Study of $\Gamma_1(p^k)$ invariants for supersingular representations of $GL_2(Q_p)$

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Abstract
We compute the dimension of $\Gamma_1(p^k)$-invariants for supersingular representations $\pi(r,0,1)$ of $GL_2(Q_p)$, when $r \not\equiv 0 \mod p-1$.

WARNING: these notes are an alpha version, and thus highly unstable. The details of the proofs (as well as simpler arguments) will be added as soon possible.

1. Introduction and notations
The aim of this note is to describe in detail the $\Gamma_1(p^k)$ invariants for supersingular representations $\pi(r,0,1)$ where $r \in \{1,\ldots,p-2\}$ and $p > 2$. The main result (Theorem 4.21) is the following:

**Theorem 1.1.** Let $r \in \{1,\ldots,p-2\}$ and $k \in \mathbb{N} \geq 1$. The dimension of the $\Gamma_1(p^k)$-invariants for the supersingular representation $\pi(r,0,1)$ is given by:

$$\dim_{\mathbb{F}_p}(\pi(r,0,1)_{\Gamma_1(p^k)}) = \begin{cases} 2(2p^{\frac{k+1}{2}} - 1) & \text{if } k \text{ is odd;} \\ 2(p^\frac{k}{2} + p^{\frac{k-2}{2}} - 2) & \text{if } k \text{ is even.} \end{cases}$$

The general strategy is completely elementary -based on the study of certain eigenspaces issued from the explicit description of $\pi(r,0,1)$- and can be outlined as follow:

1) from lemma 3.2 in [Mo] we are left to study the subspaces $\cdots \oplus R_k R_{k+1} \cdots \oplus R_{k-1} R_k$;
2) we study the $\Gamma_1(p^k)$ invariants of $R_{t-1}/R_{t-2}$, for $i \in \{0,1\}$, $k + 2 \geq t \geq 1$;
3) from 2) and left exactness of the functor $H^0(\Gamma_1(p^k), \bullet)$ we compute the spaces $$(\cdots \oplus R_{t-2} R_{t-1})_{\Gamma_1(p^k)}/(\cdots \oplus R_{t-4} R_{t-3})_{\Gamma_1(p^k)}.$$

As annonced, we will not use any sophisticated arguments, the main difficulty will be painful and boring computations (as we will see, we need to distinguish according to the reduction of $k$ modulo 4).

From now onwards, we fix an integer $r \in \{1,\ldots,p-2\}$.

1.1 Notations
For $t \geq 2$ and $\eta$ a character of $H$ we recall the $B \cap K$-equivariant isomorphism

$$\text{Ind}_{K_0(p^{t-1})}^K \eta |_{B \cap K} \simeq W^+_{t-1,\chi} \oplus W^-_{t-1,\chi}$$

for suitable subspaces $W^\pm_{t-1,\eta}$. The description of such spaces is straightforward:

**Lemma 1.2.** Let $t \geq 2$. Then

\textit{2000 Mathematics Subject Classification} 22E50, 11F85.

\textit{Keywords:} $p$-adic Langlands correspondence, supersingular representation.
i) an $\mathbf{F}_p$-base for the space $W_{l-2,\eta}^+$ is described by
\[
x_{l_0,...,l_{t-2}}(e) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \lambda_0 \left[ \begin{array}{c} \lambda_0 \\ 1 \\ 0 \end{array} \right] \cdots \sum_{\lambda_{l-2} \in \mathbf{F}_p} \lambda_{l-2} \left[ \begin{array}{c} 1 \\ p^{l-2}[\lambda_{l-2}] \\ 0 \end{array} \right] [1,e]
\]
for $l_j \in \{0,\ldots,p-1\}, j \in \{0,\ldots,t-2\}$.

ii) An $\mathbf{F}_p$-base for the space $W_{l-2,\eta}^-$ is described by the elements
\[
x'_{l_j,...,l_{t-2}}(e) \overset{\text{def}}{=} \sum_{\lambda_j \in \mathbf{F}_p} \lambda_j \left[ \begin{array}{c} 1 \\ p^{j}[\lambda_j] \\ 0 \end{array} \right] \cdots \sum_{\lambda_{l-2} \in \mathbf{F}_p} \lambda_{l-2} \left[ \begin{array}{c} 1 \\ p^{l-2}[\lambda_{l-2}] \\ 0 \end{array} \right] [1,e]
\]
where $j \in \{1,\ldots,t-3\}, l_j \in \{1,\ldots,p-1\}, l_m \in \{0,\ldots,p-1\}$ for $m \in \{j+1,\ldots,t-2\}$, and the elements
\[
x_{l_{t-2}}' \overset{\text{def}}{=} \sum_{\lambda_{l-2} \in \mathbf{F}_p} \lambda_{l-2} \left[ \begin{array}{c} 1 \\ p^{t-2}[\lambda_{l-2}] \\ 0 \end{array} \right] [1,e],[1,e]
\]
for $l_{t-2} \in \{1,\ldots,p-1\}$.

Proof. Omissis. □

We are now in the position to describe an $\mathbf{F}_p$-basis for $R_{t-1}/R_{t-2}$, where $t \geq 3$:

**Lemma 1.3 Definition.** Let $t \geq 3$. An $\mathbf{F}_p$-basis for the $K$-representation $R_{t-1}/R_{t-2}$ is described by the following elements:

i) for $j \in \{1,\ldots,r\}$ the elements
\[
x_{l_0,...,l_{t-2}}(j) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \lambda_0 \left[ \begin{array}{c} \lambda_0 \\ 1 \\ 0 \end{array} \right] \cdots \sum_{\lambda_{l-2} \in \mathbf{F}_p} \lambda_{l-2} \left[ \begin{array}{c} 1 \\ p^{l-2}[\lambda_{l-2}] \\ 0 \end{array} \right] [1,X^{r-j}Y^j]
\]
for $l_m \in \{0,\ldots,p-1\}, m \in \{0,\ldots,t-2\}$;

ii) the elements
\[
x_{l_0,...,l_{t-2}}(0) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbf{F}_p} \lambda_0 \left[ \begin{array}{c} \lambda_0 \\ 1 \\ 0 \end{array} \right] \cdots \sum_{\lambda_{l-2} \in \mathbf{F}_p} \lambda_{l-2} \left[ \begin{array}{c} 1 \\ p^{l-2}[\lambda_{l-2}] \\ 0 \end{array} \right] [1,X^r]
\]
for $l_m \in \{0,\ldots,p-1\}, m \in \{0,\ldots,t-3\}$ and $l_{t-2} \in \{r+1,\ldots,p-1\}$;

iii) for $j \in \{1,\ldots,r\}$ the elements
\[
x'_{l_j,...,l_{t-2}}(j) \overset{\text{def}}{=} \sum_{\lambda_j \in \mathbf{F}_p} \lambda_j \left[ \begin{array}{c} 1 \\ p^{j}[\lambda_j] \\ 0 \end{array} \right] \cdots \sum_{\lambda_{l-2} \in \mathbf{F}_p} \lambda_{l-2} \left[ \begin{array}{c} 1 \\ p^{l-2}[\lambda_{l-2}] \\ 0 \end{array} \right] [1,X^{r-j}Y^j]
\]
for $l_m \in \{0,\ldots,p-1\}, m \in \{1,\ldots,t-2\}$;

iv) the elements
\[
x'_{l_j,...,l_{t-2}}(0) \overset{\text{def}}{=} \sum_{\lambda_j \in \mathbf{F}_p} \lambda_j \left[ \begin{array}{c} 1 \\ p^{j}[\lambda_j] \\ 0 \end{array} \right] \cdots \sum_{\lambda_{l-2} \in \mathbf{F}_p} \lambda_{l-2} \left[ \begin{array}{c} 1 \\ p^{l-2}[\lambda_{l-2}] \\ 0 \end{array} \right] [1,X^r]
\]
for $l_m \in \{0,\ldots,p-1\}, m \in \{1,\ldots,t-3\}$ and $l_{t-2} \in \{r+1,\ldots,p-1\}$;

v) the elements
\[
[1,X^{r-j}Y^j]
\]
for $j \in \{1,\ldots,r\}$.

For $t = 2$ the description is slightly different:
Lemma 1.4 definition. An $\mathbb{F}_p$-base for $R_1/R_0$ is described as follows:

i) for $j \in \{1, \ldots, r\}$ the elements

$$x_{l_0}(j) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbb{F}_p} \lambda_0^l \left[ \begin{array}{c} \lambda_0 \\ 1 \end{array} \right] [1, X^{r-j}Y^j]$$

for $l_0 \in \{0, \ldots, p-1\}$;

ii) the elements

$$x_{l_0}(0) \overset{\text{def}}{=} \sum_{\lambda_0 \in \mathbb{F}_p} \lambda_0^l \left[ \begin{array}{c} \lambda_0 \\ 1 \end{array} \right] [1, X^{r-j}Y^j]$$

for $l_0 \in \{r, \ldots, p-1\}$;

iii) for $j \in \{1, \ldots, r\}$ the elements

$$[1, X^{r-j}Y^j].$$

We conclude the section with the main computational tools for the description of the spaces $H^0(\Gamma_1(p^k), \pi(r, 0, 1))$.

Lemma 1.5. Let $t \geq 3$, $j \in \{1, \ldots, t-2\}$ and $z' = \sum_{n=j}^{t-2} [\lambda_n]p^n$. If $m \in \mathbb{N}$ is such that $2j + m \leq t - 1$ then

$$\left[ \begin{array}{cc} 1 & p^m[\mu] \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ z' & 1 \end{array} \right] = \left[ \begin{array}{cc} 1 & 0 \\ \tilde{z}' & 1 \end{array} \right] \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$$

for suitable $p$-adic integers $a, b, c, d, \tilde{z}'$ such that:

i) $a, d \equiv 1 \text{ mod } p$ and $b = p^m[\mu]$;

ii) $\tilde{z}' = \sum_{n=j}^{t-2} [\hat{\lambda}_n]p^n$ where

a2) $\hat{\lambda}_n = \lambda_n$ for $n \in \{j, \ldots, 2j + m - 1\}$

b2) $\hat{\lambda}_n + S_{n-1}(\lambda_{n-1}) = \lambda_n$ for $n \in \{2j + m + 1, \ldots, t-2\}$ where $S_{n-1}(X) \in \mathbb{F}_p[X]$ is a polynomial of degree $p-1$ and leading coefficient $\lambda_{n-1} - \hat{\lambda}_{n-1}$;

b2) $\hat{\lambda}_{2j+m} + \lambda_{2j+m}^2 = \lambda_{2j+m}$ if $2j + m \in \{j, \ldots, t-2\}$;

b3) $S_{t-2}(X) \in \mathbb{F}_p[X]$ is a polynomial of degree zero given by $S_{t-2}(X) \in \mathbb{F}_p[X] = \mu \lambda_j^2$;

Proof. Postponed.

As we will need later on, we recall the matrix equality:

$$\left[ \begin{array}{cc} 1 + p^j[a] & 0 \\ 0 & 1 + p^j[d] \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ z' & 1 \end{array} \right] = \left[ \begin{array}{cc} z'(1 + p^j[a])^{-1}(1 + p^j[d]) & 1 \\ 0 & 1 + p^j[d] \end{array} \right]$$

where $j \in \mathbb{N}_>$, $a, d \in \mathbb{F}_p$ and $z$ is a $p$-adic integer.

