Corrigendum to “Iwasawa modules and p-modular representations of GL2”

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We want to correct the statements in [Mor17, Proposition 4.4], and in the subsequent proof.

The problem in *loc. cit.* is that in the proof we worked as if $A_{m,n}$ were $k[[\begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F/p^n\mathscr{O}_F & 1 \end{bmatrix}]]$, while we have instead $A_{m,n} \cong k[[\begin{bmatrix} 1 & 0 \\ p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F & 1 \end{bmatrix}]]$.

We give a corrected version of the statement and its proof. We freely use the notation of [Mor17] in what follows.

Proposition 0.1. *Let $n \geqslant m \geqslant 1$ and let $\underline{l} = (\underline{l}_m, \dots, \underline{l}_n) \in \left\{\{0, \dots, p-1\}^f\right\}^{(n-m)}$ be an $(n-m+1)$-tuple of f-tuples.*

Then one has the following equality in $A_{m,n}$:

$$\underline{X}^{\underline{l}} \equiv \kappa_{\underline{l}} F^{(m,n)}_{\underline{p-1}-\underline{l}_m,\dots,\underline{p-1}-\underline{l}_n} \mod \mathfrak{m}^{|\underline{l}|+(p-1)}$$

where

$$\underline{X}^{\underline{l}} = \prod_{j=0}^{f-1} X_j^{\sum_{i=m}^n p^{i-m} l_{i,j}}$$

and

$$\underline{p-1} - \underline{l_i} \overset{\text{def}}{=} (p-1-l_{i,j})_{j=0}^{f-1}$$

for all $i = m, \dots, n$.

Proof. The proof is divided into two steps: the residual case $(n-m=0)$ and a dévissage. Note that for $n-m=0$ the statement is clear up to the explicit multiplicative constant, by looking at the action of the finite torus.

If $n = m$ and $\underline{l} \in \{0, \dots, p-1\}^f$ is an f-tuple, we write $F_{\underline{l}} = F_{\underline{l}}^{(m,m)}$ not to overload notation in what follows.

Lemma 0.2. *Keep the setting of Proposition 0.1 and assume that $n-m=0$.*

For any f-tuple $\underline{l} \in \{0, \dots, p-1\}^f$ we have the following equality in $A_{m,m}$:

$$\underline{X}^{\underline{l}} = \begin{cases} \kappa_{\underline{l}} F_{\underline{p-1}-\underline{l}} & \text{if } |\underline{l}| > 0 \\ \kappa_0 F_{p-1} + (-1)^{f-1} \underline{X}^{\underline{p-1}} & \text{else} \end{cases}$$

Proof. Note first that

$$\kappa_{\underline{l}+e_i} = (p-1-l_i)\kappa_{\underline{l}} \tag{1}$$

and that $\kappa_{e_i} = 1$ for all $i \in \{0, \dots, f-1\}$. The statement is therefore an immediate induction using Lemma 0.3 below. ∎

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Lemma 0.3. *Keep the hypotheses of Lemma 0.2. Assume moreover that* $\underline{l}+e_i\leqslant \underline{p-1}$. *Then:*

$$F_{\underline{p-1}-e_i}F_{\underline{p-1}-\underline{l}}=(p-1-l_i)F_{\underline{p-1}-(\underline{l}+e_i)}.$$

Proof. By the very definition of the elements $F_{\underline{p-1}-e_i}$, $F_{\underline{p-1}-\underline{l}}$ have

$$\begin{aligned}F_{\underline{p-1}-e_i}F_{\underline{p-1}-\underline{l}}&=\sum_{\lambda,\mu\in k_F}\lambda^{\underline{p-1}-e_i}(\mu-\lambda)^{\underline{p-1}-\underline{l}}\begin{bmatrix}1&0\\ p^m[\varphi^{-m+1}(\lambda)]&1\end{bmatrix}\\&=\sum_{\underline{j}\leqslant\underline{p-1}-\underline{l}}\binom{\underline{p-1}-\underline{l}}{\underline{j}}(-1)^{\underline{j}}\sum_{\lambda\in k_F}\lambda^{\underline{p-1}-e_i+\underline{j}}F_{\underline{p-1}-\underline{l}-\underline{j}}\end{aligned}$$

and the result follows since

$$\sum_{\lambda\in k_F}\lambda^{\underline{p-1}-e_i+\underline{j}}=-\delta_{\underline{j},e_i}.$$

