not\equiv 0\,[p-1]$ *then we have a nontrivial extension*

$$0 \to \mathrm{Sym}^{\lfloor i-2j\rfloor}k^2 \otimes \det^j \to \mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j \to \mathrm{Sym}^{p-1-\lfloor i-2j\rfloor}k^2 \otimes \det^{i-j} \to 0.$$

The families

$$\{f_0,\ldots,f_{\lfloor i-2j\rfloor-1}, f_{\lfloor i-2j\rfloor} + (-1)^{i-j}[1,e]\}, \qquad \{f_{i-2j},\ldots,f_{p-1}\}$$

induce a basis for the socle and the cosocle of $\mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ *respectively.*

iii) If $i-2j \equiv 0\,[p-1]$ *then* $\mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j$ *is semi-simple and*

$$\mathrm{ind}_{\mathbf{B}(\mathbf{F}_p)}^{\mathbf{GL}_2(\mathbf{F}_p)}\chi_i^s\mathfrak{a}^j \xrightarrow{\sim} \left(1 \oplus \mathrm{Sym}^{p-1}k^2\right) \otimes \det^j.$$

The families

$$\{f_0 + (-1)^j[1,e]\}, \qquad \{f_0, f_1, \ldots, f_{p-2}, f_{p-1} + (-1)^j[1,e]\}$$

induce an k*-basis for* $\det^j$ *and* $\mathrm{Sym}^{p-1}k^2 \otimes \det^j$ *respectively.*

Proof. Omissis. Cf. [BP], Lemmas 2.5, 2.6, 2.7. □

We end this section with a technical remark on Witt polynomials, which, combined with Lemma 2.1, enables us to conclude the delicate computations needed to describe the K_t fixed vectors for supersingular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$ (§4.2). We recall ([AC], Chapitre 9, §1, partie 4) that if $\mathbf{F}$ is a finite extension of $\mathbf{F}_p$ we have the following equality in the associated ring of Witt vectors $\mathbf{W}(\mathbf{F})$:

$$[\mu] + [\lambda] \equiv [\lambda+\mu] + p[S_1(\lambda,\mu)] \bmod p^2$$

where $\mu, \lambda \in \mathbf{F}$, $[\cdot] : \mathbf{F} \to \mathbf{W}(\mathbf{F})$ is the usual Teichmüller lift and $S_1 \in \mathbf{Z}[X, Y]$ is an homogeneous polynomial of degree p:

$$S_1(X, Y) = -\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} X^{p-s} Y^s. \tag{7}$$

An immediate manipulation gives

$$S_1(X - Y, Y) = -S_1(X, -Y). \tag{8}$$

3. Structure theorems for universal representations

The aim of this section is to introduce some structure theorems for the universal representation $\pi(\sigma, 0, 1)$ of $\mathbf{GL}_2$. These results concern both the KZ-restriction and the N-restriction of $\pi(\sigma, 0, 1)$ and show that the behavior of universal representations is controlled by a certain, explicit, $k[I]$-module $R^-_{\infty,\bullet}$. If $F = \mathbf{Q}_p$ the Pontryagin dual of such module turns out to be of finite type over a suitable discrete valuation ring: this is the crucial phenomenology which gives us a complete understanding of irreducible admissible representations of $\mathbf{GL}_2(\mathbf{Q}_p)$. If F is a nontrivial finite extension of $\mathbf{Q}_p$ the situation is extremely delicate: the dual of $R^-_{\infty,\bullet}$ is defined over a complete Noetherian regular local ring of Krull dimension $[F : \mathbf{Q}_p]$ *and* is not any longer of finite type (if F is of characteristic p the module is defined over a non-noetherian profinite ring, and still not of finite type, cf. [Mo7]).

We keep the notation of the previous section, in particular σ is a fixed Serre weight. We invite the reader to refer to [Mo5], §3 for the omitted details. We remark that the results of [Mo5], §3 are formal and hold true for *any* local field with finite residual degree (cf. also [Mo5], p.1077).

3.1 Refinement of the Iwahori structure

Let $n \in \mathbf{N}_>$. Restriction of functions from K to $K_0(\varpi)$ gives a $K_0(\varpi)$-equivariant exact sequence

$$0 \to R_n^+ \to R_n \to \mathrm{ind}_{K_0(\varpi^n)}^{K_0(\varpi)}\big(\sigma^{(n)}\big) \to 0$$

which is easily checked to be split, therefore realizing the Mackey decomposition for $R_n|_{K_0(\varpi)}$. We thus define for $n \geqslant 1$

$$R_n^- \overset{\mathrm{def}}{=} \mathrm{ind}_{K_0(\varpi^n)}^{K_0(\varpi)}\big(\sigma^{(n)}\big)$$

and one verifies (cf. [Mo5], §3.1) that the partial Hecke morphisms $T_n^\pm$ give rise to a family of $K_0(\varpi)$-equivariant morphisms

$$\begin{aligned} (T_n^+)^{\mathrm{neg}} &: R_n^- \hookrightarrow R_{n+1}^-, & (T_n^+)^{\mathrm{pos}} &: R_n^+ \hookrightarrow R_{n+1}^+ \\ (T_{n+1}^-)^{\mathrm{neg}} &: R_{n+1}^- \twoheadrightarrow R_n^-, & (T_{n+1}^-)^{\mathrm{pos}} &: R_{n+1}^+ \twoheadrightarrow R_n^+. \end{aligned}$$

For technical reasons we define

$$R_0^+ \overset{\mathrm{def}}{=} R_0|_{K_0(\varpi)}, \qquad R_0^- \overset{\mathrm{def}}{=} \mathrm{cosoc}_{K_0(\varpi)}(R_1^-), \qquad R_{-1}^+ \overset{\mathrm{def}}{=} \mathrm{cosoc}_{K_0(\varpi)}(R_0^+), \qquad R_{-1}^- \overset{\mathrm{def}}{=} 0$$

as well as the operators

$$\begin{aligned} (T_0^+)^{\mathrm{neg}} &: R_0^- \xrightarrow{0} R_1^-, & (T_0^+)^{\mathrm{pos}} &: R_0^+ \hookrightarrow R_1 \twoheadrightarrow R_1^+ \\ (T_1^-)^{\mathrm{neg}} &: R_1^- \twoheadrightarrow R_0^-, & (T_1^-)^{\mathrm{pos}} &= T_1^-|_{R_1^+} : R_1^+ \twoheadrightarrow R_0^+ \\ (T_0^-)^{\mathrm{neg}} &: R_0^- \xrightarrow{0} R_{-1}^-, & (T_0^-)^{\mathrm{pos}} &: R_0^+ \twoheadrightarrow R_{-1}^+. \end{aligned}$$

We leave as an exercise to the reader to check that the morphism $(T_0^+)^{\mathrm{pos}}$ is injective and the amalgamated sum $R_{-1}^+ \oplus_{R_0^+} R_1^+$ with respect to the couple $(-(T_0^-)^{\mathrm{pos}}, (T_0^+)^{\mathrm{pos}})$ is canonically isomorphic to the image of the $K_0(\varpi)$-morphism $R_1^+ \to R_1 \to R_1/R_0$.

Following the procedures of section 2.1 we can construct inductive systems of amalgamated sums via the partial Hecke operators $(T_n^{\pm})^{\text{pos, neg}}$:

$$\left\{ R_\bullet^* \oplus_{R_{\bullet+1}^*} \cdots \oplus_{R_n^*} R_{n+1}^* \right\}_{n \in 2\mathbf{N}+\bullet+1} \tag{9}$$

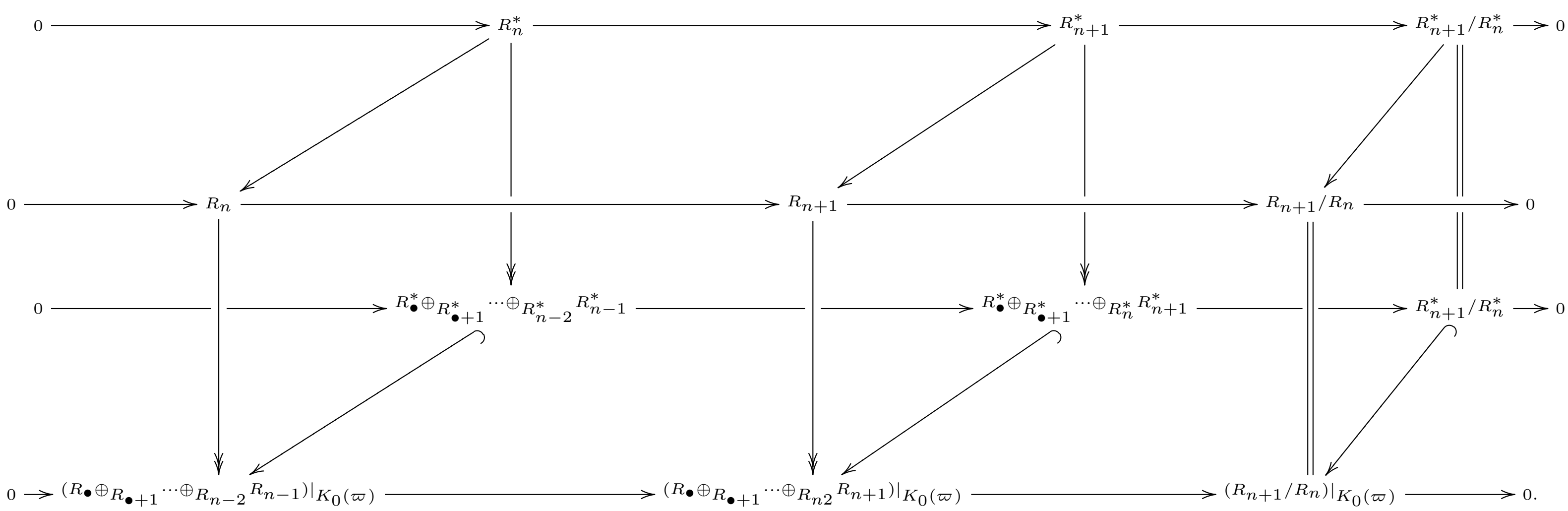
where $\bullet \in \{0, -1\}$, $* \in \{+, -\}$.

For $\bullet \in \{0, -1\}$, $* \in \{+, -\}$ we write

$$R_{\infty,\bullet}^* \overset{\text{def}}{=} \varinjlim_{n \in 2\mathbf{N}+\bullet+1} R_\bullet^* \oplus_{R_{\bullet+1}^*} \cdots \oplus_{R_n^*} R_{n+1}^*.$$

The relation between the amalgamated sums (9) and the ones defined in §2.1 is given by the following

Proposition 3.1. *Let $\bullet \in \{0, -1\}$, $* \in \{+, -\}$ and $n \in 2\mathbf{N} + \bullet + 1$, $n \geqslant 2$. We have a commutative diagram of $K_0(\varpi)$-representations, with exact rows*



Proof. This is proved in [Mo5], in the proof of Proposition 3.5, by induction. □

Remark 3.2. *We write explicitly the morphisms which initialize the inductive argument of Proposition 3.1. Concerning R_0^+, R_1^- we have the evident monomorphisms*

$$R_0^+ \overset{\sim}{\to} R_0; \qquad R_{-1}^- \oplus_{R_0^-} R_1^- = R_1^- \hookrightarrow R_1;$$

concerning R_0^- we have

$$\begin{aligned} R_0^- &\hookrightarrow R_0 \\ e &\mapsto \underline{Y}^r. \end{aligned}$$

Finally, it is easy to verify that the morphism $R_{-1}^+ \oplus_{R_0^+} R_1^+ \hookrightarrow R_1/R_0$ is induced by the couple:

$$R_1^+ \to R_1/R_0; \qquad \begin{aligned} R_{-1}^+ &\hookrightarrow R_1/R_0; \\ e &\mapsto [1, \underline{X}^r]. \end{aligned}$$

It is therefore convenient to write $\underline{Y}^r$, $\underline{X}^r$ for a linear basis for R_0^- and R_{-1}^+ respectively.

