Study of $\Gamma_1(p^k)$ invariants for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$

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Abstract

We compute the dimension of $\Gamma_1(p^n)$-invariants for supersingular representations $\pi(r,0,1)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$, when $r \not\equiv 0$ modulo $p-1$.

WARNING: these notes are an alpha version, and thus highly unstable. The details of the proofs (as well as simpler arguments) will be added as soon possible.

1. Introduction and notations

The aim of this note is to describe in detail the $\Gamma_1(p^k)$ ($k \in \mathbf{N}_>$) invariants for supersingular representations $\pi(r,0,1)$ where $r \in \{1,\dots,p-2\}$ and $p > 2$. The main result (Theorem 4.21) is the following:

Theorem 1.1. *Let $r \in \{1,\dots,p-2\}$ and $k \in \mathbf{N}_{\geqslant 1}$. The dimension of the $\Gamma_1(p^k)$-invariants for the supersingular representation $\pi(r,0,1)$ is given by:*

$$\dim_{\overline{\mathbf{F}}_p}(\pi(r,0,1)^{\Gamma_1(p^k)}) = \begin{cases} 2(2p^{\frac{k-1}{2}}-1) & \text{if } k \text{ is odd;} \\ 2(p^{\frac{k}{2}}+p^{\frac{k-2}{2}}-2) & \text{if } k \text{ is even.} \end{cases}$$

The general strategy is completely elementary -based on the study of certain eigenspaces issued from the explicit description of $\pi(r,0,1)$- and can be outlined as follow:

o) from lemma 3.2 in [Mo] we are left to study the subspaces $\cdots \oplus_{R_k} R_{k+1}$, $\cdots \oplus_{R_{k-1}} R_k$;

i) we study the $\Gamma_1(p^k)$ invariants of R_{t-1}/R_{t-2}, for $i \in \{0,1\}$, $k+2 \geqslant t \geqslant 1$;

ii) from *i*) and left exactness of the functor $H^0(\Gamma_1(p^k),\bullet)$ we compute the spaces

$$(\cdots \oplus_{R_{t-2}} R_{t-1})^{\Gamma_1(p^k)}/(\cdots \oplus_{R_{t-4}} R_{t-3})^{\Gamma_1(p^k)}.$$

As annonced, we will not use any sophisticated arguments, the main difficulty will be painful and boring computations (as we will see, we need to distingush according to the reduction of k modulo 4).

From now onwards, we fix an integer $r \in \{1,\dots,p-2\}$.

1.1 Notations

For $t \geqslant 2$ and η a character of H we recall the $B \cap K$-equivariant isomorphism

$$\mathrm{Ind}_{K_0(p^{t-1})}^K \eta|_{B\cap K} \xrightarrow{\sim} W_{t-1,\chi}^+ \oplus W_{t-1,\chi}^-$$

for suitable subspaces $W_{t-1,\eta}^\pm$. The description of such spaces is strightforward:

Lemma 1.2. *Let $t \geqslant 2$. Then*

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i) *an* $\overline{\mathbf{F}}_p$*-base for the space* $W^+_{t-2,\eta}$ *is descrbed by*

$$x_{l_0,\dots,l_{t-2}}(e)\overset{\text{def}}{=}\sum_{\lambda_0\in\mathbf{F}_p}\lambda_0^{l_0}\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}\dots\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,e]$$

for $l_j\in\{0,\dots,p-1\}$*,* $j\in\{0,\dots,t-2\}$*.*

ii) *An* $\overline{\mathbf{F}}_p$*-base for the space* $W^-_{t-2,\eta}$ *is described by the elements*

$$x'_{l_j,\dots,l_{t-2}}(e)\overset{\text{def}}{=}\sum_{\lambda_j\in\mathbf{F}_p}\lambda_j^{l_j}\begin{bmatrix}1 & 0\\ p^j[\lambda_j] & 1\end{bmatrix}\dots\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,e]$$

where $j\in\{1,\dots,t-3\}$*,* $l_j\in\{1,\dots,p-1\}$*,* $l_m\in\{0,\dots,p-1\}$ *for* $m\in\{j+1,\dots,t-2\}$*, and the elements*

$$x'_{l_{t-2}}\overset{\text{def}}{=}\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,e],[1,e]$$

for $l_{t-2}\in\{1,\dots,p-1\}$*.*

Proof. Omissis. □

We are now in the position to describe an $\overline{\mathbf{F}}_p$-basis for R_{t-1}/R_{t-2}, where $t\geqslant 3$:

Lemma 1.3 **definition**. *Let* $t\geqslant 3$*. An* $\overline{\mathbf{F}}_p$*-basis for the* K*-representation* R_{t-1}/R_{t-2} *is descrbed by the following elements:*

i) *for* $j\in\{1,\dots,r\}$ *the elements*

$$x_{l_0,\dots,l_{t-2}}(j)\overset{\text{def}}{=}\sum_{\lambda_0\in\mathbf{F}_p}\lambda_0^{l_0}\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}\dots\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,X^{r-j}Y^j]$$

for $l_m\in\{0,\dots,p-1\}$*,* $m\in\{0,\dots,t-2\}$*;*

ii) *the elements*

$$x_{l_0,\dots,l_{t-2}}(0)\overset{\text{def}}{=}\sum_{\lambda_0\in\mathbf{F}_p}\lambda_0^{l_0}\begin{bmatrix}[\lambda_0] & 1\\ 1 & 0\end{bmatrix}\dots\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,X^r]$$

for $l_m\in\{0,\dots,p-1\}$*,* $m\in\{0,\dots,t-3\}$ *and* $l_{t-2}\in\{r+1,\dots,p-1\}$*;*

iii) *for* $j\in\{1,\dots,r\}$ *the elements*

$$x'_{l_j,\dots,l_{t-2}}(j)\overset{\text{def}}{=}\sum_{\lambda_j\in\mathbf{F}_p}\lambda_j^{l_j}\begin{bmatrix}1 & 0\\ p^j[\lambda_j] & 1\end{bmatrix}\dots\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,X^{r-j}Y^j]$$

for $l_m\in\{0,\dots,p-1\}$*,* $m\in\{1,\dots,t-2\}$*;*

iv) *the elements*

$$x'_{l_j,\dots,l_{t-2}}(0)\overset{\text{def}}{=}\sum_{\lambda_j\in\mathbf{F}_p}\lambda_j^{l_j}\begin{bmatrix}1 & 0\\ p^j[\lambda_j] & 1\end{bmatrix}\dots\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{l_{t-2}}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,X^r]$$

for $l_m\in\{0,\dots,p-1\}$*,* $m\in\{1,\dots,t-3\}$ *and* $l_{t-2}\in\{r+1,\dots,p-1\}$*;*

v) *the elements*

$$[1,X^{r-j}Y^j]$$

for $j\in\{1,\dots,r\}$*.*

For $t=2$ the description is slighty different:

Lemma 1.4 **definition**. *An $\overline{\mathbf{F}}_p$-base for R_1/R_0 is descrbed as follow:*

i) *for $j \in \{1, \dots, r\}$ the elements*

$$x_{l_0}(j) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \lambda_0^{l_0} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-j}Y^j]$$

for $l_0 \in \{0, \dots, p-1\}$;

ii) *the elements*

$$x_{l_0}(0) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \begin{bmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{bmatrix} [1, X^{r-j}Y^j]$$

for $l_0 \in \{r, \dots, p-1\}$

iii) *for $j \in \{1, \dots, r\}$ the elements*

$$[1, X^{r-j}Y^j].$$

We conclude the section with the main computationals tools for the description of the spaces $H^0(\Gamma_i(p^k), \pi(r, 0, 1))$.

Lemma 1.5. *Let $t \geqslant 3$, $j \in \{1, \dots, t-2\}$ and $z' \overset{\text{def}}{=} \sum_{n=j}^{t-2} [\lambda_n] p^n$. If $m \in \mathbf{N}$ is such that $2j+m \leqslant t-1$ then*

$$\begin{bmatrix} 1 & p^m[\mu] \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \widetilde{z'} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for suitable p-adic integers $a, b, c, d, \widetilde{z'}$ such that:

i) $a, d \equiv 1 \bmod p$ *and* $b = p^m[\mu]$;

ii) $\widetilde{z'} = \sum_{n=j}^{t-2} [\widetilde{\lambda}_n] p^n$ *where*

- *a2)* $\widetilde{\lambda}_n = \lambda_n$ *for* $n \in \{j, \dots, 2j+m-1\}$
- *b2)* $\widetilde{\lambda}_n + S_{n-1}(\widetilde{\lambda}_{n-1}) = \lambda_n$ *for* $n \in \{2j+m+1, \dots, t-2\}$ *where* $S_{n-1}(X) \in \mathbf{F}_p[X]$ *is a polynomial of degree* $p-1$ *and leading coefficient* $\lambda_{n-1} - \widetilde{\lambda}_{n-1}$;
- *c2)* $\widetilde{\lambda}_{2j+m} + \lambda_j^2 \mu = \lambda_{2j+m}$ *if* $2j+m \in \{j, \dots, t-2\}$;

iii) $c = p^{t-1}[-S_{t-2}(\widetilde{\lambda}_{t-2})] + p^t *$ *for a suitable p-adic integer $* \in \mathbf{Z}_p$ and*

- *a3)* $S_{t-2}(X) \in \mathbf{F}_p[X]$ *is a polynomial of degree $p-1$ and leading coefficient $\widetilde{\lambda}_{t-2} - \lambda_{t-2}$ if $2j+m \leqslant t-2$*
- *b3)* $S_{t-2}(X) \in \mathbf{F}_p[X]$ *is a polynomial of degree zero given by* $S_{t-2}(X) \in \mathbf{F}_p[X] = \mu\lambda_j^2$.

Proof. Postponed. □

As we will need later on, we recall the matrix equality:

$$\begin{bmatrix} 1+p^j[a] & 0 \\ 0 & 1+p^j[d] \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z'(1+p^j[a])^{-1}(1+p^j[d]) & 1 \end{bmatrix} \begin{bmatrix} 1+p^j[a] & 0 \\ 0 & 1+p^j[d] \end{bmatrix} \quad (1)$$

where $j \in \mathbf{N}_>$, $a, d \in \mathbf{F}_p$ and z is a p-adic integer.

Lemma 1.6. *Let $t \geqslant 4$. We hae the following equalities in the amalgamed sum $\cdots \oplus_{R_{t-2}} R_{t-1}$:*

i)

$$\sum_{\lambda_{t-3}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{t-3}[\lambda_{t-3}+\mu] & 1\end{bmatrix}\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{r+1}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}+P_{\mu}(\lambda_{t-3})] & 1\end{bmatrix}[1,X^r]=$$
$$=\sum_{\lambda_{t-3}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{t-3}[\lambda_{t-3}] & 1\end{bmatrix}\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{r+1}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,X^r]+$$
$$+(r+1)(-1)^{r+1}\sum_{\lambda_{t-3}\in\mathbf{F}_p}P_{-\mu}(\lambda_{t-3})[1,(\lambda_{t-3}X+Y)^r];$$

ii)

$$\sum_{\lambda_{t-3}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{t-3}[\lambda_{t-3}] & 1\end{bmatrix}\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{r+1}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}+\mu] & 1\end{bmatrix}[1,X^r]=$$
$$=\sum_{\lambda_{t-3}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{t-3}[\lambda_{t-3}] & 1\end{bmatrix}\sum_{\lambda_{t-2}\in\mathbf{F}_p}\lambda_{t-2}^{r+1}\begin{bmatrix}1 & 0\\ p^{t-2}[\lambda_{t-2}] & 1\end{bmatrix}[1,X^r]+$$
$$+(r+1)(-1)^{r+1}(-\mu)\sum_{\lambda_{t-3}\in\mathbf{F}_p}[1,(\lambda_{t-3}X+Y)^r].$$

Proof. Postponed. □

Lemma 1.7. *Let k_1,k_2 be integers such that $0\leqslant k_1\leqslant p-1$ and $1\leqslant k_2$; let V be an $\overline{\mathbf{F}}_p$-vector space with a base given by*

$$\mathscr{B}=\{v_{i,j}\,|\,0\leqslant j\leqslant k_1,\,1\leqslant i\leqslant k_2\}.$$

Assume we are given, for a fixed $\mu\in\mathbf{F}_p$, an endomorphism $\phi_\mu:V\to V$ such that

$$\phi_\mu(v_{i,j})=\sum_{n=0}^{j}\binom{j}{n}(\mu)^n v_{i+n,j-n}$$

where we adopt the convention

$$v_{k,j}\overset{\text{def}}{=}v_{\lceil k\rceil,j}$$

for any $k\in\mathbf{N}_>$, $j\in\{0,\dots,k_1\}$.

