SERRE WEIGHT CONJECTURES FOR \( p \)-ADIC UNITARY GROUPS OF RANK 2

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Abstract. We prove a version of the weight part of Serre’s conjecture for mod \( p \) Galois representations attached to automorphic forms on rank 2 unitary groups which are non-split at \( p \). More precisely, let \( F/F^+ \) denote a CM extension of a totally real field such that every place of \( F^+ \) above \( p \) is unramified and inert in \( F \), and let \( \bar{\tau} : \text{Gal}(F^+/F^+) \to \mathbb{C}U_2(F_p) \) be a Galois parameter valued in the \( \mathbb{C} \)-group of a rank 2 unitary group attached to \( F/F^+ \). We assume that \( \bar{\tau} \) is semisimple and sufficiently generic at all places above \( p \). Using base change techniques and (a strengthened version of) the Taylor–Wiles–Kisin conditions, we prove that the set of Serre weights in which \( \bar{\tau} \) is modular agrees with the set of Serre weights predicted by Gee–Herzig–Savitt.

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1. Introduction

Let $p$ be a prime number. The mod $p$ local Langlands program (cf. [Bre10], [Ber11], [BM02]) predicts a correspondence between continuous Galois representations $\rho : \text{Gal}(\mathbb{Q}_p/\mathbb{Q}) \rightarrow \text{GL}_n(\mathbb{F}_p)$ and smooth admissible $\text{GL}_n(\mathbb{Q}_p)$-representations on $\mathbb{F}_p$-vector spaces. It is expected to be compatible with the classical local Langlands correspondence over $\mathbb{C}$, its geometric realization in the torsion cohomology of Shimura varieties, and classical local/global compatibility.

The case when $n = 2$ has been most extensively studied, and such a correspondence has now been established (see [Col10], [CDP14], [Eme], and the above references). However, the picture for $n > 2$ (or more general $p$-adic fields) still remains highly conjectural, and evidence suggests that such a correspondence will be much more intricate (see, for example, [BH15]). Despite this deficiency, there has been substantial progress on several expected consequences of this conjecture: the weight part of Serre’s conjecture, the Breuil–Mézard conjecture, and Breuil’s lattice conjecture ([BDJ10], [GLS14], [BM02], [GK14], [Bre14], [EGH13], [EGS15]).

In a different direction, one is also interested in the possibility of enlarging the conjectural correspondence to include more general groups. The works [Abd14] and [Koz16] give some preliminary indication that a Langlands-type correspondence might be expected to hold for the groups $\text{SL}_2(\mathbb{Q}_p)$ and $\text{U}_2(\mathbb{Q}_p)$, and reveal some new phenomena (e.g., the existence of $L$-packets in the mod $p$ setting). In general, the work of Buzzard–Gee [BG14] lays out precise statements of Langlands-type conjectures for general reductive groups by making use of an enhancement of the Langlands dual group (this will figure prominently in our considerations below). This framework reconciles the classical local Langlands correspondence with its geometric realization. These developments are also related to recent work of Gee–Herzig–Savitt: the article [GHS18] gives a formulation of the weight part of Serre’s modularity conjectures for a large class of non-classical reductive groups.

Classical Langlands correspondences (i.e., with $\mathbb{C}$-coefficients) for various reductive groups, and the relations among them, are at the core of the Langlands functoriality principle. In the specific example of unitary groups, this principle predicts that a correspondence between (packets of) automorphic representations of unitary groups on the one side and $L$-group valued Galois parameters on the other side is obtained from a correspondence on general linear groups. When the unitary group has low rank, this is studied in [Rog90, §15.1].

The goal of the present work is to give evidence for a mod $p$ Langlands correspondence for rank 2 unitary groups. Specifically, given a Galois parameter $\tau$ with values in the $C$-dual of our unitary group, we prove that the Serre weights for $\tau$ predicted by [GHS18] (which are representations of finite unitary groups) are exactly equal to the Serre weights in which $\tau$ is modular (we give a precise statement below). In order to do this, we use known instances of functoriality (in the form of classical base change results) and local/global compatibility. Thus, our methods hint at a mod $p$ principle of unitary base change.

We now introduce some notation and setup in order to state our main result. Let $K_2/K/\mathbb{Q}_p$ be unramified extensions, with $K_2/K$ quadratic. We let $U_2$ denote the unramified unitary group in two variables defined over the ring of integers $\mathcal{O}_K$ of $K$. Note that $U_2$ splits over $K_2$. We let $\mathcal{C}U_2$ denote the $\mathcal{C}$-group of $U_2$, in the terminology of [BG14] ($\mathcal{C}U_2$ is the usual Langlands $L$-group of a canonical central extension of $U_2$). An $L$-parameter is a continuous homomorphism
\( \bar{\rho} : \text{Gal}(\overline{\mathbb{Q}}_p/K) \to C\text{U}_2(\mathbb{F}_p) \), compatible with the projection \( C\text{U}_2(\mathbb{F}_p) \to \text{Gal}(K_2/K) \). The \( C \)-group also comes equipped with a canonical map \( C\text{U}_2 \to \mathcal{G}_m \), and we assume that the composite character \( \text{Gal}(\overline{\mathbb{Q}}_p/K) \xrightarrow{\bar{\rho}} C\text{U}_2(\mathbb{F}_p) \to \mathbb{F}_p^\times \) (called the multiplier of \( \bar{\rho} \)) is equal to the mod \( p \) cyclotomic character.

Inspired by the conjectures of [BG14] and the prospect of a mod \( p \) Langlands program for unitary groups, we would like to infer that the \( L \)-parameter \( \bar{\tau} \) is associated to an \( L \)-packet of smooth representations of \( \text{U}_2(K) \) over \( \mathbb{F}_p \). Unfortunately, such representations are poorly understood beyond the case \( K = \mathbb{Q}_p \) (cf. [Koz16]). A possible first step in understanding such a correspondence would be to study this question in a global context, that is, to study local/global compatibility for an \( L \)-parameter \( \bar{\tau} : \text{Gal}(\mathbb{Q}/F^+) \to C\text{U}_2(\mathbb{F}_p) \), where \( F^+/\mathbb{Q} \) is a totally real field. We assume furthermore that \( \bar{\tau} \) is associated to a nonzero Hecke eigenclass in the mod \( p \) cohomology with infinite level at \( p \) of a definite unitary group \( \mathbf{G}/\mathcal{O}_{F^+} \), which is non-split at places of \( F^+ \) above \( p \). We would like to stress that our setting differs quite markedly from the body of work related to Serre weights for unitary groups (e.g., [GLS14], [BLGG13]), wherein the group \( \mathbf{G} \) is split at places above \( p \). In particular, our Serre weights are representations of finite unitary groups, not general linear groups. We define \( W_{\text{mod}}(\bar{\tau}) \) to be the set consisting of the \( \prod_{v \mid p} \text{U}_2(\mathcal{O}_{F^+}) \)-representations appearing in the socle of the Hecke isotypic component attached to \( \tau \) of the mod \( p \) cohomology of \( \mathbf{G} \).

According to the conjectures of [GHS18], the set \( W_{\text{mod}}(\bar{\tau}) \) should be described in an explicit way by \( (\bar{\tau})|_{\text{Gal}(\overline{\mathbb{Q}}_p/F_v^+)} \) using purely representation-theoretic constructions. Let us denote \( W^2(\bar{\tau}) \) by \( \bigotimes_{v \mid p} W^2(\bar{\tau}|_{\text{Gal}(\overline{\mathbb{Q}}_p/F_v^+)} \), where \( W^2(\bar{\tau}|_{\text{Gal}(\overline{\mathbb{Q}}_p/F_v^+)} \) is the set described combinatorially in [GHS18] (thus \( W^2(\bar{\tau}) \) is again a set of representations of the group \( \prod_{v \mid p} \text{U}_2(\mathcal{O}_{F_v^+}) \)).

The main theorem of this paper is the following (we refer the reader to the bulk of the paper for any unfamiliar terminology).

**Theorem 1.1** (Corollary 7.5). Let \( F/F^+ \) be a CM field extension of \( F^+ \) which is unramified at all finite places, suppose that \( p \) is unramified in \( F^+ \) and that every place of \( F^+ \) above \( p \) is inert in \( F \). Let \( \bar{\tau} : \text{Gal}(\overline{\mathbb{Q}}/F^+) \to C\text{U}_2(\mathbb{F}_p) \) be an \( L \)-parameter with cyclotomic multiplier. Assume that:

- \( \bar{\tau}^{-1}(C\text{U}_2(\mathbb{F}_p)) = \text{Gal}(\overline{\mathbb{Q}}/F) \);
- \( \bar{\tau} \) is modular;
- \( \bar{\tau} \) is unramified outside \( p \);
- \( \bar{\tau} \) is semisimple and 4-generic at places above \( p \);
- \( \overline{\mathbb{Q}}_{\text{ker}(ad(\bar{\tau}))} \) does not contain \( F(\zeta_p) \); and
- \( \text{BC}(\bar{\tau})(\text{Gal}(\overline{\mathbb{Q}}/F)) \supseteq \text{GL}_2(F') \) for some subfield \( F' \subseteq \overline{\mathbb{F}}_p \) with \( |F'| > 6 \).

Then

\[ W^2(\bar{\tau}) = W_{\text{mod}}(\bar{\tau}). \]

In the \( \text{GL}_2 \) setting, the results of [BP12] and [EGS15] imply that, for a \( \text{GL}_2(\mathbb{F}_p) \)-valued Galois representation \( \bar{\rho}' \), the set \( W^2(\bar{\rho}') \) of modular Serre weights should be equal to the set of representations appearing in the \( \text{GL}_2(\mathcal{O}_K) \)-socle of the \( \text{GL}_2(K) \)-representation associated to \( \bar{\rho}' \) via some sort of mod \( p \) local Langlands correspondence. For \( \text{U}_2 \), the supersingular representations of \( \text{U}_2(\mathbb{Q}_p) \) constructed in [Koz16] all have simple \( \text{U}_2(\mathbb{Z}_p) \)-socle, while the set \( W^2(\bar{\rho}) \) (for generic semisimple \( \bar{\rho} \)) has size \( 2^{|K:K_0|} \). Along with the global evidence provided by Theorem 1.1, this suggests that \( W^2(\bar{\rho}) \) takes into account the \( \text{U}_2(\mathcal{O}_K) \)-socles of all \( \text{U}_2(K) \)-representations in the \( L \)-packet associated to \( \bar{\rho} \).

We obtain Theorem 1.1 by following the strategy of [GK14]. We first prove the containment \( W^2(\bar{\tau}) \supseteq W_{\text{mod}}(\bar{\tau}) \) by using a global base change argument and applying results of [Gee11]. The opposite containment follows by using a modified version of the patching functor constructed in [CEG+16] and the explicit description of \( C\text{U}_2 \)-valued local deformation rings. We explain these arguments with more details presently.
The main novelty in the unitary group setting is that for both inclusions we make use of the analogous results for $GL_2/K_2$. Firstly, we establish a compatibility between classical local base change of automorphic types (as may be deduced from work of Rogawski \cite{Rog90}) and the set of predicted Serre weights $W^2(\overline{\rho})$ (for which we introduce a notion of base change of weights). In this direction our results give the following proposition, which may be thought of as evidence towards a notion of mod $p$ base change. Recall that a tame $U_2(\mathcal{O}_K)$-type is the inflation of an irreducible $U_2(\mathbb{F}_q)$-representation over $\mathbb{F}_p$, where $\mathbb{F}_q$ denotes the residue field of $K$.

**Proposition 1.2** (Lemma 3.26, Theorem 4.9). Let $\sigma$ denote a 1-generic tame type for $U_2(\mathcal{O}_K)$, and let $V$ denote a Serre weight for $U_2(\mathcal{O}_K)$. Let $BC(\sigma)$ denote the base change of $\sigma$ (as defined in Subsection 3.5). Then

$$V \in JH(\sigma) \iff BC(V) \in JH\left(BC(\sigma)\right),$$

where $BC(V)$ is the base change of the Serre weight $V$ (as defined in Subsection 3.5) and $JH(W)$ denotes the set of Jordan-Hölder factors of the mod $p$ reduction of a $\mathbb{Z}_p$-lattice in $W$.

In particular, if $\overline{\rho}: \text{Gal}(\mathbb{Q}_p/K) \rightarrow C^\times U_2(\mathbb{F}_p)$ is a 1-generic tame $L$-parameter with cyclotomic multiplier, then the set of predicted local Serre weights $W^2(\overline{\rho})$ is of the form $JH(\overline{\rho})$, and we obtain

$$V \in W^2(\overline{\rho}) \iff BC(V) \in W^2(BC(\overline{\rho})).$$

Here $BC(\overline{\rho}): \text{Gal}(\mathbb{Q}_p/K_2) \rightarrow GL_2(\mathbb{F}_p)$ denotes the Galois representation obtained by restricting $\overline{\rho}$ to the absolute Galois group of $K_2$ and projecting onto the $GL_2$ factor.

In order to prove the inclusion $W^2(\overline{\rho}) \subseteq W_{\text{mod}}(\overline{\rho})$, we would like to employ a patching argument, which requires information regarding certain deformation rings. More precisely, let us suppose that $\overline{\rho}: \text{Gal}(\mathbb{Q}_p/K) \rightarrow C^\times U_2(\mathbb{F})$ is an $L$-parameter with $\mathbb{F}$ a finite extension of $\mathbb{F}_p$, and let $\mathcal{O}$ denote the ring of integers in some sufficiently large finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. We let $R_{\overline{\rho}}^{(1,0,1),\tau'}$ denote the deformation ring parametrizing potentially crystalline framed deformations of $\overline{\rho}$ to $\mathcal{O}$-algebras with (parallel) $p$-adic Hodge type $(1,0,1)$, inertial type $\tau'$, and cyclotomic multiplier.

In order to study the ring $R_{\overline{\rho}}^{(1,0,1),\tau'}$, we introduce the notion of Frobenius twist self-dual Kisin modules. Given this, we are able to describe the structure of $R_{\overline{\rho}}^{(1,0,1),\tau'}$ in terms of the “base changed” deformation ring $R_{BC(\overline{\rho})}^{(1,0),\tau'}$. Combining these calculations with Proposition 1.2, along with the analogous results of \cite{Gee11} for $GL_2$, we obtain the following result, which may be viewed as a “Breuil–Mézard-type” result for unitary groups.

**Proposition 1.3.** Let $\overline{\rho}: \text{Gal}(\mathbb{Q}_p/K) \rightarrow C^\times U_2(\mathbb{F})$ be a 3-generic tame $L$-parameter with cyclotomic multiplier. Let $\tau'$ be a $C^\times U_2$-valued, 2-generic inertial type for $I_K$ and $\sigma(\tau')$ the tame $U_2(\mathcal{O}_K)$-type associated to $\tau'$ via the inertial local Langlands correspondence of Theorem 4.11. Then

$$\left|W^2(\overline{\rho}) \cap JH\left(\sigma(\tau')\right)\right| = e(R_{\overline{\rho}}^{(1,0,1),\tau'} \otimes_{\mathcal{O}} \mathbb{F}),$$

where $e(\cdot)$ denotes the Hilbert–Samuel multiplicity.

To conclude, we employ a variant of the construction of \cite{CEG16} in order to produce a patching functor $M_\infty(\cdot)$ on the category of $\mathcal{O}$-modules with an action of $\prod_{v \mid p} U_2(\mathcal{O}_{F_v})$. Using the explicit structure of the rings $R_{\overline{\rho}}^{(1,0,1),\tau'}$ (namely their integrality), the properties of the patching functor $M_\infty(\cdot)$, and Proposition 1.3 we obtain the inclusion $W^2(\overline{\rho}) \subseteq W_{\text{mod}}(\overline{\rho})$ in Theorem 7.4. This is enough to prove the main Theorem 1.1.
Our results on the geometry of $R^{(1,0,1),\tau'}_{\mathbb{P}}$ in §5.3 can also be used to deduce new cases of automorphy lifting phenomena for unitary groups which are non-split at $p$. Indeed, the integrality of $R^{(1,0,1),\tau'}_{\mathbb{P}}$ (cf. §5.3.10 and Table 3) together standard Taylor–Wiles–Kisin arguments give the following Theorem (again, we refer the reader to the bulk of the paper for unfamiliar terminology):

**Theorem 1.4.** Let $F/F^+$ be a CM field extension of $F^+$ which is unramified at all finite places, suppose that $p$ is unramified in $F^+$ and that every place of $F^+$ above $p$ is inert in $F$.

Let $\rho' : \text{Gal}(\overline{Q}/F) \longrightarrow \text{GL}_2(\mathbb{Z}_p)$ be a continuous Galois representation, and let $\pi' : \text{Gal}(\overline{Q}/F) \longrightarrow \text{GL}_2(\mathbb{F}_p)$ denote the associated residual representation. Assume that

- $\rho'$ is unramified at all but finitely many places;
- we have $\rho'_{\overline{Q}^c} \cong \rho'_{\overline{Q}^v} \otimes \varepsilon^{-1}$, where $\varepsilon \in \text{Gal}(F/F^+)$ is the complex conjugation;
- for all places $v$ of $F$ above $p$, the local representation $\rho'_{\text{Gal}(\overline{Q}_p/F_v)}$ is potentially crystalline, with parallel Hodge type $(-1,0)$ and $4$-generic tame inertial type $\tau'_v$;
- for all places $v$ of $F$ above $p$, the local representation $\pi'_{\text{Gal}(\overline{Q}_p/F_v)}$ is semisimple and $4$-generic;
- $\rho'$ is unramified outside places above $p$;
- $\rho' \cong r_{i}(\pi)$ where $\pi$ is a cuspidal automorphic representation of $G(\mathbb{A}_{F^+})$, such that $\pi_{\infty}$ is trivial and for all places $v$ of $F^+$ above $p$, the local component $\pi_v$ contains the tame $U_2(\mathcal{O}_{F^+})$-representation associated to $\rho'_v$ by the inertial local Langlands correspondence (cf. Theorem 4.11);
- $\overline{Q}^v_{\ker(\text{ad}(\rho'))}$ does not contain $F(\mathcal{O}_p)$; and
- $\rho'(\text{Gal}(\overline{Q}/F)) \supseteq \text{GL}_2(\mathbb{F}')$ for some subfield $\mathbb{F}' \subseteq \mathbb{F}_p$ with $|\mathbb{F}'| > 6$.

Then $\rho'$ is automorphic.

(Recall that $\rho'$ is automorphic if $\rho' \otimes \overline{\sigma}_p$ is isomorphic to $r_{i}(\pi')$ for some cuspidal automorphic representation $\pi'$ of $G(\mathbb{A}_{F^+})$, where $r_{i}(\pi')$ is the continuous Galois representation associated to $\pi'$ as in Theorem 6.1.)

We conclude this introduction with a few remarks on natural questions which arise from the results in this paper.

In Theorems 1.1 and 1.4, the assumption that $\pi$ is unramified outside $p$ is used to simplify our arguments, and it should be possible to remove it. On the other hand removing the condition that the $L$-parameter is residually tame at places above $p$ requires further analysis of the possible set of modular weights $W^\text{mod}(\pi|_{\text{Gal}(\overline{Q}_p/F_v^+)}) \subseteq W^\text{s}(\pi|_{\text{Gal}(\overline{Q}_p/F_v^+)})$, and will depend in a subtle way on the geometry of $R^{(1,0,1),\tau'}_{\text{Gal}(\overline{Q}_p/F_v^+)}$.

In the case where $\pi|_{\text{Gal}(\overline{Q}_p/F_v^+)}$ is semisimple, the combinatorics of the set $W^\text{s}(\pi|_{\text{Gal}(\overline{Q}_p/F_v^+)})$ and the set of Jordan–Hölder constituents of tame types for $U_2(\mathcal{O}_{F_v^+})$ suggest that tame $U_2(\mathcal{O}_{F_v^+})$-representations will play the role of Breuil–Paskunas diagrams for non-split unitary groups. We expect these representations to be useful in constructing, by a purely local procedure, some mod-$p$ representations of $U_2(K)$ which naturally appear in the cohomology of Shimura curves with tame level at $p$. We hope to come back to these questions in future work.

The paper is organized as follows. In Section 2 we discuss the unitary groups over $\mathcal{O}_K$ which are relevant for this paper, namely the unramified unitary group in two variables $U_2(\mathcal{O}_K)$. In fact, in order to speak about Serre weight conjectures, we must work with a certain central extension $\widetilde{U}_2$ of $U_2$ constructed by Buzzard–Gee in [BG14]. We also define the $C$-group $C U_2$, which is the “classical” Langlands $L$-group of $\widetilde{U}_2$. We give explicit descriptions of the Galois actions on these
groups, their character groups, and their $\mathbb{F}_p$-structures. Since the groups appearing are slightly non-standard, we have attempted to give a detailed account.

Section 3 is devoted to the theory of types, that is, absolutely irreducible $\mathbf{U}_2(\mathbb{F}_q)$-representations over Frac($\mathbb{O}$), and their reductions over $\mathbb{F}$. In Subsection 3.3 we recall the notion of base change for types and compare it with local automorphic base change of smooth $\mathbf{U}_2(K)$-representations over $\mathbb{C}$. Then, in Subsections 3.4 and 3.5 we analyze the Jordan–Hölder constituents of the mod $p$ reductions of types vis-à-vis the constituents of the mod $p$ reductions of their base changes. This allows us to establish several useful properties of base change of Serre weights.

In Section 4.1 we study $L$-parameters of the form $\varrho : \text{Gal}(\overline{\mathbb{Q}}_p/K) \rightarrow C$. We relate these parameters to $\mathbf{U}_2(\mathbb{O}_K)$-representations to produce the set of (local) predicted weights $W^\prime(\varrho)$, as defined in [GHS18]. The core of this section is Subsection 4.3, which examines the compatibility between Serre weights of $L$-parameters and their base changes. To conclude, we establish Theorem 4.9 which figures in subsequent base change results.

Section 5 deals with local deformation theory of $C$-group valued $L$-parameters. We introduce the notion of Frobenius twist self-dual Kisin modules over $\mathcal{S}_R = (\mathbb{O}_K \otimes \mathbb{Z}_p \mathbb{R})[u]$ in Subsection 5.2, which are Kisin modules equipped with an isomorphism between their Frobenius pullback and their dual. Using this definition, we deduce the deformation theory of Frobenius twist self-dual Kisin modules from that of Kisin modules over $\mathcal{S}_R$ by means of base change (as in [LLHLM18]). The precise relation between deformation theory of Frobenius twist self-dual Kisin modules and $C$-group valued $L$-parameters is achieved in Subsection 5.3. In particular, we obtain an explicit presentation for the deformation rings $R_p^{(1,0,1),\tau'}$.

Sections 6 and 7 contain the main global applications, and the proof of the main theorem. In Subsections 6.1–6.3 we provide the background on algebraic automorphic forms on unitary groups which are quasi-split (but not split) at $p$, and the Galois representations associated to them, by generalizing the usual results in the literature for groups which are split at $p$ (see Theorem 6.2). We remark that the compatibility of base change of types as recalled in 3.3 and classical base change are integral to these generalizations. The main result of Section 6 is Theorem 6.7, which is the “weight elimination” statement.

In Subsections 7.1–7.3 we generalize the patching construction of [CEG+16] to our unitary groups (cf. Proposition 7.3). The modifications are largely formal, using as input the results from Subsection 6.3. The main result on “weight existence” is then obtained in Subsection 7.4, following the patching techniques of [GKL13]. The main result on automorphy lifting follows in Subsection 7.5.

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1.1. Notation. Let $p$ denote an odd prime number, and fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We denote its ring of integers by $\overline{\mathbb{Z}}_p$ and its residue field by $\overline{\mathbb{F}}_p$, and we assume that all field extensions of $\mathbb{Q}_p$ are contained in $\overline{\mathbb{Q}}_p$. Given a $p$-adic field $F$ and an element $x$ in its residue field, we define $\tilde{x}$ to be its Teichmüller lift. Throughout we will work with a finite extension $E$ of $\mathbb{Q}_p$ which will serve as our field of coefficients. We let $\mathcal{O}$ denote the ring of integers of $E$, $\varpi$ its uniformizer, and $\mathbb{F}$ its residue field. We will assume $E$ and $\mathbb{F}$ are sufficiently large as necessary.
For any field $F$, we let $\Gamma_F \overset{\text{def}}{=} \text{Gal}(\overline{F}/F)$ denote the absolute Galois group of $F$, where $\overline{F}$ is a fixed separable closure of $F$. If $F$ is a number field and $v$ is a place of $F$, we let $F_v$ denote the completion of $F$ at $v$, and use the notation $\text{Frob}_v$ to denote a geometric Frobenius element of $\Gamma_{F_v}$. If $F$ is a $p$-adic field, we let $I_F$ denote the inertia subgroup of $\Gamma_F$.

For $F$ either a number field or a $p$-adic field, we let $\varepsilon : \Gamma_F \longrightarrow \mathbb{Z}_p^\times$ denote the $p$-adic cyclotomic character, and let $\bar{\varepsilon}$ or $\omega$ denote its reduction mod $p$.

If $F$ is a $p$-adic field, $V$ a de Rham representation of $\Gamma_F$ over $E$, and $\kappa : F \hookrightarrow E$ an embedding, then we define $\text{HT}_\kappa(V)$ to be the multiset of Hodge–Tate weights with respect to $\kappa$. Thus, $\text{HT}_\kappa(V)$ contains $i$ with multiplicity $\dim_E(V \otimes_{F,\kappa} \overline{F}(i))^{\overline{F}}$. In particular, $\text{HT}_\kappa(\varepsilon) = \{-1\}$. Further, we let $\text{WD}(V)$ denote the Weil–Deligne representation associated to $V$, normalized so that $V \mapsto \text{WD}(V)$ is a covariant functor.

Let $F$ be a $p$-adic field. We let $\text{Art}_F : F^\times \longrightarrow \Gamma_F^{ab}$ denote the Artin map, which sends uniformizers to geometric Frobenius elements. Let $\text{rec}_C$ denote the Local Langlands correspondence of $[HT01]$, from isomorphism classes of smooth irreducible representations of $\text{GL}_n(F)$ over $C$ to isomorphism classes of $n$-dimensional, Frobenius-semisimple Weil–Deligne representations of the Weil group of $F$ (normalized to agree $\text{Art}_F$ in dimension 1). For a choice of isomorphism $\iota : \overline{E} \xrightarrow{\sim} C$, we define $\text{rec}_C^{\text{def}} = \iota^{-1} \circ \text{rec}_C \circ \iota$ to be the Local Langlands correspondence over $\overline{E}$.

All representations will live on vector spaces over $E$ or $\overline{E}$, or on $\mathcal{O}$-modules, unless otherwise indicated. By abuse of notation, we will generally not distinguish between a representation and its isomorphism class. If $G$ is a group, $H \leq G$ a normal subgroup, $\rho$ an $H$-representation and $g \in G$, we write $\rho^g$ to denote the $H$-representation given by $h \mapsto \rho(ghg^{-1})$.

Given a finite length representation $V$ of some group, we let $\text{JH}(V)$ denote its set of Jordan–Hölder factors. If $V$ denotes a representation of a (pro)finite group $G$ on a finite-dimensional $E$-vector space, then we may choose a $G$-stable $\mathcal{O}$-lattice $V^\circ$ inside $V$, and we write $V^{\sigma}$ for its reduction mod $\varpi$. By $[Ser77]$ Thm. 32, the set of Jordan–Hölder factors of $V^{\sigma}$ is independent of the choice of lattice $V^\circ$. We write $\text{JH}(V)$ for $\text{JH}(V^{\sigma})$.

We write matrix transposes on the right, so that $A^\top$ denotes the transpose of a matrix $A$. Given an (anti)automorphism $\theta$ of $\text{GL}_n(R)$ which commutes with the transpose, we write $A^\theta \top$ for $(A^\theta)^\top$; in particular, we write $A^{-\top}$ for $(A^{-1})^\top$.

## 2. Group-theoretic constructions

Our first task will be to introduce the groups which will be relevant to arithmetic applications. After defining unitary groups and certain central extensions in Subsections 2.1 and 2.2, we construct the dual groups with which we will be working in Subsection 2.3. For the sake of thoroughness, we also give explicit descriptions of the Galois actions and $\mathbb{F}_p$-structures. We mostly follow $[BG14]$ and $[GHS13]$, §9.

### 2.1. Unitary groups over $p$-adic fields.

#### 2.1.1. Let $f \geq 1$, and let $K$ denote the unramified extension of $\mathbb{Q}_p$ of degree $f$. We let $\mathcal{O}_K$ denote its ring of integers, with canonical uniformizer $p$, and identify its residue field with $F_q = \mathbb{F}_{p^f}$. We let $\varphi \in \Gamma_{\mathbb{Q}_p}$ denote a fixed lift of $\text{Art}_{\mathbb{Q}_p}(p) \in \Gamma_{\mathbb{Q}_p}^{ab}$; in particular, $\varphi$ is a geometric Frobenius element and we have $\varepsilon(\varphi) = 1$. The group $\Gamma_K$ is topologically generated by $\varphi^f$ and $I_K = (I_{\mathbb{Q}_p})$.

We let $K_2$ denote the unique unramified quadratic extension of $K$, and $\mathcal{O}_{K_2}$ its ring of integers. The group $\mathbf{U}_1(K) \subseteq \mathcal{O}_{K_2}^\times$ is defined as the kernel of the norm map $K_2^\times \longrightarrow K^\times$.

Fix a choice of root $\pi \overset{\text{def}}{=} (-p)^{1/(p^2-1)} \in \overline{\mathbb{Q}}_p$. We define a character $\bar{\omega}_q : \Gamma_{K_2} \longrightarrow \mathcal{O}_{K_2}^\times$ by

$$\gamma \mapsto \pi^{\gamma}.$$
We fix once and for all an embedding \( \varphi : K_2 \hookrightarrow E \), and define
\[
\tilde{\omega}_{2f} \overset{\text{def}}{=} \varphi \circ \tilde{\omega} : \Gamma K_2 \to \mathbb{O}^\times.
\]
We denote by \( \omega_{2f} \) the mod \( p \) reduction of \( \tilde{\omega}_{2f} \). Note that \( \omega_{2f}^{(p^2f-1)/(p-1)} = \omega \).

2.1.2. Let \( U_2 \) denote the algebraic group over \( \mathcal{O}_K \) given by
\[
U_2(R) = \left\{ g \in \text{GL}_2(\mathcal{O}_K \otimes_\mathbb{Z} R) : g^\top (x \oplus 1)^\top \Phi_2 g = \Phi_2 \right\},
\]
where \( R \) is an \( \mathcal{O}_K \)-algebra, and \( \Phi_2 \overset{\text{def}}{=} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \). Thus, \( U_2(\mathbb{Q}_p) \cong \text{GL}_2(\mathbb{Q}_p) \) obtains a \( \Gamma_K \)-action given by
\[
\gamma \cdot g = \begin{cases} 
  g^\gamma & \text{if } \gamma \in \Gamma K_2, \\
  (\Phi_2 g^{-\top} \Phi_2^{-1})^\gamma & \text{if } \gamma \in \Gamma_K \setminus \Gamma K_2.
\end{cases}
\]

2.1.3. Following [BG14], we set \( H \overset{\text{def}}{=} U_2 \), so that \( H \) is a canonical central extension
\[
1 \to G_m \to H \to U_2 \to 1
\]
of algebraic groups over \( \mathcal{O}_K \). The Galois action on \( G_m \) is the standard one, and \( H \) possesses a twisting element, in the terminology of \textit{op. cit.} We now recall the explicit construction of \( H \).

We proceed as follows. The group \( H \) is defined as a pushout followed by a pullback:
\[
\begin{array}{ccc}
1 & \longrightarrow & \mu_2 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G_m & \longrightarrow & \text{GL}_2 & \longrightarrow & \text{PGL}_2 & \longrightarrow & 1 \\
\downarrow & & & & & & & & & \downarrow \\
1 & \longrightarrow & G_m & \longrightarrow & H & \longrightarrow & U_2 & \longrightarrow & 1
\end{array}
\]
Concretely, \( H \) is the set of all pairs \((h, h')\), with \( h \in U_2, h' \in \text{GL}_2 \), subject to the condition that \( h \) and \( h' \) have the same image in \( \text{PGL}_2 \). The maps \( H \to U_2 \) and \( H \to \text{GL}_2 \) are the projections onto the corresponding factors, and the map \( \iota : G_m \to H \) is \( \lambda \mapsto (1, (1 0 \lambda)) \).

Note that the top two rows of the diagram above carry the standard (i.e., split) Galois action. In particular, the Galois action on the first factor of \( H \) is the one induced from \( U_2 \), while the Galois action on the second factor is the standard one.

2.1.4. Let \( T_U \) denote the diagonal maximal torus of \( U_2 \), and \( T_H \) its preimage in \( H \). Furthermore, let \( T_G, T_S, \) and \( T_P \) denote the diagonal maximal tori of \( \text{GL}_2, \text{SL}_2 \) and \( \text{PGL}_2 \), respectively. The character groups of these tori fit into a diagram:
\[
\begin{array}{ccc}
0 & \longrightarrow & X^*(T_P) \cong \mathbb{Z} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X^*(T_U) \cong \mathbb{Z}^2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X^*(T_S) \cong \mathbb{Z} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X^*(T_G) \cong \mathbb{Z}^2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X^*(T_H) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X^*(G_m) \cong \mathbb{Z} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\]

The isomorphisms appearing are the canonical ones. (The notation \( X^*(T_\bullet) \), for \( \bullet \in \{P, S, U, G, H\} \), stands for the character group of the torus \( T_\bullet \) over \( \mathbb{Q}_p \).)
We describe the remaining character group. The group $X^*(T_H)$ is a pushout, so we may identify it as

$$X^*(T_H) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X^*(T_U) \oplus X^*(T_G) \cong \mathbb{Z}^4 \right\} / \sim$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a+z & b+z \\ c+z & d+z \end{pmatrix}$$

for $z \in \mathbb{Z}$. The maps $X^*(T_U) \to X^*(T_H), X^*(T_G) \to X^*(T_H)$ are the inclusions into the corresponding factors, and the projection $X^*(T_H) \to X^*(G_m) \cong \mathbb{Z}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + d.$$  

2.1.5. We now consider cocharacter groups. The bottom two rows of the diagram above give the following commutative diagram:

$$
\begin{array}{c}
0 \to X_*(G_m) \cong \mathbb{Z} \overset{a' \mapsto (a',a')}{\longrightarrow} X_*(T_G) \cong \mathbb{Z}^2 \overset{(a',b') \mapsto a'-b'}{\longrightarrow} X_*(T_P) \cong \mathbb{Z} \to 0 \\
0 \to X_*(G_m) \cong \mathbb{Z} \longrightarrow X_*(T_H) \longrightarrow X_*(T_U) \cong \mathbb{Z}^2 \longrightarrow 0
\end{array}
$$

The isomorphisms are again the canonical ones.

We describe the remaining cocharacter group. The group $X_*(T_H)$ is a pullback, so we may identify it as

$$X_*(T_H) = \left\{ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in X_*(T_U) \oplus X_*(T_G) \cong \mathbb{Z}^4 : a' - b' = c' - d' \right\}.$$  

The maps $X_*(T_H) \to X_*(T_U), X_*(T_H) \to X_*(T_G)$ are the projections onto the corresponding factors, and the map $X_*(G_m) \cong \mathbb{Z} \to X_*(T_H)$ is

$$a' \mapsto \begin{pmatrix} 0 \\ a' \end{pmatrix}.$$  

2.1.6. The actions of $\Gamma_K$ on $X^*(T_H)$ and $X_*(T_H)$ are the ones induced from $X^*(T_U)$ and $X_*(T_U)$: they are both unramified, and we have

$$\varphi^f \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -a \\ b \\ -c \\ d \end{pmatrix}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X^*(T_H)$ and

$$\varphi^f \cdot \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} -b' \\ a' \\ -c' \\ d' \end{pmatrix}$$

for $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in X_*(T_H)$.  

The pairing $\langle -,- \rangle : X^*(T_H) \times X_*(T_H) \to \mathbb{Z}$ between characters and cocharacters is given by

$$\langle \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} \rangle = aa' + bb' + cc' + dd';$$

this is well-defined and Galois-invariant. The roots $\Phi_H \subseteq X^*(T_H)$ are given by $\{ \pm \alpha_H \}$, where

$$\alpha_H \overset{\text{def}}{=} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$
Likewise, the coroots $\Phi^\vee \subseteq X_*(T_H)$ are given by $\{ \pm \alpha_H^\vee \}$ where

$$\alpha_H^\vee \overset{\text{def}}{=} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$ 

We define the set of simple roots as $\Delta_H \overset{\text{def}}{=} \{ \alpha_H \}$, and let $B_H$ denote the corresponding Borel subgroup of $H$. We therefore have $\Delta_H^\vee = \{ \alpha_H^\vee \}$.

The group $H$ has a canonical twisting element, in the sense of [BG14]: tracing through the construction in op. cit., we obtain

$$\eta_H \overset{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in X^*(T_H).$$

This element is Galois-invariant, and $\langle \eta_H, \alpha^\vee_H \rangle = 1$.

The Weyl group of $H$ with respect to $T_H$ is denoted $W_H$; it is a cyclic group of order 2. We denote by $s$ the unique simple reflection, which generates $W_H$.

2.2. Unitary groups over $\mathbb{Q}_p$.

2.2.1. We now consider unitary groups over $\mathbb{Q}_p$. We set

$$(G, B, T) \overset{\text{def}}{=} \text{Res}_{\mathbb{Q}_K/\mathbb{Z}_p} (H, B_H, T_H),$$

all group schemes over $\mathbb{Z}_p$. We have

$$G(\mathbb{Q}_p) \cong \text{Ind}_{\mathbb{Q}_p}^{\mathbb{Q}_p} (H(\mathbb{Q}_p))$$

as $\Gamma_{\mathbb{Q}_p}$-groups, and via the evaluation maps, we have

$$(ev_1, ev_{\varphi^{-1}}, \ldots, ev_{\varphi^{f-1}}) : G(\mathbb{Q}_p) \xrightarrow{\sim} \prod_{i=0}^{f-1} H(\mathbb{Q}_p)$$

$$f \mapsto (f(\varphi^i)^{\varphi^{-i}})_{0 \leq i \leq f-1}.$$ 

Recall that $\varphi$ acts on $H$ by

$$\varphi^f \cdot (h_1, h_2) = \left( (\Phi_2 h_1^{-1} \Phi_2^{-1})^{\varphi^f}, h_2^{\varphi^f} \right).$$

Tracing through the isomorphisms above, the action of $\Gamma_{\mathbb{Q}_p}$ on the right-hand-side product is given as follows:

$$\varphi \cdot ((h_{0,1}, h_{0,2}), h_1, \ldots, h_{f-1}) = \left( h_1^\varphi, \ldots, h_{f-1}^\varphi, (\Phi_2 h_{0,1}^{-1} \Phi_2^{-1}, h_{0,2})^\varphi \right)$$

with inertia acting in the standard, diagonal way. In particular,

$$ev_1 : G(\mathbb{Q}_p) = G(\mathbb{Q}_p)^{\Gamma_{\mathbb{Q}_p}} \xrightarrow{\sim} H(\mathbb{Q}_p)^{\Gamma_K} = H(K) \cong \tilde{U}_2(K).$$

2.2.2. The character and cocharacter groups of the torus $T$ are given by

$$X^*(T) = \bigoplus_{i=0}^{f-1} X^*(T_H), \quad X_*(T) = \bigoplus_{i=0}^{f-1} X_*(T_H).$$

We will write elements of $X^*(T)$ as

$$\mu = \begin{pmatrix} a_0 & a_1 & \ldots & a_{f-1} \\ b_0 & b_1 & \ldots & b_{f-1} \\ c_0 & c_1 & \ldots & c_{f-1} \\ d_0 & d_1 & \ldots & d_{f-1} \end{pmatrix}$$

(and similarly for $X_*(T)$).
The perfect pairing \( \langle - , - \rangle: X^*(\mathbf{T}) \times X_*(\mathbf{T}) \to \mathbb{Z} \) is given by
\[
\langle \left( \begin{array}{c} a \\
\frac{b}{c} \\
\frac{d}{e} \end{array} \right), \left( \begin{array}{c} a' \\
\frac{b'}{c'} \\
\frac{d'}{e'} \end{array} \right) \rangle = \sum_{i=0}^{f-1} a_i a'_i + b_i b'_i + c_i c'_i + d_i d'_i,
\]
and the action of \( \Gamma_{Q_p} \) on \( X^*(\mathbf{T}) \) is given by
\[
\varphi \cdot \left( \begin{array}{c} a_0 \\
\frac{b_0}{c_0} \\
\frac{d_0}{e_0} \\
\frac{b_1}{c_1} \\
\frac{d_1}{e_1} \\
\cdots \\
\frac{b_{f-1}}{c_{f-1}} \\
\frac{d_{f-1}}{e_{f-1}} \end{array} \right) = \left( \begin{array}{c} a_0 \\
\frac{b_0}{c_0} \\
\frac{d_0}{e_0} \\
\frac{b_1}{c_1} \\
\frac{d_1}{e_1} \\
\cdots \\
\frac{b_{f-1}}{c_{f-1}} \\
\frac{d_{f-1}}{e_{f-1}} \end{array} \right), \quad \text{where } \varphi = \left( \begin{array}{ccc} a & b & c \\
0 & 0 & 0 \\
0 & 0 & 0 \end{array} \right). \]

An analogous action (i.e., with a “shift left”) holds for \( X_*(\mathbf{T}) \).

2.2.3. We define the simple roots \( \Delta \) as those functions \( f \) in \( \text{Ind}_{\Gamma_{K}}^{\Gamma_{Q_p}}(X^*(\mathbf{T}_H)) \) with image in \( \{ 0 \} \cup \Delta_H \), and such that \( f(\gamma) = 0 \) for all but a single coset. Explicitly, we have \( \Delta = \{ \alpha_i \}_{0 \leq i \leq f-1} \), where
\[
\alpha_i \overset{\text{def}}{=} \left( \begin{array}{ccc} 0 & 0 & 0 \\
-1 & 0 & 0 \\
\cdots & \cdots & \cdots \\
0 & 0 & 0 \end{array} \right) \in X^*(\mathbf{T}).
\]

We define \( \Delta^\vee \) analogously, and obtain \( \Delta^\vee = \{ \alpha_i^\vee \}_{0 \leq i \leq f-1} \), where
\[
\alpha_i^\vee \overset{\text{def}}{=} \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 0 & 0 \\
\cdots & \cdots & \cdots \\
0 & 0 & 0 \end{array} \right) \in X_*(\mathbf{T}).
\]

The Weyl group \( W \) of \( G \) with respect to \( \mathbf{T} \) is equal to \( W_{H}^f \). We shall write elements of \( W \) as \( w = (w_0, w_1, \ldots, w_{f-1}) \). The group \( W \) has a nontrivial Galois action given by
\[
\varphi \cdot (w_0, w_1, \ldots, w_{f-1}) = (w_{f-1}, w_{f-2}, \ldots, w_0).
\]

Finally, we define \( 1 \overset{\text{def}}{=} (1, 1, \ldots, 1) \) and \( s \overset{\text{def}}{=} (s, s, \ldots, s) \).