Before introducing the first structure theorem for universal representations of $\mathbf{GL}_2$ we need the following

Definition 3.3. *If $\sigma = \sigma_{\underline{r}}$ is a Serre weight as in (3) we write*

$$S(\sigma) \overset{\text{def}}{=} \begin{cases} Soc\Big(\mathrm{ind}_{K_0(\varpi)}^{K}\chi_{\underline{r}}^{s}\Big) \\ 1 \\ \overline{St} \end{cases}, \qquad R(\sigma) \overset{\text{def}}{=} \begin{cases} Rad\Big(\mathrm{ind}_{K_0(\varpi)}^{K}\chi_{\underline{r}}\Big) & \text{if } \dim(\sigma)\notin\{0,q\} \\ \overline{St} & \text{if } \dim(\sigma)=1 \\ 1 & \text{if } \dim(\sigma)=q. \end{cases}$$

The result is then:

Proposition 3.4. *For any $n \in 2\mathbf{N}+1$, $m \in 2\mathbf{N}+2$ we have the following exact sequence of $k[K]$-modules:*

$$0 \to R(\sigma) \to \mathrm{ind}_{K_0(\varpi)}^{K}\big(R_0^- \oplus_{R_1^-} \cdots \oplus_{R_n^-} R_{n+1}^-\big) \to R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \to 0$$
$$0 \to S(\sigma) \to \mathrm{ind}_{K_0(\varpi)}^{K}\big(R_1^- \oplus_{R_2^-} \cdots \oplus_{R_m^-} R_{m+1}^-\big) \to (R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1} \to 0$$

Proof. We start proving the first exact sequence. Recall that R_0^- is isomorphic to the character $\chi_{\underline{r}}$ so that

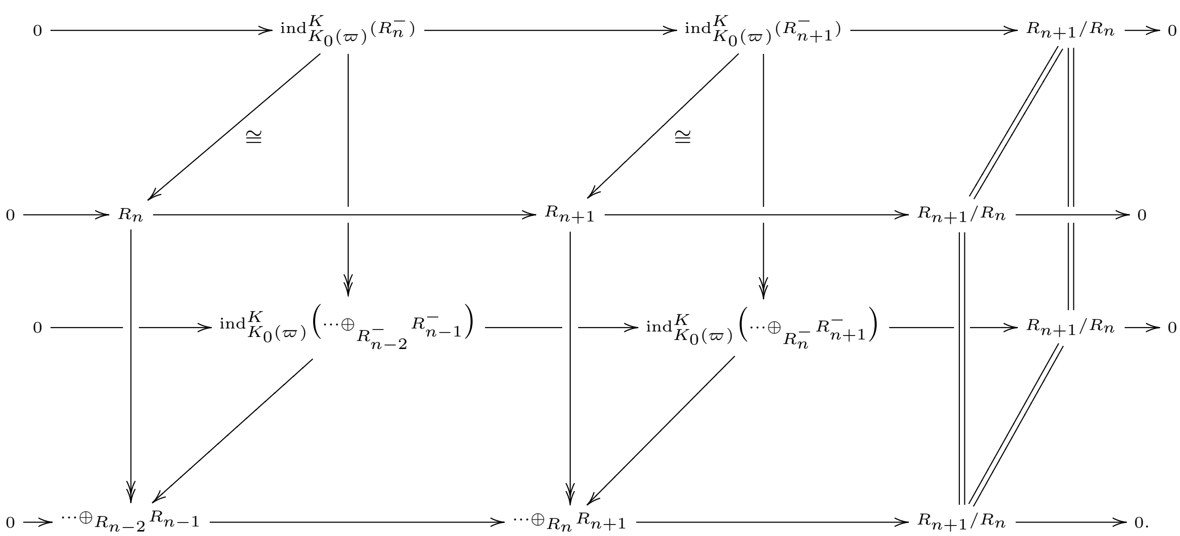
$$0 \to \mathrm{R}(\sigma) \to \mathrm{ind}_{K_0(\varpi)}^{K}\big(R_0^-\big) \to R_0 \to 0$$

is true by definition. By induction we assume the statement holds true for all $-1 \leqslant j \leqslant n-2$, j odd (the case $j=-1$ being the initialization of the inductive argument).

Recall that for all $i \in \mathbf{N}_{>}$ the natural $K_0(\varpi)$-monomorphism $R_i^- \hookrightarrow R_i$ gives rise to a K-isomorphism $\mathrm{ind}_{K_0(\varpi)}^{K} R_i^- \overset{\sim}{\to} R_i$ and, by exactness of induction, we get an exact sequence

$$0 \to \mathrm{ind}_{K_0(\varpi)}^{K}\big(R_n^-\big) \to \mathrm{ind}_{K_0(\varpi)}^{K}\big(R_{n+1}^-\big) \to R_{n+1}/R_n \to 0.$$

We therefore deduce from Proposition 3.1, using Frobenius reciprocity and exactness of induction, the following commutative diagram with exact lines



In particular we deduce

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ind}_{K_0(\varpi)}^{K}\big(\cdots \oplus_{R_{n-2}^-} R_{n-1}^-\big) & \longrightarrow & \mathrm{ind}_{K_0(\varpi)}^{K}\big(\cdots \oplus_{R_n^-} R_{n+1}^-\big) & \longrightarrow & R_{n+1}/R_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \cdots \oplus_{R_{n-2}} R_{n-1} & \longrightarrow & \cdots \oplus_{R_n} R_{n+1} & \longrightarrow & R_{n+1}/R_n & \longrightarrow & 0 \end{array}$$

and the conclusion follows from the Snake lemma and the inductive hypothesis on the morphism $\mathrm{ind}_{K_0(\varpi)}^{K}\big(\cdots \oplus_{R_{n-2}^-} R_{n-1}^-\big) \twoheadrightarrow \cdots \oplus_{R_{n-2}} R_{n-1}$.

The second exact sequence is proved in the evident, similar fashion, noticing that the K-subrepresentation of R_1/R_0 generated by $[1, \underline{X}^{\underline{r}}]$ is isomorphic to $\mathrm{coker}\big(\mathrm{S}(\sigma) \to \mathrm{ind}_{K_0(\varpi)}^{K} \chi_{\underline{r}}^{s}\big)$.

□

As a corollary, we deduce the first structure theorem for the universal representations of $\mathbf{GL}_2$:

COROLLARY 3.5. *We have the following exact sequences of $k[K]$-modules:*

$$0 \to R(\sigma) \to \mathrm{ind}_{K_0(\varpi)}^{K}\big(R_{\infty,0}^{-}\big) \to R_{\infty,0} \to 0$$
$$0 \to S(\sigma) \to \mathrm{ind}_{K_0(\varpi)}^{K}\big(R_{\infty,-1}^{-}\big) \to R_{\infty,-1} \to 0$$

Proof. The functor $\mathrm{ind}_{K_0(\varpi)}^{K}(\_)$ commutes with co-limits, as it is exact and commutes with (arbitrary) co-products. Since filtered co-limits are exact the result follow from Proposition 3.4. □

REMARK 3.6. *We point out that the same argument has been used in [AM] to get the structure theorem for the special case $F = \mathbf{Q}_p$ (*loc. cit.*, Corollaire 3.1).*

We can now introduce the second structure theorem for universal representations of $\mathbf{GL_2}$, describing the action of N, the normalizer of the Iwahori subgroup $K_0(\varpi)$, on $\pi(\sigma, 0, 1)$.

We start recalling the structure theorem for the $K_0(\varpi)$-restriction of $\pi(\sigma, 0, 1)$ (cf. [Mo5], Proposition 3.5).

PROPOSITION 3.7. *We have the following $K_0(\varpi)$-equivariant exact sequences*

$$0 \to W_1 \to R_{\infty,0}^{+} \oplus R_{\infty,0}^{-} \to R_{\infty,0}|_{K_0(\varpi)} \to 0$$
$$0 \to W_2 \to R_{\infty,-1}^{+} \oplus R_{\infty,-1}^{-} \to R_{\infty,-1}|_{K_0(\varpi)} \to 0$$

where W_1, W_2 are the 1-dimensional spaces defined by $W_1 \stackrel{\text{def}}{=} \langle (\underline{Y}^{\underline{r}}, -\underline{Y}^{\underline{r}}) \rangle$ and $W_2 \stackrel{\text{def}}{=} \langle (\underline{X}^{\underline{r}}, -\underline{X}^{\underline{r}}) \rangle$.

Proof. This is [Mo5], Proposition 3.5 (notice that, in the notation of *loc. cit.*, the elements $(-1)^{\underline{r}} F_{\underline{r}}^{(0)}(0)$ and $-[1, \underline{X}^{\underline{r}}]$ of R_1 coincide in the quotient R_1/R_0, by the definition of the operator T_0). □

In order to control the action of the normalizer N we are therefore left to study the action of the element $\alpha \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$. The result is the following:

PROPOSITION 3.8. *There exists two $K_0(\varpi)$-equivariant isomorphisms*

$$\iota_{-1} : R_{\infty,-1}^{-} \xrightarrow{\sim} \big(R_{\infty,0}^{+}\big)^{s}$$
$$\iota_{0} : R_{\infty,0}^{-} \xrightarrow{\sim} \big(R_{\infty,-1}^{+}\big)^{s}$$

such that

i) $\iota_{-1}(\underline{X}^{\underline{r}}) = \underline{Y}^{\underline{r}}$ *and* $\iota_0(\underline{Y}^{\underline{r}}) = \underline{X}^{\underline{r}}$*;*

ii) The isomorphisms ι_{-1}, ι_0 induce a commutative diagram (of k-linear spaces) with exact lines

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_1 \oplus W_2 & \longrightarrow & R_{\infty,0}^{-} \oplus R_{\infty,0}^{+} \oplus R_{\infty,-1}^{-} \oplus R_{\infty,-1}^{+} & \longrightarrow & \pi(\sigma,0,1) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & W_2 \oplus W_1 & \longrightarrow & R_{\infty,-1}^{+} \oplus R_{\infty,-1}^{-} \oplus R_{\infty,0}^{+} \oplus R_{\infty,0}^{-} & \longrightarrow & \pi(\sigma,0,1) & \longrightarrow & 0 \end{array}$$

where the right vertical arrow is the automorphism induced by the action of α on $\pi(\sigma, 0, 1)$.

Proof. We start showing that, for any $n \geqslant -1$, we have a $K_0(\varpi)$-equivariant isomorphism

$$r_n : R^-_{n+1} \xrightarrow{\sim} (R^+_n)^s.$$

The case $n=-1$ is trivial, as the spaces R^-_0, R^+_{-1} are 1-dimensional, affording the characters $\chi_{\underline{r}}$ and $\chi^s_{\underline{r}}$ respectively and we have $R^-_0 = \langle \underline{Y}^r \rangle$, $R^+_{-1} = \langle \underline{X}^r \rangle$ via the equivariant embeddings $R^-_0 \hookrightarrow R_0$, $R^+_{-1} \hookrightarrow (R_1/R_0)^+$ respectively (cf. Remark 3.2).

Assume now $n \geqslant 0$. Recall that for any $j \geqslant 0$ we have a K-equivariant isomorphism (cf. (4)):

$$\begin{aligned} \mathrm{ind}^K_{K_0(\varpi^j)}\big(\sigma^{(j)}\big) &\xrightarrow{\sim} k[K\lambda_j(\varpi)KZ] \otimes_{k[KZ]} \sigma \\ [1,v] &\mapsto \begin{bmatrix} 0 & 1 \\ \varpi^j & 0 \end{bmatrix} \otimes v. \end{aligned}$$

We deduce the following k-linear morphism:

$$R^-_{n+1} \hookrightarrow R_{n+1} \xrightarrow{\sim} k[K\lambda_{n+1}(\varpi)KZ] \otimes_{k[KZ]} \sigma \xrightarrow{\sim} k[K\lambda_n(\varpi)KZ] \otimes_{k[KZ]} \sigma \xrightarrow{\sim} R_n \twoheadrightarrow R^+_n$$

where the central arrow is induced by the action of α on the compact induction $(\mathrm{ind}^G_{KZ}\sigma)|_{KZ}$. As α normalizes $K_0(\varpi)$ we deduce that the composite arrow $r_n : R^-_{n+1} \to (R^+_n)^s$ is $K_0(\varpi)$-equivariant and an easy check shows that r_n is an epimorphism, hence an isomorphism by dimension reasons. As the Hecke operator T is equivariant, we deduce furthermore that the diagram

$$\begin{array}{ccc} R^-_{n+1} & \xrightarrow[\sim]{r_n} & (R^+_n)^s \\ {\scriptstyle (T^-_{n+1})^{\mathrm{neg}}}\downarrow & & \downarrow{\scriptstyle (T^-_n)^{\mathrm{pos}}} \\ R^-_n & \xrightarrow[\sim]{r_{n-1}} & (R^+_{n-1})^s \end{array} \tag{10}$$

commutes for all $n \geqslant 1$.

The diagram commutes also for $n=0$ and we have $r_0(\underline{X}^r) = \underline{Y}^r$ (by $K_0(\varpi)$-equivariance r_0 induces an isomorphism between the highest weight spaces of the representations R^-_1, R^+_0).

The Proposition will be completely proved once we show that for any $n \geqslant -1$ we have a $K_0(\varpi)$-equivariant isomorphism $f_n : \cdots \oplus_{R^-_n} R^-_{n+1} \to (\cdots \oplus_{R^+_{n-1}} R^+_n)^s$ which verifies the prescribed conditions on the images of $\underline{X}^r$, $\underline{Y}^r$ (for n even, odd respectively).

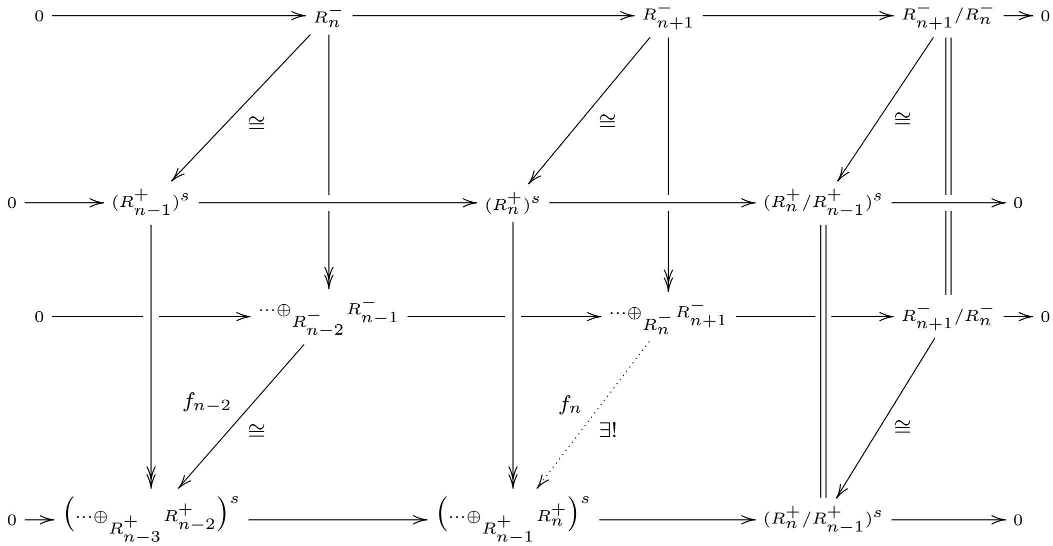
We treat the case when n is even, the other being symmetric. It is an induction on n where the case $n=0$ is given by $r_0 : R^-_1 \xrightarrow{\sim} (R^+_0)^s$.