□

We consider now the dévissage. Recall that the inclusion $p^{m+1}\mathscr{O}_F/p^{n+1}\mathscr{O}_F\hookrightarrow p^m\mathscr{O}_F/p^{n+1}\mathscr{O}_F$ induces an injective k-algebra homomorphism:

$$\begin{aligned}\iota: A_{m+1,n}&\hookrightarrow A_{m,n}\\ X_{m+1,i}&\mapsto X_{m,i}^p.\end{aligned}$$

In order to emphasize the inductive argument, we write $\mathfrak{m}$, $\mathfrak{m}_1$ to denote the maximal ideal of $A_{m,n}$, $A_{m+1,n}$ respectively (so that, in particular $\iota(\mathfrak{m}_1)=\mathfrak{m}^p$).

Given a monomial $\underline{X}^{\underline{l}}\in A_{m,n}$, we can write

$$\underline{X}^{\underline{l}}=\underline{X}^{\underline{l}^{(1)}}\iota\big(\underline{X}^{\underline{l}^{(2)}}\big)$$

for $\underline{l}^{(1)}\in\{0,\dots,p-1\}^f$, $\underline{l}^{(2)}\in\mathbf{N}^f$ verifying $\underline{l}=\underline{l}^{(1)}+p\underline{l}^{(2)}$.

By the inductive hypothesis on $A_{m+1,n}$ we have

$$\iota\big(\underline{X}^{\underline{l}^{(2)}}\big)\in\kappa_{\underline{l}^{(2)}}F^{(m+1,n)}_{\underline{p-1}-\underline{l}^{(2)}}+\iota(\mathfrak{m}_1^{|\underline{l}^{(2)}|+(p-1)})=\kappa_{\underline{l}^{(2)}}F^{(m+1,n)}_{\underline{p-1}-\underline{l}^{(2)}}+\mathfrak{m}^{p|\underline{l}^{(2)}|+p(p-1)}\tag{2}$$

and we claim that

Claim: In the situation above, we have

$$\underline{X}^{\underline{l}^{(1)}}\in\kappa_{\underline{l}^{(1)}}F^{(m)}_{\underline{p-1}-\underline{l}^{(1)}}\mod\mathfrak{m}^{|\underline{l}^{(1)}|+(p-1)}.\tag{3}$$

This will imply the statement of Proposition 0.1 since from (2) and (3) we easily get

$$\underline{X}^{\underline{l}}\equiv\kappa_{\underline{l}}F^{(m,n)}_{\underline{p-1}-\underline{l}_m,\dots,\underline{p-1}-\underline{l}_n}+\mathfrak{m}^{|\underline{l}|+(p-1)}.$$

Proof of the Claim. By Lemma 0.2 we have, in $A_{m,n}$:

$$\underline{X}^{\underline{l}^{(1)}}\in\kappa_{\underline{l}^{(1)}}F^{(m)}_{\underline{p-1}-\underline{l}^{(1)}}+\sum_{i=0}^{f-1}X_i^p\cdot A_{m,n}.\tag{4}$$

Let us consider a monomial $X_i^p\underline{X}^{\underline{t}}$ appearing with a non-zero coefficient in the sum $\sum_{i=0}^{f-1}X_i^pA_{m,n}$ in the RHS of (4). As the finite torus $\mathbf{T}(k_F)$ acts semisimply on $A_{m,n}$ and $\underline{X}^{\underline{l}^{(1)}}$, X_i^p are eigenvectors, we deduce that the f-tuple $\underline{t}\in\mathbf{N}$ verifies:

$$\sum_{j=0}^{f-1}p^jr_j\equiv\sum_{j=0}^{f-1}p^jl_j^{(1)}-p^{i+1}\mod q-1.$$

This implies $|\underline{t}| \equiv |\underline{l}^{(1)}| - 1 \bmod p-1$, hence the *Claim*. □

References

Mor17 Stefano Morra, *Iwasawa modules and p-modular representations of GL2*, Israel J. Math. **219** (2017), no. 1, 1–70. MR 3642015

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