The map \( ev_1 \) induces a bijection \( X^*(\mathbf{T})_{F_{Q_p}} \rightarrow X^*(\mathbf{T}_H)^{\Gamma_K} \). In particular, the twisting element \( \eta_H \in X^*(\mathbf{T}_H)^{\Gamma_K} \) corresponds to the twisting element
\[
\eta \overset{\text{def}}{=} \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & 1 & 0 \\
-1 & 0 & 0 \end{array} \right) \in X^*(\mathbf{T})_{\Gamma_{Q_p}}.
\]

2.3. Dual groups. We now define the relevant Langlands dual groups.

2.3.1. The based root datum of \( \mathbf{U}_2 \) (with respect to the upper-triangular Borel subgroup) is given by
\[
(\text{X}^*(\mathbf{T}_U) \cong \mathbb{Z}^2, \{ (1, -1) \}, \text{X}_*(\mathbf{T}_U) \cong \mathbb{Z}^2, \{ (1, -1) \}).
\]

Therefore, we may take \( \hat{\mathbf{U}}_2 \overset{\text{def}}{=} \text{GL}_2 \) as the dual group, which we consider as a split group scheme over \( \mathbb{Z}_p \), along with its diagonal maximal torus, upper-triangular Borel subgroup, and the fixed isomorphism between \( G_a \) and the unipotent radical of the Borel given by \( x \mapsto (\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}) \). We equip this data with the canonical isomorphism between the based root datum of \( \hat{\mathbf{U}}_2 \) and the dual based root datum of \( \mathbf{U}_2 \). In choosing this isomorphism, we obtain an induced action of \( \Gamma_K \) on \( \hat{\mathbf{U}}_2 \) given by
\[
\gamma \cdot \hat{g} = \begin{cases} 
\hat{g} & \text{if } \gamma \in \Gamma_K, \\
\Phi_2^{-1} \hat{g} \Phi_2^{-1} = \left( \begin{array}{cc} \det(\hat{g})^{-1} & 0 \\
0 & \det(\hat{g})^{-1} \end{array} \right) \hat{g} & \text{if } \gamma \in \Gamma_K \setminus \Gamma_K.
\end{cases}
\]
2.3.2. Consider now the group $H = \widehat{U}_2$. The based root datum of $H$ is given by
\[ \Psi_H \overset{\text{def}}{=} (X^*(T_H), \Delta_H, X_*(T_H), \Delta_H^\vee), \]
and therefore the dual based root datum is
\[ \Psi_H^\vee = (X_*(T_H), \Delta_H^\vee, X^*(T_H), \Delta_H). \]
We let $\widehat{H}$ denote the dual group of $H$, with maximal torus $\widehat{T}_H$ and Borel $\widehat{B}_H$ which contains $\widehat{T}_H$. By [BGL] Prop. 5.39, we have
\[ \widehat{H} \cong (\widehat{U}_2 \times G_m) / \left\langle \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right), -1 \right\rangle = \text{GL}_2 \times_{\mu_2} G_m, \]
where the Galois action on $\widehat{H}$ is the one induced from $\widehat{U}_2$. We have an isomorphism
\[ \widehat{H} = \text{GL}_2 \times_{\mu_2} G_m \overset{\sim}{\longrightarrow} \text{GL}_2 \times G_m \]
\[ [\hat{h}, a] \longmapsto \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) \hat{h}, a^2 \]
and we will identify $\widehat{H}$ with $\text{GL}_2 \times G_m$ via this isomorphism. The Galois action is then given by
\[ \gamma \cdot (\hat{h}, a) = \left\{ \begin{array}{ll} \left( \begin{array}{cc} \hat{h}, a \\ \Phi_2 \hat{h}^{-1} & \Phi_2^{-1}, a \end{array} \right) & \text{if } \gamma \in \Gamma_{K_2}, \\ \left( \begin{array}{cc} a \text{det}(\hat{h})^{-1} & 0 \\ 0 & a \text{det}(\hat{h})^{-1} \end{array} \right) \hat{h}, a & \text{if } \gamma \in \Gamma_K \setminus \Gamma_{K_2}, \end{array} \right. \]
for $(\hat{h}, a) \in \text{GL}_2 \times G_m$.
Thus, we obtain the based root datum for $\widehat{H}$
\[ \Psi_{\widehat{H}} \overset{\text{def}}{=} \left( X^*(\widehat{T}_H), \Delta, X_*(\widehat{T}_H), \Delta^\vee \right) = (Z^3, \{(1, -1, 0)\}, Z^3, \{(1, -1, 0)\}), \]
equipped with an action of $\Gamma_K$. Moreover, we obtain an isomorphism of based root data $\phi : \Psi_H^\vee \overset{\sim}{\longrightarrow} \Psi_{\widehat{H}}$:
\[ \phi : X_*(T_H) \overset{\sim}{\longrightarrow} X^*(\widehat{T}_H) \]
\[ (a', b', c', d') \longmapsto (a', b', c' - a') \]
\[ (\phi^\vee)^{-1} : X^*(T_H) \overset{\sim}{\longrightarrow} X_*(\widehat{T}_H) \]
\[ (a, b, c, d) \longmapsto (a + c, b + d, c + d) \]
where the last coordinate in the character (resp. cocharacter) group of $\widehat{T}_H$ corresponds to the $G_m$ factor of $\widehat{H}$. Note that this exchanges the roots and coroots. We use this isomorphism we identify the Weyl group of $T$ with $W_H$.

2.3.3. Finally, we define
\[ C_U \overset{\text{def}}{=} \left. \text{Gal}(F/K) \times \text{Gal}(K_2/K) \right| = (\text{GL}_2 \times G_m) \times \text{Gal}(K_2/K), \]
with the Galois group acting on $\widehat{H}$ as above. The injection $\iota : G_m \longrightarrow H$ induces a dual map $\widehat{\iota} : \left. \text{Gal}(F/K) \longrightarrow \text{Gal}(K_2/K) \right|$, which is given by $(\hat{h}, a) \times \gamma \longmapsto a$.

**Remark 2.1.** We will need to make use of the above construction in a global setting as follows. Suppose $F/F^+$ is a quadratic extension of global fields, and let $v$ denote a place of $F^+$ which is unramified and inert in $F$, and such that $F_v^+ \cong K$ and $F_v \cong K_2$. We then identify $C_U$ with
\[ \widehat{H} \times \text{Gal}(F/F^+) \]
via the isomorphism $\text{Gal}(F/F^+) \cong \text{Gal}(F_v/F_v^+) \cong \text{Gal}(K_2/K)$. 


2.3.4. We set
\[ (\tilde{G}, \tilde{B}, \tilde{T}) \stackrel{\text{def}}{=} \text{Ind}_{\Gamma_K}^{\Gamma_{Q_p}} (\tilde{H}, \tilde{B}_H, \tilde{T}_H), \]
all group schemes over \( \mathbb{Z}_p \), equipped with the induced \( \Gamma_{Q_p} \)-action. Using the (induced versions of the) isomorphisms above, we consider \( \tilde{G} \) as the dual group of \( G \), and set
\[ L G \stackrel{\text{def}}{=} \tilde{G} \rtimes \text{Gal}(K_2/\mathbb{Q}_p). \]

2.4. **An isomorphism.** We briefly digress to recall a construction of \( C^* U_2 \) from [CHT08] (see also [BG13 §8.3]).

Let \( S_2 \) denote the group scheme over \( \mathbb{Z}_p \) which is a semidirect product of \( GL_2 \times G_m \) by \( \text{Gal}(K_2/K) \), with \( \varphi^f \in \text{Gal}(K_2/K) \) acting by
\[ \varphi^f \cdot (\tilde{h}, a) = \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{h}^{-1} \right), \]
There is an isomorphism between our model \( C^* U_2 \) and \( S_2 \) given as follows:
\[
C^* U_2 \xrightarrow{\sim} S_2
\]
\[ (\tilde{h}, a) \rtimes 1 \mapsto \left( \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{h}, a^{-1} \right) \rtimes 1 \]
\[ (\tilde{h}, a) \rtimes \varphi^f \mapsto \left( \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{h} \Phi_2, a^{-1} \right) \rtimes \varphi^f \]
\[ \left( \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{h} \Phi_2, a^{-1} \right) \rtimes 1 \iff (\tilde{h}, a) \rtimes 1 \]
\[ \left( \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix} \tilde{h} \Phi_2, a^{-1} \right) \rtimes \varphi^f \iff (\tilde{h}, a) \rtimes \varphi^f. \]

The group \( S_2 \) also possesses a map \( \nu : S_2 \longrightarrow G_m \), given by \( (\tilde{h}, a) \rtimes (\varphi^f)^i \mapsto (-1)^i a \). Under the isomorphism above, this corresponds to the map \( \hat{\nu} : C^* U_2 \longrightarrow G_m \).

As in Remark 2.1, we will often identify \( S_2 \) with \( (GL_2 \times G_m) \rtimes \text{Gal}(F/F^+) \).

2.5. **\( \mathbb{F}_p \)-structures.**

2.5.1. Viewing \( G \) and \( \tilde{G} \) as group schemes over \( \mathbb{Z}_p \), we can form the \( \mathbb{F}_p \)-group schemes:
\[
(G, B, T) \stackrel{\text{def}}{=} (G, B, T) \times_{\mathbb{Z}_p} \mathbb{F}_p, \quad (G^*, B^*, T^*) \stackrel{\text{def}}{=} (\tilde{G}, \tilde{B}, \tilde{T}) \times_{\mathbb{Z}_p} \mathbb{F}_p.
\]
We denote the Frobenius structures on \( G \) and \( G^* \) by \( F \) and \( F^* \), respectively.

The map \( F \) is given by
\[
F(h_0, \ldots, h_{f-2}, (h_{f-1,1}, h_{f-1,2})) = \left( (\Phi_2 h_{f-1,1}^{-1} \Phi_2^{-1}, h_{f-1,2})^{(p)}, h_0^{(p)}, \ldots, h_{f-2}^{(p)} \right),
\]
where \( h^{(p)} \) means raising entries to the \( p \)-th power. In particular, we have \( G^F = G(\mathbb{F}_p) = H(\mathbb{F}_q) \).

Also, note that \( F \) is equal to \( p \varphi^{-1} \) on \( T \). The action of \( F \) on \( X^*(T) \) is defined by \( F(\chi) = \chi \circ F \), and therefore is explicitly given by
\[
F \left( \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \\ a_{f-1} \\ b_{f-1} \\ c_{f-1} \\ d_{f-1} \end{pmatrix} \right) = \left( \begin{pmatrix} p a_1 \\ p b_1 \\ p c_1 \\ p d_1 \\ p a_{f-1} \\ p b_{f-1} \\ p c_{f-1} \\ p d_{f-1} \end{pmatrix} \right)
\]

The map \( F^* \) is defined as \( \text{Fr} \circ \varphi \) where \( \text{Fr} \) is the relative Frobenius on the split group \( G^* \).

Therefore,
\[
F^* \left( \left( \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \\ a_{f-1} \\ b_{f-1} \\ c_{f-1} \\ d_{f-1} \end{pmatrix} \right) \right) = \left( \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \\ a_{f-1} \\ b_{f-1} \\ c_{f-1} \\ d_{f-1} \end{pmatrix} \right)^{(p)}.
\]
In particular, the action of $F^*$ on $X_\ast(T^*)$ is given by $F^*(\lambda) = F^* \circ \lambda$, and therefore is explicitly given by (after chasing through isomorphisms of root data)

$$F^* \left( \frac{a_0}{b_0} \frac{c_0}{d_0} \right) \left( \frac{a_1}{b_1} \frac{c_1}{d_1} \right) \cdots \left( \frac{a_{f-1}}{b_{f-1}} \frac{c_{f-1}}{d_{f-1}} \right) = \left( \frac{p^{a_0} \cdot b_0}{p^{b_0} \cdot c_0} \right) \cdots \left( \frac{p^{a_{f-1}} \cdot b_{f-1}}{p^{b_{f-1}} \cdot c_{f-1}} \right) \left( \frac{p^{a_{f-1}} \cdot b_{f-1}}{p^{b_{f-1}} \cdot c_{f-1}} \right) .$$

3. Representation theory

We now collect various results we will use regarding types and weights for the groups $\tilde{U}_2(F_q)$ and $GL_2(F_q)$. We give definitions of base change for both types and weights in Subsections 3.3 and 3.5, respectively, and relate the former to automorphic base change. Subsection 3.4 discusses various compatibilities between types and weights, and contains useful combinatorial properties which will be employed extensively in the applications which follow.

3.1. The group $G$.

3.1.1. Let $X_+(T)$, $X_1(T)$ and $X_0(T)$ denote respectively the subsets of $X_\ast(T)$ consisting of dominant, $p$-restricted, and inner-product-zero elements:

$$X_+(T) \overset{\text{def}}{=} \{ \mu \in X_\ast(T) : 0 \leq (\mu, \alpha_i^\vee) \text{ for all } 0 \leq i \leq f - 1 \}$$

$$X_1(T) \overset{\text{def}}{=} \{ \mu \in X_\ast(T) : 0 \leq (\mu, \alpha_i^\vee) \leq p - 1 \text{ for all } 0 \leq i \leq f - 1 \}$$

$$X_0(T) \overset{\text{def}}{=} \{ \mu \in X_\ast(T) : (\mu, \alpha_i^\vee) = 0 \text{ for all } 0 \leq i \leq f - 1 \} .$$

3.1.2. Recall that a Serre weight of $G(F_p)$ is an irreducible representation of $G(F_p)$ on an $F_p$-vector space. Given $\mu \in X_+(T)$, we let $F(\mu)$ denote the restriction to $G(F_p)$ of the algebraic $G$-representation of highest weight $\mu$. We then have the following result.

**Proposition 3.1** ([GHS18], Lemma 9.2.4). The map

$$\frac{X_1(T)}{(F - 1)X_0(T)} \xrightarrow{\mu \mapsto F(\mu)} \{ \text{Serre weights of } G(F_p) \} / \sim$$

is a well-defined bijection.

We will always assume that the coefficient field $F$ is large enough so that the representations $F(\mu)$ may be realized over $F$.

**Definition 3.2.** Given a character $\mu \in X_\ast(T)$, we say $\mu$ lies $n$-deep in the fundamental alcove if we have

$$n < (\mu + \eta, \alpha_i^\vee) < p - n$$

for all $0 \leq i \leq f - 1$. We say a Serre weight $F$ is $n$-deep if we can write $F \cong F(\mu)$ for some $n$-deep character $\mu$. (Note that this notion is independent of the choice of $\mu$.)

3.1.3. We likewise consider Deligne–Lusztig representations for the group $G(F_p)$, as in [GHS18 §9.2]. In particular, for $w \in W$ and $\mu \in X_\ast(T)$ such that $(T_w, \theta_{w,\mu})$ is maximally split, we let $R_w(\mu)$ denote the associated Deligne–Lusztig representation, a representation of $G(F_p)$ over $\overline{Q}_p$. We again assume the coefficient field $E$ is large enough so that $R_w(\mu)$ may be realized over $E$. Using the surjection $G(Z_p) \rightarrow G(F_p)$, we will occasionally view Serre weights and Deligne–Lusztig representations as representations of the compact group $G(Z_p) \cong \tilde{U}_2(\mathcal{O}_K)$.

By [Her09] §4.1, if $(w, \mu), (w', \mu') \in W \times X_\ast(T)$ are two pairs, we have an isomorphism

$$(3.1.1) \quad R_w(\mu) \cong R_{w'w}(F(w')^{-1} (w'(\mu) + F(\mu') - w'w F(w')^{-1}(\mu'))) .$$

**Definition 3.3.** Let $\sigma$ denote a Deligne–Lusztig representation. We say $\sigma$ is $n$-generic if there is an isomorphism $\sigma \cong R_w(\mu + \eta)$, where $\mu$ lies $n$-deep in the fundamental alcove.
3.1.4. We shall also need to know how the representations \( R_w(\mu) \) decompose upon reduction mod \( p \). To this end, we define the following elements of \( X^*(T) \). Fix \( w = (w_0, w_1, \ldots, w_{f-1}) \in W \), and set

\[
\rho_w \overset{\text{def}}{=} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \cdots, \quad \varepsilon_w \overset{\text{def}}{=} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \cdots, \\
\gamma_w \overset{\text{def}}{=} \cdots \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \cdots, \quad \rho \overset{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

Suppose that \( \mu \in X^*(T) \) is such that \( \mu - \eta \) is 1-deep. By the main theorem in the appendix of [Her09], we have

\[
\text{JH}\left(\overline{R_w(\mu)}\right) = \{ F_{w'}(\overline{R_w(\mu)}) \}_{w' \in W},
\]

where

\[
F_{w'}(\overline{R_w(\mu)}) \overset{\text{def}}{=} F \left( p\gamma_{w'} + w' (\mu - w \varepsilon_{\gamma_w}) + pw' - \pi(\rho) \right),
\]

and where \( \pi \) denotes the action of \( \varphi^{-1} \) on \( X^*(T) \).

**Definition 3.4.** Let \( \mu \in X_1(T) \) be a \( p \)-restricted character and \( R_w(\mu) \) a Deligne–Lusztig representation. We define the Deligne–Lusztig representation \( \beta(R_w(\mu)) \) by

\[
\beta(R_w(\mu)) \overset{\text{def}}{=} R_{2w}(\varphi(\mu - \eta) + (p-1)\eta).
\]

Note that, if \( n < \langle \mu, \alpha_i^\vee \rangle < p - n \), then \( \varphi(\mu - \eta) + (p-1)\eta \) will satisfy the same set of inequalities. Therefore if \( R_w(\mu) \) is \( n \)-generic, then \( \beta(R_w(\mu)) \) will also be \( n \)-generic.

3.1.5. Finally, suppose \( \mu \in X^*(T) \) is a character of the form

\[
\mu = \begin{pmatrix} a_0 \\ b_0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} a_{f-1} \\ b_{f-1} \\ c_{f-1} \\ 0 \end{pmatrix}
\]

(that is, suppose \( \mu \) is in the image of \( X^*(\text{Res}_{O_K/\mathbb{Z}_p}(T_H)) \hookrightarrow X^*(T) \)). Then \( R_w(\mu) \) is a representation of \( G(\mathbb{Z}_p) = \widehat{U}_2(O_K) \) on which \( i(O_K^\times) \) acts trivially, and therefore we view it as a representation of \( \widehat{U}_2(O_K)/i(O_K^\times) = U_2(O_K) \). Conversely, if \( R_w(\mu) \) is a representation of \( G(\mathbb{Z}_p) \) on which \( i(O_K^\times) \) acts trivially, then \( \mu \) is a character of the form

\[
\mu = \begin{pmatrix} a_0 \\ b_0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} a_{f-1} \\ b_{f-1} \\ c_{f-1} \\ 0 \end{pmatrix}
\]

with \( \sum_{i=0}^{f-1} c_i p^i \equiv 0 \pmod{p^f - 1} \). By applying the equivalence \( R_w(\mu) \cong R_w(\mu + (F - w)\mu) \) for an appropriately defined element \( \mu' \in X^*(T) \) and using the equivalence relation on \( X^*(T_H) \), we may assume \( \mu \) is of the form

\[
\mu = \begin{pmatrix} a'_0 \\ b'_0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} a'_1 \\ b'_1 \\ c'_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} a'_{f-1} \\ b'_{f-1} \\ c'_{f-1} \\ 0 \end{pmatrix}
\]

(one can even take \( \mu' \in X_0(T) \)).

3.2. The group \( GL_2 \).
3.2.1. Set 
\[
(G', T') \overset{\text{def}}{=} \text{Res}_{\mathbb{O}_K/\mathbb{Z}_p} (\text{GL}_2/\mathbb{O}_K) = T_{G'/\mathbb{O}_K},
\]
so that \( G'(\mathbb{F}_p) = \text{GL}_2(\mathbb{F}_q^2) \). We identify the maximal torus \( T_{G'/\mathbb{O}_K} \) of \( \text{GL}_2/\mathbb{O}_K \), with \( T_{\mathbb{U}} \times \mathbb{O}_K \mathbb{O}_K \), and therefore we identify \( X^*(T') \) with two copies of \( X^*(\text{Res}_{\mathbb{O}_K/\mathbb{Z}_p}(T_{\mathbb{U}})) \). We write elements of \( X^*(T') \) as \((\mu, \mu')\) with \( \mu, \mu' \in X^*(\text{Res}_{\mathbb{O}_K/\mathbb{Z}_p}(T_{\mathbb{U}})) \). In particular, if \( \mu \) is an element of \( X^*(T) \) of the form 
\[
\mu = \begin{pmatrix} a_{00} & a_{11} & \cdots & a_{f-1} \\ b_{00} & b_{11} & \cdots & b_{f-1} \end{pmatrix},
\]
we will identify it with 
\[
(a_{00}, b_{00}) \cdots (a_{f-1}, b_{f-1}) \in X^*(\text{Res}_{\mathbb{O}_K/\mathbb{Z}_p}(T_{\mathbb{U}})),
\]
and consider expressions such as \((\mu, \mu')\) or \((\mu, -\mathbb{Z}(\mu))\) in \( X^*(T') \).

Suppose \( \mathbb{X}_0, \mathbb{Y}_0 \in \mathbb{X}(T') \) is large enough so that \( \mathbb{X}_0 \) is 1-deep. The analog of (3.1.2) takes the following form:

\[
\text{Res}_{\mathbb{O}_K/\mathbb{Z}_p}(T_{\mathbb{U}}) \to \mathbb{X}^*(T') \quad \mathbb{X}_0 = (w_1, w_2, \ldots, w_n) \to \mathbb{X}_0 \to (w_1, w_2, \ldots, w_n).
\]

3.2.2. We define the subsets \( X_+(T'), X_1(T') \) and \( X^0(T') \) as above, and denote by \( F'(\mu) \) the restriction to \( G'(\mathbb{F}_p) \leq \text{GL}_2(\mathbb{F}_q^2) \) of the algebraic \( G' \)-representation of highest weight \( \mu \in X_+(T') \). In particular, we have the following classification result.

**Proposition 3.5** ([**GHS18**], Lemma 9.2.4). The map 
\[
\frac{X_1(T')}{(F' - 1)X^0(T')} \to \{ \text{Serre weights of } \text{GL}_2(\mathbb{F}_q^2) \}/\cong \\
\mu \to F'(\mu)
\]
is a well-defined bijection.

Once again, we will assume that \( \mathbb{F} \) is large enough so that \( F'(\mu) \) may be realized over \( \mathbb{F} \).

3.2.3. We define Deligne–Lusztig representations \( R'_w(\mu) \) for \( w \in W', \mu \in X^*(T') \) analogously to the above. We again assume that \( R'_w(\mu) \) may be realized over \( E \). Furthermore, an analog of equation (3.1.1) holds. We will often view \( F'(\mu) \) and \( R'_w(\mu) \) as representations of \( G'(Z_p) \cong \text{GL}_2(\mathbb{O}_K) \) by inflation.

3.2.4. Given \( w = (w_0, w_1, \ldots, w_2f-1) \in W' \), we define the following elements of \( X^*(T') \):

\[
\begin{array}{cccc}
\rho'_{w} & \overset{\text{def}}{=} & \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1^{w_0} & 1^{w_1} & \cdots & 1^{w_{2f-1}} \\ \end{pmatrix} \\
\varepsilon'_{w} & \overset{\text{def}}{=} & \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1^{w_0} & 1^{w_1} & \cdots & 1^{w_{2f-1}} \\ \end{pmatrix} \\
\gamma'_{w} & \overset{\text{def}}{=} & \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1^{w_0} & 1^{w_1} & \cdots & 1^{w_{2f-1}} \\ \end{pmatrix} \\
\rho' & \overset{\text{def}}{=} & \begin{pmatrix} 1 & 0 \\ 1^{w_0} & 1^{w_1} \\ \end{pmatrix}
\end{array}
\]

Suppose \( \mu \in X^*(T') \) is such that \( \mu - \rho' \) is 1-deep. The analog of (3.1.2) takes the following form:

\[
\text{JH}(\overline{R'_w(\mu)}) = \{ F'_{w'}(R'_w(\mu)) \}_{w' \in W'},
\]

where
where 
\[
F'_w(R'_w(\mu)) \overset{\text{def}}{=} F'(p\gamma'_w + w'(\mu - w(p'(\varepsilon'(s)w))) + pw'_w - \rho'),
\]
and where \(\pi'\) is the automorphism of \(X^\ast(T')\) such that \(F' = p\pi'^{-1}\).

**Definition 3.6.** Let \(\mu \in X_1(T')\) be a \(p\)-restricted character, \(w \in W'\), and \(R'_w(\mu)\) a Deligne–Lusztig representation of \(GL_2(\mathbb{F}_q^2)\). We define the Deligne–Lusztig representation \(\beta'(R'_w(\mu))\) by
\[
\beta'(R'_w(\mu)) \overset{\text{def}}{=} R'(s)(s)(\mu - \rho') + (p - 1)\rho).
\]
As with the map \(\beta\), if \(R'_w(\mu)\) is \(n\)-generic, then \(\beta'(R'_w(\mu))\) will also be \(n\)-generic.

### 3.3. Base change of types.

Our next task will be to define a notion of base change for tame types of \(U_2(\mathbb{O}_K)\). We note that this is essentially the Shintani lifting considered in [Kaw77].

**3.3.1.** We first recall the classification of irreducible representations of \(U_2(\mathbb{F}_q)\) in characteristic zero (cf. [Emm63]).

Fix a character \(\psi : \mathbb{F}_q^\times \rightarrow \mathbb{O}_K^\times\), which we also view as a character of \(B_U(\mathbb{F}_q)\) via
\[
\psi \left( \begin{pmatrix} x & y \\ 0 & x^{-q} \end{pmatrix} \right) = \psi(x),
\]
where \(x \in \mathbb{F}_q^\times\), \(y \in \mathbb{F}_q^\times\), and \(xy^q = yx^q\). (Here \(B_U\) denotes the upper triangular Borel subgroup of \(U_2\).) We let \(\text{Ind}_{B_U(\mathbb{F}_q)}^{U_2(\mathbb{F}_q)}(\psi)\) denote the induced representation. If \(\psi^{-q} \neq \psi\), then \(\text{Ind}_{B_U(\mathbb{F}_q)}^{U_2(\mathbb{F}_q)}(\psi)\) is irreducible. On the other hand, if \(\psi^{-q} = \psi\), then \(\psi\) extends to a character of \(U_2(\mathbb{F}_q)\), and we have
\[
\text{Ind}_{B_U(\mathbb{F}_q)}^{U_2(\mathbb{F}_q)}(\psi) \cong \psi \oplus (\psi \otimes E \text{St}),
\]
where \(E \text{St}\) denotes the irreducible representation \(\text{Ind}_{B_U(\mathbb{F}_q)}^{U_2(\mathbb{F}_q)}(1)/1\).

Consider now the group \(J_{\text{end}} \overset{\text{def}}{=} U_1 \times U_1\) over \(\mathbb{O}_K\), which is the unique elliptic endoscopic group of \(U_2\). Fix a character
\[
\theta = \theta_1 \otimes \theta_2 : J_{\text{end}}(\mathbb{F}_q) = U_1(\mathbb{F}_q) \times U_1(\mathbb{F}_q) \rightarrow \mathbb{O}_K^\times,
\]
\[(x, y) \mapsto \theta_1(x)\theta_2(y).
\]
We suppose that \(\theta_1 \neq \theta_2\), and let \(\sigma(\theta)\) denote the associated irreducible cuspidal representation of \(U_2(\mathbb{F}_q)\), as in [Bla10, §3.1(b)].

We have the following classification theorem.

**Theorem 3.7 ([Emm63]).** Any irreducible representation of \(U_2(\mathbb{F}_q)\) over \(E\) is isomorphic to one of the following:

- \(\psi\), where \(\psi\) is a character of \(U_2(\mathbb{F}_q)\);
- \(\psi \otimes E \text{St}\), where \(\psi\) is a character of \(U_2(\mathbb{F}_q)\);
- \(\text{Ind}_{B_U(\mathbb{F}_q)}^{U_2(\mathbb{F}_q)}(\psi)\), where \(\psi\) is a character of \(\mathbb{F}_q^\times\) which satisfies \(\psi^{-q} \neq \psi\);
- \(\sigma(\theta)\), where \(\theta = \theta_1 \otimes \theta_2\) is a character of \(U_1(\mathbb{F}_q) \times U_1(\mathbb{F}_q)\) with \(\theta_1 \neq \theta_2\).

The only isomorphisms among these representations are \(\text{Ind}_{B_U(\mathbb{F}_q)}^{U_2(\mathbb{F}_q)}(\psi) \cong \text{Ind}_{B_U(\mathbb{F}_q)}^{U_2(\mathbb{F}_q)}(\psi^{-q})\) and \(\sigma(\theta_1 \otimes \theta_2) \cong \sigma(\theta_2 \otimes \theta_1)\).

**Definition 3.8.** We define a tame type \(\sigma\) to be an irreducible \(U_2(\mathbb{O}_K)\)-representation over \(E\) which arises by inflation from an irreducible \(U_3(\mathbb{F}_q)\)-representation over \(E\). Likewise, we define a tame type over \(\mathbb{O}\) to be a representation \(\sigma\) of \(U_2(\mathbb{O}_K)\) on a finite-free \(\mathbb{O}\)-module, such that \(\sigma \otimes \mathbb{O}\) is a tame type over \(E\). We make similar definitions for the group \(GL_2(\mathbb{O}_K)\).
3.3.2. The principal series case. Consider again a character
\[ \psi : \mathbb{F}^{\times}_{q^2} \rightarrow \mathcal{O}^{\times} \]
which satisfies \( \psi^{-q} \neq \psi \), and let \( \text{Ind}_{\mathcal{B}_{U}(\mathbb{F}_{q^2})}^{\text{GL}_{2}(\mathbb{F}_{q^2})}(\psi) \) denote the (irreducible) principal series representation. We may extend the character \( \psi \) to a character \( \psi \otimes \psi^{-q} \) of \( \mathcal{B}_{U}(\mathbb{F}_{q^2}) \) as follows:

\[ \psi \otimes \psi^{-q} : \mathcal{B}_{U}(\mathbb{F}_{q^2}) \rightarrow \mathcal{O}^{\times} \]

\[ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \rightarrow \psi(x)\psi(z)^{-q}, \]

where \( x, z \in \mathbb{F}^{\times}_{q^2}, y \in \mathbb{F}_{q^2} \). We consider the (irreducible) induced representation \( \text{Ind}_{\mathcal{B}_{U}(\mathbb{F}_{q^2})}^{\text{GL}_{2}(\mathbb{F}_{q^2})}(\psi \otimes \psi^{-q}) \) of \( \mathcal{B}_{U}(\mathbb{F}_{q^2}) = \text{GL}_{2}(\mathbb{F}_{q^2}) \), and view it as a tame type of \( \text{GL}_{2}(\mathcal{O}_{K^2}) \) by inflation.

**Definition 3.9.** Let \( \psi : \mathbb{F}^{\times}_{q^2} \rightarrow \mathcal{O}^{\times} \) be a character such that \( \psi^{-q} \neq \psi \). We define the base change of \( \text{Ind}_{\mathcal{B}_{U}(\mathbb{F}_{q^2})}(\psi) \) to be the \( \text{GL}_{2}(\mathcal{O}_{K^2}) \)-type given by

\[ \text{BC} \left( \text{Ind}_{\mathcal{B}_{U}(\mathbb{F}_{q^2})}(\psi) \right) \overset{\text{def}}{=} \text{Ind}_{\mathcal{B}_{U}(\mathbb{F}_{q^2})}^{\text{GL}_{2}(\mathbb{F}_{q^2})}(\psi \otimes \psi^{-q}). \]

There is a compatibility of this definition with automorphic base change, as follows. Let \( \sigma = \text{Ind}_{\mathcal{B}_{U}(\mathbb{F}_{q^2})}(\psi) \), and suppose \( \pi \) is a smooth irreducible representation of \( \mathcal{U}_{2}(K) \) over \( \mathbb{C} \) such that \( \sigma \otimes E \mathcal{C} \subseteq \pi|_{\mathcal{U}_{2}(\mathcal{O}_{K})} \) (for some choice of morphism \( E \rightarrow \mathbb{C} \)). This implies \( \pi \) is an irreducible principal series representation, and we let \( \text{BC}(\pi) \) denote the stable base change of \( \pi \) to a representation of \( \text{GL}_{2}(K) \) (cf. [Rog90, §11.4]). Then \( \text{BC}(\pi) \) contains a unique tame type, which is isomorphic to \( \text{BC}(\sigma) \otimes_{E} \mathbb{C} \).

3.3.3. We now wish to compute the base change map on Deligne–Lusztig representations. Let \( \mu \in X^{\times}(\mathbf{T}) \) be such that

\[ \mu = \begin{pmatrix} a_0 & \cdots & a_{f-1} \\ b_0 & \cdots & b_{f-1} \\ 0 & \cdots & 0 \end{pmatrix}, \]

and suppose

\[ \sum_{i=0}^{f-1} a_ip^i - p^f \sum_{i=0}^{f-1} b_ip^i \neq \sum_{i=0}^{f-1} b_ip^i - p^f \sum_{i=0}^{f-1} a_ip^i \pmod{p^{2f} - 1}. \]

By [DL76, Prop. 8.2] we have an isomorphism of \( \mathcal{U}_{2}(\mathcal{O}_{K}) \)-representations

\[ R_{\mathbf{1}}(\mu) \cong \text{Ind}_{\mathcal{B}_{U}(\mathbb{F}_{q^2})}^{\mathcal{U}_{2}(\mathbb{F}_{q^2})}(\theta_{\mu}), \]

where

\[ \theta_{\mu} \left( \begin{pmatrix} x & y \\ 0 & x^{-q} \end{pmatrix} \right) = s_{0} \left( x^{\sum_{i=0}^{f-1} a_ip^i - p^f \sum_{i=0}^{f-1} b_ip^i} \right) \]

(recall that we identify representations of \( \mathcal{U}_{2}(\mathcal{O}_{K}) \) trivial on \( i(\mathcal{O}_{K}^{\times}) \) and representations of \( \mathcal{U}_{2}(\mathcal{O}_{K}) \)). Further, the assumption on \( \mu \) and Proposition 7.4 of op. cit. imply that \( R_{\mathbf{1}}(\mu) \) is irreducible. Consequently, the base change map becomes

\[ \text{BC} \left( R_{\mathbf{1}}(\mu) \right) = R_{\mathbf{1}}^{r_{\mathbf{1},\mathbf{1}}} \left( \mu, -s(\mu) \right). \]

Now let \( w \in W \) be an element in the \( \mathbf{F} \)-conjugacy class of \( \mathbf{1} \), and choose \( w' \in W \) such that \( w'wF(w')^{-1} = 1 \). Applying first the equivalence (3.3.1) for the element \( w' \), then the above equation, then the equivalence induced by \( (w'^{-1}, w'^{-1}) \), we obtain

\[ (3.3.1) \quad \text{BC} \left( R_{w}(\mu) \right) \cong R_{w,w'}^{r_{w,w}} \left( \mu, -s(\mu) \right). \]
3.3.4. The cuspidal case. Consider again the character \( \theta = \theta_1 \otimes \theta_2 \) of \( J_{\text{end}}(\mathbb{F}_q) \) which satisfies \( \theta_1 \neq \theta_2 \). By base change we obtain the character

\[
\tilde{\theta} : J_{\text{end}}(\mathbb{F}_q^2) = \mathbb{F}_q^2 \times \mathbb{F}_q^2 \rightarrow \mathcal{O}^\times \\
(x, y) \mapsto \theta_1(x^{-1}) \theta_2(y^{-1}).
\]

By inflation, we view this as a character of upper triangular Borel subgroup \( B_{\mathbb{U}}(\mathbb{F}_q^2) \) of \( U_2(\mathbb{F}_q^2) \cong \text{GL}_2(\mathbb{F}_q^2) \), and view the (irreducible) induced representation \( \text{Ind}_{B_{\mathbb{U}}(\mathbb{F}_q^2)}^{\text{GL}_2(\mathbb{F}_q^2)}(\tilde{\theta}) \) as a tame type of \( \text{GL}_2(\mathcal{O}_{K_2}) \).

**Definition 3.10.** Let \( \theta = \theta_1 \otimes \theta_2 : J_{\text{end}}(\mathbb{F}_q) \rightarrow \mathcal{O}^\times \) be a character such that \( \theta_1 \neq \theta_2 \). We define the base change of \( \sigma(\theta) \) to be the \( \text{GL}_2(\mathcal{O}_{K_2}) \)-type given by

\[
\text{BC}(\sigma(\theta)) \overset{\text{def}}{=} \text{Ind}_{B_{\mathbb{U}}(\mathbb{F}_q^2)}^{\text{GL}_2(\mathbb{F}_q^2)}(\tilde{\theta}).
\]

We again have a compatibility of this definition with automorphic base change. Let \( \sigma = \sigma(\theta) \), and suppose \( \pi \) is a smooth irreducible representation of \( U_2(K) \) over \( \mathbb{C} \) such that \( \sigma \otimes_E \mathbb{C} \subseteq \pi|_{U_2(\mathcal{O}_K)} \) (for some choice of morphism \( E \hookrightarrow \mathbb{C} \)). This implies that \( \pi \) is a level 0 supercuspidal representation, and we let \( \text{BC}(\pi) \) denote the stable base change of (the \( L \)-packet containing) \( \pi \). Then \( \text{BC}(\pi) \) contains a unique tame type, which is isomorphic to \( \text{BC}(\sigma) \otimes_E \mathbb{C} \) (see [Bla10, Cor. 3.6]).

3.3.5. We now wish to compute the base change map on cuspidal Deligne–Lusztig representations. Let \( \mu \in X^*(T) \) be such that

\[
\mu = \left( \begin{array}{c} a_0 \\ b_0 \\ 0 \\ b_1 \\ 0 \\ \vdots \\ a_{f-1} \\ b_{f-1} \\ 0 \end{array} \right),
\]

and suppose that \( \sum_{i=0}^{f-1} a_ip^i \neq \sum_{i=0}^{f-1} b_ip^i \pmod{p^f+1} \). By [DL76, Thm. 8.3], this assumption guarantees that \( R_{(s,1,\ldots,1)}(\mu) \) is an irreducible, cuspidal \( U_2(\mathcal{O}_K) \)-representation. Then a straightforward character computation using [Emn63, §6] and [DL76, Cor. 7.2] gives

\[
R_{(s,1,\ldots,1)}(\mu) \cong \sigma(\mu),
\]

where

\[
\theta_\mu(x, y) = \zeta_0 \left( \frac{x^{s-1} \sum_{i=0}^{f-1} a_ip^i + \sum_{i=0}^{f-1} b_ip^i}{y} \right).
\]

Consequently, the base change map becomes

\[
\text{BC} \left( R_{(s,1,\ldots,1)}(\mu) \right) = R'_{(1,1)}(\mu, -\mu).
\]

Now let \( w \in W \) be an element in the \( F \)-conjugacy class of \( (s, 1, \ldots, 1) \), and choose \( w' \in W \) such that \( w'wF(w')^{-1} = (s, 1, \ldots, 1) \). Applying first the equivalence (3.1.1) for the element \( w' \), then the above equation, then the equivalence induced by \( (w'^{-1}, 2w'^{-1}) \), we obtain

\[
\text{BC} \left( R_{w}(\mu) \right) \cong R'_{(w,w)}(\mu, -\varphi(\mu)).
\]

3.3.6. We define a base change map on the remaining irreducible representations of \( U_2(\mathbb{F}_q) \). Given a character \( \psi_0 : U_1(\mathbb{F}_q) \rightarrow \mathcal{O}^\times \), we let \( \tilde{\psi}_0 \) denote the character

\[
\tilde{\psi}_0 : \mathbb{F}_q^2 \rightarrow \mathcal{O}^\times \\
x \mapsto \psi_0(x^{1-q}).
\]

**Definition 3.11.** Let \( \psi_0 : U_1(\mathbb{F}_q) \rightarrow \mathcal{O}^\times \) denote a character of \( U_1(\mathbb{F}_q) \). We define

\[
\text{BC}(\psi_0 \circ \det) \overset{\text{def}}{=} \tilde{\psi}_0 \circ \det,
\]

\[
\text{BC}(\psi_0 \circ \det \otimes_E \text{St}) \overset{\text{def}}{=} \tilde{\psi}_0 \circ \det \otimes_E \text{St}',
\]

where \( \text{St}' \) denotes the Steinberg representation of \( \text{GL}_2(\mathbb{F}_q^2) \), inflated to \( \text{GL}_2(\mathcal{O}_{K_2}) \).
As before, the above definition is compatible with automorphic base change (cf. \cite[§11]{Rog90}).

Taken together, these definitions give a base change map on isomorphism classes of tame types. One further checks that the association \( \sigma \mapsto \text{BC}(\sigma) \) is injective on isomorphism classes.

### 3.3.7. We define an involution \( \epsilon \) on (isomorphism classes of) representations of \( \text{GL}_2(\mathbb{F}_q^2) \) by twisting a representation by the automorphism

\[
g \mapsto \left( \Phi_2 g^{-\tau} \Phi_2^{-1} \right)^{(q)}
\]

(note that the fixed points in \( \text{GL}_2(\mathbb{F}_q^2) \) of this automorphism are exactly \( \text{U}_2(\mathbb{F}_q) \)). On Deligne–Lusztig representations, this becomes

\[
\epsilon \left( R'_{(w,w')}(\mu,\mu') \right) = R'_{(w,w')} \left( -\mathfrak{s}(\mu'), -\mathfrak{s}(\mu) \right).
\]

The following result is easily checked.

**Lemma 3.12.** Let \( \sigma' \) denote a tame \( \text{GL}_2(\mathcal{O}_K)- \)type over \( E \). Then we have \( \epsilon(\sigma') \cong \sigma' \) if and only if \( \sigma' \) is of the form \( \text{BC}(\sigma) \) for a tame \( \text{U}_2(\mathcal{O}_K) \)-type \( \sigma \).

#### 3.4. Combinatorics of types and weights. For future applications to weight elimination and weight existence results, we now analyze the combinatorial properties of the set \( \text{JH}(\sigma) \) for a tame type \( \sigma \).

##### 3.4.1. Before proceeding, we make some definitions to simplify the discussion below.

**Definition 3.13.**

(i) We define \( \tilde{W} \overset{\text{def}}{=} X^*(T) \rtimes W \) to be the extended affine Weyl group. It acts on \( X^*(T) \) in the natural way, and we write elements of \( \tilde{W} \) as \( t_{\mu}w \), with \( \mu \in X^*(T) \), \( w \in W \), to underscore this action.