Lemma 1.6. Let $t \geq 4$. We have the following equalities in the amalgamed sum $\cdots \oplus_{R_{t-2}} R_{t-1}$:
i) \[
\sum_{\lambda_{t-3} \in \mathbb{F}_p} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-3}[\lambda_{t-3} + \mu] & 1 \\
\end{array} \right] \sum_{\lambda_{t-2} \in \mathbb{F}_p} \lambda_{t-2}^{r+1} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-2}[\lambda_{t-2} + \mu] & 1 \\
\end{array} \right] [1, X^r] = \\
= \sum_{\lambda_{t-3} \in \mathbb{F}_p} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-3}[\lambda_{t-3}] & 1 \\
\end{array} \right] \sum_{\lambda_{t-2} \in \mathbb{F}_p} \lambda_{t-2}^{r+1} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-2}[\lambda_{t-2}] & 1 \\
\end{array} \right] [1, X^r] + \\
+(r + 1)(-1)^{r+1} \sum_{\lambda_{t-3} \in \mathbb{F}_p} P_{\mu}(\lambda_{t-3})[1, (\lambda_{t-3}X + Y)^r];
\]

ii) \[
\sum_{\lambda_{t-3} \in \mathbb{F}_p} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-3}[\lambda_{t-3}] & 1 \\
\end{array} \right] \sum_{\lambda_{t-2} \in \mathbb{F}_p} \lambda_{t-2}^{r+1} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-2}[\lambda_{t-2} + \mu] & 1 \\
\end{array} \right] [1, X^r] = \\
= \sum_{\lambda_{t-3} \in \mathbb{F}_p} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-3}[\lambda_{t-3}] & 1 \\
\end{array} \right] \sum_{\lambda_{t-2} \in \mathbb{F}_p} \lambda_{t-2}^{r+1} \left[ \begin{array}{cc}
1 & 0 \\
p^{t-2}[\lambda_{t-2}] & 1 \\
\end{array} \right] [1, X^r] + \\
+(r + 1)(-1)^{r+1}(-\mu) \sum_{\lambda_{t-3} \in \mathbb{F}_p} [1, (\lambda_{t-3}X + Y)^r].
\]

**Proof.** Postponed. \[\square\]

**Lemma 1.7.** Let \( k_1, k_2 \) be integers such that \( 0 \leq k_1 \leq p - 1 \) and \( 1 \leq k_2 \); let \( V \) be an \( \mathbb{F}_p \)-vector space with a base given by \( \mathcal{B} = \{v_{i,j} | 0 \leq j \leq k_1, 1 \leq i \leq k_2\} \).

Assume we are given, for a fixed \( \mu \in \mathbb{F}_p \), an endomorphism \( \phi_\mu : V \to V \) such that

\[
\phi_\mu(v_{i,j}) = \sum_{n=0}^{j} \binom{j}{n} (\mu)^n v_{i+n,j-n}
\]

where we adopt the convention

\( v_{k,j} \overset{\text{def}}{=} v_{[k],j} \)

for any \( k \in \mathbb{N}_0, j \in \{0, \ldots, k_1\} \).

Then the endomorphism \( \phi_\mu \) has the scalar 1 as the only eigenvalue, and the associated eigenspace is

\( V^{\phi_\mu = 1} = \langle v_{1,0}, \ldots, v_{k_2,0} \rangle_{\mathbb{F}_p} \).

**Proof.** Postponed. \[\square\]

### 2. Study of \( R_{t-1}/R_{t-2} \)

In this section we are going to study in detail some invariant spaces of the quotients \( R_{t-1}/R_{t-2} \). More precisely, we consider the following subgroups of \( K \):

\( B \cap I_1 = \left[ \begin{array}{cc}
1 + p\mathbb{Z}_p & \mathbb{Z}_p \\
0 & 1 + p\mathbb{Z}_p \\
\end{array} \right]; K \cap U = \left[ \begin{array}{cc}
1 & \mathbb{Z}_p \\
0 & 1 \\
\end{array} \right]. \)

The obvious reason is that

i) \( (K \cap U) \cdot K_k = \Gamma_0(p^k); \)

ii) \( (B \cap I_1) \cdot K_k = \left[ \begin{array}{cc}
1 + p\mathbb{Z}_p & \mathbb{Z}_p \\
p^k\mathbb{Z}_p & 1 + p\mathbb{Z}_p \\
\end{array} \right] \) is normal in \( \Gamma_1(p^k) \), and the quotient is isomorphic to \( H \).
We recall that the study of $K_k$-invariant has been pursued in [Mo].

2.1 Concerning the action of unipotent elements

In this section we are going to describe explicitly the invariant spaces $(R_{t-1}/R_{t-2})^j$ for $j \in \mathbb{N}$, $t \geq 2$. The strategy will be elementary, using successive induction on $j$ and on the filtration defined on $R_{t-1}/R_{t-2}$; the main statement will be corollary 2.6, where we give a basis for $R_{t-1}/R_{t-2}$.

The first step is

**Lemma 2.1.** Let $t \geq 2$, $\eta$ a character of $H$ (seen as a character of $K_0(p^{t-1})$ by inflation). Let $m \in \mathbb{N}$ be such that $t - 1 \geq m \geq 0$ and define $k_0 \overset{\text{def}}{=} \frac{t - 1 - m}{2}$. Then an $\mathbb{F}_p$-basis for $(\text{Ind}^K_{K_0(p^{t-1})}\eta)$ is described as follows:

1. If $m \geq 1$, the elements $x_{l_0,...,l_{m-1},0,...,0}(e)$, with $l_j \in \{0,\ldots,p-1\}$ for $j \in \{0,\ldots,m-1\}$, while the element $x_{l_0,...,0}(e)$ if $m = 0$;
2. for $1 \leq k \leq k_0$ the elements $x'_{l_k,...,l_{k+m-1},0,...,0}(e)$ where $l_k \in \{1,\ldots,p-1\}$, $l_j \in \{0,\ldots,p-1\}$ for $k + 1 \leq j \leq 2k + m - 1$;
3. for $k_0 < k \leq t - 2$ the elements $x'_{l_k,...,l_{t-2}}(e)$ where $l_k \in \{1,\ldots,p-1\}$, $l_j \in \{0,\ldots,p-1\}$ for $k + 1 \leq j \leq t - 2$
4. the element $[1,e]$;

**Proof.** Postponed (induction on $m$).

We switch now our attention to the spaces $R_{t-1}/R_{t-1}$. We recall that the graded piece of the filtration induced by $\text{Fil}^i(R_{t-1})$ gives

$$Q(0)^{0,t-1}_{0,...,0,r+1} - \text{Ind}_{K_0(p^{t-1})}^K \chi_r \mathfrak{a} - \cdots - \text{Ind}_{K_0(p^{t-1})}^K \chi_r \mathfrak{a}^r$$

The strategy to describe the invariant spaces of $R_{t-1}/R_{t-2}$ is therefore to use lemma 2.1 and an inductive argument using the aforementioned filtration on $R_{t-1}/R_{t-2}$.

The result is the following:

**Proposition 2.2.** Let $t \geq 2$, $m \in \mathbb{N}$ such that $t - 1 \geq m \geq 0$; let moreover $i \in \mathbb{N}$ be such that $r - 1 \geq i \geq 0$. If $k_0 \overset{\text{def}}{=} \frac{t - 1 - m}{2}$ an $\mathbb{F}_p$-basis for $(R_{t-1}/\text{Fil}^i(R_{t-1}))$ is described as follows:

1. The elements $x_{l_0,...,l_{m-1},0,...,0}(i + 1)$ where $l_j \in \{0,\ldots,p-1\}$ for $j \in \{0,\ldots,m-1\}$ (with the obvious conventions if $m = 0$ or $m = t - 2$).
2. For $1 \leq k \leq k_0$, the elements $x'_{l_k,...,l_{k+m-1},0,...,0}(i + 1)$
where \( l_k \in \{1, \ldots, p-1\}, l_n \in \{0, \ldots, p-1\} \) for \( n \in \{k+1, \ldots, 2k+m-1\} \) (and the obvious convention that “there are no zeros” if \( k = k_0 \))

iii) for \( k_0 < k \leq t-2 \) the elements

\[
x_{l_k, \ldots, l_{t-2}}(j)
\]

where \( j \in \{i+1, \ldots, r\} \), \( l_k \in \{1, \ldots, p-1\} \) and \( l_n \in \{0, \ldots, p-1\} \) for \( n \in \{k+1, \ldots, t-2\} \).

iv) the elements

\[
[1, X^{r-(i+1)}Y^{i+1}], \ldots, [1, Y^r].
\]

**Proof.** Postponed (descending induction on \( i \)). Inside the proof we use a lemma.

Let us consider the \( \mathbb{F}_p \)-subspace \( U \) of \( R_{t-1}/\text{Fil}^t(R_{t-1}) \) generated by

a) \( \text{Fil}^{i+1}(R_{t-1})/\text{Fil}^i(R_{t-1}) \);
b) the elements \( x_{l_0, \ldots, l_m, 0, \ldots, 0}(i+2) \) (the indices \( l_j \) satisfying the conditions of the elements i) in the statement of the proposition)
c) for \( 1 \leq k \leq k_0 \) the elements \( x_{l_k, \ldots, l_{2k+m-1}, 0, \ldots, 0}(i+2) \) (the indices \( l_j \) satisfying the conditions of the elements ii) in the statement of the proposition)
d) for \( k_0 < k \leq t-2 \) the elements \( x_{l_k, \ldots, l_{t-2}}(j) \) with \( j \in \{i+2, \ldots, r\} \) and the indices \( l_j \) satisfying the conditions of the elements iii) in the statement of the proposition)
e) the elements \( [1, X^{r-(i+2)}Y^{i+2}], \ldots, [1, Y^r] \).

We notice that the subspace \( U' \) of \( U \) generated by the elements in d), e) is fixed under \( \begin{bmatrix} 1 & p^m \mathbb{Z}_p \\ 0 & 1 \end{bmatrix} \); if \( U'' \) is the subspace generated by the elements in a), b), c) (notice also that \( U = U' + U'' \)) we have the following lemma

**Lemma 2.3.** Under the previous assumption, let \( j \in \mathbb{N} \) be such that \( m \leq j \leq t-1 \). Then, an \( \mathbb{F}_p \)-basis for \( U'' \)

\[
\begin{bmatrix} 1 & p^m \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}
\]

is described as follow:

a) the elements

\[
x_{l_0, \ldots, l_{j-1}, 0, \ldots, 0}(i+1)
\]

(where the indices \( l_j \) satisfy the conditions of the elements i) in the statement of the proposition);

b) for \( 1 \leq n \leq \frac{t-1-j}{2} \) the elements

\[
x_{l_n, \ldots, l_{2n+j-1}, 0, \ldots, 0}(i+1)
\]

(where the indices \( l_j \) satisfy the conditions of the elements ii) in the statement of the proposition);

c) for \( \frac{t-1-j}{2} < n \leq t-2 \) the elements

\[
x_{l_n, \ldots, l_{t-2}}(i+1)
\]

(where the indices \( l_j \) satisfy the conditions of the elements iii) in the statement of the proposition);

d) for \( \frac{t-1-j}{2} < k \leq \frac{t-1-m}{2} \) the elements

\[
x_{l_k, \ldots, l_{2k+m-1}, 0, \ldots, 0}(i+2)
\]

(where the indices \( l_j \) satisfy the conditions of the elements ii) in the statement of the proposition);
We then have the following lemma.

\[\text{Proposition}\]

\[\text{Postponed. (descending induction on } j)\]

\[\text{Proof.}\]

The proposition follow applying the lemma with \( j = m \).

We are now in the position to prove the key result of this section.

**Proposition 2.4.** Let \( t \geq 2, t - 2 \geq m \geq 0 \) be integers and assume \( t + m > 3 \). Define \( k_0 \overset{\text{def}}{=} \frac{t - 1 - m}{2} \).

An \( \mathbb{F}_p \)-basis for \((R_{t-1}/R_{t-2})\) is described as follow:

i) the elements

\[x_{l_0,\ldots,l_{m-1},0,\ldots,0,r+1}(0)\]

where \( l_n \in \{0, \ldots, p-1\} \) for \( n \in \{0, \ldots, m-1\} \) (and with the obvious conventions if \( m = 0 \) or \( m = t - 2 \));

ii) for \( 1 \leq k < k_0 \) the elements

\[x'_{l_k,\ldots,l_{2k+m-1},0,\ldots,0,r+1}(0)\]

where \( l_k \in \{1, \ldots, p-1\}, l_n \in \{0, \ldots, p-1\} \) for \( n \in \{k + 1, \ldots, 2k + m - 1\} \) (if the latter is non empty; and “there ate no zeros” for \( 2k + m - 1 = t - 3 \)).

iii) for \( k_0 < k \leq t - 2 \) the elements

\[x'_{l_k,\ldots,l_{t-2}}(j)\]

where:

- for \( 1 \leq j \leq r, l_k \in \{1, \ldots, p-1\} \) and \( l_n \in \{0, \ldots, p-1\} \) where \( n \in \{k + 1, \ldots, t - 2\} \) (if non empty);
- for \( j = 0, l_{r-2} \in \{r+1, \ldots, p-1\}, l_k \in \{1, \ldots, p-1\} \) (non empty condition only if \( k < t - 2 \), and if \( k \leq t - 4, l_n \in \{0, \ldots, p-1\} \) if \( n \in \{k + 1, \ldots, t - 3\} \).

iv) the elements

\[\{1, X^{r-1}Y\}, \ldots, \{1, Y^r\}\]

v) if \( k_0 \in \mathbb{N} \), the elements

\[x'_{l_{k_0},\ldots,l_{t-2}}(i)\]

where \( i \in \{0,1\}, l_{k_0} \in \{1, \ldots, p-1\}, l_{l_{t-2}} \in \{r+1, \ldots, p-1\}, l_{t-2} \in \{0, \ldots, r\} \) and \( l_n \in \{0, \ldots, p-1\} \) where \( n \in \{k_0 + 1, \ldots, t - 3\} \) (if non empty).

