Explicit description of irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations over $\overline{\mathbf{F}}_p$

Stefano Morra

Abstract

Let p be an odd prime number. The classification of irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over $\overline{\mathbf{F}}_p$ is known thanks to the works of Barthel-Livné [BL95] and Breuil [Bre03a]. In the present paper we illustrate an exhaustive description of such irreducible representations, through the study of certain functions on the Bruhat-Tits tree of $\mathrm{GL}_2(\mathbf{Q}_p)$. In particular, we are able to detect the socle filtration for the KZ-restriction of supersingular representations, principal series and special series.

1. Introduction

Let p be a prime number, F a non-Archimedean local field, $\mathscr{O}_F$ its ring of integers and k_F the residue field, which will be assumed of characteristic p and cardinality $q = p^f$. The ℓ-adic Local Langlands correspondence (for $\ell \neq p$) provides us with a well understood dictionary between suitable representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$, n dimensional over $\mathbf{Q}_\ell$, and suitable representations of $\mathrm{GL}_n(F)$ (two independent proofs due to Harris and Taylor in [HT01] and Henniart in [Hen00]). Moreover, via a process of reduction of coefficients modulo ℓ, Vignéras deduces a semi-simple mod ℓ Local Langlands correspondence, as it results from her study in [Vig].

The theory, in the p-adic case, is far more complicated: for instance Grothendieck's ℓ-adic monodromy theorem collapses, there are not reasonable analogues of the Haar measure, there are no Whittacker models, etc... After a first conjectural approach pointed out by Breuil in [Bre04] and [Bre03b], we dispose nowadays of a p-adic local Langlands correspondence in the 2-dimensional case for $F = \mathbf{Q}_p$ by the works of many mathematicians (Berger [Ber], Berger-Breuil [BB], Colmez [Col], Paskunas [Pas1], etc...). This correspondence is compatible with the reduction of coefficients modulo p and enables us to establish a semi-simple mod p-Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ (again, such a process has been conjectured and proved in few cases by Breuil in [Bre03b] and in generality by Berger in [Ber]).

A major problem for a conjectural mod p-Langlands correspondence is represented by the lack of a complete classification for smooth irreducible admissible $\mathrm{GL}_2(\mathbf{Q}_p)$ representations over $\overline{\mathbf{F}}_p$. In [BL94] and [BL95], Barthel and Livné detect four families of such irreducible objects: besides a detailed study of principal and special series (and characters), the authors discover another class of smooth irreducible admissible representations, referred to as "supersingular", non-isomorphic to the previous ones. Recalling the notion of compact induction (see the end of the Introduction for the precise definition), a supersingular representation π is characterised up to twist as a subquotient of the cokernel of a canonical Hecke operator

$$T_{\underline{r}} \in \mathrm{End}(\text{c-ind}_{\mathrm{GL}_2(\mathscr{O}_F)F^\times}^{\mathrm{GL}_2(F)} \sigma_{\underline{r}})$$

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for a $\mathrm{GL}_2(\mathscr{O}_F)F^\times$ representation $\sigma_{\underline{r}}$ parametrised by an f-tuple of integers $\underline{r}$ (such an f-tuple depending on π).

Their nature is still very mysterious. For instance, if $F \neq \mathbf{Q}_p$, the aforementioned cokernels are not even admissible and the works of Paskunas [Pas], Breuil-Paskunas [BP] and Hu [Hu] show the existence of a huge number of supersingular representations relative to the number of Galois representations (whose classification is indeed well known).

The case $F = \mathbf{Q}_p$ is far different. The cokernels of the Hecke operators, which depend here on a single parameter $r \in \{0, \dots, p-1\}$, are irreducible and we deduce a complete description of supersingular representations for $\mathrm{GL}_2(\mathbf{Q}_p)$. The first proof of this phenomenon, due to Breuil, appears in [Bre03a]: the author is able to compute explicitly the space of I_1-invariants studying the behaviour of certain functions, denoted as X_n^0 and X_n^1, on the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$. Here I_1 denotes the pro-p-Iwahori of $\mathrm{GL}_2(\mathbf{Z}_p)$. Nowadays others ways to prove the irreducibility of $\mathrm{coker}(T_r)$ have been discovered: see for instance the papers of Ollivier ([Oll]), Emerton ([Eme08]), Berger ([Ber1]).

In the present paper we describe *completely*, through a wide generalisation of the techniques of [Bre03a], the cokernel of the Hecke operators T_r, giving their $\mathrm{GL}_2(\mathbf{Z}_p)$-socle filtration. We stress out that the techniques of this paper can be generalised to unramified extensions of $\mathbf{Q}_p$, giving the Iwahori structure for the canonical Hecke operators in terms of *euclidean structures* (see [Mo2]). As a byproduct, we give the $\mathrm{GL}_2(\mathbf{Z}_p)$-socle filtration for unramified principal series.

Using the notations of §2.2 for the characters χ_r^s and $\mathfrak{a}$ and the formalism presented in the end of this § concerning the socle filtration, the main result of the paper is the following:

Theorem 1.1 (Propositions 6.6, 7.1, 8.1, 9.1). *Let $r \in \{0, \dots, p-1\}$, p odd. Then the $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$ restriction of the supersingular representation $\mathrm{coker}(T_r)$ consists of two direct summands of infinite length, whose socle filtration is described by*

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s\mathfrak{a}^{r+1}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s\mathfrak{a}^{r+2}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s\mathfrak{a}^{r+3}) \text{---} \dots$$

and

$$\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2 \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s\mathfrak{a}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s\mathfrak{a}^2) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s\mathfrak{a}^3) \text{---} \dots$$

respectively (and I denotes the Iwahori subgroup of $\mathrm{GL}_2(\mathbf{Z}_p)$).

With suitable restriction on the value of r, Theorem 1.1 shows that the socle filtration for $\pi(r, 0, 1)|_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}$ looks as follows:

$$\begin{array}{ccccccccc}
\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 & \text{------} & \mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2 \otimes \det^{r+1} & \text{---} & \mathrm{Sym}^{r+2}\overline{\mathbf{F}}_p^2 \otimes \det^{p-2} & \text{---} & \mathrm{Sym}^{p-5-r}\overline{\mathbf{F}}_p^2 \otimes \det^{r+2} & \text{---} & \cdots \\
\bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & & \cdots \\
\mathrm{Sym}^{p-1-r}\overline{\mathbf{F}}_p^2 \otimes \det^r & \text{----} & \mathrm{Sym}^{r-2}\overline{\mathbf{F}}_p^2 \otimes \det & \text{----} & \mathrm{Sym}^{p+1-r}\overline{\mathbf{F}}_p^2 \otimes \det^{r-1} & \text{----} & \mathrm{Sym}^{r-4}\overline{\mathbf{F}}_p^2 \otimes \det^2 & \text{----} & \cdots
\end{array}$$

If moreover we write un_λ for the unramified character of $\mathbf{Q}_p$ sending the arithmetic Frobenius to $\lambda \in \overline{\mathbf{F}}_p$ and ω_1 for the cyclotomic character, we are able to prove:

Theorem 1.2 (Propositions 6.6, 10.4). *For p an odd prime number, let $\lambda \in \overline{\mathbf{F}}_p^\times$, $r \in \{0, \dots, p-1\}$ and assume $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$. The socle filtration for the $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$-restriction of the $\mathrm{GL}_2(\mathbf{Q}_p)$-principal series $\mathrm{Ind}_{B(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(un_\lambda \otimes \omega_1^r un_\lambda)$ is described by*

$$\mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(zp)}\chi_r^s\mathfrak{a}) \text{---} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi_r^s\mathfrak{a}^2) \text{---} \dots$$

The socle filtration for the $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$ *restriction of the Steinberg representation for* $\mathrm{GL}_2(\mathbf{Q}_p)$ *is*

$$\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2 \text{—} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\mathfrak{a}) \text{—} \mathrm{SocFil}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathbf{Z}_p)}\mathfrak{a}^2) \text{—} \dots$$

The strategy of the proof of Theorems 1.1 and 1.2 has been inspired by Breuil's notes [Bre] and the keypoint relies on subtle and delicate manipulations on Witt vectors. Apart from these elaborate computations, we can sum up the main ideas in the next paragraph.

Strategy of the proof[1]**.** Fix $r \in \{0, \dots, p-1\}$ and consider the algebraic representation $\sigma \overset{\text{def}}{=} \mathrm{Sym}^r\overline{\mathbf{F}}_p^2$ of $\mathrm{GL}_2(\mathbf{F}_p)$, which will be seen as a representation of $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$ in the usual way. For $n \in \mathbf{N}$ we consider the induction $R_{n+1} \overset{\text{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{\mathrm{GL}_2(\mathbf{Z}_p)}\sigma$ where $K_0(p^{n+1})$ is the subgroup of elements of $\mathrm{GL}_2(\mathbf{Z}_p)$ reducing to upper triangular matrices modulo p^{n+1}. Thus the elements of R_{n+1} are in a natural (equivariant) bijection with the functions $f \in \text{c-ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}^{\mathrm{GL}_2(\mathbf{Q}_p)}\sigma$ having support on the circle of radius $n+1$ on the Bruhat-Tits tree of $\mathrm{GL}_2(\mathbf{Q}_p)$:

Proposition 1.3 (Corollary 3.5). *We have a* $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$ *equivariant isomorphism*

$$\text{c-ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}^{\mathrm{GL}_2(\mathbf{Q}_p)}\sigma \xrightarrow{\sim} \bigoplus_{n\in\mathbf{N}} R_n$$

Therefore the canonical Hecke operator $T = T_r$ acting on the compact induction $\text{c-ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}^{\mathrm{GL}_2(\mathbf{Q}_p)}\sigma$ induces a family of operators $T_n^\pm$ on the representations R_n (§3.2):

Proposition 1.4 (Definitions 3.6, 3.7, Lemma 3.8). *For all* $n \geqslant 1$ *we have an equivariant monomorphism* T_n^+ *and an equivariant epimorphism* T_n^-*:*

$$T_n^+ : R_n \hookrightarrow R_{n+1}, \qquad\qquad T_n^- : R_n \twoheadrightarrow R_{n-1}.$$

For $n = 0$ *we have an equivariant monomorphism* $T_0^+ : R_0 \hookrightarrow R_1$.

In particular, R_n can be identified with a subrepresentation of R_{n+1} via the monomorphism T_n^+.

We will see (§4) that Propositions 1.3 and 1.4 let us deduce a natural equivariant filtration on the restriction $\mathrm{Coker}(T)|_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}$. More precisely,

Proposition 1.5 (Propositions 3.9, 4.1). *We have an equivariant isomorphism*

$$\mathrm{Coker}(T)|_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times} \xrightarrow{\sim} \pi_r \oplus \pi_{p-1-r}$$

where π_r,π_{p-1-r} *are convenient, explicit, representations of* $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times$. *Moreover* π_r *(resp.* π_{p-1-r}*) is endowed with a natural equivariant filtration* $\{Fil_n^{(r)}\}_{n\in\mathbf{N}}$ *(resp.* $\{Fil_n^{(p-1-r)}\}_{n\in\mathbf{N}}$*), the graded pieces being of the form*

$$Fil_{n+1}^{(r)}/Fil_n^{(r)} \cong R_{2n}/R_{2n-1} \qquad\qquad (\textit{resp. } Fil_{n+1}^{(p-1-r)}/Fil_n^{(p-1-r)} \cong R_{2n+1}/R_{2n}).$$

We would like to emphasize that the previous results can be generalised without much effort to any finite extension of $\mathbf{Q}_p$, see [Mo2].

Thanks to Proposition 1.5 we can first reduce to the study of the inductions R_{n+1}. Moreover, the natural $K_0(p^{n+1})$-filtration on σ induces a natural filtration $\{\mathrm{Fil}^t(R_{n+1})\}_{t\in\{0,\dots,r\}}$ on R_{n+1}, the graded pieces being isomorphic to an induction of the form $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi$ for a suitable (explicit) character χ depending on t and r.

The inductions of the form $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi$, for $0 \leqslant m \leqslant n$, are studied in §5 and §6. The keypoints of such study can be summed up as follows.

[1] in this paragraph, for the reader's convenience, we decided to use lighter notations which differ slightly from the notations used in the rest of the paper.

1) For $m \leqslant n$ we detect a family of functions $F_{l_m,\dots,l_n} \in \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi$, depending on parameters $l_m,\dots,l_n \in \{0,\dots,p-1\}$. Such functions are well behaved with respect to computations with Witt vectors and to the induction functor.
2) The parameters l_j appearing in 1) let us deduce a $\overline{\mathbf{F}}_p$-linear filtration, and the compatibility with the induction functor lets us show that such filtration is equivariant, with graded pieces of lenght one (if $m \geqslant 1$) or two (if $m = 0$).
3) Thanks to the compatibility with Witt vectors we check that the extensions between the graded pieces of the filtration in 2) are nonsplit.

Part 3) relies crucially on some explicit manipulations[2] on the ring of Witt vectors for $\mathbf{F}_p$: if $\mu, \lambda_j \in \mathbf{F}_p$ then

$$[\mu] + \sum_{j=0}^{n} p^j[\lambda_j] \equiv \sum_{j=0}^{n} p^j[\lambda_j + P_{\dots,\lambda_{j-2}}(\lambda_{j-1})] \bmod p^{n+1}$$

where $P_{\dots,\lambda_{j-2}}(\lambda_{j-1})$ is a polynomial of degree $p-1$ in λ_{j-1} and leading coefficient $P_{\dots,\lambda_{j-3}}(\lambda_{j-2})$ (and $[\cdot]$ denotes the usual Teichmüller lift). Thus:

Proposition 1.6 (Proposition 5.10). *Let $1 \leqslant m \leqslant n$ be integers and χ a smooth character of $K_0(p^{n+1})$. Then the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\chi$ is described by*

$$\chi \text{---} \chi\mathfrak{a} \text{---} \chi\mathfrak{a}^2 \text{---} \chi\mathfrak{a}^3 \text{---} \dots$$

(see the end of this § for the definition of the character $\mathfrak{a}$).

We similarly deduce :

Proposition 1.7 (Proposition 6.10). *Let χ be a smooth character of the group $K_0(p^{n+1})$. The representation $\mathrm{Ind}_{K_0(p^{n+1})}^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi$ has a natural equivariant filtration whose graded pieces are described by*

$$\mathrm{Ind}_{K_0(p)}^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi \text{---} \mathrm{Ind}_{K_0(p)}^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi\mathfrak{a} \text{---} \mathrm{Ind}_{K_0(p)}^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi\mathfrak{a}^2 \text{---} \dots$$

the extensions being non-split.

Once the socle filtration for the representations $\mathrm{Ind}_{K_0(p^{n+1})}^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi$ has been established we have to "glue" them together in order to obtain the socle filtration for the spaces R_{n+1} and, more generally, for the spaces π_r and π_{p-1-r}.

The gluening for the graded pieces $\mathrm{Fil}^t(R_{n+1})/\mathrm{Fil}^{t-1}(R_{n+1})$ is worked out in §7; the arguments are similar to those which led to the description of the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{\mathrm{GL}_2(\mathbf{Z}_p)}\chi$.

The main result is

Proposition 1.8 (Proposition 7.1). *Let $0 \leqslant j < t \leqslant r$ and let $Q \leqslant \mathrm{Fil}^j(R_{n+1})$ be a subrepresentation coming from the socle filtration for $\mathrm{Fil}^j(R_{n+1})$. Then*

$$\mathrm{soc}(\mathrm{Fil}^{t-1}(R_{n+1})/Q) = \mathrm{soc}(\mathrm{Fil}^t(R_{n+1})/Q).$$

In other words, the socle filtration of R_{n+1} is compatible with the filtration $\{\mathrm{Fil}^t(R_{n+1})\}_{t\in\{0,\dots,r\}}$ on R_{n+1}.

We are finally concerned with the socle filtration for the spaces π_r, π_{p-1-r}. As the reader will see in §8 such filtration is obtained again, by glueing, from the socle filtration of the spaces R_{n+1}/R_n.

[2]the aforementioned "delicate manipulations on Witt vectors".

The keypoint is a *compatibility of the functions*[3] $F_{l_m,\dots,l_n}$ *with the Hecke operators* $T_n^{\pm}$: we are then able to adapt in a natural way the arguments of §7 to obtain the main result.

PROPOSITION 1.9 (Proposition 8.1, 9.1). *The socle filtration for the space π_r (resp. π_{p-1-r}) is compatible with the filtration $\{Fil_n^{(r)}\}_{n\in\mathbf{N}}$ (resp. $\{Fil_n^{(p-1-r)}\}_{n\in\mathbf{N}}$) and Theorem 1.1 holds true.*

Hereafter we give the plan of the article.

In §2 we recall the structure of compact inductions $\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^{\times}}^{\mathrm{GL}_2(\mathbf{Q}_p)}$, their relations with the Bruhat-Tits tree for $\mathrm{GL}_2(\mathbf{Q}_p)$ and the structure of the Hecke algebra for compact inductions. We summarise the main properties of the parabolic induction for the finite case in §2.2, recalling in particular the description of their socle filtration.

Section 3 is devoted to the description of the $\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^{\times}$-restriction of supersingular representations in terms of simpler objects, namely the representations R_n (§3.1) and their amalgamated sums (cf. §4) by means of convenient Hecke operators $T_n^{\pm}$ on R_n (defined in §3.2). Such objects will be endowed with filtrations in §4.

Sections 5, 6, 7 and 8 are devoted to the study, and the glueing, of the socle filtations on the representations introduced in 4; in particular, in §8, such glueing are made by means of the Hecke operator T.

Finally, in §9, we make explicit how the right exactness of $\varinjlim$ makes possible to deduce the socle filtration for supersingular representations from the results in §8. The final section §10 shows how we can deduce easily the socle filtration for principal and special series using the techniques in §6.

We wish to outline that such an explicit nature for the description of supersingular $\mathrm{GL}_2(\mathbf{Q}_p)$-representations (as well as principal and special series) let us describe in greatest detail the K_t and I_t invariant elements, where K_t (resp. I_t) denotes the kernel (resp. the inverse image of upper unipotent matrices) of the reduction mod p^t morphism of elements of K (resp. of elements of K_{t-1}). Such a study has been pursued in [Mo1].

We introduce now the main notations, convention and structure of the paper.

We fix a prime number p. We write $\mathbf{Q}_p$ (resp. $\mathbf{Z}_p$) for the p-adic completion of $\mathbf{Q}$ (resp. $\mathbf{Z}$) and $\mathbf{F}_p$ the field with p elements; $\overline{\mathbf{F}}_p$ is a fixed algebraic closure of $\mathbf{F}_p$. For any $\lambda\in\mathbf{F}_p$ (resp. $x\in\mathbf{Z}_p$) we write $[\lambda]$ (resp. $\overline{x}$) for the Teichmüller lift (resp. for the reduction modulo p), defining $[0]\overset{\mathrm{def}}{=}0$.

We write $G\overset{\mathrm{def}}{=}\mathrm{GL}_2(\mathbf{Q}_p)$, $K\overset{\mathrm{def}}{=}\mathrm{GL}_2(\mathbf{Z}_p)$ the maximal compact subgroup, I the Iwahori subgroup of K (i.e. the elements of K whose reduction modulo p is upper triangular) and I_1 for the pro-p-iwahori (i.e. the elements of I whose reduction is unipotent). Moreover, let $Z\overset{\mathrm{def}}{=}Z(G)\cong\mathbf{Q}_p^{\times}$ be the center of G and $B(\mathbf{Q}_p)$ (resp. $B(\mathbf{F}_p)$) the Borel subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathbf{Q}_p)$ (resp. $\mathrm{GL}_2(\mathbf{F}_p)$).