(ii) An alcove is a connected component of

\[
X^*(T) \otimes \mathbb{R} - \left( \bigcup_{\alpha, n} \{ \langle \mu + \eta, \alpha \rangle = np \} \right).
\]

We let \( C_0 \) denote the dominant base alcove

\[
\{ \mu \in X^*(T) \otimes \mathbb{R} : 0 < \langle \mu + \eta, \alpha \rangle < p \text{ for all } 0 \leq i \leq f - 1 \}.
\]

(iii) The group \( pX^*(T) \rtimes W \subseteq \tilde{W} \) acts on the set of alcoves via the dot action • centered at \( -\eta \). We define

\[
\tilde{W}_+ \overset{\text{def}}{=} \{ \tilde{w} \in pX^*(T) \rtimes W : \tilde{w} \bullet C_0 = C_0 \}.
\]

**Remark 3.14.** One easily checks that if \( \tilde{w} = wt_{-\nu} = (w_it_{-\nu})_i \in \tilde{W}_+ \), then we must have

\[
(w_i, \nu_i) \in \{ 1 \} \times X^0(T_H) \quad \text{or} \quad (w_i, \nu_i) \in \{ s \} \times (\eta_H + X^0(T_H))
\]

for all \( 0 \leq i \leq f - 1 \).

**Lemma 3.15.** Let \( R_w(\mu + \eta) \) denote a Deligne–Lusztig representation of \( G(\mathbb{Z}_p) \), and suppose \( \mu \) is a 1-deep character. Let \( \lambda \in X_1(T) \). Then \( F(\lambda) \in \text{JH}(\tilde{R}_w(\mu + \eta)) \) if and only if there exists \( zt_{-\nu} \in \tilde{W}_+ \) such that

\[
zt_{-\nu} \bullet (\mu + w\pi(\nu)) = \mathfrak{s} \bullet (\lambda - p\eta).
\]

**Proof.** Suppose \( F(\lambda) \in \text{JH}(\tilde{R}_w(\mu + \eta)) \) for some \( \lambda \in X_1(T) \). By \cite[Prop. 10.1.8]{GHS18}, this holds if and only if there exists \( \nu \in X^*(T) \) such that

\[
z' \bullet (\mu + (w\pi - p)\nu) \uparrow \mathfrak{s} \bullet (\lambda - p\eta)
\]

for all \( i \).
for all $z' \in W$. (We refer to [Jan03] II.6.4 for the definition of $\uparrow$; since the root system is of type $A_1 \times \cdots \times A_1$, the condition $\mu' \uparrow \lambda'$ is equivalent to $\mu' \leq \lambda'$ and $\mu' \in (pX^*(T) \ltimes W) \cdot \lambda'$.) Select $z \in W$ such that

$$z(\mu + (w_\pi - p)\nu + \eta) \in X_+(T).$$

Since $z \cdot (\mu + (w_\pi - p)\nu)$ lies below $z \cdot (\lambda - p\eta)$ in the $\uparrow$ ordering, since $z(\mu + (w_\pi - p)\nu + \eta)$ is dominant, and since $z(\lambda - (p - 1)\eta)$ is $p$-restricted, we must have

$$z \cdot (\mu + (w_\pi - p)\nu) = z \cdot (\lambda - p\eta).$$

The proof of [LLHL19] Prop. 4.1.3 shows that $zt_{-p\mu} \in \tilde{W}_+$. Furthermore, we deduce a posteriori that the choice of $z$ is unique.

Conversely, if $z \cdot (\mu + (w_\pi - p)\nu) = z \cdot (\lambda - p\eta)$ for some $z$, then $z(\mu + (w_\pi - p)\nu + \eta) \in X_+(T)$, and [Jan03] II.6.4(5)] implies

$$(z')z \cdot (\mu + (w_\pi - p)\nu) \uparrow z \cdot (\mu + (w_\pi - p)\nu) = z \cdot (\lambda - p\eta)$$

for all $z' \in W$, so that $F(\lambda) \in JH(R_{w_2}(\mu + \eta))$ by [GHS18] Prop. 10.1.8].

**Proposition 3.16.** Let $\sigma_1 = R_{w_1}(\mu_1 + \eta)$, $\sigma_2 = R_{w_2}(\mu_2 + \eta)$ be two Deligne–Lusztig representation of $G(\mathbb{Z}_p)$, and suppose $\mu_1$ and $\mu_2$ are both 3-deep.

(i) We have $JH(\overline{\sigma}_1) \cap JH(\overline{\sigma}_2) \neq \emptyset$ if and only if there exists a pair $(w'_2, \mu'_2) \in W \times X^*(T)$ such that $R_{w_2}(\mu_2 + \eta) \cong R_{w'_2}(\mu'_2 + \eta)$ and

$$t_{\mu'_2}w'_2 = t_{\mu_1}w_1w$$

in $\tilde{W}$, where $w_i \in \{1, s, t_{\alpha \mu}_i\}$ for all $0 \leq i \leq f - 1$.

(ii) Suppose $JH(\overline{\sigma}_1) \cap JH(\overline{\sigma}_2) \neq \emptyset$, and let $(w'_2, \mu'_2)$ and $\tilde{w} = (\tilde{w}_i)_i$ be as in item [i] Let $\lambda \in X_1(T)$. Then $F(\lambda) \in JH(\overline{\sigma}_1) \cap JH(\overline{\sigma}_2)$ if and only if

$$z \cdot (\lambda - p\eta) = \tilde{w}_\lambda \cdot (\mu_1 + w_1\pi(\nu)) = \tilde{w}_\lambda \cdot (\mu'_2 + w'_2\pi(\nu))$$

for an element $\tilde{w}_\lambda = wt_{-p\mu} \in \tilde{W}_+$ satisfying the following conditions:

(a) if $\tilde{w}_i = s$ then $(\tilde{w}_\lambda)_{i+1} = 1$; and

(b) if $\tilde{w}_i = t_{\alpha \mu}_i s$ then $(\tilde{w}_\lambda)_{i+1} = st_{-p\mu \lambda}$.

In particular:

$$|JH(\overline{\sigma}_1) \cap JH(\overline{\sigma}_2)| = 2^{|\{i; \tilde{w}_i = 1\}|}.$$

**Proof.** We begin with item [i].

Assume $JH(\overline{\sigma}_1) \cap JH(\overline{\sigma}_2) \neq \emptyset$; by Lemma 3.15 we have

$$\mu_2 + (w_2\pi - p)\nu(2) + \eta = z^{(2)}z(1)(\mu_1 + (w_1\pi - p)\nu(1) + \eta)$$

where $z^{(j)}t_{-p\nu(\nu)} \in \tilde{W}_+$. This gives

$$R_{w_2}(\mu_2 + \eta) \cong R_{w_2}
\begin{align*}
\mu_2 + \eta + (w_2\pi - p)\nu(2) \\
\cong R_{w_2}(z^{(2)}z(1)(\mu_1 + (w_1\pi - p)\nu(1) + \eta)) \\
\cong R_{z^{(1)}z(1)=z^{(2)}z(1)w_2F(z(1)-z(2))^{-1}}(\mu_1 + (w_1\pi - p)\nu(1) + \eta) \\
\cong R_{z^{(1)}z(1)=z^{(2)}z(1)w_2F(z(1)-z(2))^{-1}}(\mu_1 + w_1\pi(\nu(1)) - w'_2\pi(\nu(1)) + \eta) \\
\cong R_{w'_2}(\mu'_2 + \eta)
\end{align*}$$

where the first isomorphism comes from adding $(w_2\pi - p)\nu(2)$, the second from $\overline{\sigma}_1$, the third from conjugation by $z^{(1)}z(1)=z^{(2)}w_2F(z(1)-z(2))^{-1}$, and the fourth by adding $(p - w'_2\pi)\nu(1)$. Here, we define $w_2 \overset{\text{def}}{=} z^{(1)}z(1)=z^{(1)}z(2)w_2F(z(1)-z(2))^{-1}$ and $\mu'_2 \overset{\text{def}}{=} \mu_1 + w_1\pi(\nu(1)) - w'_2\pi(\nu(1))$. 

We now proceed entrywise:

- If \( w_{2,i}^0 = w_{1,i} \), then by definition we have \( \mu_{2,i}^0 = \mu_{1,i} \);
- If \( w_{2,i}^0 = w_{1,i} \), then \( \mu_{2,i}^0 = \mu_{1,i} + w_{1,i}(\pi(\nu^{(1)})) - s\pi(\nu^{(1)}) \). Since \( \pi(\nu^{(1)}) \in \{0, \eta_H\} + X^0(T_H) \), we have \( \pi(\nu^{(1)}) - s\pi(\nu^{(1)}) \in \{0, \alpha_H\} \).

This means exactly that \( t_{\mu_i} w_{2}^0 = t_\mu w_1 \bar{\omega} \) in \( \widetilde{W} \), with \( \bar{\omega}_i \in \{1, s, t_{\alpha_H} s\} \).

For the converse, suppose that \( (w_{2}^0, \mu_2^0) \) satisfies \( \sigma_2 = R_{w_2}(\mu_2 + \eta) \approx R_{w_2}(\mu_2^0 + \eta) \) and

\[
t_{\mu_i} w_{2}^0 = t_{\mu_i} w_1 \bar{\omega}
\]

with \( \bar{\omega}_i \in \{1, s, t_{\alpha_H} s\} \). In particular, this implies \( \mu_2^0 \) is 1-deep. Let \( \bar{\omega}_\lambda = wt_{-\nu} \in \widetilde{W}_+ \) be any element which satisfies conditions (a), (b) in the statement of the lemma.

- If \( \bar{\omega}_i = s \), then \( w_{2,i}^0 = w_{1,i}s \) and
  \[
  \mu_{2,i}^0 = \mu_{1,i} + w_{1,i}(\nu_i - s(\nu_i - 1)) = \mu_{1,i} + w_{1,i}\pi(\nu)_i - w_{2,i}\pi(\nu)_i.
  \]

- If \( \bar{\omega}_i = t_{\alpha_H} s \), then \( w_{2,i}^0 = w_{1,i}s \) and
  \[
  \mu_{2,i}^0 = \mu_{1,i} + w_{1,i}(\alpha_H) = \mu_{1,i} + w_{1,i}(\nu_i - s(\nu_i - 1)) = \mu_{1,i} + w_{1,i}\pi(\nu)_i - w_{2,i}\pi(\nu)_i.
  \]

Finally, if \( \bar{\omega}_i = 1 \), then \( w_{2,i}^0 = w_{1,i} \) and

\[
\mu_{2,i}^0 = \mu_{1,i} + \nu_i - 1 = \nu_i.
\]

Collecting these, we obtain \( \mu_1 + w_1\pi(\nu) = \mu_2 + w_2\pi(\nu) \), i.e.

\[
\bar{\omega}_\lambda \cdot (\mu_1 + w_1\pi(\nu)) = \bar{\omega}_\lambda \cdot (\mu_2 + w_2\pi(\nu)).
\]

By Lemma 3.15 we conclude that \( F(\lambda) \in JH(\overline{\sigma_1}) \cap JH(\overline{\sigma_2}) \), where \( \lambda \) is defined by

\[
\bar{\omega}_\lambda \cdot (\lambda - p\eta) = \bar{\omega}_\lambda \cdot (\mu_1 + w_1\pi(\nu)) = \bar{\omega}_\lambda \cdot (\mu_2 + w_2\pi(\nu)).
\]

This completes the proof of item (i) and of the “if” direction in item (ii), and shows that

\[
|JH(\overline{\sigma_1}) \cap JH(\overline{\sigma_2})| \geq 2^{|\{i : \bar{\omega}_i = 1\}|}.
\]

We now conclude the proof of item (ii).

Suppose there exists some \( F(\lambda) \in JH(\overline{\sigma_1}) \cap JH(\overline{\sigma_2}) \), and let \( (w_{2}^0, \mu_2^0) \) be as in item (i). By Lemma 3.15 there exist \( \bar{\omega}_{\lambda}^{(1)} = z(i) t_{-\nu(i)} \in \widetilde{W}_+ \) such that

\[
\bar{\omega}_\lambda \cdot (\lambda - p\eta) = \bar{\omega}_{\lambda}^{(1)} \cdot (\mu_1 + w_1\pi(\nu)) = \bar{\omega}_{\lambda}^{(2)} \cdot (\mu_2 + w_2\pi(\nu)).
\]

Pairing the last equality with \( \alpha_i^\vee \) and reducing modulo 2 gives the equation

\[
\delta_{z_{1,i}^{(1)}, s} + \delta_{z_{1,i}^{(2)}, s} \equiv \delta_{z_{1,i}^{(1)}, s} + \delta_{z_{1,i}^{(2)}, s} \pmod{2}.
\]

Using this, we see that if \( z_{1,i}^{(1)} = z_{1,i}^{(2)} \), for some \( i \), then this inequality holds for all \( i \), i.e., \( z_{1} = z_{1}^{(2)} \) and \( \nu_{1} = \nu_{2} = \eta \). Substituting this into equation (3.4.2) gives a contradiction. Therefore, we must have \( z_{1} t_{-\nu(i)} = z_{2} t_{-\nu(2)} \), and equation (3.4.2) reduces to

\[
\mu_1 + w_1\pi(\nu) = \mu_2 + w_2\pi(\nu).
\]

From this equation we see that \( z_{1} t_{-\nu(i)} \) must exactly be one of the \( \bar{\omega}_\lambda \) of the statement of the lemma. This shows \( |JH(\overline{\sigma_1}) \cap JH(\overline{\sigma_2})| = 2^{|\{i : \bar{\omega}_i = 1\}|} \).

\[\Box\]

Remark 3.17. The analogous lemma holds mutatis mutandis for the group \( G' = \text{Res}_{K_2 / \mathbb{Z}_p}(GL_2 / \mathbb{O}_{K_2}) \).
3.4.2. We are now in a position to compare how intersection of Jordan–Hölder factors behaves under base change.

**Proposition 3.18.** Let $\sigma_1$, $\sigma_2$ be two 3-generic Deligne–Lusztig representations of $G(\mathbb{Z}_p)$ on which $\nu(\mathcal{O}_K^\times)$ acts trivially. Then

$$|\text{JH}(\text{BC}(\sigma_1)) \cap \text{JH}(\text{BC}(\sigma_2))| = |\text{JH}(\sigma_1) \cap \text{JH}(\sigma_2)|^2.$$  

**Proof.** Let us write $\sigma_j \cong R_{w_j}(\mu_j)$ for $j = 1, 2$, with $\mu_j - \eta$ being 3-deep. By the discussion in Subsection 3.1.5, we may assume that the last two entries of $\mu_j$ in each embedding are equal to 0.

Suppose first that $\text{JH}(\sigma_1) \cap \text{JH}(\sigma_2) \neq \emptyset$. Let $\tilde{w} = t_{\xi}v \in \tilde{W}$ be as in Lemma 3.16 so that $\sigma_2 \cong R_{w_1}(\mu_1 + w_1(\xi))$ and $|\text{JH}(\sigma_1) \cap \text{JH}(\sigma_2)| = 2^{|i;v_i=1|}$. Using (3.3.1) or (3.3.2), we get

$$\text{BC}(\sigma_1) \cong R'_{w_1}(\mu_1, -2(\mu_1)),$$

and

$$\text{BC}(\sigma_2) \cong R'_{w_1}(\mu_1 + w_1(\xi), -2(\mu_1) - 2w_1(\xi))$$

(note that $\xi_i \in \{0, \alpha^1_H\}$ for each $0 \leq i \leq j - 1$, so $-2(\xi) = \xi$). By the $GL_2$ analog of Proposition 3.16, we may obtain $BC(\sigma_2)$ from $BC(\sigma_1)$ via the element $t_{(\xi,\xi')}(v,v) \in \tilde{W}$, and therefore

$$|\text{JH}(\text{BC}(\sigma_1)) \cap \text{JH}(\text{BC}(\sigma_2))| = 2^{|i;v_i=1|} = 2^{|i;v_i=1|} = |\text{JH}(\sigma_1) \cap \text{JH}(\sigma_2)|^2.$$  

To conclude, it suffices to prove that $\text{JH}(\text{BC}(\sigma_1)) \cap \text{JH}(\text{BC}(\sigma_2)) \neq \emptyset$ implies $\text{JH}(\sigma_1) \cap \text{JH}(\sigma_2) \neq \emptyset$. Assume the former. By the $GL_2$ analog of Proposition 3.16, we may obtain $BC(\sigma_2)$ from $BC(\sigma_1)$ via an element $t_{(\xi,\xi')}(v,v') \in \tilde{W}$ with $t_{\xi_i}v_{i_1}, t_{\xi_i'}v_{i_1}' \in \{1, s, t_{\alpha^1_H}s\}$. That is, we have

$$\text{BC}(\sigma_2) \cong R'_{w_2}(\mu_2, -s(\mu_2)) \cong R_{w_1}(\mu_1, -s(\mu_1) + (w_1, w_1)(\xi, \xi')).$$

By Lemma 3.12, the isomorphism class of the representation on the right is invariant under $e$, and a straightforward calculation implies $v' = v$ and $\xi' = \xi$. Thus

$$\text{BC}(\sigma_2) \cong R_{w_1}(\mu_1, -s(\mu_1)) + (w_1, w_1)(\xi, \xi)) \cong \text{BC}(R_{w_1}(\mu_1 + w_1(\xi))).$$

Since the base change map is injective on isomorphism classes of tame types, we get

$$\sigma_2 \cong R_{w_1}(\mu_1 + w_1(\xi))$$

and consequently $\text{JH}(\sigma_1) \cap \text{JH}(\sigma_2) \neq \emptyset$ by Proposition 3.16. □

3.4.3. We introduce a metric on the set of Serre weights contained in a sufficiently generic tame type. This will turn out to be useful in the proof of Theorem 7.4.

**Definition 3.19.** Let $\sigma = R_w(\mu)$ denote a Deligne–Lusztig representation of $G(\mathbb{Z}_p)$, and suppose $\mu - \eta$ is 1-deep. Let $F(\lambda) \in \text{JH}(\sigma)$. By Lemma 3.15, there exists an element $zt_{-w\nu} \in \tilde{W}$ defined by

$$s \cdot (\lambda - \eta) = zt_{-w\nu} \cdot (\mu - \eta + w\pi(\nu)).$$

We say $z \in W$ is the label of $F(\lambda)$ with respect to $(w, \mu)$.

**Remark 3.20.** Maintain the setting of Definition 3.19. If $(w', \mu')$ is another pair such that $\sigma \cong R_{w'}(\mu')$ with $\mu' - \eta$ being 1-deep, then by equation (3.1.1) we have

$$(w', \mu') = (vwF(v^{-1}, v(\mu) + F(\nu) - vwF(v^{-1}(\nu)))$$

for some pair $(v, \nu) \in W \times X(T)$. It is easily checked that if the label of $F(\lambda)$ with respect to $(w, \mu)$ is $z$, then the label of $F(\lambda)$ with respect to $(w', \mu')$ is given by $zv^{-1}$.  

Definition 3.21. Let $\sigma$ denote a 1-generic Deligne–Lusztig representation of $G(\mathbb{Q}_p)$, and let $F, F' \in \text{JH}(\overline{\sigma})$. Choose an isomorphism $\sigma \cong R_w(\mu)$, with $\mu - \eta$ being 1-deep, and suppose that the labels of $F$ and $F'$ with respect to $(w, \mu)$ are $z$ and $z'$, respectively. We define the graph distance $\text{dgr}(F, F')$ as the number of $i$ for which $z_i \neq z'_i$ (i.e., $\text{dgr}(F, F')$ is the length $\ell(z'z^{-1})$ of $z'z^{-1}$). By Remark 3.15, the graph distance is well-defined.

Remark 3.22. Suppose that $\sigma_1$ and $\sigma_2$ are two 3-generic Deligne–Lusztig representations of $G(\mathbb{Q}_p)$, and suppose $F, F' \in \text{JH}(\overline{\sigma_1}) \cap \text{JH}(\overline{\sigma_2})$. Then the graph distance between $F$ and $F'$, computed using $\sigma_1$, agrees with the graph distance between $F$ and $F'$, computed using $\sigma_2$ (this follows from Lemma 3.20 and Proposition 3.16).

Lemma 3.23. Let $\sigma$ be a 4-generic Deligne–Lusztig representation of $G(\mathbb{Q}_p)$, and let $F, F' \in \text{JH}(\overline{\sigma})$. Then there exists a tame type $\sigma'$ such that:

- $F, F' \in \text{JH}(\overline{\sigma'})$; and
- for any $F'' \in \text{JH}(\overline{\sigma}) \cap \text{JH}(\overline{\sigma'})$ satisfying $F'' \neq F'$, we have $\text{dgr}(F, F'') < \text{dgr}(F, F')$.

Moreover, $\sigma$ and $\sigma'$ can be chosen so that $\sigma = R_w(\mu)$, $\sigma' = R_w(\mu')$ with $\mu - \eta, \mu' - \eta$ being 3-deep, and $t_{\mu' - \eta} = t_{\mu - \eta} \alpha(z)$ for an element $z \in W$ which satisfies $\ell(z) = \text{dgr}(F, F')$. (Recall that for $v \in W$ we denote $\alpha_v = \sum_{i : v_i = s} \alpha_i$.) In this case,

$$\left| \text{JH}(\overline{\sigma}) \cap \text{JH}(\overline{\sigma'}) \right| = 2^{\ell(z)} = 2^\text{dgr}(F,F').$$

Proof. Let us write $\sigma \cong R_w(\mu)$ with $\mu - \eta$ being 4-deep. By applying the equivalence (3.1.1), we may assume that the label of $F$ with respect to $(w, \mu)$ is $s$ at the cost of assuming $\mu - \eta$ is only 3-deep. Suppose that the label of $F'$ with respect to $(w, \mu)$ is $z$. By definition, we have

$$F \cong F(\mu + w(\eta) - \eta) \cong F(\mu - \alpha_w)$$

$$F' \cong F(sz(\mu + (w\pi - p)\nu) + (p - 1)\eta),$$

where $zt_{-p\nu} \in \overline{W}_+$. We define $\sigma' \equiv R_wz(\mu + w(\pi(\alpha_z)))$. We easily see that $F$ and $F'$ are Jordan–Hölder factors of $\overline{\sigma}$, whose labels with respect to $(wF(z), \mu + w\pi(\alpha_z))$ are $s$ and $z$, respectively. Moreover, by the explicit description of $\text{JH}(\overline{\sigma}) \cap \text{JH}(\overline{\sigma'})$ of Proposition 3.14(ii), we see that any element $F'' \neq F'$ of the intersection satisfies $\text{dgr}(F, F'') < \text{dgr}(F, F')$. The final part of the aforementioned proposition gives the size of the intersection.

3.5. Base change of weights. We now define a notion of base change for weights, and show that it is compatible with the notion of base change of types defined above.

3.5.1.

Definition 3.24. Let $\mu \in X_1(T)$ and let $F(\mu)$ denote a Serre weight of $G(\mathbb{Q}_p)$ on which $z(\mathcal{O}_K^\times)$ acts trivially. As in Subsection 3.1.5, we may assume $\mu$ is of the form

$$\mu = \begin{pmatrix} a_0 & a_1 & \cdots & a_{f-1} \\ b_0 & b_1 & \cdots & b_{f-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

We define the base change of $F(\mu)$ as

$$\text{BC}(F(\mu)) \equiv F'(\mu, -s(\mu)).$$

One easily checks that the map $F \mapsto \text{BC}(F)$ is well-defined and injective on isomorphism classes of Serre weights.
3.5.2. Recall the automorphism $\epsilon$ of $G'(\mathbb{F}_p) \cong GL_2(\mathbb{F}_{q^2})$ defined in Subsection 3.3.7. On Serre weights, this automorphism gives

$$\epsilon \left(F'(\mu, \mu')\right) = F'(-g(\mu'), -g(\mu)).$$

The following result is easily checked.

**Lemma 3.25.** Let $F'$ denote a Serre weight of $GL_2(\mathbb{F}_{q^2})$. Then we have $\epsilon(F') \cong F'$ if and only if $F'$ is of the form $BC(F)$ for a Serre weight $F$ of $U_2(\mathbb{F}_q)$.

3.5.3. We now wish to relate base change of types with base change of weights. The relevant result is the following.

**Lemma 3.26.** Let $\sigma$ denote a 1-generic Deligne–Lusztig representation of $G(\mathbb{Z}_p)$ on which $i(\mathcal{O}_K^\times)$ acts trivially, and let $F$ denote a Serre weight on which $i(\mathcal{O}_K^\times)$ acts trivially. We then have

$$F \in JH(\sigma) \iff BC(F) \in JH(BC(\sigma)).$$

**Proof.** Let us write $\sigma = R_w(\mu)$ where $\mu$ is of the form

$$\mu = \begin{pmatrix} a_0 & \alpha_1 & \cdots & \alpha_{j-1} \\ \beta_0 & \beta_1 & \cdots & \beta_{j-1} \end{pmatrix}$$

and $\mu - \eta$ is 1-deep. Thus $BC(\sigma) = R'_w(\mu, -g(\mu)).$

Suppose first that $F \in JH(\sigma)$. By equation (3.1.2), $F$ is of the form

$$F \cong F_w(R_w(\mu)) = F(p\gamma_w + w'(\mu - w\pi(\varepsilon_{gw'})) + p\rho_w - \pi(\rho))$$

for some $w' \in W$. Note that the parenthesized character has its last two entries equal to 0 in each embedding. Thus, we have

$$BC(F) \cong F'(p(\gamma_{w'}, -\gamma_{w'}) + (w', w')((\mu, -g(\mu)) - (w, w)\pi(\varepsilon_{gw'}), -g(\varepsilon_{gw'})))$$

$$+ p(\rho_{w'}, -g(\rho_{w'})) - (\pi(\rho), -g(\pi(\rho))).$$

A straightforward calculation shows that adding $(p - \pi')(0, 2\gamma_{w'} + \rho_{w'} + g(\rho_{w'})) \in (F' - 1)X^0(T')$ to the parenthesized character gives

$$BC(F) \cong F'(p\gamma_{w',w'} + (w', w')((\mu, -g(\mu)) - (w, w)\pi'(\varepsilon_{gw',gw'})) + p\rho_{w',w'} - \rho').$$

Hence, we obtain

$$BC(F) \cong F'_w(R'_w(\mu, -g(\mu))) \in JH(R'_w(\mu, -g(\mu))) = JH(BC(\sigma)).$$

To prove the converse we begin with an observation. Let $F'_{(w',v')}(BC(\sigma))$ be a Jordan–Hölder factor of $R'_w(\mu, -g(\mu)) = BC(\sigma)$ as in (3.2.1). Since $\epsilon(BC(\sigma)) \cong BC(\sigma)$ by Lemma 3.12, we obtain $\epsilon(F'_w(BC(\sigma))) \in JH(BC(\sigma))$. A similar argument to the one above shows that

$$\epsilon(F'_{(w',v')}(BC(\sigma))) \cong F'_{(w',v')}(BC(\sigma)).$$

Suppose now that $BC(F) \in JH(BC(\sigma))$. Then there exists $(v, v') \in W'$ such that $BC(F) \cong F'_{(v,v')}(BC(\sigma))$. Since $BC(F)$ is a base change, Lemma 3.25 and the above equation imply

$$F'_{(v',v)}(BC(\sigma)) \cong \epsilon(F'_{(v',v')}(BC(\sigma)))$$

$$\cong \epsilon(BC(F))$$

$$\cong BC(F)$$

$$\cong F'_{(v,v')}(BC(\sigma)).$$
Since the Jordan–Hölder factors of $BC(\sigma)$ are distinct, we conclude that $v' = v$. Thus, we get
\[ BC(F) \cong F'_{(v,v)}(BC(\sigma)) \cong BC(F_v(R_w(\mu))). \]
Since the base change map is injective on Serre weights, we conclude that
\[ F \cong F_v(R_w(\mu)) \in \text{JH}(R_w(\mu)) = \text{JH}(\sigma). \]
\[ \square \]

3.5.4. The following lemma will be useful in the proof of Theorem 6.1.

**Lemma 3.27.** Let $\sigma$ be a 2-generic Deligne–Lusztig representation of $G(\mathbb{Z}_p)$ on which $\iota(\mathcal{O}_K)$ acts trivially, and let $F$ denote a 3-deep Serre weight with trivial action of $\iota(\mathcal{O}_K)$ such that $F \notin \text{JH}(\overline{\sigma})$. Then there exists another Deligne–Lusztig representation $\sigma'$ of $G(\mathbb{Z}_p)$ such that $F \in \text{JH}(\overline{\sigma'})$ and $\text{JH}(\sigma) \cap \text{JH}(\overline{\sigma'}) = \emptyset$.

**Proof.** If $\sigma$ and $F$ have different central characters, then any $\sigma'$ for which $F \in \text{JH}(\overline{\sigma'})$ works. We may therefore assume that $\sigma$ and $F$ have the same central character. The remainder of the proof will be based on the combinatorics of the extension graph for Serre weights for $\mathbf{GL}_2$, as defined in [LMS] §2. We recall some of the definitions and constructions of op. cit. (and use similar notation for convenience of comparison).

Define $\Lambda'_W$ to be the weight lattice for $G^\text{der}$, the derived subgroup of $G'$, and let $\Lambda'_R$ denote the root lattice, so that $\Lambda'_W \subseteq \Lambda'_R$. Note that $\Lambda'_W \cong \mathbb{Z}^J \times \mathbb{Z}^J$ and we fix such an identification in what follows. Recall that $X^*(T')$ denotes the weight lattice for the group $G'$; we have $\Lambda'_R \subseteq X^*(T')$ and $X^*(T') \twoheadrightarrow \Lambda_W$. We further define $\widetilde{W}'_a$ (resp. $\widetilde{W}'$) to be the affine (resp. extended affine) Weyl group of $G'$, which admits a factorization $\widetilde{W}'_a \cong W' \rtimes \Lambda'_R$ (resp. $\widetilde{W}' \cong W' \rtimes X^*(T')$). The group $\widetilde{W}'_a$ (resp. $\widetilde{W}'$) is canonically isomorphic to two copies of $W \ltimes Z\Delta$ (resp. $W \ltimes X^*(\text{Res}_{\mathcal{O}_K/\mathbb{Z}_p}(T_U))$) (compare with Subsubsection 3.2.1).

We let $\widetilde{W}'_+$ denote the set of elements of $\widetilde{W}'$ which stabilize the fundamental alcove $C'_0$ of $G'$ under the $p$-dilated dot action $\cdot_p$:
\[ w' t_{\chi'} \cdot_p \mu' = w'(\mu' + \rho' + p\lambda') - \rho', \]
where $w' t_{\chi'} \in \widetilde{W}'$ and $\mu' \in X^*(T')$. Thus, $\widetilde{W}'_+$ is the analog of $\widetilde{W}_+$ of Subsection 3.4 except that translations have been scaled by a factor of $p^{-1}$. (We apologize for this inconsistency of notation.)

Let $\mu' \in X^*(T')$ satisfy $0 \leq \langle \mu', \alpha' \rangle < p - 1$ for every positive coroot $\alpha'$ of $T'$. We call such characters $p$-regular. We then define the map $\Xi_{\mu'}$ by
\[ \Xi_{\mu'} : \Lambda'_W \longrightarrow \frac{X^*(T')}{(F' - 1)X^0(T')} \]
\[ \omega' \longmapsto \tilde{\omega}' \cdot_p (\mu' + \tilde{\omega}' - \rho') \]
where $\tilde{\omega}' \in X^*(T')$ is a lift of $\omega' \in \Lambda'_W$, and $\tilde{\omega}'$ is the unique element in $\widetilde{W}'_+$ such that the class of $-\pi'^{-1}(\tilde{\omega}')$ corresponds to the class of $\tilde{w}'$ via the isomorphism $X^*(T')/\Lambda'_R \cong \widetilde{W}'_a/\widetilde{W}'_a$. Note that this is well defined. Define furthermore
\[ (3.5.1) \]
\[ \Lambda''_W \overset{\text{def}}{=} \{ \omega' \in \Lambda'_W : \omega' + \mu' - \rho' \in C'_0 \} \]
(where we consider the image of $\mu' - \rho'$ and $C'_0$ in $\Lambda'_W$), and let $\Xi_{\mu'}$ be the restriction of $\Xi_{\mu'}$ to $\Lambda''_W$. Then, as in [LLHLM] §2.1, one checks that:

(i) The image of $\Xi_{\mu'}$ is contained in the set of $p$-regular characters. Further, the map $\omega' \longmapsto F'(\Xi_{\mu'}(\omega'))$ defines a bijection between $\Lambda''_W$ and the set of $p$-regular Serre weights with the same central character as $F'(\mu' - \rho')$ (see the discussion preceding [LMS] Prop. 2.9)).
(ii) Suppose \( \mu' \in X^*(\mathcal{G}') \) is such that \( \mu' - \rho' \) is 2-deep, and consider the Deligne–Lusztig representation \( R_{\mu'}(\mu') \). Applying the analog of equation (3.1.1) for \( \mathcal{G}'(\mathbb{Z}_p) \), we obtain an isomorphism

\[
R_{\mu'}(\mu') \cong R_{\mu'}((s, s)(\mu') + pp' - w'(\rho')),
\]

where the character \((s, s)(\mu') + pp' - w'(\rho') - \rho' \) is 1-deep. Combining this isomorphism with [LMS] Prop. 2.11, Proposition 4.6 and Remark 4.7 below, we obtain

\[
JH(R_{\mu'}(\mu')) = \left\{ F'(\tau_{\mu'} + \rho') \left( t_{-\alpha_{w}(s, s)}w'(\Sigma') \right) \right\},
\]

where \( \Sigma' \subseteq \Lambda'_W \) is the subset consisting of \( \Sigma' \) of \( (s, s) \) elements of the form \((\frac{1}{0})\) or \((\frac{0}{1})\) in each embedding. (That is, \( \Sigma' \) is the image in \( \Lambda'_W \) of \( \{ \rho_{w'} \}_{w' \in \mathcal{W}'} \).

(iii) Let \( \mu \in X^*(\text{Res}_{\mathbb{Q}K/\mathbb{Z}_p}(\mathcal{T}_U)) \subseteq X^*(\mathcal{T}) \) satisfy \( 0 \leq (\mu + \rho, \alpha^\vee) < p - 1 \) for every positive coroot \( \alpha^\vee \) of \( \mathcal{T} \). We let \( \Lambda'_W \) denote the weight lattice of \( \mathcal{G}^\text{der} \) (which is a quotient of \( X^*(\text{Res}_{\mathbb{Q}K/\mathbb{Z}_p}(\mathcal{T}_U)) \)), and define \( \Lambda'^{\mu + \rho}_W \) as in (3.5.1). Consider the map

\[
\tau^G_{\mu} : \Lambda'^{\mu + \rho}_W \rightarrow \frac{X_1(\mathcal{T}')}{(F' - 1)X_0(\mathcal{T}')},
\]

\[
\omega \mapsto \tau_{\mu + \rho - \overline{\omega}(\mu) + \rho}(\omega, -\overline{g}(\omega)).
\]

Using Lemma 3.25 and item (iii), one checks that \( \tau_{\mu + \rho - \overline{\omega}(\mu) + \rho}(\omega, -\overline{g}(\omega)) \) defines a bijection between \( \Lambda'^{\mu + \rho}_W \) and \( p \)-regular Serre weights of \( \mathcal{G}'(\mathbb{Z}_p) \) which are in the image of the base change map and have the same central character as \( F'(\mu, -\overline{g}(\mu)) \).

We now proceed with the proof. Let us write \( \sigma = R_w(\mu) \), with \( \mu \) chosen as in Lemma 3.26 and \( \mu - \rho \) being 2-deep. By assumption and Lemma 3.26 we have

\[
BC(F) \not\in JH(BC(\sigma)) = JH(R_{(w,w)}(\mu - \overline{g}(\mu))).
\]

Since the character \( (\mu, -\overline{g}(\mu)) - \rho' \) is 2-deep, item (iii) above implies

\[
BC(F) \not\in \left\{ F'(\tau_{\mu + \rho - \overline{g}(\mu) + \rho}(\omega, -\overline{g}(\omega))) \right\}.
\]

Therefore, since \( F \) and \( \sigma \) have the same central character, item (iii) implies that \( BC(F) = F'(\tau^G_{\mu}(\omega)) \) for some \( \omega \in \Lambda'^{\mu + \rho}_W \cap t_{-\alpha_{w}}(s, s)w(\Sigma) \). (Here, \( \Sigma' \) is the image in \( \Lambda'_W \) of \( \{ \rho_{w'} \}_{w' \in \mathcal{W}'} \).) Since \( \Sigma' \) is a fundamental domain for the translation action of \( \Lambda'_R \) on \( \Lambda'_W \), there exists an element \( t_{(\nu - \overline{g}(\nu))} \in \Lambda'_R \subseteq \tilde{W}' \) such that

\[
(\omega, -\overline{g}(\omega)) \in t_{(\nu - \overline{g}(\nu))}t_{-\alpha_{w}}(s, s)w(\Sigma').
\]

Note that this implies \( \nu \neq 0 \) and consequently

\[
t_{-\alpha_{w}}(s, s)w(\Sigma') \cap t_{(\nu - \overline{g}(\nu))}t_{-\alpha_{w}}(s, s)w(\Sigma') = \emptyset.
\]

Recall that we have assumed \( F \) is 3-deep. Therefore the same is true of \( BC(F) \). Using the relation \( BC(F) = F'(\tau^G_{\mu}(\omega)) \), and the fact that \( \tilde{W}'_+ \) preserves \( C'_0 \) under \( -\rho_v \), we get that the character \( \mu + \omega \) is 3-deep. On the other hand, the relation (3.5.2) implies that we have

\[
\omega = \nu - \alpha_w + g(\rho_v)
\]

for some \( v \in W \). Since \( 0 \leq (\alpha_w - g(\rho_v), \alpha_i^\vee) \leq 2 \) for all \( 0 \leq i \leq f - 1 \), the relation

\[
\mu + \omega + \alpha_w - g(\rho_v) = \mu + \nu
\]

implies that \( 2 < (\mu + \nu, \alpha_i^\vee) < p - 2 \) for all \( 0 \leq i \leq f - 1 \). That is, we have that \( (\mu, -\overline{g}(\mu)) + (\nu, -\overline{g}(\nu)) - \rho' \) is 2-deep.
Now set $\sigma^\ell = R_{w}(\mu + \nu)$. By the previous paragraph and item (ii), we have

\[ JH(BC(\sigma^\ell)) = JH(R_{w}(\mu + \nu)) \]

where the last equality follows from the definition of $\Sigma^\ell$ and the fact that $(\nu, -\overline{g}(\nu)) \in \mathcal{N}_{R}$. Thus, the relation $BC(F) = F'(\Sigma^\ell)$ and equation 3.5.2 imply that $BC(F) \in JH(BC(\sigma^\ell))$, and the injectivity of $\Sigma^\ell$ and equation 3.5.3 imply $JH(BC(\sigma)) \cap JH(BC(\sigma^\ell)) = \emptyset$. We conclude by using Lemma 3.26 and Proposition 3.18.

\[ \square \]

4. Predicted Serre weights

In this section we discuss the conjectural set of weights attached to Galois parameters and their relation with base change. We give the relevant definitions in Subsection 4.1 along with a classification of mod $p$ tamely ramified $L$-parameters. We then define the set $W^I(\mathfrak{p})$ in Subsection 4.2. The main result is Theorem 4.9, which relates the sets $W^I(\mathfrak{p})$ and $W^I(BC(\mathfrak{p}))$. Finally, we state in Subsection 4.4 a version of the inertial local Langlands correspondence that we will require for local/global compatibility. Our discussion is based on [GHS18, §9].

4.1. $L$-parameters.

4.1.1. We first define the Galois representations we shall consider.

**Definition 4.1.** Let $R$ be a topological $\mathbb{Z}_p$-algebra. An $L$-parameter (with $R$-coefficients) is a continuous homomorphism $\Gamma_{\mathbb{Q}_p} \rightarrow ^L\mathbb{G}(R)$, which is compatible with the projection to $Gal(K_2/\mathbb{Q}_p)$. Likewise, we define an inertial $L$-parameter (or an inertial type) to be a continuous homomorphism $I_{\mathbb{Q}_p} \rightarrow \hat{\mathbb{G}}(R)$ which admits an extension to an $L$-parameter $\Gamma_{\mathbb{Q}_p} \rightarrow ^L\mathbb{G}(R)$. We say two (inertial) $L$-parameters are equivalent if they are $\hat{\mathbb{G}}(R)$-conjugate.

We make similar definitions for homomorphisms valued in $\mathfrak{S}_2(R)$.

By [GHS18, Lem. 9.4.1], the $\hat{\mathbb{G}}(R)$-conjugacy classes of $L$-parameters $\Gamma_{\mathbb{Q}_p} \rightarrow ^L\mathbb{G}(R)$ are in bijection with $\hat{\mathbb{H}}(R)$-conjugacy classes of $L$-parameters $\Gamma_{K} \rightarrow ^L\mathbb{H}(R) = \mathbb{C}U_2(R)$. A similar statement holds for inertial $L$-parameters (cf. op. cit., Lemma 9.4.5).

We make similar definitions of $L$-parameters $\Gamma_{F^+} \rightarrow ^C\mathbb{U}_2(R)$ if $F^+$ is a global field with a place $\nu$ satisfying $F_{\nu}^+ \cong K$ (cf. Remark 2.1).

4.1.2. The following lemma is easily checked.

**Lemma 4.2.** Let $\mathfrak{p} : \Gamma_{K} \rightarrow ^C\mathbb{U}_2(\mathbb{F})$ denote an $L$-parameter such that $\mathfrak{p}|_{\Gamma_{K_2}}$ is semisimple (or, equivalently, tamely ramified). Then, up to equivalence, $\mathfrak{p}$ is of one of the following two forms:

(i)

\[ \mathfrak{p}(h) = \left( \begin{array}{cc} \omega^r_{2f}nr_{2f,\nu}^{-1}\lambda(h) & 0 \\ 0 & \omega^{-qr+(q+1)s}_{2f,\nu}\lambda(h) \end{array} \right), \omega^s_{2f,\lambda^2}(h) \times 1, \]

\[ \mathfrak{p}(\varphi^{-f}) = \left( \begin{array}{cc} 1 & 0 \\ 0 & \nu \end{array} \right), \lambda \times \varphi^{-f}, \]

where $h \in \Gamma_{K_2}$, $0 \leq r < q^2 - 1$, $0 \leq s < q - 1$, and $\lambda, \nu \in \mathbb{F}^\times$. 
(ii) $\bar{\rho}(h) = \begin{pmatrix} \omega_{2f}^{s+(1-q)k} \nr_{2f,-\lambda}(h) & 0 \\ 0 & \omega_{2f}^{s+(1-q)\ell} \nr_{2f,-\lambda}(h) \end{pmatrix}$, $\omega_{2f}^{s\nr_{2f,\lambda}}(h) \times 1$.