**Proof.** Thanks to proposition 2.2 (and a direct space decoposition as in the proof of the latter) we see that we are led to the study of the subspace \( U'' \) of \( R_{t-1}/R_{t-2} \) generated by the elements:

a) \( Q_{0,\ldots,0,r+1}^{0,t-1}(0) \);

b) the elements

\[x_{l_0,\ldots,l_{m-1},0,\ldots,0}(1)\]

for \( l_n \in \{0, \ldots, p-1\} \), where \( n \in \{0, \ldots, m-1\} \) (if non empty);

c) for \( 1 \leq k \leq k_0 \) the elements

\[x'_{l_k,\ldots,l_{2k+m-1},0,\ldots,0}(1)\]

where \( l_k \in \{1, \ldots, p-1\} \) and \( l_n \in \{0, \ldots, p-1\} \) for \( n \in \{k + 1, \ldots, 2k + m - 1\} \) (if non empty)

We then have the following lemma.
LEMMA 2.5. In the previous situation, consider an integer $j \in \mathbb{N}$ with $t - 2 \geq j \geq m + 1$, and put $j_0 \overset{\text{def}}{=} \frac{t - 1 - j}{2}$. An $\mathbb{F}_p$-basis for $U''$ is described by:

a) the elements

$$x_{l_0,...,l_{j-1},0,...,0,r+1}(0)$$

where the indices $l_u$ verify the conditions in i);

b) for $1 \leq n < j_0$, the elements

$$x'_{l_n,...,l_{2n+j-1},0,...,0,r+1}(0)$$

where $l_n \in \{1,\ldots,p-1\}$ and $l_u \in \{0,\ldots,p-1\}$ for $u \in \{n+1,\ldots,2n+j-1\}$ (if non empty);

c) for $j_0 \leq n \leq t - 2$, the elements

$$x'_{l_n,...,l_{t-2}}(0)$$

where $l_{t-2} \in \{r+1,\ldots,p-1\}$, $l_n \in \{1,\ldots,p-1\}$ if $n < t - 2$ and, for $n \leq t - 4$, $l_u \in \{0,\ldots,p-1\}$ for $u \in \{n+1,\ldots,t-3\}$;

d) for $j_0 \leq k \leq k_0$ the elements

$$x'_{l_k,...,l_{2k+m-1},0,...,0,r+1}(1)$$

where the indices $l_u$ verify the conditions described in the point c) above.

Proof. Induction on $j$. □

Lemma 2.5 enable us to establish the inductive step for the proof of the main statement. □

As a consequence, we can describe explicitly the space of $\left[ \begin{array}{cc} 1 & Z_p \\ 0 & 1 \end{array} \right]$-invariants:

COROLLARY 2.6. Let $t \geq 4$. An $\mathbb{F}_p$-basis for $(R_{t-1}/R_{t-2})$ is described as follow:

i) the element

$$x_{0,...,0,r+1}(0);$$

ii) for $1 \leq k < \frac{t-1}{2}$ the elements

$$x'_{k,...,l_{2k-1},0,...,0,r+1}(0)$$

where $l_k \in \{1,\ldots,p-1\}$ and $l_u \in \{0,\ldots,p-1\}$ for $u \in \{k+1,\ldots,2k-1\}$ (if non empty);

iii) for $\frac{t-1}{2} \leq k \leq t - 2$ the elements

$$x'_{l_k,...,l_{t-2}}(j)$$

where

- for $1 \leq j \leq r$ we have $l_k \in \{1,\ldots,p-1\}$ and $l_u \in \{0,\ldots,p-1\}$ for $n \in \{k+1,\ldots,t-2\}$ (if non empty);
- for $j = 0$ we have $l_{t-2} \in \{r+1,\ldots,p-1\}$, $l_k \in \{1,\ldots,p-1\}$ if $k < t - 2$ and, if moreover $k \leq t - 4$, $l_u \in \{0,\ldots,p-1\}$ for $u \in \{k+1,\ldots,t-3\}$;

iv) the elements

$$[1,X^{r-1}Y],\ldots,[1,Y^r];$$

v) If $k_0 \overset{\text{def}}{=} \frac{t-1}{2} \in \mathbb{N}$ the elements

$$x'_{l_{k_0},...,l_{t-2}}(i)$$

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where $l^1_{t-2} \in \{0, \ldots, r\}$, $l^0_{t-2} \in \{r + 1, \ldots, p - 1\}$, $l_{k_0} \in \{1, \ldots, p - 1\}$ and $l_u \in \{0, \ldots, p - 1\}$ for $u \in \{k_0 + 1, \ldots, t - 3\}$ (if non empty).

The remaining cases $t = 3, t = 2$ can be detected by a direct computation.

Lemma 2.7. An $\mathbb{F}_p$-basis for $(R_2/R_1)\begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}$ is described as follow:

i) the element $x_{0,r+1}(0)$;

ii) the elements $x'_{r+1}(0), \ldots, x'_{p-1}(0)$;

iii) the elements $x'_{l_1}(1)$ where $l_1 \in \{p - 2, p - 1, \ldots, \lfloor r - 2 \rfloor\}$ (with the obvious convention on the ordering on the set $\{1, \ldots, p - 1\}$);

iv) the elements $[1, X^{r-1}Y], \ldots, [1, Y^r]$.

Proof. Postponed

Lemma 2.8. An $\mathbb{F}_p$-basis for $(R_1/R_0)\begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}$ is described as follow:

i) the element $x_r(0)$;

ii) the elements $[1, X^{r-1}Y], \ldots, [1, Y^r]$.

Proof.

3. Study of invariants in the amalgamed sum -I

The aim of this section is to describe in detail the $\begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}$-invariants of the spaces $R_i/R_{i-1} \oplus R_{i+1} \cdots \oplus R_n R_{n+1}$, for $n \geq 1$ and $i \in \{0, 1\}$. The strategy is elementary and can be summed up as follow:

1) by the left exactness of the $\begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}$-functor, it suffices to study the spaces

$$(\cdots \oplus R_{t-2} R_{t-1}) \begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix} / (\cdots \oplus R_{t-4} R_{t-3}) \begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix};$$

2) using the properties of the amalgamed sum, we dispose of a sequence of equivariant surjections

$$\cdots \rightarrow R_{t-3} / R_{t-4} \oplus R_{t-2} R_{t-1} \rightarrow R_{t-3} / \text{Fil}^{r-1} R_{t-3} \oplus R_{t-2} R_{t-1} \rightarrow R_{t-1} / R_{t-2};$$

3) by the results in section §2.1, we can use an inductive argument on the preceding sequences to deduce the description of the spaces in 1).

The following result is formal
Lemma 3.1. Let $t \geq 2$ and let $j \in \mathbb{N}$ be an integer such that $1 \leq j \leq \frac{t-2}{2}$. We have equivariant surjections

\[
R_{t-1-2j}/R_{t-2-2j} \oplus R_{t-2j} \cdot \cdot \cdot \oplus R_{t-2} R_{t-1} \twoheadrightarrow R_{t-1-2j}/\text{Fil}(R_{t-1-2j}) \oplus R_{t-2} \cdot \cdot \cdot \oplus R_{t-2} R_{t-1} \twoheadrightarrow \]

\[
\twoheadrightarrow R_{t+1-2j}/R_{t-2j} \oplus R_{t+2-2j} \cdot \cdot \cdot \oplus R_{t-2} R_{t-1}
\]

Proof. Formal consequence of the properties of the amalgamated sum.

In order to clarify the exposition, we are lead to treat separately the cases where $t$ is even or odd. From now on, we fix $t \in \mathbb{N}$; in order not to overload the notations -but not to avoid confusions as well- we adopt the following convention: the (image of the) elements of $R_{t-1}$ in the amalgamated sum will be noted by

\[
x^{(i)}_{t-2}(i);
\]

while the (image of elements) of $R_{t-1-2j}$ (where $\frac{t-1}{2} \geq j \geq 1$) will be noted by

\[
y^{(i)}_{t-1-2j}(i).
\]

We hope this will avoid confusions without making the notations too heavy.

3.1 Analysis for $t$ odd

We start with some introductory lemmas:

Lemma 3.2. Let $t \geq 5$. Fix $j \in \mathbb{N}$ an integer with $\frac{t-2}{2} \geq j \geq 1$, and define $U$ as the subspace of $R_{t-1-2j}/\text{Fil}^{-1}(R_{t-1-2j}) \oplus R_{t-2j} \cdot \cdot \cdot \oplus R_{t-2} R_{t-1}$ generated by:

a) $R_{t-1-2j}/\text{Fil}^{-1}(R_{t-1-2j})$;

b) the elements (images of elements in $R_{t+1-2j}$; we use the “$y$” notation, even if, for $j = 1$ we should have used the “$x$” notation to be consistent to what we wrote above)

\[
y^{(i)}_{t+1-2j,...,l_{t-2}}(1);
\]

where the indices $l_u$ verify conventions analogous to v) of corollary 2.6; for $1 \leq k < \frac{t+1-2j}{2}$ the elements

\[
y^{(i)}_{l_k,...,l_{2k-1},0,...,0,r+1}
\]

where the indices $l_u$ verify conventions analogous to ii) of corollary 2.6; the element

\[
y_{0,...,0,r+1}(0);
\]

c) the elements

\[
y^{(i)}_{t+3-2j,...,l_{t-1-2j},r,p-1-r,r}(1) \quad \text{(homomorphic image from } R_{t+3-2j});
\]

\[
\vdots
\]

\[
y^{(i)}_{t+3,...,l_{t-1-2j},r,p-1-r,...,p-1-r,r}(1) \quad \text{(homomorphic image from } R_{t-3});
\]

\[
x^{(i)}_{t+1,...,l_{t-1-2j},r,p-1-r,...,p-1-r,r}(1).
\]

Then, the space of \[
\begin{bmatrix}
1 & p^m \mathbb{Z}_p \\
0 & 1
\end{bmatrix}
\] -invariants of $U$, for $t - 1 - 2j \geq m \geq 1$, is described by:

a1) the space

\[
(R_{t-1-2j}/\text{Fil}^{-1}(R_{t-1-2j})) \begin{bmatrix}
1 & p^m \mathbb{Z}_p \\
0 & 1
\end{bmatrix}
\]
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b1) the elements in $c)$, as well as the elements

$$y'_{l+1,\ldots,l_{t-2}}(1);$$

(where the indices $l_u$ verify conventions analogous to $v)$ of corollary 2.6);

c1) for $\frac{t-2j-m}{2} \leq k < \frac{t+1-2j}{2}$ the elements

$$y'_{l_k,\ldots,l_{2k-1},0,\ldots,0,r+1}(0)$$

(where the indices $l_u$ verify conventions analogous to $v)$ of corollary 2.6).

Moreover, for $t - 1 - 2j > \frac{t-1}{2}$, the space of $[1 \mathbb{Z}_p^1]$-invariants of $U$ is described by

a2) the space

$$(R_{t-1-2j}/\text{Fil}^{-1}(R_{t-1-2j})) \left[ \begin{array}{cc} 1 & \mathbb{Z}_p \\ 0 & 1 \end{array} \right];$$

b2) the elements

$$y'_{\frac{t+1-2j}{2},\ldots,n_{t-2}}(1)$$

with $(l_{t-2j},l_{t-2j}) < (p-1,r)$ (in addition to the usual conventions on indices $l_u$);

c2) the elements described in $c)$, with the extra condition $l_{t-2j} \neq p-1$

Proof. Postponed. (Induction on $m$).