Assume $n \geqslant 2$ and that we have an isomorphism f_{n-2} making the following diagram commute:

$$\begin{array}{ccc} R^-_n & \twoheadrightarrow & R^-_1 \oplus_{R^-_2} \cdots \oplus_{R^-_{n-2}} R^-_{n-1} \\ \downarrow\wr & & {\scriptstyle f_{n-2}}\downarrow\wr \\ (R^+_{n-1})^s & \twoheadrightarrow & (R^+_0 \oplus_{R^+_1} \cdots \oplus_{R^+_{n-3}} R^+_{n-2})^s \end{array}$$

(with the prescribed property on the image of the element $\underline{X}^r \in R^-_1$).

Using (10) we deduce the commutative diagram with exact lines



where the morphism f_n is obtained from the universal property of $\cdots \oplus_{R_n^-} R_{n+1}^-$ (notice also that $\left(\cdots \oplus_{R_{n-1}^+} R_n^+\right)^s = \cdots \oplus_{(R_{n-1}^+)^s} (R_n^+)^s$). The morphism f_n is moreover an isomorphism, verifying the prescribed property on the image of the element $\underline{X}^r \in R_1^-$.

This completes the inductive step and, passing to co-limits, one gets the isomorphism ι_{-1} as in the statement. □

3.2 The case $F = \mathbf{Q}_p$

We specialize some of the previous constructions to the case $F = \mathbf{Q}_p$. By Corollary 3.5, Proposition 3.7 and Proposition 3.8 we see that the structure of the universal representation $\pi(\sigma, 0, 1)$ depends crucially on the modules $R_{\infty,\bullet}^-$, where $\bullet \in \{-1, 0\}$.

The dual of $R_{\infty,\bullet}^-$ is a module on the Iwasawa algebra $k[[\mathscr{O}_F]]$, and is *not* of finite type as soon as $F \neq \mathbf{Q}_p$. Moreover $k[[\mathscr{O}_F]]$ is a complete regular noetherian local ring of dimension $[F : \mathbf{Q}_p]$ if F is a finite extension of $\mathbf{Q}_p$ and is not even noetherian if $\mathrm{char}(F) = p$ (see also [Mo6], [Mo7]).

When $F = \mathbf{Q}_p$ the situation is much simpler: $R_{\infty,\bullet}^-$ is the dual of a monogenous module over a discrete valuation ring (the Iwasawa algebra of $\mathbf{Z}_p$).

We start recalling the following result:

Proposition 3.9. *Let $n \geqslant 0$. The $k[K_0(p)]$-module R_{n+1}^- is uniserial, of dimension $(r+1)p^n$, and its socle filtration is described by*

$$\chi_{\underline{r}}^s — \chi_{\underline{r}}^s \mathfrak{a} — \chi_{\underline{r}}^s \mathfrak{a}^2 — \ldots — \chi_{\underline{r}}^s \mathfrak{a}^{(r+1)p^n - 1}.$$

Proof. This is deduced from [Mo1], Proposition 5.10 and the fact that we have a $K_0(p^{n+1})$-equivariant embedding $\sigma^{(n+1)} \hookrightarrow \mathrm{ind}_{K_0(p^{n+2})}^{K_0(p^{n+1})} \chi_r^s$; moreover, for $p \geqslant 5$ it can equally be seen as a particular case of of [Mo5], Proposition 4.10.

Alternatively, the statement is a consequence of [Pas2], Propositions 4.7 and 5.9. □

The statement of Proposition 3.9 can be made more expressive.

We recall from [Mo5] that for a finite unramified extension $F/\mathbf{Q}_p$ the $k[K_0(p)]$-module R_{n+1}^- admits a linear basis $\mathscr{B}_{n+1}^-$ which is endowed with a partial order (cf. *op. cit.*, Lemma 2.6). The partial ordering on $\mathscr{B}_{n+1}^-$ induces therefore a k-linear filtration on the space R_{n+1}^- and one of the main result of [Mo5] (cf. *op. cit.*, Proposition 4.10) is to show that such filtration is $K_0(p)$-stable.

When $F = \mathbf{Q}_p$ such ordering is indeed *total* (this is linked with the aforementioned phenomenon that we are considering modules over a complete local noetherian regular ring of Krull dimension $[F:\mathbf{Q}_p]$).

Explicitly, we have a bijection

$$\begin{aligned} \{0,\dots,p-1\}^n \times \{0,\dots,r\} &\xrightarrow{\sim} \mathscr{B}^-_{n+1} \\ (l_1,\dots,l_{n+1}) &\longmapsto F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1}) \end{aligned} \tag{11}$$

where we define the element

$$F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1}) \overset{\text{def}}{=} \sum_{\lambda_1\in\mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots \sum_{\lambda_n\in\mathbf{F}_p} \lambda_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\lambda_n] & 1 \end{bmatrix} [1, X^{r-l_{n+1}}Y^{l_{n+1}}] \in R^-_{n+1}.$$

The total ordering on $\mathscr{B}^-_{n+1}$ is then induced from the order of $\mathbf{N}$ via the *injective* map

$$\begin{aligned} \mathscr{B}^-_{n+1} &\overset{P}{\hookrightarrow} \mathbf{N} \\ F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1}) &\mapsto P(F^{(1,n)}_{(l_1,\dots,l_n)}(l_{n+1})) \overset{\text{def}}{=} \sum_{j=0}^{n} p^j l_{j+1} \end{aligned}$$

(and thus coincides with the anti-lexicographical order $\prec$ on the LHS of (11)). If $F_1, F_2 \in \mathscr{B}^-_{n+1}$, we write $F_1 \prec F_2$ if $P(F_1) < P(F_2)$.

Since R^-_{n+1} is uniserial, it is easy to describe the amalgamated sum $\cdots \oplus_{R^-_n} R^-_{n+1}$:

Proposition 3.10. *Let $n \geqslant 1$. The kernel of the projection map $R^-_{n+1} \twoheadrightarrow \cdots \oplus_{R^-_n} R^-_{n+1}$ is described by:*

$$\mathit{ker}\big(R^-_{n+1} \twoheadrightarrow \cdots \oplus_{R^-_n} R^-_{n+1}\big) = \begin{cases} \langle F \in \mathscr{B}^-_{n+1},\ F \prec F^{(1,n)}_{r,p-1-r,\dots,p-1-r,r}(0)\rangle & \text{if } n \in 2\mathbf{N}+1 \\ \\ \langle F \in \mathscr{B}^-_{n+1},\ F \prec F^{(1,n)}_{p-1-r,r,,\dots,p-1-r,r}(0)\rangle & \text{if } n \in 2\mathbf{N}+2. \end{cases}$$

Proof. We consider the case where n is odd (the other is similar). Since R^-_{n+1} is uniserial and the linear filtration on R^-_{n+1} induced by the linear order on $\mathscr{B}^-_{n+1}$ coincides with the socle filtration, it will be enough to show that

$$\dim\big(\langle F \in \mathscr{B}^-_{n+1},\ F \prec F^{(1,n)}_{r,p-1-r,\dots,p-1-r,r}(0)\rangle\big) = \dim(R^-_{n+1}) - \dim(\cdots \oplus_{R^-_n} R^-_{n+1}).$$

This is a straightforward check: indeed

$$\begin{aligned} \dim\big(\langle F \in \mathscr{B}^-_{n+1},\ F \prec F^{(1,n)}_{r,p-1-r,\dots,p-1-r,r}(0)\rangle\big) &= r(\sum_{j=0}^{\frac{n-1}{2}} p^{2j}) + p(p-1-r)(\sum_{j=0}^{\frac{n-3}{2}} p^{2j}) \\ = (p-r)\frac{p^{n-1}-1}{p+1} + rp^{n-1} \end{aligned}$$

and

$$\begin{aligned} \dim(\cdots \oplus_{R^-_n} R^-_{n+1}) &= 1 + (r+1)(-1)\sum_{j=0}^{n}(-p)^j = (r+1)\frac{p^{n+1}-1}{p+1} + 1; \\ \dim(R^-_{n+1}) &= p^n(r+1). \end{aligned}$$

□

As $R^-_{\infty,\bullet}$ is a co-limit of the modules $\cdots \oplus_{R^-_n} R^-_{n+1}$, the transition maps being monomorphisms, we deduce

COROLLARY 3.11. *For $\bullet \in \{0,-1\}$ the $k[K_0(p)]$-module $R^-_{\infty,\bullet}$ is uniserial. Its socle filtration is described by*

$$(R^-_{\infty,0}): \qquad \chi_r^s\mathfrak{a}^r—\chi_r^s\mathfrak{a}^{r+1}—\chi_r^s\mathfrak{a}^{r+2}—\dots$$
$$(R^-_{\infty,-1}): \qquad \chi_r^s—\chi_r^s\mathfrak{a}—\chi_r^s\mathfrak{a}^2—\dots$$

respectively.

We write $\{\mathscr{F}_n\}_{n\in\mathbf{N}}$ for the socle filtration for $R^-_{\infty,0}$ (in particular, $\mathscr{F}_0$ is the socle of $R^-_{\infty,0}$ and $\mathscr{F}_n$ is *the* $n+1$-dimensional sub-module of $R^-_{\infty,0}$).

REMARK 3.12. *It is easy to see that for any $n \geqslant 0$ the modules R^-_{n+1} (and hence the modules $R^-_{\infty,\bullet}$) are uniserial even when restricted to the subgroup $\overline{\mathbf{U}}(p\mathbf{Z}_p)$. This follows again from [Mo1], Proposition 5.10 and can equally be deduced from the results of Paskunas, [Pas2], Proposition 4.7 and 5.9.*

4. Study of K_t and I_t invariants

In this section we assume $F = \mathbf{Q}_p$. The aim is to describe in detail the K_t and I_t invariants for supersingular representations $\pi(\sigma,0,1)$ of $\mathbf{GL}_2(\mathbf{Q}_p)$.

Thanks to the structure theorems of §3 we are essentially left to understand the invariants for the Iwasawa modules $R^-_{\infty,\bullet}$. This is developed in section 4.1: the argument follows easily from the uniserial property of $R^-_{\infty,\bullet}$, but one should carefully carry out computations in order to handle some delicate K-extensions which will appear later on in section 4.2.

The invariants for the supersingular representation $\pi(\sigma,0,1)$ will be determined in §4.2, combining the results on $R^-_{\infty,\bullet}$ with the structure theorems.

4.1 Invariants for the Iwasawa modules $R^-_{\infty,\bullet}$

We are going to describe in detail the spaces of K_t invariants (resp. I_t invariants) for the $k[K_0(p)]$-modules $R^-_{\infty,\bullet}$ (resp. $R^*_{\infty,\bullet}$).

4.1.1 *Intertwinings between the modules $R_{\infty,\bullet}$.* Recall (§1.1) that for a Serre weight σ we write $\sigma^{[s]}$ for its conjugate weight. We start from the following

PROPOSITION 4.1. *The intertwining operator $\pi(\sigma,0,1) \xrightarrow{\sim} \pi(\sigma^{[s]},0,1)$ induces a KZ-isomorphism*

$$R_{\infty,0}(\sigma) \xrightarrow{\sim} R_{\infty,-1}(\sigma^{[s]}). \tag{12}$$

Proof. We have a KZ-equivariant monomorphism

$$R_{\infty,0}(\sigma) \hookrightarrow \pi(\sigma,0,1)|_{KZ} \xrightarrow{\sim} \pi(\sigma^{[s]},0,1)|_{KZ} \xrightarrow{\sim} R_{\infty,0}(\sigma^{[s]}) \oplus R_{\infty,-1}(\sigma^{[s]}).$$

As $R_{\infty,0}(\sigma)$ and $R_{\infty,0}(\sigma^{[s]})$ have irreducible non isomorphic socles, we deduce that the composite

$$\phi_1 : R_{\infty,0}(\sigma) \hookrightarrow \pi(\sigma,0,1)|_{KZ} \xrightarrow{\sim} \pi(\sigma^{[s]},0,1)|_{KZ} \twoheadrightarrow R_{\infty,-1}(\sigma^{[s]})$$

is a KZ-equivariant monomorphism. Similarly, the composite

$$\phi_2 : R_{\infty,-1}(\sigma) \hookrightarrow \pi(\sigma,0,1)|_{KZ} \xrightarrow{\sim} \pi(\sigma^{[s]},0,1)|_{KZ} \twoheadrightarrow R_{\infty,0}(\sigma^{[s]})$$

is a KZ-equivariant monomorphism. As $\phi_1 \oplus \phi_2$ coincides (by construction) with the intertwining operator $\pi(\sigma,0,1) \xrightarrow{\sim} \pi(\sigma^{[s]},0,1)$ via the isomorphism (5) we deduce that ϕ_1, ϕ_2 are epimorphisms and the proof is complete. □

If $p \geqslant 5$ the statement of Proposition 4.1 can be sharpened, giving an isomorphism between the positive and negative parts of $R_{\infty,0}(\sigma)$, $R_{\infty,-1}(\sigma^{[s]})$:

Proposition 4.2. *Assume $p \geqslant 5$. Then the isomorphism (12) induces the $K_0(p)$-equivariant isomorphisms*

$$R^{-}_{\infty,0}(\sigma) \xrightarrow{\sim} R^{-}_{\infty,-1}(\sigma^{[s]}), \qquad\qquad R^{+}_{\infty,0}(\sigma) \xrightarrow{\sim} R^{+}_{\infty,-1}(\sigma^{[s]}).$$

Proof. We let e_0 (resp. e_0^+) be a linear generator for the space $\mathrm{soc}\big(R^{-}_{\infty,0}(\sigma)\big)$ (resp. for the space $\mathrm{soc}\big(R^{+}_{\infty,0}(\sigma)\big)$). Similarly we define the elements $e_{[s]}$, $e^+_{[s]}$.