Then the endomorphism ϕ_μ has the scalar 1 as the only eigenvalue, and the associated eigenspace is

$$V^{\phi_\mu=1}=\langle v_{1,0},\dots,v_{k_2,0}\rangle_{\overline{\mathbf{F}}_p}.$$

Proof. Postponed. □

2. Study of R_{t-1}/R_{t-2}

In this section we are going to study in detail some invariant spaces of the quotients R_{t-1}/R_{t-2}. More precisely, we consider the following subgroups of K:

$$B\cap I_1=\begin{bmatrix}1+p\mathbf{Z}_p & \mathbf{Z}_p\\ 0 & 1+p\mathbf{Z}_p\end{bmatrix};K\cap U=\begin{bmatrix}1 & \mathbf{Z}_p\\ 0 & 1\end{bmatrix}.$$

The obvoius reason is that

i) $(K\cap U)\cdot K_k=\Gamma_0(p^k)$;

ii) $(B\cap I_1)\cdot K_k=\begin{bmatrix}1+p\mathbf{Z}_p & \mathbf{Z}_p\\ p^k\mathbf{Z}_p & 1+p\mathbf{Z}_p\end{bmatrix}$ is normal in $\Gamma_1(p^k)$, and the quotient is isomorphic to H.

We recall that the study of K_k-invariant has been pursued in [Mo].

2.1 Concerning the action of unipotent elements

In this section we are going to describe explicitly the invariant spaces $(R_{t-1}/R_{t-2})^{\begin{bmatrix} 1 & p^j\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$ for $j \in \mathbf{N}$, $t \geqslant 2$. The strategy will be elementary, using succesive induction on j and on the filtration defined on R_{t-1}/R_{t-2}; the main statement will be corollary 2.6, where we give a basis for $R_{t-1}/R_{t-2}^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$.

The first step is

Lemma 2.1. *Let $t \geqslant 2$, η a character of H (seen as a character of $K_0(p^{t-1})$ by inflation). Let $m \in \mathbf{N}$ be such that $t-1 \geqslant m \geqslant 0$ and define $k_0 \overset{\text{def}}{=} \frac{t-1-m}{2}$. Then an $\overline{\mathbf{F}}_p$-basis for $(\mathrm{Ind}_{K_0(p^{t-1})}^K \eta)^{\begin{bmatrix} 1 & p^m \\ 0 & 1 \end{bmatrix}}$ is described as follow:*

i) If $m \geqslant 1$, the elements $x_{l_0,\dots,l_{m-1},0,\dots,0}(e)$, with $l_j \in \{0,\dots,p-1\}$ for $j \in \{0,\dots,m-1\}$, while the element $x_{0,\dots,0}(e)$ if $m=0$;

ii) for $1 \leqslant k \leqslant k_0$ the elements

$$x'_{l_k,\dots,l_{2k+m-1},0,\dots,0}(e)$$

where $l_k \in \{1,\dots,p-1\}$, $l_j \in \{0,\dots,p-1\}$ for $k+1 \leqslant j \leqslant 2k+m-1$;

iii) for $k_0 < k \leqslant t-2$ the elements

$$x'_{l_k,\dots,l_{t-2}}(e)$$

where $l_k \in \{1,\dots,p-1\}$, $l_j \in \{0,\dots,p-1\}$ for $k+1 \leqslant j \leqslant t-2$

iv) the element $[1,e]$;

Proof. Postponed (induction on m). □

We switch now our attention to the spaces R_{t-1}/R_{t-1}. We recall that the graded piece of the filtration induced by $Fil^i(R_{t-1})$ give

$$Q(0)^{0,t-1}_{0,\dots,0,r+1}\text{—}\mathrm{Ind}_{K_0(p^{t-1})}^K \chi_r^s \mathfrak{a}\text{—}\dots\text{—}\mathrm{Ind}_{K_0(p^{t-1})}^K \chi_r^s \mathfrak{a}^r$$

The strategy to describe the invariant spaces of R_{t-1}/R_{t-2} is therefore to use lemma 2.1 and an inductive argument using the aforementioned filtration on R_{t-1}/R_{t-2}.

The result is the following:

Proposition 2.2. *Let $t \geqslant 2$, $m \in \mathbf{N}$ such that $t-1 \geqslant m \geqslant 0$; let moreover $i \in \mathbf{N}$ be such that $r-1 \geqslant i \geqslant 0$. If $k_0 \overset{\text{def}}{=} \frac{t-1-m}{2}$ an $\overline{\mathbf{F}}_p$-basis for $(R_{t-1}/\mathrm{Fil}^i(R_{t-1}))^{\begin{bmatrix} 1 & p^m\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$ is described as follow:*

i) The elements

$$x_{l_0,\dots,l_{m-1},0,\dots,0}(i+1)$$

where $l_j \in \{0,\dots,p-1\}$ for $j \in \{0,\dots,m-1\}$ (with the obivious conventions if $m=0$ or $m=t-2$).

ii) For $1 \leqslant k \leqslant k_0$, the elements

$$x'_{l_k,\dots,l_{2k+m-1},0,\dots,0}(i+1)$$

where $l_k \in \{1,\dots,p-1\}$, $l_n \in \{0,\dots,p-1\}$ *for* $n \in \{k+1,\dots,2k+m-1\}$ *(and the obvious convention that "there are no zeros" if* $k=k_0$*)*

iii) *for* $k_0 < k \leqslant t-2$ *the elements*

$$x'_{l_k\dots,l_{t-2}}(j)$$

where $j \in \{i+1,\dots,r\}$, $l_k \in \{1,\dots,p-1\}$ *and* $l_n \in \{0,\dots,p-1\}$ *for* $n \in \{k+1,\dots,t-2\}$.

iv) *the elements*

$$[1, X^{r-(i+1)}Y^{i+1}],\dots,[1,Y^r].$$

Proof. Postponed (descending induction on i, using lemma 2.1. Inside the proof we use a lemma.

Let us consider the $\overline{\mathbf{F}}_p$-subspace U of $R_{t-1}/\mathrm{Fil}^i(R_{t-1})$ generated by

a) $\mathrm{Fil}^{i+1}(R_{t-1})/\mathrm{Fil}^i(R_{t-1})$;

b) the elements $x_{l_0,\dots,l_{m-1},0\dots,0}(i+2)$ (the indices l_j satisfying the conditions of the elements *i)* in the statement of the proposition)

c) for $1 \leqslant k \leqslant k_0$ the elements $x'_{l_k,\dots,l_{2k+m-1},0,\dots,0}(i+2)$ (the indices l_j satisfying the conditions of the elements *ii)* in the statement of the proposition)

d) for $k_0 < k \leqslant t-2$ the elements $x'_{l_k,\dots,l_{t-2}}(j)$ with $j \in \{i+2,\dots,r\}$ and the indices l_j satisfying the conditions of the elements *iii)* in the statement of the proposition)

e) the elements $[1, X^{r-(i+2)}Y^{i+2}],\dots,[1,Y^r]$.

We notice that the subspace U' of U generated by the elements in *d), e)* is fixed under $\begin{bmatrix} 1 & p^m\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$; if U'' is the subspace generated by the elements in *a), b), c)* (notice also that $U = U' + U''$) we have the following lemma

Lemma 2.3. *Under the previous assumption, let* $j \in \mathbf{N}$ *be such that* $m \leqslant j \leqslant t-1$*. Then, an* $\overline{\mathbf{F}}_p$*-basis for* $U''^{\begin{bmatrix} 1 & p^j\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$ *is described as follow:*

a) *the elements*

$$x_{l_0,\dots,l_{j-1},0,\dots,0}(i+1)$$

(where the indices l_j *satisfy the conditions of the elements* i) *in the statement of the proposition);*

b) *for* $1 \leqslant n \leqslant \frac{t-1-j}{2}$ *the elements*

$$x'_{l_n,\dots,l_{2n+j-1},0,\dots,0}(i+1)$$

(where the indices l_j *satisfy the conditions of the elements* ii) *in the statement of the proposition);*

c) *for* $\frac{t-1-j}{2} < n \leqslant t-2$ *the elements*

$$x'_{l_n,\dots,l_{t-2}}(i+1)$$

(where the indices l_j *satisfy the conditions of the elements* iii) *in the statement of the proposition);*

d) *for* $\frac{t-1-1}{2} < k \leqslant \frac{t-1-m}{2}$ *the elements*

$$x'_{l_k,\dots,l_{2k+m-1},0,\dots,0}(i+2)$$

(where the indices l_j *satisfy the conditions of the elements* ii) *in the statement of the proposition);*

e) *the element* $[1, X^{r-(i+1)}Y^{i+1}]$.

Proof. Postponed. (descending induction on j) $\square$

The proposition follow applying the lemma with $j = m$. $\square$

We are now in the position to prove the key result of this section.

Proposition 2.4. *Let $t \geqslant 2$, $t-2 \geqslant m \geqslant 0$ be integers and assume $t+m > 3$. Define $k_0 \overset{\text{def}}{=} \frac{t-1-m}{2}$. An $\overline{\mathbf{F}}_p$-basis for* $(R_{t-1}/R_{t-2})^{\begin{bmatrix} 1 & p^m\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$ *is described as follow:*

i) *the elements*

$$x_{l_0,\dots,l_{m-1},0\dots,0,r+1}(0)$$

where $l_n \in 0,\dots,p-1$ for $n \in \{0,\dots,m-1\}$ (and with the obvious conventions if $m = 0$ or $m = t-2$);

ii) *for $1 \leqslant k < k_0$ the elements*

$$x'_{l_k,\dots,l_{2k+m-1},0,\dots,0,r+1}(0)$$

where $l_k \in \{1,\dots,p-1\}$, $l_n \in \{0,\dots,p-1\}$ for $n \in \{k+1,\dots,2k+m-1\}$ (if the latter is non empty; and "there ate no zeros" for $2k+m-1 = t-3$).

iii) *for $k_0 < k \leqslant t-2$ the elements*

$$x'_{l_k,\dots,l_{t-2}}(j)$$

where:

- *for $1 \leqslant j \leqslant r$, $l_k \in \{1,\dots,p-1\}$ and $l_n \in \{0,\dots,p-1\}$ where $n \in \{k+1,\dots,t-2\}$ (if non empty);*
- •• *for $j = 0$, $l_{t-2} \in \{r+1,\dots,p-1\}$, $l_k \in \{1,\dots,p-1\}$ (non empty condition only if $k < t-2$), and if $k \leqslant t-4$, $l_n \in \{0,\dots,p-1\}$ if $n \in \{k+1,\dots,t-3\}$.*

iv) *the elements*

$$[1, X^{r-1}Y], \dots, [1, Y^r]$$

v) *if $k_0 \in \mathbf{N}$, the elements*

$$x'_{l_{k_0},\dots,l^i_{t-2}(i)}$$

where $i \in \{0,1\}$, $l_{k_0} \in \{1,\dots,p-1\}$, $l^0_{t-2} \in \{r+1,\dots,p-1\}$, $l^1_{t-2} \in \{0,\dots,r\}$ and $l_n \in \{0,\dots,p-1\}$ where $n \in \{k_0+1,\dots,t-3\}$ (if non empty).