Remark 3.3. The second part of the statement of lemma 3.2 holds also for $t - 1 - 2j = \frac{t-1}{2}$, where the extra condition on the elements $x'_{l_{t-2j},\ldots,l_{t-2j},r,p-1-r,\ldots,p-1-r,r,1}(1)$ is instead $l_{k_0} \neq |p-3|$. We now state the key result of the section.

Lemma 3.4. Let $t \geq 5$, put $k_0 \overset{\text{def}}{=} \frac{t-1}{2}$ and let $j \in \mathbb{N}$ be such that $t - 1 - 2j > k_0 + 1$. The space of $[1 \mathbb{Z}_p^1]$-invariants inside $R_{t-1-2j}/R_{t-2-2j} \oplus \cdots \oplus R_{t-2} R_{t-2}$ is described as follow:

i) the elements

$$x'_{l_k,\ldots,l_{t-2}}(j)$$

the indices $j$, $l_u$ satisfying the conventions described in iii) of corollary 2.6;

the elements

$$[1,X^{r-1}Y],\ldots,[1,Y^r];$$

the elements

$$x'_{l_{k_0},\ldots,l_{t-2}}(0)$$

ii) the elements

$$x'_{l_{k_0},\ldots,l_{t-2}}(1)$$

where the indices $l_u$ verify the condition of $v)$ in corollary 2.6, together with $(l_{t-2-2j},\ldots,l_{t-2}) \preceq (r,p-1-r,\ldots,p-1-r,r)$; moreover such elements are invariant in $R_0 \oplus R_1 \oplus \cdots \oplus R_{t-2}$ $R_{t-1}$ if $(l_{t-2-2j},\ldots,l_{t-2}) \preceq (r,p-1-r,\ldots,p-1-r,r)$.
\[ \begin{align*}
\text{iii) elements of the form} & \quad y_{t-3}^{j}, \ldots, t_{t-2}, r, p-1-r, \ldots, p-1-r, r(1) \quad \text{(homomorphic image from } R_{t-3}); \\
\text{\vdots} & \\
\text{iii) the space} & \quad (R_{t-1-2j}/R_{t-2-2j}) \left[ \begin{array}{cc} 1 & Z_p \\ 0 & 1 \end{array} \right]; \\
\text{iv) homomorphic image of elements inside} & \quad (R_0 \oplus R_1 \cdots \oplus R_{t-2} R_{t-1}) \left[ \begin{array}{cc} 1 & Z_p \\ 0 & 1 \end{array} \right].
\end{align*} \]

**Proof.** It is an induction on \( j \), using the results in lemma 3.2.

We define, for \( t \geq 2 \) the space

\[ V_{t-1} \overset{\text{def}}{=} (R_0 \oplus R_1 \cdots \oplus R_{t-2} R_{t-1}) \left[ \begin{array}{cc} 1 & Z_p \\ 0 & 1 \end{array} \right]/(R_0 \oplus R_1 \cdots \oplus R_{t-4} R_{t-3}) \left[ \begin{array}{cc} 1 & Z_p \\ 0 & 1 \end{array} \right]. \]

To complete the description of \( V_{t-1} \) in the case \( t \) odd we have to distinguish two situations.

**3.1.1 Analysis for \( k_0 \) even.** We assume now \( k_0(\overset{\text{def}}{=} t-1) \) even. We therefore have to consider the chain of epimorphisms (where we assume \( t \geq 5 \))

\[ R_{k_0}/R_{k_0-1} \oplus R_{k_0+1} \cdots \oplus R_{t-2} R_{t-1} \rightarrow R_{k_0}/\text{Fil}^{r-1}(R_{k_0}) \oplus R_{k_0+1} \cdots \oplus R_{t-2} R_{t-1} \rightarrow \]

\[ \rightarrow R_{k_0+2}/R_{k_0+1} \oplus R_{k_0+3} \cdots \oplus R_{t-2} R_{t-1}. \]

Thanks to lemma 3.4 and lemma 3.2 we deduce

**Proposition 3.5.** Let \( t \geq 5 \) be such that \( k_0 \in 2N \). An \( \overline{F}_p \)-basis for \( V_{t-1} \) is described by:

a) for \( k_0 < k \leq t-2 \) the elements

\[ x^{j}_{l_0, \ldots, l_{t-2}}(j) \]

where the indices \( j, l_u \) verify the conditions described in iii) of corollary 2.6;

b) the elements

\[ [1, X^{r-1}Y], \ldots, [1, Y^{r}]; \]

c) the elements

\[ x^{j}_{l_0, \ldots, l_{t-2}}(0) \]

where the indices \( l_u \) verify the conditions described in v) of corollary 2.6;

d) the elements

\[ x^{j}_{l_0, \ldots, l_{t-2}}(1) \]

where \( l_0 \in \{1, \ldots, p-1\} \) and \((l_{k_0+1}, \ldots, l_{t-2}) \prec (r, p-1-r, \ldots, p-1-r, r)\);

e) for \( l_0 \in \{p-2, p-1, 1, \ldots, [p-3-r] - 1\} \) (if non empty, and with the obvious convention on the ordering on the set \( \{1, \ldots, p-1\} \)) the elements

\[ x^{j}_{l_0, \ldots, l_{t-2}}(1) \]

together with the element

\[ x^{j}_{[p-3-r], \ldots, r}(1) + ca y^{[p-3], p-1-r, \ldots, r}(1) \]
for a suitable constant $c_0 \in \mathbb{F}_p$. 

**Proof.** Postponed. 

### 3.1.2 Analysis for $k_0$ odd

We assume now $k_0 \overset{\text{def}}{=} \frac{t-1}{2}$ odd. We therefore have to consider the chain of epimorphisms (where we assume $t \geq 7$)

$$
\begin{align*}
R_{k_0+1}/R_{k_0} \oplus R_{k_0+2} \oplus \cdots \oplus R_{t-2} R_{t-1} &\rightarrow R_{k_0+1}/\text{Fil}^{r-1}(R_{k_0+1}) \oplus R_{k_0+2} \oplus \cdots \oplus R_{t-2} R_{t-1} \\
&\rightarrow R_{k_0+3}/R_{k_0+2} \oplus R_{k_0+4} \oplus \cdots \oplus R_{t-2} R_{t-1}.
\end{align*}
$$

Thanks to lemma 3.4 and lemma 3.2 we deduce

**Proposition 3.6.** Let $t \geq 5$ be such that $k_0 \in 2N + 1$. An $\mathbb{F}_p$-basis for $V_{t-1}$ is described by:

a) for $k_0 < k \leq t - 2$ the elements

$$
x'_{t_k,\ldots,t_{t-2}}(j)
$$

where the indices $j, l_u$ verify the conditions described in iii) of corollary 2.6;

b) the elements

$$
[1, X^{r-1}Y], \ldots, [1, Y^r];
$$

c) the elements

$$
x'_{l_{k_0},\ldots,l_{t-2}}(0)
$$

where the indices $l_u$ verify the conditions described in v) of corollary 2.6;

d) the elements

$$
x'_{l_{k_0},\ldots,l_{t-2}}(1)
$$

where $l_{k_0} \in \{1, \ldots, p - 1\}$ and $(l_{k_0+1}, \ldots, l_{t-2}) \prec (p - 1 - r, r, \ldots, p - 1 - r, r)$;

e) for $l_{k_0} \in \{p - 2, p - 1, 1, \ldots, \lceil r - 2 \rceil - 1\}$ (if non empty, and with the obvious convention on the ordering on the set $\{1, \ldots, p - 1\}$) the elements

$$
x'_{l_{k_0},p-1-r,\ldots,r}(1)
$$

and the element

$$
x'_{[r-2],p-1-r,\ldots,r}(1) + c_0 y_{[p-3-r],r,\ldots,r}(1)
$$

for a suitable constant $c_0 \in \mathbb{F}_p$.

The case $t = 3$ requires some extra care and is treated below:

**Lemma 3.7.** An $\mathbb{F}_p$-basis for $V_2$ is described by:

i) the elements

$$
[1, X^{r-1}Y], \ldots, [1, Y^r];
$$

ii) the elements

$$
x'_{r+1}(0), \ldots, x'_{p-1}(0);
$$

iii) for $l_1 \in \{p - 2, p - 1, 1, \ldots, \lceil r - 2 \rceil - 1\}$ the elements

$$
x'_{l_1}(1)
$$

and the element

$$
x'_{[r-2]}(1) + XY^{r-1}
$$

(where $XY^{r-1} \in R_0$)

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We are now left to count the dimensions of such spaces.

**Lemma 3.8.** Let \( t \geq 1 \) be an odd integer and put \( k_0 \triangleq \frac{t-1}{2} \). The dimension of \( V_{t-1} \) is then:

\[
\dim_{\mathbb{F}_p}(V_{t-1}) = \begin{cases} 
  p^{k_0-1}(p-1) + (p-1)[(p-r)p^{k_0-1} - (p-1-r)p^{k_0-1}] + (p-1-r) & \text{if } k_0 \text{ is even} \\
  p^{k_0-1}(p-1) + (p-1)(r+1)p^{k_0-1} + r & \text{if } k_0 \text{ is odd}
\end{cases}
\]

for \( t \geq 3 \) and

\[
\dim_{\mathbb{F}_p}(V_0) = 1
\]

The dimension of \( \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} \)-invariants of \( R_0 \oplus R_1 \cdots \oplus R_{t-2} R_{t-1} \) is given by:

\[
\dim_{\mathbb{F}_p}(R_0 \oplus R_1 \cdots \oplus R_{t-2} R_{t-1}) \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} = \begin{cases} 
  p^{k_0} + (r+1)p^{k_0-1} & \text{if } k_0 \geq 0 \text{ is even} \\
  p + r + p(p^{k_0-1} - 1) + p(r+1)p^{k_0-1} & \text{if } k_0 \text{ is odd}
\end{cases}
\]

*Proof.* Computation. \( \square \)

### 3.2 Analysis for \( t \) even

In this paragraph, we fix an even integer \( t \in 2\mathbb{N} \). The analysis of \( \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} \)-invariants for \( R_1/R_0 \oplus R_2 \cdots \oplus R_{t-2} R_{t-1} \) follows closely the arguments seen in paragraph §3.1. In particular, the proofs will mostly be left to the reader.

We recall the sequence of equivariant epimorphisms

\[
(R_1/R_0) \oplus R_2 \cdots \oplus R_{t-2} R_{t-1} \rightarrow (R_1/Fil^r_1(R_1)) \oplus R_2 \cdots \oplus R_{t-2} R_{t-1} \rightarrow (R_3/R_2) \oplus R_4 \cdots \oplus R_{t-2} R_{t-1} \rightarrow \cdots
\]

and that, for \( t \geq 4 \), an \( \mathbb{F}_p \)-basis for \( (R_{t-1}/R_{t-2}) \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} \) is described as follow:

- **a)** the element \( x_{t-1,0,0,\ldots,0}^{(0)} \);
- **b)** for \( 1 \leq k \leq k_0' \) the elements

\[
x_{t-1,0,\ldots,0}^{(l_k,0,\ldots,0,0,\ldots,0)}
\]

with \( l_k \in \{1, \ldots, p-1\} \) and \( l_u \in \{0, \ldots, p-1\} \) for \( u \in \{k+1, \ldots, 2k-1\} \) (if non empty);
- **c)** for \( k_0' + 1 \leq k \leq t-2 \) the elements

\[
x_{t-1,0,\ldots,0}^{(l_k,0,\ldots,0,0,\ldots,0)}(j)
\]

where the indices \( j, l_u \) verify the conditions of corollary 2.6-iii
- **d)** the elements

\[
[1, X^{r-1}Y], \ldots, [1, Y^r],
\]

where we defined

\[
k_0' \triangleq \frac{t-2}{2}.
\]

We notice that the elements of the form c), d) are certainly invariant in the amalgamed sum (as they are homomorphic image of invariant elements of \( R_{t-1} \)).