By Proposition 3.7 we can write the following equivariant exact sequences

$$0 \to \langle (e_0^+, e_0)\rangle \to R^{+}_{\infty,0}(\sigma) \oplus R^{-}_{\infty,0}(\sigma) \to R_{\infty,0}(\sigma)|_{K_0(p)} \to 0$$
$$0 \to \langle (e^+_{[s]}, e_{[s]})\rangle \to R^{+}_{\infty,-1}(\sigma^{[s]}) \oplus R^{-}_{\infty,-1}(\sigma^{[s]}) \to R_{\infty,-1}(\sigma^{[s]})|_{K_0(p)} \to 0$$

(up to replace the elements e_0, $e_{[s]}$ by suitable nonzero scalar multiples), hence obtaining the induced isomorphisms

$$R^{-}_{\infty,0}(\sigma)/\langle e_0\rangle \oplus R^{+}_{\infty,0}(\sigma)/\langle e_0^+\rangle \xrightarrow{\sim} R_{\infty,0}(\sigma)/\langle \overline{e}\rangle$$
$$R^{-}_{\infty,-1}(\sigma^{[s]})/\langle e_{[s]}\rangle \oplus R^{+}_{\infty,-1}(\sigma^{[s]})/\langle e^+_{[s]}\rangle \xrightarrow{\sim} R_{\infty,-1}(\sigma^{[s]})/\langle \overline{e}^{[s]}\rangle$$

where $\overline{e}$ is a linear basis for the image of the subspace $\langle (0,e_0), (e_0^+,0)\rangle \leqslant R^{+}_{\infty,0}(\sigma) \oplus R^{-}_{\infty,0}(\sigma)$ in $R_{\infty,0}(\sigma)$ and $\overline{e}^{[s]}$ is defined in the evident, analogous way.

By Corollary 3.11 we note that the $K_0(p)$-socle of $R_{\infty,-1}(\sigma^{[s]})/\langle \overline{e}^{[s]}\rangle$ is described by

$$\mathrm{soc}\big(R_{\infty,-1}(\sigma^{[s]})/\langle \overline{e}^{[s]}\rangle\big) = \chi_r\mathfrak{a} \oplus \chi_r\mathfrak{a}^{-1}$$

which is multiplicity free if $p \geqslant 5$.

As $\overline{e}$, $\overline{e}^{[s]}$ are fixed under the action of the pro-p Sylow of $K_0(p)$, we deduce from Lemma 4.3 that the isomorphism (12) induces a $K_0(p)$-equivariant isomorphism

$$R_{\infty,0}(\sigma)/\langle \overline{e}\rangle \xrightarrow{\sim} R_{\infty,-1}(\sigma^{[s]})/\langle \overline{e}^{[s]}\rangle.$$

Since the representations $R^{\pm}_{\infty,0}(\sigma)$, $R^{\pm}_{\infty,-1}(\sigma^{[s]})$ are uniserial and $\mathrm{soc}\big(R_{\infty,-1}(\sigma^{[s]})/\langle \overline{e}^{[s]}\rangle\big)$ is multiplicity free, one deduces the isomorphisms

$$R^{-}_{\infty,0}(\sigma)/\langle e_0\rangle \xrightarrow{\sim} R^{-}_{\infty,-1}(\sigma^{[s]})/\langle e_{[s]}\rangle, \qquad\qquad R^{+}_{\infty,0}(\sigma)/\langle e_0^+\rangle \xrightarrow{\sim} R^{+}_{\infty,-1}(\sigma^{[s]})/\langle e^+_{[s]}\rangle.$$

The statement follows. □

The following result is well known (it is an immediate consequence of [Bre03a], Théorème 3.2.4 and Corollaire 4.1.4), but we decided to give here a self contained argument:

Lemma 4.3. *In the hypotheses of Proposition 4.2 we have $\dim\big(R_{\infty,0}(\sigma)\big)^{K_1(p)} = 1$, where $K_1(p)$ is the pro-p Sylow of $K_0(p)$.*

Proof. We use the notations appearing in the proof of Proposition 4.2.

Define $Z_1 \stackrel{\mathrm{def}}{=} K_1(p) \cap Z$. Then Z_1 acts trivially on $R_{\infty,0}(\sigma)$ and the exact sequence

$$0 \to \langle \overline{e}_0\rangle \to R_{\infty,0}(\sigma) \to R^{-}_{\infty,0}(\sigma)/\langle e_0\rangle \oplus R^{+}_{\infty,0}(\sigma)/\langle e_0^+\rangle \to 0$$

yields the exact sequence of cohomology

$$0 \to \langle \overline{e}_0\rangle \to \big(R_{\infty,0}(\sigma)\big)^{K_1(p)/Z_1} \to \big(R^{-}_{\infty,0}(\sigma)/\langle e_0\rangle\big)^{K_1(p)/Z_1} \oplus \big(R^{+}_{\infty,0}(\sigma)/\langle e_0^+\rangle\big)^{K_1(p)/Z_1} \to$$
$$\to H^1(K_1(p)/Z_1, \langle \overline{e}_0\rangle).$$

Recall that, as $K_1(p)/Z_1$ acts trivially on $\langle \overline{e}_0\rangle$, the space $H^1(K_1(p)/Z_1, \langle \overline{e}_0\rangle)$ is naturally identified with the space of continuous group homomorphisms $\mathrm{Hom}(K_1(p)/Z_1, k)$.

By Corollary 3.11 and since the space $\mathrm{Ext}^1_{K_0(p)/Z_1}(\chi_r^s\mathfrak{a}^{r+1}, \chi_r^s\mathfrak{a}^r)$ is one dimensional ([Pas2], Proposition 5.4), one checks that the image of the composite map

$$\big(R^-_{\infty,0}(\sigma)/\langle e_0\rangle\big)^{K_1(p)/Z_1} \hookrightarrow \big(R^-_{\infty,0}(\sigma)/\langle e_0\rangle\big)^{K_1(p)/Z_1} \oplus \big(R^+_{\infty,0}(\sigma)/\langle e_0^+\rangle\big)^{K_1(p)/Z_1} \to \mathrm{Hom}(K_1(p)/Z_1, k)$$

coincides with the linear subspace generated by morphism

$$\begin{aligned} K_0(p)/Z_1 &\to k \\ \begin{bmatrix} a & b \\ pc & d \end{bmatrix} &\mapsto \overline{c}. \end{aligned}$$

Similarly, the image of the subspace $\big(R^+_{\infty,0}(\sigma)/\langle e_0^+\rangle\big)^{K_1(p)/Z_1}$ coincides with the linear subspace generated by morphism

$$\begin{aligned} K_0(p)/Z_1 &\to k \\ \begin{bmatrix} a & b \\ pc & d \end{bmatrix} &\mapsto \overline{b}. \end{aligned}$$

From [Pas2], Proposition 5.2 we deduce that the connection homomorphism is surjective, hence an isomorphism as the $K_0(p)$-representations $R^-_{\infty,0}(\sigma)$, $R^+_{\infty,0}(\sigma)$ are uniserial.

The conclusion follows. ∎

By virtue of Proposition 4.1 (resp. Proposition 3.8) it will be enough to study the K_t invariants (resp. I_t invariants) for the Iwasawa module $R^-_{\infty,0}$ (resp. $R^-_{\infty,0}$ and $R^-_{\infty,-1}$).

Recall that $R^-_{\infty,0}$ is unserial, and we denoted by $\{\mathscr{F}_n\}_{n\in\mathbf{N}}$ its socle filtration (cf. Corollary 3.11). The K_t invariants of $R^-_{\infty,0}$ are then described by the following

Proposition 4.4. *Let $t \geqslant 1$. We have a $K_0(p)$-equivariant exact sequence*

$$0 \to \big(R^-_{\infty,0}\big)^{K_t} \to \mathscr{F}_{p^{t-1}} \to \chi_r^s\mathfrak{a}^{r+1} \to 0.$$

Moreover, for any lift $e_1 \in \mathscr{F}_{p^{t-1}}$ of a linear basis of $\chi_r^s\mathfrak{a}^{r+1}$ we have

$$\left(\begin{bmatrix} 1+p^ta & p^tb \\ p^tc & 1+p^td \end{bmatrix} - 1\right)\cdot e_1 = \overline{c}\kappa_{e_1}e_0 \tag{13}$$

where $a, b, c, d \in \mathbf{Z}_p$, $\kappa_{e_1} \in k^\times$ is a suitable nonzero scalar depending only on e_1 and e_0 is a linear generator of $\mathrm{soc}\big(R^-_{\infty,0}\big)$.

Proof. As $R^-_{\infty,0}$ is admissible uniserial and K_t is normal in $K_0(p)$ we deduce that $\big(R^-_{\infty,0}\big)^{K_t} = \mathscr{F}_{n(t)}$ where $n(t) \in \mathbf{N}$ is defined by

$$n(t) = \max\Big\{n \in \mathbf{N}, \quad \text{s.t. } \mathscr{F}_n = \big(\mathscr{F}_n\big)^{K_t}\Big\}$$

and hence we are left to prove that $n(t) = p^{t-1} - 1$ (an elementary computation shows that the graded piece $\mathscr{F}_{p^{t-1}}/\mathscr{F}_{p^{t-1}-1}$ affords the character $\chi_r^s\mathfrak{a}^{r+1}$).

This will be a careful computation, using the properties of Witt polynomials. We remark that the cases where $r \in \{0, p-1\}$ are slightly more delicate to verify.

We start from the following

Lemma 4.5. *Let $K_1(p^{t+2})$ be the maximal pro-p subgroup of $K_0(p^{t+2})$. Let $z \stackrel{\text{def}}{=} \sum_{j=1}^{t} p^j[\lambda_j] \in \mathbf{Z}_p$ and $a,b,c,d \in \mathbf{Z}_p$. Then we have*

$$\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} \kappa'$$

for a suitable element $\kappa' \in K_1(p^{t+2})$ and

$$z' = \sum_{j=1}^{t-1} p^j[\lambda_j] + p^t[\lambda_t + \overline{c}] + p^{t+1}[S_1(\lambda_t, \overline{c}) + r(\lambda_1)]$$

where $S_1(\lambda_t, \overline{c})$ is the specialization of the Witt polynomial (7) and $r \in \mathbf{F}_p[\lambda_1]$ is a linear polynomial in λ_1 depending on a,b,c,d.

Proof. We have

$$\begin{bmatrix} 1 & 0 \\ -z' & 1 \end{bmatrix} \begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1+p^t(a+bz) & p^t b \\ w & 1+p^t(d-bz') \end{bmatrix}$$

where $w \stackrel{\text{def}}{=} -z'(1+p^t(a+bz)) + z + p^t(c+dz)$. Thus $z' \equiv (z+p^t c+p^t dz)(1+p^t a)^{-1} \bmod p^{t+2}$ (notice that $p^t bzz' \equiv 0 \bmod p^{t+2}$) and we deduce

$$\begin{aligned} z' &\equiv (z+p^t c+p^t dz)(1-p^t a) \bmod p^{t+2} \\ &\equiv z+p^t c+p^t dz - p^t az - p^{2t}ac \bmod p^{t+2} \end{aligned}$$

(the first line is deduced noticing that $(z+p^t c+p^t dz)p^{2t} \equiv 0 \bmod p^{t+2}$ and the second noticing that $p^{2t}z \equiv 0 \bmod p^{t+2}$).

The result follows from an immediate computation on Witt vectors. □

In order to complete the proof of the Proposition we now distinguish two cases.

Case A: t is odd.

It suffices to show that $(R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-)^{K_t}$ is a proper sub-$k[K_0(p)]$-module of dimension p^{t-1} sitting inside $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$ (notice that $\dim(R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-) \geqslant p^{t-1}+1$ is verified for all values of $t \geqslant 1$, $p \geqslant 3$ and $r \in \{0,\dots,p-1\}$).

We recall that, for a t-tuple $(l_1,\dots,l_t) \in \{0,\dots,p-1\}^t$, we have

$$F^{(1,t)}_{l_1,\dots,l_t}(0) \equiv \begin{cases} 0 & \text{if } (l_1,\dots,l_t) \prec (r,p-1-r,\dots,p-1-r,r) \\ F^{(1,t)}_{r,p-1-r,\dots,p-1-r,r}(0) & \text{if } (l_1,\dots,l_t) = (r,p-1-r,\dots,p-1-r,r) \end{cases} \tag{14}$$

in $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$, by Proposition 3.10.

We again have to distinguish two situations

Sub-case A1: $r < p-1$.

By the unseriality of $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$ and the compatibility between the $K_0(p)$-action and the linear ordering on $\mathscr{B}^-_{n+1}$ we deduce that *the* $(p^{t-1}+1)$-dimensional sub-module of $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$ is generated by the element $F^{(1,t)}_{r,p-1-r,\dots,p-1-r,r+1}(0) \in \mathscr{B}^-_{t+1}$

If $(l_1,\dots,l_t) \preceq (r,p-1-r,r,\dots,p-1-r,r+1)$ is a t-tuple we deduce, from Lemma 4.5 and

(14), the following equality in $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$:

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F^{(1,t)}_{l_1,\dots,l_t}(0) =$$
$$= \sum_{j=1}^{l_t} \binom{l_t}{j} (-\bar{c})^j F^{(1,t)}_{l_1,\dots,l_{t-1},l_t-j}(0) \equiv -l_t \bar{c}^j F^{(1,t)}_{l_1,\dots,l_{t-1},l_t-1}(0) \qquad \text{(as } l_t \leqslant r+1\text{)}$$
$$\equiv \begin{cases} 0 & \text{if } (l_1,\dots,l_t) \prec (r,p-1-r,r,\dots,p-1-r,r+1) \\ -(r+1)\bar{c}F^{(1,t)}_{r,p-1-r,\dots,p-1-r,r}(0) & \text{if } (l_1,\dots,l_t) = (r,p-1-r,r,\dots,p-1-r,r+1) \end{cases}$$

This proves the Proposition for t odd and $r < p-1$.

Sub-case A2: $r = p-1$.

This situation is slightly more delicate and we need to know the properties of the homogeneous degree of the Witt polynomial $S_1(X,Y)$ defined in (7).

As in case $A1$ we see that the $(p^{t-1}+1)$-dimensional sub-module of $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_t^-} R_{t+1}^-$ is generated by the element $F^{(1,t)}_{r,p-1-r,\dots,p-1-r,0}(1) \in \mathscr{B}^-_{t+1}$.