Proof. Thanks to proposition 2.2 (and a direct space decoposition as in the proof of the latter) we see that we are led to the study of the subspace U'' of R_{t-1}/R_{t-2} generated by the elements:

a) $Q^{0,t-1}_{0,\dots,0,r+1}(0)$;

b) the elements

$$x_{l_0,\dots,l_{m-1},0,\dots,0}(1)$$

for $l_n \in \{0,\dots,p-1\}$, where $n \in \{0,\dots,m-1\}$ (if non empty);

c) for $1 \leqslant k \leqslant k_0$ the elements

$$x'_{l_k,\dots,l_{2k+m-1},0,\dots,0}(1)$$

where $l_k \in \{1,\dots,p-1\}$ and $l_n \in \{0,\dots,p-1\}$ for $n \in \{k+1,\dots,2k+m-1\}$ (if non empty)

We then have the following lemma.

Lemma 2.5. *In the previous situation, consider an integer $j \in \mathbf{N}$ with $t-2 \geqslant j \geqslant m+1$, and put $j_0 \overset{\text{def}}{=} \frac{t-1-j}{2}$. An $\overline{\mathbf{F}}_p$-basis for $U''^{\left[\begin{smallmatrix} 1 & p^j\mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right]}$ is described by:*

a) *the elements*

$$x_{l_0,\dots,l_{j-1},0,\dots,0,r+1}(0)$$

where the indices l_u verify the conditions in i);

b) *for $1 \leqslant n < j_0$, the elements*

$$x'_{l_n,\dots,l_{2n+j-1},0\dots,0,r+1}(0)$$

where $l_n \in \{1,\dots,p-1\}$ and $l_u \in \{0,\dots,p-1\}$ for $u \in \{n+1,\dots,2n+j-1\}$ (if non empty);

c) *for $j_0 \leqslant n \leqslant t-2$, the elements*

$$x'_{l_n,\dots,l_{t-2}}(0)$$

where $l_{t-2} \in \{r+1,\dots,p-1\}$, $l_n \in \{1,\dots,p-1\}$ if $n < t-2$ and, for $n \leqslant t-4$, $l_u \in \{0,\dots,p-1\}$ for $u \in \{n+1,\dots,t-3\}$;

d) *for $j_0 \leqslant k \leqslant k_0$ the elements*

$$x'_{l_k,\dots,l_{2k+m-1},0,\dots,0,r+1}(1)$$

where the indices l_u verify the conditions described in the point c) above.

Proof. Induction on j. □

Lemma 2.5 enable us to establish the inductive step for the proof of the main statement. □

As a consequence, we can describe explicitly the space of $\left[\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right]$-invariants:

Corollary 2.6. *Let $t \geqslant 4$. An $\overline{\mathbf{F}}_p$-basis for $(R_{t-1}/R_{t-2})^{\left[\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right]}$ is described as follow:*

i) *the element*

$$x_{0,\dots,0,r+1}(0);$$

ii) *for $1 \leqslant k < \frac{t-1}{2}$ the elements*

$$x'_{l_k,\dots,l_{2k-1},0,\dots,0,r+1}(0)$$

where $l_k \in \{1,\dots,p-1\}$ and $l_u \in \{0,\dots,p-1\}$ for $u \in \{k+1,\dots,2k-1\}$ (if non empty);

iii) *for $\frac{t-1}{2} < k \leqslant t-2$ the elements*

$$x'_{l_k,\dots,t-2}(j)$$

where

- *for $1 \leqslant j \leqslant r$ we have $l_k \in \{1,\dots,p-1\}$ and $l_u \in \{0,\dots,p-1\}$ for $n \in \{k+1,\dots,t-2\}$ (if non empty);*

•• *for $j = 0$ we have $l_{t-2} \in \{r+1,\dots,p-1\}$, $l_k \in \{1,\dots,p-1\}$ if $k < t-2$ and, if moreover $k \leqslant t-4$, $l_u \in \{0,\dots,p-1\}$ for $u \in \{k+1,\dots,t-3\}$;*

iv) *the elements*

$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

v) *If $k_0 \overset{\text{def}}{=} \frac{t-1}{2} \in \mathbf{N}$ the elements*

$$x'_{l_{k_0},\dots,l^i_{t-2}}(i)$$

where $l_{t-2}^{(1)} \in \{0, \dots, r\}$, $l_{t-2}^{(0)} \in \{r+1, \dots, p-1\}$, $l_{k_0} \in \{1, \dots, p-1\}$ *and* $l_u \in \{0, \dot{;} p-1\}$ *for* $u \in \{k_0+1, \dots, t-3\}$ *(if non empty).*

The remaining cases $t = 3, t = 2$ can be detected by a direct computation.

Lemma 2.7. *An* $\overline{\mathbf{F}}_p$*-basis for* $(R_2/R_1)^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$ *is described as follow:*

i) *the element*
$$x_{0,r+1}(0);$$
ii) *the elements*
$$x'_{r+1}(0), \dots, x'_{p-1}(0);$$
iii) *the elements*
$$x'_{l_1}(1)$$
where $l_1 \in \{p-2, p-1, 1, \dots, \rceil r-2 \lceil\}$ *(with the obvious convention on the ordering on the set* $\{1, \dots, p-1\}$*);*

iv) *the elements*
$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

Proof. Postponed □

Lemma 2.8. *An* $\overline{\mathbf{F}}_p$*-basis for* $(R_1/R_0)^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$ *is described as follow:*

i) *the element*
$$x_r(0);$$
ii) *the elements*
$$[1, X^{r-1}Y], \dots, [1, Y^r].$$

Proof. □

3. Study of invariants in the amalgamed sum -I

The aim of this section is to describe in detail the $\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariants of the spaces $R_i/R_{i-1} \oplus_{R_{i+1}} \cdots \oplus_{R_n} R_{n+1})$, for $n \geqslant 1$ and $i \in \{0, 1\}$. The stategy is elementary and can be summed up as follow:

1) by the left exactness of the $\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-functor, it sufficies to study the spaces
$$(\cdots \oplus_{R_{t-2}} R_{t-1})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}} / (\cdots \oplus_{R_{t-4}} R_{t-3})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}};$$
2) using the properties of the amalgamed sum, we dispose of a sequence of equivariant surjections
$$\cdots \twoheadrightarrow R_{t-3}/R_{t-4} \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow R_{t-3}/\mathrm{Fil}^{r-1}(R_{t-3}) \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow R_{t-1}/R_{t-2}.$$
3) by the results in section §2.1, we can use an inductive argument on the preceeding sequences to deduce the description of the spaces in 1).

The following result is formal

Lemma 3.1. *Let $t \geqslant 2$ and let $j \in \mathbf{N}$ be an integer such that $1 \leqslant j \leqslant \frac{t-2}{2}$. We have equivariant surjections*

$$R_{t-1-2j}/R_{t-2-2j} \oplus_{R_{t-2j}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow R_{t-1-2j}/\mathrm{Fil}(R_{t-1-2j}) \oplus_{R_{t-2j}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow$$
$$\twoheadrightarrow R_{t+1-2j}/R_{t-2j} \oplus_{R_{t+2+2j}} \cdots \oplus_{R_{t-2}} R_{t-1}$$

Proof. Formal consequence of the properties of the amalgamed sum. □

In order to clarify the exposition, we are lead to treat separately the cases where t is even or odd. From now on, we fix $t \in \mathbf{N}$; in order not to overload the notations -but not to avoid confusions as well- we adopt the following convention: the (image of the) elements of R_{t-1} in the amalgamed sum will be noted by

$$x^{(\prime)}_{\dots,l_{t-2}}(i);$$

while the (image of elements) of R_{t-1-2j} (where $\frac{t-1}{2} \geqslant j \geqslant 1$) will be noted by

$$y^{(\prime)}_{\dots,l_{t-2-2j}}(i).$$

We hope this will avoid confusions without making the notations too heavy.

3.1 Analysis for t odd

We start with some introductory lemmas:

Lemma 3.2. *Let $t \geqslant 5$. Fix $j \in \mathbf{N}$ an integer with $\frac{t-2}{2} \geqslant j \geqslant 1$, and define U as the subspace of $R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j}) \oplus_{R_{t-2j}} \cdots \oplus_{R_{t-2}} R_{t-1}$ generated by:*

a) $R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j})$;

b) *the elements (images of elements in R_{t+1-2j}; we use the "y" notation, even if, for $j = 1$ we should have used the "x" notation to bo consistent to what we wrote above)*

$$y'_{l_{\frac{t+1-2j}{2}},\dots,l^1_{t-2}}(1);$$

where the indices l_u verify conventionsanalogous to v) of corollary 2.6;
for $1 \leqslant k < \frac{t+1-2j}{2}$ the elements

$$y'_{l_k,\dots,l_{2k-1},0,\dots,0,r+1}$$

where the indices l_u verify conventions analogous to ii) of corollary 2.6;
the element

$$y_{0,\dots,0,r+1}(0);$$

c) *the elements*

$$y'_{l_{\frac{t+3-2j}{2}},\dots,l_{t-1-2j},r,p-1-r,r}(1) \quad \text{(homomorphic image from } R_{t+3-2j}\text{)};$$
$$\vdots$$
$$y'_{l_{\frac{t-3}{2}},\dots,l_{t-1-2j},r,p-1-r,\dots,p-1-r,r}(1) \quad \text{(homomorphic image from } R_{t-3}\text{)};$$
$$x'_{l_{\frac{t-1}{2}},\dots,l_{t-1-2j},r,p-1-r,\dots,p-1-r,r}(1).$$

Then, the space of $\begin{bmatrix} 1 & p^m\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$*-invariants of U, for $t-1-2j \geqslant m \geqslant 1$, is described by:*

a1) *the space*

$$(R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j}))^{\begin{bmatrix} 1 & p^m\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$$

$b1)$ *the elements in $c)$, as well as the elements*

$$y'_{l_{\frac{t+1-2j}{2}},\dots,l^1_{t-2}}(1);$$

(where the indices l_u verify conventionsanalogous to $v)$ of corollary 2.6);

$c1)$ *for $\frac{t-2j-m}{2}\leqslant k<\frac{t+1-2j}{2}$ the elements*

$$y'_{l_k,\dots,l_{2k-1},0\dots,0,r+1}(0)$$

(where the indices l_u verify conventions analogous to $ii)$ of corollary 2.6).

Moreover, for $t-1-2j>\frac{t-1}{2}$, the space of $\left[\begin{array}{cc}1 & \mathbf{Z}_p\\ 0 & 1\end{array}\right]$-invariants of U is described by

$a2)$ *the space*

$$(R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j}))^{\left[\begin{array}{cc}1 & \mathbf{Z}_p\\ 0 & 1\end{array}\right]};$$

$b2)$ *the elements*

$$y'_{l_{\frac{t+1-2j}{2}},\dots,l^1_{t-2}}(1)$$

with $(l_{t-1-2j},l_{t-2j})\prec(p-1,r)$ (in addition to the usual conventions on indices l_u);

$c2)$ *the elements described in $c)$, with the extra condition $l_{t-1-2j}\neq p-1$*

Proof. Postponed. (Induction on m). □

Remark 3.3. *The second part of the statement of lemma 3.2 holds also for $t-1-2j=\frac{t-1}{2}$, where the extra condition on the elements $x'_{l_{\frac{t-1}{2}},\dots,l_{t-1-2j},r,p-1-r,\dots,p-1-r,r}(1)$ is instead $l_{k_0}\neq\rceil p-3\rceil$.*

We now state the key result of the section.