The following results are completely analogous to lemmas 3.2 and 3.4.
Lemma 3.9. Let \( j \in \mathbb{N} \geq 1 \). We consider the subspace \( U \) of \((R_{t-1-2j}/\text{Fil}^{r-1}(R_{t-1-2j}))+\cdots+R_{t-2} R_{t-1}\) generated by the following elements:

a) \( R_{t-1-2j}/\text{Fil}^{r-1}(R_{t-1-2j}) \);

b) the homomorphic image from \( R_{t+1-2j} \) of the elements \(^1\) for \( 1 \leq k < \frac{t-1-2j}{2} \) the elements (homomorphic image from \( R_{t+1-2j} \))

\[
y_{k,\ldots,l_{2k-1},0,\ldots,0,r+1}^{t-1-2j}(0)
\]

where the indices \( l_u \) verify conventions analogous to ii) of corollary 2.6;

the element

\[
y_{0,\ldots,0,r+1}(0)
\]

(homomorphic image from \( R_{t+1-2j} \));

c) the elements

\[
y_{l-1-2j,\ldots,l_{t-1-2j},r+1}(0)
\]

(homomorphic image from \( R_{t+1-2j} \));

c) the elements

\[
y_{l-1-2j,\ldots,l_{t-1-2j},r+1}(0)
\]

Then, the space of \( \begin{bmatrix} 1 & p^m \mathbb{Z}_p \\ 0 & 1 \end{bmatrix} \)-invariants of \( U \), for \( t-1-2j \geq m \geq 1 \), is described by:

a1) the space

\[
(R_{t-1-2j}/\text{Fil}^{r-1}(R_{t-1-2j})) \begin{bmatrix} 1 & p^m \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}
\]

b1) the elements in c);

c1) for \( \frac{t-2j-m}{2} \leq k < \frac{t-1-2j}{2} \) the elements

\[
y_{k,\ldots,l_{2k-1},0,\ldots,0,r+1}(0)
\]

(where the indices \( l_u \) verify conventions analogous to ii) of corollary 2.6).

Moreover, for \( t-1-2j > \frac{t-1}{2} \), the space of \( \begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix} \)-invariants of \( U \) is described by

a2) the space

\[
(R_{t-1-2j}/\text{Fil}^{r-1}(R_{t-1-2j})) \begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix}
\]

b2) the elements described in c), with the extra condition \( l_{t-1-2j} \neq p-1 \)

Proof. Postponed. (Induction on \( m \)).

Remark 3.10. The second part of the statement of lemma 3.9 holds also for \( t-1-2j = \frac{t-2}{2} \), where the extra condition on the elements \( x_{l_0}^{t-1-2j,\ldots,l_{t-1-2j},r+1}(0) \) is instead \( l_{t_0}^{r} \neq [p-3] \).

Similarly, we have:

\(^1\)Once again we use the “\( y \)” notation, even if, for \( j = 1 \) we should have used the “\( z \)” notation to be consistent with our notations. The same remark holds for the element \( y_{k,\ldots,l_{2k-1},0,\ldots,0,r+1}(0) \) described in c) below.
LEMMA 3.11. Let $t \geq 4$ and let $j \in \mathbb{N}_{\geq 1}$ be such that $t - 1 - 2j > k'_0 + 1$. The space of invariants inside $R_{t-1-2j}/R_{t-2-2j} \oplus \cdots \oplus R_{t-2} R_{t-1}$ is described as follows:

i) for $k_0 < k \leq t - 2$ the elements

$$x'_{t_1, \ldots, t_{t-2}}(j)$$

the indices $j, l_u$ satisfying the conventions described in iii) of corollary 2.6, as well as the elements

$$[1, X^{r-1}Y], \ldots, [1, Y^r];$$

ii) the elements

$$x'_{t_0, \ldots, t_{t-3}, r+1}(0)$$

where the indices $l_u$ verify the condition of i) in corollary 2.6, together with $(l_{t-2-2j}, \ldots, t_{t-3}) \preceq (r, p-1-r, \ldots, p-1-r)$; moreover such elements are invariant in $R_{t-1}/R_0 \oplus \cdots \oplus R_{t-2} R_{t-1}$ if $(l_{t-2-2j}, \ldots, t_{t-3}) < (r, p-1-r, \ldots, p-1-r);$  

iii) elements of the form

$$y'_{t_1-4, \ldots, t_{t-3-2j}, r, p-1-r, \ldots, p-1-r, r+1}(0)$$

(homomorphic image from $R_{t-3}$);

$$\vdots$$

$$y'_{t_1-2j, \ldots, t_{t-3-2j}, r, p-1-r, r+1}(0)$$

(homomorphic image from $R_{t+1-2j}$);

iv) the space

$$(R_{t-1-2j}/R_{t-2-2j}) \left[ \begin{array}{c} 1 \\ 0 \\ Z_p \end{array} \right];$$

v) homomorphic image of other suitable elements inside $(R_1/R_0 \oplus \cdots \oplus R_{t-1}) \left[ \begin{array}{c} 1 \\ 0 \\ Z_p \end{array} \right]$.  

Proof. Postponed.  

As in section 3.1, we define, for $t \geq 2$ the space

$$V_{t-1} \overset{\text{def}}{=} ((R_{t-1}/R_0) \oplus R_{t-2} \cdots \oplus R_{t-4} R_{t-1}) \left[ \begin{array}{c} 1 \\ 0 \\ Z_p \end{array} \right] /((R_{t-1}/R_0) \oplus R_{t-2} \cdots \oplus R_{t-4} R_{t-1}) \left[ \begin{array}{c} 1 \\ 0 \\ Z_p \end{array} \right].$$

Again, to complete the description of $V_{t-1}$ in the case $t$ even we have to distinguish two situations.

3.2.1 Analysis for $k'_0$ odd. We assume now $k'_0$ odd. We therefore have to consider the chain of epimorphisms (where we assume $t \geq 4$)

$$(R_{k'_0}/R_{k'_0-1}) \oplus R_{k'_0+1} \cdots \oplus R_{t-2} R_{t-1} \rightarrow (R_{k'_0}/\text{Fil}^{r-1}(R_{k'_0})) \oplus R_{k'_0+1} \cdots \oplus R_{t-2} R_{t-1} \rightarrow$$

$$(R_{k'_0+2}/R_{k'_0+1}) \oplus R_{k'_0+3} \cdots \oplus R_{t-2} R_{t-1}.$$ 

Thanks to lemma 3.11 and lemma 3.9 we deduce

PROPOSITION 3.12. Let $t \geq 4$ be such that $k'_0$ is odd, and $k'_0 > 1$. An $F_p$-basis for $V_{t-1}$ is described by:

a) for $k_0 < k \leq t - 2$ the elements

$$x'_{t_1, \ldots, t_{t-2}}(j)$$

where the indices $j, l_u$ verify the conditions described in iii) of corollary 2.6;
b) the elements
\[ [1, X^{r-1}Y], \ldots, [1, Y^r]; \]

c) the elements
\[ x'_{l_k', \ldots, l_{t-3}, r+1}(0) \]
where \( l_k' \in \{1, \ldots, p-1\} \) and \((l_{k_0+1}, \ldots, l_{t-3}) < (r, p-1-r, \ldots, p-1-r)\);

d) for \( l_k' \in \{p-2, p-1, 1, \ldots, [p-3-r] - 1\} \) (if non empty, and with the obvious convention on the ordering on the set \( \{1, \ldots, p-1\} \)) the elements
\[ x'_{l_k, r, \ldots, p-1-r, r+1}(0) \]
together with the element
\[ x'_{[p-3-r], r, \ldots, r+1}(0) + c_0 y_{[p-3]} r, p-1-r, \ldots, p-1-r, r+1(0) \]
for a suitable constant \( c_0 \in \mathbb{F}_p \).

Proof. Postponed.

With some extra care, we deduce the same result for \( t = 4 \):

Lemma 3.13. Let \( t = 4 \). Then an \( \mathbb{F}_p \)-basis for \((R_1/R_0) \oplus R_2 R_3\)
\[ \begin{bmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{bmatrix} \]
is described by:

a) an \( \mathbb{F}_p \)-basis of \((R_1/R_0)\);  
b) the elements
\[ x'_{l_1, r+1}(0) \]
where \( l_1 \in \{p-2, p-1, 1, \ldots, [p-3-r] - 1\} \) (with the obvious convention on the ordering on the set \( \{1, \ldots, p-1\} \));  
c) the element
\[ x'_{[p-3-r], r+1}(0) + c_0 x_{r+1}(0) \]
for a suitable constant \( c_0 \in \mathbb{F}_p \);  
d) the elements
\[ x'_{l_2}(j) \]
where the indices \( j, l_2 \) verify the conditions of \( iii \) in corollary 2.6, as well as the elements
\[ [1, X^{r-1}Y], \ldots, [1, Y^r]. \]

Proof. Postponed.

3.2.2 Analysis for \( k'_0 \) even. We assume now \( k'_0 \) even. We therefore have to consider the chain of epimorphisms (where we assume \( t \geq 4 \))
\[ (R_0 R_{k_0+1} R_{k_0+2} R_{t-2} R_{t-1} \rightarrow (R_{k_0+1}/\text{Fil}^{r-1}(R_{k_0+1})) \oplus R_{k_0+2} \cdots \oplus R_{t-2} R_{t-1} \rightarrow)
\]
\[ \rightarrow (R_{k_0+3}/R_{k_0+2}) \oplus \cdots \oplus R_{t-2} R_{t-1}. \]

Thanks to lemma 3.11 and lemma 3.9 we deduce

Proposition 3.14. Let \( t \geq 4 \) be such that \( k'_0 \) is even. An \( \mathbb{F}_p \)-basis for \( V_{t-1} \) is described by:
a) for $k_0 < k \leq t - 2$ the elements
\[ x_{l_k,\ldots,l_{t-2}}^r(j) \]
where the indices $j, l_u$ verify the conditions described in iii) of corollary 2.6;

b) the elements
\[ [1, X^{r-1}Y], \ldots, [1, Y^r]; \]
c) the elements
\[ x_{l_{k_0}^0,\ldots,l_{t-3},r+1}^r(0) \]
where $l_{k_0}^0 \in \{1, \ldots, p - 1\}$ and $(l_{k_0+1}, \ldots, l_{t-3}) \prec (p - 1 - r, r, \ldots, p - 1 - r);

d) for $k_0 \in \{p - 2, p - 1, 1, \ldots, [r - 2] - 1\}$ (if non empty, and with the obvious convention on the ordering on the set \{1, \ldots, p - 1\}) the elements
\[ x_{l_{k_0},p-1-r,\ldots,p-1-r,r+1}^r(0) \]
together with the element
\[ x_{[r-2],p-1-r,\ldots,p-1-r,r+1}^r(0) + c_0 y_{[p-3-r],r,\ldots,p-1-r,r+1}(0) \]
for a suitable constant $c_0 \in \mathbb{F}_p$.

**Proof.** Postponed. \qed

We are now left to count the dimensions of such spaces.

**Lemma 3.15.** Let $t \geq 1$ be an even integer and put $k_0' \overset{\text{def}}{=} \frac{t-1}{2}$.

The dimension of $V_{t-1}$ is then:
\[ \dim_{\mathbb{F}_p}(V_{t-1}) = \begin{cases} p^{k_0'-1}(p-1)(r+1) + (p-1)(r+1)p^{k_0'-1} - rp^{k_0'-1} & \text{if } k_0' \text{ is even} \\ p^{k_0'-1}(p-1)(r+1) + (p-1)(p-r)p^{k_0'-1} + (p-1-r) & \text{if } k_0' \text{ is odd} \end{cases} \]

for $t \geq 4$ and
\[ \dim_{\mathbb{F}_p}(V_1) = r + 1 \]

The dimension of \[ \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} \]-invariants of $R_1/R_0 \oplus R_2 \cdots \oplus R_{t-2}$ $R_{t-1}$ is given by:
\[ \dim_{\mathbb{F}_p}(R_0 \oplus R_1 \cdots \oplus R_{t-2} R_{t-1}) = \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} = \begin{cases} p^{k_0} + r + p(r+1)\frac{p^{k_0'-1}}{p+1} & \text{if } k_0 \geq 0 \text{ is even} \\ (p-1)(r+2) + (r+1)p^2\frac{p^{k_0'-1}}{p+1} + p(p^{k_0'-1} - 1) & \text{if } k_0 \text{ is odd} \end{cases} \]

**Proof.** Computation. \qed

4. Study of invariants in the amalgamated sum -II

In the present section we are going to complete our study of $\Gamma_1(p^k)$-invariants for supersingular representations $\pi(r, 0, 1)$ of $\text{GL}_2(\mathbb{Q}_p)$, with $r \neq 0, p - 1$.