$\bar{\rho}(\varphi^{-f}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda \times \varphi^{-f}$,

where $h \in \Gamma_{K_2}$, $0 \leq k, \ell < q + 1$, $0 \leq s < q - 1$, and $\lambda \in \mathbb{F}^\times$.

In both cases $\nr_{2f,x}$ denotes the unramified character of $\Gamma_{K_2}$ sending $\varphi^{-2f}$ to $x$.

4.1.3.

Definition 4.3. Let $R$ denote a topological $\mathbb{Z}_p$-algebra.

(i) Let $\rho : \Gamma_K \longrightarrow \mathbb{C}U_2(R)$ denote an $L$-parameter, and write $\rho|_{\Gamma_{K_2}} = \rho_2 \oplus \rho_1$, where $\rho_2 : \Gamma_{K_2} \longrightarrow \GL_2(R)$, $\rho_1 : \Gamma_{K_2} \longrightarrow \mathbb{G}_m(R) = \mathbb{R}^\times$. We define the base change of $\rho$ to be

$$BC(\rho) \overset{\text{def}}{=} \rho_2.$$  

(ii) Let $\rho : \Gamma_K \longrightarrow \mathfrak{S}_2(R)$ denote an $L$-parameter valued in $\mathfrak{S}_2$, and write $\rho|_{\Gamma_{K_2}} = \rho_2 \oplus \rho_1$, where $\rho_2 : \Gamma_{K_2} \longrightarrow \GL_2(R)$, $\rho_1 : \Gamma_{K_2} \longrightarrow \mathbb{G}_m(R) = \mathbb{R}^\times$. We define the base change of $\rho$ to be

$$BC'(\rho) \overset{\text{def}}{=} \rho_2.$$  

We make similar definitions if $F^+$ is a global field with a place $v$ satisfying $F^+_v \cong K$ (cf. Remark 2.1).

The two notions of base change are related as follows. Let $\rho : \Gamma_K \longrightarrow \mathbb{C}U_2(R)$ denote an $L$-parameter, and let $\theta$ denote the continuous character $\hat{\tau} \circ \rho : \Gamma_K \longrightarrow \mathbb{R}^\times$. Using the isomorphism of Subsection 2.4 we get an isomorphism of $\GL_2(R)$-valued Galois representations

(4.1.1) $BC'(\rho) \cong BC(\rho) \otimes \theta^{-1}$.

4.1.4. Recall from Subsubsection 2.3.2 that we have a map $(\phi^v)^{-1} : X^*(\mathbf{T}_H) \xrightarrow{\sim} X_*(\mathbf{T}_H)$, which induces an isomorphism $X^*(\mathbf{T}) \xrightarrow{\sim} X_*(\mathbf{T})$. Given $\mu \in X^*(\mathbf{T})$ (viewed as an element of $X_*(\mathbf{T})$) and $w \in W$, we define a tamely ramified inertial $L$-parameter $\tau(w, \mu) : I_K \longrightarrow \widehat{H}(\mathbb{F})$ by

$$\tau(w, \mu) \overset{\text{def}}{=} \prod_{i=0}^{2f-1} (F^* \circ w^{-1})^i(\mu(\omega_{2f})).$$

We define $BC(\phi)$ to be the canonical identification of the dual root datum of the split group $\GL_{2/0_{K_2}}$ with the root datum of its dual group. Given this, we make an analogous definition of tamely ramified inertial $L$-parameters $\tau'((w, w'), (\mu, \mu')) : I_{K_2} \longrightarrow \GL_2(\mathbb{F})$.

Lemma 4.4. Suppose $\bar{\rho} : \Gamma_K \longrightarrow \mathbb{C}U_2(\mathbb{F})$ is a tamely ramified $L$-parameter which satisfies $\hat{\tau} \circ \bar{\rho} = \omega$. Via the identification of [GHS18, Lem. 9.4.5] we have

$$\bar{\rho}|_{I_K} \cong \tau(w, \mu + \eta)$$

with $w \in W$ and $\mu \in X^*(\mathbf{T})$ of the form

$$\mu = \begin{pmatrix} a_0 \\ b_0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} a_{f-1} \\ b_{f-1} \\ 0 \end{pmatrix}.$$  

Furthermore, we have

$$BC(\bar{\rho})|_{I_{K_2}} \cong \tau'((w, w'), (\mu, -\zeta(\mu) + \rho)).$$

Proof. The proof is a straightforward exercise using the definitions. □
4.1.5. We will also need a definition of genericity to study the relation between $L$-parameters, the set of conjectural associated weights and local deformations.

**Definition 4.5.** Suppose $\overline{\rho} : \Gamma_K \rightarrow {}^G\mathbf{U}_2(\mathbb{F})$ is a tamely ramified $L$-parameter. We say $\overline{\rho}$ is $n$-generic if, via the identification of $\text{[GHS18], Lem. 9.4.5}$, we can write

$$\overline{\rho}|_{I_K} \cong \tau(w, \mu + \eta)$$

where $w \in W$ and $\mu \in X^*(T)$ is $n$-deep.

4.2. The set $W^\gamma$. We now give a description of the set $W^\gamma$. We refer to $\text{[GHS18], §9}$ for the definition, and to op. cit. Proposition 9.2.1 for the definition of $V_\phi$.

**Proposition 4.6.** Let $\overline{\rho} : \Gamma_K \rightarrow {}^G\mathbf{U}_2(\mathbb{F})$ be a 1-generic tamely ramified $L$-parameter, and write $\overline{\rho}|_{I_K} \cong \tau(w, \mu + \eta)$ as in Definition 4.5, with $\mu$ being 1-deep. Let $V_\phi(\overline{\rho}) = R_\phi(\mu + \eta)$ be the associated Deligne–Lusztig representation of $G(\mathbb{F}_p)$ as in $\text{[GHS18, Props. 9.2.1 and 9.2.2]}$. Then

$$W^\gamma(\overline{\rho}) = JH\left(\beta(R_\phi(\mu + \eta))\right).$$

**Proof.** By definition of $W^\gamma(\overline{\rho})$, we must prove that

$$\mathcal{R}\left(JH\left(R_\phi(\mu + \eta)\right)\right) = JH\left(\beta(R_\phi(\mu + \eta))\right),$$

where $\mathcal{R}$ is the reflection operator defined in $\text{[GHS18, §9]}$. We use Equation (3.1.2). We claim that

$$\mathcal{R}(F_{w'}(R_\phi(\mu + \eta))) \cong F_{w'}(\beta(R_\phi(\mu + \eta)))$$

for all $w' \in W$. Note first that $\pi$ and $s$ commute as operators on $X^*(T)$, and the group $W$ is commutative. Therefore, in order to prove (4.2.1), it suffices to show

$$p\gamma_w + s\gamma_w'(\mu + \eta) - s\gamma_w'w\pi(\varepsilon\gamma_w') + ps(\rho_{w'}) - \pi(s(\rho)) + s(\rho) - p\gamma_w(\eta) - \rho \equiv$$

$$p\gamma_w + w'(s(\mu + \eta) - s(\eta) + (p - 1)\eta - s\pi(\varepsilon\gamma_{w'})) + p\gamma_{w'} - \pi(\rho),$$

the equivalence being taken modulo $(F - 1)X^0(T)$.

One easily checks that $s(\gamma_w) = \gamma_w$ and $-\pi(s(\rho)) + s(\rho) - \rho = -\pi(\rho)$, and hence (4.2.1) will be satisfied if we show that

$$p\gamma_{w'} - p\gamma_w \equiv -w's(\eta) + (p - 1)\gamma_w'(\eta) + pp_{w'},$$

modulo $(F - 1)X^0(T)$.

Expanding the left-hand side gives

$$p\gamma_{w'} - p\gamma_w = \cdots \left(\begin{array}{c} 0 \\ 0 \end{array}\right) \cdots \left(\begin{array}{c} 0 \\ 0 \end{array}\right) \cdots,$$

while expanding the right-hand side gives

$$-w's(\eta) + (p - 1)\gamma_w'(\eta) + pp_{w'} = \cdots \left(\begin{array}{c} 0 \\ p \end{array}\right) \cdots \left(\begin{array}{c} 0 \\ p \end{array}\right) \cdots.$$

In particular, adding

$$(F - 1)\left(\begin{array}{c} 0 \\ 1 \end{array}\right) = \left(\begin{array}{c} 0 \\ p \end{array}\right)$$
Let Proposition 4.8 gives
\[
\begin{array}{c}
\cdots \left( \begin{array}{cc}
0 & 0 \\
p-1 & p-1
\end{array} \right) \cdots \left( \begin{array}{cc}
0 & 0 \\
p-1 & p-1
\end{array} \right) \cdots = \cdots \\
w_i' = 1 & w_i' = s
\end{array}
\]
where the equality follows form the equivalence relation on \( X^*(\mathbf{T}) \). This gives the claim. \( \square \)

**Remark 4.7.** The above proposition and its proof carry over mutatis mutandis to the group \( \text{GL}_2(\mathcal{O}_{K_2}) \) and a tamely ramified Galois parameter \( \Gamma_{K_2} \rightarrow \text{GL}_2(\mathbb{F}) \) (cf. [Dia07]).

### 4.3. Base change and \( W^? \)

This section contains the main result on compatibility between the set \( W^? \) and base change of \( L \)-parameters (Theorem 4.9).

#### 4.3.1.

**Proposition 4.8.** Let \( \varphi : \Gamma_K \rightarrow C \text{U}_2(\mathbb{F}) \) be a 1-generic tamely ramified \( L \)-parameter which satisfies \( \widehat{\tau} \circ \varphi = \omega \). Then the subgroup \( \iota(\mathcal{O}_K^*) \) acts trivially on \( \beta(V_\phi(\overline{p})) \), and
\[
\beta'(V_{BC(\phi)}(BC(\varphi))) \cong BC(\beta(V_\phi(\overline{p}))).
\]

**Proof.** By Lemma 4.4 we may write
\[
\overline{p}|_{I_K} \cong \tau(w, \mu + \eta),
\]
with \( \mu \) being 1-deep and of the form
\[
\mu = \left( \begin{array}{c}
0_0 \\
b_0 \\
0
\end{array} \right) \cdots \left( \begin{array}{c}
0_0 \\
b_f-1 \\
0
\end{array} \right).
\]

Applying the map \( \beta \) to \( V_\phi(\overline{p}) \cong R_w(\mu + \eta) \) gives
\[
\beta(V_\phi(\overline{p})) \cong R_{zw}(s(\mu) + (p - 1)\eta).
\]
Notice that \( \iota(\mathcal{O}_K^*) \) acts trivially on this representation. In order to apply the base change map, the character appearing inside the Deligne–Lusztig representation must have its last two entries equal to zero. Using the equivalence given by adding the element \( -(\mathbf{F} - \overline{zw})(\eta) \), we get
\[
\beta(V_\phi(\overline{p})) \cong R_{zw}(s(\mu) - \eta + \overline{zw}(\eta)) = R_{zw}(s(\mu) - \sum_{w_i = 1}^{1} \alpha_i),
\]
and by (3.3.1) or (3.3.2), we obtain
\[
BC(\beta(V_\phi(\overline{p}))) \cong R'_{(zw, \overline{zw})}(s(\mu) - \sum_{w_i = 1}^{1} \alpha_i, -\mu - \sum_{w_i = 1}^{1} \alpha_i).
\]
On the other hand, Lemma 4.4 gives
\[
BC(\overline{p})|_{I_{K_2}} \cong \tau'(((w, w), (\mu, -\overline{zw}(\mu)) + \rho'),
\]
and therefore
\[
V_{BC(\phi)}(BC(\varphi)) \cong R'_{(w, w)}((\mu, -\overline{zw}(\mu)) + \rho').
\]
Applying the map \( \beta' \) gives
\[
\beta'(V_{BC(\phi)}(BC(\varphi))) \cong R'_{(zw, \overline{zw})}((s(\mu), -\mu) + (p - 1)\rho').
\]
Finally, using the equivalence given by adding \( -(\mathbf{F}' - (\overline{zw}, \overline{zw}))(\rho') \) we obtain
\[
\beta'(V_{BC(\phi)}(BC(\varphi))) \cong R'_{(zw, \overline{zw})}\left( s(\mu), -\mu - \rho' + (\overline{zw}, \overline{zw})(\rho') \right) \cong R'_{(zw, \overline{zw})}\left( s(\mu) - \sum_{w_i = 1}^{1} \alpha_i, -\mu - \sum_{w_i = 1}^{1} \alpha_i \right).
\]
\( \square \)
4.3.2. The main result of this section concerns local functoriality of predicted Serre weights.

**Theorem 4.9.** Let $\bar{p}: \Gamma_K \to C\mathbf{U}_2(\mathbb{F})$ be a $1$-generic tamely ramified $L$-parameter which satisfies $\hat{\iota} \circ \bar{p} = \omega$, and let $F$ denote a Serre weight of $G(\mathbb{Z}_p)$ on which $i(\mathcal{O}_K^\times)$ acts trivially. Then

$$F \in W^I(\bar{p}) \iff BC(F) \in W^I(BC(\bar{p})).$$

**Proof.** This follows by combining Lemma 3.26 and Propositions 4.6 and 4.8. $\square$

4.4. Inertial Local Langlands. In this subsection we discuss the inertial local Langlands correspondence which will be used in the rest of the paper. Recall that a *tame inertial type* $\tau'$ is a homomorphism $\tau': I_{K_2} \to \text{GL}_2(\mathcal{O})$ with open kernel, which is tamely ramified, and such that $\tau'$ extends to a representation of the Weil group of $K_2$.

We set $L \overset{\text{def}}{=} K_2((-p)^{1/2}((-2)^{-1}))$, and assume for simplicity that $\tau'$ factors as $\tau': I_{K_2} \to \text{Gal}(L/K_2) \to \text{GL}_2(\mathcal{O})$.

This implies that $\tau'$ is of the form

$$\tau' \cong \tilde{\omega}_{2f}^c \oplus \tilde{\omega}_{2f}^b.$$ 

If $a \not\equiv b \pmod{p^{2f} - 1}$, we call such a type a *principal series tame (inertial) type*. By Henniart’s appendix to [BM02], the inertial type $\tau'$ is associated to the tame type $\sigma'(\tau') \overset{\text{def}}{=} \text{Ind}_{B_U(\mathbb{F}_p^2)}^{\text{GL}_2(\mathbb{F}_p^2)}(\theta_a \otimes \theta_b)$ if $a \not\equiv b \pmod{p^{2f} - 1}$, and

$$\sigma'(\tau') \overset{\text{def}}{=} \theta_a \circ \text{det}$$

if $a \equiv b \pmod{p^{2f} - 1}$, where we use the notation $\theta_z(x) = \zeta_0(z)$. We view $\sigma'(\tau')$ as a representation of $\text{GL}_2(\mathcal{O}_{K_2})$ by inflation.

Suppose now that $(\tau')^{\varphi - f} \cong \tau'^\vee$, where $\tau'^\vee$ denotes the dual type (this condition means exactly that $\tau'$ extends to a map $\rho: \Gamma_K \to C\mathbf{U}_2(\mathcal{O})$ such that $\text{BC}(\rho)|_{I_{K_2}} \cong \tau'$ and $(\hat{\iota} \circ \rho)|_{I_{K}}$ is the trivial character). In this case $\tau'$ is of the form

$$\tilde{\omega}_{2f}^c \oplus \tilde{\omega}_{2f}^{-qc} \text{ or } \tilde{\omega}_{2f}^{(1-q)a} \oplus \tilde{\omega}_{2f}^{(1-q)b},$$

so that $\sigma'(\tau')$ is of the form

$$\text{Ind}_{B_U(\mathbb{F}_p^2)}^{\text{GL}_2(\mathbb{F}_p^2)}(\theta_c \otimes \theta_{-qc}), \quad \text{Ind}_{B_U(\mathbb{F}_p^2)}^{\text{GL}_2(\mathbb{F}_p^2)}(\theta_{(1-q)a} \otimes \theta_{(1-q)b}), \quad \text{or } \theta_{(1-q)a} \circ \text{det}.$$ 

In particular, these tame types come via base change from tame types of $\mathbf{U}_2(\mathcal{O}_K)$. We therefore make the following definition.

**Definition 4.10.** Let $\tau'$ denote a tame inertial type which factors through $\text{Gal}(L/K_2)$, and suppose furthermore that $(\tau')^{\varphi - f} \cong \tau'^\vee$.

(i) If $\tau' \cong \tilde{\omega}_{2f}^c \oplus \tilde{\omega}_{2f}^{-qc}$ with $c \not\equiv -qc \pmod{p^{2f} - 1}$, we set

$$\sigma(\tau') \overset{\text{def}}{=} \text{Ind}_{B_U(\mathbb{F}_q)}^{\mathbf{U}_2(\mathbb{F}_q)}(\theta_c),$$

which we view as a representation of $\mathbf{U}_2(\mathcal{O}_K)$ via inflation.

(ii) If $\tau' \cong \tilde{\omega}_{2f}^{(1-q)a} \oplus \tilde{\omega}_{2f}^{(1-q)b}$ with $a \not\equiv b \pmod{p^{2f} - 1}$, we set

$$\sigma(\tau') \overset{\text{def}}{=} \sigma(\theta_a \otimes \theta_b),$$

where we view $\theta_a$ and $\theta_b$ as characters of $\mathbf{U}_1(\mathbb{F}_q)$ by restriction, and where we view $\sigma(\tau')$ as a representation of $\mathbf{U}_2(\mathcal{O}_K)$ via inflation.
(iii) If \( \tau' \cong \tilde{\omega}_{2f}^{(1-q)a} \oplus \tilde{\omega}_{2f}^{(1-q)b} \), we set
\[
\sigma(\tau') \overset{\text{def}}{=} \theta_a \circ \det,
\]
where we view \( \theta_a \) as a character of \( U(\mathbb{F}_q) \) by restriction, and where we view \( \sigma(\tau') \) as a representation of \( U_2(\mathcal{O}_K) \) via inflation.

By construction, we have \( BC(\sigma(\tau')) \cong \sigma'(\tau') \).

We may now state a version of the inertial Local Langlands correspondence.

**Theorem 4.11.** Let \( \tau' : I_K \to GL_2(\mathbb{O}) \) be a tame inertial type as in Definition 4.10 so that in particular \( (\tau')^{\varphi^{\text{f}}} \cong \tau'^{\text{f}} \). Let \( \pi \) denote a smooth irreducible representation of \( U_2(K) \) over \( E \), and let \( \pi^{\text{f}} \) denote the direct sum of all representations appearing in the \( L \)-packet containing \( \pi \). Let \( BC(\pi) \) denote the stable base change of the \( L \)-packet containing \( \pi \). Then \( \pi^{\text{f}} \mid U_2(\mathcal{O}_K) \) contains \( \sigma(\tau') \) if and only if \( \text{rec}_{E}(BC(\pi)) \mid I_K \cong \tau' \) and \( N = 0 \) on \( \text{rec}_{E}(BC(\pi)) \). In this case, we have
\[
\dim_{\mathbb{F}} \text{Hom}_{U_2(\mathcal{O}_K)}(\sigma(\tau'), \pi^{\text{f}} \mid U_2(\mathcal{O}_K)) = 1.
\]

**Proof.** This follows from Henniart’s inertial local Langlands correspondence ([BM02]; see also [CEG+16 3.7 Thm.]) and the properties of the stable base change map ([Rog90 §11.4]). The statement about multiplicity may be deduced by restricting to the derived subgroup and using results of Nevins ([Nev05 Thm. 1] and [Nev13 Thm. 5.3]). \( \square \)

## 5. Local deformations

In this section we compute potentially crystalline deformation rings for certain \( L \)-parameters \( \bar{\rho} : \Gamma_K \to GL_2(\mathbb{F}). \) The main result is Corollary 5.24 which relates Hilbert–Samuel multiplicities of such rings with the set \( W^{\{0,1\}}(\bar{\rho}) \). This will be used to prove the “weight existence” direction of Corollary 7.5.

We follow [LLHLLM18 §6], adapting the base change techniques to our setting (see also [CDM18]). Subsection 5.1 contains the background on Kisin modules for \( GL_2 \), together with their classification by shapes. In Subsection 5.2, we introduce the notion of polarized (or Frobenius twist self-dual) Kisin modules and use a base change technique to compute their deformations. We then relate the deformation problems of polarized Kisin modules and of \( L \)-parameters to obtain the desired description of the potentially crystalline deformations rings.

### 5.1. Kisin modules

Throughout this subsection, we let \( R \) denote a complete local Noetherian \( \mathcal{O} \)-algebra with residue field \( \mathbb{F} \). We start by defining the relevant categories of Kisin modules with tame descent data \( Y^{\mu, \tau}(R) \subseteq Y^{[0,1], \tau'}(R) \) ([CL18 §5], see also [Le19 §3]).

#### 5.1.1. The ring \( \mathcal{S}_R \overset{\text{def}}{=} (\mathcal{O}_{K_2} \otimes_{\mathcal{O}_p} R)[[u]] \) is equipped with a Frobenius map \( \varphi : \mathcal{S}_R \to \mathcal{S}_R \) which is the arithmetic Frobenius on \( \mathcal{O}_{K_2} \) (i.e., \( \varphi = \varphi^{-1} \) on \( \mathcal{O}_{K_2} \)), which is trivial on \( R \), and which sends \( u \) to \( u^p \).

**Definition 5.1.** A Kisin module with height in \([0,1]\) over \( R \) is a finitely generated projective \( \mathcal{S}_R \)-module \( \mathfrak{M} \) together with an \( \mathcal{S}_R \)-linear map \( \phi_{\mathfrak{M}} : \varphi^* \mathfrak{M} \overset{\text{def}}{=} \mathcal{S}_R \otimes_{\mathcal{S}_R} \varphi \mathfrak{M} \to \mathfrak{M} \) such that
\[
E(u)\mathfrak{M} \subseteq \phi_{\mathfrak{M}}(\varphi^* \mathfrak{M}) \subseteq \mathfrak{M},
\]
where \( E(u) \) denotes the Eisenstein polynomial of \((-p)^{1/(p^2f-1)}\) over \( K_2 \), i.e. \( E(u) = u^{p^2f-1} + p \).

We often write \( \mathfrak{M} \) for a Kisin module, the Frobenius map \( \phi_{\mathfrak{M}} \) being implicit.
5.1.2. Recall that $\pi = (-p)^{1/(p^{2f-1})} \in \overline{Q}_p$, and set $L \overset{\text{def}}{=} K_2(\pi)$. For $g \in \text{Gal}(L/K_2)$, we have defined

$$\tilde{\omega}_\pi(g) = \frac{\pi^g}{\pi} \in \mathcal{O}^\times_{K_2}.$$  

(Note that reducing $\tilde{\omega}_\pi$ mod $p$ induces an isomorphism $\text{Gal}(L/K_2) \overset{\sim}{\rightarrow} \mathbb{F}_p^{\times_{2f}}$.) Given $g \in \text{Gal}(L/K_2)$, we let $\hat{g}$ denote the $\mathcal{O}_{K_2} \otimes \mathbb{Z}_p$ $R$-linear automorphism of $\mathfrak{S}_R$ given by $u \mapsto (\tilde{\omega}_\pi(g) \otimes 1)u$. Note that $\overline{\varphi} \circ \hat{g} = \hat{g} \circ \overline{\varphi}.$

**Definition 5.2.** Let $\mathfrak{M}$ denote a Kisin module over $R$.

(i) A semilinear action of $\text{Gal}(L/K_2)$ on $\mathfrak{M}$ is a collection $\{\hat{g}\}_{g \in \text{Gal}(L/K_2)}$ of $\hat{g}$-semilinear additive bijections $\hat{g} : \mathfrak{M} \rightarrow \mathfrak{M}$ such that $\hat{g} \circ \hat{h} = \hat{g}h$ for all $g, h \in \text{Gal}(L/K_2)$.

(ii) A Kisin module with descent datum over $R$ is a Kisin module together with a semilinear action of $\text{Gal}(L/K_2)$ given by $\{\hat{g}\}_{g \in \text{Gal}(L/K_2)}$ which commutes with $\phi_{\mathfrak{M}}$, i.e., we have

$$\hat{g} \circ \phi_{\mathfrak{M}} = \phi_{\mathfrak{M}} \circ \overline{\varphi}^\ast \hat{g}$$

for all $g \in \text{Gal}(L/K_2)$.

5.1.3. Any Kisin module $\mathfrak{M}$ admits a decomposition

$$\mathfrak{M} = \bigoplus_{i=0}^{2f-1} \mathfrak{M}^{(i)},$$

where $\mathfrak{M}^{(i)}$ is the $R[u]$-submodule of $\mathfrak{M}$ such that $(x \otimes 1)m = (1 \otimes \varsigma_0 \circ \varphi^i(x))m$ for $m \in \mathfrak{M}^{(i)}$ and $x \in \mathcal{O}_{K_2}$.

We let

$$\tau' : I_{K_2} \rightarrow \text{Gal}(L/K_2) \rightarrow \text{GL}_2(\mathcal{O})$$

denote a tamely ramified inertial type which factors through $\text{Gal}(L/K_2)$. Recall that this implies $\tau'$ can be written $\tau' = \tilde{\omega}_2^a + \tilde{\omega}_2^b$.

**Definition 5.3.** Suppose $\mathfrak{M}$ is a Kisin module with descent datum over $R$. We say the descent datum is of type $\tau'$ if we have $\mathfrak{M}^{(i)} / u \mathfrak{M}^{(i)} \cong \tau' \otimes_\mathcal{O} R$ as representations of $\text{Gal}(L/K_2)$ for every $0 \leq i \leq 2f - 1$.

5.1.4. We now define the categories of Kisin modules that will be relevant for us. Let $\mu \overset{\text{def}}{=} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ denote the standard minuscule cocharacter of $G' = \text{Res}_{\mathcal{O}_{K_2}/\mathbb{Z}_p} \text{GL}_2/\mathcal{O}_{K_2}$.

**Definition 5.4.** Fix a principal series tame type $\tau'$.

(i) We define $Y^{[0,1],\tau'}(R)$ to be the category of Kisin modules over $R$ of rank 2, with height in $[0,1]$, and descent datum of type $\tau'$.

(ii) We define $Y^{\mu,\tau'}(R)$ to be the full subcategory of $Y^{[0,1],\tau'}(R)$ consisting of Kisin modules such that

$$E(u) \det \mathfrak{M} = \phi_{\mathfrak{M}}(\overline{\varphi}^\ast (\det \mathfrak{M})).$$

Note that the definition of $Y^{\mu,\tau'}(R)$ above is consistent with the construction of [CL18 §5], thanks to Theorem 5.13 and Corollary 5.12 of op. cit.. See also [LLHLM18 Thm. 4.18].

5.1.5. We fix some notation, following [LLHLM18 §2.1]. Let $\tau'$ be a principal series tame type of $I_{K_2}$. We may write

$$\tau' = \eta_1 \oplus \eta_2 = \tilde{\omega}_2^{-\sum_{i=0}^{2f-1} a_1,i \varphi^i} \oplus \tilde{\omega}_2^{-\sum_{i=0}^{2f-1} a_2,i \varphi^i},$$

\(\text{E}(u)\) det \(\mathfrak{M} = \phi_{\mathfrak{M}}(\overline{\varphi}^\ast (\det \mathfrak{M}))\).
with \(0 \leq a_{k,i} \leq p-1\) for all \(i\). Until the end of this paper we assume that neither \(\eta_1\) nor \(\eta_2\) are trivial, i.e. \((a_{k,i})_i \notin \{(p-1, \ldots, p-1), (0, \ldots, 0)\}\) for \(k = 1, 2\). Set \(a_1 \overset{\text{def}}{=} (a_{1,i})_i\), \(a_2 \overset{\text{def}}{=} (a_{2,i})_i\), and given \(0 \leq j \leq 2f-1\), define the shifted sums

\[
a_1^{(j)} \overset{\text{def}}{=} \sum_{i=0}^{2f-1} a_{1,i-j} p^i, \quad a_2^{(j)} \overset{\text{def}}{=} \sum_{i=0}^{2f-1} a_{2,i-j} p^i,
\]

so that, in particular, \(\eta_1 = \tilde{\omega}_2 a_1^{(0)}\), \(\eta_2 = \tilde{\omega}_2 a_2^{(0)}\).

**Definition 5.5.** Let \(n \geq 0\). We say the pair \((a_1, a_2)\) is \(n\)-generic if

\[
n < |a_{1,i} - a_{2,i}| < p - n
\]

for every \(0 \leq i \leq 2f-1\). If \(\tau'\) is associated to \((a_1, a_2)\) as above, we say \(\tau'\) is \(n\)-generic if the pair \((a_1, a_2)\) is.

This agrees with the notion of genericity given in Definition 4.5.

5.1.6.

**Definition 5.6.** Let \(\tau'\) denote a principal series tame type of \(I_{K_2}\), and let \((a_1, a_2)\) denote the associated pair. Suppose \(\tau'\) is 2-generic. An orientation of \(\tau'\) is an element \(w = (w_i)_i \in S_{2f}^2\) such that

\[
a_{w_i(1)}^{(i)} \geq a_{w_i(2)}^{(i)}
\]

for all \(0 \leq i \leq 2f-1\).

(We view \(S_2\) as a subgroup of \(\text{GL}_2(\mathbb{Z})\) via the standard embedding as permutation matrices. Since \(S_{2f}^2 \cong W'\), we also view orientations as elements of \(W'\) when convenient.) We note that an orientation depends on the ordering of the characters \(\eta_1, \eta_2\). Since we take \(\tau'\) to be 2-generic, the orientation is unique, and \(w_i\) depends only on the pair \((a_{1,2f-1-i}, a_{2,2f-1-i})\).

5.1.7. In what follows, we use the notation \(v \overset{\text{def}}{=} u^{2f-1}\).

**Definition 5.7.** Let \(\tau' = \eta_1 \oplus \eta_2\) denote a 2-generic principal series tame type, \(\mathfrak{M} \in Y^{[0,1], \tau'(R)}\), and let \(\mathfrak{M} = \bigoplus_{i=0}^{2f-1} \mathfrak{M}^{(i)}\) be the decomposition of \(\mathfrak{M}\) as in Subsection 5.1.3.

(i) We let \(\mathfrak{M}_1^{(i)}\) (resp. \(\mathfrak{M}_2^{(i)}\)) denote the \(R[v]\)-submodule of \(\mathfrak{M}^{(i)}\) on which \(\text{Gal}(L/K_2)\) acts by \(\eta_1\) (resp. \(\eta_2\)).

(ii) We define \(\overline{\mathfrak{M}}_1^{(i)}\) (resp. \(\overline{\mathfrak{M}}_2^{(i)}\)) to be the \(R[v]\)-submodule of \(\overline{\mathfrak{M}}(\mathfrak{M}^{(i)}) = (\overline{\mathfrak{M}} \mathfrak{M})^{(i+1)}\) on which \(\text{Gal}(L/K_2)\) acts by \(\eta_1\) (resp. \(\eta_2\)).

(iii) We define an eigenbasis \(\beta \overset{\text{def}}{=} \{\beta^{(i)}\}_i\) of \(\mathfrak{M}\) to be a collection of ordered bases \(\beta^{(i)} = (f_1^{(i)}, f_2^{(i)})\) of each \(\mathfrak{M}^{(i)}\) such that \(f_1^{(i)} \in \mathfrak{M}_1^{(i)}\) and \(f_2^{(i)} \in \mathfrak{M}_2^{(i)}\).

Now let \(\tau'\) be a 2-generic principal series tame type, with orientation \(w = (w_i)_i\). We have a commutative diagram

\[
\begin{array}{ccc}
\overline{\mathfrak{M}}^{(i-1)}_{w_i(2)} & \overset{u p^{2f-1-(a_{w_i(1)}^{(i)}-a_{w_i(2)}^{(i)})}}{\longrightarrow} & \overline{\mathfrak{M}}^{(i-1)}_{w_i(1)} \\
\downarrow \phi_{\mathfrak{M}, w_i(2)}^{(i-1)} & & \downarrow \phi_{\mathfrak{M}, w_i(1)}^{(i-1)} \\
\mathfrak{M}^{(i)}_{w_i(2)} & \overset{u p^{2f-1-(a_{w_i(1)}^{(i)}-a_{w_i(2)}^{(i)})}}{\longrightarrow} & \mathfrak{M}^{(i)}_{w_i(1)} \\
\downarrow \phi_{\mathfrak{M}, w_i(2)}^{(i)} & & \downarrow \phi_{\mathfrak{M}, w_i(1)}^{(i)} \\
\mathfrak{M}^{(i)}_{w_i(2)} & \overset{u p^{2f-1-(a_{w_i(1)}^{(i)}-a_{w_i(2)}^{(i)})}}{\longrightarrow} & \mathfrak{M}^{(i)}_{w_i(1)} \\
\end{array}
\]

Here, \(\phi_{\mathfrak{M}, k}^{(i-1)}\) denotes the restriction of \(\phi_{\mathfrak{M}}\) to \(\overline{\mathfrak{M}}^{(i-1)}_k\).
5.1.8. Fix a principal series 2-generic tame type $\tau'$ and $\mathfrak{M} \in Y^{[0,1],\tau'}(R)$. Let $w = (w_i)_i$ denote the orientation of $\tau'$, and let $\beta = \{\beta^{(i)}\}$ denote an eigenbasis for $\mathfrak{M}$. We define
\[
\beta^{(i)}_{w_i(2)} \overset{\text{def}}{=} \left( a^{(i)}_{w_i(1)} - a^{(i)}_{w_i(2)}, f^{(i)}_{w_i(1)}, f^{(i)}_{w_i(2)} \right),
\]
the first is an $R[v]$-basis for $\mathfrak{M}_{w_i(2)}$, the second is an $R[v]$-basis for $\mathfrak{M}_{w_i(2)}$. We then define the matrix $A^{(i)} \in \text{Mat}_{2 \times 2}(R[v])$ by the condition
\[
(5.1.2) \quad \phi^{(i)}_{\mathfrak{M}_{w_i+1}(2)} \left( \varphi^{(i)}_{w_i+1(2)}(w) \right) = \beta^{(i+1)}_{w_i+1(2)} A^{(i)}.
\]
We say that $A^{(i)}$ is the matrix of the partial Frobenius of $\mathfrak{M}$ (at embedding $i$, with respect to $\beta$).

5.1.9. We now find a more convenient expression for the data of the matrices $(A^{(i)})_i$.

We define the extended affine Weyl group of $\text{GL}_2$ as
\[
\tilde{W} \overset{\text{def}}{=} \text{N}_{\text{GL}_2}(\mathbf{T}_G)(\mathbb{F}(v))/\mathbf{T}_G(\mathbb{F}[v]),
\]
where $\mathbf{T}_G$ denotes the torus dual to $T_G/\mathcal{O}_{K_2}$. We have an exact sequence
\[
0 \longrightarrow X_*(\mathbf{T}_G) \longrightarrow \tilde{W} \longrightarrow S_2 \longrightarrow 0,
\]
where the first nontrivial map sends a cocharacter to its value on $v$. Furthermore, we have a Bruhat decomposition
\[
\text{GL}_2(\mathbb{F}(v)) = \bigsqcup_{\tilde{w} \in \tilde{W}} J\tilde{w}J,
\]
where $J$ denotes the standard Iwahori subgroup of $\text{GL}_2(\mathbb{F}[v])$, that is, the set of matrices which are upper triangular mod $v$.

Using the canonical identification $X_*(\mathbf{T}_G) \cong X^*(T_G/\mathcal{O}_{K_2})$, we identify $\tilde{W}^{J}$ with the extended affine Weyl group $\tilde{W}'$ of $G'$.

**Definition 5.8.** Let $\tilde{w} = (\tilde{w}_i)_i \in \tilde{W}'$, let $\tau'$ be a principal series 2-generic tame type, and let $w = (w_i)_i \in W'$ denote the orientation of $\tau'$. Let $\mathfrak{M} \in Y^{[0,1],\tau'}(\mathbb{F})$.

1. We say $\mathfrak{M}$ has shape $\tilde{w}$ if for some eigenbasis $\beta$, the matrices $(A^{(i)})_i$ (defined by (5.1.2), with respect to $\beta$) have the property that $A^{(i)} \in J\tilde{w}_i J$.
2. As in the discussion following [LLHLM18, Def. 2.17], the notion of shape does not depend on the choice of eigenbasis. We define $Y^{\mu,\tau'}(\mathbb{F})$ to be the full subcategory of $Y^{\mu,\tau'}(\mathbb{F})$ consisting of Kisin modules of shape $\tilde{w}$.

5.1.10. Upon choosing the dominant chamber corresponding to $J$ in $X_*(\mathbf{T}_G) \otimes _\mathbb{Z} \mathbb{R}$, we obtain a Bruhat order $\leq$ on $\tilde{W}$. Given a cocharacter $\lambda \in X_*(\mathbf{T}_G)$, we define the $\lambda$-admissible set as
\[
\text{Adm}(\lambda) \overset{\text{def}}{=} \left\{ \tilde{w} \in \tilde{W} : \tilde{w} \leq t_{w(\lambda)} \text{ for some } w \in S_2 \right\}.
\]
In particular, we have
\[
\text{Adm}(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \left\{ \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} \right\}.
\]
We denote these elements by $t$, $t'$ and $w$, respectively. Given $\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we define
\[
\text{Adm}(\mu) \overset{\text{def}}{=} \prod_{i=0}^{2f-1} \text{Adm}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}),
\]
which we call the $\mu$-admissible set. As in [LLHLM18 Cor. 2.19], we have that $Y^\mu_{\bar{w}}(F)$ is nonempty if and only if $\bar{w} \in \text{Adm}(\mu)$.

We now have the analog of [LLHLM18 Thm. 2.21], using op. cit., Lemma 2.20.

**Lemma/Definition 5.9.** Suppose $\bar{w} = (\bar{w}_i)_{i \in \hat{W}}$ is $\mu$-admissible and $\tau'$ is a 2-generic principal series tame type. Let $\mathfrak{M} \in Y^\mu_{\bar{w}}(F)$. Then there is an eigenbasis $\beta$ for $\mathfrak{M}$ such that the matrix of partial Frobenius $A^{(i)}$ has the form given in Table 1. We call such an eigenbasis a gauge basis.

**Table 1. Shapes of Kisin modules over $F$**

Here we have $c_{j,k} \in F$ and $\bar{c}_{j,k} \in F^\times$.

<table>
<thead>
<tr>
<th>$\bar{w}_i$</th>
<th>$t$</th>
<th>$t'$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{(i)}$</td>
<td>$\left( \begin{array}{cc} v\bar{c}<em>{1,1} &amp; 0 \ v\bar{c}</em>{2,1} &amp; \bar{c}_{2,2} \end{array} \right)$</td>
<td>$\left( \begin{array}{cc} \bar{c}<em>{1,1} &amp; \bar{c}</em>{1,2} \ 0 &amp; v\bar{c}_{2,2} \end{array} \right)$</td>
<td>$\left( \begin{array}{cc} 0 &amp; \bar{c}<em>{1,2} \ v\bar{c}</em>{2,1} &amp; 0 \end{array} \right)$</td>
</tr>
</tbody>
</table>

5.1.11. Now fix $\mathfrak{M} \in Y^\mu_{\bar{w}}(F)$, and fix a gauge basis $\beta$ for $\mathfrak{M}$. We denote by $Y^\mu_{\mathfrak{M}}(R)$ the category of pairs $(\mathfrak{M}, j)$, where $\mathfrak{M} \in Y^\mu_{\bar{w}}(R)$ and $j$ is an isomorphism $j: \mathfrak{M} \otimes_R F \xrightarrow{\sim} \mathfrak{M}$.

**Definition 5.10.** Let $(\mathfrak{M}, j) \in Y^\mu_{\mathfrak{M}}(R)$. A gauge basis of $(\mathfrak{M}, j)$ is an eigenbasis $\beta$ lifting $\beta$ via $j$ such that the matrix of partial Frobenius $A^{(i)}$ satisfies the degree conditions given in Table 2.

Note that a gauge basis for $(\mathfrak{M}, j) \in Y^\mu_{\mathfrak{M}}(R)$ exists by the analog of [LLHLM18 Thm. 4.1], and the set of gauge bases for $(\mathfrak{M}, j)$ is in bijection with the set of eigenbases of $\mathfrak{M}/u\mathfrak{M}$ lifting $\beta \mod u$ by the analog of [LLHLM18 Thm. 4.16]. (See also the cases $A_1$, $A_2$ of [Le19 Thm. 3.3], where a detailed proof of the cases $t$ and $w$ above is given.)

**Table 2. Deforming Kisin modules by shape**

Here, $\text{deg}(A^{(i)})$ denotes the degree of the polynomial in each entry. We write $n^*$ to denote a polynomial entry of degree $n$ whose leading coefficient is a unit. We have $c_{j,k} \in R$ and $\bar{c}_{j,k} \in R^\times$.

Row 3 is deduced from row 2 by imposing condition 5.1.11.

<table>
<thead>
<tr>
<th>$\bar{w}_i$</th>
<th>$t$</th>
<th>$t'$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{deg}(A^{(i)})$</td>
<td>$\begin{cases} 1^* &amp; \text{if } v \leq 0 \ 0^* &amp; \text{if } v &gt; 0 \end{cases}$</td>
<td>$\begin{cases} 0^* &amp; \text{if } v(0^<em>) \leq 0 \ \infty &amp; \text{if } v(0^</em>) &gt; 0 \end{cases}$</td>
<td>$\begin{cases} \leq 0 &amp; \text{if } v \leq 0 \ 0^* &amp; \text{if } v &gt; 0 \end{cases}$</td>
</tr>
<tr>
<td>$A^{(i)}$</td>
<td>$\left( \begin{array}{cc} (v + p)c_{1,1} &amp; 0 \ vc_{2,1} &amp; c_{2,2} \end{array} \right)$</td>
<td>$\left( \begin{array}{cc} c_{1,1} &amp; c_{1,2} \ 0 &amp; c_{2,2}(v + p) \end{array} \right)$</td>
<td>$\left( \begin{array}{cc} c_{1,1} &amp; c_{1,2} \ vc_{2,1} &amp; c_{2,2} \end{array} \right)$</td>
</tr>
</tbody>
</table>

5.2. Duality. We introduce the notion of Frobenius twist self-dual Kisin modules over $K$ and study their relation with usual Kisin modules over $K_2$ via the theory of base change (as in [LLHLM18 §6]). The main result of this section (Lemma 5.17) describes the matrix of partial Frobenius on Frobenius twist self-dual Kisin modules.