Lemma 3.4. *Let $t\geqslant 5$, put $k_0\stackrel{\text{def}}{=}\frac{t-1}{2}$ and let $j\in\mathbf{N}$ be such that $t-1-2j>k_0+1$. The space of $\left[\begin{array}{cc}1 & \mathbf{Z}_p\\ 0 & 1\end{array}\right]$-invariants inside $R_{t-1-2j}/R_{t-2-2j}\oplus\cdots\oplus_{R_{t-2}}R_{t-2}$ is described as follow:*

$i)$ *the elements*

$$x'_{l_k,\dots,l_{t-2}}(j)$$

the indices j, l_u satisfying the conventions described in $iii)$ of corollary 2.6;
the elements

$$[1,X^{r-1}Y],\dots,[1,Y^r];$$

the elements

$$x'_{l_{k_0},\dots,l^0_{t-2}}(0)$$

$ii)$ *the elements*

$$x'_{l_{k_0},\dots,l^1_{t-2}}(1)$$

where the indices l_u verify the condition of $v)$ in corollary 2.6, toghether with $(l_{t-2-2j},\dots,l_{t-2})\preceq(r,p-1-r,\dots,p-1-r,r)$; moreover such elements are invariant in $R_0\oplus_{R_1}\oplus\cdots\oplus_{R_{t-2}}R_{t-1}$ if $(l_{t-2-2j},\dots,l_{t-2})\prec(r,p-1-r,\dots,p-1-r,r)$;

iii) *elements of the form*

$$y'_{l_{\frac{t-3}{2}},\dots,l_{t-3-2j},r,p-1-r,\dots,p-1-r,r}(1)\quad \text{(homomorphic image from } R_{t-3});$$

$$\vdots$$

$$y'_{l_{\frac{t+1-2j}{2}},\dots,l_{t-3-2j},r,p-1-r,r}(1)\quad \text{(homomorphic image from } R_{t+1-2j});$$

iii) *the space*

$$(R_{t-1-2j}/R_{t-2-2j})\left[\begin{array}{cc} 1 & \mathbf{Z}_p \\ 0 & 1 \end{array}\right];$$

iv) *homomorphic image of elements inside* $(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1})\left[\begin{array}{cc} 1 & \mathbf{Z}_p \\ 0 & 1 \end{array}\right]$.

Proof. It is an induction on j, using the results in lemma 3.2 $\square$

We define, for $t \geqslant 2$ the space

$$V_{t-1} \overset{\text{def}}{=} (R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1})\left[\begin{array}{cc} 1 & \mathbf{Z}_p \\ 0 & 1 \end{array}\right]/(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-4}} R_{t-3})\left[\begin{array}{cc} 1 & \mathbf{Z}_p \\ 0 & 1 \end{array}\right].$$

To complete the description of V_{t-1} in the case t odd we have to distinguish two situations.

3.1.1 *Analysis for k_0 even.* We assume now $k_0(\overset{\text{def}}{=} \frac{t-1}{2})$ even. We therefore have to consider the chain of epimorphisms (where we assume $t \geqslant 5$)

$$\begin{aligned} R_{k_0}/R_{k_0-1} \oplus_{R_{k_0+1}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow R_{k_0}/\mathrm{Fil}^{r-1}(R_{k_0}) \oplus_{R_{k_0+1}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow \\ \twoheadrightarrow R_{k_0+2}/R_{k_0+1} \oplus_{R_{k_0+3}} \cdots \oplus_{R_{t-2}} R_{t-1}. \end{aligned}$$

Thanks to lemma 3.4 and lemma 3.2 we deduce

Proposition 3.5. *Let $t \geqslant 5$ be such that $k_0 \in 2N$. An $\overline{\mathbf{F}}_p$-basis for V_{t-1} is described by:*

a) *for $k_0 < k \leqslant t-2$ the elements*

$$x'_{l_k,\dots,l_{t-2}}(j)$$

where the indices j, l_u verify the conditions described in iii) of corollary 2.6;

b) *the elements*

$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

c) *the elements*

$$x'_{l_{k_0},\dots,l^0_{t-2}}(0)$$

where the indices l_u verify the conditions described in v) of corollary 2.6;

d) *the elements*

$$x'_{l_{k_0},\dots,l_{t-2}}(1)$$

where $l_{k_0} \in \{1, \dots, p-1\}$ and $(l_{k_0+1}, \dots, l_{t-2}) \prec (r, p-1-r, dots, p-1-r, r)$;

e) *for $l_{k_0} \in \{p-2, p-1, 1, \dots, \lceil p-3-r \rceil -1\}$ (if non empty, and with the obvious convention on the ordering on the set $\{1, \dots, p-1\}$) the elements*

$$x'_{l_{k_0},r,\dots,r}(1)$$

together with the element

$$x'_{\lceil p-3-r\rceil,r,\dots,r}(1) + c_0 y_{\lceil p-3\rceil,p-1-r,r,\dots,r}(1)$$

for a suitable constant $c_0 \in \overline{\mathbf{F}}_p$.

Proof. Postponed. $\square$

3.1.2 *Analysis for* k_0 *odd* We assume now $k_0(\overset{\text{def}}{=} \frac{t-1}{2})$ odd. We therefore have to consider the chain of epimorphisms (where we assume $t \geqslant 7$)

$$R_{k_0+1}/R_{k_0} \oplus_{R_{k_0+2}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow R_{k_0+1}/\mathrm{Fil}^{r-1}(R_{k_0+1}) \oplus_{R_{k_0+2}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow$$
$$\twoheadrightarrow R_{k_0+3}/R_{k_0+2} \oplus_{R_{k_0+4}} \cdots \oplus_{R_{t-2}} R_{t-1}.$$

Thanks to lemma 3.4 and lemma 3.2 we deduce

Proposition 3.6. *Let* $t \geqslant 5$ *be such that* $k_0 \in 2N+1$. *An* $\overline{\mathbf{F}}_p$*-basis for* V_{t-1} *is described by:*

a) *for* $k_0 < k \leqslant t-2$ *the elements*

$$x'_{l_k,\dots,l_{t-2}}(j)$$

where the indices j, l_u *verify the conditions described in* iii) *of corollary 2.6;*

b) *the elements*

$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

c) *the elements*

$$x'_{l_{k_0},\dots,l^0_{t-2}}(0)$$

where the indices l_u *verify the conditions described in* v) *of corollary 2.6;*

d) *the elements*

$$x'_{l_{k_0},\dots,l_{t-2}}(1)$$

where $l_{k_0} \in \{1,\dots,p-1\}$ *and* $(l_{k_0+1},\dots,l_{t-2}) \prec (p-1-r, r, \dots, p-1-r, r)$*;*

e) *for* $l_{k_0} \in \{p-2, p-1, 1, \dots, \lceil r-2 \rceil - 1\}$ *(if non empty, and with the obvious convention on the ordering on the set* $\{1,\dots,p-1\}$*) the elements*

$$x'_{l_{k_0},p-1-r,r,\dots,r}(1)$$

together with the element

$$x'_{\lceil r-2 \rceil,p-1-r,r,\dots,r}(1) + c_0 y_{\lceil p-3-r \rceil,r,\dots,r}(1)$$

for a suitable constant $c_0 \in \overline{\mathbf{F}}_p$.

The case $t = 3$ requires some extra care and is treated below:

Lemma 3.7. *An* $\overline{\mathbf{F}}_p$*-basis for* V_2 *is described by:*

i) *the elements*

$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

ii) *the elements*

$$x'_{r+1}(0), \dots, x'_{p-1}(0);$$

iii) *for* $l_1 \in \{p-2, p-1, 1, \dots, \lceil r-2 \rceil - 1\}$ *the elements*

$$x'_{l_1}(1)$$

and the element

$$x'_{\lceil r-2 \rceil}(1) + XY^{r-1}$$

(where $XY^{r-1} \in R_0$*)*

We are now left to count the dimensions of such spaces.

Lemma 3.8. *Let $t \geqslant 1$ be an odd integer and put $k_0 \overset{\text{def}}{=} \frac{t-1}{2}$.*
The dimension of V_{t-1} is then:

$$\dim_{\overline{\mathbf{F}}_p}(V_{t-1}) = \begin{cases} p^{k_0-1}(p-1) + (p-1)[(p-r)\frac{p^{k_0}-1}{p+1} - (p-1-r)p^{k_0-1}] + (p-1-r) & \text{if } k_0 \text{ is even} \\ p^{k_0-1}(p-1) + (p-1)(r+1)\frac{p^{k_0-1}-1}{p+1} + r & \text{if } k_0 \text{ is odd} \end{cases}$$

for $t \geqslant 3$ and

$$\dim_{\overline{\mathbf{F}}_p}(V_0) = 1$$

The dimension of $\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariants of $R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1}$ is given by:

$$\dim_{\overline{\mathbf{F}}_p}(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}} = \begin{cases} p^{k_0} + (r+1)\frac{p^{k_0}-1}{p+1} & \text{if } k_0 \geqslant 0 \text{ is even} \\ p + r + p(p^{k_0-1}-1) + p(r+1)\frac{p^{k_0-1}-1}{p+1} & \text{if } k_0 \text{ is odd} \end{cases}$$

Proof. Computation. ◻

3.2 Analysis for t even

In this paragraph, we fix an even integer $t \in 2\mathbf{N}$. The analysis of $\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariants for $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1}$ follows closely the arguments seen in paragraph §3.1. In particular, the proofs will mostly be left to the reader.

We recall the sequence of equivariant epimorphisms

$$(R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow (R_1/\mathrm{Fil}^{r-1}(R_1)) \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow (R_3/R_2) \oplus_{R_4} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow \dots$$
$$(R_{t-3}/\mathrm{Fil}^{r-1}(R_{t-3})) \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow R_{t-1}/R_{t-2}$$

and that, for $t \geqslant 4$, an $\overline{\mathbf{F}}_p$-basis for $(R_{t-1}/R_{t-2})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$ is described as follow:

a) the element $x_{0,\dots,0,r+1}(0)$;

b) for $1 \leqslant k \leqslant k_0'$ the elements
$$x'_{l_l,\dots,l_{2k-1},0,\dots,0,r+1}$$
with $l_k \in \{1,\dots,p-1\}$ and $l_u \in \{0,\dots,p-1\}$ for $u \in \{k+1,\dots,2k-1\}$ (if non empty);

c) for $k_0'+1 \leqslant k \leqslant t-2$ the elements
$$x'_{l_k,\dots,l_{t-2}}(j)$$
where the indices j, l_u verify the conditions of corollary 2.6-*iii*)

d) the elements
$$[1, X^{r-1}Y], \dots, [1, Y^r],$$

where we defined
$$k_0' \overset{\text{def}}{=} \frac{t-2}{2}.$$

We notice that the elements of the form *c*), *d*) are certanly invariant in the amalgamed sum (as they are homomorphic image of invariant elements of R_{t-1}).

The followng results are completely analogous to lemmas 3.2 and 3.4.

Lemma 3.9. *Let $j \in \mathbf{N}_{\geqslant 1}$. We consider the subspace U of $(R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j})) \oplus \cdots \oplus_{R_{t-2}} R_{t-1}$ generated by the following elements:*

a) $R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j})$;

b) the homomorphic image from R_{t+1-2j} of the elements [1] *for $1 \leqslant k < \frac{t-1-2j}{2}$ the elements (homomorphic image from R_{t+1-2j})*

$$y'_{l_k,\ldots,l_{2k-1},0,\ldots,0,r+1}$$

where the indices l_u verify conventions analogous to $ii)$ of corollary 2.6;
the element

$$y_{0,\ldots,0,r+1}(0)$$

(homomorphic image from R_{t+1-2j});

c) the elements

$$y'_{l_{\frac{t-2j}{2}},\ldots,l_{t-1-2j},r+1}(0) \quad \text{(homomorphic image from } R_{t+1-2j});$$

$$\vdots$$

$$y'_{l_{\frac{t-4}{2}},\ldots,l_{t-1-2j},r,p-1-r,\ldots,p-1-r,r+1}(0) \quad \text{(homomorphic image from } R_{t-3});$$

$$x'_{l_{\frac{t-2}{2}},\ldots,l_{t-1-2j},r,p-1-r,\ldots,p-1-r,r+1}(0).$$

Then, the space of $\begin{bmatrix} 1 & p^m\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariants of U, for $t-1-2j \geqslant m \geqslant 1$, is described by:

a1) the space

$$(R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j}))^{\begin{bmatrix} 1 & p^m\mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$$

b1) the elements in c);

c1) for $\frac{t-2j-m}{2} \leqslant k < \frac{t-1-2j}{2}$ the elements

$$y'_{l_k,\ldots,l_{2k-1},0\ldots,0,r+1}(0)$$

(where the indices l_u verify conventions analogous to $ii)$ of corollary 2.6).

