SOCLE FILTRATION FOR SOME MODULAR REPRESENTATIONS OF $\mathrm{GL}_2(\mathbf{F}_p)$

STEFANO MORRA

ABSTRACT. Let p be a prime number and let $k \in \mathbf{N}$ be an integer. We show an algorithm describing the socle filtration of the modular representations $\mathbf{Sym}^k\overline{\mathbf{F}}_p^2$ of $\mathrm{GL}_2(\mathbf{F}_p)$.

1. INTRODUCTION

The study of modular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over a field of characteristic p intensified greatly in the last few years. Such an interest comes from the research of a -still conjectural- modulo p Langlands correspondence, whose most up to date reference is the work of Breuil-Paskunas [7]. The first hypothesis about the possibility of such a correspondence are in Breuil's work [6]. In the latter, the author conjectures an explicit correspondence between the semisimplified of some 2-dimensional crystalline irreducible Gaois representations V_{k,a_p} of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ over $\overline{\mathbf{Q}_p}$ and the semisimplified of certain modular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$, proving it with suitable restrictions on the Hodge-Tate weights of V_{k,a_p} (see Théorème 3.3.6). The recents works of Berger-Breuil [5] and Colmez [10] let us dispose of a p-adic Local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ and we know thanks to the recent results of Berger [4] that such a correspondence is compatible with the reduction modulo p in the sense specified by Breuil (definition 1.1 in [6]).

Now, two questions arise naturally:

-) Given an irreducible 2-dimensional crystalline representation V_{k,a_p} of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ over $\overline{\mathbf{Q}_p}$, is it possible to know explicitely its "reduction modulo p", dependingly from k, a_p?

-) Is it possible to generalize Breuil's correspondence modulo p when we consider finite and unramified extention F of $\overline{\mathbf{Q}_p}$?

Concerning the first question, a recent work of Buzzard and Gee [9] gives the answer when $k \neq 3 \bmod p-1$ and the p-adic valuation $val(a_p)$ of the slope a_p is in $]0,1[$. Concerning the second, the works of Breuil-Paskunas [7], and Hu [11] show that the situation is far more complicate than for $\mathrm{GL}_2(\mathbf{Q}_p)$ and nowadays we don't dispose of satisfactory means to describe smooth irreducible admissible representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbf{F}}_p$ for F finite and unramified extention of $\mathbf{Q}_p$.

On the other hand, modular representations of $\mathrm{GL}_2(\mathbf{F}_q)$ and $\mathrm{SL}_2(\mathbf{F}_q)$ (where $q = p^n$) are widely studied since the '70. For instance, a complete description of the principal indecomposable modules for $\mathrm{SL}_2(\mathbf{F}_q)$ can be found in the works of [3] and [13], the periodicity of the Weil modules attached to the representations $\mathrm{Sym}^m\overline{\mathbf{F}}_p^2$ of $\mathrm{SL}_2(\mathbf{F}_q)$ is given in [2], and a detailed study of the representations

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$\mathrm{Sym}^m\overline{\mathbf{F}}_p^2$ of $\mathrm{GL}_2(\mathbf{F}_p)$ is given by the paper of Glover [12].

In particular, the possibility of having an explicit decomposition of the representation $\mathrm{Sym}^m\overline{\mathbf{F}}_p^2$ of $\mathrm{GL}_2(\mathbf{F}_p)$ has became a suitable tool in the study of explicit reduction of modular represenions of $\mathrm{GL}_2(\mathbf{Q}_p)$, cf. [6], lemme 5.1.3 or [8], lemme 4.1.4. It's therefore quite a surprise that we don't find in the literature a reference concerning that thanks to [12] we do dispose of the socle filtration of the representation $\mathrm{Sym}^m\overline{\mathbf{F}}_p^2$!

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2. Description of the socle filtration

We remind the standard notations. We fix a prime number p, $\mathbf{F}_p$ the field with p elements, and $\overline{\mathbf{F}}_p$ a fixed algebraic closure of $\mathbf{F}_p$. The group $\mathrm{GL}_2(\mathbf{F}_p)$ acts naturally on $\overline{\mathbf{F}}_p^2$ from which we deduce a natural action of G over the symmetric algebra $\mathrm{Sym}(\overline{\mathbf{F}}_p^2)$, compatible with the natural grading. Given $m \in \mathbf{N}$, we will write V_m for the G-representation given by the $m-1$ homogeneous component $\mathrm{Sym}^{m-1}\overline{\mathbf{F}}_p^2$ of $\mathrm{Sym}(\overline{\mathbf{F}}_p^2)$ (with the convention that $V_0 = 0$). Withouth any explicit remark, all the objects will be G-representations over $\overline{\mathbf{F}}_p$, and all morphisms will be G equivariant.

We remind the description of irreducible G-representations over $\overline{\mathbf{F}}_p$:

Lemma 2.1. *Given $m \in \{1, \dots, p\}$, $n \in \{0, \dots, p-2\}$, the representations*

$$V_m \otimes \det^n$$

are pairwise non isomorphic, and give all the irreducible represntations of G over $\overline{\mathbf{F}}_p$.

Proof: Omissis. Cf. [12], (6.2). ♯

We recall that for each irreducible G-representation V, we can associate the principal indecomposable module $P(V)$ (PIM for short), which is a projective indecomposable G-representation, whose isomorphism class is determined by its socle (cf. [1], §2). Then, given $m \in \{1, \dots, p\}$ and $n \in \{0, \dots, p-2\}$ we write $P[m, n]$ for the PIM of socle $V_m \otimes \det^n$.