5.2.1. We now collect the relevant properties of Cartier duality which we will need.

**Definition 5.11.** (cf. [Bro08 § 3.4.1]) Suppose $\tau'$ is a tame principal series type, $R$ is a local Artinian $\mathcal{O}$-algebra with residue field $F$, and let $\mathfrak{M} \in Y^\mu_{\bar{w}}(R)$. We define the Cartier dual of $\mathfrak{M}$ to be

$$\mathfrak{M}^\vee \overset{\text{def}}{=} \text{Hom}_{\mathcal{O}_R}(\mathfrak{M}, \mathfrak{S}_R),$$
which we equip with a Frobenius map by
\[ 1 \otimes f \longmapsto \phi_{\mathcal{S}} \circ (1 \otimes f) \circ \phi_{\mathcal{S}}^{-1} \circ E(u), \]
where \( 1 \otimes f \in \mathfrak{p}^* \text{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{S} \mathcal{R}) \cong \text{Hom}_{\mathcal{S}}(\mathfrak{p}^* \mathcal{M}, \mathfrak{p}^* \mathcal{S} \mathcal{R}). \) (Note that the map \( \phi_{\mathcal{S}} \) is injective by [Kis09 Lem. 1.2.2(1)].) We also equip \( \mathcal{M}^\vee \) with a descent datum, given by
\[ \widehat{g} : \mathcal{M}^\vee \rightarrow \mathcal{M}^\vee \]
\[ f \longmapsto \widehat{g} \circ f \circ g^{-1} \]
(the right-hand \( g^{-1} \) denotes the semilinear action of \( \text{Gal}(L/K_2) \) on \( \mathcal{M} \), while the left-hand \( \widehat{g} \) denotes the semilinear action on \( \mathcal{S} \mathcal{R} \).) With this definition, one easily checks that the descent datum of \( \mathcal{M}^\vee \) is of type \( \tau^\vee \), where \( \tau^\vee \) is the type dual to \( \tau \), so that \( \mathcal{M}^\vee \in Y^{\mu, \tau^\vee}(R) \).

Before proceeding with the proof of the proposition below, we introduce some notation. We define \( \{ p_n \}_{n \geq 0} \) to be a sequence of elements of \( \overline{Q} \) which satisfy \( p_{n+1} = p_n \) and \( p_0 = -p \), and define \( K_\infty \overset{\text{def}}{=} \bigcup_{n \geq 0} K(p_n) \) and \( K_2, \infty \overset{\text{def}}{=} \bigcup_{n \geq 0} K_2(p_n) \). Note that \( \text{Gal}(K_2, \infty/K_\infty) \cong \text{Gal}(K_2/K) \).

**Proposition 5.12.** Suppose \( R \) is a local Artinian \( O \)-algebra with residue field \( F \). Let \( \tau' : I_{K_2} \rightarrow GL_2(\mathcal{O}) \) be a principal series tame type. Then \( \mathcal{M} \mapsto \mathcal{M}^\vee \) defines an involutive functor \( Y^{\mu, \tau'}(R) \rightarrow Y^{\mu, \tau^\vee}(R) \), which enjoys the following properties:

- We have \( T_{\text{dd}}(\mathcal{M}) \cong T_{\text{dd}}(\mathcal{M}^\vee) \otimes \varepsilon \) as \( K_{2, \infty} \)-representations, where the functor \( T_{\text{dd}}^* \) is as defined in [LLHLM18 §2.3].
- Let \( \beta = \{ \beta^{(i)} \}_{i} \) be an eigenbasis of \( \mathcal{M} \) as in Definition 5.7. Let \( C^{(i)} \overset{\text{def}}{=} \text{Mat}_\beta(\phi_{\mathcal{M}}^{(i)}) \in \text{Mat}_{2 \times 2}(R[u]) \) denote the matrix of the Frobenius on \( \mathfrak{p}^* \mathcal{M}(\mathcal{M}) \), defined by
  \[ \phi_{\mathcal{M}}^{(i)} \left( 1 \otimes f_1^{(i)}, 1 \otimes f_2^{(i)} \right) = \left( f_1^{(i+1)}, f_2^{(i+1)} \right) C^{(i)} . \]

Then the matrix of Frobenius on \( \mathcal{M}^\vee \) with respect to the dual basis \( \beta^\vee \) is given by
\[ \text{Mat}_{\beta^\vee}(\phi_{\mathcal{M}^\vee}^{(i)}) = E(u)(C^{(i)})^{-\top} . \]

**Proof.** The first point follows from [Dro08 Prop. 3.4.1.7], while the second point follows from an explicit calculation. \( \square \)

**5.2.2.** We explain how orientations and shapes change under duality. Suppose \( \tau' = \eta_1 \oplus \eta_2 \) is a 2-generic principal series tame type, which is associated to the pair \( (a_1, a_2) \) and orientation \( w = (w_i) \in S_2^f \). We fix an ordering on the characters of \( \tau^\vee \) so that \( \tau^\vee = \eta_1^{-1} \oplus \eta_2^{-1} \) is associated to the pair \( (p - 1 - a_1, p - 1 - a_2) \) and orientation \( (s, g)w \) (recall that we view elements of \( S_2^f \cong W \) as pairs of elements of \( S_2^f \cong W \) as in [3.2.1]). Note that \( \tau' \) is \( n \)-generic if and only if \( \tau^\vee \) is \( n \)-generic.

Assume that \( R \) is a local Artinian \( O \)-algebra with residue field \( F \), \( \mathcal{M} \in Y^{\mu, \tau'}(R) \), and let \( \beta \) denote an eigenbasis of \( \mathcal{M} \). The matrix \( C^{(i)} \) of Frobenius on \( \mathfrak{p}^* \mathcal{M}(\mathcal{M}) \) (as in the above proposition) and the matrix \( A^{(i)} \) of the partial Frobenius (as in Subsubsection 5.1.8) are related by the equation
\[ C^{(i)} = w_{i+1} \left[ \begin{array}{cc} a_{w_{i+1}} & 0 \\ 0 & a_{w_{i+1}} \end{array} \right] A^{(i)} \left[ \begin{array}{cc} -a_{w_{i+1}} & 0 \\ 0 & -a_{w_{i+1}} \end{array} \right] w_{i+1}^{-1} \]
(for the proof, see [LLHLM18 Prop 2.13]). Using this relation for the dual Kisin module \( \mathcal{M}^\vee \) and dual type \( \tau^\vee \) (ordered as in the previous paragraph), along with Proposition 5.12, we conclude that the matrix of partial Frobenius on \( \mathcal{M}^\vee \), with respect to \( \beta^\vee \) at embedding \( i \), is equal to
\[ E(u)s(A^{(i)})^{-\top} s = \left( \begin{array}{cc} v + p & 0 \\ 0 & v + p \end{array} \right) s(A^{(i)})^{-\top}s. \]
Now suppose $\mathcal{M} \in Y^{\mu,\tau'}(\mathbb{F})$. The above relation shows that $\mathcal{M}$ has shape $\bar{w}_i$ at embedding $i$ if and only if $\mathcal{M}^\vee$ has shape $(\begin{smallmatrix} \bar{g} & 0 \\ -1 & 1 \end{smallmatrix}) s \bar{w}_{i}^{-1} s$ at embedding $i$. In particular this involution on $\bar{W}'$ fixes $\text{Adm}(\mu)$ pointwise, and thus Cartier duality induces an involutive functor

$$Y_{\bar{w}}^{\mu,\tau'}(\mathbb{F}) \rightarrow Y_{\bar{w}}^{\mu,\tau''}(\mathbb{F}).$$

Furthermore, equation (5.2.1) shows that $\mathcal{M}^\vee$ is a gauge basis for $\mathcal{M}$ if and only if $\mathcal{M}^\vee$ is a gauge basis for $\mathcal{M}^\vee$. Similarly, if $(\mathcal{M}, j) \in Y_{\mathcal{M}}^{\mu,\tau'}(R)$, then $\beta$ is a gauge basis for $(\mathcal{M}, j)$ if and only if $\beta^\vee$ is a gauge basis for $(\mathcal{M}^\vee, (j')^{-1})$.

5.2.3. In the following, we use the notation $\sigma$ to denote the automorphism of $\mathcal{G}_R$ which is the arithmetic Frobenius on $\mathcal{O}_{K_2}$ and which acts trivially on $R$ and the variable $u$. Thus, given a Kisin module $\mathcal{M}$, we may form the pullback $\sigma^*\mathcal{M} := \mathcal{G}_R \otimes_{\sigma, \mathcal{G}_R} \mathcal{M}$ along $\sigma$, equipped with Frobenius $\phi_{\sigma^*\mathcal{M}} \overset{\text{def}}{=} \sigma^* \phi_{\mathcal{M}}$. One easily checks that $(\sigma^*\mathcal{M})^{(i)} = \sigma^* (\mathcal{M}^{(i-1)})$. If $\mathcal{M}$ comes equipped with a descent datum, $\sigma^*\mathcal{M}$ obtains a descent datum via the canonical identification

$$\tilde{g}^\sigma : (\sigma^*\mathcal{M}) \overset{\sim}{\rightarrow} \sigma^* (\tilde{g}^\mathcal{M})$$

(here $\tilde{g}^\mathcal{M}$ denotes the pullback of $\mathcal{M}$ along the automorphism $\tilde{g}$ of $\mathcal{G}_R$, and similarly for $\tilde{g}^\sigma$).

We note that if $\tau = \eta_1 \oplus \eta_2$ is a principal series type and $\mathcal{M}$ has type $\tau'$, then $\sigma^*\mathcal{M}$ has type $(\tau')^\varphi = \eta_1^p \oplus \eta_2^p$. Thus, Frobenius twisting gives a functor

$$\sigma^* : Y^{\mu,\tau'}(R) \rightarrow Y^{\mu, (\tau')^\varphi}(R).$$

We make similar definitions for iterates of $\sigma$.

We briefly describe how the Frobenius twist transforms certain objects associated to Kisin modules. Twisting changes the principal series tame type $\tau'$ into $(\tau')^\varphi$. Thus, it also transforms the associated pair

$$(a_1, a_2) = ((a_{1,0}, a_{1,1}, \ldots, a_{1,2f-1}), (a_{2,0}, a_{2,1}, \ldots, a_{2,2f-1}))$$

into

$$(a_{1,1}, a_{1,2}, \ldots, a_{1,2f-1}, a_{1,0}, (a_{2,1}, a_{2,2}, \ldots, a_{2,2f-1}, a_{2,0}),$$

and transforms the orientation $w = (w_0, w_1, \ldots, w_{2f-1})$ into $(w_{2f-1}, w_0, \ldots, w_{2f-2})$. Further, given an eigenbasis $\beta = \{(f_1, f_2)\}_i$ for $\mathcal{M}$, the elements $\sigma^* \beta := \{(1 \otimes f_1, 1 \otimes f_2)\}_i$ form an eigenbasis of $\sigma^* \mathcal{M}$. Therefore, by their definition, the Frobenius twist transforms the matrices $(A^{(0)}, A^{(1)}, \ldots, A^{(2f-1)})$ of partial Frobenius (with respect to $\beta$) into $(A^{(2f-1)}, A^{(0)}, \ldots, A^{(2f-2)})$, and if $\mathcal{M} \in Y^{\mu,\tau'}(\mathbb{F})$ has shape $\bar{w} = (\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{2f-1})$, then $\sigma^* \mathcal{M} \in Y^{\mu, (\tau')^\varphi}(\mathbb{F})$ will have shape $(\bar{w}_{2f-1}, \bar{w}_0, \ldots, \bar{w}_{2f-2})$. Finally, we obtain an isomorphism on the associated $\Gamma_{K_2, \infty}$-representation

$$T^\varphi_{\bar{w}}(\sigma^* \mathcal{M}) \cong T^\varphi_{\bar{w}}(\mathcal{M})^\varphi,$$

where we recall that the superscript $\varphi$ denotes the twist of the representation by $\varphi$.

5.2.4. Suppose now that $\tau'$ is a 2-generic principal series tame type which satisfies $(\tau')^{\varphi^{-1}} = \tau''$ (and note that $(\tau')^{\varphi^{-1}} = (\tau')^{\varphi^1}$). As in Subsection 4.4, this implies that $\tau'$ is of the form

$$\tau' = \bar{w}_c^{−e} \oplus \bar{w}_d^{−f} \quad \text{or} \quad \tau' = \bar{w}_2^{−(1+p^f)} \oplus \bar{w}_2^{−(1+p^f)} \oplus \bar{w}_2^{−(1+p^f)} \oplus \bar{w}_2^{−(1+p^f)} \oplus \bar{w}_2^{−(1+p^f)} \oplus \bar{w}_2^{−(1+p^f)}.$$

If $\tau'$ is 2-generic, the orientation on $\tau'$ has the form $(z, z)$ for $z \in W$ in the first case, while in the second case the orientation has the form $(z, z, z)$ for $z \in W$.

The discussion above gives the following:
Lemma 5.13. Assume $\tau' = \eta_1 \oplus \eta_2$ is a 2-generic principal series tame type, and let $w = (w_i)_i \in W'$ denote its orientation. Suppose that $(\tau')^{\varphi^{-f}} \cong \tau'^{\vee}$. Then

$$\eta_{w_{i+j}(2)}^{\tau'} = \eta_{w_{i+j}(2)}^{-1}$$

for every $0 \leq i \leq 2f - 1$.

Proof. This may be proved casewise, using the possible orientations on $\tau'$.

5.2.5. As in [LLHLMS §6.1], we define Kisin modules which are Frobenius-twist self-dual:

Definition 5.14. Let $R$ denote a local Artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, and let $\tau'$ denote a principal series tame type which satisfies $(\tau')^{\varphi^{-f}} \cong \tau'^{\vee}$. We define

$$Y_{\mu,\tau'}^\text{pol}(R) \defeq \left\{ (\mathcal{M}, \iota) : \mathcal{M} \in Y_{\mu,\tau'}^\text{pol}(R), \quad \iota : (\sigma^f)^*\mathcal{M} \rightarrow (\sigma^f)^*\mathcal{M}^{\vee} \right\},$$

where $\iota$ is a map of Kisin modules with descent data, such that the composite morphism

$$\mathcal{M} \xrightarrow{\text{can}} (\sigma^f)^*((\sigma^f)^*\mathcal{M}) \xrightarrow{((\sigma^f)^*\iota)^*} (\sigma^f)^*\mathcal{M} \xrightarrow{\text{can}} ((\sigma^f)^*\mathcal{M})^{\vee} \xrightarrow{(\iota^{\vee})^{-1}} \mathcal{M}$$

is $-1$ on $\mathcal{M}$. We call $\iota$ a polarization of $\mathcal{M}$. A morphism $(\mathcal{M}_1, \iota_1) \rightarrow (\mathcal{M}_2, \iota_2)$ in $Y_{\mu,\tau'}^\text{pol}(R)$ is a morphism $\alpha : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ in $Y_{\mu,\tau'}^\text{pol}(R)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{M}_1 & \xrightarrow{(\sigma^f)^*\iota_1} & (\sigma^f)^*\mathcal{M}_2 \\
\downarrow{\iota_1} & & \downarrow{\iota_2} \\
\mathcal{M}_1^{\vee} & \xleftarrow{\alpha^{\vee}} & \mathcal{M}_2^{\vee}
\end{array}
$$

Definition 5.15. Let $R$ be a local Artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, suppose $\tau'$ is a 2-generic principal series tame type, and let $(\mathcal{M}, \iota) \in Y_{\mu,\tau'}^\text{pol}(R)$. A gauge basis of $(\mathcal{M}, \iota)$ is a gauge basis $\beta$ of $\mathcal{M} \in Y_{\mu,\tau'}(R)$ which is compatible with $\iota$, meaning $\iota((\sigma^f)^*\beta) = (1_{\mathcal{M}}^{\vee})\beta^{\vee}$.

We now discuss the effect of adding a gauge basis.

Proposition 5.16. Let $R$ be a local Artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, and let $\tau'$ be a 2-generic principal series tame type. Let $(\mathcal{M}, \iota) \in Y_{\mu,\tau'}^\text{pol}(R)$. Then the set of gauge bases of $(\mathcal{M}, \iota)$ is a torsor for $\widehat{T}_G(\mathcal{O}_{K_2} \otimes_{\mathcal{O}_p} R)^{\sigma^f = \text{inv}}$, where inv denotes the homomorphism $t \mapsto t^{-1}$.

Proof. The proof follows the argument of [LLHLMS Prop. 6.12]. Let $\beta$ be a gauge basis of $\mathcal{M} \in Y_{\mu,\tau'}(R)$. Then $\iota((\sigma^f)^*\beta)$ is a gauge basis of $\mathcal{M}^{\vee}$ and by [LLHLMS Thm. 4.16], the set of gauge bases of $\mathcal{M}^{\vee}$ are uniquely determined up to scaling and are exactly $\widehat{T}_G(\mathcal{O}_{K_2} \otimes_{\mathcal{O}_p} R)^{\beta^{\vee}}$. Thus $\iota((\sigma^f)^*\beta) = c\beta^{\vee}$ for a unique $c \in \widehat{T}_G(\mathcal{O}_{K_2} \otimes_{\mathcal{O}_p} R)$, and the cocycle condition satisfied by $\iota$ is equivalent to $c^{-1}\sigma^f(c) = -1$. Further, given $t \in \widehat{T}_G(\mathcal{O}_{K_2} \otimes_{\mathcal{O}_p} R)$, we have $\iota((\sigma^f)^{*}(t\beta)) = \sigma^f(t)\iota((\sigma^f)^*\beta) = \sigma^f(t)c\beta^{\vee}$. Since the basis on $\mathcal{M}^{\vee}$ dual to $t\beta$ is $t^{-1}\beta^{\vee}$, we conclude that the set of gauge bases of $(\mathcal{M}, \iota)$ is exactly the set of solutions $t \in \widehat{T}_G(\mathcal{O}_{K_2} \otimes_{\mathcal{O}_p} R)$ to the equation $(1_{\mathcal{M}}^{\vee})t^{-1} = \sigma^f(t)c$. The conclusion follows as in [LLHLMS Prop. 6.12]: using that $\text{Res}_{K_2/\mathcal{O}_p}$ splits over $\mathcal{O}$, we have that the equation has a solution, and the solution set is a $\widehat{T}_G(\mathcal{O}_{K_2} \otimes_{\mathcal{O}_p} R)^{\sigma^f = \text{inv}}$ torsor.
5.2.6.

**Lemma 5.17.** Let $R$ be a local Artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, and $\tau'$ a 2-generic principal series tame type which satisfies $(\tau')^{\phi^{-1}} \cong \tau'^{\phi}$. Let $w = (w_i)_i \in W'$ denote the orientation of $\tau'$.

(i) Let $(\mathcal{M}, \iota) \in Y^{\mu, \tau'}(R)$ and let $\beta$ denote a gauge basis for $(\mathcal{M}, \iota)$. Let $A^{(i)}$ be the matrix of partial Frobenius of $\mathcal{M} \in Y^{\mu, \tau'}(R)$ with respect to $\beta$. We then have

\[
A^{(i-f)} = \begin{cases} 
E(u)s(A^{(i)}) - T & \text{if } i \neq f - 1, 2f - 1, \\
E(u)s(A^{(i)}) - T & \text{if } i = f - 1, 2f - 1.
\end{cases}
\]

In particular, if $R = \mathbb{F}, (\mathcal{M}, \iota) \in Y^{\mu, \tau'}(\mathbb{F})$, and $\mathcal{M}$ has shape $\bar{w} = (\bar{w}_i)_i \in \bar{W}'$, then

\[
\bar{w}_{i-f} = \bar{w}_i.
\]

(ii) Conversely if $\mathcal{M} \in Y^{\mu, \tau'}(R)$ and the matrices $A^{(i)}$ of partial Frobenius satisfy the condition $\overline{5.2.2}$ for a gauge basis $\beta$ of $\mathcal{M}$, then there exists a polarization $\iota$ on $\mathcal{M}$ such that $(\mathcal{M}, \iota) \in Y^{\mu, \tau'}(R)$, and such that $\beta$ is a gauge basis for $(\mathcal{M}, \iota)$.

**Proof.** (i) We follow [LLHLM18, §2.1, 6.1]. Let $j : (\sigma^f)^*\mathcal{M} \to \mathcal{M}$ denote the $\sigma^{-f}$-semilinear bijection sending $s \otimes m$ to $\sigma^{-f}(s)m$. We have a commutative diagram of $R[v]$-modules:

\[
\begin{array}{cccccc}
\mathcal{M}^{(i-f)}_{\eta_{w_i-f+1}^{(2)}} & \xrightarrow{\phi^{(i-f)}} & (\sigma^f)^*\mathcal{M}^{(i)}_{\eta_{w_i-f+1}^{(2)}} & \xrightarrow{\phi^{(i)}} & (\sigma^f)^*\mathcal{M}^{(i)}_{\eta_{w_i-f+1}^{(2)}} & \xrightarrow{\phi^{(i)}} & (\sigma^f)^*\mathcal{M}^{(i)}_{\eta_{w_i-f+1}^{(2)}} \\
\mathcal{M}^{(i-f+1)}_{\eta_{w_i-f+1}^{(2)}} & \xrightarrow{j} & ((\sigma^f)^*\mathcal{M})^{(i+1)}_{\eta_{w_i-f+1}^{(2)}} & \xrightarrow{\phi^{(i)}} & (\sigma^f)^*\mathcal{M}^{(i+1)}_{\eta_{w_i-f+1}^{(2)}} & \xrightarrow{\phi^{(i)}} & (\sigma^f)^*\mathcal{M}^{(i+1)}_{\eta_{w_i-f+1}^{(2)}} \\
\end{array}
\]

(here the subscripts denote isotypic components). The left square commutes by [LLHLM18, Lem. 6.2], the center square commutes by Lemma 5.13, and the right square commutes by definition of polarization. By Subsubsection 5.2.2, we see that the matrix of partial Frobenius on $\mathcal{M}^{\psi}$ at embedding $i$ is $E(u)s(A^{(i)})^{-T}s$. Since $\beta$ is a gauge basis which is compatible with the polarization, the above commutative diagram implies that $A^{(i-f)}$ is of the form stated above.

(ii) We may define $\iota : (\sigma^f)^*\mathcal{M} \sim \to \mathcal{M}^\psi$ by the condition

\[
\iota(1 \otimes f^{(i)}_{w_i(k)}) = \begin{cases} 
-f_{w_i-f+1}^{(i-f),\psi} & \text{if } 0 \leq i \leq f - 1, \\
f_{w_i-f+1}^{(i-f),\psi} & \text{if } f \leq i \leq 2f - 1,
\end{cases}
\]

where $k = 1, 2$, and where $f^{(i-f),\psi}_{w_i-f+1} s(k)$ denotes the basis vector of $\mathcal{M}^\psi$ dual to $f^{(i-f)}_{w_i-f+1} s(k)$. The relation [5.2.2] guarantees that $\iota$ is a morphism of Kisin modules. \hfill \square

5.3. **Deformations.** In this subsection we describe deformations of Frobenius twist self-dual Kisin modules and relate them to deformations of local $L$-parameters. The main result is Corollary 5.24 giving a description of the special fiber of the Galois deformation ring in terms of Serre weights.

Throughout the discussion, we fix a tamely ramified $L$-parameter $\overline{\rho} : \Gamma_K \to \mathcal{G}U_2(\mathbb{F})$ such that $\overline{\rho} \circ \overline{\rho} = \overline{\varepsilon}$, and let $\tau' : I_{K^2} \to \mathcal{G}L_2(\mathcal{O})$ be a principal series tame type satisfying $(\tau')^{\phi^{-1}} \cong \tau'^{\phi}$. 

\[
\overline{\rho} \circ \overline{\rho} = \overline{\varepsilon}, \quad \text{and let } \tau' : I_{K^2} \to \mathcal{G}L_2(\mathcal{O}) \text{ be a principal series tame type satisfying } (\tau')^{\phi^{-1}} \cong \tau'^{\phi}.
\]
5.3.1. We begin with Kisin modules. Fix $(\mathfrak{M}, \tau) \in Y^\mu_{\text{pol}}(F)$ and let $\bar{w} = (\bar{w}_i)_i \in \bar{W}$ denote the shape of $\mathfrak{M}$. We also fix a compatible gauge basis $\overline{\beta}$, and assume that $\tau'$ is $2$-generic. Given a local Artinian $\mathcal{O}$-algebra $R$ with residue field $\mathbb{F}$, we let

$$Y_{\mathfrak{M}, \text{pol}}^\mu,\tau'(R) \overset{\text{def}}{=} \left\{ (\mathfrak{M}_R, \iota_R, J_R) : \begin{array}{c} \diamond (\mathfrak{M}_R, \iota_R, J_R) \in Y_{\mathfrak{M}, \text{pol}}^\mu,\tau'(R) \\
\diamond J_R : \mathfrak{M}_R \otimes_R \mathbb{F} \xrightarrow{\sim} \mathfrak{M} \\
\diamond (J_R)^{-1} \circ (\iota_R \otimes_R \mathbb{F}) = \iota \circ (\sigma^f)^* J_R \end{array} \right\}$$

and

$$D^\tau',\overline{\beta}_{\mathfrak{M}, \text{pol}}(R) \overset{\text{def}}{=} \left\{ (\mathfrak{M}_R, \iota_R, J_R, R') : \begin{array}{c} \diamond (\mathfrak{M}_R, \iota_R, J_R) \in Y_{\mathfrak{M}, \text{pol}}^\mu,\tau'(R) \\
\diamond \iota_R \text{ is a gauge basis of } (\mathfrak{M}_R, \iota_R) \text{ lifting } \overline{\beta} \end{array} \right\}.$$ Using [LLHM18] Thms. 4.16, 4.17 along with Lemma 5.17 and Proposition 5.16 we see that $D^\tau',\overline{\beta}_{\mathfrak{M}, \text{pol}}$ is a $\mathbb{G}^2_{\text{aff}}$-torsor, and in particular is representable by a formal Artin stack, since $Y_{\mathfrak{M}, \text{pol}}^\mu,\tau'$ is. As $D^\tau',\overline{\beta}_{\mathfrak{M}, \text{pol}}$ has no nontrivial automorphisms we conclude that $D^\tau',\overline{\beta}_{\mathfrak{M}, \text{pol}}$ is representable by a complete local Noetherian $\mathcal{O}$-algebra $R^\tau',\overline{\beta}_{\mathfrak{M}, \text{pol}}$. The act of deforming a polarized Kisin module $(\mathfrak{M}, \bar{w}, \tau, \overline{\beta})$ with $\mathfrak{M} \in Y_{\mathfrak{M}, \text{pol}}^\mu,\tau'(F)$ and a gauge basis on it is equivalent to deforming the collection of associated matrices $(A(\tilde{\theta}) \big|_{0 \leq i \leq 2f - 1})$ subject to the degree conditions of Table 2 and (5.2.2). We conclude that:

**Theorem 5.18.** Let $\tau'$ be a $2$-generic principal series tame type which satisfies $(\tau')^{\varphi - f} \cong \tau^{\psi'}$, and let $\mathfrak{M}, \bar{w}, \tau, \overline{\beta}$ be as above. Then

$$R^\tau',\overline{\beta}_{\mathfrak{M}, \text{pol}} \cong \bigotimes_{i \in \{0, \ldots, f - 1\}} R^\text{expl}_{\bar{w}_i}$$

where $R^\text{expl}_{\bar{w}_i}$ is as in Table 3, and the completed tensor product is taken over $\mathcal{O}$. In particular $R^\tau',\overline{\beta}_{\mathfrak{M}, \text{pol}}$ is an integral domain.

**Table 3. Deformation rings by shape**

<table>
<thead>
<tr>
<th>$w_i$</th>
<th>$t$</th>
<th>$\tau'$</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{w}_i$</td>
<td>$\mathcal{O}[c_{2,1}, x_{1,1}^<em>, x_{2,2}^</em>]$</td>
<td>$\mathcal{O}[c_{1,2}, x_{1,1}^<em>, x_{2,2}^</em>]$</td>
<td>$\mathcal{O}[x_{1,1}, y_{2,2}, x_{1,2}^<em>, x_{2,2}^</em>]/(x_{1,1}y_{2,2} + p)$</td>
</tr>
</tbody>
</table>

5.3.2. We now discuss deformations of $L$-parameters.

We recall a result from [CHT08] in a language more suited for our purposes. Let $R$ be a topological $\mathbb{Z}_p$-algebra. By Lemma 2.1.1 of op. cit., there is a bijection between

- $L$-parameters $\rho : \Gamma_K \rightarrow C\mathbf{U}_2(R)$; and
- triples $(\rho', \theta, \alpha)$, where $\rho' : \Gamma_K \rightarrow \mathbf{GL}_2(R)$ is a continuous homomorphism, $\theta : \Gamma_K \rightarrow R^\times$ is a continuous character, and $\alpha$ is a compatible polarization, that is, $\alpha : (\rho')^{\varphi - f} \sim \rho'^\psi \otimes \theta$ such that the composite map $\rho' \xrightarrow{v} (\rho')^{\varphi - f} \xrightarrow{\alpha^{\varphi - f}} (\rho' \otimes \theta)^{\varphi - f} \xrightarrow{\text{can}} (\rho' \otimes \theta^{-1})^{\varphi - f} \psi \xrightarrow{(\alpha^{-1}) \psi} \rho'$ is equal to multiplication by $-\theta(\varphi - f)$.

The correspondence is given by sending $\rho : \Gamma_K \rightarrow C\mathbf{U}_2(R)$ to $(BC(\rho), \tilde{\tau} \circ \rho, \alpha)$, where $\rho(\varphi - f) = (A, \theta(\varphi - f)) \times \varphi - f$ and $\alpha(v) = \Phi_2^{-1} A^{-1} v$.

In what follows, we will usually fix $\theta = \varepsilon$ (hence $-\theta(\varphi - f) = -1$), so that $L$-parameters $\rho : \Gamma_K \rightarrow C\mathbf{U}_2(R)$ with $\tilde{\tau} \circ \rho = \varepsilon$ correspond bijectively to pairs $(\rho', \alpha)$ where $\rho' : \Gamma_K \rightarrow \mathbf{GL}_2(R)$
is a continuous homomorphism and $\alpha$ is a compatible polarization. In particular, our fixed $\overline{p}$ is associated to $(\overline{\rho}(\overline{p}), \overline{\alpha})$.

5.3.3. We introduce several deformation problems for Galois representations. Let $R^\varphi_{\overline{p}}$ denote the universal framed deformation ring of $\overline{p}$. By [BC19, §§3.2–3.3], there exists a unique $\mathcal{O}$-flat quotient $R^\varphi_{\overline{p}}$ of $R^\varphi_{\overline{p}}$ with the property that if $B$ is a finite local $E$-algebra, then a morphism $x : R^\varphi_{\overline{p}} \rightarrow B$ factors through $R^\varphi_{\overline{p}}$ if and only if the corresponding $L$-parameter $\rho_x : \Gamma_K \rightarrow ^\varphi \mathbb{U}_2(B)$ is potentially crystalline with $p$-adic Hodge type $(1, 0, 1) \in X_*(\hat{T})$, inertial type $\tau'$ and cyclotomic multiplier $\hat{\tau} \circ \rho_x = \varepsilon$. We recall the terminology used above:

- An $L$-parameter $\Gamma_K \rightarrow ^\varphi \mathbb{U}_2(B)$ is potentially crystalline if and only if it is so after composition with any faithful algebraic representation $^\varphi \mathcal{U}_2 \hookrightarrow \mathbf{GL}_n$.
- Suppose the $L$-parameter $\rho : \Gamma_K \rightarrow ^\varphi \mathbb{U}_2(B)$ has cyclotomic multiplier $\hat{\tau} \circ \rho = \varepsilon$. Then $\rho$ has $p$-adic Hodge type $(1, 0, 1)$ if and only if $\text{BC}(\rho)$ has $p$-adic Hodge type $\mu = \left(\frac{1}{2}\right)$, that is, if $\text{BC}(\rho)$ has Hodge–Tate weights $\{-1, 0\}$.
- An $L$-parameter $\rho : \Gamma_K \rightarrow ^\varphi \mathbb{U}_2(B)$ has inertial type $\tau'$ if $\text{WD}(\rho)|_{I_K} \cong (\tau' \oplus 1_{I_K}) \otimes_E \overline{E}$ (by $\text{WD}(\rho)$ we mean the $\overline{E}$-points of the torus whose construction is contained in [BC19, Section 2.8, Lemma 2.6.6, and Definition 2.1.1]). Assuming $\rho$ has cyclotomic multiplier, this is equivalent to $\text{WD}(\text{BC}(\rho))|_{I_{K_2}} \cong \tau'$.

(In this section, we will always be working with framed deformations with $p$-adic Hodge type $(1, 0, 1)$ and cyclotomic multiplier, so we omit $(1, 0, 1)$, $\varepsilon$ and $\square$ from the notation.) We write $D_{\text{BC}(\overline{p})}^\varphi = \text{Spf} R^\varphi_{\overline{p}}$. Similarly, we let $R^\varphi_{\text{BC}(\overline{p})}$ be the framed potentially crystalline deformation ring parametrizing lifts of $\text{BC}(\overline{p})$ with $p$-adic Hodge type $\mu$ and inertial type $\tau'$. We write $D_{\text{BC}(\overline{p})}^\varphi = \text{Spf} R^\varphi_{\text{BC}(\overline{p})}$.

5.3.4. Let $R$ denote a local Artinian $\mathcal{O}$-algebra with residue field $\overline{F}$. We define

$$D_{\text{BC}(\overline{p}), \text{pol}}^\varphi(R) \overset{\text{def}}{=} \left\{ (\rho'_R, \alpha_R) : \begin{array}{l}
\rho'_R \in D_{\text{BC}(\overline{p})}^\varphi(R) \\
\alpha_R \text{ is a compatible polarization of } \rho'_R \text{ lifting } \overline{\alpha}
\end{array} \right\}$$

We have natural maps

$$D_{\overline{p}}^\varphi \overset{\sim}{\rightarrow} D_{\text{BC}(\overline{p}), \text{pol}}^\varphi \overset{\sim}{\rightarrow} D_{\text{BC}(\overline{p})}^\varphi$$

where the first isomorphism follows from Subsubsection 5.3.2.

5.3.5. Our next task will be to relate deformations of $L$-parameters to deformations of Kisin modules. Before considering further deformation problems we record the following result.

**Lemma 5.19.** Let $\overline{p} : \Gamma_K \rightarrow ^\varphi \mathbb{U}_2(\overline{F})$ be a tamely ramified $L$-parameter satisfying $\hat{\tau} \circ \overline{p} = \overline{\varepsilon}$ and $\tau'$ a 2-generic principal series tame type satisfying $({\tau'})^{\varphi} \cong \tau''$. Then there exists at most one Kisin module $\overline{M} \in Y^{\mu, \tau'}(\overline{F})$ such that $T_{\text{dd}}^*(\overline{M}) \cong \text{BC}(\overline{p})|_{\Gamma_{K_2, \infty}}$. If such an $\overline{M}$ exists, then there is a unique polarization $\overline{\tau}$ on $\overline{M}$ such that $(\overline{M}, \overline{\tau}) \in Y^{\mu, \tau'}(\overline{F})$, and such that $\overline{\tau}$ is compatible with the polarization $\overline{\alpha}$ on $\text{BC}(\overline{p})|_{\Gamma_{K_2, \infty}}$ via $T_{\text{dd}}^*$.\[Proof.\] The first part of the Lemma is [LLHLM18, Thm. 3.2]. Assume that $\overline{M} \in Y^{\mu, \tau'}(\overline{F})$ satisfies $T_{\text{dd}}^*(\overline{M}) \cong \text{BC}(\overline{p})|_{\Gamma_{K_2, \infty}}$, and let $M \overset{\text{def}}{=} \overline{M} \otimes_{\overline{\mathbb{F}}} \mathcal{O}_{\mathbb{F}}$ denote the associated étale $\varphi$-module (where $\mathcal{O}_{\mathbb{F}} \overset{\text{def}}{=} \mathcal{O}_{\mathbb{E}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}$ and $\mathcal{O}_{\mathbb{E}}$ is the $p$-adic completion of $\mathcal{O}_{K_2}[u][1/\overline{u}]$). Since the category of $\Gamma_{K_2, \infty}$-representations is equivalent to the category of étale $\varphi$-modules, and since $\text{BC}(\overline{p})$ is essentially conjugate self dual, we have an isomorphism

$$\iota : (\sigma^f)^* M \overset{\sim}{\rightarrow} M^\vee$$
Lemma 5.21. (since a non-zero morphism \( \tau \) shape of the polarizations on both sides. We define the conclude that the map \( \iota \overset{\text{def}}{=} \iota_{|\sigma} \) factors through an isomorphism \( (\sigma')^* M \overset{\sim}{\longrightarrow} M' \), giving a polarization on \( M' \).

We now claim that if \( \bar{\tau}_1, \bar{\tau}_2 \) are polarizations on \( M' \) which are compatible with the polarization \( \bar{\sigma} \) on \( BC(\bar{\rho})|_{\Gamma_{K,\infty}} \) then \( \bar{\tau}_1 = \bar{\tau}_2 \). Since \( T_{dd}(\bar{\tau}_1) = T_{dd}(\bar{\tau}_2) \) we deduce that \( (\bar{\tau}_1 - \bar{\tau}_2) \otimes_{\mathcal{O}_F} \mathcal{O}_{E,F} = 0 \) and hence \( \text{im}(\bar{\tau}_1 - \bar{\tau}_2) \) is a \( u \)-torsion \( \mathcal{O}_F \)-submodule of \( M' \). Since \( M' \) is a projective \( \mathcal{O}_F \)-submodule we conclude that \( \bar{\tau}_1 - \bar{\tau}_2 = 0 \).

We may now introduce the following definition:

**Definition 5.20.** Let \( \bar{\rho} : \Gamma_K \longrightarrow ^C U_2(\mathbb{F}) \) be a tamely ramified \( L \)-parameter such that \( \hat{\rho} \circ \bar{\rho} = \bar{\tau} \) and let \( \tau' \) be a 2-generic principal series tame type satisfying \( (\tau')^\vee \overset{\sim}{\cong} \tau' \). Assume that there exists \( (M, \bar{\tau}) \in Y_{\text{pol}}(\mathbb{F}) \) together with an isomorphism \( T_{dd}(\bar{M}) \overset{\sim}{\longrightarrow} BC(\bar{\rho})|_{\Gamma_{K,\infty}} \) compatible with the polarizations on both sides. We define the *shape of \( \rho \) with respect to \( \tau' \) to be the shape of \( M' \), and denote it by \( \bar{\omega}(\bar{\rho}, \tau') \).

Whenever we invoke the shape of an \( L \)-parameter with respect to a 2-generic type \( \tau' \) (with \( \rho \) and \( \tau' \) as above), we implicitly assume that there exists a (necessarily unique) polarized Kisin module \( (\bar{M}, \bar{\tau}) \in Y_{\text{pol}}(\mathbb{F}) \) such that \( T_{dd}(\bar{M}) \overset{\sim}{\longrightarrow} BC(\bar{\rho})|_{\Gamma_{K,\infty}} \) compatible with the polarizations on both sides.

5.3.6. In what follows, we fix a polarized Kisin module \( (\bar{M}, \bar{\tau}) \in Y_{\text{pol}}(\mathbb{F}) \) and an isomorphism \( \bar{\delta} : T_{dd}(\bar{M}) \overset{\sim}{\longrightarrow} BC(\bar{\rho})|_{\Gamma_{K,\infty}} \), which is compatible with the polarizations on both sides (with \( \bar{\rho} \) and \( \tau' \) as above). (The existence of such a \( (\bar{M}, \bar{\tau}) \) is a necessary condition for the ring \( R_{\rho}^{\tau'} \) to be nonzero, since a non-zero morphism \( x : R_{\rho}^{\tau'} \longrightarrow \mathcal{O} \) gives rise to an element of \( Y_{\text{pol}}(\mathcal{O}) \) which reduces to \( (\bar{M}, \bar{\tau}) \) modulo \( \bar{\sigma} \), by the analogue of [Kis06, Theorem (0.1)] with coefficients and descent data.)

Let \( R \) denote a local Artinian \( \mathcal{O} \)-algebra with residue field \( \mathbb{F} \). We define

\[
D_{\bar{M}, BC(\bar{\rho})}^{\tau'}(R) \overset{\text{def}}{=} \left\{ \begin{array}{c}
\diamond (M_R, J_R) \in Y_{\text{pol}}^{\mu, \tau'}(R) \\
\diamond \rho_R' \in D_{BC(\bar{\rho})}^{\tau'}(R) \\
\diamond \delta_R : T_{dd}(M_R) \overset{\sim}{\longrightarrow} \rho_R'|_{\Gamma_{K,\infty}} \text{ lifts } \bar{\delta} \\
\end{array} \right\}
\]

\[
D_{\bar{M}, \text{pol}, \mathbb{F}}^{\tau'}(R) \overset{\text{def}}{=} \left\{ \begin{array}{c}
\diamond (M_R, \iota_R, J_R, \rho_R, \delta_R) : \diamond (M_R, J_R) \in Y_{\text{pol}}^{\mu, \tau'}(R) \\
\diamond \rho_R \in D_{\bar{M}, \mathbb{F}}^{\tau'}(R) \\
\diamond \delta_R : T_{dd}(M_R) \overset{\sim}{\longrightarrow} BC(\rho_R)|_{\Gamma_{K,\infty}} \text{ lifts } \bar{\delta} \\
\text{compatibly with the polarizations} \\
\end{array} \right\}
\]

The forgetful functor \( (M_R, \iota_R) \longrightarrow M_R \) along with the base change map \( \rho_R \longrightarrow BC(\rho_R) \) induces a morphism \( D_{\bar{M}, \text{pol}, \mathbb{F}}^{\tau'} \longrightarrow D_{\bar{M}, BC(\bar{\rho})}^{\tau'}(R) \) which is compatible with \( T_{dd} \).

**Lemma 5.21.** Let \( \bar{\rho} \) and \( \tau' \) be as above, so that in particular \( \tau' \) is 2-generic and satisfies \( (\tau')^\vee \overset{\sim}{\cong} \tau' \). Then the natural map \( D_{\bar{M}, \text{pol}, \mathbb{F}}^{\tau'} \longrightarrow D_{\bar{\rho}}^{\tau'} \) is an isomorphism.