Moreover, for $t-1-2j > \frac{t-1}{2}$, the space of $\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariants of U is described by

a2) the space

$$(R_{t-1-2j}/\mathrm{Fil}^{r-1}(R_{t-1-2j}))^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}};$$

b2) the elements described in c), with the extra condition $l_{t-1-2j} \neq p-1$

Proof. Postponed. (Induction on m). □

Remark 3.10. *The second part of the statement of lemma 3.9 holds also for $t-1-2j = \frac{t-2}{2}$, where the extra condition on the elements $x'_{l_{k'_0},\ldots,l_{t-1-2j},r,p-1-r,\ldots,p-1-r,r+1}(0)$ is instead $l_{k'_0} \neq \lceil p-3 \rceil$.*

Similarly, we have:

[1] once again we use the "y" notation, even if, for $j=1$ we should have used the "x" notation to be consistent with our notations. The same remark holds for the element $y'_{l_{\frac{t-2j}{2}},\ldots,l_{t-1-2j},r+1}(0)$ described in $c)$ below.

Lemma 3.11. *Let $t \geqslant 4$ and let $j \in \mathbf{N}_{\geqslant 1}$ be such that $t-1-2j > k_0'+1$. The space of $\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$-invariants inside $R_{t-1-2j}/R_{t-2-2j} \oplus \cdots \oplus_{R_{t-2}} R_{t-1}$ is described as follow:*

i) for $k_0 < k \leqslant t-2$ the elements

$$x'_{l_k,\dots,l_{t-2}}(j)$$

the indices j, l_u satisfying the conventions described in iii) of corollary 2.6, as well as the elements

$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

ii) the elements

$$x'_{l_{k_0'},\dots,l_{t-3},r+1}(0)$$

where the indices l_u verify the condition of ii) in corollary 2.6, toghether with $(l_{t-2-2j}, \dots, l_{t-3}) \preceq (r, p-1-r, \dots, p-1-r)$; moreover such elements are invariant in $R_1/R_0 \oplus \cdots \oplus_{R_{t-2}} R_{t-1}$ if $(l_{t-2-2j}, \dots, l_{t-3}) \prec (r, p-1-r, \dots, p-1-r)$;

iii) elements of the form

$$y'_{l_{\frac{t-4}{2}},\dots,l_{t-3-2j},r,p-1-r,\dots,p-1-r,r+1}(0) \quad \text{(homomorphic image from } R_{t-3});$$
$$\vdots$$
$$y'_{l_{\frac{t-2j}{2}},\dots,l_{t-3-2j},r,p-1-r,r+1}(0) \quad \text{(homomorphic image from } R_{t+1-2j});$$

iv) the space

$$(R_{t-1-2j}/R_{t-2-2j})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}};$$

v) homomorphic image of other suitable elements inside $(R_1/R_0 \cdots \oplus_{R_{t-2}} R_{t-1})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}$.

Proof. Postponed. □

As in section 3.1, we define, for $t \geqslant 2$ the space

$$V_{t-1} \overset{\text{def}}{=} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}} / ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_{t-4}} R_{t-3})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}}.$$

Again, to complete the description of V_{t-1} in the case t even we have to distinguish two situations.

3.2.1 *Analysis for k_0' odd.* We assume now k_0' odd. We therefore have to consider the chain of epimorphisms (where we assume $t \geqslant 4$)

$$(R_{k_0'}/R_{k_0'-1}) \oplus_{R_{k_0'+1}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow (R_{k_0'}/\mathrm{Fil}^{r-1}(R_{k_0'})) \oplus_{R_{k_0'+1}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow$$
$$\twoheadrightarrow (R_{k_0'+2}/R_{k_0'+1}) \oplus_{R_{k_0'+3}} \cdots \oplus_{R_{t-2}} R_{t-1}.$$

Thanks to lemma 3.11 and lemma 3.9 we deduce

Proposition 3.12. *Let $t \geqslant 4$ be such that k_0' is odd, and $k_0' > 1$. An $\overline{\mathbf{F}}_p$-basis for V_{t-1} is described by:*

a) for $k_0 < k \leqslant t-2$ the elements

$$x'_{l_k,\dots,l_{t-2}}(j)$$

where the indices j, l_u verify the conditions described in iii) of corollary 2.6;

b) *the elements*

$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

c) *the elements*

$$x'_{l_{k'_0},\dots,l_{t-3},r+1}(0)$$

where $l_{k'_0} \in \{1, \dots, p-1\}$ *and* $(l_{k_0+1}, \dots, l_{t-3}) \prec (r, p-1-r, \dots, p-1-r)$*;*

d) *for* $l_{k_0} \in \{p-2, p-1, 1, \dots, \lceil p-3-r\rceil - 1\}$ *(if non empty, and with the obvious convention on the ordering on the set* $\{1, \dots, p-1\}$*) the elements*

$$x'_{l_{k_0},r,\dots,p-1-r,r+1}(0)$$

together with the element

$$x'_{\lceil p-3-r\rceil,r,\dots,r+1}(0) + c_0 y_{\lceil p-3\rceil,p-1-r,r,\dots,p-1-r,r+1}(0)$$

for a suitable constant $c_0 \in \overline{\mathbf{F}}_p$*.*

Proof. Postponed. □

With some extra care, we deduce the same result for $t = 4$:

Lemma 3.13. *Let* $t = 4$*. Then an* $\overline{\mathbf{F}}_p$*-basis for* $(R_1/R_0 \oplus_{R_2} R_3)^{\left[\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right]}$ *is described by:*

a) *an* $\overline{\mathbf{F}}_p$*-basis of* $(R_1/R_0)^{\left[\begin{smallmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right]}$*;*

b) *the elements*

$$x'_{l_1,r+1}(0)$$

where $l_1 \in \{p-2, p-1, 1, \dots, \lceil p-3-r\rceil - 1\}$ *(with the obvious convention on the ordering on the set* $\{1, \dots, p-1\}$*);*

c) *the element*

$$x'_{\lceil p-3-r\rceil,r+1}(0) + c_0 x_{r+1}(0)$$

for a suitable constant $c_0 \in \mathbf{F}_p$*;*

d) *the elements*

$$x'_{l_2}(j)$$

where the indices j*,* l_2 *verify the conditions of iii) in corollary 2.6, as well as the elements*

$$[1, X^{r-1}Y], \dots, [1, Y^r].$$

Proof. Postponed. □

3.2.2 *Analysis for* k'_0 *even.* We assume now k'_0 even. We therefore have to consider the chain of epimorphisms (where we assume $t \geqslant 4$)

$$(R_{k'_0+1}/R_{k'_0}) \oplus_{R_{k'_0+2}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow (R_{k'_0+1}/\mathrm{Fil}^{r-1}(R_{k'_0+1})) \oplus_{R_{k'_0+2}} \cdots \oplus_{R_{t-2}} R_{t-1} \twoheadrightarrow$$
$$\twoheadrightarrow (R_{k'_0+3}/R_{k'_0+2}) \oplus \cdots \oplus_{R_{t-2}} R_{t-1}.$$

Thanks to lemma 3.11 and lemma 3.9 we deduce

Proposition 3.14. *Let* $t \geqslant 4$ *be such that* k'_0 *is even. An* $\overline{\mathbf{F}}_p$*-basis for* V_{t-1} *is described by:*

a) *for* $k_0 < k \leqslant t-2$ *the elements*
$$x'_{l_k,\dots,l_{t-2}}(j)$$
where the indices j, l_u *verify the conditions described in* iii) *of corollary 2.6;*

b) *the elements*
$$[1, X^{r-1}Y], \dots, [1, Y^r];$$

c) *the elements*
$$x'_{l_{k'_0},\dots,l_{t-3},r+1}(0)$$
where $l_{k'_0} \in \{1, \dots, p-1\}$ *and* $(l_{k_0+1}, \dots, l_{t-3}) \prec (p-1-r, r, \dots, p-1-r)$*;*

d) *for* $l_{k_0} \in \{p-2, p-1, 1, \dots, \lceil r-2 \rceil - 1\}$ *(if non empty, and with the obvious convention on the ordering on the set* $\{1, \dots, p-1\}$*) the elements*
$$x'_{l_{k_0},p-1-r,\dots,p-1-r,r+1}(0)$$
together with the element
$$x'_{\lceil r-2 \rceil,p-1-r,\dots,p-1-r,r+1}(0) + c_0 y_{\lceil p-3-r \rceil,r,\dots,p-1-r,r+1}(0)$$
for a suitable constant $c_0 \in \overline{\mathbf{F}}_p$*.*

Proof. Postponed. □

We are now left to count the dimensions of such spaces.

Lemma 3.15. *Let* $t \geqslant 1$ *be an even integer and put* $k'_0 \overset{\text{def}}{=} \frac{t-1}{2}$*.*
The dimension of V_{t-1} *is then:*
$$\dim_{\overline{\mathbf{F}}_p}(V_{t-1}) = \begin{cases} p^{k'_0-1}(p-1)(r+1) + (p-1)[(r+1)\frac{p^{k'_0}-1}{p+1} - rp^{k'_0-1}] + r & \text{if } k'_0 \text{ is even} \\ p^{k'_0-1}(p-1)(r+1) + (p-1)(p-r)\frac{p^{k'_0-1}-1}{p+1} + (p-1-r) & \text{if } k'_0 \text{ is odd} \end{cases}$$
for $t \geqslant 4$ *and*
$$\dim_{\overline{\mathbf{F}}_p}(V_1) = r+1$$
The dimension of $\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}$*-invariants of* $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_{t-2}} R_{t-1}$ *is given by:*
$$\dim_{\overline{\mathbf{F}}_p}(R_0 \oplus_{R_1} \cdots \oplus_{R_{t-2}} R_{t-1})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}} = \begin{cases} p^{k'_0} + r + p(r+1)\frac{p^{k'_0}-1}{p+1} & \text{if } k_0 \geqslant 0 \text{ is even} \\ \\ (p-1)(r+2) + 1 + (r+1)p^2\frac{p^{k'_0-1}-1}{p+1} + p(p^{k'_0-1}-1) \\ \quad \text{if } k_0 \text{ is odd} \end{cases}$$

Proof. Computation. □

4. Study of invariants in the amalgamed sum -II

In the present section we are going to complete our study of $\Gamma_1(p^k)$-invariants for supersingular representations $\pi(r,0,1)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$, with $r \neq 0, p-1$.

To be more precise, for $k \in \mathbf{N}_{\geqslant 1}$ we describe in detail the spaces
$$W_k \overset{\text{def}}{=} (\cdots \oplus_{R_k} R_{k+1})^{\Gamma_1(p^k)}/(\cdots \oplus_{R_{k-2}} R_{k-1})^{\Gamma_1(p^k)}$$
$$\widetilde{W}_k \overset{\text{def}}{=} (\cdots \oplus_{R_{k-1}} R_k)^{\Gamma_1(p^k)}/(\cdots \oplus_{R_{k-3}} R_{k-2})^{\Gamma_1(p^k)};$$

together with the results in section §3 we will then be able to compute the dimension of $\Gamma_1(p^k)$-invariants (proposition 4.21).

We start with the following, elementary, observation:

$$\Gamma_1(p^k) = \begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+p^k\mathbf{Z}_p & p^k\mathbf{Z}_p \\ 1+p^k\mathbf{Z}_p & p^k\mathbf{Z}_p \end{bmatrix} \quad \text{for} \quad k \geqslant 1; \tag{2}$$

$$(\cdots \oplus_{R_{k-2-i}} R_{k-1-i})^{\Gamma_1(p^k)} = (\cdots \oplus_{R_{k-2-i}} R_{k-1-i})^{\begin{bmatrix} 1 & \mathbf{Z}_p \\ 0 & 1 \end{bmatrix}} \quad \text{for } i \in \{0,1\}. \tag{3}$$

We are now lead to the analysis of the two cases W_k and $\widetilde{W}_k$.