For $r\in\{0,\dots,p-1\}$ we denote by σ_r the algebraic representation $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2$ (endowed with the natural action of $\mathrm{GL}_2(\mathbf{F}_p)$). Explicitly, if we consider the identification $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2\cong\overline{\mathbf{F}}_p[X,Y]_r^h$ (where $\overline{\mathbf{F}}_p[X,Y]_r^h$ means the graded component of degree r for the natural grading on $\overline{\mathbf{F}}_p[X,Y]$) then

$$\sigma_r(\begin{bmatrix} a & b\\ c & d\end{bmatrix})X^{r-i}Y^i\overset{\mathrm{def}}{=}(aX+cY)^{r-i}(bX+dY)^i$$

[3] more precisely, natural lifts inside π_r, π_{p-1-r} of the functions $F_{l_m,\dots,l_n}$

for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{F}_p)$, $i \in \{0, \dots, r\}$. We then endow σ_r with the action of K obtained by inflation $K \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_p)$ and, by imposing a trivial action of $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$, we get a smooth KZ-representation. Such a representation is still denoted σ_r, not to overload the notations.

If H stands for the maximal torus of $\mathrm{GL}_2(\mathbf{F}_p)$ and $\chi : H \to \overline{\mathbf{F}}_p^\times$ is a multiplicative character we will write χ^s for the conjugate character defined by $\chi^s(h) \stackrel{\text{def}}{=} \chi(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} h \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ for $h \in H$. Characters of H will be seen as characters of $B(\mathbf{F}_p)$ or, by inflation, as characters of any subgroup of K which reduces to $B(\mathbf{F}_p)$ modulo p, without any commentary.

By "representation" we always mean a smooth representation with central character with coefficients in $\overline{\mathbf{F}}_p^\times$. If V is a $\widetilde{K}$-representation, for $\widetilde{K}$ a subgroup of K, and $v \in V$, we write $\langle \widetilde{K} \cdot v \rangle$ to denote the sub-$\widetilde{K}$ representation of V generated by v. For a $\widetilde{K}$-representation V we write $\mathrm{soc}_{\widetilde{K}}(V)$ (or $\mathrm{soc}(V)$, or $\mathrm{soc}^1(V)$ if $\widetilde{K}$ is clear from the context) to denote the maximal semisimple sub-representation of V. Inductively, the subrepresentation $\mathrm{soc}^i(V)$ of V being defined, we define $\mathrm{soc}^{i+1}(V)$ as the inverse image of $\mathrm{soc}^1(V/\mathrm{soc}^i(V))$ via the projection $V \twoheadrightarrow V/\mathrm{soc}^i(V)$. We therefore obtain an increasing filtration $\{\mathrm{soc}^n(V)\}_{n \in \mathbf{N}_>}$ which will be referred to as the socle filtration for V; we will say that a subrepresentation W of V "comes from the socle filtration" if we have $W = \mathrm{soc}^n(V)$ for some $n \in \mathbf{N}_>$ (with the convention that $\mathrm{soc}^0(V) \stackrel{\text{def}}{=} 0$). The sequence of the graded pieces of the socle filtration for V will be shortly denoted by

$$\mathrm{SocFil}(V) \stackrel{\text{def}}{=} \mathrm{soc}^1(V) \text{---} \mathrm{soc}^1(V)/\mathrm{soc}^0(V) \text{---} \dots \text{---} \mathrm{soc}^{n+1}(V)/\mathrm{soc}^n(V) \text{---} \dots$$

We finally recall the Kroneker delta: if S is any set, and $s_1, s_2 \in S$ we define

$$\delta_{s_1, s_2} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad s_1 \neq s_2 \\ 1 & \text{if} \quad s_1 = s_2. \end{cases}$$

2. Preliminaries and definitions

The aim of this section is to recall some classical facts concerning compact inductions of p-adic representations (§2.1 and §2.2), and to give some explicit computations in the ring of p-adic integers $\mathbf{Z}_p$ (§2.3): such computations will play a key role in the rest of the article.

2.1 Compact induction of KZ-representations

For the details and proofs, the reader is invited to see [Ser77] or ([Bre03a], §2).

We write $\mathscr{T}$ for the tree of $\mathrm{GL}_2(\mathbf{Q}_p)$. It is well known that we have an explicit G-equivariant bijection (with respect to the natural left G-action defined on the two sets) between the vertices $\mathscr{V}$ of $\mathscr{T}$ and the right cosets of G/KZ. We define the following elements of G:

$$\alpha \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}, \qquad w \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and recall the Cartan decomposition

$$G = \coprod_{n \in \mathbf{N}} KZ\alpha^{-n}KZ;$$

then, for all $n \in \mathbf{N}$, the classes in $KZ\alpha^{-n}KZ/KZ$ correspond to the vertices of the tree at distance n from the central vertex.

We set $I_0 \stackrel{\text{def}}{=} \{0\}$ and for $n \in \mathbf{N}_>$ we define the following subset of $\mathbf{Z}_p$:

$$I_n \stackrel{\text{def}}{=} \{\sum_{j=0}^{n-1} p^j[\mu_j] \quad \text{for } \mu_j \in \mathbf{F}_p\}.$$

For $n \geqslant 1$ we have a set-theoretic map

$$\begin{aligned} &[\cdot]_{n-1} : I_n \to I_{n-1} \\ &\sum_{j=0}^{n-1} p^j[\mu_j] \mapsto \sum_{j=0}^{n-2} p^j[\mu_j]. \end{aligned}$$

Moreover for $n \in \mathbf{N}$, $\mu \in I_n$ we put

$$\begin{aligned} g^0_{n,\mu} &\stackrel{\text{def}}{=} \begin{bmatrix} p^n & \mu \\ 0 & 1 \end{bmatrix} \\ g^1_{n,\mu} &\stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ p\mu & p^{n+1} \end{bmatrix}. \end{aligned}$$

We have then the following family of representatives for G/KZ:

$$G = \coprod_{n\in\mathbf{N},\, \mu\in I_n} g^0_{n,\mu}KZ \coprod \coprod_{n\in\mathbf{N},\, \mu\in I_n} g^1_{n,\mu}KZ; \tag{1}$$

more precisely, we have

$$KZ\alpha^{-n}KZ = \coprod_{\mu\in I_n} g^0_{n,\mu}KZ \coprod \coprod_{\mu\in I_{n-1}} g^1_{n-1,\mu}KZ$$

for $n \in \mathbf{N}_>$. Heuristically, the $g^0_{n,\mu}$'s correspond to the vertices at distance n from the central vertex, located in the "positive part" of the tree, while the $g^1_{n-1,\mu}$'s correspond to the vertices at distance n from the central vertex, located in the "negative" part of the tree.

Let σ be a smooth KZ-representation over $\overline{\mathbf{F}}_p$, V_σ the underlying $\overline{\mathbf{F}}_p$-vector space. The *induced representation from* σ, noted by

$$\mathrm{Ind}^G_{KZ}\sigma,$$

is defined as the $\overline{\mathbf{F}}_p$-vector space of functions $f : G \to V_\sigma$, compactly supported modulo Z and verifying the condition $f(\kappa g) = \sigma(\kappa) \cdot f(g)$ for any $\kappa \in KZ$, $g \in G$, this space being endowed with a left G-action defined by right translation of functions (i.e. $(g \cdot f)(t) \stackrel{\text{def}}{=} f(tg)$ for any $g, t \in G$). It turns out that $\mathrm{Ind}^G_{KZ}\sigma$ is again a smooth representation of G over $\overline{\mathbf{F}}_p$. For $g \in G$, $v \in V_\sigma$, we define the element $[g, v] \in \mathrm{Ind}^G_{KZ}\sigma$ as follows:

$$\begin{aligned} [g, v](t) &\stackrel{\text{def}}{=} \sigma(tg) \cdot v \quad \text{if} \quad t \in KZg^{-1} \\ [g, v](t) &\stackrel{\text{def}}{=} 0 \quad \text{if} \quad t \notin KZg^{-1}. \end{aligned}$$

Then we have the equalities $g_1 \cdot [g_2, v] = [g_1g_2, v]$ and $[g\kappa, v] = [g, \sigma(\kappa) \cdot v]$ for $g_1, g_2, g \in G$ and $\kappa \in KZ$. Moreover:

PROPOSITION 2.1. *Let $\mathscr{B}$ an $\overline{\mathbf{F}}_p$-basis of V_σ, and $\mathscr{G}$ a system of representatives for the left cosets of G/KZ. Then, the family*

$$\mathscr{I} \stackrel{\text{def}}{=} \{[g, v], \text{for } g \in \mathscr{G},\, v \in \mathscr{B}\}$$

is an $\overline{\mathbf{F}}_p$-basis for the induced representation $\mathrm{Ind}^G_{KZ}\sigma$.

Proof: Omissis (cf. [BH06], Lemma 2.5 or [Bre], Lemma 3.5). $\sharp$

If $f \in \mathrm{Ind}_{KZ}^{G}\sigma$, the $\mathscr{T}-$*support* (or simply the support) of f is defined as the set of vertices gKZ of the tree $\mathscr{T}$ such that $f(g^{-1}) \neq 0$; this notion does not depend on the chosen representative g of the vertex gKZ. We define for $n \in \mathbf{N}$ the following subspace of $\mathrm{Ind}_{KZ}^{G}\sigma$:

$$W(n) \stackrel{\mathrm{def}}{=} \{f \in \mathrm{Ind}_{KZ}^{G}\sigma, \quad \text{the support of } f \text{ is contained in } KZ\alpha^{-n}KZ\}.$$

We see (by Cartan decomposition) that the subspaces $W(n)$ are KZ-stable, for all $n \in \mathbf{N}$, and therefore

Lemma 2.2. *There exists a family $\{\Psi_n\}_{n\in\mathbf{N}}$ of natural KZ-equivariant epimorphisms*

$$\Psi_n : \mathrm{Ind}_{KZ}^{G}\sigma \twoheadrightarrow W(n)$$

inducing a natural KZ-equivariant isomorphism

$$\mathrm{Ind}_{KZ}^{G}\sigma \stackrel{\sim}{\to} \bigoplus_{n\in\mathbf{N}} W(n).$$

Proof: Obvious.♯

Some Hecke Operators. The Hecke algebra for the induced representation from σ is defined by

$$\mathcal{H} \stackrel{\mathrm{def}}{=} \mathrm{End}_G(\mathrm{Ind}_{KZ}^{G}\sigma).$$

It is an $\overline{\mathbf{F}}_p$ algebra; moreover, there exists a canonical operator $T \in \mathcal{H}$ which induces an isomorphism of $\overline{\mathbf{F}}_p$-algebras

$$\mathcal{H} \stackrel{\sim}{\to} \overline{\mathbf{F}}_p[T]$$

(cf. [BL95], §3). If we specialise to the case $\sigma = \sigma_r$ for $0 \leqslant r \leqslant p-1$ we have the following explicit description of the Hecke operator T:

Lemma 2.3. *For $n \in \mathbf{N}_>$, $\mu \in I_n$ and $0 \leqslant j \leqslant r$ we have:*

$$T([g^0_{n,\mu}, X^{r-j}Y^j]) = \sum_{\mu_n \in \mathbf{F}_p} [g^0_{n+1,\mu+p^n[\mu_n]}, (-\mu_n)^j X^r] + [g^0_{n-1,[\mu]_{n-1}}, \delta_{j,r}(\mu_{n-1}X+Y)^r]$$

$$T([g^1_{n,\mu}, X^{r-j}Y^j]) = \sum_{\mu_n \in \mathbf{F}_p} [g^1_{n+1,\mu+p^n[\mu_n]}, (-\mu_n)^{r-j} Y^r] + [g^0_{n-1,[\mu]_{n-1}}, \delta_{j,0}(X+\mu_{n-1}Y)^r].$$

For $n = 0$, $0 \leqslant j \leqslant r$ we have

$$T([1_G, X^{r-j}Y^j]) = \sum_{\mu_0 \in \mathbf{F}_p} [g^0_{1,[\mu_0]}, (-\mu_0)^j X^r] + [\alpha, \delta_{j,r}Y^r]$$

$$T([\alpha, X^{r-j}Y^j]) = \sum_{\mu_1 \in \mathbf{F}_p} [g^1_{1,[\mu_1]}, (-\mu_1)^{r-j} Y^r] + [1_G, \delta_{j,0}X^r]$$

.

Proof: Cf. [Bre03a], §2.5 and lemme 3.1.1♯

We are going to fix the notations for supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$: if $r \in \{0, \dots, p-1\}$ we write

$$\pi(r,0,1) \stackrel{\mathrm{def}}{=} \mathrm{coker}(T : \mathrm{Ind}_{KZ}^{G}\sigma_r \to \mathrm{Ind}_{KZ}^{G}\sigma_r).$$

2.2 Induction of $B(\mathbf{F}_p)$-representations

For details and proofs we invite the reader to see §1 and §2 in Breuil and Paskunas's article [BP].

Let η be an $\overline{\mathbf{F}}_p$-character of the Borel subgroup $B(\mathbf{F}_p)$; it is by inflation a character of the

Iwahori subgroup $K_0(p)$ of K and we have a natural isomorphism

$$\mathrm{Ind}_{K_0(p)}^K\eta \xrightarrow{\sim} \mathrm{Ind}_{B(\mathbf{F}_p)}^{\mathrm{GL}_2(\mathbf{F}_p)}\eta.$$

For $i \in \mathbf{N}$ we define the following $\overline{\mathbf{F}}_p$-characters of the Borel subgroup $B(\mathbf{F}_p)$:

$$\chi_i^s : B(\mathbf{F}_p) \to \overline{\mathbf{F}}_p$$
$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto d^i$$

and

$$\mathfrak{a} : B(\mathbf{F}_p) \to \overline{\mathbf{F}}_p$$
$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto ad^{-1}.$$

If e_η is an $\overline{\mathbf{F}}_p$-basis of η, the element $[1_K, e_\eta]$ is a K-generator of $\mathrm{Ind}_{K_0(p)}^K\eta$. The structure of the induced representations $\mathrm{Ind}_{K_0(p)}^K\eta$ is completely known, and the following proposition collects the main results which will be needed in the rest of the paper. We introduce the following notation: for any $x \in \mathbf{Z}$, define $\lceil x\rceil \in \{1, \dots, p-1\}$ (resp. $\lfloor x\rfloor \in \{0, \dots, p-2\}$) by $x \equiv \lceil x\rceil \bmod p-1$ (resp. $x \equiv \lfloor x\rfloor \bmod p-1$).

Proposition 2.4. *Let $i, j \in \{0, \dots, p-1\}$, $\chi \overset{\mathrm{def}}{=} \chi_i^s\mathfrak{a}^j$. Then the induction $\mathrm{Ind}_{K_0(p)}^K\chi$ has length 2, with components:*

i) $\mathrm{Sym}^{\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2 \otimes \det^j$, *which is isomorphic to the K-subrepresentation generated by the element* $\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1_K, e_\chi]$;

ii) $\mathrm{Sym}^{p-1-\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2 \otimes \det^{i-j}$.

Moreover

i′) if $\chi \neq \chi^s$ the short exact sequence

$$0 \to \mathrm{Sym}^{\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2 \otimes \det^j \to \mathrm{Ind}_{K_0(p)}^K\chi \to \mathrm{Sym}^{p-1-\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2 \otimes \det^{i-j} \to 0$$

is nonsplit;

ii′) if $\chi = \chi^s$ (i.e. $i-2j \equiv 0 \bmod [p-1]$) then $\mathrm{Ind}_{K_0(p)}^K\chi$ is semisimple and $\mathrm{Sym}^{p-1-\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes\det^{i-j}$ (i.e. $\det^j$) is the K-subrepresentation of $\mathrm{Ind}_{K_0(p)}^K\chi$ generated by

$$\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1_K, e_\chi] + (-1)^j[1_K, e_\chi].$$

Proof: It is a well known result about representations of $\mathrm{GL}_2(\mathbf{F}_p)$ over $\overline{\mathbf{F}}_p$. See also [BP], Lemmas 2.2, 2.6, 2.7 ♯

Remark 2.5. *It is possible to detect an $\overline{\mathbf{F}}_p$ basis of H-eigenvector for the irreducible fators of the induction $\mathrm{Ind}_{K_0(p)}^K\chi$ described in Proposition 2.4 (see [BP], Lemmas 2.6 and 2.7). Indeed, an $\overline{\mathbf{F}}_p$-basis of H-eigenvectors for the subrepresentation $\mathrm{Sym}^{\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2 \otimes \det^j$ is given by the elements*

$$\sum_{\mu_0\in\mathbf{F}_p} \mu_0^l \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1_K, e_\chi] \quad \text{for } 0 \leqslant l < \lceil i-2j\rceil$$

$$\sum_{\mu_0\in\mathbf{F}_p} \mu_0^l \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1_K, e_\chi] + (-1)^{i-j}[1_K, e_\chi] \quad \text{for } l = \lceil i-2j\rceil,$$

while the homomorphic image of the elements

$$\sum_{\mu_0\in\mathbf{F}_p}\mu_0^l\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, e_\chi]\quad \text{for } \lceil i-2j\rceil\leqslant l\leqslant p-1$$

describes an $\overline{\mathbf{F}}_p$*-basis of* H*-eigenvectors in the quotient* $\mathrm{Ind}_{K_0(p)}^K\chi/\mathrm{Sym}^{\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes\det^j$ *(which is naturally isomorphic to* $\mathrm{Sym}^{p-1-\lceil i-2j\rceil}\overline{\mathbf{F}}_p^2\otimes\det^{i-j}$*).*

The next lemma will play a crucial role in the sequel.

Lemma 2.6. *Let* $0\leqslant r\leqslant p-1$, $0\leqslant t\leqslant p-2$ *be integers, and consider the natural projection*

$$\mathrm{Ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^t\overset{\pi}{\twoheadrightarrow}\mathrm{Sym}^{p-1-\lfloor r-2t\rfloor}\overline{\mathbf{F}}_p^2\otimes\det^{r-t}.$$

If $f\in\mathrm{Ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^t$ *is such that*

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}f=a^{r-(t+1)}d^{t+1}$$

for any $a,d\in\mathbf{F}_p^\times$ *then* $\pi(f)$ *is of the following form (up to multiplication by a scalar multiple):*

i) if $r-2t\not\equiv 0,1\,[p-1]$ *then* $\pi(f)=0$*;*

ii) if $r-2t\equiv 1\,[p-1]$ *then* $\pi(f)=X^{p-2}$*;*

iii) if $r-2t\equiv 0\,[p-1]$ *then* $\pi(f)=X^{p-2}Y$*. More precisely, the image of* f *via the isomorphism*

$$\mathrm{Ind}_{K_0(p)}^K\det^t\overset{\sim}{\to}\det^t\oplus\mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2\otimes\det^t$$

is $(0,X^{p-2}Y)$*.*

Proof: The H-eigencharacters of $\mathrm{Sym}^{p-1-\lfloor r-2t\rfloor}\overline{\mathbf{F}}_p^2\otimes\det^{r-t}$ are

$$a^{p-1-(r-2t)+r-t-j}d^{r-t+j}$$

for $j\in\{0,\dots,p-1-\lfloor r-2t\rfloor\}$, each of them corresponding respectively to the H-eigenvector $X^{p-1-\lfloor r-2t\rfloor-j}Y^j$. Therefore, the condition on $\pi(f)$ to be an H-eigencharacter gives

$$a^{t-j}d^{r-t+j}=a^{r-t-1}d^{t+1}$$

for a suitable $j\in\{0,\dots,p-1-\lfloor r-2t\rfloor\}$ and for all $a,d\in\mathbf{F}_p^\times$; in other words

$$p-1-\lfloor r-2t\rfloor\equiv j-1\,[p-1]$$

for some $j\in\{0,\dots,p-1-\lfloor r-2t\rfloor\}$. This is possible iff $j=0$ and $r-2t\equiv 1\,[p-1]$ or $j=1$ and $r-2t\equiv 0\,[p-1]$. $\sharp$

2.3 Computations on Witt vectors

In this section we are going to describe the p-adic expansion of some elements in $\mathbf{Z}_p$. The explicit description of Lemmas 2.7 and 2.8 is one of the key arguments to describe the socle filtration for the KZ-restriction of supersingular representations. The main reference for this section is [Ser63], Ch. II.