**Proof.** Let \( R \) be a local Artinian \( \mathcal{O} \)-algebra with residue field \( \mathbb{F} \), and let \( \rho_R \in D_{\bar{M}, \mathbb{F}}^{\tau'}(R) \). Recall that the data of \( \rho_R \) is equivalent to the data of \( (BC(\rho_R), \alpha_R) \), with \( \alpha_R \) a compatible polarization. By [LLHLM18, Cor. 3.6], the representing rings \( R_{\bar{M}, BC(\bar{\rho})}^{\tau'} \) and \( R_{BC(\bar{\rho})}^{\tau'} \) are isomorphic, and hence there
exists a unique pair \((\mathcal{M}_R, \delta_R)\), where \(\mathcal{M}_R \in \text{Y}^{\rho^\prime}(R)\) and \(\delta_R : T_{\text{dd}}(\mathcal{M}_R) \sim \to \text{BC}(\rho_R)|_{\Gamma_{K_2,\infty}}\) lifts \(\delta\). It remains to construct a unique polarization on \(\mathcal{M}_R\) compatible with \(\alpha_R\). By the equivalence of categories between \(\text{étale} \varphi\)-modules and \(\Gamma_K\text{-}\alpha\)-representations, the polarization \(\alpha_R\) induces a polarization \(\iota_R : (\sigma^f)^*\mathcal{M}_R \sim \to \mathcal{M}_R\), where \(\mathcal{M}_R\) denotes the \(\text{étale} \varphi\)-module associated to \(\mathcal{M}_R\). The uniqueness of \(\mathcal{M}_R\) implies that \(\iota_R\) carries \((\sigma^f)^*\mathcal{M}_R\) to \(\mathcal{M}_R\). Finally, the fact that \(\iota_R\) is unique follows exactly as in the proof of Lemma 5.19.

5.3.7. We now fix a gauge basis \(\beta\) on \((\mathcal{M}, \iota)\). For a local Artinian \(\mathcal{O}\)-algebra \(R\) with residue field \(\mathbb{F}\), we define

\[
D_{\mathcal{M}, \text{pol}, \beta}(R) \overset{\text{def}}{=} \left\{ \begin{array}{l}
(\mathcal{M}_R, \iota_R, J_R, \beta_R, \rho_R, \delta_R) \in D_{\mathcal{M}, \text{pol}, \beta}(R) \\
\beta_R \text{ is a gauge basis for } (\mathcal{M}_R, \iota_R) \text{ lifting } \beta \end{array} \right\}.
\]

We see by Proposition 5.16 that the forgetful map \(D_{\mathcal{M}, \text{pol}, \beta}(R) \to D_{\mathcal{M}, \text{pol}, \beta}(R)\) is a representable formal torus torsor of relative dimension \(2f\). We denote by \(R_{\mathcal{M}, \text{pol}, \beta}(R) \to R_{\mathcal{M}, \text{pol}, \beta}(R)\) the corresponding map of deformation rings. It is a formally smooth morphism of relative dimension \(2f\) between complete local Noetherian \(\mathcal{O}\)-algebras.

Finally, we define the deformation problem

\[
D_{\mathcal{M}, \text{pol}, \beta}(R) \overset{\text{def}}{=} \left\{ (\mathcal{M}_R, \iota_R, J_R, \beta_R, \rho_R, e_R) : \begin{array}{l}
(\mathcal{M}_R, \iota_R, J_R, \beta_R) \in D_{\mathcal{M}, \text{pol}, \beta}(R) \\
\rho_R \text{ is an isomorphism, } e_R \text{ is a basis for } T_{\text{dd}}(\mathcal{M}_R) \\
\text{lifting the (pullback via } \delta \text{ of the)} \\
\text{standard basis on } \text{BC}(\mathcal{P})|_{\Gamma_{K_2,\infty}}
\end{array} \right\}
\]

In particular, if \((\mathcal{M}_R, \iota_R, J_R, \beta_R, e_R) \in D_{\mathcal{M}, \text{pol}, \beta}(R)\), then \((T_{\text{dd}}(\mathcal{M}_R), e_R)\) is a framed deformation of \(\text{BC}(\mathcal{P})|_{\Gamma_{K_2,\infty}}\). We let \(R_{\mathcal{M}, \text{pol}, \beta}(R)\) denote the deformation ring corresponding to the above deformation problem.

5.3.8. The relationships between the various deformation problems are summarized in the following diagram, where “f.s.” stands for formally smooth.

\[
\begin{array}{ccc}
\text{Spf } R_{\mathcal{P}} \ & \sim & \text{Spf } R_{\mathcal{M}, \text{pol}, \beta}(R) \\
\text{f.s.} & & \text{Spf } R_{\mathcal{M}, \text{pol}, \beta}(R) & \sim & \text{Spf } R_{\mathcal{M}, \text{pol}, \beta}(R) \to \text{Spf } R_{\mathcal{M}, \text{pol}, \beta}(R)
\end{array}
\]

The maps which are formally smooth correspond to forgetting either a gauge basis on the (polarized) Kisin module or a framing on the Galois representation. The former is formally smooth of relative dimension \(2f\) while the latter is formally smooth of relative dimension \(4\). The isomorphism follows from Lemma 5.21.

Our next goal will be to show that the remaining map \(\text{Spf } R_{\mathcal{M}, \text{pol}, \beta}(R) \to \text{Spf } R_{\mathcal{M}, \text{pol}, \beta}(R)\) is an isomorphism. This will follow from some calculations with Galois cohomology.

5.3.9. Given the tamely ramified \(L\)-parameter \(\mathcal{P} : \Gamma_K \to \text{U}_2(\mathbb{F})\) with \(\iota \circ \mathcal{P} = \mathcal{E}\), we set \(\text{ad}^0(\mathcal{P}) \overset{\text{def}}{=} \mathfrak{gl}_2(\mathbb{F})\). It is a direct summand of the Lie algebra of \(\text{U}_2\) endowed with the adjoint action of \(\Gamma_K\) via \(\mathcal{P}\). Explicitly the action of \(\Gamma_K\) on the direct summand \(\text{ad}^0(\mathcal{P})\) is given as follows: \(\Gamma_K\) acts by the adjoint action (via \(\text{BC}((\mathcal{P}))\)), and \(\mathcal{P}(\varphi^{-f}) = (A, 1) \times \varphi^{-f}\) acts by

\[
X \mapsto -A \Phi_2 X^\top \Phi_2^{-1} A^{-1}.
\]

Lemma 5.22. Suppose \(\mathcal{P}\) is \(1\)-generic. Then the restriction map on cocycles

\[
Z^1(\Gamma_K, \text{ad}^0(\mathcal{P})) \to Z^1(\Gamma_{K_\infty}, \text{ad}^0(\mathcal{P}))
\]

is injective.
Proposition 5.23. Suppose $\overline{p}$ is $1$-generic. Then the natural map $\text{Spf } R_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}, \square} \rightarrow \text{Spf } R_{\mathfrak{m}, \text{pol}}^{\rho, \tau}$ is an isomorphism.

Proof. By considering tangent spaces and using the above lemma, the map in question is a closed immersion (compare [LLHL18, Prop. 5.11]). Therefore it suffices to prove it is surjective on $R$-points. This is obtained following the argument of the proof of [LLHL18 Theorem 5.12], noting that in our situation, the monodromy condition in op. cit. is empty and the $p$-adic Hodge type is $(1,0)$ in all embeddings. (Alternatively, one can invoke [CDM18 Thm. 2.1.12]: the cited theorem implies that if $(\mathfrak{M}_R, \iota_R, J_R, \beta_R, e_R) \in D_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}}(R)$, then we may extend the framed deformation $(T_{\text{fr}}(\mathfrak{M}_R), e_R)$ of $BC(\overline{p})$ to a framed deformation of $BC(\overline{p})$; the claim about functoriality in op. cit. implies that the polarization $T_{\text{fr}}(\iota_R)$ also extends.)

5.3.10. By Theorem 5.18, Proposition 5.23 and (5.3.1), we finally conclude that:

(5.3.2) $R_{\overline{p}}^{\overline{\rho}}[S_1, \ldots, S_2f] \cong R_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}} \cong R_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}}[T_1, \ldots, T_4] \cong \left( \bigotimes_{i \in \{0, \ldots, f-1\}} R_{\overline{w}_i}^{\text{pol}} \right) [T_1, \ldots, T_4]$

where $\overline{w} = (\overline{w}_i)_{\overline{i}} = \overline{w}(\overline{\rho}, \overline{\tau}) \in \overline{W}'$ is the shape of $\overline{p}$ with respect to $\overline{\tau}'$.

5.3.11. The following corollary is the main result on the local Galois side.

Corollary 5.24. Let $\overline{p} : \Gamma_K \rightarrow \text{GU}_2(\mathbb{F})$ be a $3$-generic tamely ramified $L$-parameter which satisfies $\hat{T} \circ \overline{p} = \overline{\tau}$. Let $\tau'$ denote a $3$-generic principal series tame type which satisfies $(\tau')^{\overline{\tau} - f} \cong \tau'^f$, and let $\sigma(\tau')$ denote the tame type associated to $\tau'$ via Theorem 4.11. We view $\sigma(\tau')$ as a Deligne–Lusztig representation of $G(\mathbb{F}_p)$ on which $i(\mathbb{F}_p) \times$ acts trivially. Assume that there exists $(\mathfrak{M}, \overline{\tau}) \in Y_{\text{pol}}^{\mu, \tau'}(\mathbb{F})$ together with an isomorphism $T_{\text{fr}}(\mathfrak{M}) \cong BC(\overline{p})$ compatible with the polarizations on both sides.

We then have

$$|W'(\overline{p}) \cap \text{JH}((\sigma(\tau')))| = e(R_{\overline{p}}^{\overline{\rho}} \otimes_{\mathfrak{m}} \mathbb{F}),$$

where $e(-)$ denotes the Hilbert–Samuel multiplicity.

Proof. Let $(\mathfrak{M}, \overline{\tau}) \in Y_{\text{pol}}^{\mu, \tau'}(\mathbb{F})$ correspond to $\overline{p}$, let $\overline{\rho}$ denote a gauge basis for $(\mathfrak{M}, \overline{\tau})$, and let $\overline{w} = (\overline{w}_i)_{\overline{i}} = \overline{w}(\overline{\rho}, \overline{\tau}') \in \overline{W}'$ denote the shape of $\overline{p}$ with respect to $\overline{\tau}'$. The isomorphism (5.3.2) above implies that

$$e(R_{\overline{p}}^{\overline{\rho}} \otimes_{\mathfrak{m}} \mathbb{F}) = e(R_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}} \otimes_{\mathfrak{m}} \mathbb{F}) = 2^{\lfloor 0 \leq i \leq f-1; \overline{w}_i = w \rfloor},$$

where the last equality follows from Table 3.

By the $\text{GL}_2$-analog of the discussion in [LLHL18 §5.2], we see that $R_{BC(\overline{p})}^{\overline{\rho}}$ is a formally smooth modification of $R_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}}$, where the latter ring represents the functor sending a local Artinian $\mathcal{O}$-algebra $R$ with residue field $\mathbb{F}$ to the set of triples $(\mathfrak{M}_R, J_R, \beta_R)$, where $(\mathfrak{M}_R, J_R) \in Y_{\mathfrak{m}}^{J_R}(\mathbb{F})$ and $\beta_R$ is a gauge basis of $(\mathfrak{M}_R, J_R)$ lifting $\overline{\tau}$. Further, the structure of $R_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}}$ is obtained by removing the restriction "$i \in \{0, \ldots, f-1\}$" in the right-hand side of Theorem 5.18 (this is the $\text{GL}_2$-analog of [LLHL18 Thm. 4.17]). Thus, Lemma 5.17(i) implies

$$e(R_{BC(\overline{p})}^{\overline{\rho}} \otimes_{\mathfrak{m}} \mathbb{F}) = e(R_{\mathfrak{m}, \text{pol}}^{\overline{\rho}, \overline{\tau}} \otimes_{\mathfrak{m}} \mathbb{F}).$$
irrelevant for the construction and the basic properties of the group 
compact unitary group 
$U^{\kappa}$.

Finally, for an embedding $\iota$ which satisfies $\iota(\overline{w}) = w$, where $\overline{w}$ is an element in the dual group subgroup of $\mathbb{F}$

which is a totally definite unitary group, quasi-split at all finite places. Explicitly, we have

where $U^{\iota}$ such that $U^{\iota}$ is the quasi-split unitary group over $\mathbb{F}$

This implies that $\mathbb{F}$ is a totally definite unitary group, quasi-split at all finite places.

Hence, it is enough to prove that

$$|W'(BC(\mathfrak{p})) \cap JH (BC(\sigma(\tau')))| = e(R'_{\mathcal{O}} \otimes \mathbb{F})^2.$$ 

This follows from Propositions 3.18 (applied to $\beta(V_{\mathfrak{p}}(\mathfrak{p}))$ and $\sigma(\tau')$), 4.6 and 4.8 $\square$

6. Global applications I

In this section we apply the results of Sections 3 and 4 in a global context. Our main references will be [CHT08] and [CEG+16]; as such, we will be considering Galois representations valued in the group $G_2$. (We will translate these results back to the group $C\text{U}_2$ at the end of Section 7.) After preliminaries on automorphic forms on unitary groups and their associated Galois representations (Theorem 6.2), we give the main result on weight elimination in Theorem 6.7 building on the compatibility of base change of tame types and $L$-parameters.

We caution the reader that some of the notation below differs from previous sections.

6.1. Unitary groups.

6.1.1. Let $F$ be an imaginary CM field with maximal totally real subfield $F^+$. We suppose:

- $F^+/\mathbb{Q}$ is unramified at $p$;
- $F/F^+$ is unramified at all finite places; and
- $\sigma$ every place of $F^+$ above $p$ is inert in $F$.

This implies that $\mathbb{F}$ is even (cf. [GK14, §3.1]), and there exists a reductive group $G/\mathcal{O}_{F^+}$, which is a totally definite unitary group, quasi-split at all finite places. Explicitly, we have

$$G(R) = \{ g \in GL_2(\mathcal{O}_F \otimes \mathcal{O}_{F^+}, R) : g(c \cdot 1)^\top g = 1 \},$$

where $R$ is an $\mathcal{O}_{F^+}$-algebra, and where we write $c \in \text{Gal}(F/F^+)$ for the complex conjugation.

Note that this group is different from the group $G$ from Subsection 2.2.1.

The group $G$ is equipped with an isomorphism $\iota : G \times \mathcal{O}_{F^+} \mathcal{O}_F \sim \mathcal{O}_F \otimes \mathcal{O}_F$, which satisfies $\iota \circ (1 \otimes c) \circ \iota^{-1}(g) = g^{-c\top}$. For all places $v$ of $F^+$ which split in $F$ as $v = wv'$, we obtain an induced isomorphism $\iota_v : G(\mathcal{O}_{F^+}) \sim \mathcal{O}_{F_v} \otimes \mathcal{O}_{F_v}$ such that $\iota_v \circ \iota_v^{-1}(g) = g^{-c_v\top}$. If $v$ is a place of $F^+$ which is inert in $F$, then we have an isomorphism $\iota_v : G(\mathcal{O}_{F^+}) \sim \mathcal{O}_{F_v} \otimes \mathcal{O}_{F_v}$, where $\mathcal{U}_2$ is the quasi-split unitary group over $\mathcal{O}_{F^+}$ defined Subsection 2.1. This isomorphism is given by $g \mapsto \left( \begin{smallmatrix} b & \overline{b} \\ \overline{b} & b \end{smallmatrix} \right)$ $\left( \begin{smallmatrix} 1 & b \\ \overline{b} & 1 \end{smallmatrix} \right)^{-1}$, where $b \in \mathcal{O}_F^\times$ is an element which satisfies $bb' = -1$ and $b \not\in \mathcal{O}_F^\times$. Finally, for an embedding $\kappa^+ : F^+ \hookrightarrow \mathbb{R}$, the group $G(F^+_{\kappa})$ is compact, and isomorphic to the compact unitary group $U_2(\mathbb{R})$.

(We note that the running hypothesis in [GK14] that $v$ splits in $F$ for $v$ a place of $F^+$ above $p$ is irrelevant for the construction and the basic properties of the group $G$.)
6.1.2. Set $F_p^+ \defeq F^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{O}_{F^+,p} \defeq \mathcal{O}_{F \otimes \mathbb{Z}} \mathbb{Z}_p$. Recall that $E$ is our coefficient field, with ring of integers $\mathfrak{o}$, uniformizer $\varpi$, and residue field $\mathbb{F}$. We assume $E$ is sufficiently large; in particular, we will assume that the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}$ is contained in $E$.

We write $\Sigma^+_p$ (resp. $\Sigma_p$) for the set of places of $F^+$ (resp. $F$) lying above $p$. Restriction from $F$ to $F^+$ gives a bijection between $\Sigma_p$ and $\Sigma^+_p$, and we will often identify these two sets. Similarly, we let $I^+_p$ (resp. $I_p$) denote the set of embeddings $\kappa^+ : F^+ \hookrightarrow E$ (resp. $\kappa : F \hookrightarrow E$). We fix a subset $\tilde{I}_p \subseteq I_p$ such that $I_p = \tilde{I}_p \cup \tilde{I}_p^c$. Then restriction from $F$ to $F^+$ gives a bijection between $\tilde{I}_p$ and $I^+_p$. Further, composing $\kappa^+ \in I^+_p$ (resp. $\kappa \in \tilde{I}_p$) with the valuation on $E$ gives an element of $\Sigma^+_p$ (resp. $\Sigma_p$), and we let $v(\kappa^+)$ (resp. $v(\kappa)$) denote the place induced from the embedding $\kappa^+ \in I^+_p$ (resp. $\kappa \in \tilde{I}_p$). This gives the following diagram:

$$
\begin{array}{ccc}
I_p & \hookrightarrow & \tilde{I}_p \\
\downarrow \text{res} & & \downarrow \text{res} \\
I^+_p & \longrightarrow & \Sigma^+_p \ni v(\kappa^+)
\end{array}
$$

(6.1.1)

For a finite place $v$ of $F^+$ (resp. $F$), we let $\mathbb{F}_v^+$ (resp. $\mathbb{F}_v$) denote the residue field of $v$. We have $G(\mathbb{F}_v^+) \cong U_2(\mathbb{F}_v^+)$ for all $v \in \Sigma^+_p$ by construction.

6.2. Algebraic automorphic forms on unitary groups.

6.2.1. Let $K = \prod_v K_v$ be a compact open subgroup of $G(\mathbb{A}_{F^+}^\infty)$. We set

$$
K_p \defeq \prod_v K_v, \quad K^p \defeq \prod_{v \notin \Sigma^+_p} K_v,
$$

and if $k \in K$, we write $k_p$ for the projection of $k$ to $K_p$. We say that the level $K$ is sufficiently small if for all $t \in G(\mathbb{A}_{F^+}^\infty)$, the finite group $t^{-1}G(F^+) \cap K$ does not contain an element of order $p$.

6.2.2. Let $K = \prod_v K_v \subseteq G(\mathbb{A}_{F^+}^\infty) \times G(\mathcal{O}_{F^+,p})$ be a compact open subgroup, and suppose $W$ is an $\mathcal{O}$-module endowed with an action of $G(\mathcal{O}_{F^+,p})$. The space of algebraic automorphic forms on $G(\mathbb{A}_{F^+}^\infty)$ of level $K$ and coefficients in $W$ is defined as the $\mathcal{O}$-module

$$
S_G(K,W) \defeq \{ f : G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) \rightarrow W : f(gk) = k_p^{-1}f(g) \forall g \in G(\mathbb{A}_{F^+}^\infty), k \in K \}.
$$

Given a compact open subgroup $K$ as above, we have

$$
G(\mathbb{A}_{F^+}^\infty) = \bigsqcup_i G(F^+)t_i K
$$

for some finite set $\{t_i\}_i$. This induces an isomorphism of $\mathcal{O}$-modules

$$
S_G(K,W) \overset{\sim}{\rightarrow} \bigoplus_i W^{K \cap t_i^{-1}G(F^+)t_i}
$$

$$
f \rightarrow (f(t_i))_i.
$$

In particular we have inclusions $S_G(K,W) \subseteq S_G(K',W)$ for $K' \subseteq K$. If we assume that $K$ is sufficiently small or $A$ is a flat $\mathcal{O}$-algebra, we further have

$$
(6.2.1) \quad S_G(K,W) \otimes_{\mathcal{O}} A \cong S_G(K,W \otimes_{\mathcal{O}} A).
$$
6.2.3. Suppose that \( J = \prod_v J_v \subseteq G(\mathbb{A}_{F,+}^{\infty}) \times G(\mathcal{O}_{F,+}) \) is a compact subgroup. We define
\[
S_G(J,W) \overset{\text{def}}{=} \lim_{K \supseteq J} S_G(K,W),
\]
where \( K \) runs over compact open subgroups containing \( J \), for which \( K_p \subseteq G(\mathcal{O}_{F,+}) \). If \( g \in G(\mathbb{A}_{F,+}^{\infty}) \) is such that \( g_p \in G(\mathcal{O}_{F,+}) \) then \((g.f)(h) = g_p.f(h)\) defines an element \( g.f \) of \( S_G(gJg^{-1},W) \). Hence, we obtain an action of \( g \) on \( S_G(J,W) \) as soon the relation \( J \subseteq gJg^{-1} \) is satisfied. In particular, if \( J = \prod_v J_v \subseteq G(\mathbb{A}_{F,+}^{\infty}) \times G(\mathcal{O}_{F,+}) \) is any compact subgroup, then \( J \) acts on \( S_G(\{1\},W) \), and we have
\[
(6.2.2) \quad S_G(\{1\}, W)^J = S_G(J,W).
\]

6.2.4. Recall the map \( I_p \to \Sigma_p \) defined by \( \kappa \mapsto v(\kappa) \). This gives a bijection \( I_p \overset{\sim}{\to} \bigsqcup_{v \in \Sigma_p} \text{Hom}(F_v,E) \) and we identify embeddings \( F_v \hookrightarrow E \) with elements in \( I_p \) without further comment. Let \( v \in \Sigma_p \).

We define \( \overline{\text{Hom}}(F_v,E) \subseteq \text{Hom}(F_v,E) \) by the condition
\[
\overline{I}_p \overset{\sim}{\to} \bigsqcup_{v \in \Sigma_p} \overline{\text{Hom}}(F_v,E)
\]
where the map is given as restriction of the map \( I_p \overset{\sim}{\to} \bigsqcup_{v \in \Sigma_p} \text{Hom}(F_v,E) \). Note that \( \kappa \mapsto \kappa \circ c \) defines a non-trivial involution on \( \text{Hom}(F_v,E) \) and hence \( |\text{Hom}(F_v,E)| = \frac{1}{2} |\text{Hom}(F_v,E)| \).

6.2.5. Let \( \mathbb{Z}_+^2 \) denote the set of all pairs of integers \((\lambda_1,\lambda_2)\) such that \( \lambda_1 \geq \lambda_2 \). (Thus, for \( v \in \Sigma_p^+ \), we may identify \((\mathbb{Z}_+^2)_{\text{Hom}(F_v^+,E)}\) with \( X_+(\text{Res}_{\mathcal{O}_{F_v^+}}/\mathbb{Z}_p(\mathbf{T}_U)) \), where \( \mathbf{T}_U \) denotes the torus of the group \( U_2 \) defined in Subsubsection 2.1.4 with \( K = F_v^+ \). Note that the discussion in Subsection 3.1 works equally well for the group \( U_2 \) and its restriction of scalars.) Given \( \lambda_v = (\lambda_\kappa)_\kappa \in (\mathbb{Z}_+^2)_{\text{Hom}(F_v,E)} \), we let \( W_{\lambda_v} \) denote the free \( \mathcal{O} \)-module
\[
W_{\lambda_v} \overset{\text{def}}{=} \bigotimes_{\kappa \in \text{Hom}(F_v,E)} \text{det}^{\lambda_\kappa,1} \otimes_{\mathcal{O}_{F_v}} \text{Sym}^{\lambda_\kappa,2}(\mathcal{O}_{F_v}^2) \otimes_{\mathcal{O}_{F_v^+,E}} \mathcal{O},
\]
which, by restriction and using the isomorphism \( t_v \), has an action of \( G(\mathcal{O}_{F_v^+}) \). Given an element \( \lambda = (\lambda_\kappa)_\kappa \in (\mathbb{Z}_+^2)_{\overline{I}_p} = \bigoplus_{v \in \Sigma_p^+} (\mathbb{Z}_+^2)_{\text{Hom}(F_v,E)} \), we set
\[
W_\lambda \overset{\text{def}}{=} \bigotimes_{v \in \Sigma_p^+} W_{\lambda_v},
\]
which is a free \( \mathcal{O} \)-module with an action of \( \prod_{v \in \Sigma_p^+} G(\mathcal{O}_{F_v^+}) = G(\mathcal{O}_{F,+}) \).

Since \( F_v^+ \) is unramified over \( \mathbb{Q}_p \), for every \( v \in \Sigma_p^+ \), restriction and reduction mod \( p \) give bijections
\[
\overline{\text{Hom}}(F_v,E) \overset{\sim}{\to} \text{Hom}(F_v^+,E) \overset{\sim}{\to} \text{Hom}(F_v^+,\mathbb{F}_p).
\]
For an element \( \lambda = (\lambda_v)_{v \in \Sigma_p} \in (\mathbb{Z}_+^2)_{\overline{I}_p} = \bigoplus_{v \in \Sigma_p^+} (\mathbb{Z}_+^2)_{\text{Hom}(F_v,E)} \), we let \( \overline{\lambda}_v = (\overline{\lambda}_v)_{v \in \Sigma_p} \) denote its image in \( \bigoplus_{v \in \Sigma_p^+} (\mathbb{Z}_+^2)_{\text{Hom}(F_v^+,E)} \). Let \( \mathbb{Z}_+^{2,p} \) denote the subset of \( \mathbb{Z}_+^2 \) consisting of elements \((\lambda_1,\lambda_2)\) satisfying \( \lambda_1 - \lambda_2 \leq p - 1 \). Then the image of \((\mathbb{Z}_+^{2,p})_{\overline{I}_p} \) in \( \bigoplus_{v \in \Sigma_p^+} (\mathbb{Z}_+^2)_{\text{Hom}(F_v^+,E)} \) gives rise to the irreducible mod \( p \) representations of \( G(\mathcal{O}_{F,+}) \), in a manner similar to Proposition 3.1. More precisely, under the identification of \((\mathbb{Z}_+^2)_{\text{Hom}(F_v^+,E)} \) with \( X_+(\text{Res}_{\mathcal{O}_{F_v^+}}/\mathbb{Z}_p(\mathbf{T}_U)) \), the set \((\mathbb{Z}_+^{2,p})_{\text{Hom}(F_v^+,E)} \) is identified
with $X_1(\text{Res}_{O_{F^+}/O}(T_U))$. In particular, if $\lambda = (\lambda_v)_{v \in \Sigma_p} \in (\mathbb{Z}_{+}^2)^{\hat{\tau}_p} = \bigoplus_{v \in \Sigma_p} (\mathbb{Z}_+^2)^{\text{Hom}(F_v,E)}$, we have

$$W_{\lambda} \otimes_O F \cong \bigotimes_{v \in \Sigma_p} F(\bar{\lambda}_v)$$

as mod $p$ representations of $G(O_{F^+})$.

6.2.6. We now relate the spaces $S_G(K,W)$ to spaces of classical automorphic forms.

We let $\mathcal{A}$ denote the space of automorphic forms on $G(\mathbb{A}_{F^+})$ (see, e.g., [GS11 §§1.5 - 1.8]). Since $G$ is totally definite, $\mathcal{A}$ decomposes as a $G(\mathbb{A}_{F^+})$-representation as

$$(6.2.3) \quad \mathcal{A} \cong \bigoplus_{\pi} m(\pi)\pi$$

defined by $\iota_*: (\mathbb{Z}_+^2)^{\hat{\tau}_p} \xrightarrow{\sim} (\mathbb{Z}_+^2)^{\text{Hom}(F^+,\mathbb{R})}$

where $\pi$ runs through the isomorphism classes of irreducible admissible representations of $G(\mathbb{A}_{F^+})$ and $m(\pi)$ is the (finite) multiplicity of $\pi$ in $\mathcal{A}$ ([Gue11 §2.2], [BC09 §6.2.3]).

Fix an isomorphism $\iota: F \xrightarrow{\sim} \mathbb{C}$. This gives an identification

$$\iota_*: (\mathbb{Z}_+^2)^{\hat{\tau}_p} \xrightarrow{\sim} (\mathbb{Z}_+^2)^{\text{Hom}(F^+,\mathbb{R})}$$

where $\iota_*: (\mathbb{Z}_+^2)^{\hat{\tau}_p} \xrightarrow{\sim} (\mathbb{Z}_+^2)^{\text{Hom}(F^+,\mathbb{R})}$.

The set $(\mathbb{Z}_+^2)^{\text{Hom}(F^+,\mathbb{R})}$ parametrizes irreducible complex representations of $G(F_\infty^+)$; given $\mu \in (\mathbb{Z}_+^2)^{\text{Hom}(F^+,\mathbb{R})}$, we let $W_\mu$ denote the associated irreducible complex $G(F_\infty^+)$-representation.

For $\lambda \in (\mathbb{Z}_+^2)^{\hat{\tau}_p}$, the space $W_\lambda \otimes_{O,1} \mathbb{C}$ is a complex representation of $G(F_p^+)$. We let

$$\theta: W_\lambda \otimes_{O,1} \mathbb{C} \xrightarrow{\sim} W_{\iota_\lambda}$$

denote a $G(F^+)$-equivariant isomorphism.

6.2.7. From now onwards we let $\sigma^0 = \bigotimes_{v \in \Sigma_p} \sigma^0_v$ denote a smooth $G(O_{F^+,p})$-representation on a finite free $O$-module such that $\sigma^0 \otimes_O E$ is a tame $G(O_{F^+,p})$-type. (By abuse of language, we say that $\sigma^0$ is a tame $G(O_{F^+,p})$-type over $\mathcal{O}$.)

Fix $\lambda \in (\mathbb{Z}_+^2)^{\hat{\tau}_p}$. By letting $G(F^+)$ act trivially on the second tensor factor of $W_{\iota_\lambda} \otimes_O (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee$ we define an isomorphism

$$(6.2.4) \quad S_G(\{1\}, (W_\lambda \otimes_{O,1} \mathbb{C}) \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})) \xrightarrow{\sim} \text{Hom}_{G(F^+)}(W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee, \mathcal{A})$$

as follows. Let $f: G(F^+) \backslash G(\mathbb{A}_{F^+}) \rightarrow (W_\lambda \otimes_{O,1} \mathbb{C}) \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})$ be an element of the left hand side. We send this element to a homomorphism $\tilde{f}: W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee \rightarrow \mathcal{A}$ defined by

$$\tilde{f}(w^\vee)(g) = w^\vee ((\xi_\infty(g^{-1}) \otimes 1) \circ (\theta \otimes 1) \circ (\xi_p(g_p) \otimes 1).f(g^\infty)) \ ,$$

where $g \in G(\mathbb{A}_{F^+})$, $w^\vee \in (W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C}))^\vee \cong W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee$, $\xi_p$ denotes the action of $G(F_p^+)$ on $W_{\iota_\lambda} \otimes \mathbb{C}$, and $\xi_\infty$ denotes the action of $G(F_\infty^+)$ on $W_{\iota_\lambda}$. One easily checks that this isomorphism is well defined and $G(\mathbb{A}_{F^+}) \times G(O_{F^+,p})$-equivariant. Therefore if $J = \prod_v J_v \subseteq G(\mathbb{A}_{F^+}) \times G(O_{F^+,p})$ is a compact subgroup we have

$$S_G(J, W_\lambda \otimes_O \sigma^0) \otimes_{O,1} \mathbb{C} \cong S_G(J, (W_\lambda \otimes_{O,1} \mathbb{C}) \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})) \cong S_G(\{1\}, (W_\lambda \otimes_{O,1} \mathbb{C}) \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C}))^J$$

$$(6.2.4) \quad \cong \text{Hom}_{G(F^+)}(W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee, \mathcal{A})^J$$

$$(6.2.3) \quad \cong \text{Hom}_{G(F^+)}(W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee, \mathcal{A})^J$$

$$(6.2.2) \quad \cong \text{Hom}_{G(F^+)}(W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee, \mathcal{A})$$

$$(6.2.1) \quad \cong \text{Hom}_{G(F^+)}(W_{\iota_\lambda} \otimes_C (\sigma^0 \otimes_{O,1} \mathbb{C})^\vee, \mathcal{A})$$
which factors as in Definition 4.10 and satisfies

\[(\text{Fix } \tau_i)\text{ for } \lambda = (\lambda\kappa)\text{ for which}
\]

\[\lambda_{\kappa,i} = -\lambda_{\kappa,3-i}\]

for \(i = 1, 2\). Note that the restriction map induces a bijection

\[(\mathbb{Z}_+^2)^I \cong (\mathbb{Z}_+^2)^I.\]

We use the following notation in the theorem below. Throughout, we fix an isomorphism \(\iota : \overline{E} \xrightarrow{\sim} \mathbb{C}\), and recall that \(\text{rec}_\overline{E}\) denotes the Local Langlands correspondence over \(\overline{E}\). We define \(|\det|^{-1/2}\) to be the \(\overline{E}\)-valued character whose composition with \(\iota\) is the square root of \(|\det|^{-1}\) which takes positive values.

**Theorem 6.1.** Fix \(\lambda \in (\mathbb{Z}_+^2)^I\), and for every \(v \in \Sigma_p^+\), let \(\tau_v\) denote a tame inertial type of \(I_{F_v}\), which factors as in Definition 4.10 and satisfies \((\tau_v)\iota^{-1|\mathfrak{p}^+}_v \equiv \tau_v^\vee\). Let \(\xi \equiv \bigotimes_{v \in \Sigma_p^+} \sigma(\tau_v)\) and let \(\Xi\) be an irreducible \(G(A_F^\infty)^G(\overline{A}_p^\infty)\)-subrepresentation of \(S_G(\mathcal{O}_{F^+, p}, (W_{\lambda} \otimes \sigma^\circ) \otimes \sigma^\vee)\otimes_E \overline{E}\). Then there exists a cuspidal automorphic representation \(\pi\) of \(G(A^\infty_F)\) such that \(\pi_v \equiv \Xi_v \otimes_{\overline{E}, \mathbb{C}} C\) for all finite places \(v \notin \Sigma_p^+\), \(\pi_{\infty} \cong W_{\lambda, \mathbb{C}}^\vee\), and \(\pi_\mathfrak{p}|_{G(\mathcal{O}_{F^+, p})}\) contains \(\sigma \otimes_{E, \mathbb{C}} C\). Furthermore, there exists a unique continuous semisimple representation

\(r_\iota(\pi) : I_F \rightarrow \text{GL}_2(E)\)

satisfying the following properties:

(i) We have an isomorphism

\[r_\iota(\pi)^C \cong r_\iota(\pi)^\vee \otimes \varepsilon^{-1}.\]

(ii) If \(v\) is a finite place of \(F^+\) which splits as \(v = vv^c\) in \(F\), then

\[\text{WD}(r_\iota(\pi)|_{I_{F_v}})^{\text{F-ss}} \cong \text{rec}_\overline{E}(\Xi_v \otimes \varepsilon_{v^c}^{-1}) \otimes |\det|^{-1/2}.\]

(iii) If \(v \notin \Sigma_p^+\) is a finite place of \(F^+\) which is inert in \(F\), then

\[\text{WD}(r_\iota(\pi)|_{I_{F_v}})^{\text{F-ss}} \cong \text{rec}_\overline{E}(\text{BC}_{F_v/F^+_v}(\Xi_v) \otimes |\det|^{-1/2}),\]

where \(\text{BC}_{F_v/F^+_v}\) denotes the stable local base change.

(iv) Let \(v \in \Sigma_p^+\). Then \(r_\iota(\pi)\) is potentially crystalline at \(v\) (viewed as a place of \(F\)), and we have

\[\text{WD}(r_\iota(\pi)|_{I_{F_v}})|_{I_{F_v}} \cong \tau'_v.\]

If \(\kappa \in I_p\) satisfies \(v(\kappa) = v\), then

\[\text{HT}_\kappa(r_\iota(\pi)|_{I_{F_v}}) = \{\lambda_{\kappa, 1} + 1, \lambda_{\kappa, 2}\}\]

(where we view \(\lambda\) as an element of \((\mathbb{Z}_+^2)^I\) via the bijection preceding the theorem). In particular, \(r_\iota(\pi)|_{I_{F_v}}\) is Hodge–Tate regular.
11.5.1] instead of [Lab11, Cor. 5.3] in order to control what happens above p.

Let \( G^* \) denote the quasi-split unitary group in two variables over \( F^+ \), defined as in [Rog90 §1.9]. Theorem 1.7.1 of [KMSW] implies that there exists a Jacquet–Langlands transfer from \( L \)-packets on \( G(\mathbb{A}_{F^+}) \) to \( L \)-packets on \( G^*(\mathbb{A}_{F^+}) \), which induces isomorphisms at all finite places of the constituents of the \( L \)-packets. (Alternatively, we may appeal to either of the following methods to prove this result: (1) noting that \( G^\text{der} \cong \text{SL}_1(D) \) and \( G^*\text{,der} \cong \text{SL}_2 \) (where \( D \) denotes the quaternion algebra over \( F^+ \) which is ramified exactly at the infinite places of \( F^+ \)), we proceed in a similar fashion as [LL79] §7, p. 781; (2) we may embed \( G \) and \( G^* \) into their respective similitude groups, which are isomorphic to \( \text{GL}_1(D) \times \mathbb{G}_m \text{Res}_{F/F^+} G_m \) and \( \text{GL}_2 \times \mathbb{G}_m \text{Res}_{F/F^+} G_m \), and apply the results of [LS19] along with the classical Jacquet–Langlands correspondence between \( \text{GL}_1(D) \) and \( \text{GL}_2 \).)

Let \( BC_{F/F^+} \) denote the global stable base change map (cf. [Rog90 §11.5]), and put \( \Pi \overset{\text{def}}{=} BC_{F/F^+}(\text{JL}(\pi)) \), where \( \pi \) denotes the \( L \)-packet containing \( \pi \). Then \( \Pi \) is an automorphic representation of \( \text{GL}_2(\mathbb{A}_F) \) which enjoys the following properties:

- \( \Pi \) is conjugate self-dual;
- \( \Pi_{\infty} \) is cohomological of weight \( t_+ \lambda \) (viewed as an element of \( (\mathbb{Z}_+^2)^{\text{Hom}(F,\mathbb{C})} \));
- If \( w \) is a place of \( F \) which is split over a place \( v \) of \( F^+ \), then
  \[ \Pi_w \cong BC_{F_w/F^+_w}(\pi_v) = \pi_v \circ t_w^{-1} \]
  where \( BC_{F_w/F^+_w} \) denotes the local base change (cf. [Gue11 §2.4]);
- If \( v \) is a place of \( F \) lying over an inert place \( v \) of \( F^+ \), then
  \[ \Pi_v \cong BC_{F_v/F^+_v}(\pi_v), \]
  where \( BC_{F_v/F^+_v} \) denotes the local base change (described explicitly in [Rog90 Prop. 11.4.1], and in further detail in [Bla10 Cor. 3.6 and Thm. 4.4]);
- If \( v \in \Sigma_p \), then we have an injection
  \[ \sigma'(\tau'_v) \otimes_{E,\iota} \mathbb{C} \hookrightarrow \Pi_v|\text{GL}_2(\mathcal{O}_{F_v}). \]

Hence, if \( \tau'_v \) is a principal series tame type, \( \Pi_v \) is a principal series representation.

The construction of \( r_\pi(\pi) \) now follows just as in [Gue11 Thm. 2.3], appealing to [Rog90 Thm. 11.5.1] instead of [Lab11 Cor. 5.3] in order to control what happens above \( p \). All the properties listed follow from [Gue11 Thm. 0.1], [Car12 Thm. 1.1], [Car14 Thm. 1.1], and Theorem 4.11.

Fix a sufficiently small compact open subgroup \( K = \prod_p K_p \subseteq G(\mathbb{A}_{F^+}) \), and let \( T \) denote a finite set of finite places of \( F^+ \), which contains all inert places \( v \) for which \( K_v \) is not hyperspecial and all split places \( v \) for which \( K_v \neq G(\mathcal{O}_{F^+_v}) \). We define the abstract Hecke algebra \( \mathbb{T}_T \) to be the commutative polynomial \( \mathcal{O} \)-algebra generated by formal variables \( T_w^{(i)} \) for \( i = 1, 2 \), and \( w \) a place of \( F \) split over a place of \( F^+ \) such that \( w|_{F^+} \notin T \).

Fix \( \lambda \in (\mathbb{Z}_+^2)_{\mathbb{T}_T} \) and let \( \tau' \overset{\text{def}}{=} \{ \tau'_v \}_{v \in \Sigma_p} \) and \( \sigma^o \) denote a \( G(\mathcal{O}_{F^+_p}) \)-stable \( \mathcal{O} \)-lattice in \( \sigma = \bigotimes_{v \in \Sigma_p^+} \sigma(\tau'_v) \). Given \( K \) as above, with \( K_v \subseteq G(\mathcal{O}_{F^+_v}) \) for all \( v \in \Sigma_p^+ \), the Hecke operator \( T_w^{(i)} \) acts on the space \( S_G(K, W_{\lambda} \otimes_{\mathcal{O}} \sigma^{o, V}) \) via the characteristic function of double coset

\[ K_v t_w^{-1} \left( \begin{array}{cc} w^{1} & 1_i \\ 1_{2-i} & K_v \end{array} \right) \]
(here $\varpi_w$ denotes a choice of an uniformizer of $F_w$, and $v = w|_{F^+}$). The image of $\mathbb{T}^T$ in $\text{End}_\mathcal{O}(S_{G}(K,W_\lambda \otimes_\mathcal{O} \sigma^{0,\nu}))$ will be denoted $\mathbb{T}_{\lambda_s,\tau}(K)$. The algebra $\mathbb{T}_{\lambda_s,\tau}(K)$ is reduced, finite free over $\mathcal{O}$, and thus a semi-local ring. Furthermore, note that we have $T^{(i)}_w = T^{(2-i)}_w (T^{(2)}_w)^{-1}$ in $\mathbb{T}_{\lambda_s,\tau}(K)$.

Recall that we have

$$S_{G}(K,W_\lambda \otimes_\mathcal{O} \sigma^{0,\nu}) \otimes_\mathcal{O} \mathcal{E} \cong \bigoplus_{\Xi} M_\Xi \otimes \Xi^{K^p},$$

where the direct sum runs over all irreducible constituents $\Xi$ of $S_{G}(\mathcal{O}(\mathcal{O}_{F^+,p}),W_\lambda \otimes_\mathcal{O} \sigma^{0,\nu}) \otimes_\mathcal{O} \mathcal{E}$ for which $\Xi \subsetneq p$ and this localization annihilates all the direct summands of (6.3.1) for which $\text{ker}(\lambda^{\mathcal{O}}) = 0$, and where $M_\Xi$ is a multiplicity space. The Hecke algebra $\mathbb{T}_{\lambda_s,\tau}(K)$ acts on each $\Xi^{K^p}$ by scalars, and we obtain a homomorphism

$$\lambda_\Xi : \mathbb{T}_{\lambda_s,\tau}(K) \longrightarrow \mathcal{E}.$$ 

The ideal $\text{ker}(\lambda_\Xi)$ is a minimal prime ideal, and every minimal prime of $\mathbb{T}_{\lambda_s,\tau}(K)$ arises in this way.