We now have, for a $(t-1)$-tuple $(l_1,\dots,l_{t-1})$

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F^{(1,t)}_{l_1,\dots,l_{t-1},0}(1) = \sum_{\lambda_1 \in \mathbf{F}_p} \lambda_1^{l_1} \begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix} \cdots$$
$$\cdots \sum_{\lambda_{t-1} \in \mathbf{F}_p} \lambda_{t-1}^{l_{t-1}} \begin{bmatrix} 1 & 0 \\ p^{t-1}[\lambda_{t-1}] & 1 \end{bmatrix} \sum_{\lambda_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\lambda_t + \bar{c}] & 1 \end{bmatrix} \big(S_1(\lambda_t,\bar{c}) + r(\lambda_1)\big)[1,X^r]$$

where $S_1(\lambda_1,\bar{c}) + r(\lambda_1)$ is defined as in Lemma 4.5. Thanks to (8) we can write

$$\sum_{\lambda_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\lambda_t + \bar{c}] & 1 \end{bmatrix} \big(S_1(\lambda_t,\bar{c}) + r\big)[1,X^r] = \sum_{\lambda_t \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^t[\lambda_t] & 1 \end{bmatrix} \big(-S_1(\lambda_t,-\bar{c}) + r'\big)[1,X^r]$$

where $r' \in \mathbf{F}_p[\lambda_1]$ is a convenient polynomial of degree 1 in λ_1 (depending on a,b,c,d). In particular, $-S_1(\lambda_t,-\bar{c}) + r' = (-\bar{c})\lambda_t^{p-1} + P(\lambda_t)$ for a convenient polynomial of degree $p-2$ in λ_t and hence, by (14),

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1 \right) \cdot F^{(1,t)}_{l_1,\dots,l_{t-1},0}(1) \equiv (-\bar{c}) F^{(1,t)}_{l_1,\dots,l_{t-1},p-1}(0).$$

Again, we have

$$F^{(1,t)}_{l_1,\dots,l_{t-1},p-1}(0) \equiv \begin{cases} 0 & \text{if } (l_1,\dots,l_{t-1}) \prec (r,p-1-r,\dots,p-1-r) \\ -(r+1)\bar{c}F^{(1,t)}_{r,p-1-r,\dots,p-1-r,r}(0) & \text{if } (l_1,\dots,l_{t-1}) = (r,p-1-r,\dots,p-1-r). \end{cases}$$

This let us conclude the case $r = p-1$ (the K_t invariance of the elements $F^{(1,t)}_{l_1,\dots,l_t}(0)$ is clear).

Case B: t is even.

The argument are completely analogous to those of Case A and the details are left to the reader. We distinguish again two situations.

Sub-case $B1$: $r > 0$.

We now consider the element $F^{(1,t-1)}_{r,p-1-r,\dots,p-1-r,r}(1) \in \mathscr{B}^-_t$ as a linear generator for the $(p^{t-1}+1)$-dimensional sub-module of $R_0^- \oplus_{R_1^-} \cdots \oplus_{R_{t-1}^-} R_t^-$.

As we have seen for the Case $A1$ we have

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right) \cdot F^{(1,t-1)}_{l_1,\dots,l_{t-1}}(1) \equiv$$

$$\equiv \begin{cases} 0 & \text{if } (l_1,\dots,l_{t-1}) \prec (r, p-1-r,\dots,p-1-r,r) \\ \bar{c}F^{(1,t-1)}_{r,p-1-r,\dots,p-1-r,r}(0) & \text{if } (l_1,\dots,l_{t-1}) = (r,p-1-r,\dots,p-1-r,r) \end{cases}$$

(the K_t invariance of the elements $F^{(1,t-1)}_{l_1,\dots,l_{t-1}}(0)$ is clear).

Sub-case $B1$: $r=0$.

In this situation we have to consider the element $F^{(1,t+1)}_{r,p-1-r,\dots,r,0,1}(0) \in \mathscr{B}^-_{t+2}$ as a linear generator for the $(p^{t-1}+1)$-dimensional sub-module of $R^-_0 \oplus_{R^-_1} \cdots \oplus_{R^-_{t+1}} R^-_{t+2}$.

A direct computation together with an argument on Witt polynomials (as in Case $A2$) shows that

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right) \cdot F^{(1,t+1)}_{l_1,\dots,l_{t-1},0,1}(0) \equiv$$

$$\equiv \begin{cases} 0 & \text{if } (l_1,\dots,l_{t-1}) \prec (r, p-1-r,\dots,p-1-r,r) \\ \bar{c}F^{(1,t+1)}_{r,p-1-r,\dots,p-1-r,r}(0) & \text{if } (l_1,\dots,l_{t-1}) = (r,p-1-r,\dots,p-1-r,r) \end{cases}$$

□

We turn now our attention to the analysis of I_t invariants for the modules $R^-_{\infty,\bullet}$. The result is the following:

Proposition 4.6. *If either $t \geqslant 1$ and $p \geqslant 5$ or $t \geqslant 2$ and $p=3$ the action of $\mathbf{U}(p^{t-1}\mathbf{Z}_p)$ is trivial on $\mathscr{F}_{p^{t-1}}$.*

In particular

$$\left(R^-_{\infty,0}\right)^{I_t} = \left(R^-_{\infty,0}\right)^{K_t} = \mathscr{F}_{p^{t-1}-1}$$

and for any $x \in \mathscr{F}_{p^{t-1}}$ we have

$$\left(\begin{bmatrix} 1+p^t a & p^{t-1} b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right) \cdot x = \bar{c}\kappa_x e_0$$

where $a,b,c,d \in \mathbf{Z}_p$, $\kappa_x \in k$ is an appropriate scalar depending only on x and e_0 is a linear generator of $\mathrm{soc}\left(R^-_{\infty,0}\right)$.

Proof. Once we show that $\mathscr{F}_{p^{t-1}}$ is fixed under the action of $\mathbf{U}(p^{t-1}\mathbf{Z}_p)$, the second part of the statement follows easily by the Iwahori decomposition and Proposition 4.4.

Again, we can check the $\mathbf{U}(p^{t-1}\mathbf{Z}_p)$-invariance of $\mathscr{F}_{p^{t-1}}$ by an explicit argument on Witt vectors. We notice that, for $z = \sum_{j=1}^{t+1} p^j[\lambda_j]$ and $b \in \mathbf{Z}_p$ we have

$$\begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix}\kappa'$$

where $\kappa' \in K_1(p^{t+2})$ (the maximal pro-p subgroup of $K_0(p^{t+2})$) and $z' = \sum_{j=1}^{t} p^j[\lambda_j] + p^{t+1}[\lambda_{t+1} - \lambda_1^2\bar{b}]$.

We distinguish several cases according to the values of t and r.

Case A: t is odd.

A direct computation gives the following equality inside R_{t+1}:

$$\left(\begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix} - 1\right) F^{(1,t)}_{l_1,\dots,l_t}(l_{t+1}) = \begin{cases} 0 & \text{if } l_{t+1}=0 \\ -\bar{b}F^{(1,t)}_{\lceil l_1+2\rceil,l_2,\dots,l_t}(0) & \text{if } l_{t+1}=1. \end{cases}$$

This proves the result for $R^-_{\infty,0}$ when $t \geqslant 1$ is odd and $r < p-1$, since $\mathscr{F}_{p^{t-1}}$ is linearly generated by the elements $F \in \mathscr{B}^-_{t+1}$ verifying $F \preceq F^{(1,t)}_{r,p-1-r,\dots,p-1-r,r+1}(0)$ (cf. case $A1$ in the proof of Proposition 4.4).

As far as the case $r = p-1$ is concerned, we recall (Proposition 3.10) that $F^{(1,t)}_{\lceil l_1+2\rceil,l_2,\dots,l_{t-1},0}(0) \equiv 0$ inside the amalgamated sum $R^-_0 \oplus_{R^-_1} \cdots \oplus_{R^-_t} R^-_{t+1}$ as soon as $t > 1$. If $t = 1$ we have $F^{(1)}_2(0) \equiv 0$ as soon as $2 < r$. This let us deduce the required result for $R^-_{\infty,0}$ when $r = p-1$ and $t \geqslant 2$ (or $t = 1$ and $p > 3$), since $\mathscr{F}_{p^{t-1}}$ is linearly generated by the elements $F \in \mathscr{B}^-_{t+1}$ verifying $F \preceq F^{(1,t)}_{r,p-1-r,\dots,p-1-r,0}(1)$ (cf. case $A2$ in the proof of Proposition 4.4).

Case B: t is even.

Since

$$\left(\begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix} - 1\right) F^{(1,t-1)}_{l_1,\dots,l_{t-1}}(l_t) = 0$$

inside R^-_t, the result is clear for $r > 0$ via the description of $\mathscr{F}_{p^{t-1}}$ (again, cf. the case $B1$ in the proof of Proposition 4.4).

Concerning the case $r = 0$ we have

$$\left(\begin{bmatrix} 1 & p^{t-1}b \\ 0 & 1 \end{bmatrix} - 1\right) F^{(1,t+1)}_{l_1,\dots,l_{t-1},0,1}(0) = \overline{b} F^{(1,t+1)}_{\lceil l_1+2\rceil,\dots,l_{t-1},0,0}(0)$$

which is zero in the amalgamated sum $R^-_0 \oplus_{R^-_1} \cdots \oplus_{R^-_{t+1}} R^-_{t+2}$; the conclusion follows again from the explicit description of $\mathscr{F}_{p^{t-1}}$ (cf. the case $B2$ in the proof of Proposition 4.4). □

Remark 4.7. *The statements of Propositions 4.4 and 4.6 hold if we replace $R^-_{\infty,0}$ with $R^-_{\infty,-1}$. If $p \geqslant 5$ this follows immediately from the generality of the Serre weight σ and Proposition 4.2.*

Otherwise, one can pedantically repeat the direct arguments in the proofs of Propositions 4.4 and 4.6, noticing that now the $(p^{t-1}+1)$ dimensional submodule of $R^-_{\infty,-1}$ is generated by the element $F^{(1,t)}_{p-1-r,r,\dots,p-1-r,r+1}(0)$ if t is even *and $r < p-1$, by the element $F^{(1,t)}_{p-1-r,r,\dots,p-1-r,0}(1)$ if t is* even *and $r = p-1$, etc...*

The tedious details are left to the interested reader.

Remark 4.8. *From the equality (13) we deduce that the exact sequence of Proposition 4.4*

$$0 \to \mathscr{F}_{p^{t-1}-1} \to \mathscr{F}_{p^{t-1}} \to \chi_r^s \mathfrak{a}^{r+1} \to 0$$

is non-split even when restricted to $\overline{\mathbf{U}}(p^t\mathbf{Z}_p)$. Hence, by Remark 3.12, the space of $\overline{\mathbf{U}}(p^t\mathbf{Z}_p)$-fixed vectors in $R^-_{\infty,0}$ is precisely $\mathscr{F}_{p^{t-1}-1}$

4.2 Invariants for supersingular representations.

We are now in the position to determine precisely the space of K_t, I_t fixed vectors for supersingular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$.

The case of I_t fixed vectors is an immediate consequence of Proposition 3.8 and Proposition 4.6:

Proposition 4.9. *Let $t \geqslant 1$ and $p \geqslant 5$ (or $t \geqslant 2$ and $p = 3$). We have a short exact sequence of $k[K_0(p)]$-modules*

$$0 \to W_1 \to (R^-_{\infty,0})^{I_t} \oplus (R^+_{\infty,0})^{I_t} \to (R_{\infty,0})^{I_t} \to 0 \tag{15}$$

(where W_1 is the 1-dimensional space defined in Proposition 3.7).

In particular, for any $t \geqslant 1$ and $p \geqslant 3$ we have

$$\dim(\pi(\sigma,0,1))^{I_t} = 2(2p^{t-1}-1).$$

Proof. Write e_0 (resp. e_0^+) for a linear generator of $\mathrm{soc}_{K_0(p)}(R^-_{\infty,0})$ (resp. $\mathrm{soc}_{K_0(p)}(R^+_{\infty,0})$). Up to replace e_0, e_0^+ by appropriate scalar multiples, we have an equivariant exact sequence (cf. Proposition 3.7)

$$0 \to \langle (e_0, e_0^+)\rangle \to R^-_{\infty,0} \oplus R^+_{\infty,0} \to R_{\infty,0}|_{K_0(p)} \to 0.$$

From the associated long exact sequence in cohomology, we see that the exactness of (15) is established once we prove that the natural morphism

$$H^1(I_t, \langle (e_0, e_0^+)\rangle) \to H^1(I_t, R^-_{\infty,0}) \oplus H^1(I_t, R^+_{\infty,0})$$

is injective. Recall that, as I_t acts trivially on $\langle (e_0, e_0^+)\rangle$, we have a canonical isomorphism

$$H^1(I_t, \langle (e_0, e_0^+)\rangle) \cong \mathrm{Hom}(I_t, k)$$

(where the Hom denotes the space of continuous group homomorphisms).

Assume that $\gamma \in \mathrm{Hom}(I_t, k)$ has trivial image in $H^1(I_t, R^-_{\infty,0}) \oplus H^1(I_t, R^+_{\infty,0})$. This means that there exists an element $(x^-, x^+) \in R^-_{\infty,0} \oplus R^+_{\infty,0}$ such that

$$\gamma(g)(e_0, e_0^+) = (g-1)(x^-, x^+) \in \langle (e_0, e_0^+)\rangle \qquad (16)$$

for all $g \in I_t$. In particular $x^- \in \big(R^-_{\infty,0}/\langle e_0\rangle\big)^{I_t}$, $x^+ \in \big(R^+_{\infty,0}/\langle e_0^+\rangle\big)^{I_t}$ and by Lemma 4.10 below it follows that x^- (resp. x^+) belongs to the $(p^{t-1}+1)$-dimensional submodule of $R^-_{\infty,0}$ (resp. $R^+_{\infty,0}$), *i. e.* that $x^- \in \mathscr{F}_{p^{t-1}}$.