To be more precise, for $k \in \mathbb{N}_{\geq 1}$ we describe in detail the spaces
\[ W_k = (\cdots \oplus R_k \; R_{k+1})^{\Gamma_1(p^k)}/(\cdots \oplus R_{k-2} \; R_{k-1})^{\Gamma_1(p^k)} \]
\[ \tilde{W}_k = (\cdots \oplus R_{k-1} \; R_k)^{\Gamma_1(p^k)}/(\cdots \oplus R_{k-2} \; R_{k-3})^{\Gamma_1(p^k)} \]
together with the results in section §3 we will then be able to compute the dimension of $\Gamma_1(p^k)$-invariants (proposition 4.21).

We start with the following, elementary, observation:

$$\Gamma_1(p^k) = \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 + p^k Z_p & p^k Z_p \\ 1 + p^k Z_p & p^k Z_p \end{bmatrix} \text{ for } k \geq 1;$$

(\cdots \oplus_{R_{k-2-i}} R_{k-1-i})^{\Gamma_1(p^k)} = (\cdots \oplus_{R_{k-2-i}} R_{k-1-i}) \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} \text{ for } i \in \{0, 1\}.

We are now lead to the analysis of the two cases $W_k$ and $\tilde{W}_k$.

4.1 Study of $W_k$

An immediate consequence of corollary 2.6 and proposition 3.5 in [Mo] is that

**Lemma 4.1.** Let $k \geq 2$. Then an $\overline{F}_p$-basis for $(R_{k+1}/R_k)^{\Gamma_1(p^k)}$ is described by:

a) the element $x_{0,0,0,r+1}(0)$;

b) for $1 \leq n \leq k+1/2$ the elements

$$x'_{l_1,\ldots,l_{2n-1},0\ldots,0,r+1}(0)$$

where $l_n \in \{1, \ldots, p-1\}$ and $l_u \in \{0, \ldots, p-1\}$ for $u \in \{n+1, \ldots, 2n-1\}$ (if non empty);

c) for $k+1/2 \leq n \leq k$ the elements

$$x'_{l_1,\ldots,l_{k-1},r+1}(0)$$

where, if $n < k$, we convene that $l_n \in \{1, \ldots, p-1\}$ and $l_u \in \{0, \ldots, p-1\}$ for $u \in \{n+1, \ldots, k-1\}$ (if non empty)

We can now describe an $\overline{F}_p$-basis for the subspace $V_{k+1} \land (R_{k+1}/R_k)^{\Gamma_1(p^k)}$:

**Proposition 4.2.** Let $k \geq 2$ be an integer. An $\overline{F}_p$-basis for $V_{k+1} \land (R_{k+1}/R_k)^{\Gamma_1(p^k)}$ is described as follow:

1) for $k$ odd the elements:

$$x'_{l_{k+1}/2,\ldots,l_{k-1},r+1}(0)$$

where $l_u \in \{0, \ldots, p-1\}$ for $u \in \{k+1/2, \ldots, k-1\}$.

2) Assume $k$ even. Then the basis is described by the elements

$$x'_{l_{k+2}/2,\ldots,l_{k-1},r+1}(0)$$

where $l_u \in \{0, \ldots, p-1\}$ for $u \in \{k+2/2, \ldots, k-1\}$, and the elements

a2) if $k/2$ is odd the elements

$$x'_{l_{k+1}/2,\ldots,l_{k-1},r+1}(0)$$

with $l_{k/2} \in \{1, \ldots, p-1\}$ and $(l_{k+2/2}, \ldots, l_{k-1}) \prec (r, p-1-r, \ldots, p-1-r)$; the elements

$$x'_{l_{k/2},r,p-1-r,\ldots,p-1-r,r+1}(0)$$

for $l_{k/2} \in \{p-2, p-1, 1, \ldots, [p-3-r] - 1\}$ together with

$$x'_{[p-3-r],r,p-1-r,\ldots,p-1-r,r+1}(0) + c_0 y_{[p-3],p-1-r,\ldots,p-1-r,r+1}(0)$$

where $c_0 \in \overline{F}_p$ is a suitable constant;
b2) if $\frac{k}{2}$ is even the elements

$$x'_{l_k, \ldots, l_{k-1}, r+1}(0)$$

with $l_k \in \{1, \ldots, p - 1\}$ and $(l_{k+2}, \ldots, l_{k-1}) \prec (p - 1 - r, \ldots, p - 1 - r)$; the elements

$$x_{l_k, p-1-r, \ldots, p-1-r, r+1}(0)$$

for $l_k \in \{p - 2, p - 1, 1, \ldots, [r - 2] - 1\}$ together with

$$x'_{[r-2], p-1-r, \ldots, p-1-r, r+1}(0) + c_0 y_{[p-3-r], r, \ldots, p-1-r, r+1}(0)$$

where $c_0 \in \mathbf{F}_p$ is a suitable constant.

Proof. Postponed. □

For sake of completeness, we recall the results for $k = 1$.

**Lemma 4.3.** For $k = 1$ the space $V_2 \cap (R_2/R_1)^{\Gamma_1(p)}$ is 1-dimensional, and a basis is given by the element

$$x'_{r+1}(0).$$

Let $v \in (\cdots \oplus R_k R_{k+1})$ be the canonical lift of an element $\overline{v} \in V_k \cap (R_{k+1}/R_k)^{\Gamma_1(p_k)}$. If we write $pr$ for the map

$$(\cdots \oplus R_k R_{k+1})^{\Gamma_1(p_k)} \xrightarrow{pr} R_{k+1}/R_k$$

then we see that $\overline{v}$ is in the image of $pr$ iff it exists $y \in \cdots \oplus R_{k-2} R_{k-1}$ such that $y + v \in (\cdots \oplus R_k R_{k+1})^{\Gamma_1(p_k)}$ which is equivalent to $v \in (\cdots \oplus R_k R_{k+1})^{\Gamma_1(p_k)}$ since $v$ is $K_{k+1}$-invariant and $y$ is $K_k$-invariant in the amalgamed sum.

We outline the elementary result:

**Lemma 4.4.** Let $k \geq 1$. The action of

$$\begin{bmatrix} 1 + p^k \mathbf{Z}_p & 0 \\ 0 & 1 + p^k \mathbf{Z}_p \end{bmatrix}$$

is trivial on the canonical lifts of the elements in $V_k \cap (R_{k+1}/R_k)^{\Gamma_1(p_k)}$. Moreover if $1 \leq n \leq k - 1$ we have

$$\begin{bmatrix} 1 \\ p^k \mu \end{bmatrix} x'_{l_n, \ldots, l_{k-1}, r+1}(0) = x'_{l_n, \ldots, l_{k-1}, r+1}(0) +
+ (r + 1)(-1)^{r+1}(-\mu)(\kappa(l_{k-1})) y'_{l_n, \ldots, l_{k-2}}(r - (p - 1 - l_{k-1}))$$

where we define

$$\kappa(l_{k-1}) \overset{\text{def}}{=} \ \begin{cases} 0 & \text{if } l_{k-1} < p - 1 - r; \\
\neq 0 & \text{if } l_{k-1} \geq p - 1 - r. 
\end{cases}$$

(with the convention that, for $n = k - 1$, $y'_{l_{k-1}}(x) = [1, X^{r-x} Y^x]$).

Proof. Postponed. □

We define $\mathcal{U}$ as the $\mathbf{F}_p$-subspace of $(\cdots \oplus R_k R_{k+1})$ generated by the canonical lifts of $V_k \cap (R_{k+1}/R_k)^{\Gamma_1(p_k)}$. Then $(\cdots \oplus R_{k-2} R_{k-1}) + \mathcal{U}$ is a $\begin{bmatrix} 1 & 0 \\ p^k \mathbf{Z}_p & 1 \end{bmatrix}$-stable subspace of $(\cdots \oplus R_k R_{k+1})$.

4.1.1 **The case $k$ odd.** Assume now $k \geq 2$, $k$ odd. We have the following result:

**Lemma 4.5.** Let $k \geq 2$, $k$ odd. We consider $j \in \mathbb{N}$ such that $k - 2j - 1 > \frac{k+1}{2}$. Then the $\begin{bmatrix} 1 & 0 \\ p^k \mathbf{Z}_p & 1 \end{bmatrix}$-invariants of $((R_{k-2j-1}/R_{k-2j-2}) \oplus \cdots \oplus R_{k-1}) + \mathcal{U}$ are described by:
a) the space \( (R_{k-2j_1}/R_{k-2j_2}) \oplus \cdots \oplus R_{k-1} \); 
b) the elements
\[
x'_{l_{k+1}, \ldots, l_{k-1}, r+1}(0)
\]
where \( (l_{k-2j_1}, \ldots, l_{k-1}) \leq (r, p-1-r, \ldots, p-1-r) \) and \( (l_{k+1}, \ldots, l_{k-2j_2}) \in \{0, \ldots, p - 1 \}^{k-2j_2 - \frac{k+1}{2}} \).

**Proof.** Postponed. (induction on \( j \)).

We therefore deduce:

**PROPOSITION 4.6.** Let \( k \geq 2 \) be odd. An \( \mathbb{F}_p \)-basis for \( W_k \) is described by the elements
\[
x'_{l_{k+1}, \ldots, l_{k-1}, r+1}(0)
\]
where
\[
(l_{k+1}, \ldots, l_{k-1}, r+1) \in \begin{cases} 
(p - 1 - r, \ldots, r, p - 1 - r) & \text{if } k+1 \in 2\mathbb{N} \\
(r, p - 1 - r, \ldots, r, p - 1 - r) & \text{if } k+1 \in 2\mathbb{N} + 1.
\end{cases}
\]

**Proof.** Postponed.

For \( k = 1 \) we get

**LEMMA 4.7.** For \( k = 1 \) we have
\[
\dim_{\mathbb{F}_p}(W_1) = 0.
\]

**Proof.** Postponed.

### 4.1.2 The case \( k \) even. In this section we assume that \( k \in \mathbb{N} \) is an even integer. We have then

**LEMMA 4.8.** Let \( j \in \mathbb{N} \) be such that \( k - 2j - 1 > \frac{k}{2} + 1 \). The space of \( \begin{bmatrix} 1 & 0 \\
p^k \mathbb{Z}_p & 1 \end{bmatrix} \)-invariants of
\[
((R_{k-2j_1}/R_{k-2j_2}) \oplus \cdots \oplus R_{k-1}) + U
\]
a) the space \( ((R_{k-2j_1}/R_{k-2j_2}) \oplus \cdots \oplus R_{k-1}) \);

b) the elements described in 2-a2) (resp. 2-b2)) of proposition 4.2 if \( \frac{k}{2} \) is odd (resp. even);

c) the elements
\[
x'_{l_{k+1}, \ldots, l_{k-1}, r+1}(0)
\]
where \( (l_{k-2j_1}, \ldots, l_{k-1}) \leq (r, p-1-r, \ldots, p-1-r) \) and \( (l_{k+1}, \ldots, l_{k-2j_2}) \in \{0, \ldots, p - 1 \}^{k-2j_2 - \frac{k}{2}} \). Moreover, if we have \( (l_{k-2j_1}, \ldots, l_{k-1}) \leq (r, p-1-r, \ldots, p-1-r) \), the element is invariant in the amalgamand sum
\[
\lim_{n \to \infty} ((R_1/R_0) \oplus R_2 \cdots \oplus R_n / R_{n+1})
\]

**Proof.** Postponed. (Induction on \( j \)).

We are now able to describe \( W_k \) for \( k \) even:

**PROPOSITION 4.9.** Let \( k \geq 2 \) be an even integer. An \( \mathbb{F}_p \)-basis for \( W_k \) is described as follow:

1) if \( \frac{k}{2} \) is odd, the elements
\[
x'_{l_{k+1}, \ldots, l_{k-1}, r+1}(0)
\]

2) if \( \frac{k}{2} \) is even, the elements
\[
x'_{l_{k+1}, \ldots, l_{k-1}, r+1}(0)
\]

3) the elements described in 2-a2) (resp. 2-b2)) of proposition 4.2 if \( \frac{k}{2} \) is odd (resp. even).
where \((l_{\frac{k}{2}+1}, \ldots, l_{k-1}) < (r, \ldots, p-1-r)\) and \(l_{\frac{k}{2}} \in \{0, \ldots, p-1\}\) together with the following \(p-2-r\)-elements
\[
\begin{align*}
x'_{p-1,r,\ldots,p-1-r,r+1}(0) + c_1 x'_{r,p-1-r,\ldots,p-1-r,r+1}(0); \\
x'_{1,r,\ldots,p-1-r,r+1}(0); \\
\vdots \\
x'_{[p-3-r]-1,r,\ldots,p-1-r,r+1}(0); \\
x'_{[p-3-r],r,\ldots,p-1-r,r+1}(0) + c_0 y_{[p-3-r],r,\ldots,p-1-r,r+1}(0).
\end{align*}
\]

2) If \(\frac{k}{2}\) is even, the elements
\[
x'_{l_{\frac{k}{2}},\ldots,l_{k-1},r+1}(0)
\]

where \((l_{\frac{k}{2}+1}, \ldots, l_{k-1}) < (p-1-r, r, \ldots, p-1-r)\) and \(l_{\frac{k}{2}} \in \{0, \ldots, p-1\}\) together with the following \(r-1\)-elements
\[
\begin{align*}
x'_{p-1,p-1-r,\ldots,p-1-r,r+1}(0) + c_1 x'_{p-1-r,\ldots,p-1-r,r+1}(0); \\
x'_{1,p-1-r,\ldots,p-1-r,r+1}(0); \\
\vdots \\
x'_{[r-2]-1,p-1-r,\ldots,p-1-r,r+1}(0); \\
x'_{[r-2],p-1-r,\ldots,p-1-r,r+1}(0) + c_0 y_{[p-3-r],r,\ldots,p-1-r,r+1}(0).
\end{align*}
\]

We can sum up the results, giving the dimensions of the spaces \(W_k\).