Fix a maximal ideal $\mathfrak{m} \subseteq \mathbb{T}_{\lambda_s,\tau}(K)$. Then we have

$$S_{G}(K,W_\lambda \otimes_\mathcal{O} \sigma^{0,\nu})_\mathfrak{m} \otimes_\mathcal{O} \mathcal{E} \neq 0,$$

and this localization annihilates all the direct summands of (6.3.1) for which $\text{ker}(\lambda_\Xi) \not\subseteq \mathfrak{m}$. Let $\mathfrak{p} \subseteq \mathfrak{m}$ denote a minimal prime ideal, corresponding to an irreducible constituent $\Xi$ of (6.3.1). We choose an invariant lattice in $r_\mathfrak{p}(\pi)$ (for $\pi$ associated to $\Xi$ as in Theorem 6.1), reduce modulo $\mathfrak{p}$, and semisimplify to obtain a representation $\bar{\tau}_\mathfrak{m}$; by the density argument of Theorem 6.1 this is independent of the choice of $\mathfrak{p}$ and $\Xi$.

**Theorem 6.2.** Fix $\lambda \in (\mathbb{Z}^+)^T$ and let $\tau' = \{\tau'_v\}_{v \in \Sigma_\mathfrak{p}}$ be as in Theorem 6.1. Suppose that $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}_{\lambda_s,\tau}(K)$ such that the residue field $\mathbb{T}_{\lambda_s,\tau}(K)/\mathfrak{m}$ is equal to $\mathbb{F}$. Suppose further that $\bar{\tau}_\mathfrak{m}$ is absolutely irreducible. Then $\bar{\tau}_\mathfrak{m}$ has an extension to a continuous homomorphism

$$\bar{\tau}_\mathfrak{m} : \Gamma_{F^+} \longrightarrow \mathcal{G}(\mathbb{F}).$$

Choose such an extension. There exists a continuous lifting

$$r_\mathfrak{m} : \Gamma_{F^+} \longrightarrow \mathcal{G}(\mathbb{T}_{\lambda_s,\tau}(K)_\mathfrak{m})$$

satisfying the following properties. Note that properties (i) and (iii) characterize $r_\mathfrak{m}$ up to isomorphism.

(i) The representation $r_\mathfrak{m}$ is unramified at all but finitely many places.

(ii) We have $\nu \circ r_\mathfrak{m} = \epsilon^{-1}$.

(iii) If $v \not\in T$ is a finite place of $F^+$ which splits as $v = wu = \mathfrak{c}$ in $F$, then $r_\mathfrak{m}$ is unramified at $w$ and $\text{BC}'(r_\mathfrak{m})(\text{Frob}_w)$ has characteristic polynomial

$$X^2 - T^{(1)}_w X + N(w)T^{(2)}_w.$$

(iv) If $v \not\in \Sigma_\mathfrak{p}^+$ is an inert place, then $r_\mathfrak{m}$ is unramified at $v$.

(v) Given $v \in \Sigma_\mathfrak{p}^+$ and a homomorphism $x : \mathbb{T}_{\lambda_s,\tau}(K)_\mathfrak{m} \longrightarrow \mathcal{E}$, the representation $x \circ r_\mathfrak{m}|_{\Gamma_{F_v^+}}$ is potentially crystalline, and we have

$$\text{WD}(x \circ r_\mathfrak{m}|_{\Gamma_{F_v^+}}) \cong r'_v \oplus 1_{F_v^+}.$$ 

If $\kappa \in I_\mathfrak{p}^+$ satisfies $v(\kappa) = v$, then

$$\text{HT}_\kappa(\text{BC}'(x \circ r_\mathfrak{m})|_{\Gamma_{F_v}}) = \{\lambda_{\kappa,1} + 1, \lambda_{\kappa,2}\}.$$ 

**Proof.** This follows exactly as in [CHT08 Prop. 3.4.4], using Theorem 6.1. The fact that $\nu \circ r_\mathfrak{m} = \epsilon^{-1}$ in (ii) (instead of $\epsilon^{-1}\delta_\mathfrak{m}|_{F/F^+}$) follows from the main result of [BC11].
6.3.3. We recall one more well known lemma on the space of algebraic automorphic forms.

**Lemma 6.3.** Let $K = \prod_v K_v \subseteq \mathbb{G}(\mathbb{A}^\infty_{p^+})$ be a sufficiently small compact open subgroup as above, and let $W$ be a finite, $p$-torsion free $\mathcal{O}$-module endowed with an action of $\mathbb{G}(\mathcal{O}_{F^+})$. Fix a maximal ideal $m$ of $\mathbb{T}$. Then

$$S_G(K, W \otimes_\mathcal{O} F)_m \neq 0 \iff S_G(K, W \otimes_\mathcal{O} E)_m \neq 0.$$ 

**Proof.** This is standard (see, for example, the proof of [CHT08, Lem. 3.4.1]). More precisely, the fact that $K$ is sufficiently small gives the isomorphism (6.2.1), and implies $S_G(K, W)_m$ is $p$-torsion free. We therefore obtain

$$S_G(K, W \otimes_\mathcal{O} F)_m \cong S_G(K, W)_m \otimes_\mathcal{O} F \neq 0 \iff S_G(K, W \otimes_\mathcal{O} E)_m \cong S_G(K, W)_m \otimes_\mathcal{O} E \neq 0.$$ 

□

6.4. **Weight elimination.**

6.4.1.

**Definition 6.4.** A Serre weight for $G$ is an isomorphism class of smooth, absolutely irreducible representations of $\prod_{v \in \Sigma^+} G(\mathbb{F}^+_v)$ over $\mathbb{F}$, inflated to $G(\mathcal{O}_{F^+})$. If $v \in \Sigma^+$, a Serre weight at $v$ is an isomorphism class of smooth, absolutely irreducible representations of $G(\mathbb{F}^+_v)$ over $\mathbb{F}$, inflated to $G(\mathcal{O}_{F^+})$.

Note that any Serre weight $V$ for $G$ is of the form $V \cong \bigotimes_{v \in \Sigma^+} V_v$, where $V_v$ are Serre weights at $v$.

**Definition 6.5.** Let $\bar{\tau} : \Gamma_{F^+} \rightarrow \mathbb{G}_2(\mathbb{F})$ be a continuous representation such that

- $\nu \circ \bar{\tau} = \bar{\tau}^{-1}$,
- $\bar{\tau}^{-1}(GL_2(\mathbb{F}) \times G_m(\mathbb{F})) = \Gamma_F$,
- $BC'\bar{\tau} : \Gamma_F \rightarrow GL_2(\mathbb{F})$ is absolutely irreducible.

Let $K = \prod_v K_v \subseteq \mathbb{G}(\mathbb{A}^\infty_{p^+})$ be a compact open subgroup, $T$ a finite set of places as in Subsubsection 6.3.2 and suppose $\bar{\tau}$ is unramified at all finite places $v$ of $F^+$ which split in $F$ and for which $v \not\in T$. We define a maximal ideal $m_\bar{\tau}$ of $\mathbb{T}^T$ by

$$m_\bar{\tau} \overset{\text{def}}{=} \left( \varpi, T_w^{(1)} - \text{tr}(BC'(\bar{\tau})(\text{Frob}_w)), T_w^{(2)} - N(w)^{-1} \det(BC'(\bar{\tau})(\text{Frob}_w)) \right)_{w|_{F^+} \not\in T}$$

where $w|_{F^+} = v \not\in T$ splits as $v = uw^e$ in $F$.

**Definition 6.6.** Let $\bar{\tau}$ be as in Definition 6.5 and let $V$ be a Serre weight for $G$. Let $K = \prod_v K_v \subseteq \mathbb{G}(\mathbb{A}^\infty_{p^+})$ be a sufficiently small compact open subgroup with $K_v$ hyperspecial for $v$ inert in $F$ and $K_v = \mathbb{G}(\mathcal{O}_{F^+})$ for $v \in \Sigma^+_{p^+}$, and let $T$ be a finite subset as in Subsubsection 6.3.2 such that $\bar{\tau}$ is unramified at each split place not in $T$. We say that $\bar{\tau}$ is modular of weight $V$ and level $K$ (or that $V$ is a Serre weight of $\bar{\tau}$ at level $K$) if

$$S_G(K, V^\vee)_{m_\bar{\tau}} \neq 0.$$ 

We say that $\bar{\tau}$ is modular of weight $V$ (or that $V$ is a Serre weight of $\bar{\tau}$) if $\bar{\tau}$ is modular of weight $V$ and level $K$, for some sufficiently small compact open subgroup $K \subseteq \mathbb{G}(\mathbb{A}^\infty_{p^+})$ as above. We denote by $W_{\text{mod}}(\bar{\tau})$ for the set of all Serre weights of $\bar{\tau}$. We say that $\bar{\tau}$ is modular if $W_{\text{mod}}(\bar{\tau}) \neq \emptyset$. 

6.4.2. Fix $\tau$ as in Definition 6.5 and for $v \in \Sigma_p^+$ define $\tilde{\theta}_v \overset{\text{def}}{=} \tau|_{\Gamma_{F_v^+}}$. By Proposition 4.6 we have a set of conjectural Serre weights $W'(\tilde{\theta}_v)$ at $v$ for every $v \in \Sigma_p^+$. (Here we use the isomorphism $C U_2 \cong \mathcal{O}_2$ of Subsection 2.4.) Moreover, the condition $\nu \circ \tau = \tilde{\tau}^{-1}$ implies that the $U_2(\mathcal{O}_v^+)$-representations appearing in $W'(\tilde{\theta}_v)$ descend to $U_2(\mathbf{F}_v^+) \cong G(\mathbf{F}_v^+)$, see for instance Proposition 4.8.) Thus, we can attach to $\tau$ a set $W'(\tau)$ of predicted Serre weights for $G$ defined as

$$W'(\tau) \overset{\text{def}}{=} \left\{ \bigotimes_{v \in \Sigma_p^+} V_v : V_v \in W'(\tilde{\theta}_v) \text{ for all } v \in \Sigma_p^+ \right\}.$$ 

In a similar fashion we define the set $W'(BC(\tau))$ of conjectural weights attached to $BC(\tau)$. (Note that, under the isomorphism in Subsection 2.4, we have $BC(\tau) \cong BC'(\tau) \otimes \tilde{\tau}$.)

If $\sigma = \bigotimes_{v \in \Sigma_p^+} \sigma_v$ is a tame $G(\mathcal{O}_F{+},p)$-type, we define the base change of $\sigma$ as

$$BC(\sigma) \overset{\text{def}}{=} \bigotimes_{v \in \Sigma_p^+} BC_v(\sigma_v),$$

where $BC_v$ denotes the local base change of types.

**Theorem 6.7.** Let $\tau : \Gamma_{F^+} \longrightarrow \mathcal{O}_2(\mathbf{F})$ be a continuous representation such that

- $\nu \circ \tau = \tilde{\tau}^{-1}$;
- $\tilde{\tau}^{-1}(GL_2(\mathbf{F}) \times G_m(\mathbf{F})) = \Gamma_F$;
- $BC'(\tau) : \Gamma_F \longrightarrow GL_2(\mathbf{F})$ is absolutely irreducible;
- $\tau$ is modular;
- $\tilde{\tau}|_{\Gamma_{F_v^+}}$ is tamely ramified and 3-generic for every $v \in \Sigma_p^+$.

Then

$$W_{\text{mod}}(\tau) \subseteq W'(\tau).$$

**Proof.** Let $V \in W_{\text{mod}}(\tau)$, and assume by contradiction that $V \notin W'(\tau)$. By Proposition 4.6 and Lemma 3.27 there exists a tame $U_2(\mathcal{O}_F{+},p)$-type $\sigma = \bigotimes_{v \in \Sigma_p^+} \sigma_v$ such that

- (i) $V \in JH(\tau)$;
- (ii) $JH(\tau) \cap W'(\tau) = \emptyset$.

(Note that if $\theta_v$ is $n$-generic, so is $\beta(V_\phi(\theta_v))$.) We define $\tau'_v$ to be the tame principal series type such that $\sigma_v \cong \sigma(\tau'_v)$ (so that, in particular, $(\tau'_v)^{\cong \mathcal{O}_v{+},p} \cong \tau_v^{\mathcal{O}_v{+}}$).

By definition of modularity, there exists a sufficiently small compact open subgroup $K = \prod_v K_v$ such that $K_v$ is hyperspecial for $v$ inert in $F$ and $K_v = G(\mathcal{O}_F{+})$ for $v \in \Sigma_p^+$, and a finite set of places $T$ such that

$$S_G(K, V_v^\vee) \neq 0.$$ 

Since $K$ is sufficiently small, the functor of algebraic automorphic forms is exact, so item (i) implies

$$S_G(K, \sigma_v^\vee V_v) \neq 0,$$

and Lemma 6.3 gives

$$S_G(K, \sigma_v^\vee \otimes_{\mathcal{O}_v} E) \neq 0.$$

By the discussion preceding Theorem 6.2 and upon choosing an isomorphism $\iota : E \overset{\sim}{\longrightarrow} \mathcal{O}_2$, there exists a cuspidal automorphic representation $\pi$ of $G(\mathcal{O}_F{+})$ such that

- $\pi_\infty$ is the trivial representation of $G(F_\infty{+})$;
- $Hom_{K_v}(\sigma \otimes_{\mathcal{O}_v} \mathcal{O}_v, \pi_v) \neq 0$;
- for each place $v$ of $F^+$ which is split in $F$ and not contained in $T$, the local constituent $\pi_v$ is an unramified principal series with Satake parameters determined by a minimal prime of $T_{0,\tau} = (K_{\text{mr}})$ via $\iota$. 

As in the proof of Theorem 6.1, we obtain a continuous representation
\[ r_\varepsilon(\pi) : \Gamma_F \rightarrow GL_2(\overline{E}) \]
such that
- \( r_\varepsilon(\pi) \) lifts \( BC'(\overline{\pi}) \);
- for each \( v \in \Sigma_p, r_\varepsilon(\pi)|_{\Gamma_{F_v}} \) is potentially crystalline, and
  \[ \text{WD}(r_\varepsilon(\pi)|_{\Gamma_{F_v}})|_{I_{F_v}} \cong \tau'_\varepsilon; \]
- for each \( \kappa \in I_p \), we have
  \[ \text{HT}_\kappa(r_\varepsilon(\pi)|_{\Gamma_{F_{\varepsilon}(\kappa)}}) = \{1, 0\}. \]

Consequently, the representation \( r_\varepsilon(\pi) \otimes \varepsilon \) has the following properties:
- \( r_\varepsilon(\pi) \otimes \varepsilon \) lifts \( BC'(\overline{\pi}) \otimes \varepsilon \cong BC(\overline{\pi}) \);
- for each \( v \in \Sigma_p, (r_\varepsilon(\pi) \otimes \varepsilon)|_{\Gamma_{F_v}} \) is potentially crystalline, and
  \[ \text{WD}((r_\varepsilon(\pi) \otimes \varepsilon)|_{\Gamma_{F_v}})|_{I_{F_v}} \cong \tau'_\varepsilon; \]
- for each \( \kappa \in I_p \), we have
  \[ \text{HT}_\kappa((r_\varepsilon(\pi) \otimes \varepsilon)|_{\Gamma_{F_{\varepsilon}(\kappa)}}) = \{0, -1\}. \]

By the above, we see that \( BC(\overline{\pi}_v) \) has a potentially Barsotti–Tate lift of type \( \tau'_\varepsilon \) for every \( v \in \Sigma_p \), namely \( (r_\varepsilon(\pi) \otimes \varepsilon)|_{\Gamma_{F_v}} \) (with notation as in [Gee11, Def. 2.3]). Therefore, Proposition 3.12 of op. cit. implies that
\[ JH(BC(\sigma(\tau'_\varepsilon))) \cap W^2(BC(\overline{\pi}_v)) \neq \emptyset \]
for all \( v \in \Sigma_p \). By Propositions 3.18, 4.6 and 4.8, we obtain
\[ JH(\sigma(\tau'_\varepsilon)) \cap W^2(\overline{\pi}_v) \neq \emptyset \]
for all \( v \in \Sigma_p^+ \). However, this contradicts item (ii), which concludes the proof. \( \square \)

7. Global Applications II

In this section, we use patching techniques to prove the existence of Serre weights for \( L \)-parameters, using the results on local deformation theory obtained in Section 5. We first adapt the patching construction of [CEG+16] to the case of unitary groups which are not split at places above \( p \), and state the necessary properties in Proposition 7.3. This allows us to deduce the main results on weight existence in Theorem 7.4 and automorphy lifting in Theorem 7.7.

7.1. Setup.

7.1.1. Suppose that \( \overline{\pi} : \Gamma_{F^+} \rightarrow G_2(\overline{F}) \) is a fixed Galois representation such that
- \( \overline{\pi}^{-1}(GL_2(\overline{F}) \times G_m(\overline{F})) = \Gamma_F \);
- \( \nu \circ \overline{\pi} = \overline{\pi}^{-1} \);
- \( \overline{\pi} \) is modular;
- \( \overline{\pi} \) is unramified outside \( \Sigma_p^+ \);
- \( \overline{\pi}|_{\Gamma_{F_v^+}} \) is tamely ramified and 4-generic for all \( v \in \Sigma_p^+ \);
- \( \overline{F}\ker(\text{ad}(\overline{\pi})) \) does not contain \( F(\zeta_p) \); and
- \( BC'(\overline{\pi})(\Gamma_F) \supseteq GL_2(\overline{F}') \) for some subfield \( \overline{F}' \subseteq \overline{F} \) with \( |\overline{F}'| > 6 \).

The last condition implies that \( BC'(\overline{\pi})(\Gamma_F(\zeta_p)) \) is adequate (cf. [BLGG] Prop. 6.5), and that \( BC'(\overline{\pi}) \) is absolutely irreducible. Furthermore, the argument in [CEG+16] shows that this condition also guarantees the existence of a place \( v_1 \) of \( F^+ \) such that
- \( v_1 \) splits as \( \overline{v}_1 \overline{v}_1 \) in \( F' \);
- \( v_1 \) does not split completely in \( F(\zeta_p) \);
BC′(τ)(\text{Frob}_{\bar{v}_1}) has distinct \mathbb{F}\text{-rational eigenvalues, whose ratio is not equal to } N(v_1)^{\pm 1}.

7.1.2. Let \lambda \in (\mathbb{Z}_p^2)^{\tilde{I}_p} and for every \nu \in \Sigma^+_p, let \tau'_\nu denote a tame inertial type which satisfies (\nu')_{\uparrow \mathcal{M}} \cong \nu'. Set T \overset{\text{def}}{=} \Sigma^+_p \cup \{v_1\} and \bar{T} \overset{\text{def}}{=} \Sigma_p \cup \{\bar{v}_1\}. We consider a slight generalization of the global deformation problems of [CHT08 §2.3]:

\begin{align*}
&S \overset{\text{def}}{=} \left( F/F^+, T, \bar{T}, \mathcal{O}, F, \varepsilon^{-1}, \{\bar{R}_v^{\square}\}_{v \in \Sigma^+_p} \cup \{\bar{R}_{\bar{v}_1}^{\square}\} \right), \\
&S_{\lambda, \tau'} \overset{\text{def}}{=} \left( F/F^+, T, \bar{T}, \mathcal{O}, F, \varepsilon^{-1}, \{R_v^{\square, \lambda_v, \tau'_\nu}\}_{v \in \Sigma^+_p} \cup \{\bar{R}_{\bar{v}_1}^{\square}\} \right).
\end{align*}

The difference here is that we allow places in \( T \) to be inert in \( F \). In this notation, \( \tilde{R}_v^{\square} \) denotes the maximal reduced and \( p \)-torsion free quotient of the universal framed deformation ring parametrizing lifts \( \rho \) of \( \bar{\pi}|_{\Gamma_{F_v^+}} \) such that \( \nu \circ \rho = \varepsilon^{-1} \). Further, the ring \( R_v^{\square, \lambda_v, \tau'_\nu} \) denotes the unique quotient of \( \tilde{R}_v^{\square} \) with the property that if \( B \) is a finite local \( E \)-algebra, then \( x : \tilde{R}_v^{\square} \rightarrow B \) factors through \( R_v^{\square, \lambda_v, \tau'_\nu} \) if and only if the corresponding representation \( r_x : \Gamma_{F_v^+} \rightarrow \mathcal{G}_2(B) \) is potentially crystalline, and satisfies \( \nu \circ r_x = \varepsilon^{-1} \), \( \text{HT}_\kappa(\text{BC}(r_x)) = \{\lambda_{v,1} + 1, \lambda_{v,2}\} \), and \( \text{WD}(\text{BC}(r_x))|_{I_{F_v}} \cong \tau'_\nu \). (Again, the existence of such a quotient follows from [BG19 §3.2–3.3].) In particular, if \( \lambda = 0 \), then by applying the isomorphism of Subsection 2.4 we obtain an isomorphism of deformation rings \( R_v^{\square, 0, \nu} \cong R_{p_v}^{\tau} \), where the second ring is as in Subsubsection 5.3.3.

We note also that \( \tilde{R}_v^{\square} \) is formally smooth over \( \mathcal{O} \) of relative dimension 4, and all of the corresponding Galois representations lifting \( \bar{\pi}|_{\Gamma_{F_v^+}} \) are unramified (see [CEG+16 Lem. 2.5]).

We let \( R_S^{\text{univ}} \) be the complete local Noetherian \( \mathcal{O} \)-algebra representing the functor of deformations of \( \bar{\pi} \) of type \( S \), and let \( R_S^{\square} \) denote the \( \mathcal{O} \)-algebra representing \( T \)-framed deformations of \( \bar{\pi} \) of type \( S \). (We define a framing at places in \( \Sigma^+_p \) just as in [CHT08 Def. 2.2.1], i.e., as an element of \( I_2 + \text{Mat}_{2 \times 2}(\mathcal{O}_R) \subseteq \ker(\mathcal{G}_2(R) \rightarrow \mathcal{G}_2(\mathbb{F})) \).) We have similar notation for the deformation problem \( S_{\lambda, \tau'} \).

7.1.3. Set

\[ T \overset{\text{def}}{=} 0[X_{v, i, j} : v \in T, 1 \leq i, j \leq 2]. \]

Choose a lift \( r_S^{\text{univ}} \) representing the universal deformation of type \( S \), and form the tuple

\[ \left( r_S^{\text{univ}}, \left\{ \left( \begin{array}{cc} 1 + X_{v,1,1} & X_{v,1,2} \\
X_{v,2,1} & 1 + X_{v,2,2} \end{array} \right) \right\}_{v \in T} \right). \]

This gives a representative of the universal \( T \)-framed deformation of type \( S \), and we obtain

\[ R_S^{\square, T} \cong R_S^{\text{univ}} \otimes_0 T \cong R_S^{\text{univ}}[X_{v, i, j}] \]

(and similarly for \( S_{\lambda, \tau'} \)).

We set

\[ R^{\text{loc}} \overset{\text{def}}{=} \bigotimes_{v \in \Sigma^+_p} \tilde{R}_v^{\square} \otimes \bar{\mathcal{O}}_{\bar{v}_1}, \]

\[ R^{\text{loc}}_{\lambda, \tau'} \overset{\text{def}}{=} \bigotimes_{v \in \Sigma^+_p} R_v^{\square, \lambda_v, \tau'_\nu} \otimes \bar{\mathcal{O}}_{\bar{v}_1}, \]

where all completed tensor products are taken over \( \mathcal{O} \).

**Proposition 7.1.** Assume that \( R_v^{\square, \lambda_v, \tau'_\nu} \) has a non-zero \( \mathcal{O} \)-point for all \( v \in \Sigma^+_p \). Then \( R^{\text{loc}}_{\lambda, \tau'}[1/p] \) is regular. If moreover \( (\lambda_v, \tau'_\nu) = (0, \tau'_\nu) \) with \( \tau'_\nu \) being 2-generic and \( \bar{\pi}|_{\Gamma_{F_v^+}} \) being 1-generic and semisimple, then \( R^{\text{loc}}_{0, \tau'}[1/p] \) is formally smooth, and \( R^{\text{loc}}_{0, \tau'} \) is equidimensional of dimension \( 1 + 4|T| + |F^+ : \mathbb{Q}|. \)
The fact that $R_{\lambda,\tau}^{\text{loc}}[1/p]$ is regular follows from [BG19, Thm. 3.3.7], formal smoothness of $\tilde{R}_v^\square$, and [CEG+16 Cor. A.2]. When $\lambda = 0$, formal smoothness of $R_{\lambda,\tau}^{\text{loc}}[1/p]$ follows from the results of [5.3.10] formal smoothness of $\tilde{R}_v^\square$, and [Kis09 Lem. (3.4.12)]. The claim about dimensions then follows from [BG19 Thm. 3.3.7], the fact that $\tilde{R}_v^\square$ is of relative dimension 4 over $\mathcal{O}$, and [BLGHT11 Lem. 3.3].

7.1.4. We now relate the above constructions to spaces of automorphic forms. We fix a compact open subgroup $K_m = \prod_v K_{m,v} \subseteq G(A_{F_v}^\infty)$ satisfying the following properties:

- if $v$ is a place of $F^+$ which is inert in $F$ and $v \not\in \Sigma_p^+$, then $K_{m,v}$ is a maximal hyperspecial subgroup of $G(F_v^+)$;
- if $v$ is a place of $F^+$ which is split in $F$ and $v \neq v_1$, then $K_{m,v} = G(O_{F_v^+})$;
- if $v \in \Sigma_p^+$, then $K_{m,v} = \ker(G(O_{F_v^+}) \to G(O_{F_v^+}/\mathfrak{m}_v^{m,v}))$;
- if $v = v_1$ and $\tilde{v}_1$ is the fixed place of $F$ above $v_1$, then $K_{m,v_1}$ is the preimage under $\tilde{v}_1$ of the upper-triangular pro-$\tilde{v}_1$-Iwahori subgroup of $GL_2(O_{F_{v_1}})$.

These assumptions guarantee that $K_m$ is sufficiently small. We define $K \overset{\text{def}}{=} K_0$.

Before proceeding, we will need the following level-lowering result.

**Proposition 7.2.** Suppose $\tau$ satisfies the hypotheses from the beginning of Subsection 7.1, so that in particular $\tau$ is modular, unramified outside $p$, and $\tau|_{\Gamma_{F_v^+}}$ is tamely ramified and 4-generic for all $v \in \Sigma_p^+$. Then $\tau$ is modular of level $K_0$.

**Proof.** Suppose $\tau$ is modular of weight $V = \bigotimes_{v \in \Sigma_p^+} V_v$. Thus, there exists a finite set of finite places $T'$ and a sufficiently small level $K' = \prod_v K'_v \subseteq G(A_{F_v}^\infty)$ (with $K'_v = G(O_{F_v^+})$ for $v \in \Sigma_p^+$ and split $v \not\in T'$, and $K'_v$ is a maximal hyperspecial subgroup for all other inert $v$) satisfying

$$S_G(K', V^\vee) \neq 0.$$ 

For each $v \in \Sigma_p^+$, we choose a principal series tame type $\tau'_v$ such that $(\tau'_v)^{\rho-\rho|_{\mathcal{F}_p}} \cong \tau_v^{K'_v}$ and such that $V \in JH(\sigma)$, where $\sigma := \bigotimes_{v \in \Sigma_p^+} \sigma(\tau'_v)$. By the genericity hypotheses and Theorem 6.7, $V_v$ is 3-deep for every $v \in \Sigma_p^+$, and consequently $\tau'_v$ is 2-generic. Since $K'$ is sufficiently small, Lemma 6.3 implies

$$S_G(K', \sigma^\vee) \neq 0.$$

As in the proof of Theorem 6.7, there exists an automorphic representation $\pi$ of $G(A_{F^+})$, and (after choosing an isomorphism $\iota : E \sim \mathbb{C}$) an associated Galois representation

$$r_\iota(\pi) : \Gamma_F \to GL_2(E)$$

which lifts $BC'(\tau) \otimes_F \mathbb{F}_p$.

Let $\Pi$ denote the automorphic representation of $GL_2(A_F)$ obtained from $\pi$ by base change (as in the proof of Theorem 6.1), and let $\Sigma^+_{\text{ram}}$ denote the set of prime-to-$p$ places of $F^+$ at which $\pi$ is ramified (note that every place of $\Sigma^+_{\text{ram}}$ is split in $F$, and if $\Pi$ is ramified at some place $w$, then $w|_{F^+} \in \Sigma^+_{\text{ram}}$). Adjusting the place $v_1$ if necessary, we may assume $v_1 \not\in \Sigma^+_{\text{ram}}$. We choose a totally real extension $L^+$ of $F^+$ such that the following conditions hold:

- 4 divides $[L^+ : \mathbb{Q}]$;
- $L^+/F^+$ is Galois and solvable;
- $L \overset{\text{def}}{=} L^+ F$ is linearly disjoint from $\mathcal{F}^{\kappa(\tau)}(\zeta_p)$ over $F$;
- $L/L^+$ is everywhere unramified;
- $p$ is unramified in $L$;
- $v_1$ splits completely in $L$;
- if $w$ is a place of $L^+$ lying over a place in $\Sigma^+_{\text{ram}}$, then $N(w) \equiv 1 \pmod{p}$;
If $\Pi_L$ denotes the base change of $\Pi$ to an automorphic representations of $GL_2(\mathbb{A}_L)$ and $w$ is a place of $L$ lying over a place in $\Sigma^+_{\text{ram}}$, then $\Pi^{\text{Iw}_{L,w}}_L \neq 0$, where $\text{Iw}_w \subseteq GL_2(\mathbb{O}_{L,w})$ denotes the standard upper-triangular Iwahori subgroup.

We use the following notation in what follows: if $L/F$ is a finite extension of number fields and $\tilde{T}$ is a finite set of finite places of $\tilde{F}$, we let $\text{BC}_{L/F}(\tilde{T})$ (or $\text{BC}(\tilde{T})$ when the context is clear) denote the set of places of $\tilde{L}$ lying above $\tilde{T}$.

Let $\pi_{L+}$ denote a descent of $\Pi_L$ to an automorphic representation of $G(\mathbb{A}_{L+})$. If $w$ is a place of $L^+$ which splits as $\tilde{w}\tilde{w}'$ in $L$, then $\pi_{L+,w} \cong \Pi^{\text{Iw}_{\tilde{w},\tilde{w}'},w}_{L,\tilde{w}}$, where $\text{Iw}_{\tilde{w},\tilde{w}'}$ is an isomorphism $G(L^+_{\tilde{w}}) \cong GL_2(L_{\tilde{w}'})$ which identifies groups of integral points. In particular, if $\Pi_{L,\tilde{w}}$ is unramified, so is $\pi_{L+,w}$. Suppose now that $w$ is a place of $L^+$ which is inert in $L$ and $\Pi_{L,w}$ is unramified. By the explicit description of local base change found in [Rog90], Ch. 11, we see that $\pi_{L+,w}$ is unramified relative to a hyperspecial maximal compact subgroup of $G(L^+_w)$, which is equal to $G(\mathbb{O}_{L,w})$ for all but finitely many inert primes $w$. Thus, we define $K_{L+} = \prod_w K_{L+,w} \subseteq G(\mathbb{A}_{L+}^{\infty})$ to be the compact open subgroup satisfying the following conditions:

- If $w$ is a place of $L^+$ which is inert in $L$ and $w \notin BC_{L+/F}(\Sigma_p^+)$, then $K_{L+,w}$ denotes a hyperspecial maximal compact subgroup of $G(\mathbb{O}_{L,w})$ relative to which $\pi_{L+,w}$ is unramified, chosen to be equal to $G(\mathbb{O}_{L,w})$ for all but finitely many such $w$;
- If $w \in BC_{L+/F}(\Sigma_p^+)$, then $K_{L+,w} = G(\mathbb{O}_{L,w})$;
- If $w$ is a place of $L^+$ which is split in $L$ and $w \notin BC_{L+/F}(\Sigma^+_{\text{ram}} \cup \{v_1\})$, then $K_{L+,w} = G(\mathbb{O}_{L,w})$;
- If $w \in BC_{L+/F}(\Sigma^+_{\text{ram}})$, then $K_{L+,w}$ is the preimage under $\iota_{\tilde{w}}$ of $\text{Iw}_{\tilde{w}}$, where $\tilde{w}$ is a fixed choice of place of $L$ lying over $w$;
- If $w \in BC_{L+/F}(\{v_1\})$, then $K_{L+,w}$ is the preimage under $\iota_{\tilde{w}}$ of the upper-triangular pro-$\tilde{w}$-Iwahori subgroup.

Note that $K_{L+}$ is sufficiently small. The representation $\pi_{L+}$ then contributes to the space

$$S_{G_{\partial L^+}}(K_{L+}, \sigma^\vee_{L+})_{m_{\partial_{L+}}},$$

where $\sigma_{L+} = \bigotimes_{w \in BC_{L+/F}(\Sigma^+_{\text{ram}})} \sigma(\iota_{\tilde{w}}|_{L_{w}^{\infty}})$ and $m_{\partial_{L+}}$ is the maximal ideal of $T^{\text{BC}(T')}$ defined as in $v_{w}\in\Sigma^+$.

**Definition 6.5.** Here $T' = \Sigma_p^+ \cup \Sigma^+_{\text{ram}} \cup \{v_1\}$.

For every place $w \in BC_{L+/F}(\Sigma^+_{\text{ram}})$, we fix two distinct tame characters $\psi_{w,1}, \psi_{w,2} : \mathbb{O}_{L,w}^{\infty} \to \mathbb{O}_{L,w}^{\times}$ of $p$-power order, and define $\psi_w : K_{L+,w} \to \mathbb{O}_{L,w}^{\times}$ by

$$\psi_w\left(\iota_{\tilde{w}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \psi_{w,1}(a)\psi_{w,2}(d),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{Iw}_{\tilde{w}}$. (Note that such a choice of characters is possible by the choice of $L^+$.)

We define $\psi = \bigotimes_{w \in BC(\Sigma^+_{\text{ram}})} \psi_w$, so that the reduction mod $\varpi$ of $\psi$ is the trivial character of $\prod_{w \in BC(\Sigma^+_{\text{ram}})} K_{L+,w}$. Let

$$S_{G_{\partial L^+}}(K_{L+}, \psi \otimes E \sigma^\vee_{L+})$$

denote the space of algebraic automorphic forms with nebentypus $\psi$ above $\Sigma^+_{\text{ram}}$ (defined as in Subsection 6.2 except that the component $\prod_{w \in BC(\Sigma^+_{\text{ram}})} K_{L+,w}$ acts by $\psi$).

We claim $S_{G_{\partial L^+}}(K_{L+}, \psi \otimes E \sigma^\vee_{L+})_{m_{\partial_{L+}}} \neq 0$. Indeed, after possibly replacing $E$ by a finite extension, the representation $\pi_{L+}$ gives a morphism

$$\theta : \tau^{\text{BC}(T')}_{0, \text{BC}(T')(K_{L+})} \to \mathbb{O},$$
(where $BC(\tau')$ denotes the collection $\{\tau_v|_{I_{L_v}}\}_{w|_F=v\in\Sigma_w}$) and by reduction modulo $\varpi$ we obtain

$$\theta \otimes_0 F : T_{0,BC(\tau')}(K_{L^+}) \otimes_0 F \longrightarrow F.$$ 

Let $T_{0,BC(\tau')}(K_{L^+}, F)$ denote the image of the universal Hecke algebra $T_{BC(\tau')}$ in

$$\text{End}_0 \left( S_{\text{GL}_2(L^+, K_{L^+}, \sigma_{L^+})} \right),$$

where $\sigma_{L^+}$ denotes a choice of $K_{L^+, p}$-stable $\Theta$-lattice in $\sigma_{L^+}$. Since the kernel of $T_{0,BC(\tau')}(K_{L^+}) \otimes_0 F \longrightarrow T_{0,BC(\tau')}(K_{L^+}, F)$ is nilpotent, $\theta \otimes F$ factors through a map

$$\overline{\theta} : T_{0,BC(\tau')}(K_{L^+}, F) \longrightarrow F.$$

Now let $T_{0,BC(\tau')}(K_{L^+}, \psi, F)$ denote the image of the universal Hecke algebra $T_{BC(\tau')}$ in

$$\text{End}_0 \left( S_{\text{GL}_2(L^+, \psi \otimes \sigma_{L^+}^\vee)} \right),$$

and define $T_{0,BC(\tau')}(K_{L^+}, \psi, F)$ analogously. Since $\psi$ is of $p$-power order, we have $T_{0,BC(\tau')}(K_{L^+}, \psi, F) = T_{0,BC(\tau')}(K_{L^+}, F)$, and by pulling $\overline{\theta}$ back we get

$$\theta' : T_{0,BC(\tau')}(K_{L^+}, \psi) \longrightarrow T_{0,BC(\tau')}(K_{L^+}, \psi) \otimes_0 F \longrightarrow T_{0,BC(\tau')}(K_{L^+}, \psi, F) \longrightarrow \overline{\theta},$$

We view $\ker(\theta')$ as a prime ideal lying over the ideal $(\varpi)$ relative to the finite flat extension $\mathcal{O} \longrightarrow T_{0,BC(\tau')}(K_{L^+}, \psi)$. By the going-down theorem, there exists a prime $p \subseteq \ker(\theta')$ lying over $(0) \subseteq \mathcal{O}$.

The minimal prime $p$ constructed above corresponds to an automorphic representation $\pi'_{L^+}$ contributing to $S_{\text{GL}_2(L^+, \pi \otimes \sigma_{L^+}^\vee)}$. Let $\Pi_L'$ denote the base change of $\pi'_{L^+}$ to $\text{GL}_2(\mathbb{A}_L)$. Then for every place $w$ of $L^+$ which is inert in $L$ and for which $w \not\in BC_{L^+/F}(\Sigma^+_p)$, the representation $\Pi_L'|_{I_w}$ is unramified. For every place $w$ of $L^+$ which splits as $w = \overline{w}\overline{w}$ in $L$ and for which $w \not\in BC_{L^+/F}(\Sigma^+_p \cup \{v_1\})$, the representation $\Pi_L'|_{I_{w, \overline{w}}} = \pi'_{L^+/w} \circ l_{\overline{w}}^{-1}$ is unramified. Finally, if $w$ splits in $L$ and $w \in BC_{L^+/F}(\Sigma^+_\text{ram})$, then $(\Pi_L'|_{I_{w, \overline{w}}} \circ w_{\text{ram}} \circ \overline{w}_{\text{ram}})^{-1} \neq 0$. By choice of the characters $\psi_w$, the latter condition implies that $\Pi_L'|_{I_{w, \overline{w}}}$ must be a principal series representation.

Associated to $\pi'_{L^+}$ (or $\Pi_L'$) we have a Galois representation

$$r_{\pi'_{L^+}} : \Gamma_L \longrightarrow \text{GL}_2(E)$$

lifting $BC(\overline{\tau})|_{I_{\mathfrak{m}} \otimes E \mathbb{F}_p}$. By the discussion in the previous paragraph and local/global compatibility, the representation $r_{\pi'_{L^+}}$ is unramified outside $BC_{L^+/F}(\Sigma^+_p \cup \Sigma^+_{\text{ram}})$ (recall that all deformations at places above $v_1$ are unramified), and tamely ramified at $BC_{L^+/F}(\Sigma^+_{\text{ram}})$. Further, if $w \in BC_{L^+/F}(\Sigma^+_p)$, then $r_{\pi'_{L^+}}|_{I_{w, \overline{w}}}$ is potentially crystalline with (parallel) Hodge–Tate weights $(0, 0)$ and inertial type $\tau_{L^+}$. Thus, if we let

$$r_{\Pi'_M} : \Gamma_M \longrightarrow \text{GL}_2(E)$$

denote the Galois representation associated to $\Pi_M'$, then we see that $r_{\Pi'_M}$ is a lift of $BC(\overline{\tau})|_{I_{\mathfrak{m}} \otimes E \mathbb{F}_p}$. Moreover, $r_{\Pi'_M}$ is unramified outside $BC_{M/F}(\Sigma^+_p)$ and if $w \in BC_{M/F}(\Sigma^+_p)$, then $r_{\Pi'_M}|_{I_{w, \overline{w}}}$ is potentially crystalline with (parallel) Hodge–Tate weights $(0, 0)$ and inertial type $\tau_{L^+}$. Thus, if we let

$$r_{\Pi'_M} : \Gamma_M \longrightarrow \text{GL}_2(E)$$

denote the Galois representation associated to $\Pi_M'$, then we see that $r_{\Pi'_M}$ is a lift of $BC(\overline{\tau})|_{I_{\mathfrak{m}} \otimes E \mathbb{F}_p}$.
Recall that we have defined a deformation problem $S_{0,r'}$. We define $S_M$ to be the “base changed” deformation problem, so that

$$S_M \overset{\text{def}}{=} \left( \frac{M}{M^+}, \text{BC}_{M/F}(\Sigma^+ \cup \{v_1\}), \text{BC}_{M/F}(\Sigma_0 \cup \{\bar{v}_1\}), 0, \tau|_{\Gamma_{M^+}}, \varepsilon^{-1}, \right.$$  

$$\left. \{R_w^{\square,0,T}(\Sigma^+ \cup \{\bar{v}_1\}) \}_{w \in \text{BC}_{M/F}(\Sigma_0 \cup \{\bar{v}_1\})} \right).$$

Thus, we see that the extension of $r_i(\Pi'_{M})$ to $\Gamma_{M^+}$ corresponds to an $E$-point of $R_{S_M}^{\text{univ}}$. A variant of the patching construction of [Gue11] Thm. 3.4 with potentially Barsotti–Tate deformation rings (see also the argument which follows in subsequent sections) shows that $(R_{S_M}^{\text{univ}})^{\text{red}}$ is isomorphic to an appropriate localized Hecke algebra, and consequently $R_{S_M}^{\text{univ}}$ is finite over $\mathcal{O}$. Just as in the proof of [BLGGT14] Lem. 1.2.3(1), we have that $R_{S_M}^{\text{univ}}$ is finite over $R_{S_M}^{\text{univ}}$. Combining these facts with the dimension calculation in [CHT08] Cor. 2.3.5, we see that $R_{S_M}^{\text{univ}}$ is a finite $\mathcal{O}$-module of positive rank.