Hence, for $g = \begin{bmatrix} 1+p^t a & p^{t-1}b \\ p^t c & 1+p^t d \end{bmatrix} \in I_t$ we deduce from Proposition 4.6 that

$$(g-1)x^- = \overline{c}\kappa^- e_0$$

where $\kappa^- \in k$ depends only on x^-. By symmetry (see Remark 4.7 and Proposition 3.8) we similarly have

$$(g-1)x^+ = \overline{b}\kappa^+ e_0^+$$

for an appropriate scalar $\kappa^+ \in k$ depending only on x^+.

It follows from (16) that $\kappa^-\overline{c} = \kappa^+\overline{b}$ for any choice of $c, b \in \mathbf{Z}_p$ and this implies $\kappa^- = \kappa^+ = 0$, i.e. γ is the zero homomorphism.

Thanks to Proposition 4.6, Remark 4.7 and the isomorphism $R^-_{\infty,-1} \xrightarrow{\sim} (R^+_{\infty,0})^s$ (Proposition 3.8) we deduce from (15) that

$$\dim(R_{\infty,0})^{I_t} = (2p^{t-1} - 1)$$

if either $t \geqslant 1$ and $p \geqslant 5$ or $t \geqslant 2$ and $p = 3$, and hence (by generality of σ and Proposition 4.1)

$$\dim(\pi(\sigma, 0, 1))^{I_t} = 2(2p^{t-1} - 1).$$

if either $t \geqslant 1$ and $p \geqslant 5$ or $t \geqslant 2$ and $p = 3$. The remaining case $t = 1$ and $p = 3$ is covered in [Bre03a] and the proof is complete. □

Lemma 4.10. *In the notations and hypotheses of Proposition 4.9 we have*

$$\dim\big(R^-_{\infty,0}/\langle e_0\rangle\big)^{I_t} = p^{t-1} = \dim\big(R^+_{\infty,0}/\langle e_0^+\rangle\big)^{I_t}.$$

Proof. The equivariant exact sequence

$$0 \to \langle e_0\rangle \to R^-_{\infty,0} \to R^-_{\infty,0}/\langle e_0\rangle \to 0$$

yields the exact sequence

$$0 \to \langle e_0\rangle \to \big(R^-_{\infty,0}\big)^{\overline{\mathbf{U}}(p^t\mathbf{Z}_p)} \to \big(R^-_{\infty,0}/\langle e_0\rangle\big)^{\overline{\mathbf{U}}(p^t\mathbf{Z}_p)} \to H^1(p^t\mathbf{Z}_p, \langle e_0\rangle).$$

We have $H^1(p^t\mathbf{Z}_p, \langle e_0\rangle) \cong \mathrm{Hom}(p^t\mathbf{Z}_p, k)$ and, as k is p-elementary abelian and the Frattini quotient of $p^t\mathbf{Z}_p$ is $\mathbf{Z}/(p)$, we deduce that $\mathrm{Hom}(p^t\mathbf{Z}_p, k)$ is one dimensional. We therefore deduce by Remark 4.8 that $\dim\big(R^-_{\infty,0}/\langle e_0\rangle\big)^{I_t} \leqslant p^{t-1}$.

Finally, the element $e_1 \in R^-_{\infty,0}$ defined in Proposition 4.4 is I_t-fixed in $R^-_{\infty,0}/\langle e_0\rangle$ (Proposition 4.6) and linearly independent with the elements in $\big(R^-_{\infty,0}\big)^{I_t}$: it follows that $\dim\big(R^-_{\infty,0}/\langle e_0\rangle\big)^{I_t} = p^{t-1}$.

By Remark 4.7 we have an analogous result for $R^-_{\infty,-1}$, and the equality

$$\dim\big(R^+_{\infty,0}/\langle e_0^+\rangle\big)^{I_t} = p^{t-1}$$

follows then from the intertwinings of Proposition 3.8. □

We turn our attention to the analysis of K_t fixed vectors for supersingular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$. We start recalling some results (cf. [Mo1]) concerning the KZ-socle filtration for the representations $\mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0})$ and $\pi(\sigma,0,1)$.

The $K_0(p)$-socle filtration $R^-_{\infty,0}$ induces a K-equivariant filtration on $\mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0})$ (hence on $R_{\infty,0}$); the extensions between its first graded pieces look as follow:

$$\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^r \text{---} \mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+1} \text{---} \mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+2} \text{---} \mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+3} \text{---} \dots \tag{17}$$

One can show ([Mo1], Lemmas 6.8 and 6.9 or [AM]) that the KZ-equivariant filtration on $\mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0})$ obtained by the evident refinement of (17) *is indeed* the KZ-socle filtration for $\mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0})$ (cf. [Mo1], Theorem 1.1).

In particular, if $e_1 \in \mathscr{F}_{p^{t-1}}$ is a linear generator for the socle

$$\mathrm{soc}_{K_0(p)}\bigg(R^-_{\infty,0}/\mathscr{F}_{(p^{t-1}-1)}\bigg) = \chi_r^s\mathfrak{a}^{r+1}$$

we see that

$$\mathrm{soc}_K\bigg(\mathrm{ind}_{K_0(p)}^K\big(R^-_{\infty,0}/\mathscr{F}_{(p^{t-1}-1)}\big)\bigg) = \mathrm{soc}_K\big(\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+1}\big),$$

where the finite induction of the RHS is generated, under K, by the image of the element $[1, e_1]$ via the map $\mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0}) \twoheadrightarrow \mathrm{ind}_{K_0(p)}^K\big(R^-_{\infty,0}/\mathscr{F}_{(p^{t-1}-1)}\big)$. Notice moreover that if $R_{\infty,t}$ denotes the image of $\mathrm{ind}_{K_0(p)}^K(\mathscr{F}_{(p^{t-1}-1)})$ inside $R_{\infty,0}$ we have, for $t \geqslant 1$

$$\mathrm{ind}_{K_0(p)}^K\big(R^-_{\infty,0}/\mathscr{F}_{(p^{t-1}-1)}\big) \xrightarrow{\sim} R_{\infty,0}/R_{\infty,t}$$

via the epimorphism of Corollary 3.5. We recall that the main properties of the element $e_1 \in \mathscr{F}_{p^{t-1}}$ were described in Proposition 4.4 and 4.6 and we define the following element of $\mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0})$:

$$f(e_1) \overset{\text{def}}{=} \sum_{\lambda_0\in\mathbf{F}_p}\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}[1, e_1] - \delta_{r,p-3}[1,e_1] \in \mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0}).$$

By Proposition 2.1 we see that

LEMMA 4.11. *Via the natural epimorphism* $\mathrm{ind}_{K_0(p)}^K(R^-_{\infty,0}) \twoheadrightarrow \mathrm{ind}_{K_0(p)}^K\big(R^-_{\infty,0}/\mathscr{F}_{(p^{t-1}-1)}\big)$ *the element* $f(e_1)$ *maps to a highest weight vector for the Serre weight* $\mathrm{Sym}^{\lfloor p-3-r\rfloor}k^2 \otimes \det^{r+1}$ *appearing in* $\mathrm{soc}_K\big(\mathrm{ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^{r+1}\big)$, *unless* $(r,p) = (p-1,3)$ *(in which case it maps to a highest weight vector for* $\overline{St}\otimes\det$*).*

Since K_t is normal in K, the sub-module $R_{\infty,t}$ is formed by K_t fixed vectors. Nevertheless, for $r \leqslant p-3$, it is strictly contained in $(R_{\infty,0})^{K_t}$:

Lemma 4.12. *Assume $r \leqslant p-3$. The natural morphism*

$$\left\langle \mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_{(p^{t-1}-1)}\big), f(e_1)\right\rangle_{k[K]} \hookrightarrow \mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$$

factors through $(R_{\infty,0})^{K_t} \hookrightarrow R_{\infty,0}$.

Proof. It suffices to show that for any $\kappa \in K_t$ we have

$$(\kappa - 1)\cdot f(e_1) \in \ker\big(\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}\big).$$

For $a,b,c,d \in \mathbf{Z}_p$ we have

$$\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix}\begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+p^t a' & p^t b' \\ p^t c' & 1+p^t d' \end{bmatrix} \tag{18}$$

with $\overline{c'} = \overline{b} + (\overline{a-d})\lambda_0 - \overline{c}\lambda_0^2$.

We therefore deduce from Proposition 4.4 the following equality in $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-)$:

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right)\cdot f(e_1) = \overline{b} f_0(e_0) + (\overline{a-d}) f_1(e_0) - \overline{c}(f_2(e_0) + \delta_{r,p-3}[1,e_0]).$$

where e_0 is a convenient linear generator of $\mathrm{soc}_{K_0(p)}(R_{\infty,0}^-)$.

Since the kernel of the epimorphism $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$ is linearly generated by the elements

$$\ker\big(\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}\big) = \langle f_0(e_0), \dots, f_{p-2-r}(e_0), f_{p-1-r}(e_0) + [1,e_0]\rangle_k$$

(cf. Corollary 3.5 and Proposition 2.1) the required result follows. □

We can now describe completely the K_t fixed vectors of $R_{\infty,0}$:

Proposition 4.13. *Let $t \geqslant 1$. The space of K_t fixed vectors of $R_{\infty,0}$ is given by*[1]

$$(R_{\infty,0})^{K_t} = \begin{cases} \left\langle \mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_{(p^{t-1}-1)}\big), f(e_1)\right\rangle_{k[K]} & \text{if } r \leqslant p-3 \\ \mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_{(p^{t-1}-1)}\big) & \text{if } r \in \{p-2, p-1\}. \end{cases} \tag{19}$$

Proof. In order to ease notations we define the $k[K]$-module

$$M_0 \overset{\mathrm{def}}{=} \begin{cases} \left\langle \mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_{(p^{t-1}-1)}\big), f(e_1)\right\rangle_{k[K]} & \text{if } r \leqslant p-3 \\ \mathrm{ind}_{K_0(p)}^K\big(\mathscr{F}_{(p^{t-1}-1)}\big) & \text{if } r \in \{p-2, p-1\} \end{cases}$$

and write M for its image in $R_{\infty,0}$ under the epimorphism $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$.

Assume that $R_{\infty,0}^{K_t}/M \neq \{0\}$.

Then, by the description of the K-socle filtration for $R_{\infty,0}$, we see that

$$\mathrm{soc}\big(R_{\infty,0}^{K_t}/M\big) = \mathrm{soc}\big(R_{\infty,0}/M\big) = \begin{cases} \mathrm{cosoc}(\mathrm{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+1}) & \text{if } r \leqslant p-4 \\ \overline{St} \otimes \det^{-1} & \text{if } r = p-3 \\ \mathrm{soc}(\mathrm{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+1}) & \text{if } r \in \{p-2, p-1\}. \end{cases}$$

[1] The scrupulous reader will notice a slight abuse of notation: in the RHS of (19) we have $k[K]$-submodules of $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-)$ and one should consider their images under the epimorphism $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$. This abuse should cause no confusion, avoiding instead an overload of notations.

Moreover, the following elements of $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-)$

$$\begin{cases} f_{p-3-r}(e_1) & \text{if } r \leqslant p-3 \\ f_0(e_1) & \text{if } r \in \{p-2, p-1\} \text{ and } (r,p) \neq (p-1,3) \\ f_0(e_1),\ f_0(e_1) - [1, e_1] & \text{if } (r,p) = (p-1,3). \end{cases} \tag{20}$$

are mapped to a linear basis for the highest weight space of $\mathrm{soc}\big(R_{\infty,0}/M\big)$ (in the case $(r,p) = (p-1,3)$ then $f_0(e_1)$, $f_0(e_1) - [1, e_1]$ are mapped to the highest weight space of $\overline{St} \otimes \det$, $\det$ respectively).

Since M is formed by K_t fixed vectors the elements described in (20) should be K_t fixed vectors of $R_{\infty,0}$. This is absurd, as we show in the following lines.

We treat first the case $r \leqslant p-3$. Thanks to (18) and Proposition 4.4 we have the following equality in $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-)$:

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right) \cdot f_{p-3-r}(e_1) = \bar{b} f_{p-3-r}(e_0) + (\overline{a-d}) f_{p-2-r}(e_0) - \bar{c} f_{p-1-r}(e_0)$$

(where e_0 is again a convenient linear generator of $\mathrm{soc}_{K_0(p)}(R_{\infty,0}^-)$).

Via Proposition 2.1 and the epimorphism of Corollary 3.5 we deduce the following equality in $R_{\infty,0}$:

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right) \cdot f_{p-3-r}(e_1) = -\bar{c} f_{p-1-r}(e_0).$$

But $f_{p-1-r}(e_0)$ is a linear generator for the highest weight space of $\mathrm{soc}(R_{\infty,0})$ (Proposition 2.1) and hence $f_{p-3-r}(e_1)$ can not be a K_t fixed vector in $R_{\infty,0}$.

The case $r \in \{p-2, p-1\}$ is completely analogous: we have

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right) \cdot f_0(e_1) = \bar{b} f_0(e_0) + (\overline{a-d}) f_1(e_0) - \bar{c} f_2(e_0) \tag{21}$$

(resp.

$$\left(\begin{bmatrix} 1+p^t a & p^t b \\ p^t c & 1+p^t d \end{bmatrix} - 1\right) \cdot (f_0(e_1) - [1, e_1]) = \bar{b} f_0(e_0) + (\overline{a-d}) f_1(e_0) - \bar{c}(f_2(e_0) + [1, e_0]) \tag{22}$$

when $(r,p) = (p-1,3)$) and one verifies by Proposition 2.1 that the RHS of (21) (resp. of (22)) is mapped to a linearly independent family inside $\mathrm{soc}(R_{\infty,0})$ via the epimorphism $\mathrm{ind}_{K_0(p)}^K(R_{\infty,0}^-) \twoheadrightarrow R_{\infty,0}$.