4.1 Study of W_k

An immedate consequence of corollary 2.6 and proposition 3.5 in [Mo] is that

Lemma 4.1. *Let $k \geqslant 2$. Then an $\overline{\mathbf{F}}_p$-basis for $(R_{k+1}/R_k)^{\Gamma_1(p^k)}$ is described by:*

a) the element $x_{0,\dots,0,r+1}(0)$;

b) for $1 \leqslant n \leqslant \frac{k+1}{2}$ the elements

$$x'_{l_n,\dots,l_{2n-1},0\dots,0,r+1}(0)$$

where $l_n \in \{1,\dots,p-1\}$ and $l_u \in \{0,\dots,p-1\}$ for $u \in \{n+1,\dots,2n-1\}$ (if non empty);

c) for $\frac{k+1}{2} \leqslant n \leqslant k$ the elements

$$x'_{l_n,\dots,l_{k-1},r+1}(0)$$

where, if $n < k$, we convene that $l_n \in \{1,\dots,p-1\}$ and $l_u \in \{0,\dots,p-1\}$ for $u \in \{n+1,\dots,k-1\}$ (if non empty)

We can now describe an $\overline{\mathbf{F}}_p$-basis for the subspace $V_{k+1} \wedge (R_{k+1}/R_k)^{\Gamma_1(p^k)}$:

Proposition 4.2. *Let $k \geqslant 2$ be an integer. An $\overline{\mathbf{F}}_p$-basis for $V_{k+1} \wedge (R_{k+1}/R_k)^{\Gamma_1(p^k)}$ is described as follow:*

1) for k odd the elements:

$$x'_{l_{\frac{k+1}{2}},\dots,l_{k-1},r+1}(0)$$

where $l_u \in \{0,\dots,p-1\}$ for $u \in \{\frac{k+1}{2},\dots,k-1\}$.

2) Assume k even. Then the basis is described by the elements

$$x'_{l_{\frac{k+2}{2}},\dots,l_{k-1},r+1}(0)$$

where $l_u \in \{0,\dots,p-1\}$ for $u \in \{\frac{k+2}{2},\dots,k-1\}$, and the elements

a2) if $\frac{k}{2}$ is odd the elements

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1},r+1}(0)$$

with $l_{\frac{k}{2}} \in \{1,\dots,p-1\}$ and $(l_{\frac{k+2}{2}},\dots,l_{k-1}) \prec (r, p-1-r,\dots,p-1-r)$; the elements

$$x'_{l_{\frac{k}{2}},r,p-1-r,\dots,p-1-r,r+1}(0)$$

for $l_{\frac{k}{2}} \in \{p-2, p-1, 1, \dots, \lceil p-3-r\rceil - 1\}$ together with

$$x'_{\lceil p-3-r\rceil,r,p-1-r,\dots,p-1-r,r+1}(0) + c_0 y_{\lceil p-3\rceil,p-1-r,\dots,p-1-r,r+1}(0)$$

where $c_0 \in \mathbf{F}_p$ is a suitable constant;

b2) if $\frac{k}{2}$ is even the elements

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1},r+1}(0)$$

with $l_{\frac{k}{2}}\in\{1,\dots,p-1\}$ and $(l_{\frac{k+2}{2}},\dots,l_{k-1})\prec(p-1-r,\dots,p-1-r)$; the elements

$$x'_{l_{\frac{k}{2}},p-1-r,\dots,p-1-r,r+1}(0)$$

for $l_{\frac{k}{2}}\in\{p-2,p-1,1,\dots,\lceil r-2\rceil-1\}$ together with

$$x'_{\lceil r-2\rceil,p-1-r,\dots,p-1-r,r+1}(0)+c_0y_{\lceil p-3-r\rceil,r,\dots,p-1-r,r+1}(0)$$

where $c_0\in\mathbf{F}_p$ is a suitable constant.

Proof. Postponed. □

For sake of completeness, we recall the results for $k=1$.

Lemma 4.3. *For $k=1$ the space $V_2\wedge(R_2/R_1)^{\Gamma_1(p)}$ is 1-dimensional, and a basis is given by the element*

$$x'_{r+1}(0).$$

Let $v\in(\cdots\oplus_{R_k}R_{k+1})$ be the canonical lift of an element $\overline{v}\in V_k\wedge(R_{k+1}/R_k)^{\Gamma_1(p^k)}$. If we write pr for the map

$$(\cdots\oplus_{R_k}R_{k+1})^{\Gamma_1(p^k)}\overset{pr}{\to}R_{k+1}/R_k$$

then we see that $\overline{v}$ is in the image of pr iff it exists $y\in\cdots\oplus_{R_{k-2}}R_{k-1}$ such that $y+v\in(\cdots\oplus_{R_k}R_{k+1})^{\Gamma_1(p^k)}$ which is equivalent to $v\in(\cdots\oplus_{R_k}R_{k+1})^{\Gamma_1(p^k)}$ since v is $\begin{bmatrix}1&\mathbf{Z}_p\\0&1\end{bmatrix}$-invariant and y is K_k-invariant in the amalgamed sum.

We outline the elementary result:

Lemma 4.4. *Let $k\geqslant 1$. The action of $\begin{bmatrix}1+p^k\mathbf{Z}_p&0\\0&1+p^k\mathbf{Z}_p\end{bmatrix}$ is trivial on the canonical lifts of the elements in $V_k\wedge(R_{k+1}/R_k)^{\Gamma_1(p^k)}$. Moreover if $1\leqslant n\leqslant k-1$ we have*

$$\begin{bmatrix}1&0\\p^k[\mu]&1\end{bmatrix}x'_{l_n,\dots,l_{k-1},r+1}(0)=x'_{l_n,\dots,l_{k-1},r+1}(0)+$$
$$+(r+1)(-1)^{r+1}(-\mu)(\kappa(l_{k-1}))y'_{l_n,\dots,l_{k-2}}(r-(p-1-l_{k-1}))$$

where we define

$$\kappa(l_{k-1})\overset{def}{=}\begin{cases}0 & \text{if}\quad l_{k-1}<p-1-r;\\ \neq 0 & \text{if}\quad l_{k-1}\geqslant p-1-r.\end{cases}$$

(with the convention that, for $n=k-1$, $y'_{l_{-1}}(x)=[1,X^{r-x}Y^x]$).

Proof. Postponed. □

We define $\mathcal{U}$ as the $\overline{\mathbf{F}}_p$-subspace of $(\cdots\oplus_{R_k}R_{k+1})$ generated by the canonical lifts of $V_k\wedge(R_{k+1}/R_k)^{\Gamma_1(p^k)}$. Then $(\cdots\oplus_{R_{k-2}}R_{k-1})+\mathcal{U}$ is a $\begin{bmatrix}1&0\\p^k\mathbf{Z}_p&1\end{bmatrix}$-stable subspace of $(\cdots\oplus_{R_k}R_{k+1})$.

4.1.1 *The case k odd.* Assume now $k\geqslant 2$, k odd. We have the following result:

Lemma 4.5. *Let $k\geqslant 2$, k odd. We consider $j\in\mathbf{N}$ such that $k-2j-1>\frac{k+1}{2}$. Then the $\begin{bmatrix}1&0\\p^k\mathbf{Z}_p&1\end{bmatrix}$-invariants of $((R_{k-2j-1}/R_{k-2j-2})\oplus\cdots\oplus_{R_{k-2}}R_{k-1})+\mathcal{U}$ are described by:*

a) *the space* $((R_{k-2j-1}/R_{k-2j-2}) \oplus \cdots \oplus_{R_{k-2}} R_{k-1})$*;*

b) *the elements*

$$x'_{l_{\frac{k+1}{2}},\dots,l_{k-1},r+1}(0)$$

where $(l_{k-2-2j},\dots,l_{k-1}) \preceq (r,p-1-r,\dots,p-1-r)$ *and* $(l_{\frac{k+1}{2}},\dots,l_{k-2j-3}) \in \{0,\dots,p-1\}^{k-2j-2-\frac{k+1}{2}}$.

Proof. Postponed. (induction on j). □

We therefore deduce:

Proposition 4.6. *Let $k \geqslant 2$ be odd. An $\overline{\mathbf{F}}_p$-basis for W_k is described by the elements*

$$x'_{l_{\frac{k+1}{2}},\dots,l_{k-1},r+1}(0)$$

where

$$(l_{\frac{k+1}{2}},\dots,l_{k-1},r+1) \prec \begin{cases} (p-1-r,r,\dots,r,p-1-r) & \text{if } \frac{k+1}{2} \in 2\mathbf{N} \\ (r,p-1-r,\dots,r,p-1-r) & \text{if } \frac{k+1}{2} \in 2\mathbf{N}+1. \end{cases}$$

Proof. Postponed. □

For $k=1$ we get

Lemma 4.7. *For $k=1$ we have*

$$\dim_{\overline{\mathbf{F}}_p}(W_1) = 0.$$

Proof. Postponed. □

4.1.2 *The case k even.* In this section we assume that $k \in \mathbf{N}$ is an even integer. We have then

Lemma 4.8. *Let $j \in \mathbf{N}$ be such that $k-2j-1 > \frac{k}{2}+1$. The space of $\begin{bmatrix} 1 & 0 \\ p^k\mathbf{Z}_p & 1 \end{bmatrix}$-invariants of $((R_{k-2j-1}/R_{k-2j-2}) \oplus \cdots \oplus_{R_{k-2}} R_{k-1}) + \mathcal{U}$ is described by*

a) *the space* $((R_{k-2j-1}/R_{k-2j-2}) \oplus \cdots \oplus_{R_{k-2}} R_{k-1})$*;*

b) *the elements described in $2-a2)$ (resp. $2-b2)$) of proposition 4.2 if $\frac{k}{2}$ is odd (resp. even);*

c) *the elements*

$$x'_{l_{\frac{k}{2}+1},\dots,l_{k-1},r+1}(0)$$

where $(l_{k-2-2j},\dots,l_{k-1}) \preceq (r,p-1-r,\dots,p-1-r)$ *and* $(l_{\frac{k}{2}+1},\dots,l_{k-3-2j}) \in \{0,\dots,p-1\}^{\frac{k}{2}-2j-2}$*. Moreover, if we have* $(l_{k-2-2j},\dots,l_{k-1}) \prec (r,p-1-r,\dots,p-1-r)$*, the element is invariant in the amalgamd sum* $\varinjlim_{n\,\text{even}} ((R_1/R_0) \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1})$.

Proof. Postponed. (Induction on j). □

We are now able to describe W_k for k even:

Proposition 4.9. *Let $k \geqslant 2$ be an even integer. An $\overline{\mathbf{F}}_p$-basis for W_k is described as follow:*

1) *if $\frac{k}{2}$ is odd, the elements*

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1},r+1}(0)$$

where $(l_{\frac{k}{2}+1},\ldots,l_{k-1})\prec(r,\ldots,p-1-r)$ *and* $l_{\frac{k}{2}}\in\{0,\ldots,p-1\}$ *together with the following* $p-2-r$*-elements*

$$\begin{aligned}
&x'_{p-1,r,\ldots,p-1-r,r+1}(0)+c_1x'_{r,p-1-r,\ldots,p-1-r,r+1}(0);\\
&x'_{1,r,\ldots,p-1-r,r+1}(0);\\
&\vdots\\
&x'_{\lceil p-3-r\rceil-1,r,\ldots,p-1-r,r+1}(0);\\
&x'_{\lceil p-3-r\rceil,r,\ldots,p-1-r,r+1}(0)+c_0y_{\lceil p-3\rceil,p-1-r,r,\ldots,p-1-r,r+1}(0).
\end{aligned}$$

2) *If* $\frac{k}{2}$ *is even, the elements*

$$x'_{l_{\frac{k}{2}},\ldots,l_{k-1},r+1}(0)$$

where $(l_{\frac{k}{2}+1},\ldots,l_{k-1})\prec(p-1-r,r,\ldots,p-1-r)$ *and* $l_{\frac{k}{2}}\in\{0,\ldots,p-1\}$ *together with the following* $r-1$*-elements*

$$\begin{aligned}
&x'_{p-1,p-1-r,\ldots,p-1-r,r+1}(0)+c_1x'_{p-1-r,\ldots,p-1-r,r+1}(0);\\
&x'_{1,p-1-r,\ldots,p-1-r,r+1}(0);\\
&\vdots\\
&x'_{\lceil r-2\rceil-1,p-1-r,\ldots,p-1-r,r+1}(0);\\
&x'_{\lceil r-2\rceil,p-1-r,\ldots,p-1-r,r+1}(0)+c_0y_{\lceil p-3-r\rceil,r,\ldots,p-1-r,r+1}(0).
\end{aligned}$$

We can sum up the results, giving the dimensions of the spaces W_k.