Now, let $r : \Gamma_{F^+} \rightarrow \mathfrak{S}_2(\mathcal{E})$ correspond to an $E$-point of $R_{S_M}^{\text{univ}}$, so that in particular $r$ is a lift of $\tau \otimes_F \mathbb{F}_p$ which is unramified outside of $\Sigma^+$. The restriction $r|_{\Gamma_{M^+}}$ corresponds to an $E$-point of $R_{S_M}^{\text{univ}}$, which necessarily factors through the reduced ring. Thus, $\text{BC}^r(r)|_{\Gamma_{M^+}}$ is automorphic, and by solvable base change $\text{BC}^r(r)$ is also automorphic. This implies that $S_{\mathcal{G}}(K_0, \sigma^{0,r}) \neq 0$, and therefore by Lemma 6.3 we have $S_{\mathcal{G}}(K_0, V^{\Gamma}(\mathfrak{m}_r) \mathfrak{m} = 0$ for some $V^{\Gamma} \in \mathcal{H}(\mathfrak{f})$. □

Recall that we have defined a maximal ideal $\mathfrak{m} \overset{\text{def}}{=} \mathfrak{m}_r \subseteq \mathbb{T}^T$ associated to $\tau$ (Definition 6.5). Proposition 7.2 shows that $S_{\mathcal{G}}(K_{\mathfrak{m}}, \sigma^{0,r}) \neq 0$ for $m \geq 1$, where $\sigma^{0}$ is a $\mathcal{G}(\mathcal{O}_{F^+, p})$-stable $\mathcal{O}$-lattice in a tame type. Since the $p$-component of $K_m$ acts trivially on $\sigma^{0}$ for $m \geq 1$, we have $S_{\mathcal{G}}(K_{\mathfrak{m}}, \sigma^{0,r}) \cong S_{\mathcal{G}}(K_{\mathfrak{m}}, \mathcal{O}) \otimes \sigma^{0,r}$. Thus, the image of $\mathfrak{m}$ in $\mathbb{T}^T_{0,1}(K_{\mathfrak{m}})$ (which will be denoted by the same symbol $\mathfrak{m}$) is a maximal ideal. By Theorem 6.2 we have a continuous lift of $\tau$ given by

$$r_{\mathfrak{m}} \otimes \mathcal{O}/\mathcal{O}^r : \Gamma_{F^+} \rightarrow \mathfrak{S}_2(\mathbb{T}^T_{0,1}(K_{\mathfrak{m}}) \otimes \mathcal{O}/\mathcal{O}^r),$$

which is of type $\mathfrak{s}$. Therefore, we obtain a surjection

$$R_{S_M}^{\text{univ}} \twoheadrightarrow \mathbb{T}^T_{0,1}(K_{\mathfrak{m}}) \otimes \mathcal{O}/\mathcal{O}^r.$$  

In particular, $S_{\mathcal{G}}(K_{\mathfrak{m}}, \mathcal{O}/\mathcal{O}^r) \mathfrak{m}$ is a finite $R_{S_M}^{\text{univ}}$-module.

7.2. Auxiliary primes.

7.2.1. Let $q$ denote the maximum of $\left|F^+: \mathbb{Q}\right|$ and $\dim_{\mathbb{Q}} H^1_{\varepsilon, \tau}(\Gamma_{F^+, T}, \text{ad}^0(\tau)(1))$ (defined as in [CHT08] §2.3); note that the latter cohomology group is the usual $H^1(\Gamma_{F^+, T}, \text{ad}^0(\tau)(1))$ since “$S = T$” in the notation of op. cit.). The proof of [Tho12] Prop. 4.4 remains valid, and thus for each $N \geq 1$ we can find a tuple $(Q_N, \bar{Q}_N, \{\bar{v}_v, \bar{v}'_v\}_{v \in Q_N})$ such that

- $Q_N$ is a finite set of places of $F^+$ which split in $F$;
- $|Q_N| = q$;
- $Q_N$ is disjoint from $T$;
- $Q_N$ consists of a single place $\bar{v}$ of $F$ above each place $v$ of $Q_N$;
- if $v \in Q_N$ then $N(v) \equiv 1 \pmod{p^N}$; and
- if $v \in Q_N$, then $\text{BC}(\bar{\tau})(\mathcal{O}_{F_v} \cong \bar{\psi}_v \oplus \bar{\psi}'_v$ where $\bar{\psi}_v$ and $\bar{\psi}'_v$ are distinct unramified characters.

For $v \in Q_N$. We let $R_{F_v}^{\square}$ denote the quotient of $R_{F_v}^{\square}$ corresponding to lifts $\Gamma_{F_v} \rightarrow \text{GL}_2(R)$ of $\text{BC}(\bar{\tau})(\mathcal{O}_{F_v})$ which are $1 + \text{Mat}_{2 \times 2}(\mathfrak{m}_{R'})$-conjugate to a lift of the form $\psi \oplus \psi'$, where $\psi$ lifts $\bar{\psi}_v$, $\psi'$
7.2.3. Let \( \psi \) lifts \( \bar{\psi}_v \), and \( \psi' \) is unramified. We then obtain a deformation problem

\[
S_{Q_N} \overset{\text{def}}{=} \left( F/F^+, T \cup Q_N, \bar{T} \cup \bar{Q}_N, \mathcal{O}, \tau, v^{-1}, \{ R_{v, \psi} \}_{v \in \Sigma_p^+} \cup \{ R_{v, \psi'} \}_{v \in Q_N} \right).
\]

From this data we obtain the associated universal (resp. \( T \)-framed) deformation ring \( R_{S_{Q_N}}^{\text{univ}} \) (resp. \( R_{S_{Q_N}}^{\text{loc}} \)) of type \( S_{Q_N} \). By [Tho12 Prop. 4.4], the ring \( R_{S_{Q_N}}^{\text{loc}} \) can be topologically generated over \( R_{\text{loc}} \) by \( q - [F^+ : \mathbb{Q}] \) elements.

7.2.2. We now shrink the subgroup \( K \) at places in \( Q_N \). For \( w \) a place of \( F \), denote by \( K_0(w) \) and \( K_1(w) \) the following subgroups:

\[
K_0(w) \overset{\text{def}}{=} \{ g \in \mathbf{GL}_2(\mathcal{O}_w) : g \equiv (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \pmod{w} \} \\
K_1(w) \overset{\text{def}}{=} \ker(K_0(w) \to \mathbf{F}_w^\times(p))
\]

where \( \mathbf{F}_w^\times(p) \) denotes the maximal \( p \)-power order quotient of \( \mathbf{F}_w^\times \), and the map in question sends \( (a, b) \) to the image of \( d \) (mod \( w \)) in \( \mathbf{F}_w^\times(p) \). For \( i = 0, 1 \), define

\[
K_i(Q_N)_m \overset{\text{def}}{=} K_m^{Q_N} \cdot \prod_{v \in Q_N} \nu_v^{-1}(K_i(\bar{v})).
\]

7.2.3. Let \( \mathbb{T}^{T \cup Q_N} \subset \mathbb{T}^T \) denote the universal Hecke algebra away from \( T \cup Q_N \), and define \( m_{Q_N} \overset{\text{def}}{=} m_T \cap \mathbb{T}^{T \cup Q_N} \). As in [CEG+16 §2.6] (which is based on [Tho12 Prop. 5.9]), we have a projection operator

\[
\text{pr} \in \text{End}_0 \left( S_G(K_i(Q_N)_m, \mathcal{O}/\mathcal{O}^r)_{m_{Q_N}} \right)
\]

for \( i = 0, 1 \). This operator induces an isomorphism

\[
\text{pr} : S_G(K_m, \mathcal{O}/\mathcal{O}^r)_m \overset{\sim}{\longrightarrow} \text{pr} \left( S_G(K_0(Q_N)_m, \mathcal{O}/\mathcal{O}^r)_{m_{Q_N}} \right),
\]

which commutes with the action of \( G(\mathcal{O}_{F^+, p}) \).

7.2.4. Define

\[
\Gamma_m \overset{\text{def}}{=} \prod_{v \in \Sigma_p^+} G(\mathcal{O}_{F_v^+, \mathcal{O}^m_v})
\]

and

\[
\Delta_{Q_N} \overset{\text{def}}{=} K_0(Q_N)_m / K_1(Q_N)_m \cong \prod_{v \in Q_N} K_0(\bar{v}) / K_1(\bar{v}),
\]

a finite \( p \)-group.

The space \( \text{pr}(S_G(K_1(Q_N)_m, \mathcal{O}/\mathcal{O}^r)_{m_{Q_N}}) \) has commuting actions of \( \Gamma_m \) and \( \Delta_{Q_N} \), under which it becomes a projective \( (\mathcal{O}/\mathcal{O}^r)[\Delta_{Q_N}][\Gamma_m] \)-module (this follows from [CHT08 Lem. 3.3.1]). From this, we obtain \( \Gamma_m \)-equivariant isomorphisms

\[
\text{pr} \left( S_G(K_1(Q_N)_m, \mathcal{O}/\mathcal{O}^r)_{m_{Q_N}} \right)^{\Delta_{Q_N}} \cong \text{pr} \left( S_G(K_0(Q_N)_m, \mathcal{O}/\mathcal{O}^r)_{m_{Q_N}} \right) \\
\cong S_G(K_m, \mathcal{O}/\mathcal{O}^r)_m
\]

where the last isomorphism follows from the previous subsection.
7.2.5. Let \( T_{0,1}^{\mathbb{Q}^N}(K_i(Q_N)_m, \mathcal{O}/\mathfrak{w}^r)_m \mathbb{Q}_N \) denote the image of \( T_{0,1}^{\mathbb{Q}^N} \) in \( \text{End}_0(\text{pr}(S_G(K_i(Q_N)_m, \mathcal{O}/\mathfrak{w}^r)_m \mathbb{Q}_N)) \). (This is the mod \( \mathfrak{w}^r \) reduction of the image of \( T_{0,1}^{\mathbb{Q}^N}(K_i(Q_N)_m) \) in \( \text{End}_0(\text{pr}(S_G(K_i(Q_N)_m, \mathcal{O})_m \mathbb{Q}_N)) \).) We let \( r_{\mathbb{Q}_N}^{\text{pr}} : \Gamma_F \rightarrow \mathbb{G}_2 \left( T_{0,1}^{\mathbb{Q}^N}(K_i(Q_N)_m, \mathcal{O}/\mathfrak{w}^r)_m \mathbb{Q}_N \right) \) denote the Galois representation obtained by pushing forward the representation of Theorem 6.2 to \( T_{0,1}^{\mathbb{Q}^N}(K_i(Q_N)_m, \mathcal{O}/\mathfrak{w}^r)_m \mathbb{Q}_N \). Using the construction of \( r_{\mathbb{Q}_N}^{\text{pr}} \), the local/global compatibility statements of Theorem 6.1 and the properties of the auxiliary primes (along with [Tho12, Prop. 5.12]), we see that \( r_{\mathbb{Q}_N}^{\text{pr}} \) is a lift of type \( \mathcal{S}_N \). In particular, \( \text{pr}(S_G(K_i(Q_N)_m, \mathcal{O}/\mathfrak{w}^r)_m \mathbb{Q}_N) \) is a finite \( R_{\mathcal{S}_N}^{\text{univ}} \)-module.

7.2.6. We identify the group \( \Delta_{\mathbb{Q}_N} \) with the image of \( \prod_{v \in \mathbb{Q}_N} I_{F_v} \) in the maximal abelian \( p \)-power order quotient of \( \prod_{v \in \mathbb{Q}_N} \Gamma_{F_v} \). This gives rise to a homomorphism \( \Delta_{\mathbb{Q}_N} \rightarrow R_{\mathcal{S}_N}^{\text{univ}, \times} \) as follows: let \( R_{\mathcal{S}_N}^{\text{univ}} \) denote any choice of universal deformation, and consider the map

\[
\prod_{v \in \mathbb{Q}_N} \Gamma_{F_v} \xrightarrow{r_{\mathcal{S}_N}^{\text{univ}}} \prod_{v \in \mathbb{Q}_N} \text{GL}_2(R_{\mathcal{S}_N}^{\text{univ}}) \xrightarrow{\text{det}} R_{\mathcal{S}_N}^{\text{univ}, \times}.
\]

Thus, we obtain morphisms \( \mathcal{O}[\Delta_{\mathbb{Q}_N}] \rightarrow R_{\mathcal{S}_N}^{\text{univ}} \rightarrow R_{\mathcal{S}_N}^{\mathbb{C}^T} \). This gives an induced \( \mathcal{O}[\Delta_{\mathbb{Q}_N}] \)-module structure on \( \text{pr}(S_G(K_i(Q_N)_m, \mathcal{O}/\mathfrak{w}^r)_m \mathbb{Q}_N) \), which agrees with the action of \( \Delta_{\mathbb{Q}_N} \) via diamond operators. These morphisms also lead to natural isomorphisms

\[
R_{\mathcal{S}_N}^{\text{univ}}/a_{\mathbb{Q}_N} \cong R_{\mathbb{S}_N}^{\text{univ}} \quad \text{and} \quad R_{\mathcal{S}_N}^{\mathbb{C}^T}/a_{\mathbb{Q}_N} \cong R_{\mathbb{S}_N}^{\mathbb{C}^T},
\]

where \( a_{\mathbb{Q}_N} \) denotes the augmentation ideal of \( \mathcal{O}[\Delta_{\mathbb{Q}_N}] \) (this follows again from [Tho12, Prop. 5.12]; cf. [GK14, §4.3.7]).

7.2.7. For each \( N \), we choose a lift \( r_{\mathcal{S}_N}^{\text{univ}} \) representing the universal deformation of type \( \mathcal{S}_N \), with \( r_{\mathcal{S}_N}^{\text{univ}} \pmod{a_{\mathbb{Q}_N}} = r_{\mathbb{S}_N}^{\text{univ}} \). The choice of \( r_{\mathcal{S}_N}^{\text{univ}} \) gives an isomorphism \( R_{\mathcal{S}_N}^{\mathbb{C}^T} \cong R_{\mathcal{S}_N}^{\text{univ}} \otimes_{\mathcal{O}} \mathcal{T} \), which reduces modulo \( a_{\mathbb{Q}_N} \) to the isomorphism \( R_{\mathbb{S}_N}^{\mathbb{C}^T} \cong R_{\mathbb{S}_N}^{\text{univ}} \otimes_{\mathcal{O}} \mathcal{T} \).

7.3. Patching.

7.3.1. Let \( q \) be as in Section 7.2 and define

\[
\Delta_{\infty} \overset{\text{def}}{=} \mathbb{Z}_p^q, \\
S_{\infty} \overset{\text{def}}{=} I[\Delta_{\infty}] \cong \mathcal{O}[z_1, \ldots, z_{4|\mathcal{T}|}, y_1, \ldots, y_q] \\
R_{\infty} \overset{\text{def}}{=} R^{\text{loc}}[x_1, \ldots, x_q-[F^+:\mathbb{Q}]] \\
R_{\lambda^r,\infty} \overset{\text{def}}{=} R^{\text{loc}}[x_1, \ldots, x_q-[F^+:\mathbb{Q}]]
\]

We let \( a \subseteq S_{\infty} \) denote the augmentation ideal of \( S_{\infty} \). For each \( N \geq 1 \), fix a surjection \( \Delta_{\infty} \twoheadrightarrow \Delta_{\mathbb{Q}_N} \); passing to completed group algebras, we get a surjective map \( S_{\infty} = I[\Delta_{\infty}] \twoheadrightarrow I[\Delta_{\mathbb{Q}_N}] \). We view \( R_{\mathcal{S}_N}^{\mathbb{C}^T} \) as an \( S_{\infty} \)-algebra via \( S_{\infty} \twoheadrightarrow I[\Delta_{\mathbb{Q}_N}] \rightarrow R_{\mathcal{S}_N}^{\mathbb{C}^T} \), which gives \( R_{\mathcal{S}_N}^{\mathbb{C}^T}/a \cong R_{\mathbb{S}_N}^{\text{univ}} \).

Recall that \( R_{\mathcal{S}_N}^{\mathbb{C}^T} \) can be topologically generated over \( R^{\text{loc}} \) by \( q-[F^+:\mathbb{Q}] \) variables. Therefore, we can choose a surjection of \( R^{\text{loc}} \)-algebras

\[
R_{\infty} \twoheadrightarrow R_{\mathcal{S}_N}^{\mathbb{C}^T}.
\]
7.3.2. We may now proceed exactly as in [CEG+16 §2.8] and patch together (certain quotients of) the spaces

\[(7.3.1) \quad \text{pr} \left( S_G(K_i(Q_N)_{2N}, \mathcal{O}/\mathcal{W}^N)_{mQ_N} \right) \overset{\vee}{\otimes} R_{S\mathcal{Q}_N}^{\cont}, \]

where $\vee$ denotes the Pontryagin dual. (In our setup, we are omitting the Hecke operators at $v_1$, and we ignore the maps $\alpha_N$ of op. cit.) Thus, we obtain a profinite topological $S_\infty[G(O_{F^+,p})]$-module $M_\infty$, with a commuting action of $R_\infty$. Furthermore, $M_\infty$ enjoys the following properties:

- $S_\infty$ acts on (7.3.1) via the map $S_\infty \to T[A\mathcal{Q}_N] \to R_{S\mathcal{Q}_N}^{\cont}$ of Subsection 7.3.1 and this action factors through an $\mathcal{O}$-algebra morphism $S_\infty \to R_\infty$. Since the image of $S_\infty$ in $\text{End}_{S_\infty}(M_\infty)$ is closed, this implies we may factor the action of $S_\infty$ on $M_\infty$ through an $\mathcal{O}$-algebra morphism $S_\infty \to R_\infty$.
- The argument at the bottom of p. 29 of [CEG+16] implies that $M_\infty$ is a finite $S_\infty[G(O_{F^+,p})]$-module, and thus it is a finite $R_\infty[G(O_{F^+,p})]$-module.
- As in [CEG+16] 2.10 Prop., $M_\infty$ is projective over $S_\infty[G(O_{F^+,p})]$.

7.3.3. Using the patched module $M_\infty$, we define a patching functor $M_{\infty}(-)$ from the category of finitely generated $\mathcal{O}$-modules with an action of $G(O_{F^+,p})$ to the category of $R_\infty$-modules by

$$M_{\infty}(W) \overset{\text{def}}{=} \text{Hom}_{G(O_{F^+,p})}^{\cont}(W, M_{\infty}^\vee).$$

By projectivity of $M_\infty$ (in the category of pseudocompact $G(O_{F^+,p})$-modules), the functor $M_{\infty}(-)$ is exact. Moreover, if $W$ is $p$-torsion free, then we have

$$M_{\infty}(W) \cong \text{Hom}_{G(O_{F^+,p})}^{\cont}(W, M_{\infty}^{\text{d}}),$$

where "d" denotes the Schikhof dual.

**Proposition 7.3.**

(i) We have a $G(O_{F^+,p})$-equivariant isomorphism

$$M_{\infty}/a \cong \left( \lim_{\rightarrow} S_G(K^p, \mathcal{O}/\mathcal{W}^n)_{m} \right) ,$$

which is compatible with the action of $\bigotimes_{v \in \Sigma_p^+} \hat{R}_v$ on both sides (the action on the right-hand side is given by the maps

$$\bigotimes_{v \in \Sigma_p^+} \hat{R}_v \rightarrow R_{S\mathcal{Q}_N}^{\cont} \rightarrow \lim_{\rightarrow} \mathbb{T}_{0,1}(K_m).$$

(ii) If $W$ is a free $\mathcal{O}$-module of finite type (resp. a free $\mathbb{F}$-module of finite type) with a continuous $G(O_{F^+,p})$-action, then $M_{\infty}(W)$ is a free $S_\infty$-module of finite type (resp. a free $S_\infty \otimes \mathbb{F}$-module of finite type).

(iii) Let $\lambda \in (Z_+^2)^{\vee}$ and let $\tau' = \{ \tau'_v \}_{v \in \Sigma_p^+}$ be a collection of tame inertial types satisfying $\tau'_v \sim \{ \tau'_v \}_{v \in \Sigma_p^+}$. Let $\sigma = \bigotimes_{v \in \Sigma_p^+} \sigma(\tau'_v)$ and $W \overset{\text{def}}{=} W_\lambda \otimes \sigma^\circ$. Then we have an isomorphism

$$M_{\infty}(W)/a \cong S_G(K, W^{\text{d}}),$$

compatible with the surjection $R_{\infty}/a \rightarrow R_{S\mathcal{Q}_N}^{\cont}$ (note that $R_{S\mathcal{Q}_N}^{\cont}$ acts on the right hand side via $R_{S\mathcal{Q}_N}^{\cont} \rightarrow \mathbb{T}_{\lambda,\tau}(K_m)$).
Suppose \( V \) is a Serre weight. Then we have an isomorphism
\[
M_\infty(V)/a \cong S_G(K, V^\vee_{m'})
\]
compatible with the surjection \( R_\infty/a \rightarrow R^\text{univ}_S \) (and the action on the right hand side is obtained as in item (iii)).

Let \( \lambda \in (\mathbb{Z}_p^2)^\hat{p} \) and let \( \tau'_v = \{ \tau'_v \}_{v \in \Sigma_p^+} \) be a collection of tame inertial types satisfying \( (\tau'_v)^{-1} \cdot (p^\frac{1}{p}\mathbb{Z}_p^+) \cong \tau_v^\vee \). Set \( \sigma \triangleq \bigotimes_{v \in \Sigma_p^+} \sigma(\tau'_v) \). Then the \( R_\infty \)-action on \( M_\infty(W_\lambda^\vee \otimes_\mathbb{Q}_p \sigma) \) factors through \( R_{\lambda, \tau'_v, \infty} \). Further, if \( M_\infty(W_\lambda^\vee \otimes_\mathbb{Q}_p \sigma) \neq 0 \), then it is maximal Cohen-Macaulay over \( R_{\lambda, \tau'_v, \infty} \), and the support of \( M_\infty(W_\lambda^\vee \otimes_\mathbb{Q}_p \sigma) \) is a union of components of \( \text{Spec}R_{\lambda, \tau'_v, \infty} \).

Let \( \overline{R}_{\lambda, \tau'_v, \infty} \) denote the quotient of \( R_{\lambda, \tau'_v, \infty} \) which acts faithfully on \( M_\infty(W_\lambda^\vee \otimes_\mathbb{Q}_p \sigma) \). Then \( M_\infty(W_\lambda^\vee \otimes_\mathbb{Q}_p \sigma)[1/p] \) is locally free of rank 2 over \( \overline{R}_{\lambda, \tau'_v, \infty}[1/p] \).

Let \( V \) be a Serre weight with highest weight \( \lambda \in (\mathbb{Z}_p^2)^{\hat{p}} \subseteq (\mathbb{Z}_p^2)^{\hat{p}} \). Then \( M_\infty(V) \neq 0 \) if and only if \( \pi \) is modular of weight \( V \). In this case, the \( R_\infty \)-action on \( M_\infty(V) \) factors through \( R_{\lambda,1, \infty} \otimes_\mathbb{Q}_p \mathbb{F} \) and \( M_\infty(V) \) is maximal Cohen-Macaulay over \( R_{\lambda,1, \infty} \otimes_\mathbb{Q}_p \mathbb{F} \).

Proof. (i) This follows from the patching construction (see [CEG+16, §2.8]). The argument of 2.11 Corollary of op. cit. shows \( \bigotimes_{v \in \Sigma_p^+} \overline{R}_v \)-equivariance.

(ii) The module \( M_\infty \) is a finite projective \( S_\infty[\mathcal{G}(\mathcal{O}_{F^+, p})] \)-module. If \( W \) is \( p \)-torsion free, then the proof of [CEG+16, 4.18 Lem.] implies that \( M_\infty(W) \) is a finite free \( S_\infty \)-module. The same argument applies when \( W \) is a free \( \mathbb{F} \)-module of finite type.

(iii) Using part (i), we have
\[
(M_\infty(W)/a)^d \cong \text{Hom}_{\mathcal{G}(\mathcal{O}_{F^+, p})}^\text{cts}(W, M_\infty^d)[a]
\]
\[
\cong \text{Hom}_{\mathcal{G}(\mathcal{O}_{F^+, p})}^\text{cts}(W, (M_\infty/a)^d)
\]
\[
\cong S_G(K, W^d)_m.
\]

The statement about the action of the deformation ring follows in a manner analogous to the proof of [HLM17, Thm. 5.2.1(iii)].

(iv) This follows by applying the previous point to a free \( \mathcal{O} \)-module whose reduction mod \( p \) is \( V \), and reducing mod \( p \).

(v) This follows exactly as the proof of [CEG+16, 4.18 Lem.], using [BG19, Thm. 3.3.7] instead of [Kis18, Thm. 3.3.8]. For the precise value of the rank, it suffices to show that \( S_G(K, W_\lambda \otimes_\mathbb{Q}_p \sigma_{0, \lambda})[1/p] \) is locally free over \( \mathbb{T}^T_{\lambda, \tau'_v}(K)[1/p] \) of rank 2 (compare [GK14, §4.1.2]). A prime ideal \( \mathfrak{p} \) of \( \mathbb{T}^T_{\lambda, \tau'_v}(K)[1/p] \) corresponds to a Hecke eigensystem \( \lambda_\mathfrak{p} : \mathbb{T}^T_{\lambda, \tau'_v}(K)[1/p] \rightarrow \mathbb{E} \), and therefore we obtain
\[
(S_G(K, W_\lambda \otimes_\mathbb{Q}_p \sigma_{0, \lambda})[1/p])_\mathfrak{p} \otimes_{E, a} \mathbb{C} \cong \bigoplus_{\pi_\infty \in \Pi_{W_\lambda}, \lambda_\mathfrak{p} = \pi_\infty} m(\pi) \text{Hom}_{\mathcal{G}(\mathcal{O}_{F^+, p})}^\text{cts}(\sigma_\mathfrak{p} \otimes_\mathbb{C} \mathbb{C}_p) \otimes_{\mathbb{C}} (\pi_\infty \circ \mathbb{F})^K_{\mathfrak{p}},
\]
where \( \lambda_\mathfrak{p} : \mathbb{T}^T_{\lambda, \tau'_v}(K)[1/p] \rightarrow \mathbb{C} \) is the Hecke eigensystem corresponding to \( \pi_\mathfrak{p} \). Since the base change map is injective on \( L \)-packets, strong multiplicity one for \( \mathcal{G}_p \) implies that there is at most one \( L \)-packet contributing to the direct sum above. Further, the base change map is determined by local base changes of local \( L \)-packets. We have that the condition \( \pi_\mathfrak{p}^{\nu} \neq 0 \) for \( \nu \in \Sigma_p^+ \) inert in \( F \) determines a unique member of the local \( L \)-packet at \( v \), and the condition \( \text{Hom}_{\mathcal{G}(\mathcal{O}_{F^+, p})}^\text{cts}(\sigma_\mathfrak{p} \otimes_\mathbb{C} \mathbb{C}_p) \neq 0 \) along with the multiplicity one property of Theorem 4.11 also determines a unique member of the local \( L \)-packet at \( v \in \Sigma_p^+ \). Therefore, there is at most one
automorphic representation $\pi$ contributing to the direct sum above. For this $\pi$, we know that $r_\pi(\pi)$ is irreducible (being a lift of $BC'(\pi)$), and therefore the base change of $\pi$ to $\text{GL}_2(\mathbb{A}_F)$ is cuspidal. This implies that the $L$-packet $\text{JL}(\pi)$ on $G^+_*({\mathbb{A}_F})$ is stable, and therefore $m(\pi^*) = 1$ for any $\pi^* \in \text{JL}(\pi)$ by [Rog90], Thm. 11.5.1(c). Using an analog of the relation “$n(\pi) = n(\tau) \prod_v c(\pi_v)$” in [LL79, p. 781], we obtain $m(\pi) = 1$ (see also [KMSW], Thm. 1.7.1). To conclude, we note that $\dim \text{Hom}_G(\sigma^\circ \otimes \mathcal{O}_F, \pi) = 1$ by Theorem 4.11 and $\dim (\pi^\infty \circ K^\circ) = 2$ by [Tay06], Lem. 1.6(2)) (since we have omitted Hecke operators at $v_1$).

(vi) Let $V$ be a Serre weight. By point (iv) and Nakayama’s lemma, $M_\infty(V) \neq 0$ if and only if $\overline{\tau}$ is modular of weight $V$ and level $K$. Therefore, in order to conclude it suffices to show that if $\overline{\tau}$ is modular of weight $V$, then $\overline{\tau}$ is modular of weight $V$ and level $K$. This follows from Proposition 7.2: in that proof, if we choose $\sigma$ so that $\text{JH}(\sigma) \cap \text{W}^\circ(\overline{\tau}) = \{ V \}$, then Theorem 6.7 and exactness of the functor of algebraic automorphic forms guarantees that the $V'$ appearing at the end of the proof is equal to $V$.

The claim about $M_\infty(V)$ being maximal Cohen–Macaulay follows exactly as in the previous point.

\[ \square \]

### 7.4. Weight Existence.

**Theorem 7.4.** Let $\overline{\tau} : \Gamma_{F^+} \rightarrow \mathfrak{g}_2(\mathbb{F})$ be a continuous representation such that

- $\nu(\overline{\tau}) = \overline{\varepsilon}^{-1}$
- $\overline{\tau}^{-1}(\text{GL}_2(\mathbb{F}) \times \mathbb{G}_m(\mathbb{F})) = \Gamma_F$.
- $\text{BC}'(\pi)(\Gamma_F) \supseteq \text{GL}_2(\mathbb{F}')$ for some subfield $\mathbb{F}' \subseteq \mathbb{F}$ with $|\mathbb{F}'| > 6$.
- $\overline{\tau}$ is modular.
- $\overline{\tau}|_{\Gamma^+_{F'}}$ is tamely ramified and 4-generic for all $\nu \in \Sigma_\mathbb{F}^+$.
- $\overline{\tau}$ is unramified outside $\Sigma_\mathbb{F}^+$.
- $\mathbb{F}^{\text{ker(ad}(\pi))}$ does not contain $F(\zeta_p)$.

Then

$$W^\circ(\overline{\tau}) \subseteq W_{\text{mod}}(\overline{\tau}).$$

**Proof.** Let $V \in W^\circ(\overline{\tau})$ and $V' \in W_{\text{mod}}(\overline{\tau})$. We will prove that $e(M_\infty(V)) = 2$ by induction on $d \overset{\text{def}}{=} \text{dgr}(V, V') = \sum_{\nu \in \Sigma_\mathbb{F}^+} \text{dgr}(V_\nu, V'_\nu)$. We write $e(M_\infty(V))$ to denote $d!$ times the coefficient of degree $d$ of the Hilbert–Samuel polynomial of $M_\infty(V)$ as a module over $R_\infty/\text{Ann}_{R_\infty}(M_\infty(V))$, where $d$ denotes the Krull dimension of $R_\infty/\text{Ann}_{R_\infty}(M_\infty(V))$.

By Lemma 3.23 there exists a tame $U_2(\mathcal{O}_{F^+, \mathbb{F}})$-type $\sigma = \bigotimes_{\nu \in \Sigma_\mathbb{F}^+} \sigma_\nu$ such that:

- (i) $V, V' \in \text{JH}(\sigma)$;
- (ii) $|\text{JH}(\sigma) \cap \text{W}^\circ(\overline{\tau})| = 2^d$;
- (iii) for any $V'' \in \text{JH}(\sigma) \cap W^\circ(\overline{\tau})$ satisfying $V'' \neq V$ one has $\text{dgr}(V'', V') < \text{dgr}(V, V')$.

We define $\tau'_\nu$ to be the tame principal series type such that $\sigma_\nu \cong \sigma(\tau'_\nu)$. We note that in this case, we have isomorphisms

$$\left( \prod_{\nu \in \Sigma_\mathbb{F}^+} R_{\tau'_\nu} \right) \left[ x_1, \ldots, x_{q-|F^+:\mathbb{Q}|} \right] \cong R_{0, \tau'_\nu, \infty} = \overline{R}_{0, \tau'_\nu, \infty},$$

(the last equality follows from Proposition 7.3) and the fact that each $R_{\tau'_\nu}$ is integral, cf. Table 3.

We thus have

$$2(2^d - 1) + e(M_\infty(V)) = \sum_{V'' \in \text{JH}(\sigma) \cap \text{W}^\circ(\overline{\tau})} e(M_\infty(V'')) = e(M_\infty(\sigma^\circ))$$
The first equality follows from the inductive hypothesis and item \( \text{(iii)} \). For the second, we note that \( M_\infty(-) \) is exact, and if \( V'' \) is a Serre weight such that \( V'' \notin W'(\overline{\pi}) \), then Theorem \( 6.7 \) and Proposition \( 7.3(\text{vi}) \) imply \( M_\infty(V'') = 0 \). For the third we use Proposition \( 7.3(\text{v}) \) above, and the fourth follows by Corollary \( 5.24 \). Hence, we obtain \( e(M_\infty(V)) = 2 \), and in particular \( M_\infty(V) \neq 0 \). Thus \( V \in W_{\text{mod}}(\overline{\pi}) \) by Proposition \( 7.3(\text{vi}) \). □

Combining Theorems \( 6.7 \) and \( 7.4 \) along with the isomorphism in Subsection \( 2.4 \) we obtain the following.

**Corollary 7.5.** Let \( \tau : \Gamma_{F^+} \rightarrow GL_2(\mathbb{F}) \) be a continuous \( L \)-parameter such that
- \( \bar{\gamma} \circ \overline{\pi} = \overline{\pi} \)
- \( \tau^{-1}(GL_2(\mathbb{F}) \times G_m(\mathbb{F})) = \Gamma_F \);
- \( BC(\tau)(\Gamma_F) \supseteq GL_2(\mathbb{F}') \) for some subfield \( \mathbb{F}' \subseteq \mathbb{F} \) with \( |\mathbb{F}'| > 6 \);
- \( \overline{\pi} \) is modular;
- \( \overline{\rho}_{\Gamma_{F_v}}^+ \) is tamely ramified and \( 4 \)-generic for all \( v \in \Sigma_p^+ \);
- \( \overline{\rho} \) is unramified outside \( \Sigma_p^+ \);
- \( \overline{F}_{\text{ker}(\text{ad}(\overline{\pi}))} \) does not contain \( F(\zeta_p) \).

Then
\[
W^2(\overline{\pi}) = W_{\text{mod}}(\overline{\pi}).
\]

### 7.5. Automorphy lifting

We now discuss our other main global application.

**Definition 7.6.** Let \( F/F^+ \) and \( G \) be as in Subsection \( 6.1 \), and suppose \( r' : \Gamma_F \rightarrow GL_2(E) \) is a continuous Galois representation. We say \( r' \) is **automorphic** if there exists a cuspidal automorphic representation \( \pi \) of \( GL_2(\mathbb{A}_{F^+}) \) such that \( r' \otimes E \pi \cong r_\pi \), where \( r_\pi \) is as in Theorem \( 6.1 \).

**Theorem 7.7.** Let \( F/F^+ \) and \( G \) be as in Subsection \( 6.1 \). Let \( r' : \Gamma_F \rightarrow GL_2(\mathbb{O}) \) be a Galois representation and let \( \overline{\pi} : \Gamma_F \rightarrow GL_2(\mathbb{F}) \) denote the associated residual representation. Assume that
- \( r' \) is unramified at all but finitely many places;
- we have \( r'^c \cong r'^{\nu} \otimes \varepsilon^{-1} \);
- for all \( \kappa \in \hat{I}_p \), the local representation \( r'|_{\Gamma_{F_v}(\kappa)} \) is potentially crystalline, with \( H_{K}(r'|_{\Gamma_{F_v}(\kappa)}) = \{1, 0\} \) and \( 4 \)-generic tame inertial type \( \tau_v^{(\kappa)} \);
- \( \overline{\pi} \) is unramified outside \( \Sigma_p \);
- for all \( v \in \Sigma_p \), the local representation \( \overline{\pi}|_{\Gamma_{F_v}} \) is tamely ramified and \( 4 \)-generic;
- \( \overline{\pi} \cong \overline{r_\pi(\pi)} \) where \( \pi \) is a cuspidal automorphic representation of \( GL_2(\mathbb{A}_{F^+}) \) with \( \pi_\infty \) trivial and such that for all \( v \in \Sigma_p^+ \), \( \pi_v|G(\mathbb{O}_{F_v}) \) contains the tame \( GL_2(\mathbb{O}_{F_v}) \)-representation associated to \( \tau_v^{(\kappa)} \) by the inertial Local Langlands correspondence (cf. Theorems \( 4.11 \) and \( 6.1 \));
- \( \overline{F}_{\text{ker}(\text{ad}(\overline{\pi}))} \) does not contain \( F(\zeta_p) \);
- \( \overline{\pi}(\Gamma_F) \supseteq GL_2(\mathbb{F}') \) for some subfield \( \mathbb{F}' \subseteq \mathbb{F} \) with \( |\mathbb{F}'| > 6 \).

Then \( r' \otimes \mathbb{O} \) is automorphic.

**Proof.** Let \( \Sigma_{\text{ram}} \) denote the set of prime-to-\( p \) places of \( F \) at which \( r' \) is ramified, and \( \Sigma_{\text{ram}}^+ \) the set of places of \( F^+ \) which are the restriction to \( F^+ \) of places in \( \Sigma_{\text{ram}} \). By the proof of Proposition \( 7.2 \) we can assume that \( r_\pi \) is unramified outside \( \Sigma_p \), and in particular outside \( \Sigma_{\text{ram}} \cup \Sigma_p \). We
view $r_i(\pi)$ as being valued in $GL_2(\mathcal{O})$. Let $\overline{\pi} : \Gamma_{F^+} \rightarrow S_2(\mathcal{O})$ and $\overline{r_i}(\pi) : \Gamma_{F^+} \rightarrow S_2(\mathcal{O})$ denote respectively the extensions of $r_i'$ and $r_i(\pi)$ to $\Gamma_{F^+}$ which satisfy $\nu \circ \overline{\pi} = \epsilon^{-1} = \nu \circ \overline{r_i}(\pi)$. We also let $\tilde{\pi}$ denote the reduction mod $\mathfrak{p}$ of $\overline{\pi}$ or $\overline{r_i}(\pi)$. For $v \in \Sigma_{ram}^+$ let $\tilde{R}_v$ denote the maximal reduced and $p$-torsion free quotient of the universal framed deformation ring parametrizing lifts $\rho$ of $\overline{\pi}|_{\Gamma_{F^+_v}}$ such that $\nu \circ \rho = \epsilon^{-1}$.

We let $S_{\Sigma_{ram}, r'_i}$ denote the deformation problem given by

$$S_{\Sigma_{ram}, r'_i} \overset{\text{def}}{=} \left\{ \frac{F}{F^+, \Sigma_p^+ \cup \{v_1\}} \cup \Sigma_{ram}^+, \Sigma_p \cup \{\overline{v_1}\} \cup \Sigma_{ram}^+, \emptyset, \tilde{\pi}, \epsilon^{-1}, \{\tilde{R}_v^{0, \tilde{r}'_i}\}_{v \in \Sigma_p^+} \cup \{\tilde{R}_{v_1}\} \cup \{\tilde{R}_v\}_{v \in \Sigma_{ram}^+} \right\},$$

and let $R^{\text{univ}}_{\Sigma_{ram}, r'_i}$ denote the complete local Noetherian $O$-algebra representing the functor of deformations of $\tilde{\pi}$ of type $S_{\Sigma_{ram}, r'_i}$. By the conditions on $r'_i$ and $r_i(\pi)$, both $\overline{\pi}$ and $\overline{r_i}(\pi)$ are deformations of $\tilde{\pi}$ of type $R^{\text{univ}}_{\Sigma_{ram}, r'_i}$, and therefore the $\ker(S_2(\mathcal{O}) \rightarrow S_2(F))$-conjugacy classes of $\overline{\pi}$ and $\overline{r_i}(\pi)$ give rise to morphisms $\iota : R^{\text{univ}}_{\Sigma_{ram}, r'_i} \rightarrow \emptyset$ and $\zeta_{r'_i} : R^{\text{univ}}_{\Sigma_{ram}, r'_i} \rightarrow \emptyset$, respectively.

Now let $\sigma \overset{\text{def}}{=} \bigotimes_{v \in \Sigma_p^+} \sigma(v')$ denote the tame type associated to the collection $\tau' = \{\tau'_v\}_{v \in \Sigma_p^+}$ by Theorem 4.11 and let $\sigma^0$ denote a $G(\mathcal{O}_{F^+, p})$-stable $O$-lattice in $\sigma$. Let

$$R_{0, \tau', \Sigma_{ram}, \infty} \overset{\text{def}}{=} \left( \bigotimes_{v \in \Sigma_p^+} R^{0, \tilde{r}'_i}_{v_1} \right) \otimes_{\tilde{R}_{v_1}} \left( \bigotimes_{v \in \Sigma_{ram}^+} \tilde{R}_v \right),$$

A variant of the patching construction above (see the proof of the Sublemma of Theorem 3.6.1, specifically p. 430, in [BLGG11]) provides us with the following data:

(i) A ring $R_{0, \tau', \Sigma_{ram}, \infty}$ which a formal power series ring over $R^{\text{loc}}_{0, \tau', \Sigma_{ram}}$, together with a surjection $R_{0, \tau', \Sigma_{ram}, \infty} \twoheadrightarrow R^{\text{univ}}_{\Sigma_{ram}, r'_i}$

(ii) an $R^{\text{univ}}_{\Sigma_{ram}, r'_i}$-module $M_{\infty}(\sigma^0)$ whose support is a union of irreducible components of $\text{Spec} R_{0, \tau', \Sigma_{ram}, \infty}$; and

(iii) the mod $\mathfrak{a}$ reduction of $M_{\infty}(\sigma^0)$ is isomorphic to $S_G(K', (\sigma^0)_d)^{\text{d}}$ compatibly with the morphism $R_{0, \tau', \Sigma_{ram}, \infty}/\mathfrak{a} \rightarrow R^{\text{univ}}_{\Sigma_{ram}, r'_i} \twoheadrightarrow T_{\Sigma_{ram}^+} K'$ (here $K'$ is an appropriately chosen compact open subgroup of $G(F_{\infty, p})$).

The pullbacks of $\iota'$ and $\zeta_{r'_i}$ via the surjection of item (i) define $O$-points of $\text{Spec} R_{0, \tau', \Sigma_{ram}, \infty}$, lying in the same irreducible component, since the rings $\tilde{R}_{v_1}^{0, \tilde{r}'_i}$ and $\tilde{R}_v^{0, \tilde{r}'_i}$, for each $v \in \Sigma_p^+$, are integral domains (by Table 3, equation (5.3.2) and the choice of $v_1$) and for each $v \in \Sigma_{ram}$ the rings $\tilde{R}_v$ satisfy Property ($\ast$) of [BLGG11] §3.5 (by virtue of Lemma 3.5.1, in op. cit.). We conclude by item (ii) that the pullbacks of $\iota'$ and $\zeta_{r'_i}$ via the surjection of item (i) are in the support of $\tilde{M}_{\infty}(\sigma^0)$. In particular the pullback of $\iota'$ lies in the support of $\tilde{M}_{\infty}(\rho)$, which implies that $\tau'$ is automorphic by item (iii) and standard arguments (see, e.g., the final paragraph of the proof of [Tay08] Thm. 4.1) or [Geel, p. 39).}

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