This completes the proof. □

As a corollary, we get the desired structure for K_t fixed vectors of supersingular representations of $\mathbf{GL}_2(\mathbf{Q}_p)$:

COROLLARY 4.14. *Let $t \geqslant 1$. The space of K_t fixed vectors for the supersingular representation $\pi(\sigma, 0, 1)$ decomposes into the direct sum of two $k[K]$-modules $(\pi(\sigma,0,1))^{K_t} = (R_{\infty,0})^{K_t} \oplus (R_{\infty,-1})^{K_t}$, whose socle filtration is respectively described by:*

$$(R_{\infty,0})^{K_t}: \quad \mathrm{Sym}^r k^2 \text{---} \mathrm{socfil}(\mathrm{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+1}) \text{---} \mathrm{socfil}(\mathrm{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+2}) \text{---} \dots \text{---} \mathrm{socfil}(\mathrm{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r}) \text{---} \mathrm{Sym}^{p-3-r} k^2 \otimes \det{}^{r+1}$$

$$(R_{\infty,-1})^{K_t}: \quad \mathrm{Sym}^{p-1-r} k^2 \otimes \det{}^r \text{---} \mathrm{socfil}(\mathrm{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}) \text{---} \mathrm{socfil}(\mathrm{ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^2) \text{---} \dots \text{---} \mathrm{socfil}(\mathrm{ind}_{K_0(p)}^K \chi_r^s) \text{---} \mathrm{Sym}^{r-2} k^2 \otimes \det$$

where we have $p^{t-1} - 1$ parabolic inductions in each line and the weight $\mathrm{Sym}^{p-3-r} k^2 \otimes \det^{r+1}$ in the first line (resp. $\mathrm{Sym}^{r-2} k^2 \otimes \det$ in the second line) appears only if $p-3-r \geqslant 0$ (resp. $r-2 \geqslant 0$).

Proof. The statement concerning the direct summand $(R_{\infty,0})^{K_t}$ follows immediately from Corollary 3.5 and Proposition 4.13. By the generality of σ and Proposition 4.1 one deduces the result for $(R_{\infty,-1})^{K_t}$. □

In particular, we have

Corollary 4.15. *Let $t \geqslant 1$. The dimension of K_t invariant for the supersingular representation $\pi(\sigma, 0, 1)$ is given by*

$$\dim((\pi(\sigma,0,1))^{K_t}) = (p+1)(2p^{t-1}-1) + \begin{cases} p-3 & \text{if } r \notin \{0, p-1\} \\ p-2 & \text{if } r \in \{0, p-1\}. \end{cases}$$

5. The case of principal and special series

In order to complete the picture concerning K_t and I_t invariants for irreducible admissible representations of $\mathbf{GL}_2(\mathbf{Q}_p)$ we are left to treat the case of principal and special series.

Recall ([BL94], [Her2]) that the irreducible principal series for $\mathbf{GL}_2(\mathbf{Q}_p)$ are described by the parabolic induction

$$\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})$$

where $\mu \in \overline{k}^{\times}$, un_μ is the unramified character of $\mathbf{Q}_p^{\times}$ verifying $\mathrm{un}_\mu(p) = \mu$, $r \in \{0, \ldots, p-1\}$ and $(r, \mu) \notin \{(0, \pm 1), (p-1, \pm 1)\}$. The special series are described (up to twist) by the short exact sequence

$$0 \to 1 \to \mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)} 1 \to \mathrm{St} \to 0. \tag{23}$$

It is easy to see that we have K-equivariant isomorphisms (see for instance [Mo1], §10):

$$\big(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})\big)|_K \cong \mathrm{ind}_{K_0(p^\infty)}^{K} \chi_r^s \cong \varinjlim_{n\geqslant 1} \big(\mathrm{ind}_{K_0(p^{n+1})}^{K} \chi_r^s\big)$$

where $K_0(p^\infty) \stackrel{\text{def}}{=} \mathbf{B}(\mathbf{Z}_p)$ and the transition morphisms for the co-limit in the RHS are obtained inducing the natural monomorphisms of $K_0(p^n)$-representations

$$\chi_r^s \hookrightarrow \mathrm{ind}_{K_0(p^{n+1})}^{K_0(p^n)} \chi_r^s.$$

Moreover by the Bruhat-Iwahori and Mackey decompositions, we have a $K_0(p)$-equivariant split exact sequence

$$0 \to \big(\mathrm{ind}_{K_0(p^\infty)}^{K} \chi_r^s\big)^+ \to \mathrm{ind}_{K_0(p^\infty)}^{K} \chi_r^s \to \mathrm{ind}_{K_0(p^\infty)}^{K_0(p)} \chi_r^s \to 0.$$

The following results are formal.

Lemma 5.1. *Let $\mu \in \overline{k}^{\times}$ and $r \in \{0, \ldots, p-1\}$. We have a K-equivariant isomorphism*

$$\big(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})\big)|_K \cong \mathrm{ind}_{K_0(p)}^{K} \big(\varinjlim_{n\geqslant 1} \big(\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)} \chi_r^s\big)\big).$$

The action of $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$ on the principal series $\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}})$ induces an isomorphism

$$\big(\mathrm{ind}_{K_0(p^\infty)}^{K_0(p)} \chi_r^s\big)^s \xrightarrow{\sim} \big(\mathrm{ind}_{K_0(p^\infty)}^{K} \chi_r^s\big)^+.$$

Proof. The first isomorphism comes from the continuity and transitivity of the induction functor $\mathrm{ind}_{K_0(p)}^{K}(\_)$. The second comes from a direct computation on the explicit isomorphism

$$(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu \otimes \omega^r \mathrm{un}_{\mu^{-1}}))|_{K_0(p)} \cong \mathrm{ind}_{K_0(p^\infty)}^{K_0(p)} \chi_r^s \oplus \big(\mathrm{ind}_{K_0(p^\infty)}^{K} \chi_r^s\big)^+$$

given by Mackey decomposition (recalling that $\begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix}$ normalizes $K_0(p)$). □

The K_t, I_t fixed vectors for the co-limit $\varinjlim_{n\geqslant 1}\big(\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_r^s\big)$ are described by the

PROPOSITION 5.2. *Let $t \geqslant 1$ and $r \in \{0, \dots, p-2\}$. Then*

$$\big(\mathrm{ind}_{K_0(p^\infty)}^{K_0(p)}\chi_r^s\big)^{K_t} = \mathrm{ind}_{K_0(p^t)}^{K_0(p)}\chi_r^s = \big(\mathrm{ind}_{K_0(p^\infty)}^{K_0(p)}\chi_r^s\big)^{I_t}$$

Proof. We know that $\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_r^s$ is uniserial for all $n \geqslant 1$, in particular the co-limit $\varinjlim_{n\geqslant 1}\big(\mathrm{ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_r^s\big)$ is uniserial. It is therefore sufficient to prove the result in the statement replacing the co-limit by $\mathrm{ind}_{K_0(p^{t+1})}^{K_0(p)}\chi_r^s$.

In this case, we have again an explicit linear basis $\mathscr{B}_{t+1}^-$ for the induced representation $\mathrm{ind}_{K_0(p^{t+1})}^{K_0(p)}\chi_r^s$, endowed with a linear ordering which is compatible with the $K_0(p)$-socle filtration (see [Mo1] §5 or [Mo5], §4):

$$\mathscr{B}_{t+1}^- \ni F_{l_1,\dots,l_t}^{(1,t)} \overset{\text{def}}{=} \sum_{\lambda_1\in\mathbf{F}_p}\lambda_1^{l_1}\begin{bmatrix} 1 & 0 \\ p[\lambda_1] & 1 \end{bmatrix}\dots\sum_{\lambda_t\in\mathbf{F}_p}\lambda_t^{l_t}\begin{bmatrix} 1 & 0 \\ p^t[\lambda_t] & 1 \end{bmatrix}[1,e]$$

where $(l_1,\dots,l_t)\in\{0,\dots,p-1\}^t$ and e is a linear basis for the character χ_r^s (again $\mathscr{B}_{t+1}^-$ is endowed with the lexicographical order).

The statement can be now verified directly, as for Proposition 4.4, 4.6, but the computations are much easier. □

We finally deduce:

PROPOSITION 5.3. *Let $\mu \in \overline{k}^\times$ and $r \in \{0,\dots,p-1\}$ and let $t \geqslant 1$. The K-socle filtration for the K_t invariants of the principal series $\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu\otimes\omega^r\mathrm{un}_{\mu^{-1}})$ is described by*

$$\mathrm{socfil}(\mathrm{ind}_{K_0(p)}^{K}\chi_r^s)\text{---}\mathrm{socfil}(\mathrm{ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a})\text{---}\mathrm{socfil}(\mathrm{ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^2)\text{---}\dots\text{---}\mathrm{socfil}(\mathrm{ind}_{K_0(p)}^{K}\chi_r^s)$$

where the number of parabolic induction is p^{t-1}.

Moreover, the I_t fixed vectors for the principal series $\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu\otimes\omega^r\mathrm{un}_{\mu^{-1}})$ are described by

$$\big(\mathrm{ind}_{\mathbf{B}(\mathbf{Q}_p)}^{\mathbf{GL}_2(\mathbf{Q}_p)}(\mathrm{un}_\mu\otimes\omega^r\mathrm{un}_{\mu^{-1}})\big)^{I_t} \cong \mathrm{ind}_{K_0(p^t)}^{K_0(p)}\chi_{\underline{r}}^s \oplus \big(\mathrm{ind}_{K_0(p^t)}^{K_0(p)}\chi_{\underline{r}}^s\big)^s.$$

The Steinberg representation verifies in particular

$$(St)^{K_t}: \qquad \overline{St}\text{---}\mathrm{socfil}(\mathrm{ind}_{K_0(p)}^{K}\mathfrak{a})\text{---}\mathrm{socfil}(\mathrm{ind}_{K_0(p)}^{K}\mathfrak{a}^2)\text{---}\dots\text{---}\mathrm{socfil}(\mathrm{ind}_{K_0(p)}^{K}1)$$

(where the number of parabolic induction is $p^{t-1}-1$) and

$$(St)^{I_t} \cong \mathrm{ind}_{K_0(p^t)}^{K_0(p)}1 \oplus_1 \big(\mathrm{ind}_{K_0(p^t)}^{K_0(p)}1\big)^s$$

where the amalgamated sum on the RHS is defined through the natural $K_0(p)$-equivariant morphism $1 \hookrightarrow \mathrm{ind}_{K_0(p^t)}^{K_0(p)}1$.

Proof. This is an immediate consequence of Lemma 5.1 and Proposition 5.2 (using the fact that K_t is normal in K).

The statement concerning the Steinberg representation is clear from the exact sequence (23). □

Notice that from the uniseriality of $\mathrm{ind}_{K_0(p^\infty)}^{K_0(p)}\chi_r^s$ and Lemma 5.1 we have an isomorphism

$$\big(\mathrm{ind}_{K_0(p^t)}^{K_0(p)}\chi_{\underline{r}}^s\big)^s \xrightarrow{\sim} \big(\mathrm{ind}_{K_0(p^{t-1})}^{K}\chi_{\underline{r}}^s\big)^+$$

for any $t \geqslant 1$ and any smooth character $\chi_{\underline{r}}^s$ (by a counting dimension argument).

6. Global applications

In this section we describe the relation between the results of §4.2 and the local-global compatibility of the p-modular Langlands correspondence recently established by Emerton. We first need to recall some of the constructions of [Eme10] (see also [Bre11]).

Let $\mathbf{A}_f$ be the ring of finite adeles of $\mathbf{Q}$, $G_{\mathbf{Q}}$ be the absolute Galois group of $\mathbf{Q}$ and write $G_{\mathbf{Q}_\ell}$ for its decomposition group at a rational prime ℓ.

For a compact open subgroup K_f of the adelic group $\mathbf{GL}_2(\mathbf{A}_f)$ we write $Y(K_f)$ to denote the modular curve (defined over $\mathbf{Q}$) whose complex points are

$$Y(K_f)(\mathbf{C}) = \mathbf{GL}_2(\mathbf{Q})\backslash\big((\mathbf{C}\setminus\mathbf{R})\times\mathbf{GL}_2(\mathbf{A}_f)/K_f\big).$$

For $A \in \{\mathscr{O}, k\}$ we consider the first étale cohomology group

$$H^1(K_f)_A \overset{\text{def}}{=} H^1_{\acute{e}t}(Y(K_f)_{\overline{\mathbf{Q}}}, A)$$

where $Y(K_f)_{\overline{\mathbf{Q}}}$ is the base change of $Y(K_f)$ to $\overline{\mathbf{Q}}$.

For a fixed compact open subgroup K^p of $\mathbf{GL}_2(\mathbf{A}_f^p)$ we introduce the following modules, endowed with commuting actions of $G_{\mathbf{Q}}$ and $\mathbf{GL}_2(\mathbf{Q}_p)$:

$$H^1(K^p)_k \overset{\text{def}}{=} \varinjlim_{K_p} H^1(K_pK^p)_k, \qquad \text{and} \qquad \widehat{H}^1(K^p)_{\mathscr{O}} \overset{\text{def}}{=} \big(\varinjlim_{K_p} H^1(K_pK^p)_{\mathscr{O}}\big)^\wedge$$

where K_p runs over the compact open subgroups of $\mathbf{GL}_2(\mathbf{Q}_p)$ and the hat $\wedge$ denotes the p-adic completion of the $\mathscr{O}$-module $\varinjlim_{K_p} H^1(K_pK^p)_{\mathscr{O}}$.