For $\lambda,\mu\in\mathbf{F}_p$ we define the following element of $\mathbf{F}_p$:

$$-P_\lambda(\mu)\overset{\text{def}}{=}\sum_{j=1}^{p-1}\frac{\binom{p}{j}}{p}\lambda^{p-j}\mu^j.$$

Note that $P_\lambda(\mu)$ is a polynomial in μ, of degree $p-1$ and whose leading coefficient is $-\lambda$. We have the

Lemma 2.7. *Let $\lambda, \mu \in \mathbf{F}_p$. Then*

i) the following equality holds in $\mathbf{Z}_p$:

$$[\lambda] + [\mu] = [\lambda + \mu] + p[P_\lambda(\mu)] + p^2 t_{\lambda,\mu}$$

where $t_{\lambda,\mu} \in \mathbf{Z}_p$ is a suitable p-adic integer depending only on λ, μ;

ii) the following equality holds in $\mathbf{F}_p$

$$P_\lambda(\mu - \lambda) = -P_{-\lambda}(\mu).$$

Proof: Omissis.♯

We can use Lemma 2.7 to deduce more general results.

Lemma 2.8. *Let $\lambda \in \mathbf{F}_p$, $\sum_{j=0}^{n} p^j[\mu_j] \in I_{n+1}$. Then the following equality holds in $\mathbf{Z}_p/(p^{n+1})$:*

$$[\lambda] + \sum_{j=0}^{n} p^j[\mu_j] \equiv [\lambda + \mu_0] + p[\mu_1 + P_\lambda(\mu_0)] + \cdots + p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1})]$$

where, for all $j = 1, \dots, n-2$, the $P_{\lambda,\dots,\mu_j}(X)$'s (resp. $P_{\lambda,\mu_0}(X)$, resp. $P_\lambda(X)$) are suitable polynomials in $\mathbf{F}_p[X]$, of degree $p-1$, depending only on $\lambda, \dots, \mu_j$ (resp. on λ, μ_0, resp. on λ), and whose leading coefficient is $-P_{\lambda,\dots,\mu_{j-1}}(\mu_j)$ (resp. $-P_\lambda(\mu_0)$, resp. $-\lambda$).

Proof: It is an immediate induction using Lemma 2.7-*i*). ♯

Lemma 2.9. *Let $\lambda \in \mathbf{F}_p$, $z \overset{\text{def}}{=} \sum_{j=1}^{n} p^j[\mu_j]$ and let $k \geqslant 0$. There exists a p-adic integer $z' = \sum_{j=1}^{n} p^j[\mu'_j] \in \mathbf{Z}_p$ such that*

$$z \equiv z'(1 + zp^k[\lambda]) \bmod p^{n+1}.$$

Furthermore, for $j = k+3, \dots, n$ (resp. $j = k+2$, resp. $j \leqslant k+1$) we have the following equality in $\mathbf{F}_p$:

$$\mu_j = \mu'_j + \mu_{j-k-1}\mu'_1\lambda + \cdots + \mu_1\mu_{j-k-1}\lambda + S_{j-2}(\mu_{j-1})$$

(resp. $\mu_{k+2} = \mu'_{k+2} + \mu'_1\mu_1\lambda$ if $j = k+2$, resp. $\mu_j = \mu'_j$ if $j \leqslant k+1$) where $S_{j-2}(X) \in \mathbf{F}_p[X]$ is a polynomial of degree $p-1$, depending only on $\lambda, \dots, \mu_{j-2}$ and leading coefficient $-s_{\lambda,\dots,\mu_{j-2}} \overset{\text{def}}{=} \mu'_{j-1} - \mu_{j-1}$.

Proof: Exercise on Witt vectors.♯

To conclude this section we recall two elementary results which will be used in the rest of the paper:

Lemma 2.10. *i) For $0 \leqslant j \leqslant p-1$ we have the equality in $\mathbf{F}_p$:*

$$\sum_{\mu \in \mathbf{F}_p} \mu^j = -\delta_{j,p-1}.$$

ii) Let V be an $\overline{\mathbf{F}}_p$-vector space and let $v_0, \dots, v_{p-1} \in V$ be any p-tuple of elements of V. The sub $\overline{\mathbf{F}}_p$-vector space of V generated by $\sum_{j=0}^{p-1} \mu^j v_j$ for μ varying in $\mathbf{F}_p$ coincide with the $\overline{\mathbf{F}}_p$-subvector space of V generated by the elements $v_0, \dots, v_{p-1}$.

Proof: The assertions are both elementary; the second comes from the fact that the Vandermonde matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^{p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p-1 & (p-1)^2 & \dots & (p-1)^{p-1} \end{bmatrix}$$

is invertible modulo p. ♯

3. Reinterpretation of the KZ-restriction of supersingular representations: the KZ-representations R_n

The goal of this section is to give a precise description of the KZ-restriction of supersingular representations $\pi(r,0,1)|_{KZ}$; the main result is then Proposition 3.9, whose formulation is due to Breuil ([Bre], §4.2). To be more precise, the first step is to introduce, in §3.1, the K-representations R_n, from which we get an alternative description of the compact induction $\mathrm{Ind}_{KZ}^G\sigma$ (cf. Proposition 3.5). Subsequently, we endow the R_n's with suitable Hecke operators $T_n^{\pm}: R_n \to R_{n\pm 1}$ which let us define the amalgamated sums (4); Proposition 3.9 will then be a formal consequence.

3.1 Defining the K-representations R_n

For all $n \in \mathbf{N}$ we define the following subgroup of K:

$$K_0(p^n) \overset{\text{def}}{=} \{\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K, \text{ where } c \in \mathbf{Z}_p\}$$

(in particular, $K_0(p^0) = K$ and $K_0(p)$ is the Iwahori subgroup). For $0 \leqslant r \leqslant p-1$ and $n \in \mathbf{N}$ we define the following $K_0(p^n)$-representation σ_r^n over $\overline{\mathbf{F}}_p$: the associated $\overline{\mathbf{F}}_p$-vector space of σ_r^n is $\mathrm{Sym}^r\overline{\mathbf{F}}_p^2$, while the left action of $K_0(p^n)$ is given by

$$\sigma_r^n(\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix}) \cdot X^{r-j}Y^j \overset{\text{def}}{=} \sigma_r(\begin{bmatrix} d & c \\ p^n b & a \end{bmatrix}) \cdot X^{r-j}Y^j$$

for any $\begin{bmatrix} a & b \\ p^n c & d \end{bmatrix} \in K_0(p^n)$, $0 \leqslant j \leqslant r$; in particular, σ_r^0 is isomorphic to σ_r. Finally, we define

$$R_r^n \overset{\text{def}}{=} \mathrm{Ind}_{K_0(p^n)}^K \sigma_r^n.$$

If r is clear from the context, we will write simply R_n instead of R_r^n.

In order to establish the relation between the R_r^n's and the compact induction $\mathrm{Ind}_{KZ}^G\sigma_r$ we need the following elementary lemma:

Lemma 3.1. *Fix $n \in \mathbf{N}$. Right translation by $\alpha^n w$ induces a bijection*

$$K/K_0(p^n) \overset{\sim}{\to} KZ\alpha^{-n}KZ/KZ.$$

Proof: Elementary, noticing that $(\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} KZ \begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix}) \cap K = K_0(p^n)$. ♯

For any $n \in \mathbf{N}_>$, $\mu \in I_n$ and $\mu' \in I_{n-1}$ we see that

$$g_{n,\mu}^0 = \begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix} \alpha^n w, \qquad g_{n-1,\mu'}^1 w = \begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix} \alpha^n w$$

from which we deduce the following corollaries.

COROLLARY 3.2. *Let $n \in \mathbf{N}_>$. We have the following decomposition for K:*

$$K = \coprod_{\mu \in I_n} \begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix} K_0(p^n) \coprod \coprod_{\mu' \in I_{n-1}} \begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix} K_0(p^n).$$

Proof: Immediate from the decomposition given in (1). $\sharp$

COROLLARY 3.3. *Let $0 \leqslant r \leqslant p-1$, $n \in \mathbf{N}_>$. The family*

$$\mathcal{R}_r^n \overset{\text{def}}{=} \{[\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j], [\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^{r-j}Y^j] \,\text{for}\, \mu \in I_n,\ \mu' \in I_{n-1}, 0 \leqslant j \leqslant r\}$$

is an $\overline{\mathbf{F}}_p$-basis for the representations R_n. Moreover, the element

$$[1_{KZ}, Y^r] \in R_r^n$$

is a K-generator for the representation R_r^n.

Proof: Immediate from Proposition 2.1 and Corollary 3.2. $\sharp$

The following result is the key to establish the relation between the compact induction $\mathrm{Ind}_{KZ}^G \sigma_r$ and the R_n's.

PROPOSITION 3.4. *Let $0 \leqslant r \leqslant p-1$, $n \in \mathbf{N}$ and let $W(n)$ be the KZ subrepresentation of $\mathrm{Ind}_{KZ}^G \sigma_r$ defined in §2.1. We have a KZ-equivariant isomorphism*

$$\Phi_n : W(n) \xrightarrow{\sim} R_n$$

such that for all $0 \leqslant j \leqslant r$

$$\Phi_n([g^0_{n,\mu}, X^{r-j}Y^j]) = [\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]$$

$$\Phi_n([g^1_{n-1,\mu'}, X^{r-j}Y^j]) = [\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^jY^{r-j}]$$

if $n > 0$ and

$$\Phi_0([1_G, X^{r-j}Y^j]) = X^jY^{r-j}$$

if $n = 0$.

Proof: We fix an index $n \geqslant 1$ (the case $n = 0$ is immediately verified). Thanks to Proposition 2.1 it is clear that Φ_n is an $\overline{\mathbf{F}}_p$-linear isomorphism. Concerning the KZ-equivariance, we fix $\kappa \in K$, $l \in \mathbf{N}$ and, for $i \in \{0,1\}$, $g^i_{n-i,\mu}$ and $\mu \in I_{n-i}$. Then $\kappa p^l g^i_{n-i,\mu} = g^{i(\kappa)}_{n-i(\kappa),\mu(\kappa)} \kappa_1 p^{l_1}$ for some $\kappa_1 \in K$, $l_1 \in \mathbf{N}$ while $i(\kappa) \in \{0,1\}$ and $\mu(\kappa) \in I_{n-i(\kappa)}$ depend only on κ. If $g_{i,\mu}$ (resp. $g_{i(\kappa),\mu(\kappa)}$) is the representative of $K/K_0(p^n)$ corresponding to $g^i_{n-i,\mu}$ (resp. $g^{i(\kappa)}_{n-i(\kappa),\mu(\kappa)}$) via the bijection of Lemma 3.1 we get:

$$\begin{cases} \kappa g_{i,\mu} = g_{i(\kappa),\mu(\kappa)} \kappa_2 \\ \kappa p^l g^i_{n-i,\mu} = g_{i(\kappa),\mu(\kappa)} \kappa_1 p^{l_1} \end{cases}$$

for some $\kappa_2 \in K_0(p^n)$ and since $g^i_{n-i,\mu} = g_{i,\mu} \begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} w^i$ (and similarly for $g^{i(\kappa)}_{n-i(\kappa),\mu(\kappa)}$, $g_{i(\kappa),\mu(\kappa)}$) we conclude

$$\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} \kappa_2 \begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix} w^i = w^{i(\kappa)} \kappa_1 p^{n+l_1-l}.$$

We finally need the equality

$$\sigma_r(\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix}\kappa_2\begin{bmatrix} 0 & 1 \\ p^n & 0 \end{bmatrix}) = \sigma_r^n(\kappa_2).$$

to see that

$$\Phi_n(\kappa p^l \cdot [g^i_{n,\mu}, v]) = \kappa \cdot \Phi_n([g_{i,n}, w \cdot v])$$

and the proof is complete. ♯

We deduce immediately the main result of this section:

Corollary 3.5. *Let $r \in \{0, \ldots, p-1\}$. We have a KZ equivariant isomorphism*

$$\mathrm{Ind}_{KZ}^G \sigma_r \xrightarrow{\sim} \bigoplus_{n\in\mathbf{N}} R_r^n$$

3.2 Hecke operators on the R_n's, description of $\pi(r,0,1)|_{KZ}$

In this section we are going to define some Hecke operators T_n^+, T_n^- on the representations R_n's which allow us to give a description of the KZ-restriction of a supersingular representation $\pi(r,0,1)|_{KZ}$ in terms of the R_n, T_n^+, T_n^-. The main result will be Proposition 3.9.

We start from the definition of the Hecke operators on the R_n's.

Definition 3.6. *Let $n \in \mathbf{N}_>$. We define the $\overline{\mathbf{F}}_p$-linear morphism $T_n^+ : R_n \to R_{n+1}$ by the conditions*

$$T_n^+([\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]) \overset{\mathrm{def}}{=} \sum_{\mu_n\in\mathbf{F}_p}[\begin{bmatrix} \mu + p^n[\mu_n] & 1 \\ 1 & 0 \end{bmatrix}, (-\mu_n)^j X^r]$$

$$T_n^+([\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^jY^{r-j}]) \overset{\mathrm{def}}{=} \sum_{\mu_n\in\mathbf{F}_p}[\begin{bmatrix} 1 & 0 \\ p(\mu' + [\mu_n]p^{n-1}) & 1 \end{bmatrix}, (-\mu_n)^{r-j} X^r]$$

for $\mu \in I_n$, $\mu' \in I_{n-1}$ and $0 \leqslant j \leqslant r$.
We define the $\overline{\mathbf{F}}_p$-linear morphism $T_0^+ : R_0 \to R_1$ by the condition:

$$T_0^+([1_K, X^{r-j}Y^j]) \overset{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}[\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix}, (-\mu_0)^{r-j} X^r] + [1_K, \delta_{j,0}X^r]$$

for $0 \leqslant j \leqslant r$.

Identifying R_n with $W(n)$ via the isomorphism described in Proposition 3.4 and using the results of §2.1 we see that

$$T_n^+([g, v]) = \Psi_{n+1}(T([g, v])) \tag{2}$$

for all $g \in KZ\alpha^{-n}KZ$, $v \in \sigma_r$ (i.e. $T_n^+([g, v]$ is described as the projection of $T([g, v])$ on the $W(n+1)$ component of the compact induction).

Similarly, we have

Definition 3.7. *Let $n \in \mathbf{N}$, $n \geqslant 2$. We define the $\overline{\mathbf{F}}_p$-linear morphism $T_n^- : R_n \to R_{n-1}$ by the conditions:*

$$T_n^-([\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]) \overset{\mathrm{def}}{=} [\begin{bmatrix} [\mu]_{n-1} & 1 \\ 1 & 0 \end{bmatrix}, \delta_{j,r}(\mu_{n-1}X + Y)^r]$$

$$T_n^-([\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^jY^{r-j}]) \overset{\mathrm{def}}{=} [\begin{bmatrix} 1 & 0 \\ p[\mu']_{n-2} & 1 \end{bmatrix}, \delta_{j,0}(\mu_{n-2}X + Y)^r]$$

for $\mu \in I_n$, $\mu' \in I_{n-1}$ and $0 \leqslant j \leqslant r$.
For $n = 1$ we define $T_1^- : R_1 \to R_0$ by the conditions:

$$T_1^-([\begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix}, X^{r-j}Y^j]) \stackrel{\text{def}}{=} \delta_{j,r}(X + \mu_0 Y)^r$$
$$T_1^-([1_K, X^j Y^{r-j}]) \stackrel{\text{def}}{=} \delta_{j,0} Y^r.$$

for $\mu_0 \in \mathbf{F}_p$, $0 \leqslant j \leqslant r$.

Again, identifying R_n with $W(n)$ via the isomorphism described in Proposition 3.4 and using the results of §2.1 we see

$$T_n^-([g, v]) = \Psi_{n-1}(T([g, v])) \tag{3}$$

for all $g \in KZ\alpha^{-n}KZ$, $v \in \sigma_r$ and $n \in \mathbf{N}_>$ (i.e. $T_n^-([g, v]$ is described as the projection of $T([g, v])$ on the $W(n-1)$ component of the compact induction).

Thanks to the isomorphism of Proposition 3.4, we deduce the following properties of the Hecke operators $T_n^{\pm}$:

LEMMA 3.8. *The operators $T_n^{\pm}$ enjoy the following properties:*

1) *For all $n \in \mathbf{N}_>$, the morphisms is T_n^+, T_n^- are K-equivariant; for $n = 0$, the morphism T_0^+ is K-equivariant.*
2) *For all $n \geqslant 0$ the morphism T_n^+ is injective.*
3) *For all $n \geqslant 1$ the morphism T_n^- is surjective.*

Proof: *i*). We recall that the KZ-action on the tree preserves the distances from the central vertex. The assertion is then clear from the KZ-equivariance of T and the equalities (2), (3).

ii) and *iii*). We recall that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r & r^2 & \dots & r^r \end{bmatrix}$$

is invertible modulo p. This implies, for any fixed $i \in \{0, 1\}$, the following facts:

-) by support reasons the condition $T_n^+([g_{i,\mu}, v]) = 0$ forces $v = 0$ for any choice $\mu \in I_{n-i}$;
-) if $n \geqslant 1+i$ and $\mu \in I_{n-1-i}$ the $\overline{\mathbf{F}}_p$-subvector space of R_{n-1} generated by $T_n^-([g_{i,p^i\mu+p^{n-1}[\mu_{n-1}]}, Y^r])$ for $\mu_{n-1} \in \mathbf{F}_p$ coincide with the $\overline{\mathbf{F}}_p$-subvector space of R_{n-1} generated by $[g_{i,p^i\mu}, X^{r-j}Y^j]$ for $j \in \{0, \dots, r\}$.

This ends the proof. ♯

From now onwards we will consider R_n as a K-subrepresentation of R_{n+1} via the monomorphism T_n^+, for any $n \in \mathbf{N}$, without any further comment.

We can use the Hecke operators $T_n^{\pm}$ in order to construct a sequence of amalgamated sums of the R_n's. We define $R_0 \oplus_{R_1} R_2$ as the amalgamated sum

$$\begin{tikzcd} R_1 \arrow[r, hook, "T_1^+"] \arrow[d, "-T_1^-"'] & R_2 \arrow[d, dotted, "pr_2"] \\ R_0 \arrow[r, dotted] & R_0 \oplus_{R_1} R_2 \end{tikzcd}$$

where the second projection pr_2 is epi by base change. For any odd integer $n \in \mathbf{N}_>$ we define inductively the amalgamated sum $R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}$ as:

$$\begin{tikzcd} R_n \arrow[r, hook, "T_n^+"] \arrow[d, "-pr_{n-1}\circ T_n^-"'] & R_{n+1} \arrow[d, dotted, "pr_{n+1}"] \\ R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, dotted] & R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_n} R_{n+1}; \end{tikzcd} \tag{4}$$

once again, the second projection pr_{n+1} is epi by base change.

For any even positive integer $m \in \mathbf{N}_>$ we define the amalgamated sum $R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1}$ in the evident similar way.