Proposition 4.10. *Let* $k\in\mathbf{N}_{\geqslant 1}$. *The dimension of the space* W_k *is then given by*

1) *for* k *odd, we have*

$$\dim_{\overline{\mathbf{F}}_p}(W_k)=\begin{cases}(p-1-r)\frac{p^{\frac{k+1}{2}-1}}{p^2-1}+pr\frac{p^{\frac{k-3}{2}-1}}{p^2-1} & \text{if}\quad \frac{k+1}{2}\in 2\mathbf{N}\\ (p-r)\frac{p^{\frac{k-1}{2}-1}}{p+1} \quad \text{if}\quad \frac{k+1}{2}\in 2\mathbf{N}+1\end{cases}$$

2) *For* k *even, we have*

$$\dim_{\overline{\mathbf{F}}_p}(W_k)=\begin{cases}p(p-r)\frac{p^{\frac{k}{2}-1}-1}{p+1}+(p-2-r) \quad \text{if}\quad \frac{k}{2}\in 2\mathbf{N}+1\\ p[(p-1-r)\frac{p^{\frac{k}{2}}-1}{p^2-1}+pr\frac{p^{\frac{k}{2}-2}-1}{p^2-1}]+(r-1) \quad \text{if}\quad \frac{k+1}{2}\in 2\mathbf{N}\end{cases}$$

4.2 Study of $\widetilde{W}_k$

In this section, we follow closely the steps which led us to the description of W_k in paragraph 4.1.

Again, we use corollary 2.6 and proposition 3.5 in [Mo] to get

Lemma 4.11. *Let* $k\geqslant 3$ *be an integer. An* $\overline{\mathbf{F}}_p$*-basis for* $(R_k/R_{k-1})^{\Gamma_1(p^k)}$ *is described as follow:*

a) *the element* $x_{0,\ldots,0,r+1}(0)$

b) *for* $1\leqslant n<\frac{k}{2}$ *the elements*

$$x'_{l_n,\ldots,l_{2n-1},0,\ldots,0,r+1}(0)$$

where the indices l_u *verify the conditions in* *ii*) *of proposition 2.6;*

c) *for* $n\in\{\frac{k}{2},\frac{k+1}{2}\}\cap\mathbf{N}$ *the elements*

$$x'_{l_n,\ldots,l_{k-2},l^{(i)}_{k-1}}(i)$$

where $i \in \{0,1\}$, $l_{k-1}^{(0)} \in \{r+1,\dots,p-1\}$, $l_{k-1}^{(1)} \in \{0,\dots,r\}$ and $(l_n,\dots,l_{k-2}) \in \{0,\dots,p-1\}^{k-1-n}$.

For $k=2$ an $\overline{\mathbf{F}}_p$-basis for $(R_2/R_1)^{\Gamma_1(p^2)}$ is given by

a2) *the element $x_{0,r+1}(0)$;*

b2) *the elements*

$$x'_{r+1}(0),\dots,x'_{p-1}(0);$$

c2) *the elements*

$$x'_0(1),\dots,x'_{\lceil r-2\rceil}(1)$$

together with the element $x'_{p-2}(1)$ if $r=p-2$.

For $k=1$ an $\overline{\mathbf{F}}_p$-basis for $(R_1/R_0)^{\Gamma_1(p)}$ is given by

$$x_r(0).$$

We deduce an $\overline{\mathbf{F}}_p$-basis for the space $V_k \wedge (R_k/R_{k-1})^{\Gamma_1(p^k)}$:

LEMMA 4.12. *Let $k \in \mathbf{N}$, $k \geqslant 3$. An $\overline{\mathbf{F}}_p$-basis for the space $V_k \wedge (R_k/R_{k-1})^{\Gamma_1(p^k)}$ is described as follow.*

1) *Assume k even. Then we have the elements*

a1)

$$x'_{l_{\frac{k}{2}+1},\dots,l_{k-2},l_{k-1}^{(i)}}(i)$$

where $i \in \{0,1\}$, $l_{k-1}^{(0)} \in \{r+1,\dots,p-1\}$, $l_{k-1} \in \{0,\dots,r\}$ and $(l_{\frac{k}{2}+1},\dots,l_{k-2}) \in \{0,\dots,p-1\}^{\frac{k}{2}-2}$;

b1)

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(0)$$

where $l_{k-1} \in \{r+1,\dots,p-1\}$, $l_{\frac{k}{2}} \in \{1,\dots,p-1\}$ and $(l_{\frac{k}{2}+1},\dots,l_{k-2}) \in \{0,\dots,p-1\}^{\frac{k}{2}-2}$;

c1) *According to the parity of $\frac{k}{2}$ we have*

c1.1) *if $\frac{k}{2}$ is even the elements*

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(1)$$

where $l_{\frac{k}{2}} \in \{1,\dots,p-1\}$ and $(l_{\frac{k}{2}+1},\dots,l_{k-1}) \prec (r,\dots,r)$, together with the elements

$$\begin{aligned}
&x'_{p-2,r,\dots,r}(1);\\
&x'_{p-1,r,\dots,r}(1);\\
&x'_{1,r,\dots,r}(1);\\
&\vdots\\
&x'_{\lceil p-3-r\rceil-1,r,\dots,r}(1);\\
&x'_{\lceil p-3-r\rceil,r,\dots,r}(1)+c_0 y'_{\lceil p-3\rceil,p-1-r,r,\dots,r}(1);
\end{aligned}$$

(with $c_0 \in \mathbf{F}_p$ a suitable constant);

c1.2) *if $\frac{k}{2}$ is odd the elements*

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(1)$$

where $l_{\frac{k}{2}} \in \{1, \dots, p-1\}$ *and* $(l_{\frac{k}{2}+1}, \dots, l_{k-1}) \prec (p-1-r, \dots, r)$, *together with the elements*

$$\begin{aligned}
&x'_{p-2,r,\dots,r}(1);\\
&x'_{p-1,r,\dots,r}(1);\\
&x'_{1,r,\dots,r}(1);\\
&\vdots\\
&x'_{\lceil r-2\rceil-1,r,\dots,r}(1);\\
&x'_{\lceil r-2\rceil,r,\dots,r}(1) + c_0 y'_{\lceil p-3-r\rceil,r,\dots,r}(1);
\end{aligned}$$

(with $c_0 \in \mathbf{F}_p$ *a suitable constant);*

2) *Assume* k *odd. Then we have the elements*

$a2)$

$$x'_{l_{\frac{k+1}{2}},\dots,l_{k-2},l^{(i)}_{k-1}}(i)$$

where $i \in \{0,1\}$, $l^{(0)}_{k-1} \in \{r+1, \dots, p-1\}$, $l^{(i)}_{k-1} \in \{0, \dots, r\}$ *and* $(l_{\frac{k+1}{2}}, \dots, l_{k-2}) \in \{0, \dots, p-1\}^{\frac{k+1}{2}-2}$;

$b2)$ *According to the parity of* $\frac{k-1}{2}$ *we have:*

$b2.1)$ *if* $\frac{k-1}{2}$ *is odd, the elements*

$$x'_{l_{\frac{k-1}{2}},\dots,l_{k-2},r+1}(0)$$

where $(l_{\frac{k+1}{2}}, \dots, l_{k-2}) \prec (r, \dots, p-1-r)$, $l_{\frac{k-1}{2}} \in \{1, \dots, p-1\}$ *together with the elements*

$$\begin{aligned}
&x'_{p-2,r,\dots,p-1-r,r+1}(0);\\
&x'_{p-1,r,\dots,p-1-r,r+1}(0);\\
&x'_{1,r,\dots,p-1-r,r+1}(0);\\
&\vdots\\
&x'_{\lceil p-3-r\rceil-1,r,\dots,p-1-r,r+1}(0);\\
&x'_{\lceil p-3-r\rceil,r,\dots,p-1-r,r+1}(0) + c_0 y'_{\lceil p-3\rceil,p-1-r,r,\dots,p-1-r,r+1}(0);
\end{aligned}$$

(with $c_0 \in \mathbf{F}_p$ *a suitable constant, and, for* $k=3$, $y'_{\dots}$ *is remplaced by* $y_{r+1}(0)$*);*

$b2.2)$ *if* $\frac{k-1}{2}$ *is even, the elements*

$$x'_{l_{\frac{k-1}{2}},\dots,l_{k-2},r+1}(0)$$

where $(l_{\frac{k+1}{2}}, \dots, l_{k-2}) \prec (p-1-r, \dots, p-1-r)$, $l_{\frac{k-1}{2}} \in \{1, \dots, p-1\}$ *together with* *the elements*

$$\begin{array}{l}
x'_{p-2,p-1-r,\dots,p-1-r,r+1}(0);\\
x'_{p-1,p-1-r,\dots,p-1-r,r+1}(0);\\
x'_{1,p-1-r,\dots,p-1-r,r+1}(0);\\
\vdots\\
x'_{\lceil r-2\rceil-1,p-1-r,\dots,p-1-r,r+1}(0);\\
x'_{\lceil r-2\rceil,p-1-r,\dots,p-1-r,r+1}(0)+c_0y'_{\lceil p-3-r\rceil,r,\dots,p-1-r,r+1}(0);
\end{array}$$

(with $c_0\in\mathbf{F}_p$ a suitable constant).

Proof. Postponed. □

For sae of completeness, we have cover the cases $k\in\{1,2\}$:

Lemma 4.13. *For $k=2$ an $\overline{\mathbf{F}}_p$-basis for $V_2\wedge(R_2/R_1)^{\Gamma_1(p^2)}$ is described by the elements $b2)$, $c2)$ of lemma 4.11; for $k=1$ an $\overline{\mathbf{F}}_p$-basis for $V_1\wedge(R_1/R_0)^{\Gamma_1(p)}$ is described by the element $x_r(0)$.*

We are lead to distinguish two situations, according to the parity of k.

4.2.1 *The case k even.* In this paragraph we fix $k\in 2\mathbf{N}$, $k\geqslant 2$. We start with the following observation

Lemma 4.14. *In the amalgamed sum $\varinjlim_{n,\mathrm{odd}} R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}$ the action of $\begin{bmatrix}1+p^k\mathbf{Z}_p & 0\\ 0 & 1+p^k\mathbf{Z}_p\end{bmatrix}$ is trivial on the lifs of the elements 1) in proposition 4.12, as well as on the elements described in lemma 4.13.*

The action of $\begin{bmatrix}1 & 0\\ p^k\mathbf{Z}_p & 1\end{bmatrix}$ is trivial on the lifts of the elements

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(0)$$

where $(l_{\frac{k}{2}},\dots,l_{k-2})\in\{0,\dots,p-1\}^{\frac{k}{2}-1}$ and $l_{k-1}\in\{r+1,\dots,p-1\}$.
Finally, let $n\in\{\frac{k}{2}+1,\frac{k+1}{2}\}\cap\mathbf{N}$ and assume $k\geqslant 6$. We have the following equality in the amalgamed sum:

$$\begin{bmatrix}1 & 0\\ p^k[\mu] & 1\end{bmatrix}x'_{l_{\frac{k}{2}+1},\dots,l_{k-1}}(1)=x'_{l_{\frac{k}{2}+1},\dots,l_{k-1}}(1)+$$
$$+\delta_{r,l_{k-1}}(r+1)(-1)^{r+1}\mu\kappa(l_{k-2})y_{l_{\frac{k}{2}+1},\dots,l_{k-3}}(r-(p-1-l_{k-2}))$$

where we define

$$\kappa(l_{k-2})\overset{\mathrm{def}}{=}\begin{cases}0 \quad \text{if} & l_{k-2}<p-1-r;\\ \neq 0\text{if} & l_{k-2}\geqslant p-1-r.\end{cases}$$

(and with the convention that, for $k=6$, $y'_{l_{-1}}(x)=[1,X^{r-x}Y^x]$).