Let Σ_0 be a finite set of non-Archimedean places of $\mathbf{Q}$, not containing p and let $\Sigma \overset{\text{def}}{=} \Sigma_0 \cup \{p\}$. We will be interested in compact open subgroups of $\mathbf{GL}_2(\mathbf{A}_f^p)$ of the form $K_{\Sigma_0}K_0^\Sigma$, where K_{Σ_0} is a compact open subgroup of $G_{\Sigma_0} \overset{\text{def}}{=} \prod_{\ell\in\Sigma_0}\mathbf{GL}_2(\mathbf{Q}_\ell)$ and $K_0^\Sigma \overset{\text{def}}{=} \prod_{\ell\notin\Sigma}\mathbf{GL}_2(\mathbf{Z}_\ell)$; we will write for short

$$H^1(K_{\Sigma_0})_k \overset{\text{def}}{=} H^1(K_{\Sigma_0}K_0^\Sigma)_k, \qquad \text{and} \qquad \widehat{H}^1(K_{\Sigma_0})_{\mathscr{O}} \overset{\text{def}}{=} \widehat{H}^1(K_{\Sigma_0}K_0^\Sigma)_{\mathscr{O}}.$$

For a compact open subgroup K_p in $\mathbf{GL}_2(\mathbf{Q}_p)$ we write $\mathbf{T}(K_pK_{\Sigma_0}K_0^\Sigma)$ for the sub $\mathscr{O}$-algebra of

$$\mathrm{End}_{\mathscr{O}[G_{\mathbf{Q}}]}(H^1(K_pK_{\Sigma_0}K_0^\Sigma)_{\mathscr{O}})$$

generated by the Hecke operators T_ℓ, S_ℓ for those primes $\ell\notin\Sigma$.

If $K_p' \leqslant K_p$ are compact open in $\mathbf{GL}_2(\mathbf{Q}_p)$ we have a (surjective) transition homomorphism $\mathbf{T}(K_p'K_{\Sigma_0}K_0^\Sigma) \twoheadrightarrow \mathbf{T}(K_pK_{\Sigma_0}K_0^\Sigma)$, which is compatible, in the evident sense, with the actions on the étale cohomologies. We deduce a $G_{\mathbf{Q}}\times\mathbf{GL}_2(\mathbf{Q}_p)$ equivariant action of

$$\mathbf{T}(K_{\Sigma_0}) \overset{\text{def}}{=} \varprojlim_{K_p}\mathbf{T}(K_pK_{\Sigma_0}K_0^\Sigma)$$

on the module $\widehat{H}^1(K_{\Sigma_0})_{\mathscr{O}}$, hence ([Eme10], (5.1.2)) on $H^1(K_{\Sigma_0})_k$.

By construction, the action of $\mathbf{T}(K_{\Sigma_0})$ on the sub-module $H^1(K_pK_{\Sigma_0}K_0^{\Sigma})_{\mathscr{O}}$ (resp. $H^1(K_pK_{\Sigma_0}K_0^{\Sigma})_k$) factors through the surjection $\mathbf{T}(K_{\Sigma_0}) \twoheadrightarrow \mathbf{T}(K_pK_{\Sigma_0}K_0^{\Sigma})$.

Let $\overline{\rho} : G_{\overline{\mathbf{Q}}} \to \mathbf{GL}_2(k)$ be a continuous, absolutely irreducible Galois representation. We assume moreover that $\overline{\rho}$ is modular and we define Σ_0 to be the set of primes dividing the Artin conductor of $\overline{\rho}$ ([Ser87], §1.2).

We recall that a compact open subgroup K_{Σ_0} of G_{Σ_0} is an *allowable level* for $\overline{\rho}$ if there exists a maximal ideal $\mathfrak{m}$ of $\mathbf{T}(K_{\Sigma_0})$, having residue field k and such that

$$T_\ell \equiv \operatorname{tr}(\overline{\rho}(\mathrm{Frob}_\ell)) \bmod \mathfrak{m}, \qquad S_\ell \equiv \ell^{-1}\det(\overline{\rho}(\mathrm{Frob}_\ell)) \bmod \mathfrak{m}.$$

Since $\overline{\rho}$ is modular we deduce from the level part of Serre conjecture that any compact open subgroup in the Σ_0-component of $\ker\big(\mathbf{GL}_2(\widehat{\mathbf{Z}}) \to \mathbf{GL}_2(\widehat{\mathbf{Z}}/(N))\big)$ is an allowable level for $\overline{\rho}$.

If K_{Σ_0} is allowable and $\mathfrak{m}$ is a maximal ideal associated to $\overline{\rho}$ in the previous sense, we consider the following $\mathfrak{m}$-adic completion:

$$\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}} \overset{\text{def}}{=} \mathbf{T}(K_{\Sigma_0})_{\mathfrak{m}}, \qquad \widehat{H}^1(K_{\Sigma_0})_{\mathscr{O},\overline{\rho}} \overset{\text{def}}{=} \big(\widehat{H}^1(K_{\Sigma_0})_{\mathscr{O}}\big)_{\mathfrak{m}}, \qquad H^1(K_{\Sigma_0})_{k,\overline{\rho}} \overset{\text{def}}{=} \big(H^1(K_{\Sigma_0})_k\big)_{\mathfrak{m}}.$$

The action of the completed Hecke algebra $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ on the $G_{\overline{\mathbf{Q}}} \times \mathbf{GL}_2(\mathbf{Q}_p)$-modules $\widehat{H}^1(K_{\Sigma_0})_{\mathscr{O},\overline{\rho}}$, $H^1(K_{\Sigma_0})_{k,\overline{\rho}}$ is equivariant. Moreover for an inclusion of allowable levels $K'_{\Sigma_0} \leqslant K_{\Sigma_0}$ we have a surjective transition homomorphism $\mathbf{T}(K'_{\Sigma_0})_{\overline{\rho}} \twoheadrightarrow \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ which is compatible, in the evident sense, with the actions on the completed étale cohomologies.

Therefore, the co-limit $\widehat{H}^1_{\mathscr{O},\overline{\rho},\Sigma} \overset{\text{def}}{=} \varinjlim_{K_{\Sigma_0}} \widehat{H}^1(K_{\Sigma_0})_{\mathscr{O},\overline{\rho}}$, taken over all allowable levels K_{Σ_0} in G_{Σ_0}, is naturally a module over the $\mathscr{O}$-algebra $\mathbf{T}_{\overline{\rho},\Sigma} \overset{\text{def}}{=} \varprojlim_{K_{\Sigma_0}} \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ and the same holds for the co-limit $H^1_{k,\overline{\rho},\Sigma} \overset{\text{def}}{=} \varinjlim_{K_{\Sigma_0}} H^1(K_{\Sigma_0})_{k,\overline{\rho}}$ ([Eme10], (5.3.4)).

The modules $\widehat{H}^1_{\mathscr{O},\overline{\rho},\Sigma}$, $H^1_{k,\overline{\rho},\Sigma}$ are furthermore endowed with a linear action of $G_{\overline{\mathbf{Q}}} \times \mathbf{GL}_2(\mathbf{Q}_p) \times G_{\Sigma_0}$ which turns out to be $\mathbf{T}_{\overline{\rho},\Sigma}$-linear. Notice again that, by construction, the action of $\mathbf{T}_{\overline{\rho},\Sigma}$ on the sub-module $\widehat{H}^1(K_{\Sigma_0})_{\mathscr{O},\overline{\rho}}$ (resp. $H^1(K_{\Sigma_0})_{k,\overline{\rho}}$) factors through the surjection of local $\mathscr{O}$-algebras $\mathbf{T}_{\overline{\rho},\Sigma} \twoheadrightarrow \mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$.

We can now introduce a local-global application of the results in section 4.2.

PROPOSITION 6.1. *Let $p \geqslant 3$ and $\overline{\rho} : G_{\mathbf{Q}} \to \mathbf{GL}_2(k)$ be an odd, continuous, absolutely irreducible Galois representation such that $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ is absolutely irreducible. Let Σ_0 be the set of primes dividing the Artin conductor of $\overline{\rho}$ and let κ be the minimal weight associated to $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ (cf. [Ser87], §2.2).*

Let K_{Σ_0} be an allowable level for $\overline{\rho}$ and define

$$d \overset{\text{def}}{=} \dim_k \Big(\bigotimes_{\ell \in \Sigma_0} \pi(\overline{\rho}|_{G_{\mathbf{Q}_\ell}})\Big)^{K_{\Sigma_0}} \tag{24}$$

where $\pi(\overline{\rho}|_{G_{\mathbf{Q}_\ell}})$ is the smooth p-modular representations of $\mathbf{GL}_2(\mathbf{Q}_\ell)$ attached to $\overline{\rho}|_{G_{\mathbf{Q}_\ell}}$ via the p-modular Langlands correspondence of Emerton-Helm ([EH]).

Then, if either $t \geqslant 1$ and $p \geqslant 5$ or $t \geqslant 2$ and $p = 3$ we have

$$\dim_k \big(H^1_{\acute{e}t}(Y(K_tK_{\Sigma_0}K_0^{\Sigma})_{\overline{\mathbf{Q}}}, k)[\mathfrak{m}]\big) = 2d\big(2p^{t-1}(p+1) - 3\big) \qquad \text{if } \kappa - 2 \equiv 0 \bmod p+1$$
$$\dim_k \big(H^1_{\acute{e}t}(Y(K_tK_{\Sigma_0}K_0^{\Sigma})_{\overline{\mathbf{Q}}}, k)[\mathfrak{m}]\big) = 2d\big(2p^{t-1}(p+1) - 4\big) \qquad \text{if } \kappa - 2 \not\equiv 0 \bmod p+1$$
$$\dim_k \big(H^1_{\acute{e}t}(Y(I_tK_{\Sigma_0}K_0^{\Sigma})_{\overline{\mathbf{Q}}}, k)[\mathfrak{m}]\big) = 4d\big(2p^{t-1} - 1\big).$$

where $K_0^{\Sigma} \overset{\text{def}}{=} \prod_{\ell\notin(\Sigma_0\cup\{p\})} \mathbf{GL}_2(\mathbf{Z}_\ell)$ *and* $\mathfrak{m}$ *is a maximal ideal associated to* $\overline{\rho}$ *in the Hecke algebra* $\mathbf{T}(K_{\Sigma_0})$.

Proof. Let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra $\mathbf{T}(K_{\Sigma_0})$ associated to $\overline{\rho}$. We will use the same notation $\mathfrak{m}$ for the maximal ideals of the local $\mathscr{O}$-algebras $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$, $\mathbf{T}_{\overline{\rho},\Sigma}$.

Assume that either $t \geqslant 1$ and $p \geqslant 5$ or $t \geqslant 2$ and $p = 3$. Then for $K_\mathfrak{p} \in \{K_t, I_t\}$, the congruence subgroup $K_\mathfrak{p} \prod_{\ell\neq p} \mathbf{GL}_2(\mathbf{Z}_\ell)$ is neat[2] in the sense of [Eme10], Definition 5.3.7 (the proof of Lemme 2 (1) in §5.5 of Carayol's article [Car] applies line to line).

We can therefore use [Eme10] Lemma 5.3.8 (2) and the equivariance of the Hecke action on the cohomology spaces to obtain

$$\left(H^1_{k,\overline{\rho},\Sigma}\right)^{K_\mathfrak{p} K_{\Sigma_0}}[\mathfrak{m}] = H^1(K_\mathfrak{p} K_{\Sigma_0} K_0^{\Sigma})_{k,\overline{\rho}}[\mathfrak{m}] = H^1(K_\mathfrak{p} K_{\Sigma_0} K_0^{\Sigma})_k[\mathfrak{m}]$$

(where $\mathfrak{m}$ is seen as an ideal of $\mathbf{T}_{\overline{\rho},\Sigma}$, $\mathbf{T}(K_{\Sigma_0})_{\overline{\rho}}$ or $\mathbf{T}(K_\Sigma)$ thanks to the compatibility of the Hecke action on the sub-modules of $H^1_{k,\overline{\rho},\Sigma}$, $H^1(K_\mathfrak{p} K_{\Sigma_0} K_0^{\Sigma})_k$).

Let $\overline{\kappa-2} \in \{0, \ldots, p\}$ be defined by $\overline{\kappa-2} \equiv \kappa - 2 \bmod p+1$ (we know from [Ser87] that $\overline{\kappa-2} < p-1$). From the proof of Proposition 6.1.20 in [Eme10] we have an equivariant isomorphism

$$H^1_{k,\overline{\rho},\Sigma}[\mathfrak{m}] \cong \overline{\rho} \otimes \pi_p \otimes \pi_{\Sigma_0}(\overline{\rho})$$

where $\pi_{\Sigma_0}(\overline{\rho}) \overset{\text{def}}{=} \otimes_{\ell\in\Sigma_0} \pi(\overline{\rho}|_{G_{\mathbf{Q}_l}})$ and π_p is a supersingular representation whose KZ-socle contains, up to twist, the weight $\sigma \overset{\text{def}}{=} \mathrm{Sym}^{\overline{\kappa-2}} k^2$ (more precisely π_p is, up to twist, the supersingular representation attached to $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ in [Bre03a]).

We deduce

$$H^1(K_\mathfrak{p} K_{\Sigma_0} K_0^{\Sigma})_k[\mathfrak{m}] \cong \overline{\rho} \otimes \left(\pi_p\right)^{K_\mathfrak{p}} \otimes \left(\pi_{\Sigma_0}(\overline{\rho})\right)^{K_{\Sigma_0}}$$

and the result follows from Proposition 4.9, Corollary 4.15 and the definition of d, noticing that $\pi_p \cong \pi(\sigma, 0, 1)$ up to twist. □

Remark 6.2. *By the level part of the refined Serre conjecture ([Ser87]) one expects the subgroup*

$$K_{1,\Sigma_0}(N) \overset{\text{def}}{=} \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \prod_{\ell\in\Sigma_0} \mathbf{GL}_2(\mathbf{Z}_\ell) |\ c \equiv d-1 \equiv 0 \bmod N \right\}$$

to be an allowable level for which $d = 1$ *in (24), at least if the semi-simplifications* $\overline{\rho}|^{ss}_{G_{\mathbf{Q}_\ell}}$ *are not twists of* $1 \oplus |\cdot|$.

In the classical ℓ*-adic correspondence this is indeed the compatibility between the Artin and adelic conductor but in the* ℓ*-modular case, such compatibility does not seem to appear in the current literature.*

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