We are now ready to state the main result of this section

Proposition 3.9. *Let $0 \leqslant r \leqslant p-1$. We have a KZ equivariant isomorphism*

$$\pi(r,0,1)|_{KZ} \xrightarrow{\sim} \varinjlim_{n\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}) \oplus \varinjlim_{m\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1}).$$

Proof: We have the following commutative diagram, with KZ-equivariant arrows:

$$\begin{tikzcd} (\mathrm{Ind}_{KZ}^G \sigma_r)|_{KZ} \arrow[r, "T|_{KZ}"] \arrow[d, "\wr"] & (\mathrm{Ind}_{KZ}^G \sigma_r)|_{KZ} \arrow[d, "\wr"] \\ \bigoplus_{n\in\mathbf{N}} R_n \arrow[r, "T_0^+ + \sum_{n\geqslant 1}(T_n^+ + T_n^-)"] & \bigoplus_{n\in\mathbf{N}} R_n; \end{tikzcd}$$

as the restriction functor is exact, we deduce that the isomorphism of corollary 3.5 induces an isomorphism $\pi(r,0,1)|_{KZ} \cong \mathrm{coker}(T_0^+ + \sum_{n\geqslant 1}(T_n^+ + T_n^-))$. We dispose of the evident inductive systems:

$$\Big\{\sum_{j=1,\,j\,\mathrm{odd}}^{n} T_j^+ + T_j^- : \bigoplus_{j=1,\,j\,odd}^{n} R_j \to \bigoplus_{i=0,\,i\,even}^{n+1} R_i\Big\}_{n\in\mathbf{N},\,n\,\mathrm{odd}}$$

$$\Big\{T_0^+ + \sum_{j=1,\,j\,\mathrm{even}}^{n} T_j^+ + T_j^- : \bigoplus_{j=0,\,j\,even}^{n} R_j \to \bigoplus_{i=0,\,i\,odd}^{n+1} R_i\Big\}_{n\in\mathbf{N},\,n\,\mathrm{even}}$$

so that, by the right exactness of the functor $\varinjlim$, the isomorphism of corollary 3.5 gives

$$\pi(r,0,1)|_{KZ} \cong \varinjlim_{n,\,\mathrm{odd}} \Big(\mathrm{coker}\big(\sum_{j=1,\,j\,\mathrm{odd}} T_j^+ + T_j^-\big)\Big) \oplus \varinjlim_{n,\,\mathrm{even}} \Big(\mathrm{coker}\big(T_0^+ + \sum_{j=1,\,j\,\mathrm{even}}^{n} T_j^+ + T_j^-\big)\Big).$$

It follows finally from the definitions of the amalgamated sum (and an immediate induction) that

$$(\mathrm{coker}(\sum_{j=1,\, j\,\mathrm{odd}} T_j^+ + T_j^-)) = R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$$

$$(\mathrm{coker}(T_0^+ + \sum_{j=1,\, j\,\mathrm{even}}^{n} T_j^+ + T_j^-)) = R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$$

and the proof is complete. $\sharp$

4. Defining the filtrations on the spaces R_n, $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$

In this section, we fix once for all an integer $r \in \{0, \ldots, p-1\}$. Our aim is to to point out, in definition 4.3, a filtration on $\varinjlim_{n\,\mathrm{odd}} R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ (resp. $\varinjlim_{n\,\mathrm{even}} R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}$) which will let us describe explicitly the socle filtration for the KZ-restriction of the supersingular representation $\pi(r, 0, 1)|_{KZ}$.

Proposition 4.1. *For any odd integer $n \in \mathbf{N}_>$ we have a natural commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d, "-pr_{n-1}\circ T_n^-"] & R_{n+1} \arrow[r] \arrow[d] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r] & R_{n+1}/R_n \arrow[r] & 0
\end{tikzcd}$$

with exact lines. We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N}\setminus\{0\}}.$$

Proof: The proof is by induction. We dispose of the commutative diagram:

$$\begin{tikzcd}
R_n \arrow[r, hook, "T_n^+"] \arrow[d, "-pr_{n-1}\circ T_n^-"] & R_{n+1} \arrow[d, dotted, "pr_{n+1}"] \\
R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, dotted] & R_0 \oplus_{R_1} \oplus_{R_n} R_{n+1}
\end{tikzcd}$$

where the morphism $-pr_{n-1}\circ T_n^-$ is epi by the inductive hypothesis; it follows then from the universal property of the amalgamated sum that the morphism pr_{n+1} is epi too. Moreover, since the forgetful functor $For : \mathrm{Rep}_K \to \mathrm{Vect}_{\overline{\mathbf{F}}_p}$ is right exact we deduce, by the injectivity of T_n^+ and base change in the category $\mathrm{Vect}_{\overline{\mathbf{F}}_p}$, that the morphism $R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \to R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ is injective too.

From the universal property of the amalgamated sum we get the natural commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r] \arrow[d] & R_{n+1} \arrow[r] \arrow[d] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\
& R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r, hook] \arrow[rr, bend right, "0"'] & R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \arrow[r, dotted, "\exists !"] & R_{n+1}/R_n &
\end{tikzcd}$$

where the first line is exact. The exactness of the second line is then an immediate diagram chase. $\sharp$

From the proof of Proposition 4.1 we see that we have actually a much stronger result: if $0 \leqslant j \leqslant n-2$ is odd and Q_{j+1} is any quotient of R_{j+1} we can still define the amalgamated sums

$Q_{j+1} \oplus_{R_{j+2}} \cdots \oplus_{R_n} R_{n+1}$ as in 4; then

Corollary 4.2. *Let $0 \leqslant j \leqslant n-2$ be odd, Q_{j+1} be a quotient of R_{j+1}. We have a natural commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d] & R_{n+1} \arrow[r] \arrow[d] & R_{n+1}/R_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q_{j+1} \oplus_{R_{j+2}} \cdots \oplus_{R_{n-2}} R_{n-1} \arrow[r] & Q_{j+1} \oplus_{R_{j+2}} \cdots \oplus_{R_n} R_{n+1} \arrow[r] & R_{n+1}/R_n \arrow[r] & 0
\end{tikzcd}$$

with exact lines (and with the obvious convention $Q_{j+1} \oplus_{R_j} R_{j+1} \overset{\text{def}}{=} Q_{j+1}$).

We have an analogous result concerning the family

$$\{R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_n} R_{n+1}\}_{n \in 2\mathbf{N}\setminus\{0\}}.$$

For each $n \in \mathbf{N}$ we look at a natural filtration on R_{n+1}. The definition is the following:

Definition 4.3. *Let $n \in \mathbf{N}$, $0 \leqslant t \leqslant r$. We define $\mathrm{Fil}^t(R_{n+1})$ as the K-subrepresentation of R_{n+1} generated by $[1_K, X^{r-t}Y^t]$. For $t=-1$, we define $\mathrm{Fil}^{-1}(R_{n+1}) \overset{\text{def}}{=} 0$.*

We note that

Lemma 4.4. *Let $n \in \mathbf{N}$. The family*

$$\{\mathrm{Fil}^t(R_{n+1})\}_{t=-1}^{t=r}$$

defines a separated and exhaustive decreasing filtration on R_{n+1}. Moreover, for each $t \in \{0, \dots, r\}$, the family

$$\mathscr{B}_{n+1,t} \overset{\text{def}}{=} \left\{ [\begin{bmatrix} \mu & 1 \\ 1 & 0 \end{bmatrix}, X^{r-i}Y^i], [\begin{bmatrix} 1 & 0 \\ p\mu' & 1 \end{bmatrix}, X^{r-i}Y^i],\ \mu \in I_{n+1},\ \mu' \in I_n,\ 0 \leqslant i \leqslant t \right\}$$

is an $\overline{\mathbf{F}}_p$ basis for $\mathrm{Fil}^t(R_{n+1})$; in particular $\mathrm{Fil}^t(R_{n+1})$ has dimension $(p+1)p^n(t+1)$ over $\overline{\mathbf{F}}_p$.

Proof: It is immediate from corollary 3.3 and the definition of the σ_r^{n+1}'s. ♯

By Frobenius reciprocity, we have an explicit description of the graded pieces of the filtration defined in 4.3:

Lemma 4.5. *Let $n \in \mathbf{N}$, and fix $-1 \leqslant t \leqslant r$. Then, we have a K-equivariant isomorphism:*

$$\mathrm{Fil}^t(R_{n+1})/\mathrm{Fil}^{t-1}(R_{n+1}) \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t.$$

where the characters χ_r^s, $\mathfrak{a}$, defined in §2.2, are seen as characters on $K_0(p^{n+1})$ by inflation $K_0(p^{n+1}) \twoheadrightarrow B(\mathbf{F}_p)$.

Proof: As the image of the element $[1_K, X^{r-t}Y^t]$ is a K-generator of the graded piece $\mathrm{Fil}^t(R_{n+1})/\mathrm{Fil}^{t-1}(R_{n+1})$, and $K_0(p^{n+1})$ acts on it by the character $\chi_r^s\mathfrak{a}^t$ we deduce by Frobenius reciprocity a K-equivariant epimorphism:

$$\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t \twoheadrightarrow \mathrm{Fil}^t(R_{n+1})/\mathrm{Fil}^{t-1}(R_{n+1}).$$

As the two spaces have the same $\overline{\mathbf{F}}_p$-dimension, the latter is indeed an isomorphism. ♯

We then see that the first step to understand the nature of $\pi(r,0,1)|_{KZ}$ consists in the study of the induced representations $\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^t$ for $n \in \mathbf{N}$, $0 \leqslant t \leqslant r$; such a study will be the object of the following two sections (§5, §6).

5. Study of an Induction-I

In this section, we will fix two integers $1 \leqslant m \leqslant n+1$ and η a character of $B(\mathbf{F}_p)$ (which will be considered as a continuous character of $K_0(p^{n+1})$ by inflation), and we will fix a basis $\{e_\eta\}$ for η. The object of this section is then (cf. Proposition 5.10) to describe explicitly the socle filtration for

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta$$

and the proof will be essentially an induction on the length $n+1-m$ (§5.1, §5.2).

For $1 \leqslant m \leqslant n+1$ define a subset I_{n+1}/I_m of $\mathbf{Z}_p$:

$$I_{n+1}/I_m \overset{\mathrm{def}}{=} \{\sum_{j=m}^{n} p^j[\mu_j],\, \mu_j \in \mathbf{F}_p\}.$$

We have the following elementary lemmas.

LEMMA 5.1. *For $1 \leqslant m \leqslant n+1$ we have the decomposition*

$$K_0(p^m)/K_0(p^{n+1}) = \coprod_{x \in I_{n+1}/I_m} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} K_0(p^{n+1}).$$

In particular, the family

$$\mathscr{I}_{m,n+1} \overset{\mathrm{def}}{=} \{[\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, e_\eta],\, x \in I_{n+1}/I_m\}$$

is an $\overline{\mathbf{F}}_p$-basis for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta$ and $\dim_{\overline{\mathbf{F}}_p}\left(\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta\right) = p^{n+1-m}$.

Proof: Immediate from corollary 3.3. ♯

LEMMA 5.2. *Let $1 \leqslant m \leqslant n+1$ be integers and η a character of $B(\mathbf{F}_p)$. Then we have a $K_0(p^m)$-equivariant canonical isomorphism:*

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}\eta \overset{\sim}{\to} (\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1) \otimes \eta$$

where η is seen (by inflation) as a character of $K_0(p^{n+1})$ and $K_0(p^m)$ in the left hand side and in the right hand side respectively.

Proof: The assignment, for $x \in I_{n+1}/I_m$,

$$[\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, e_\eta] \mapsto [\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, e_1] \otimes e_\eta$$

defines an $\overline{\mathbf{F}}_p$-isomorphism which is actually $K_0(p^m)$-equivariant, as $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \in K_1$ for all $x \in I_{n+1}/I_m$. ♯

In particular, by Lemma 5.2, we can assume $\eta = 1$.

5.1 The case $m = n$

We establish here the first step concerning the inductive description of the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)}1$; fix once for all an $\overline{\mathbf{F}}_p$-basis $\{e\}$ for the underlying vector space of the trivial character 1. We introduce the objects:

DEFINITION 5.3. *Let $n \in \mathbf{N}_>$ and $0 \leqslant l_n \leqslant p-1$. Then:*

i) *we define the following element of* $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)}1$*:*

$$F_{l_n}^{(n)} \overset{\mathrm{def}}{=} \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} [\begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix}, e];$$

we define formally $F_{-1}^{(n)}, F_p^{(n)} \overset{\mathrm{def}}{=} 0$*;*

ii) *we define the following quotient of* $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)}1$*:*

$$Q_{l_n}^{(n,n+1)} \overset{\mathrm{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)}1 / \langle F_0^{(n)}, \dots, F_{l_n-1}^{(n)} \rangle_{\overline{\mathbf{F}}_p};$$

we define formally $Q_p^{(n,n+1)} \overset{\mathrm{def}}{=} 0$.

For any $0 \leqslant l_n, l'_n \leqslant p-1$ we will often commit the abuse to use the same notation for $F_{l_n}^{(n)}$ and its image in the quotient $Q_{l'_n}^{(n,n+1)}$. The meaning will be clear according to the context.

The next computation is the main tool to describe the socle filtration for $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)}1$.

Lemma 5.4. *Let* $g \in K_0(p^{n+1})$*,* $\lambda \in \mathbf{F}_p$ *and* $0 \leqslant l_n \leqslant p-1$*. Then we have the equalities in* $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^n)}1$*:*

i) $g \cdot F_{l_n}^{(n)} = \mathfrak{a}^{l_n}(g) F_{l_n}^{(n)}$*;*

ii) $\begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} F_{l_n}^{(n)} = \sum_{j=0}^{l_n} \binom{l_n}{j} (-\lambda)^j F_{l_n-j}^{(n)}$.

Proof: *i)*. If $g = \begin{bmatrix} a & b \\ p^{n+1}c & d \end{bmatrix}$, then we can write

$$g \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p^n[\mu_n \overline{a}^{-1} \overline{d}] & 1 \end{bmatrix} \begin{bmatrix} a' & b \\ p^{n+1}c' & d' \end{bmatrix}$$

where $a', c', d' \in \mathbf{Z}_p$ and $a' \equiv a\,[p]$, $d' \equiv d\,[p]$. Thus,

$$gF_{l_n}^{(n)} = \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} [\begin{bmatrix} 1 & 0 \\ p^n[\mu_n \overline{a}^{-1} \overline{d}] & 1 \end{bmatrix}, e] = (\overline{a}\overline{d}^{-1})^{l_n} F_{l_n}^{(n)}.$$

Since $[\lambda] + [\mu_n] \equiv [\lambda + \mu_n]$ modulo p, we deduce

$$\begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} F_{l_n}^{(n)} = \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} [\begin{bmatrix} 1 & 0 \\ p^n[\mu_n + \lambda] & 1 \end{bmatrix}, e].$$

The result follows. ♯

As a consequence, we get the corollaries:

Corollary 5.5. *For any* $0 \leqslant l_n \leqslant p-1$*, the sub-*$K_0(p^n)$ *representation of* $Q_l^{(n,n+1)}$ *generated by* $F_{l_n}^{(n)}$ *is isomorphic to* $\mathfrak{a}^{l_n}$*.*

Proof: For any $g \in K_0(p^n)$ we can write $g = \begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} \kappa$ with suitable elements $\lambda \in \mathbf{F}_p$, $\kappa \in K_0(p^{n+1})$ (Lemma 5.1). The result comes from Lemma 5.4 and the definition of $Q_{l_n}^{(n,n+1)}$. ♯

Corollary 5.6. *For any* $0 \leqslant l_n \leqslant p-1$ *we have* $K_0(p^n)$*-equivariant exact sequence*

$$0 \to \langle F_{l_n}^{(n)} \rangle \to Q_{l_n}^{(n,n+1)} \to Q_{l_n+1}^{(n,n+1)} \to 0$$

which is nonsplit if $l_n \leqslant p-2$. Moreover,

$$\dim_{\overline{\mathbf{F}}_p}(Q_{l_n}^{(n,n+1)}) = p - l_n.$$

Proof: The exact sequence is clear. Furthermore, if $\phi : Q_{l_n}^{(n,n+1)} \to \langle F_{l_n}^{(n)} \rangle$ is any $K_0(p^n)$-equivariant morphism, we see that

$$\phi(F_{l_n}^{(n)}) = \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} \phi([1_{K_0(p^n)}, e]) = \phi([1_{K_0(p^n)}, e]) \sum_{\mu_n \in \mathbf{F}_p} \mu_n^{l_n}.$$

Thus, there cannot be any $K_0(p^n)$ equivariant sections for $\langle F_{l_n}^{(n)} \rangle \to Q_{l_n}^{(n,n+1)}$ if $0 \leqslant l_n \leqslant p-2$. The assertion concerning the dimension is immediate by induction.$\sharp$

COROLLARY 5.7. *Let $0 \leqslant l_n \leqslant p-1$. Then the socle of $Q_{l_n}^{(n,n+1)}$ is given by:*

$$\mathrm{soc}(Q_{l_n}^{(n,n+1)}) = \langle F_{l_n}^{(n)} \rangle.$$

Proof: We have $Q_{p-1}^{(n,n+1)} \cong \langle F_{p-1}^{(n)} \rangle$, as the two spaces are 1-dimensional. By a decreasing induction, assume $\mathrm{soc}(Q_{l_n+1}^{(n,n+1)}) = \langle F_{l_n+1}^{(n)} \rangle$ for $l_n \leqslant p-2$ and consider the exact sequence

$$0 \to \langle F_{l_n}^{(n)} \rangle \to Q_{l_n}^{(n,n+1)} \to Q_{l_n+1}^{(n,n+1)} \to 0.$$

If τ is an irreducible $K_0(p^n)$-subrepresentation of $Q_{l_n}^{(n,n+1)}$ such that $\tau \cap \langle F_{l_n}^{(n)} \rangle = 0$, we deduce that $F_{l_n+1}^{(n)} + c_1 F_{l_n}^{(n)} \in \tau$ for a suitable $c_1 \in \overline{\mathbf{F}}_p$. From the equality

$$\begin{bmatrix} 1 & 0 \\ p^n[\lambda] & 1 \end{bmatrix} (F_{l_n+1}^{(n)} + c_1 F_{l_n}^{(n)}) = F_{l_n+1}^{(n)} - (l_n+1)\lambda F_{l_n}^{(n)} + c_1 F_{l_n}^{(n)}$$

in $Q_{l_n}^{(n,n+1)}$ (where $\lambda \in \mathbf{F}_p^\times$), we find $F_{l_n}^{(n)} \in \tau$, contradiction. $\sharp$

5.2 The general case

Fix two integers $1 \leqslant m \leqslant n+1$. In this section we establish the inductive step which lets us describe the socle filtration for the representation $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$. We recall the following result:

PROPOSITION 5.8. *Let $1 \leqslant m \leqslant n+1$. For any $m \leqslant j \leqslant n+1$ we have a canonical isomorphism:*

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1 \xrightarrow{\sim} \mathrm{Ind}_{K_0(p^j)}^{K_0(p^m)} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^j)} 1.$$

For any two $(n+1-m)$-tuples $(j_m, \dots, j_n)$, $(l_m, \dots, l_n) \in \{0, \dots, p-1\}^{n-m+1}$ we define inductively

$$(j_m, \dots, j_n) \prec (l_m, \dots, l_n)$$

if either $(j_{m+1}, \dots, j_n) \prec (l_{m+1}, \dots, l_n)$ or $(j_{m+1}, \dots, j_n) = (l_{m+1}, \dots, l_n)$ and $j_m < l_m$. We can therefore introduce the objects:

DEFINITION 5.9. *Let $(l_m, \dots, l_n) \in \{0, \dots, p-1\}^{n-m+1}$ be an $(n+1-m)$-tuples. Then:*

i) we define inductively the following element of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$:

$$F_{l_m}^{(m)} * \dots * F_{l_n}^{(n)} \stackrel{\mathrm{def}}{=} \sum_{\mu_m \in \mathbf{F}_p} \mu_m^{l_m} \begin{bmatrix} 1 & 0 \\ p^m[\mu_m] & 1 \end{bmatrix} [1_{K_0(p^m)}, F_{l_{m+1}}^{(m+1)} * \dots * F_{l_n}^{(n)}]$$

where we adopt the convention $F_{l_m+1}^{(m)} * \dots * F_{l_n}^{(n)} \stackrel{\mathrm{def}}{=} F_0^{(m)} * F_{l_{m+1}+1}^{(m+1)} * \dots * F_{l_n}^{(n)}$ if $l_m = p-1$.

ii) We define the following quotient of $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1$*:*

$$Q^{(m,n+1)}_{l_m,\dots,l_n} \overset{\mathrm{def}}{=} \mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^m)} 1/\langle F^{(m)}_{j_m} * \dots * F^{(n)}_{j_n} \text{ for } (j_m,\dots,j_n) \prec (l_m,\dots,l_n)\rangle_{\overline{\mathbf{F}}_p}$$

where we adopt the convention $Q^{(m,n+1)}_{l_m+1,\dots,l_n} \overset{\mathrm{def}}{=} Q^{(m,n+1)}_{0,l_{m+1}+1,\dots,l_n}$ *if* $l_m = p-1$.