**Proposition 4.10.** Let \(k \in \mathbb{N}_{\geq 1}\). The dimension of the space \(W_k\) is then given by

1) for \(k\) odd, we have
\[
\dim_{\mathbb{F}_p}(W_k) = \begin{cases} 
(p - 1 - r) \frac{p^{k+1} - 1}{p^2-1} + pr \frac{p^{k-1}-1}{p^2-1} & \text{if } \frac{k+1}{2} \in 2\mathbb{N} \\
(p - r) \frac{p^{k-1} - 1}{p^2-1} & \text{if } \frac{k+1}{2} \in 2\mathbb{N} + 1
\end{cases}
\]

2) For \(k\) even, we have
\[
\dim_{\mathbb{F}_p}(W_k) = \begin{cases} 
p(p - r) \frac{p^{k-1} - 1}{p+1} + (p - 2 - r) & \text{if } \frac{k}{2} \in 2\mathbb{N} + 1 \\
p[(p - 1 - r) \frac{p^{k-1} - 1}{p^2-1} + pr \frac{p^{k-2} - 1}{p^2-1}] + (r - 1) & \text{if } \frac{k+1}{2} \in 2\mathbb{N}
\end{cases}
\]

**4.2 Study of \(\tilde{W}_k\)**

In this section, we follow closely the steps which led us to the description of \(W_k\) in paragraph 4.1.

Again, we use corollary 2.6 and proposition 3.5 in [Mo] to get

**Lemma 4.11.** Let \(k \geq 3\) be an integer. An \(\mathbb{F}_p\)-basis for \((R_k/R_{k-1})^{\Gamma_1(p^k)}\) is described as follow:

a) the element \(x_{0,\ldots,0,r+1}(0)\)

b) for \(1 \leq n < \frac{k}{2}\) the elements
\[
x'_{l_n,\ldots,l_{2n-1},0,\ldots,0,r+1}(0)
\]
where the indices \(l_n\) verify the conditions in ii) of proposition 2.6;

c) for \(n \in \{\frac{k}{2}, \frac{k+1}{2}\} \cap \mathbb{N}\) the elements
\[
x'_{l_n,\ldots,l_{k-2},l_{k-1}^{(i)}}(i)
\]


where \( i \in \{0, 1\}, l_{k-1}^{(0)} \in \{r + 1, \ldots, p - 1\}, l_{k-1}^{(1)} \in \{0, \ldots, r\}\) and \((l_n, \ldots, l_{k-2}) \in \{0, \ldots, p - 1\}^{k-1-n}.

For \( k = 2 \) an \( \overline{F}_p \)-basis for \((R_2/R_1)^{\Gamma_1(p^2)}\) is given by

\[
\begin{align*}
\text{a2)} & \quad \text{the element } x_{0,r+1}(0); \\
\text{b2)} & \quad \text{the elements } \\
& \quad x'_r(0), \ldots, x'_{p-1}(0); \\
\text{c2)} & \quad \text{the elements } \\
& \quad x'_0(1), \ldots, x'_{p-2}(1)
\end{align*}
\]

[...]

For \( k = 1 \) an \( \overline{F}_p \)-basis for \((R_1/R_0)^{\Gamma_1(p)}\) is given by

\[x_r(0)\]

We deduce an \( \overline{F}_p \)-basis for the space \( V_k \wedge (R_k/R_{k-1})^{\Gamma_1(p^k)}\):

**Lemma 4.12.** Let \( k \in \mathbb{N}, k \geq 3 \). An \( \overline{F}_p \)-basis for the space \( V_k \wedge (R_k/R_{k-1})^{\Gamma_1(p^k)}\) is described as follows.

1) Assume \( k \) even. Then we have the elements

\[
\begin{align*}
\text{a1)} & \quad x'_{l_{k+1}^{(0)} \ldots l_{k-2}^{(1)}}(i) \\
& \quad \text{where } i \in \{0, 1\}, l_{k-1}^{(0)} \in \{r + 1, \ldots, p - 1\}, l_{k-1}^{(1)} \in \{0, \ldots, r\}\) and \((l_{k+1}^{(0)}, \ldots, l_{k-2}) \in \{0, \ldots, p - 1\}^{k-2}; \\
\text{b1)} & \quad x'_{l_{k+1}^{(0)} \ldots l_{k-1}^{(0)}}(0) \\
& \quad \text{where } l_{k-1} \in \{r + 1, \ldots, p - 1\}, l_{k} \in \{1, \ldots, p - 1\} \text{ and } (l_{k+1}^{(0)}, \ldots, l_{k-2}) \in \{0, \ldots, p - 1\}^{k-2}; \\
\text{c1)} & \quad \text{According to the parity of } \frac{k}{2} \text{ we have} \\
\text{c1.1)} & \quad \text{if } \frac{k}{2} \text{ is even the elements } \\
& \quad x'_{l_{k+1}^{(0)} \ldots l_{k-1}^{(1)}}(1) \\
& \quad \text{where } l_{k} \in \{1, \ldots, p - 1\} \text{ and } (l_{k+1}^{(0)}, \ldots, l_{k-2}) < (r, \ldots, r), \text{ together with the elements} \\
& \quad x'_{p-2,r,\ldots,r}(1); \\
& \quad x'_{p-1,r,\ldots,r}(1); \\
& \quad x'_{1,r,\ldots,r}(1); \\
& \quad \ldots \\
& \quad x'_{p-3-r\ldots,1,r,\ldots,r}(1); \\
& \quad x'_{p-3-r\ldots,r}(1) + c_0 y'_{[p-3],p-1-r,\ldots,r}(1); \\
& \quad \text{(with } c_0 \in \mathbb{F}_p \text{ a suitable constant);} \\
\text{c1.2)} & \quad \text{if } \frac{k}{2} \text{ is odd the elements } \\
& \quad x'_{l_{k+1}^{(0)} \ldots l_{k-1}^{(1)}}(1)
\end{align*}
\]
where \( l_k \in \{1, \ldots, p-1\} \) and \( (l_{k+1}^{(i)}, \ldots, l_{k-1}^{(i)}) \prec (p-1-r, \ldots, r) \), together with the elements

\[
x'_{p-2},r,\ldots,r(1); \\
x'_{p-1},r,\ldots,r(1); \\
x'_{1},r,\ldots,r(1); \\
\vdots \\
x'_{[r-2]-1},r,\ldots,r(1); \\
x'_{[r-2]},r,\ldots,r(1) + c_0 y'_{[p-3-r],r,\ldots,r}(1);
\]

(with \( c_0 \in \mathbb{F}_p \) a suitable constant);

2) Assume \( k \) odd. Then we have the elements

a2)

\[
x'_{l_{k+1}^{(i)},\ldots,l_{k-2}^{(i)}}(i)
\]

where \( i \in \{0, 1\}, l_{k-1}^{(0)} \in \{r+1, \ldots, p-1\}, l_{k-1}^{(i)} \in \{0, \ldots, r\} \) and \( (l_{k+1}^{(i)}, \ldots, l_{k-2}^{(i)}) \in \{0, \ldots, p-1\} \);

b2) According to the parity of \( \frac{k-1}{2} \) we have:

b2.1) if \( \frac{k-1}{2} \) is odd, the elements

\[
x'_{l_{k+1}^{(i)},\ldots,l_{k-2}^{(i)}}(0)
\]

where \( (l_{k+1}^{(i)}, \ldots, l_{k-2}^{(i)}) \prec (r, \ldots, p-1-r), l_{k-1}^{(i)} \in \{1, \ldots, p-1\} \) together with the elements

\[
x'_{p-2},r,\ldots,p-1-r,r+1(0); \\
x'_{p-1},r,\ldots,p-1-r,r+1(0); \\
x'_{1},r,\ldots,p-1-r,r+1(0); \\
\vdots \\
x'_{[p-3-r]-1},r,\ldots,p-1-r,r+1(0); \\
x'_{[p-3-r],r,\ldots,p-1-r,r+1}(0) + c_0 y'_{[p-3-r],p-1-r,r,\ldots,p-1-r,r+1}(0);
\]

(with \( c_0 \in \mathbb{F}_p \) a suitable constant, and, for \( k = 3 \), \( y'_{\ldots} \) is remplaced by \( y_{r+1}(0) \));

b2.2) if \( \frac{k-1}{2} \) is even, the elements

\[
x'_{l_{k-1}^{(i)},\ldots,l_{k-2}^{(i)}}(0)
\]

where \( (l_{k+1}^{(i)}, \ldots, l_{k-2}^{(i)}) \prec (p-1-r, \ldots, p-1-r), l_{k-1}^{(i)} \in \{1, \ldots, p-1\} \) together with
the elements

\[
\begin{align*}
x'_{p-2,p-1-r,\ldots,p-1-r,r+1}(0); \\
x'_{p-1,p-1-r,\ldots,p-1-r,r+1}(0); \\
x'_{1,p-1-r,\ldots,p-1-r,r+1}(0); \\
\vdots \\
x'_{[r-2],p-1-r,\ldots,p-1-r,r+1}(0); \\
x'_{[r-2],[r-3],\ldots,p-1-r,r+1}(0) + c_0 y'_{[p-3-r],r,\ldots,p-1-r,r+1}(0);
\end{align*}
\]

(with \(c_0 \in \mathbb{F}_p\) a suitable constant).

\textit{Proof.} Postponed. \qed

For sake of completeness, we have cover the cases \(k \in \{1, 2\} \):

\textbf{Lemma 4.13.} For \(k = 2\) an \(\mathbb{F}_p\)-basis for \(V_2 \wedge (R_2/R_1)_{\Gamma_1(p^2)}\) is described by the elements \(b2\), \(c2\) of lemma 4.11; for \(k = 1\) an \(\mathbb{F}_p\)-basis for \(V_1 \wedge (R_1/R_0)_{\Gamma_1(p)}\) is described by the element \(x_r(0)\).

We are lead to distinguish two situations, according to the parity of \(k\).

\textbf{4.2.1 The case \(k\) even.} In this paragraph we fix \(k \in 2\mathbb{N}, k \geq 2\). We start with the following observation

\textbf{Lemma 4.14.} In the amalgamed sum \(\lim_{n, \text{odd}} R_0 \oplus R_1 \cdots \oplus R_n, R_{n+1}\) the action of \(
\begin{bmatrix}
1 + p^k \mathbb{Z}_p & 0 \\
0 & 1 + p^k \mathbb{Z}_p
\end{bmatrix}
\)
is trivial on the lifts of the elements \(1)\) in proposition 4.12, as well as on the elements described in lemma 4.13.