For $k\geqslant 4$, let $\mathcal{U}$ be the subspace of $R_0\oplus_{R_1}\cdots\oplus_{R_{k-1}}R_k$ generated by the (canonical lift of the) following elements:

a) the elements $c1.1)$ (resp. $c1.2)$) of lemma 4.12-1) if $\frac{k}{2}$ is even (resp. odd);

b) the elements

$$x'_{l_{\frac{k}{2}+1},\dots,l_{k-1}}(1)$$

where $l_{k-1}\in\{0,\dots,r\}$ and $(l_{\frac{k}{2}+1},\dots,l_{k-2})\in\{0,\dots,p-1\}^{\frac{k}{2}-2}$.

As in §4.1.1 we start with a lemma

Lemma 4.15. *Let $k\geqslant 4$ be an even integer, and let $j\in\mathbf{N}$ be such that $k-2j-2>\frac{k}{2}+1$. Then, the space of $\begin{bmatrix}1 & 0\\ p^k\mathbf{Z}_p & 1\end{bmatrix}$-invariants of $((R_{k-2j-2}/R_{k-2j-3})\oplus\cdots\oplus_{R_{k-3}}R_{k-2})+\mathcal{U}$ is described by*

a) *the space $((R_{k-2j-2}/R_{k-2j-3})\oplus\cdots\oplus_{R_{k-3}}R_{k-2})$;*

b) *the elements c1.1) (resp. c1.2)) of lemma 4.12-1) if $\frac{k}{2}$ is even (resp. odd);*

c) *the elemets*

$$x'_{l_{\frac{k}{2}+1},\dots,l_{k-1}}(1)$$

where $(l_{k-2j-3},\dots,l_{k-1})\preceq(r,\dots,r)$ and $(l_{\frac{k}{2}+1},\dots,l_{k-2j-4})\in\{0,\dots,p-1\}^{\frac{k}{2}-2j-4}$. Moreover, if $(l_{k-2j-3},\dots,l_{k-1})\prec(r,\dots,r)$, such elements are invariant in the amalgamed sum $\varinjlim_{n,\mathrm{odd}} R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}$.

Thanks to the preceeding lemma, we are able to describe an $\overline{\mathbf{F}}_p$-basis for $\widetilde{W}_k$, when k is even.

Proposition 4.16. *Let $k\in 2\mathbf{N}$ be a non zero even integer. An $\overline{\mathbf{F}}_p$-basis for the space $\widetilde{W}_k$ is described as follow.*

a) *The elements*

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(0)$$

where $l_{k-1}\in\{r+1,\dots,p-1\}$ and $(l_{\frac{k}{2}},\dots,l_{k-2})\in\{0,\dots,p-1\}^{\frac{k}{2}-1}$;

b) *according to the parity of $\frac{k}{2}$ the elements*

b1) *if $\frac{k}{2}$ is odd, the $r-1$ elements*

$$\begin{aligned}&x'_{0,p-1-r,\dots,r}(1);\\ &x'_{1,p-1-r,\dots,r}(1);\\ &\vdots\\ &x'_{\lceil r-2\rceil-1,p-1-r,\dots,r}(1);\\ &x'_{\lceil r-2\rceil,p-1-r,\dots,r}(1)+c_0y'_{\lceil p-3-r\rceil,r,\dots,r}(1)\end{aligned}$$

(where $y'_{\dots}$ has to be replaced by $XY^{r-1}\in R_0$ if $k=2$ and $c_0\in\mathbf{F}_p$ is a suitable constant) together with the elements

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(1)$$

where $(l_{\frac{k}{2}+1},\dots,l_{k-1})\prec(p-1-r,\dots,r)$ and $l_{\frac{k}{2}}\in\{0,\dots,p-1\}$.

b2) if $\frac{k}{2}$ is even, the $p-2-r$ elements

$$\begin{aligned}&x'_{0,r,\dots,r}(1);\\ &x'_{1,r,\dots,r}(1);\\ &\vdots\\ &x'_{\lceil p-3-r\rceil-1,r,\dots,r}(1);\\ &x'_{\lceil p-3-r\rceil,r,\dots,r}(1)+c_0 y'_{\lceil p-3\rceil,p-1-r,r,\dots,r}(1)\end{aligned}$$

(where $c_0\in\mathbf{F}_p$ is a suitable constant) together with the elements

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(1)$$

where $(l_{\frac{k}{2}+1},\dots,l_{k-1})\prec(r,\dots,r)$ and $l_{\frac{k}{2}}\in\{0,\dots,p-1\}$.

4.2.2 *The case k odd* Assume now k an odd integer. As the element $x_r(0)\in R_1/R_0$ is clearly $\Gamma_1(p)$-invariant, we will assume $k\geqslant 3$ throught this paragraph.

As in the previous section we have

Lemma 4.17. *In the amalgamed sum $\varinjlim_{n,\text{even}}(R_1/R_0)\oplus_{R_2}\cdots\oplus_{R_n}R_{n+1}$ the action of $\begin{bmatrix}1+p^k\mathbf{Z}_p & 0\\ 0 & 1+p^k\mathbf{Z}_p\end{bmatrix}$ is trivial on the lifs of the elements 2) in proposition 4.12.*

The action of $\begin{bmatrix}1 & 0\\ p^k\mathbf{Z}_p & 1\end{bmatrix}$ is trivial on the lifts of the elements

$$x'_{l_{\frac{k}{2}},\dots,l_{k-1}}(0)$$

where $(l_{\frac{k+1}{2}},\dots,l_{k-2})\in\{0,\dots,p-1\}^{\frac{k+1}{2}-2}$ and $l_{k-1}\in\{r+1,\dots,p-1\}$.

We therefore define $\mathcal{U}$ as the $\overline{\mathbf{F}}_p$-subspace of $(R_1/R_0)\oplus_{R_2}\cdots\oplus_{R_{k-1}}R_k$ generated by the (canonical lifts of the) elements

$$x'_{l_{\frac{k+1}{2}},\dots,l_{k-1}}(1)$$

where $l_{k-1}\in\{0,\dots,r\}$ and $(l_{\frac{k+1}{2}},\dots,l_{k-2})\in\{0,\dots,p-1\}^{\frac{k+1}{2}-2}$.

We have

Lemma 4.18. *Let $k\geqslant 3$ be an od integer and let $j\in\mathbf{N}$ be such that $k-2j-2>\frac{k+1}{2}$. The space of $\begin{bmatrix}1 & 0\\ p^k\mathbf{Z}_p & 1\end{bmatrix}$-invariants of $((R_{k-2j-2}/R_{k-2j-3})\oplus\cdots\oplus_{R_{k-3}}R_{k-2})+\mathcal{U}$ is described by:*

a) the space $((R_{k-2j-2}/R_{k-2j-3})\oplus\cdots\oplus_{R_{k-3}}R_{k-2})$;

b) the elements

$$x'_{l_{\frac{k+1}{2}},\dots,l_{k-1}}(1)$$

where $(l_{k-2j-3},\dots,l_{k-1})\preceq(r,\dots,r)$ and $(l_{\frac{k+1}{2}},\dots,l_{k-2j-4})\in\{0,\dots,p-1\}^{\frac{k-1}{2}-2j-3}$. Moreover, such elements are invariant in the amalgamed sum $(R_1/R_0)\oplus_{R_2}\cdots\oplus_{R_{k-1}}R_k$ if $(l_{k-2j-3},\dots,l_{k-1})\prec(r,\dots,r)$.

Proof. Postponed. □

We finally get the description of $\widetilde{W}_k$ for $k\geqslant 3$, k odd.

PROPOSITION 4.19. *Let $k \in \mathbf{N}$ be an odd integer, and assume $k \geqslant 3$. An $\overline{\mathbf{F}}_p$-basis for $\widetilde{W}_k$ is described as follow.*

a) the elements

$$x'_{l_{\frac{k+1}{2}},\ldots,l_{k-1}}(0)$$

where $l_{k-1} \in \{r+1,\ldots,p-1\}$ and $(l_{\frac{k+1}{2}},\ldots,l_{k-2}) \in \{0,\ldots,p-1\}^{\frac{k+1}{2}-2}$;

b) according to the parity of $\frac{k-1}{2}$ we have

b1) if $\frac{k-1}{2}$ is even, the elements in $b2.2)$ of lemma 4.12 together with the elements

$$x'_{l_{\frac{k+1}{2}},\ldots,l_{k-1}}(1)$$

with $(l_{\frac{k+1}{2}},\ldots,l_{k-1}) \prec (p-1-r,\ldots,r)$;

b2) if $\frac{k-1}{2}$ is odd, the elements in $b2.1)$ of lemma 4.12 together with the elements

$$x'_{l_{\frac{k+1}{2}},\ldots,l_{k-1}}(1)$$

with $(l_{\frac{k+1}{2}},\ldots,l_{k-1}) \prec (r,\ldots,r)$.

Proof. Postponed. □

We sum up what We can sum up the results, giving the dimensions of the spaces W_k.

PROPOSITION 4.20. *Let $k \in \mathbf{N}_{\geqslant 3}$. The dimension of the space $\widetilde{W}_k$ is then given by*

1) for k odd, we have

$$\dim_{\overline{\mathbf{F}}_p}(\widetilde{W}_k) = \begin{cases} p((p-1-r)\frac{p^{\frac{k-1}{2}}-1}{p^2-1} + pr\frac{p^{\frac{k-5}{2}}-1}{p^2-1}) + r + (p-1)p^{\frac{k-3}{2}} \\ \quad \text{if} \quad \frac{k-1}{2} \in 2\mathbf{N} \\ \\ p(p-r)\frac{p^{\frac{k-3}{2}}-1}{p+1} + (p-1-r) + (p-1)p^{\frac{k-3}{2}} \\ \quad \frac{k-1}{2} \in 2\mathbf{N}+1 \end{cases}$$

2) For k even, we have

$$\dim_{\overline{\mathbf{F}}_p}(\widetilde{W}_k) = \begin{cases} (p-1-r)p^{\frac{k}{2}-1} + p(r+1)\frac{p^{\frac{k}{2}-1}-1}{p+1} + (r-1) \\ \quad \text{if} \quad \frac{k}{2} \in 2\mathbf{N}+1 \\ \\ (p-1-r)p^{\frac{k}{2}-1} + p(r\frac{p^{\frac{k}{2}}-1}{p^2-1} + p(p-1-r)\frac{p^{\frac{k-4}{2}}-1}{p^2-1}) + (p-2-r) \\ \quad \text{if} \quad \frac{k+1}{2} \in 2\mathbf{N} \end{cases}$$

We are finally able to compute the dimension of $\Gamma_1(p^k)$-invariants, using propositions 3.15, 4.10, 4.20:

THEOREM 4.21. *Let $k \in \mathbf{N}_{\geqslant 1}$ be an integer and $r \in \{1,\ldots,p-1\}$. Then the dimension of $\Gamma_1(p^k)$-invariants for the supersingular representation $\pi(r,0,1)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ is described as follow:*

$$\dim_{\overline{\mathbf{F}}_p}(\pi(r,0,1)^{\Gamma_1(p^k)}) = \begin{cases} 2(2p^{\frac{k-1}{2}}-1) & \text{if } k \text{ is odd;} \\ 2(p^{\frac{k}{2}} + p^{\frac{k-2}{2}} - 2) & \text{if } k \text{ is even.} \end{cases}$$

Proof. Postponed. □

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