We give here the statement of the main result.

Proposition 5.10. *Let* $1 \leqslant m \leqslant n+1$ *be integers, and* $(l_m,\dots,l_n) \in \{0,\dots,p-1\}^{n-m+1}$ *a* $(n-m+1)$*-tuple. Then*

i) The $K_0(p^m)$*-subrepresentation of* $Q^{(m,n+1)}_{l_m,\dots,l_n}$ *generated by* $F^{(m)}_{l_m} * \dots * F^{(n)}_{l_n}$ *is isomorphic to* [4] $\mathfrak{a}^{l_m} \otimes \dots \otimes \mathfrak{a}^{l_n} = \mathfrak{a}^{l_m+\dots+l_n}$*;*

ii) we have a $K_0(p^m)$*-equivariant exact sequence:*

$$0 \to \langle F^{(m)}_{l_m} * \dots * F^{(n)}_{l_n}\rangle \to Q^{(m,n+1)}_{l_m,\dots,l_n} \to Q^{(m,n+1)}_{l_m+1,\dots,l_n} \to 0 \tag{5}$$

which is nonsplit if $(l_m,\dots,l_n) \neq (p-1,\dots,p-1)$*. Moreover*

$$Q^{(m,n+1)}_{0,l_{m+1},\dots,l_n} = \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1},\dots,l_n}$$

and

$$\dim_{\overline{\mathbf{F}}_p}(Q^{(m,n+1)}_{l_m,\dots,l_n}) = p^{n-m+1} - \sum_{j=0}^{n-m} p^{n-m-j} l_{n-j}.$$

iii) The socle of $Q^{(m,n+1)}_{l_m,\dots,l_n}$ *is given by*

$$\mathrm{soc}(Q^{(m,n+1)}_{l_m,\dots,l_n}) = \langle F^{(m)}_{l_m} * \dots * F^{(n)}_{l_n}\rangle.$$

As we said, the proof is an induction on the length $n+1-m$, the case $m=n$ being proved in the previous section; in what follows, we will therefore assume Proposition 5.10 for any length l with $l < n+1-m$. We first need the following tools.

Lemma 5.11. *Let* $(l_m,\dots,l_n) \in \{0,\dots,p-1\}^{n-m+1}$ *be an* $(n-m+1)$*-tuple. The following diagrams are commutative with exact lines*

i)

$$\begin{tikzcd}
0 \arrow[r] & \langle F^{(m)}_{l_m} * \dots * F^{(n-1)}_{l_{n-1}}\rangle \otimes \mathfrak{a}^{l_n} \arrow[r] \arrow[d, "\wr"] & Q^{(m,n)}_{l_m,\dots,l_{n-1}} \otimes \mathfrak{a}^{l_n} \arrow[r] \arrow[d, hook] & Q^{(m,n)}_{l_m+1,\dots,l_{n-1}} \otimes \mathfrak{a}^{l_n} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \langle F^{(m)}_{l_m} * \dots * F^{(n)}_{l_n}\rangle \arrow[r] & Q^{(m,n+1)}_{l_m,\dots,l_n} \arrow[r] & Q^{(m,n+1)}_{l_m+1,\dots,l_n} \arrow[r] & 0;
\end{tikzcd}$$

ii)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} F^{(m+1)}_{l_{m+1}} * \dots * F^{(n)}_{l_n} \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1},\dots,l_n} \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1}+1,\dots,l_n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q^{(m,m+1)}_{l_m} \otimes \mathfrak{a}^{l_{m+1}} \otimes \dots \otimes \mathfrak{a}^{l_n} \arrow[r] & Q^{(m,n+1)}_{l_m,\dots,l_n} \arrow[r] & \mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)} Q^{(m+1,n+1)}_{l_{m+1}+1,\dots,l_n} \arrow[r] & 0.
\end{tikzcd}$$

[4] as remarked by the referee, the notation with the tensor product may be confusing as it can be interpreted as a character of $(n+1-m)$ copies of $K_0(p^m)$. As stressed in the statement of Proposition 5.10, the tensor product $\mathfrak{a}^{l_m} \otimes \dots \otimes \mathfrak{a}^{l_n}$ we mean here is the classical tensor product of $K_0(p^m)$-representation, see for instance [Alp], **II** §5.

Proof: The proof will be an induction on the $(n+1-m)$-tuple $(l_m,\dots,l_n)\in\{0,\dots,p-1\}^{n+1-m}$.

$i)$ From corollary 5.6 and the exactness of the induction functor we dispose of the following exact sequence for any $0\leqslant l_n\leqslant p-1$:

$$0\to \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}\langle F_{l_n}^{(n)}\rangle\to \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}Q_{l_n}^{(n,n+1)}\to \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}Q_{l_n+1}^{(n,n+1)}\to 0$$

and $\langle F_{l_n}^{(n)}\rangle\cong\mathfrak{a}^{l_n}$. We assume, inductively, to have the commutative diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}1\otimes\mathfrak{a}^{l_n} \arrow[r]\arrow[d] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}Q_{l_n}^{(n,n+1)} \arrow[r]\arrow[d] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}Q_{l_n+1}^{(n,n+1)} \arrow[r]\arrow[d, equal] & 0\\
0 \arrow[r] & Q_{l_m,\dots,l_{n-1}}^{(m,n)}\otimes\mathfrak{a}^{l_n} \arrow[r] & Q_{l_m,\dots,l_n}^{(m,n+1)} \arrow[r] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}Q_{l_n+1}^{(n,n+1)} \arrow[r] & 0.
\end{tikzcd}$$

We can invoke Proposition 5.10 for $\mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}1\otimes\mathfrak{a}^{l_n}$ deducing the diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d]\\
\langle F_{l_m}^{(m)}*\cdots*F_{l_{n-1}}^{(n-1)}\rangle\otimes\mathfrak{a}^{l_n} \arrow[r, hook]\arrow[d, "\wr"] & Q_{l_m,\dots,l_{n-1}}^{(m,n)}\otimes\mathfrak{a}^{l_n} \arrow[d]\\
\langle F_{l_m}^{(m)}*\cdots*F_{l_n}^{(n)}\rangle \arrow[r, hook]\arrow[d] & Q_{l_m,\dots,l_n}^{(m,n+1)} \arrow[d]\\
0 \arrow[r]\arrow[d] & \mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}Q_{l_n+1}^{(n,n+1)} \arrow[d]\\
0 & 0
\end{tikzcd}$$

and we are left to use the snake lemma to conclude the induction (notice that if $(l_m,\dots,l_{n-1})=(p-1,\dots,p-1)$ we just deduce the isomorphism $\mathrm{Ind}_{K_0(p^n)}^{K_0(p^m)}Q_{l_n+1}^{(n,n+1)}\cong Q_{0,\dots,0,l_n+1}^{(m,n+1)}$).

$ii)$. It is similar to $i)$. The details are left to the reader. ♯

LEMMA 5.12. *Fix two integers $1\leqslant m\leqslant n+1$, let $(l_m,\dots,l_n)\in\{0,\dots,p-1\}^{n-m+1}$ be an $(n-m+1)$-tuple and assume $(l_m,\dots,l_n)\prec(p-1,\dots,p-1)$. Moreover, let $\lambda\in\mathbf{F}_p$ and $t=\sum_{j\in\mathbf{N}}p^j[t_j]\in\mathbf{Z}_p$ be a p-adic integer.*

Then, the action of $\begin{bmatrix}1&0\\ p^m[\lambda]+p^{m+1}t&1\end{bmatrix}$ on $F_{l_m+1}^{(m)}*F_{l_{m+1}}^{(m+1)}*\cdots*F_{l_n}^{n}$ inside $Q_{l_m,\dots,l_n}^{(m,n+1)}$ is described by

$$\begin{aligned}\begin{bmatrix}1&0\\ p^m[\lambda]+p^{m+1}t&1\end{bmatrix}&\cdot F_{l_m+1}^{(m)}*\cdots*F_{l_n}^{(n)}=\\ &=F_{l_m+1}^{(m)}*\cdots*F_{l_n}^{(n)}+(l_j+1)(-1)^{j-m+1}\lambda F_{l_m}^{(m)}*\cdots*F_{l_n}^{(n)}\end{aligned}$$

where $j\in\{m,\dots,n\}$ is minimal with respect to the property that $l_j+1\not\equiv 0\bmod p$.

Proof: The case $m=n$ is an immediate computation, and it is left to the reader. In order to establish the general step, we need to distinguish two cases:

Situation A). Assume $l_m\leqslant p-2$. It follows from Proposition 5.10 applied to $\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p^{m+1})}1$ that $\begin{bmatrix}1&0\\ p^{m+1}\mathbf{Z}_p&1\end{bmatrix}$ acts trivially on $F_{l_{m+1}}^{(m+1)}*\cdots*F_{l_n}^{(n)}$ in $Q_{l_{m+1},\dots,l_n}^{(m+1,n+1)}$, and we deduce the following equalities in $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)}Q^{(m+1,n+1)}_{l_{m+1},\dots,l_n}$:

$$\begin{aligned}\begin{bmatrix}1 & 0\\ p^m[\lambda]+p^{m+1}t & 1\end{bmatrix}\sum_{\mu_m\in\mathbf{F}_p}\mu_m^{l_m+1}\begin{bmatrix}1 & 0\\ p^m[\mu_m] & 1\end{bmatrix}[1,F^{(m+1)}_{l_{m+1}}*\dots*F^{(n)}_{l_n}] &=\\ =\sum_{\mu_m\in\mathbf{F}_p}\mu_m^{l_m+1}\begin{bmatrix}1 & 0\\ p^m[\lambda+\mu_m] & 1\end{bmatrix}[1,F^{(m+1)}_{l_{m+1}}*\dots*F^{(n)}_{l_n}] &=\\ =\sum_{j=0}^{l_m+1}\binom{l_m+1}{j}(-\lambda)^j[1,F^{(m+1)}_{l_{m+1}}*\dots*F^{(n)}_{l_n}].\end{aligned}$$

We conclude using the projection $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)}Q^{(m+1,n+1)}_{l_{m+1},\dots,l_n}\twoheadrightarrow Q^{(m,n+1)}_{l_m,\dots,l_n}$.

Situation B). Assume $l_m=p-1$; therefore $F^{(m)}_{l_m+1}*\dots*F^{(n)}_{l_n}=F^{(m)}_0*F^{(m+1)}_{l_{m+1}+1}*\dots*F^{(n)}_{l_n}$. Lemma 2.7 and the inductive hypothesis applied to $F^{(m+1)}_{l_{m+1}+1}*\dots*F^{(n)}_{l_n}\in Q^{(m+1,n+1)}_{l_{m+1},\dots,l_n}$ let us deduce the following equalities inside $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)}Q^{(m+1,n+1)}_{l_{m+1},\dots,l_n}$:

$$\begin{aligned}&\begin{bmatrix}1 & 0\\ p^m[\lambda]+p^{m+1}t & 1\end{bmatrix}\sum_{\mu_m\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^m[\mu_m] & 1\end{bmatrix}[1,F^{(m+1)}_{l_{m+1}+1}*\dots*F^{(n)}_{l_n}]=\\ &=\sum_{\mu_m\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^m[\mu_m+\lambda] & 1\end{bmatrix}[1,F^{(m+1)}_{l_{m+1}+1}*\dots*F^{(n)}_{l_n}]+\\ &\quad+(l_j+1)(-1)^{j-m}\sum_{\mu_m\in\mathbf{F}_p}(P_\lambda(\mu_m)+t_0)\begin{bmatrix}1 & 0\\ p^m[\lambda+\mu_m] & 1\end{bmatrix}[1,F^{(m+1)}_{l_{m+1}+1}*\dots*F^{(n)}_{l_n}]\\ &=F^{(m)}_{l_m+1}*\dots*F^{(n)}_{l_n}+(l_j+1)(-1)^{j-m}(t_0F^{(m)}_0*F^{(m+1)}_{l_{m+1}}*\dots*F^{(n)}_{l_n}+\\ &\quad+\sum_{s=1}^{p-1}\frac{\binom{p}{s}}{p}(-\lambda)^{p-s}F^{(m)}_s*F^{(m+1)}_{l_{m+1}}*\dots*F^{(n)}_{l_n})\end{aligned}$$

where $j\in\{m+1,\dots,n\}$ is minimal with respect to the property that $l_j<p-1$. The conclusion comes using the projection $\mathrm{Ind}_{K_0(p^{m+1})}^{K_0(p^m)}Q^{(m+1,n+1)}_{l_{m+1},\dots,l_n}\twoheadrightarrow Q^{(m,n+1)}_{l_m,\dots,l_n}$. ♯

We are now able to deduce easily Proposition 5.10.

Proof of Proposition 5.10:

i) From Lemma 5.11-*i*) we have an isomorphism $\langle F^{(m)}_{l_m}*\dots*F^{(n-1)}_{l_{n-1}}\rangle\otimes\mathfrak{a}^{l_n}\xrightarrow{\sim}\langle F^{(m)}_{l_m}*\dots*F^{(n)}_{l_n}\rangle$ and we have $\langle F^{(m)}_{l_m}*\dots*F^{(n-1)}_{l_{n-1}}\rangle\cong\mathfrak{a}^{l_m}\otimes\dots\otimes\mathfrak{a}^{l_{n-1}}$ by the inductive hypothesis.

ii) As in corollary 5.6, we see that for any $K_0(p^m)$-equivariant morphism $\phi:Q^{(m,n+1)}_{l_m,\dots,l_n}\to\langle F^{(m)}_{l_m}*\dots*F^{(n)}_{l_n}\rangle$ we have

$$\phi(F^{(m)}_{l_m}*\dots*F^{(n)}_{l_n})=(-\delta_{p-1,l_m})\dots(-\delta_{p-1,l_n})\phi([1_{K_0(p^n)},e])$$

so that there cannot be any splitting for $\langle F^{(m)}_{l_m}*\dots*F^{(n)}_{l_n}\rangle\to Q^{(m,n+1)}_{l_m,\dots,l_n}$ if $(l_m,\dots,l_n)\prec(p-1,\dots,p-1)$. The identity

$$\dim_{\overline{\mathbf{F}}_p}(Q^{(m,n+1)}_{l_m,\dots,l_n})=p^{n-m+1}-\sum_{j=0}^{n-m}p^{n-m-j}l_{n-j}$$

is now an immediate induction.

iii) The case $(l_m,\dots,l_n)=(p-1,\dots,p-1)$ is trivial. We will prove the general case by a descending induction on the $(n+1-m)$-tuple $(l_m,\dots,l_n)$. Consider the exact sequence

$$0\to\langle F_{l_m}^{(m)}*\dots*F_{l_n}^{(n)}\rangle\to Q_{l_m,\dots,l_n}^{(m,n+1)}\to Q_{l_{m+1},\dots,l_n}^{(m,n+1)}\to 0$$

and let $\tau\leqslant Q_{l_m,\dots,l_n}^{(m,n+1)}$ be an irreducible subrepresentation such that $\tau\cap\langle F_{l_m}^{(m)}*\dots*F_{l_n}^{(n)}\rangle=0$. The inductive hypothesis $\mathrm{soc}(Q_{l_{m+1},\dots,l_n}^{(m,n+1)})=\langle F_{l_{m+1}}^{(m)}*\dots*F_{l_n}^{(n)}\rangle$ lets us conclude that

$$\tau=\langle F_{l_m+1}^{(m)}*\dots*F_{l_n}^{(n)}+c_1F_{l_m}^{(m)}*\dots*F_{l_n}^{(n)}\rangle\cong\mathfrak{a}^{l_m+1}\otimes\dots\mathfrak{a}^{l_n}$$

for a suitable $c_1\in\overline{\mathbf{F}}_p$. But by Lemma 5.12 we have the equalities in $Q_{l_m,\dots,l_n}^{(m,n+1)}$:

$$\begin{aligned}\begin{bmatrix}1&0\\p^m[\lambda]&1\end{bmatrix}&(F_{l_m+1}^{(m)}*\dots*F_{l_n}^{(n)}+c_1F_{l_m}^{(m)}*\dots*F_{l_n}^{(n)})=\\&=(F_{l_m+1}^{(m)}*\dots*F_{l_n}^{(n)}+c_1F_{l_m}^{(m)}*\dots*F_{l_n}^{(n)})+\\&\quad+\lambda(l_j+1)(-1)^{j-m+1}F_{l_m}^{(m)}*\dots*F_{l_n}^{(n)}\end{aligned}$$

(where $j\in\{m,\dots,n\}$ is defined as in Lemma 5.12) from which $F_{l_m}^{(m)}*\dots*F_{l_n}^{(n)}\in\tau$ if $\lambda\neq 0$, contradiction. $\sharp$.

6. Study of an Induction -II

Throughout this section we consider integers r,t with $0\leqslant r\leqslant p-1$, $0\leqslant t\leqslant p-2$ and $n\in\mathbf{N}_>$. Our aim is to describe the socle filtration of the induction

$$\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^t$$

using the results of section §5; the main result is then Proposition 6.6.

We start by fixing the following elements of $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^t$.

Definition 6.1. *Let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple, and let $t'\overset{\mathrm{def}}{=}\sum_{i=1}^n l_i$. We define*

$$F_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\overset{\mathrm{def}}{=}\begin{cases}\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}]\\ \qquad\text{if } r-2(t+t')\not\equiv 0\,[p-1];\\[2ex] \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}]+(-1)^{t+t'}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}]\\ \qquad\text{if } r-2(t+t')\equiv 0\,[p-1]\end{cases}$$

$$F_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\overset{\mathrm{def}}{=}\begin{cases}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}]\\ \qquad\text{if } r-2(t+t')\not\equiv 0\,[p-1];\\[2ex] \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1_K,F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}]\\ \qquad\text{if } r-2(t+t')\equiv 0\,[p-1].\end{cases}$$

If $(j_1,\dots,j_n)$, $(j'_1,\dots,j'_n)\in\{0,\dots,p-1\}^n$ are two n-tuples and $i,i'\in\{0,1\}$ we define

$$(i,j_1,\dots,j_n)\prec(i',j'_1,\dots,j'_n)$$

iff either $(j_1,\dots,j_n)\prec(j'_1,\dots,j'_n)$ or $(j_1,\dots,j_n)=(j'_1,\dots,j'_n)$ and $i<i'$. Finally

Definition 6.2. *Let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple, $i\in\{0,1\}$ and let $t'\overset{\text{def}}{=}\sum_{j=1}^n l_j$. We define the quotient $Q^{(0,n+1)}_{i,l_1,\dots,l_n}$ of $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^t$ as*

$$Q^{(0,n+1)}_{i,l_1,\dots,l_n}\overset{\text{def}}{=}\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^t/(\sum_{(j,j_1,\dots,j_n)\prec(i,l_1,\dots,l_n)}\langle K\cdot F_j^{(0)}*\dots F_{j_n}^{(n)}\rangle)$$

where

$$\sum_{(j,j_1,\dots,j_n)\prec(i,l_1,\dots,l_n)}\langle K\cdot F_j^{(0)}*\dots F_{j_n}^{(n)}\rangle$$

denotes the sub-K-representation of $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\mathfrak{a}^t$ generated by the elements $F_j^{(0)}*\dots F_{j_n}^{(n)}$ for $(j,j_1,\dots,j_n)\prec(i,l_1,\dots,l_n)$.