Proposition 2.2. *Let $m \in \{1, \dots, p\}$, $n \in \{0, \dots, p-2\}$. The socle filtration associated to $P[m, n]$ is then*

i) $V_1 \otimes \det^n$ —— $V_{p-2} \otimes \det^{n+1}$ —— $V_1 \otimes \det^n$ *if $m = 1$;*

ii) $V_m \otimes \det^n - V_{p-1-m} \otimes \det^{m+n} \oplus V_{p+1-m} \otimes \det^{m+n-1} - V_m \otimes \det^n$ *if $2 \leq m \leq p-1$;*

iii) $V_p \otimes \det^n$ *if $m = p$.*

Proof: Omissis. Cf. [12], (6.1).♯

We notice that for sufficiently small weights the decomposition in PIM of projective G-representations is completely explicit. More precisely:

Proposition 2.3. *Let $k \in \{0, \dots, p-1\}$. Then we have*

$$V_{kp} \cong \bigoplus_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} P[p-k+1+2j, k-1-j].$$

Proof: Omissis. Cf. [12], (6.3).♯

We recall ([12]) that for any $m \in \mathbf{N}$, we have a decomposition

$$V_m \cong \mathrm{Pr}(V_m) \oplus \mathrm{In}(V_m)$$

where $\mathrm{Pr}(V_m)$ is projective and $\mathrm{In}(V_m)$ is either zero or non projective indecomposable; such a decomposition is essentially unique thanks to the Krull-Schmidt theorem.
For small weights, we can describe in detail the socle filtration of $\mathrm{In}(V_m)$ and the decomposition in PIM of $\mathrm{Pr}(V_m)$.

Proposition 2.4. *Let $k \in \{0, \dots, p-2\}$, $r \in \{0, \dots, p-1\}$.*
The socle filtration of $\mathrm{In}(V_{pk+r})$ is given by:

$$i)\ \mathrm{soc}(\mathrm{In}(V_{kp+r})) = \begin{cases} \bigoplus_{i=0}^{\min\{k+1,r\}-1} V_{k+r-2i} \otimes \det^i & \text{if } k+r<p \\ \bigoplus_{i=1}^{p-\max\{k+1,r\}} V_{2p-k-r-2i} \otimes \det^{k+r+i-p} & \text{if } k+r \geq p. \end{cases}$$

$$ii)\ \mathrm{In}(V_{kp+r})/\mathrm{soc}(\mathrm{In}(V_{kp+r})) = \begin{cases} \bigoplus_{i=0}^{\min\{p-k,r\}-1} V_{p-k+r-1-2i} \otimes \det^{k+i} & \text{if } k-r \geq 0 \\ \bigoplus_{i=0}^{p-1-\max\{p-k,r\}} V_{p+k-r-1-2i} \otimes \det^{r+i} & \text{if } r>k. \end{cases}$$

The decomposition into PIM of V_{kp+r} is given by:

$$\mathrm{Pr}(V_{kp+r}) = \begin{cases} \bigoplus_{j=0}^{\lfloor \frac{k-r-1}{2} \rfloor} P[p-k+r+1+2j, k-1-j] \\ \text{if } k-r \geq 0 \text{ and } k+r<p \\ \\ \bigoplus_{j=0}^{\lfloor \frac{k-r-1}{2} \rfloor} P[p-k+r+1+2j, k-1-j] \oplus \bigoplus_{j=0}^{\lfloor \frac{k+r-p}{2} \rfloor} P[2p-k-r+2j, k+r-1-j] \\ \text{if } k-r \geq 0 \text{ and } k+r \geq p \\ \\ 0 \text{ if } r-k>0 \text{ and } k+r<p \\ \\ \bigoplus_{j=0}^{\lfloor \frac{k+r-p}{2} \rfloor} P[2p-k-r+2j, k+r-1-j] \\ \text{if } r-k>0 \text{ and } k+r \geq p \end{cases}$$

Proof: Omissis. Cf. [12], (6.4). ♯

Let $k \in \{0, \dots, p-1\}$. We set

$$Q_k \overset{\text{def}}{=} \bigoplus_{j=0}^{\lfloor \frac{p-k-2}{2} \rfloor} P[k+2(j+1), -1-j] \oplus \bigoplus_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} P[p-k+1+2j, k-1-j]$$

and

$$Q_p \stackrel{\text{def}}{=} \bigoplus_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} P[1+2j,-j].$$

Then, we have

Theorem 2.5. *Let $m \in \mathbf{N}$, and write $m = n(p+1)+k$, with $k \in \{0,\dots,p\}$. We then have an isomorphism*

$$V_{m+p(p-1)} \cong V_m \oplus (Q_k \otimes \det^n).$$

Proof: Omissis. Cf. [12], (6.7) and (6.8).♯

We finally notice that such an explicit description of V_m gives us an algorithm to compute its socle filtration for any $m \in \mathbf{N}$. Indeed, we know the socle filtration of each PIM, and since the functor $V \mapsto \mathrm{soc}(V)$ is additive, theorem 2.5 let us reduce to the case where $m < p(p-1)$. But then, we can conclude thanks to the explicit descrition given by proposition 2.4.

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Laboratoire Mathématiques de Versailles
Université de Versailles, 45 Avenue des Etas-Unis, 78035 Versailles, France
E-mail address: `stefano.morra@math.uvsq.fr`