The action of \(\begin{bmatrix}
1 + p^k \mathbb{Z}_p & 0 \\
0 & 1 + p^k \mathbb{Z}_p
\end{bmatrix}\) is trivial on the lifts of the elements

\(x^f_{l_{1,2},\ldots,l_{k-1}}(0)\)

where \((l_{1,2},\ldots,l_{k-2}) \in \{0, \ldots, p - 1\}^{k-1} \) and \(l_{k-1} \in \{r + 1, \ldots, p - 1\}\).

Finally, let \(n \in \left\{\frac{k}{2} + 1, \frac{k+1}{2}\right\} \cap \mathbb{N}\) and assume \(k \geq 6\). We have the following equality in the amalgamed sum:

\[
\begin{bmatrix}
1 + p^k [j] & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x^f_{l_{1,2},\ldots,l_{k-1}}(1)
\end{bmatrix}
= \begin{bmatrix}
x^f_{l_{1,2},\ldots,l_{k-1}}(1) + \delta_{r,l_{k-1}}(r + 1)(-1)^{r+1} \mu \kappa(l_{k-2}) y_{l_{1,2},\ldots,l_{k-3}}(r - (p - 1 - l_{k-2}))
\end{bmatrix}
\]

where we define

\[
\kappa(l_{k-2}) \overset{\text{def}}{=} \begin{cases}
0 & \text{if } l_{k-2} < p - 1 - r; \\
\neq 0 & \text{if } l_{k-2} \geq p - 1 - r.
\end{cases}
\]

(and with the convention that, for \(k = 6\), \(y_{l_{1,2}}(x) = [1, X^{r-x}Y^{x}]\)).

For \(k \geq 4\), let \(U\) be the subspace of \(R_0 \oplus R_1 \cdots \oplus R_{k-1} R_k\) generated by the (canonical lift of the) following elements:

a) the elements \(c1.1\) (resp. \(c1.2\)) of lemma 4.12-1) if \(\frac{k}{2}\) is even (resp. odd);
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b) the elements

\[ x'_{l_{k-1}}(1) \]

where \( l_{k-1} \in \{0, \ldots, r\} \) and \((l_{k-2+1}, \ldots, l_{k-2}) \in \{0, \ldots, p-1\}\).

As in §4.1.1 we start with a lemma

**Lemma 4.15.** Let \( k \geq 4 \) be an even integer, and let \( j \in \mathbb{N} \) be such that \( k - 2j - 2 > \frac{k}{2} + 1 \). Then, the space of \( \left[ \begin{array}{c} 1 \\ p^{k/2} \mathbb{Z}_p \\ 1 \end{array} \right] \)-invariants of \((R_{k-2j-2}/R_{k-2j-3}) \oplus \cdots \oplus R_{k-2} + U \) is described by

a) the space \((R_{k-2j-2}/R_{k-2j-3}) \oplus \cdots \oplus R_{k-2}\);

b) the elements c1.1) (resp. c1.2)) of lemma 4.12-1) if \( k \) is even (resp. odd);

c) the elements

\[ x'_{l_{k-1}}(1) \]

where \((l_{k-2j-3}, \ldots, l_{k-1}) \preceq (r, \ldots, r)\) and \((l_{k-2+1}, \ldots, l_{k-2j-4}) \in \{0, \ldots, p-1\}\). Moreover, if \((l_{k-2j-3}, \ldots, l_{k-1}) \prec (r, \ldots, r)\), such elements are invariant in the amalgamed sum

\[ \lim_{n, \text{odd}} R_0 \oplus R_1 \cdots \oplus R_n R_{n+1}. \]

Thanks to the preceding lemma, we are able to describe an \( \mathbb{F}_p \)-basis for \( \widetilde{W}_k \), when \( k \) is even.

**Proposition 4.16.** Let \( k \in 2\mathbb{N} \) be a non zero even integer. An \( \mathbb{F}_p \)-basis for the space \( \widetilde{W}_k \) is described as follow.

a) The elements

\[ x'_{l_{k-1}}(0) \]

where \( l_{k-1} \in \{r+1, \ldots, p-1\} \) and \((l_{k-2+1}, \ldots, l_{k-2}) \in \{0, \ldots, p-1\}\);

b) according to the parity of \( \frac{k}{2} \) the elements

b1) if \( \frac{k}{2} \) is odd, the \( r-1 \) elements

\[ x'_{0,p-1-r,\ldots,r}(1); \]
\[ x'_{1,p-1-r,\ldots,r}(1); \]
\[ \vdots \]
\[ x'_{[r-2]-1,p-1-r,\ldots,r}(1); \]
\[ x'_{[r-2],p-1-r,\ldots,r}(1) + c_0 y'_{[p-3]-r,\ldots,r}(1) \]

(where \( y'_{\ldots} \) has to be replaced by \( X Y^{r-1} \in R_0 \) if \( k = 2 \) and \( c_0 \in \mathbb{F}_p \) is a suitable constant) together with the elements

\[ x'_{l_{k-1}}(1) \]

where \((l_{k-2+1}, \ldots, l_{k-1}) \prec (p-1-r, \ldots, r)\) and \( l_{k-2} \in \{0, \ldots, p-1\} \).
b2) if \( \frac{k}{2} \) is even, the \( p-2-r \) elements

\[
\begin{align*}
x_0', r, \ldots, r(1); \\
x_1', r, \ldots, r(1); \\
& \vdots \\
x_{[p-3-r]-1}', r, \ldots, r(1); \\
x_{[p-3-r]}', r, \ldots, r(1) + c_0 y_{[p-3], p-1-r, \ldots, r}(1)
\end{align*}
\]

(where \( c_0 \in \mathbb{F}_p \) is a suitable constant) together with the elements

\[ x_{l_{\frac{k}{2}}, \ldots, l_{k-1}}(1) \]

where \((l_{\frac{k}{2}+1}, \ldots, l_{k-1}) \prec (r, \ldots, r)\) and \( l_{\frac{k}{2}} \in \{0, \ldots, p-1\} \).

4.2.2 The case \( k \) odd Assume now \( k \) an odd integer. As the element \( x_r(0) \in R_1/R_0 \) is clearly \( \Gamma_1(p) \)-invariant, we will assume \( k \geq 3 \) throughout this paragraph.

As in the previous section we have

**Lemma 4.17.** In the amalgamed sum \( \lim_{n\to\infty} (R_1/R_0) \oplus R_2 \cdots \oplus R_n R_{n+1} \) the action of

\[
\begin{bmatrix}
1 + p^k \mathbb{Z}_p & 0 \\
0 & 1 + p^k \mathbb{Z}_p
\end{bmatrix}
\]

is trivial on the lifts of the elements 2) in proposition 4.12.

The action of

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

is trivial on the lifts of the elements

\[ x_{l_{\frac{k}{2}}, \ldots, l_{k-1}}(0) \]

where \((l_{\frac{k}{2}+1}, \ldots, l_{k-2}) \in \{0, \ldots, p-1\}^{\frac{k+1}{2}-2} \) and \( l_{k-1} \in \{r+1, \ldots, p-1\} \).

We therefore define \( \mathcal{U} \) as the \( \overline{\mathbb{F}}_p \)-subspace of \((R_1/R_0) \oplus R_2 \cdots \oplus R_{k-1} R_k \) generated by the (canonical lifts of the) elements

\[ x_{l_{\frac{k+1}{2}}, \ldots, l_{k-1}}(1) \]

where \( l_{k-1} \in \{0, \ldots, r\} \) and \((l_{\frac{k+1}{2}}, \ldots, l_{k-2}) \in \{0, \ldots, p-1\}^{\frac{k+1}{2}-2} \).

We have

**Lemma 4.18.** Let \( k \geq 3 \) be an odd integer and let \( j \in \mathbb{N} \) be such that \( k - 2j - 2 > \frac{k+1}{2} \). The space of

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]-invariants of \((R_{k-2j-2}/R_{k-2j-3}) \oplus \cdots \oplus R_{k-3} R_{k-2}) + \mathcal{U} \) is described by:

a) the space \((R_{k-2j-2}/R_{k-2j-3}) \oplus \cdots \oplus R_{k-3} R_{k-2})\);

b) the elements

\[ x_{l_{\frac{k+1}{2}}, \ldots, l_{k-1}+1}(1) \]

where \((l_{k-2j-3}, \ldots, l_{k-1}) \preceq (r, \ldots, r) \) and \((l_{\frac{k+1}{2}}, \ldots, l_{k-2j-4}) \in \{0, \ldots, p-1\}^{\frac{k+1}{2}-2j-3} \). Moreover, such elements are invariant in the amalgamed sum \((R_1/R_0) \oplus R_2 \cdots \oplus R_{k-1} R_k \) if \((l_{k-2j-3}, \ldots, l_{k-1}) \prec (r, \ldots, r)\).

**Proof.** Postponed.

We finally get the description of \( \overline{W}_k \) for \( k \geq 3, k \) odd.
Proposition 4.19. Let $k \in \mathbb{N}$ be an odd integer, and assume $k \geq 3$. An $\mathbb{F}_p$-basis for $\tilde{W}_k$ is described as follows.

a) the elements

$$x'_{l_{k+1}, \ldots, l_{k-1}}(0)$$

where $l_{k-1} \in \{r+1, \ldots, p-1\}$ and $(l_{k+1}, \ldots, l_{k-2}) \in \{0, \ldots, p-1\}^{k+1-2}$;

b) according to the parity of $\frac{k-1}{2}$ we have

b1) if $\frac{k-1}{2}$ is even, the elements in b2.2) of lemma 4.12 together with the elements

$$x'_{l_{k+1}, \ldots, l_{k-1}}(1)$$

with $(l_{k+1}, \ldots, l_{k-1}) \prec (p-1-r, \ldots, r)$;

b2) if $\frac{k-1}{2}$ is odd, the elements in b2.1) of lemma 4.12 together with the elements

$$x'_{l_{k+1}, \ldots, l_{k-1}}(1)$$

with $(l_{k+1}, \ldots, l_{k-1}) \prec (r, \ldots, r)$.

Proof. Postponed.

We sum up what we can sum up the results, giving the dimensions of the spaces $W_k$.

Proposition 4.20. Let $k \in \mathbb{N}_{\geq 3}$. The dimension of the space $\tilde{W}_k$ is then given by

1) for $k$ odd, we have

$$\dim_{\mathbb{F}_p}(\tilde{W}_k) = \begin{cases} p((p-1-r)p^{\frac{k-1}{2}-1} + pr^{\frac{k-3}{2}-1}) + r + (p-1)p^{\frac{k-3}{2}} & \text{if } \frac{k-1}{2} \in 2\mathbb{N} \\ p(p-r)p^{\frac{k-3}{2}} + (p-1-r) + (p-1)p^{\frac{k-3}{2}} & \text{if } \frac{k-1}{2} \in 2\mathbb{N} + 1 \end{cases}$$

2) For $k$ even, we have

$$\dim_{\mathbb{F}_p}(\tilde{W}_k) = \begin{cases} (p-1-r)p^{\frac{k}{2}-1} + p(r+1)p^{\frac{k-1}{2}-1} + (r-1) & \text{if } \frac{k}{2} \in 2\mathbb{N} + 1 \\ (p-1-r)p^{\frac{k}{2}-1} + p(r) + (p-1-r)p^{\frac{k-4}{2}} + (p-2-r) & \text{if } \frac{k+1}{2} \in 2\mathbb{N} \end{cases}$$

We are finally able to compute the dimension of $\Gamma_1(p^k)$-invariants, using propositions 3.15, 4.10, 4.20:

Theorem 4.21. Let $k \in \mathbb{N}_{\geq 1}$ be an integer and $r \in \{1, \ldots, p-1\}$. Then the dimension of $\Gamma_1(p^k)$-invariants for the supersingular representation $\pi(r, 0, 1)$ of $GL_2(\mathbb{Q}_p)$ is described as follows:

$$\dim_{\mathbb{F}_p}(\pi(r, 0, 1)^{\Gamma_1(p^k)}) = \begin{cases} 2(2p^{\frac{k-1}{2}} - 1) & \text{if } k \text{ is odd;} \\ 2(p^\frac{k}{2} + p^{\frac{k+2}{2}} - 2) & \text{if } k \text{ is even.} \end{cases}$$

Proof. Postponed.
Study of $\Gamma_1(p^k)$ invariants for supersingular representations of $GL_2(\mathbb{Q}_p)$

References

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