As usual, we adopt the convention

$$Q^{(0,n+1)}_{i+1,l_1,\dots,l_n}\overset{\text{def}}{=}Q^{(0,n+1)}_{0,l_1+1,\dots,l_n}$$

if $i=1$. We remark that in the previous definitions we do not keep track of the integers r,t: we adopted this choice in order not to overload the notations. We believe the values of r, t will be clear from the context (cf. §7, §8).

The study of the socle filtration starts from the following elementary lemma:

Lemma 6.3. *If $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ is an n-tuple, we have the following commutative diagrams with exact rows:*

i)

$$\begin{tikzcd}
0 \arrow[r] & \langle K\cdot F_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle \arrow[r] \arrow[d, equal] & \mathrm{Ind}_{K_0(p)}^K\langle F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle \arrow[r] \arrow[d, two heads] & \langle K\cdot F_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle \arrow[r] \arrow[d, hook] & 0\\
0 \arrow[r] & \langle K\cdot F_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle \arrow[r] & Q^{(0,n+1)}_{0,l_1,\dots,l_n} \arrow[r] & Q^{(0,n+1)}_{1,l_1,\dots,l_n} \arrow[r] & 0;
\end{tikzcd}$$

ii)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_{K_0(p)}^K\langle F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle \arrow[r] \arrow[d] & Q^{(0,n+1)}_{0,l_1,\dots,l_n} \arrow[r] \arrow[d] & Q^{(0,n+1)}_{0,l_1+1,\dots,l_n} \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & \langle K\cdot F_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle \arrow[r] & Q^{(0,n+1)}_{1,l_1,\dots,l_n} \arrow[r] & Q^{(0,n+1)}_{0,l_1+1,\dots,l_n} \arrow[r] & 0.
\end{tikzcd}$$

Proof: It is an induction on the n-tuple $(l_1,\dots,l_n)$. By Proposition 5.10 and the exactness of the induction functor we have the exact sequence

$$0\to\mathrm{Ind}_{K_0(p)}^K\langle F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\to\mathrm{Ind}_{K_0(p)}^KQ^{(1,n+1)}_{l_1,\dots,l_n}\to\mathrm{Ind}_{K_0(p)}^KQ^{(1,n+1)}_{l_1+1,\dots,l_n}\to0$$

and we dispose of the exact sequence (cf. Lemma 2.4)

$$0\to\langle K\cdot F_0^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\to\mathrm{Ind}_{K_0(p)}^K\langle F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\to\langle K\cdot F_1^{(0)}*F_{l_1}^{(1)}*\dots*F_{l_n}^{(n)}\rangle\to0.$$

The conclusion comes applying the snake lemma to the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\langle K\cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \arrow[r, hook] \arrow[d, equal] & \mathrm{Ind}_{K_0(p)}^{K}\langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \arrow[d] \\
\langle K\cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \arrow[r, hook] \arrow[d] & \mathrm{Ind}_{K_0(p)}^{K} Q_{l_1,\dots,l_n}^{(1,n+1)} \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathrm{Ind}_{K_0(p)}^{K} Q_{l_1+1,\dots,l_n}^{(1,n+1)} \arrow[d] \\
0 & 0
\end{tikzcd}$$

assuming inductively that $\mathrm{Ind}_{K_0(p)}^{K} Q_{l_1,\dots,l_n}^{(1,n+1)} = Q_{0,l_1,\dots,l_n}^{(0,n+1)}$. $\sharp$

We deduce the following two corollaries:

Corollary 6.4. *Let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple. Then:*

i) The K-subrepresentation of $Q_{0,l_1,\dots,l_n}^{(0,n+1)}$ generated by $F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ is isomorphic to

$$\begin{aligned}\langle K\cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle &\xrightarrow{\sim} \mathrm{Sym}^{\lfloor r-2(t+t')\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{t+t'}\\ F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} &\mapsto X^{\lfloor r-2(t+t')\rfloor}.\end{aligned}$$

If, moreover, $r-2(t+t')\equiv 0[p-1]$, then the K-subrepresentation of $Q_{0,l_1,\dots,l_n}^{(0,n+1)}$ generated by $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ is isomorphic to

$$\begin{aligned}\langle K\cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle &\xrightarrow{\sim} \mathrm{Sym}^{p-1}\overline{\mathbf{F}}_p^2\otimes\det{}^{t+t'}\\ F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} &\mapsto X^{p-1}.\end{aligned}$$

ii) The K-subrepresentation of $Q_{1,l_1,\dots,l_n}^{(0,n+1)}$ generated by $F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ is isomorphic to

$$\begin{aligned}\langle K\cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle &\xrightarrow{\sim} \mathrm{Sym}^{p-1-\lfloor r-2(t+t')\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{r-(t+t')}\\ F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)} &\mapsto X^{p-1-\lfloor r-2(t+t')\rfloor}.\end{aligned}$$

Proof: As $\langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \cong \chi_r^s\mathfrak{a}^{t+t'}$ the statement is an immediate consequence of Lemma 6.3 and Proposition 2.4. $\sharp$

Corollary 6.5. *Let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple. Then:*

i) If $(l_1,\dots,l_n)\neq(p-1,\dots,p-1)$ the exact sequences:

$$\begin{aligned}0\to\langle K\cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \to Q_{0,l_1,\dots,l_n}^{(0,n+1)}\to Q_{1,l_1,\dots,l_n}^{(0,n+1)}\to 0;\\ 0\to\langle K\cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle \to Q_{1,l_1,\dots,l_n}^{(0,n+1)}\to Q_{0,l_1+1,\dots,l_n}^{(0,n+1)}\to 0\end{aligned}$$

are non split.

ii) *If* $(l_1,\dots,l_n)=(p-1,\dots,p-1)$ *the exact sequence*

$$0\to\langle K\cdot F_0^{(0)}*F_{p-1}^{(1)}*\cdots*F_{p-1}^{(n)}\rangle\to Q^{(0,n+1)}_{0,p-1,\dots,p-1}\to Q^{(0,n+1)}_{1,p-1,\dots,p-1}\to 0$$

is nonsplit iff $r-2t\equiv 0[p-1]$.

iii) *The dimension of the quotients* $Q^{(0,n+1)}_{i,l_1,\dots,l_n}$ *for* $i\in\{0,1\}$ *is:*

$$\dim_{\overline{\mathbf{F}}_p}(Q^{(0,n+1)}_{0,l_1,\dots,l_n})=(p+1)p^n-(p+1)(\sum_{j=1}^n p^{j-1}l_j)$$

$$\dim_{\overline{\mathbf{F}}_p}(Q^{(0,n+1)}_{1,l_1,\dots,l_n})=(p+1)p^n-(p+1)(\sum_{j=1}^n p^{j-1}l_j)-(\lfloor r-2(t+t')\rfloor+1).$$

Proof: *i)* and *ii)*. As the action of K_1 on $\langle K\cdot F_i^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle$ is trivial (for $i\in\{0,1\}$), we deduce as in Proposition 5.10-*ii)* that

$$\phi(F_i^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)})=0$$

for any K-equivariant morphism $Q^{(0,n+1)}_{i,l_1,\dots,l_n}\to\langle K\cdot F_i^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle$ and for any $(n+1)$-tuple $(i,l_1,\dots,l_n)\in\{0,1\}\times\{0,\dots,p-1\}^n$ such that $(l_1,\dots,l_n)\prec(p-1,\dots,p-1)$. The assertion *ii)* is then immediate from Proposition 2.4.

The proof on *iii)* is finally an obvious induction. ♯

6.2 Study of the socle filtration

The present section is devoted to the proof of the following result:

Proposition 6.6. *Assume* p *is odd; let* $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ *be an* n*-tuple, and let* $t'\overset{\text{def}}{=}\sum_{i=1}^n l_i$. *Then*

i) *the socle of* $Q^{(0,n+1)}_{1,l_1,\dots,l_n}$ *is described by*

$$\mathrm{soc}(Q^{(0,n+1)}_{1,l_1,\dots,l_n})=\langle KF_1^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle$$

ii) *the socle of* $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$ *is described by*

$$\mathrm{soc}(Q^{(0,n+1)}_{0,l_1,\dots,l_n})=\begin{cases}\langle K\cdot F_0^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle \text{ if } r-2(t+t')\not\equiv 0[p-1];\\[2ex] \langle K\cdot F_0^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle\oplus\langle K\cdot F_1^{(0)}*F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}\rangle\\ \text{if } r-2(t+t')\equiv 0[p-1].\end{cases}$$

The proof is a descending induction on the n-tuple $(l_1,\dots,l_n)$, the statement being clear if $(l_1,\dots,l_n)=(p-1,\dots,p-1)$.

We prove the result for a fixed n-tuple $(l_1,\dots,l_n)$, assuming it true for $Q^{(0,n+1)}_{0,l_1+1,\dots,l_n}$ (resp. for $Q^{(0,n+1)}_{1,l_1,\dots,l_n}$).

Study of $\mathrm{soc}(Q^{(0,n+1)}_{1,l_1+1,\dots,l_n})$**.** We dispose of the following commutative diagram with exact lines (cf. Lemma 6.3):

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}^K_{K_0(p)}\langle F^{(1)}_{l_1} * \dots * F^{(n)}_{l_n}\rangle \arrow[r] \arrow[d, "pr_1"] & Q^{(0,n+1)}_{0,l_1,\dots,l_n} \arrow[r] \arrow[d, "pr_2"] & Q^{(0,n+1)}_{0,l_1+1,\dots,l_n} \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & \langle K\cdot F^{(0)}_1 * F^{(1)}_{l_1} * \dots * F^{(n)}_{l_n}\rangle \arrow[r] & Q^{(0,n+1)}_{1,l_1,\dots,l_n} \arrow[r] & Q^{(0,n+1)}_{0,l_1+1,\dots,l_n} \arrow[r] & 0.
\end{tikzcd}$$

We define the elements of $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$:

$$\begin{aligned}
x &\stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, F^{(1)}_{l_1+1} * \dots * F^{(n)}_{l_n}]\\
x' &\stackrel{\mathrm{def}}{=} [1_K, F^{(1)}_{l_1+1} * \dots * F^{(n)}_{l_n}]\\
y &\stackrel{\mathrm{def}}{=} x + (-1)^{t+t'+1}x';
\end{aligned}$$

the behaviour of the elements x, x' in $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$ is the object of the next

Lemma 6.7. *We have the following equalities in $Q^{(0,n+1)}_{0,l_1,\dots,l_n}$ for p odd*[5]*:*

i) if $a, d\in\mathbf{F}_p^\times$ then

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix} x = a^{r-(t+t'+1)}d^{t+t'+1}x;$$
$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix} x' = a^{t+t'+1}d^{r-(t+t'+1)}x'.$$

ii) Let $j\in\{1,\dots,n\}$ be minimal with respect to the property that $l_j\leqslant p-2$ and let $\lambda\in\mathbf{F}_p$. Then

$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix} x = x + (l_j+1)(-1)^j\sum_{\mu_0\in\mathbf{F}_p} - P_{-\lambda}(\mu_0)\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, F^{(1)}_{l_1} * \dots * F^{(n)}_{l_n}];$$
$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix} x' = x' + (l_j+1)(-1)^j\delta_{p,3}(1-\delta_{1,j})\lambda[1_K, F^{(1)}_{l_1} * \dots * F^{(n)}_{l_n}].$$

Proof: $i)$ Follows easily from the definition of the elements x, x' and the equalities

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}\begin{bmatrix}z & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}z[ad^{-1}] & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}[d] & 0\\ 0 & [a]\end{bmatrix}$$
$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}\begin{bmatrix}1 & 0\\ z & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ z[a^{-1}d] & 1\end{bmatrix}\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}$$

for $z\in\mathbf{Z}_p$, $a, d\in\mathbf{F}_p^\times$

$ii)$ The first equality is immediately deduced from Lemma 5.12 and the relation:

$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}[\lambda+\mu_0] & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ p[P_\lambda(\mu_0)]+p^2h & 1\end{bmatrix}$$

for $\lambda, \mu_0\in\mathbf{F}_p$ and $h\in\mathbf{Z}_p$ a suitable p-adic integer.

The second equality is more delicate. From Lemma 2.9 we deduce

$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ p[\mu_1]+\dots+p^n[\mu_n] & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ p[\mu'_1]+\dots+p^n[\mu'_n] & 1\end{bmatrix}\Lambda$$

[5] this is required only for the equality concerning x' in $ii)$

where $\Lambda \in K_0(p^{n+1})$ is upper unipotent modulo p and, for $i \geqslant 3$ we have

$$\mu_i = \mu_i' + \mu_{i-1}'\mu_1\lambda + \cdots + \mu_1'\mu_{i-1}\lambda + S_{i-2}(\mu_{i-1})$$

where $S_{i-2} \in \mathbf{F}_p[X]$ is a polynomial of degree $p-1$ and leading coefficient $-s_{i-2} \stackrel{\text{def}}{=} \mu_{i-1}' - \mu_{i-1}$, while, for $i \in \{1,2\}$ we have

$$\mu_2 = \mu_2' + \mu_1\mu_1'\lambda, \qquad\qquad \mu_1 = \mu_1'.$$

If $j \in \{1,\dots,n\}$ is as in the statement we can write

$$F_{l_1+1}^{(1)} * \cdots * F_{l_n}^{(n)} = F_0^{(1)} * \cdots * F_0^{(j-1)} * F_{l_j+1}^{(j)} * \cdots * F_{l_n}^{(n)}.$$

(with the obvious convention if $j=1$) and a direct computation in $\mathrm{Ind}_{K_0(p^n)}^{K_0(p)}\chi_r^s\mathfrak{a}^t$ gives:

$$\begin{aligned} v &\stackrel{\text{def}}{=} \begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} F_{l_1+1}^{(1)} * \cdots * F_{l_n}^{(n)} \\ &= \sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \cdots \sum_{\mu_{j-1}\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{j-1}[\mu_{j-1}'] & 1 \end{bmatrix} \sum_{\mu_j\in\mathbf{F}_p} \mu_j^{l_j+1} \begin{bmatrix} 1 & 0 \\ p^j[\mu_j'] & 1 \end{bmatrix} \cdots \\ &\qquad \cdots \sum_{\mu_n\in\mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n'] & 1 \end{bmatrix} [1,e]. \end{aligned}$$

If $j < n$ we can now use the recursive property of the s_{i-1}'s for $i = j,\dots,n-1$ and project v successively via the epimorphisms

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K_0(p)}\chi_r^s\mathfrak{a}^t \twoheadrightarrow \mathrm{Ind}_{K_0(p^n)}^{K_0(p)}Q_{l_n}^{(n,n+1)} \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{Ind}_{K_0(p^{j+1})}^{K_0(p)}Q_{l_{j+1},\dots,l_n}^{(j+1,n+1)}.$$

We see that v is sent to the following element $\widetilde{v}$ of $\mathrm{Ind}_{K_0(p^{j+1})}^{K_0(p)}Q_{l_{j+1},\dots,l_n}^{(j+1,n+1)}$ (with the convention that if $j=n$, we just have $v=\widetilde{v}$ and $Q_{l_{j+1},\dots,l_n}^{(j+1,n+1)} \stackrel{\text{def}}{=} \chi_r^s\mathfrak{a}^t$):

$$\begin{aligned} \widetilde{v} = &\sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \cdots \\ &\cdots \sum_{\mu_{j-1}\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{j-1}[\mu_{j-1}'] & 1 \end{bmatrix} \sum_{\mu_j\in\mathbf{F}_p} (\mu_j' + s_{j-1})^{l_j+1} \begin{bmatrix} 1 & 0 \\ p^j[\mu_j'] & 1 \end{bmatrix} \sum_{\mu_{j+1}\in\mathbf{F}_p} \mu_{j+1}^{l_{j+1}} \begin{bmatrix} 1 & 0 \\ p^{j+1}[\mu_{j+1}] & 1 \end{bmatrix} \cdots \\ &\cdots \sum_{\mu_n\in\mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1,e]. \end{aligned}$$

This lets us deduce the statement if $j=1$, while, if $j \geqslant 2$ we map $\widetilde{v}$ in $\mathrm{Ind}_{K_0(p^j)}^{K_0(p)}Q_{l_j,\dots,l_n}^{(j,n+1)}$ via the epimorphism $\mathrm{Ind}_{K_0(p^{j+1})}^{K_0(p)}Q_{l_{j+1},\dots,l_n}^{(j+1,n+1)} \twoheadrightarrow \mathrm{Ind}_{K_0(p^j)}^{K_0(p)}Q_{l_j,\dots,l_n}^{(j,n+1)}$ to get:

$$\begin{aligned} &F_{l_1+1}^{(1)} * \cdots * F_{l_n}^{(n)} + (l_j+1) \sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \cdots \\ &\quad \cdots \sum_{\mu_{j-1}\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{j-1}[\mu_{j-1}'] & 1 \end{bmatrix} s_{j-1} \sum_{\mu_j\in\mathbf{F}_p} \mu_j^{l_j} \begin{bmatrix} 1 & 0 \\ p^j[\mu_j] & 1 \end{bmatrix} \cdots \\ &\quad\quad \cdots \sum_{\mu_n\in\mathbf{F}_p} \mu_n^{l_n} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1,e]. \end{aligned}$$

We use again the recursive property of the s_{i-1}'s for $i = 2,\dots,j$ and the chain of epimorphisms

$$\mathrm{Ind}_{K_0(p^j)}^{K_0(p)}Q_{l_j,\dots,l_n}^{(j,n+1)} \twoheadrightarrow \mathrm{Ind}_{K_0(p^{j-1})}^{K_0(p)}Q_{p-1,\dots,l_n}^{(j-1,n+1)} \twoheadrightarrow \cdots \twoheadrightarrow Q_{l_1,\dots,l_n}^{(1,n+1)}$$

to see that the image of v in $Q^{(1,n+1)}_{l_1,\dots,l_n}$ is
$$F^{(1)}_{l_1+1}*\dots*F^{(n)}_{l_n}+(l_j+1)(-1)^j\lambda\delta_{p,3}F^{(1)}_{l_1}*\dots*F^{(n)}_{l_n}.$$
This let us conclude the proof. ♯

We can now prove the main result of this paragraph (i.e. the proof of i) of Proposition 6.6)

LEMMA 6.8. *Assume p is odd. Let $(l_1,\dots,l_n)\in\{0,\dots,p-1\}^n$ be an n-tuple and assume that the statement of Proposition 6.6-ii) holds true for the n-tuple $(l_1+1,\dots,l_n)$.*

Then
$$\mathrm{soc}(Q^{(0,n+1)}_{1,l_1,\dots,l_n})=\langle K\cdot F^{(0)}_1*F^{(1)}_{l_1}*\dots*F^{(n)}_{l_n}\rangle.$$

Proof: Assume false. Let τ be an irreducible K-subrepresentation of $Q^{(0,n+1)}_{1,l_1,\dots,l_n}$ such that $\tau\cap\langle K\cdot F^{(0)}_1*F^{(1)}_{l_1}*\dots*F^{(n)}_{l_n}\rangle=0$. Therefore the natural projection $Q^{(0,n+1)}_{1,l_1,\dots,l_n}\twoheadrightarrow Q^{(0,n+1)}_{0,l_1+1,\dots,l_n}$ induces an isomorphism of τ onto an irreducible summand of $\mathrm{soc}(Q^{(0,n+1)}_{0,l_1+1,\dots,l_n})$. Assuming that Proposition 6.6-ii) holds true for the n-tuple $(l_1+1,\dots,l_n)$ we can distinguish the situations:

A) the subrepresentation τ maps isomorphically into the K-subrepresentation of $Q^{(0,n+1)}_{0,l_1+1,\dots,l_n}$ generated by (the image of) x.

B) We have $r-2(t+t'+1)\equiv 0\,[p-1]$ and the subrepresentation τ maps isomorphically into the K-subrepresentation of $Q^{(0,n+1)}_{0,l_1+1,\dots,l_n}$ generated by (the image of) y.

Study of case A. Let $f\in\mathrm{Ind}^K_{K_0(p)}F^{(1)}_{l_1}*\dots*F^{(n)}_{l_n}$ be such that $pr_2(x+f)\in\tau$. The induced isomorphism $\tau\xrightarrow{\sim}\langle K\cdot x\rangle$ and the behaviour of x in $\mathrm{soc}(Q^{(0,n+1)}_{0,l_1+1,\dots,l_n})$ let us deduce the necessary conditions:

1) for all $a,d\in\mathbf{F}_p^\times$,
$$\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}(x+f)-a^{r-(t+t'+1)}d^{t+t'+1}(x+f)\in\ker(pr_2);$$

2) for all $\lambda\in\mathbf{F}_p$
$$\begin{bmatrix}1&[\lambda]\\0&1\end{bmatrix}(f+x)-(f+x)\in\ker(pr_2).$$

Condition 1) and Lemma 6.7-i) give $\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}f-a^{r-(t+t'+1)}d^{t+t'+1}f\in\ker pr_1$ so that, by Lemma 2.6, we deduce
$$\begin{bmatrix}1&[\lambda]\\0&1\end{bmatrix}pr_1(f)-pr_1(f)=\begin{cases}0\,\text{if}\,r-2(t+t')\not\equiv 0\,[p-1]\\[2ex] c_1\lambda\displaystyle\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1,F^{(1)}_{l_1}*\dots*F^{(n)}_{l_n}]\\ \text{if}\,r-2(t+t')\equiv 0\,[p-1]\end{cases}$$
for some $c_1\in\overline{\mathbf{F}}_p$. Thus, condition 2) and Lemma 6.7-ii) let us conclude that
$$(l_j+1)(-1)^j\sum_{i=1}^{p-1}\frac{\binom{p}{i}}{p}(-\lambda)^{p-i}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^i\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1,F^{(1)}_{l_1}*\dots*F^{(n)}_{l_n}]+$$
$$+c_1\delta_{0,r-2(t+t')}\lambda\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1,F^{(1)}_{l_1}*\dots*F^{(n)}_{l_n}]\in\ker pr_1$$

for any $\lambda \in \mathbf{F}_p$, and by Lemma 2.10-ii) we can deduce in particular

$$\sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{p-1} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \in \ker pr_1 \quad \text{for } r - 2(t+t') \not\equiv 0,$$

$$\sum_{\mu_0 \in \mathbf{F}_p} \mu_0 \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \in \ker pr_1 \quad \text{for } r - 2(t+t') \equiv 0.$$

Thanks to Remark 2.5 we see that both conditions are absurd, for the case $r - 2(t+t') \not\equiv 0\,[p-1]$ and $r - 2(t+t') \equiv 0\,[p-1]$ respectively. $\sharp$

Study of case B***.*** Let $f \in \operatorname{Ind}_{K_0(p)}^{K} F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}$ be such that $pr_2(y+f) \in \tau$. The induced isomorphism $\tau \xrightarrow{\sim} \langle K \cdot y\rangle$ and the behaviour of y in $\operatorname{soc}(Q_{0,l_1+1,\dots,l_n}^{(0,n+1)})$ let us deduce the necessary conditions:

1) for all $a, d \in \mathbf{F}_p^{\times}$,

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (y+f) - (ad)^{t+t'+1}(y+f) \in \ker(pr_2);$$

2) for all $\lambda \in \mathbf{F}_p$

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (f+y) - (f+y) \in \ker(pr_2).$$

We deduce from condition 1) and Lemma 6.7-i) that $pr_1(f)$ is an H-eigenvector for $\langle K \cdot F_1^0 * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle$ with associated eigencharacter $a^{r-(t+t'+1)}d^{t+t'+1}$. Thus, by Lemma 2.10, we have

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} pr_1(f) = \begin{cases} 0 \text{ if } r - 2(t+t') \not\equiv 0\,[p-1] \text{ i.e. } p \neq 3 \\[2ex] c_1 \lambda \displaystyle\sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}] \\ \text{if } r - 2(t+t') \equiv 0\,[p-1] \text{ i.e. } p = 3. \end{cases}$$

for some $c_1 \in \overline{\mathbf{F}}_p$. The conclusion follows again from Lemma 6.7-ii), similarly to case A). $\sharp$

The proof of Lemma 6.8 is therefore complete. $\sharp$

Study of $\operatorname{soc}(Q_{0,l_1,\dots,l_n}^{(0,n+1)})$***.*** We have the following commutative diagram with exact lines (cf. Lemma 6.3):

$$\begin{array}{ccccccccc}
0 \to & \langle K \cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle & \to & \operatorname{Ind}_{K_0(p)}^{K}\langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle & \to & \langle K \cdot F_1^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle & \to 0 \\
& \| & & \downarrow & & \downarrow & \\
0 \to & \langle K \cdot F_0^{(0)} * F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle & \longrightarrow & Q_{0,l_1,\dots,l_n}^{(0,n+1)} & \longrightarrow & Q_{1,l_1,\dots,l_n}^{(0,n+1)} & \longrightarrow 0.
\end{array}$$

Lemma 6.9. *Assume p is odd. Let $(l_1,\dots,l_n) \in \{0,\dots,p-1\}^n$ be an n-tuple and assume that the statement of Proposition 6.6-i) holds true for the representation $Q_{1,l_1,\dots,l_n}^{(0,n+1)}$.*

Then

$$\operatorname{soc}(Q_{0,l_1,\dots,l_n}^{(0,n+1)}) = \operatorname{soc}(\operatorname{Ind}_{K_0(p)}^{K}\langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle).$$

Proof: Assume false. Let τ be an irreducible K-subrepresentation of $Q_{0,l_1,\dots,l_n}^{(0,n+1)}$ and assume

$$\tau \cap \operatorname{Ind}_{K_0(p)}^{K}\langle F_{l_1}^{(1)} * \cdots * F_{l_n}^{(n)}\rangle = 0.$$

In particular, the natural projection $Q^{(0,n+1)}_{0,l_1,\dots,l_n} \twoheadrightarrow Q^{(0,n+1)}_{1,l_1,\dots,l_n}$ induces an isomorphism $\tau \xrightarrow{\sim} \mathrm{soc}(Q^{(0,n+1)}_{1,l_1,\dots,l_n})$. Assuming Proposition 6.6-*i*) for the representation $Q^{(0,n+1)}_{1,l_1,\dots,l_n}$, we deduce that it exists $f \in \langle K \cdot F^{(0)}_0 * F^{(1)}_{l_1} * \dots * F^{(n)}_{l_n}\rangle$ such that

$$f + F^{(0)}_1 * F^{(1)}_{l_1} * \dots * F^{(n)}_{l_n} \in \tau$$

is a K-generator of τ, contradiction. $\sharp$

***End of the proof of Proposition** 6.6.* The statement of Proposition 6.6 is trivially true for the n-tuple $(l_1, \dots, l_n) = (p-1, \dots, p-1)$, since

$$Q^{(0,n+1)}_{0,p-1,\dots,p-1} \cong \mathrm{Ind}^K_{K_0(p)}\langle F^{(1)}_{p-1} * \dots * F^{(n)}_{p-1}\rangle \cong \mathrm{Ind}^K_{K_0(p)}\chi^s_r\mathfrak{a}^t.$$

The general case follows then from a descending induction, using Lemmas 6.8 and 6.9. $\sharp$

A weaker result. We can state a similar, although weaker, result concerning the structure of $\mathrm{Ind}^K_{K_0(p^{n+1})}\chi^s_r\mathfrak{a}^t$. Indeed, by exactness of the functor $\mathrm{Ind}^K_{K_0(p)}$ and Proposition 5.10 we have a natural equivariant filtration on $\mathrm{Ind}^K_{K_0(p^{n+1})}\chi^s_r\mathfrak{a}^t$, whose graded pieces are isomorphic to finite inductions of characters, depending explicitely on $\chi^s_r\mathfrak{a}^t$ and on the graded piece. The fact that the extensions between the graded pieces are non split can be deduced with the same techniques used for Proposition 6.6 and we get

PROPOSITION 6.10. *Let $r \in \{0, \dots, p-1\}$, $t \in \{0, \dots, p-2\}$ and $n \in \mathbf{N}$. The representation $\mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Z}_p)}_{K_0(p^{n+1})}\chi^s_r\mathfrak{a}^t$ has a natural equivariant filtration whose graded pieces are described by*

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Z}_p)}_{K_0(p)}\chi^s_r\mathfrak{a}^t \text{—} \mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Z}_p)}_{K_0(p)}\chi^s_r\mathfrak{a}^{t+1} \text{—} \mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Z}_p)}_{K_0(p)}\chi^s_r\mathfrak{a}^{t+2} \text{—} \dots \text{—} \mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Z}_p)}_{K_0(p)}\chi^s_r\mathfrak{a}^{t-1} \text{—} \mathrm{Ind}^{\mathrm{GL}_2(\mathbf{Z}_p)}_{K_0(p)}\chi^s_r\mathfrak{a}^t$$

the extensions being non-split. Moreover, the number of finite parabolic inductions is p^n.

Proof. Left to the reader. $\square$

7. Socle filtration for the spaces R_n

In this section we will use the results of §6 to give an exhaustive description of the socle filtration for the R_n's, for any $n \in \mathbf{N}$. The precise statement is the following:

PROPOSITION 7.1. *Assume p odd; let $1 \leqslant r \leqslant p-1$, $n \in \mathbf{N}_>$ and $1 \leqslant t \leqslant r$ be integers. Then*

$$\mathrm{soc}(\mathrm{Fil}^{t-1}(R_{n+1})) = \mathrm{soc}(\mathrm{Fil}^t(R_{n+1})).$$

More generally, we have

$$\mathrm{soc}(\mathrm{Fil}^{t-1}(R_{n+1})/Q) = \mathrm{soc}(\mathrm{Fil}^t(R_{n+1})/Q)$$

for any subrepresentation Q of $\mathrm{Fil}^j(R_{n+1})$, $0 \leqslant j \leqslant t-1$ coming from the socle filtration of $\mathrm{Fil}^j(R_{n+1})$.

The rest of the paragraph is devoted to its proof, which is very similar to the proof of Proposition 6.6. For a notational convenience, we will prove the result concerning the representations $\mathrm{Fil}^{t-1}(R_{n+1})$, $\mathrm{Fil}^t(R_{n+1})$. In order to obtain the general result we just have repeat the same arguments replacing $\mathrm{Fil}^{t-1}(R_{n+1})$ and $\mathrm{Fil}^t(R_{n+1})$ by $\mathrm{Fil}^{t-1}(R_{n+1})/Q$ and $\mathrm{Fil}^t(R_{n+1})/Q$ respectively (and other similar formal adjustments which will be clear to the reader).

We fix integers $0 \leqslant r \leqslant p-1$, $n \in \mathbf{N}$, $1 \leqslant t \leqslant r$, and we define the elements of $\mathrm{Fil}^t(R_{n+1})$:

$$x \overset{\mathrm{def}}{=} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots \sum_{\mu_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1_K, X^{r-t}Y^t] \in \mathrm{Fil}^t(R_{n+1});$$

$$x' \overset{\mathrm{def}}{=} \sum_{\mu_1 \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_n \in \mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} [1_K, X^{r-t}Y^t] \in \mathrm{Fil}^t(R_{n+1});$$

$$y \overset{\mathrm{def}}{=} x + (-1)^t x'.$$

Moreover, we consider the map

$$pr : \mathrm{Fil}^{t-1}(R_{n+1}) \twoheadrightarrow \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^{t-1} \twoheadrightarrow Q_{0,p-1,\dots,p-1}^{(0,n+1)} \overset{\sim}{\to} \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t-1}$$

where the first arrow is the natural projection given by the reduction modulo $\mathrm{Fil}^{t-2}(R_{n+1})$ and the second arrow is more precisely described by the commutative diagram (cf. also Lemma 5.11)

$$\begin{tikzcd}
\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \mathfrak{a}^{t-1} \arrow[rr] \arrow[d, two heads] & & Q_{0,p-1,\dots,p-1}^{(0,n+1)} \\
Q_{0,\dots,0,p-1}^{(0,n+1)} \cong \mathrm{Ind}_{K_0(p^n)}^K \chi_r^s \mathfrak{a}^{t-1} \arrow[d, two heads] & & \\
\vdots \arrow[d, two heads] & & \\
Q_{0,p-1}^{(0,2)} \cong \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t-1}. \arrow[uuurr, two heads] & &
\end{tikzcd}$$

We finally set

$$pr_{\mathrm{tot}} : \mathrm{Fil}^{t-1}(R_{n+1}) \overset{pr}{\twoheadrightarrow} \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{t-1} \overset{\pi}{\twoheadrightarrow} \mathrm{Sym}^{p-1-\lfloor r-2(t-1) \rfloor} \overline{\mathbf{F}}_p^2 \otimes \det{}^{r-(t-1)}$$

where π is the natural projection defined in Lemma 2.6. We start from the following computational lemma.

Lemma 7.2. *We have the following equalities in* $\mathrm{Fil}^t(R_{n+1})$ *for* p *odd* [6]*:*

i) *For all* $a, d \in \mathbf{F}_p^\times$,

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} x = a^{r-t} d^t x$$

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} x' = a^t d^{r-t} x'.$$

ii) *For all* $\lambda \in \mathbf{F}_p$ *then* $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x - x$ *and* $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x' - x'$ *are in* $\mathrm{Fil}^{t-1}(R_{n+1})$ *and*

$$pr(\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x - x) = t(-1)^n \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} (-P_{-\lambda,}(\mu_0))[1_K, X^{r-(t-1)}Y^{t-1}]$$

$$pr(\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x' - x') = t(-1)^n \lambda \delta_{p,3} [1_K, X^{r-(t-1)}Y^{t-1}]$$

(where $P_{-\lambda}(\mu_0)$ *has been defined in* §*2.3)*

[6] the requirement p odd is used for the equality concerning x' in *ii)*

Proof: *i*) It is analogous to the proof of Proposition 6.6-*i*).

ii) From Lemma 2.8 we deduce

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x =$$

$$= \sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\lambda+\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_n\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1})] & 1 \end{bmatrix} [1, X^{r-t}(P_{\lambda,\dots,\mu_{n-1}}(\mu_n)X+Y)^t] =$$

$$= x + t\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\lambda+\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_n\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1})] & 1 \end{bmatrix} P_{\lambda,\dots,\mu_{n-1}}(\mu_n)[1, X^{r-(t-1)}Y^{t-1}] + q$$

for a suitable $q \in \mathrm{Fil}^{t-2}(R_{n+1})$ and where the elements $P_{\lambda,\dots,\mu_{j-1}}(\mu_j)$ for $j \in \{1,\dots,n\}$ (resp. $P_\lambda(\mu_0)$) are defined in Lemma 2.8. We are now left to map the element $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x - x \in \mathrm{Fil}^{t-1}(R_{n+1})$ in $\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^{t-1}$ to get

$$t\sum_{\mu_0\in\mathbf{F}_p} \begin{bmatrix} [\lambda+\mu_0] & 1 \\ 1 & 0 \end{bmatrix} \dots$$

$$\dots \sum_{\mu_n\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n + P_{\lambda,\dots,\mu_{n-2}}] & 1 \end{bmatrix} (\mu_{n-1})P_{\lambda,\dots,\mu_{n-1}}(\mu_n)[1, X^{r-(t-1)}Y^{t-1}]$$

and the result follows using the chain of epimorphisms

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^{t-1} \twoheadrightarrow Q_{0,\dots,0,p-1}^{(0,n+1)} \twoheadrightarrow \dots \twoheadrightarrow Q_{0,p-1,\dots,p-1}^{(0,n+1)}$$

and the recursive property of the polynomials $P_{\lambda,\dots,\mu_{j-1}}(X) \in \mathbf{F}_p[X]$ for $j \in \{1,\dots,n\}$.

Similarly, from Lemma 2.9 we deduce the following equality in $\mathrm{Fil}^t(R_{n+1})$:

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x' =$$

$$= x' + t\sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1] & 1 \end{bmatrix} \dots \sum_{\mu_n\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n] & 1 \end{bmatrix} (-s_{\lambda,\dots,\mu_n})[1, X^{r-(t-1)}Y^{t-1}] + q'$$

for some $q' \in \mathrm{Fil}^{t-2}(R_{n+1})$. We map the element $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x' - x' \in \mathrm{Fil}^{t-1}(R_{n+1})$ in $\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^{t-1}$ to get

$$t\sum_{\mu_1\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p[\mu_1'] & 1 \end{bmatrix} \dots \sum_{\mu_n\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^n[\mu_n'] & 1 \end{bmatrix} (-s_{\lambda,\dots,\mu_n})[1, X^{r-(t-1)}Y^{t-1}]$$

and the result follows using the chain of epimorphisms

$$\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^{t-1} \twoheadrightarrow Q_{0,\dots,0,p-1}^{(0,n+1)} \twoheadrightarrow \dots \twoheadrightarrow Q_{0,p-1,\dots,p-1}^{(0,n+1)}$$

and the recursive property of the s_i for $i \in \{1,\dots,n\}$ (here we need $p \geqslant 3$). $\sharp$

End of the proof of Proposition *7.1.* Let now τ be an irreducible K-subrepresentation of $\mathrm{Fil}^t(R_{n+1})$, and assume $\tau \cap \mathrm{Fil}^{t-1}(R_{n+1}) = 0$. Therefore the natural projection $\mathrm{Fil}^t(R_{n+1}) \twoheadrightarrow$

$\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t$ induces an isomorphism of τ onto an irreducible factor of $\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t)$. As

$$\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t)=\mathrm{soc}(Q_{0,\dots,0}^{(0,n+1)})=\mathrm{soc}(\mathrm{Ind}_{K_0(p)}^{K}\langle x'\rangle)$$

by Proposition 6.6, we distinguish two situations:

A) the subrepresentation τ maps isomorphically into the K-subrepresentation of $\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t$ generated by (the image of) x.

B) We have $r-2t\equiv 0\,[p-1]$ and the subrepresentation τ maps isomorphically into the K-subrepresentation of $\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t$ generated by (the image of) y.

***Study of case** A.* Let $f\in\mathrm{Fil}^{t-1}(R_{n+1})$ be such that $x+f\in\tau$. From the induced isomorphism $\tau\overset{\sim}{\to}\langle K\cdot x\rangle$ and the behaviour of x in $\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t)$ we deduce the following necessary conditions:

1) for all $a,d\in\mathbf{F}_p^{\times}$ we have

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}(x+f)-a^{r-t}d^t(x+f)=0$$

inside $\mathrm{Fil}^t(R_{n+1})$;

2) for all $\lambda\in\mathbf{F}_p$ we have

$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}(x+f)-(x+f)=0$$

inside $\mathrm{Fil}^t(R_{n+1})$.

Condition 1) and Lemma 7.2-*i*) imply in particular that $pr_{\mathrm{tot}}(f)$ is an H-eigenvector of

$$\mathrm{Sym}^{p-1-\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{r-(t-1)}\cong\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{t-1}/\mathrm{Sym}^{\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{t-1}$$

of associated eigencharacter $a^{r-t}d^t$. It follows then from Lemma 2.6 that

$$\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}pr_{\mathrm{tot}}(f)-pr_{\mathrm{tot}}(f)=\begin{cases}0\text{ if }r-2(t-1)\not\equiv 0\,[p-1]\\[2ex] c_1\displaystyle\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-(t-1)}Y^{t-1}]\\ \text{if }r-2(t-1)\equiv 0\,[p-1]\end{cases}$$

for a suitable $c_1\in\overline{\mathbf{F}}_p$. We conclude from condition 2) and Lemma 7.2-*ii*)

$$t(-1)^n\sum_{j=1}^{p-1}\frac{\binom{p}{j}}{p}(-\lambda)^{p-j}\sum_{\mu_0\in\mathbf{F}_p}\mu_0^j\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-(t-1)}Y^{t-1}]+$$
$$+\delta_{0,r-2(t-1)}c_1\lambda\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-(t-1)}Y^{t-1}]=0$$

inside $\mathrm{Sym}^{p-1-\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det^{r-(t-1)}$, and this is clearly impossible: by Lemma 2.10-*ii*) we would get in particular

$$\sum_{\mu_0\in\mathbf{F}_p}\mu_0^{p-1}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-(t-1)}Y^{t-1}]=0\quad\text{for }r-2(t-1)\not\equiv 0$$

$$\sum_{\mu_0\in\mathbf{F}_p}\mu_0\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,X^{r-(t-1)}Y^{t-1}]=0\quad\text{for }r-2(t-1)\equiv 0$$

which gives an absurd for $r-2(t-1)\not\equiv 0\,[p-1]$ and $r-2(t-1)\equiv 0\,[p-1]$ respectively (cf. Remark 2.5). $\sharp$

Study of case B. Let $f\in \mathrm{Fil}^{t-1}(R_{n+1})$ be such that $y+f\in\tau$. From the induced isomorphism $\tau\xrightarrow{\sim}\langle Ky\rangle$ and the behaviour of y in $\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^{K}\chi_r^s\mathfrak{a}^t)$ we deduce the following necessary conditions:

1) for all $a,d\in\mathbf{F}_p^{\times}$ we have
$$\begin{bmatrix}[a]&0\\0&[d]\end{bmatrix}(y+f)-a^{r-t}d^t(y+f)=0$$
inside $\mathrm{Fil}^t(R_{n+1})$;
2) for all $\lambda\in\mathbf{F}_p$ we have
$$\begin{bmatrix}1&[\lambda]\\0&1\end{bmatrix}(y+f)-(y+f)=0$$
inside $\mathrm{Fil}^t(R_{n+1})$.

We deduce from condition 1) and Lemma 7.2 that $pr_{\mathrm{tot}}(f)$ is an H-eigenvector of
$$\mathrm{Sym}^{p-1-\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{r-(t-1)}\cong\mathrm{Ind}_{K_0(p)}^{K}\chi_r^s\mathfrak{a}^{t-1}/\mathrm{Sym}^{\lfloor r-2(t-1)\rfloor}\overline{\mathbf{F}}_p^2\otimes\det{}^{t-1}$$
of associated eigencharacter $a^{r-t}d^t$ and therefore, by Lemma 2.6
$$\begin{bmatrix}1&[\lambda]\\0&1\end{bmatrix}pr_{\mathrm{tot}}(f)-pr_{\mathrm{tot}}(f)=\begin{cases}0\text{ if }r-2(t-1)\not\equiv 0\,[p-1]\ (\text{i.e. }p\neq 3)\\[2ex] c_1\displaystyle\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1,X^{r-(t-1)}Y^{t-1}]\\ \text{if }r-2(t-1)\equiv 0\,[p-1]\ (\text{i.e. }p=3)\end{cases}$$
for a suitable $c_1\in\overline{\mathbf{F}}_p$. The conclusion follows from Lemma 7.2, similarly to the previous case. $\sharp$

The proof of Proposition 7.1 is therefore complete. $\sharp$

8. Socle filtration for the spaces $R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}$

We are finally ready to describe the socle filtration for the K-representations
$$\varinjlim_{n\,\text{even}}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}),\qquad \varinjlim_{m\,\text{odd}}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_m}R_{m+1}).$$
The main statement is the following:

Proposition 8.1. *Assume p is odd; let $n\in\mathbf{N}_>$ (resp. $m\in\mathbf{N}_>$) be an odd (resp. even) integer, $0\leqslant r\leqslant p-2$. Then :*

i)
$$\begin{aligned}\mathrm{soc}(R_0\oplus_{R_1}\cdots\oplus_{R_{n-2}}R_{n-1})&=\mathrm{soc}(R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1})\\ (\text{resp.}\quad \mathrm{soc}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_{m-2}}R_{m-1})&=\mathrm{soc}(R_1/R_0\oplus_{R_2}\cdots\oplus_{R_m}R_{m+1}))\end{aligned}$$
where we formally define $R_0\oplus_{R_{-1}}R_0\overset{\mathrm{def}}{=}R_0$ (resp. $R_1/R_0\oplus_{R_0}R_1\overset{\mathrm{def}}{=}R_1/R_0$).

ii) More generally, if $0\leqslant j\leqslant n-1$ is even (resp. $1\leqslant j'\leqslant m-1$ is odd) and Q is a K-subrepresentation of R_j/R_{j-1} (resp. $R_{j'}/R_{j'-1}$) coming from the socle filtration of R_j/R_{j-1}

(resp. $R_{j'}/R_{j'-1}$), then

$$\mathrm{soc}((R_j/Q)\oplus_{R_{j+1}}\cdots\oplus_{R_{n-2}}R_{n-1})=\mathrm{soc}((R_j/Q)\oplus_{R_{j+1}}\cdots\oplus_{R_n}R_{n+1})$$
$$(\text{resp.}\quad \mathrm{soc}((R_{j'}/Q)\oplus_{R_{j'+1}}\cdots\oplus_{R_{m-2}}R_{m-1})=\mathrm{soc}((R_{j'}/Q)\oplus_{R_{j'+1}}\cdots\oplus_{R_m}R_{m+1}))$$

where we formally define $(R_j/Q)\oplus_{R_{n-2}}R_{n-1}\overset{\mathrm{def}}{=}(R_j/Q)$ if $j=n-1$ (resp. $(R_{j'}/Q)\oplus_{R_{m-2}}R_{m-1}$ if $j'=m-1$).

The rest of the paragraph is devoted to its proof, starting with the following lemmas.

LEMMA 8.2. *Let $n\geqslant 2$ be an integer and $0\leqslant r\leqslant p-1$. The composite map $T_2^-\circ\cdots\circ T_n^-:R_n\twoheadrightarrow R_1$ induces an isomorphism:*

$$\begin{tikzcd}
R_n \arrow[rr, two heads, "T_2^-\circ\cdots\circ T_n^-"] \arrow[d] & & R_1/\mathrm{Fil}^{r-1}(R_1) \\
R_n/\mathrm{Fil}^{r-1}(R_n)\cong \mathrm{Ind}_{K_0(p^n)}^K\chi_r \arrow[d] & & \\
Q_{0,p-1,\dots,p-1}^{(0,n)}\cong \mathrm{Ind}_{K_0(p)}^K\chi_r \arrow[uurr, "\cong"'] & &
\end{tikzcd}$$

Moreover, if $r\neq 0,p-1$ the composite map $T_1^-\circ\cdots\circ T_n^-:R_n\twoheadrightarrow R_0$ induces an isomorphism:

$$\begin{tikzcd}
R_n \arrow[rr, two heads, "T_1^-\circ\cdots\circ T_n^-"] \arrow[d] & & R_0 \\
R_n/\mathrm{Fil}^{r-1}(R_n)\cong \mathrm{Ind}_{K_0(p^n)}^K\chi_r \arrow[d] & & \\
Q_{1,p-1,\dots,p-1}^{(0,n)}\cong \mathrm{Sym}^r\overline{\mathbf{F}}_p^2 \arrow[uurr, "\cong"'] & &
\end{tikzcd}$$

Proof: First of all, notice that for any $m\geqslant 1$ we have a factorisation:

$$\begin{tikzcd}
R_m \arrow[r, two heads, "T_m^-"] \arrow[d] & R_{m-1} \\
R_m/\mathrm{Fil}^{r-1}(R_m) \arrow[ur, dotted, two heads] &
\end{tikzcd}$$

Thus, by the very definition of the operators T_j^-'s and Lemma 2.10-i) we deduce

$$R_n/\mathrm{Fil}^{r-1}(R_n)\twoheadrightarrow R_1/\mathrm{Fil}^{r-1}(R_1)$$
$$[1,F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}]\mapsto(-\delta_{p-1,l_1})\dots(-\delta_{p-1,l_n})\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0]&1\\1&0\end{bmatrix}[1,Y^r]$$

(where we put

$$[1,F_{l_1}^{(1)}*\cdots*F_{l_n}^{(n)}]\overset{\mathrm{def}}{=}\sum_{\mu_1\in\mathbf{F}_p}\mu_1^{l_1}\begin{bmatrix}1&0\\p[\mu_1]&1\end{bmatrix}\dots\sum_{\mu_n\in\mathbf{F}_p}\mu_n^{l_n}\begin{bmatrix}1&0\\p^n[\mu_n]&1\end{bmatrix}[1,Y^r]\,).$$

The previous epimorphism factors then through

$$R_n/\mathrm{Fil}^{r-1}(R_n)\cong\mathrm{Ind}_{K_0(p^n)}^K\chi_r\twoheadrightarrow Q_{0,p-1,\dots,p-1}^{(0,n)}$$

and such a factorisation is indeed an isomorphism as the spaces $Q^{(0,n)}_{0,p-1,\dots,p-1}$ and $R_1/\mathrm{Fil}^{r-1}(R_1)$ have the same dimension.

Moreover, if $r\neq 0,p-1$, we see that

$$T_1^-(\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1,Y^r])=0$$

and therefore the morphism

$$R_n/\mathrm{Fil}^{r-1}(R_n)\twoheadrightarrow R_1/\mathrm{Fil}^{r-1}(R_1)\overset{T_1^-}{\twoheadrightarrow} R_0$$

factors through

$$R_n/\mathrm{Fil}^{r-1}(R_n)\cong \mathrm{Ind}^K_{K_0(p^n)}\chi_r\twoheadrightarrow Q^{(0,n)}_{1,p-1,\dots,p-1};$$

again such a factorisation is an isomorphism by dimensional reasons. ♯

LEMMA 8.3. *Let $n\geqslant 1$ (resp. $n=0$), and $0\leqslant r\leqslant p-2$. Then the natural map $\mathrm{Fil}^0(R_{n+1})\overset{\sim}{\to}\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s$ induces an isomorphism*

$$\mathrm{Fil}^0(R_{n+1})/R_n\overset{\sim}{\to}Q^{(0,n+1)}_{0,\dots,0,r+1}$$
$$(\mathrm{Fil}^0(R_1)/R_0\overset{\sim}{\to}\mathrm{Sym}^{p-1-\lfloor r\rfloor}\overline{\mathbf{F}}_p^2\qquad\text{resp.})$$

Proof: Assume $n\geqslant 1$. For any $(n-1)$-tuple $(l_1,\dots,l_{n-1})\in\{0,\dots,p-1\}^{n-1}$ and any $j\in\{0,\dots,r\}$ we have

$$T_n^+(\sum_{\mu_1\in\mathbf{F}_p}\begin{bmatrix}1&0\\ p[\mu_1]&1\end{bmatrix}\cdots\sum_{\mu_{n-1}\in\mathbf{F}_p}\mu_{n-1}^{l_{n-1}}\begin{bmatrix}1&0\\ p^{n-1}[\mu_{n-1}]&1\end{bmatrix}[1,X^{r-j}Y^j])=$$
$$(-1)^j\sum_{\mu_1\in\mathbf{F}_p}\begin{bmatrix}1&0\\ p[\mu_1]&1\end{bmatrix}\cdots\sum_{\mu_{n-1}\in\mathbf{F}_p}\mu_{n-1}^{l_{n-1}}\begin{bmatrix}1&0\\ p^{n-1}[\mu_{n-1}]&1\end{bmatrix}\sum_{\mu_n\in\mathbf{F}_p}\mu_n^j\begin{bmatrix}1&0\\ p^n[\mu_n]&1\end{bmatrix}[1,X^r].$$

We thus conclude that the natural map

$$\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^r\twoheadrightarrow\mathrm{Fil}^0(R_{n+1})/R_n$$

factors through $\mathrm{Ind}^K_{K_0(p^{n+1})}\chi_r^s\twoheadrightarrow Q^{(0,n+1)}_{0,\dots,0,r+1}$. Such a factorisation is indeed an isomorphism by dimensional reasons. The case $n=0$ is similar and left to the reader. ♯

We are now ready to prove Proposition 8.1 and the strategy will be analogous to the one used in the proof of Proposition 7.1. Once again, we will give a detailed proof for statement i). Statement ii) is obtained exactly in the same way, with formal adjustments which will be clear to the reader (e.g. replace $R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1}$ with $(R_j/Q)\oplus_{R_{j+1}}\cdots\oplus_{R_n}R_{n+1}$, adjustment of the source of the morphism π_{n-1} below according to Q, etc...).

Let us fix integers $n\geqslant 3$, n odd, $0\leqslant r\leqslant p-2$; the case $n=1$ or $m\geqslant 2$, m even will be treated exactly in the same manner and will be left to the reader. We recall the commutative diagram with exact lines (cf. Proposition 4.1):

$$\begin{tikzcd}
0 \arrow[r] & R_n \arrow[r, "T_n^+"] \arrow[d, "-T_n^-"] & R_{n+1} \arrow[r] \arrow[dd, "pr_{n+1}"] & R_{n+1}/R_n \arrow[r] \arrow[dd, equal] & 0\\
 & R_{n-1} \arrow[d, "pr_{n-1}"] & & & \\
0 \arrow[r] & R_0\oplus_{R_1}\cdots\oplus_{R_{n-2}}R_{n-1} \arrow[r] & R_0\oplus_{R_1}\cdots\oplus_{R_n}R_{n+1} \arrow[r] & R_{n+1}/R_n \arrow[r] & 0.
\end{tikzcd}$$

We write π_{n-1} for the natural epimorphism

$$\pi_{n-1}: R_{n-1} \twoheadrightarrow R_{n-1}/\mathrm{Fil}^{r-1}(R_{n-1}) \twoheadrightarrow Q^{(0,n-1)}_{0,p-1,\dots,p-1} \xrightarrow{\sim} R_1/\mathrm{Fil}^{r-1}(R_1)$$

where the last isomorphism is the one described in Lemma 8.2. As we did in §7 we define the following elements in R_{n+1}:

$$\begin{aligned}
x &\stackrel{\mathrm{def}}{=} \sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}\dots\sum_{\mu_{n-1}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{n-1}[\mu_{n-1}] & 1\end{bmatrix}\sum_{\mu_n\in\mathbf{F}_p}\mu_n^{r+1}\begin{bmatrix}1 & 0\\ p^n[\mu_n] & 1\end{bmatrix}[1_K, X^r]\\
x' &\stackrel{\mathrm{def}}{=} \sum_{\mu_1\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p[\mu_1] & 1\end{bmatrix}\dots\sum_{\mu_{n-1}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{n-1}[\mu_{n-1}] & 1\end{bmatrix}\sum_{\mu_n\in\mathbf{F}_p}\mu_n^{r+1}\begin{bmatrix}1 & 0\\ p^n[\mu_n] & 1\end{bmatrix}[1_K, X^r]\\
y &\stackrel{\mathrm{def}}{=} x+(-1)^{r+1}x'.
\end{aligned}$$

A direct computation gives the key result:

Lemma 8.4. *Assume p is odd* [7]*; let $a, d\in\mathbf{F}_p^{\times}$, $\lambda\in\mathbf{F}_p$. Then:*

i) we have the following equalities in R_{n+1}:

$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x = a^{-1}d^{r+1}x$$
$$\begin{bmatrix}[a] & 0\\ 0 & [d]\end{bmatrix}x' = a^{r+1}d^{-1}x'$$

ii) the elements $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x - x$ and $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x' - x'$ are in R_n and we have:

$$\pi_{n-1}\circ(-T_n^-)\left(\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x - x\right) = (r+1)(-1)^{r+1}\sum_{\mu_0\in\mathbf{F}_p}P_{-\lambda}(\mu_0)\begin{bmatrix}[\mu_0] & 1\\ 1 & 0\end{bmatrix}[1_K, Y^r]$$
$$\pi_{n-1}\circ(-T_n^-)\left(\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x' - x'\right) = (r+1)(-1)^{r+1}(-\lambda)\delta_{p,3}[1_K, Y^r]$$

(where $P_{-\lambda}(\mu_0)$ has been defined in §2.3).

Proof: *i)* It is analogous to the proof of Lemma 7.2-*i*).

ii). First of all, we study the action of $\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}$ on x inside R_{n+1}. As $\begin{bmatrix}1 & 0\\ p^{n+1}\mathbf{Z}_p & 1\end{bmatrix}$ acts trivially on $[1, X^r]\in R_{n+1}$ we deduce from Lemma 2.8:

$$\begin{aligned}
&\begin{bmatrix}1 & [\lambda]\\ 0 & 1\end{bmatrix}x =\\
&=\sum_{j=0}^{r+1}\binom{r+1}{j}\sum_{\mu_0\in\mathbf{F}_p}\begin{bmatrix}[\mu_0+\lambda] & 1\\ 1 & 0\end{bmatrix}\dots\\
&\dots\sum_{\mu_{n-1}\in\mathbf{F}_p}\begin{bmatrix}1 & 0\\ p^{n-1}[\mu_{n-1}+P_{\lambda,\dots,\mu_{n-3}}(\mu_{n-2})] & 1\end{bmatrix}(-P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1}))^j\,\cdot\\
&\cdot\sum_{\mu_n\in\mathbf{F}_p}\mu_n^{r-(j-1)}\begin{bmatrix}1 & 0\\ p^n[\mu_n] & 1\end{bmatrix}[1_K, X^r]
\end{aligned}$$

[7] such a requirement is needed for the equality concerning x' in *ii*)

and therefore

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} x - x = T_n^+(v)$$

where $v \in R_n$ is defined as

$$v \stackrel{\text{def}}{=} \sum_{j=1}^{r+1} \binom{r+1}{j} (-1)^{r+(j-1)} \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0+\lambda] & 1 \\ 1 & 0 \end{bmatrix} \cdots$$

$$\cdots \sum_{\mu_{n-1} \in \mathbf{F}_p} (-P_{\lambda,\dots,\mu_{n-2}}(\mu_{n-1}))^j \begin{bmatrix} 1 & 0 \\ p^{n-1}[\mu_{n-1} + P_{\lambda,\dots,\mu_{n-3}}(\mu_{n-2})] & 1 \end{bmatrix} [1_K, X^{j-1}Y^{r-(j-1)}].$$

We are now left to study the image of $-T_n^-(v) \in R_{n-1}$ via the epimorphism π_{n-1}: a direct computation using the recursive property of the Witt polynomials $P_{\lambda,\dots,\mu_{j-2}}(X) \in \mathbf{F}_p[X]$ (for $j \in \{2,\dots,n\}$) together with Lemma 2.10-i) yields finally the result.

The behaviour of the element $x' \in R_{n+1}$ can be described in a similar way, using now Lemma 2.9 and the recursive property of the $s_{\lambda,\dots,\mu_{j-1}}$'s for $j \in \{2,\dots,n\}$. The details are left to the reader. $\sharp$

***End of the proof of Proposition** 8.1.* Fix an irreducible K-subrepresentation τ of $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1}$ such that $\tau \cap R_0 \oplus_{R_1} \cdots \oplus_{R_{n-2}} R_{n-1} = 0$; therefore the natural projection $R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1} \twoheadrightarrow R_{n+1}/R_n$ induces an isomorphism of τ onto an irreducible factor of $\mathrm{soc}(R_{n+1}/R_n)$. Thanks to Proposition 7.1, Lemma 8.3 and Proposition 6.6 we distinguish two situations:

A) the subrepresentation τ maps isomorphically into the K-subrepresentation of R_{n+1}/R_n generated by (the image of) x.

B) We have $r = p-3$ and the subrepresentation τ maps isomorphically into the K-subrepresentation of R_{n+1}/R_n generated by (the image of) y.

***Study of case** A.* Let $f \in R_n$ be such that $pr_{n+1}(x + T_n^+(f)) \in \tau$. From the induced isomorphism $\tau \xrightarrow{\sim} \langle K \cdot x \rangle$ and the behaviour of x in R_{n+1}/R_n we deduce the following necessary conditions:

1) for all $a, d \in \mathbf{F}_p^\times$ we have

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (x + T_n^+(f)) - a^{-1}d^{r+1}(x + T_n^+(f)) \in \ker(pr_{n+1})$$

2) for all $\lambda \in \mathbf{F}_p$ we have

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (x + T_n^+(f)) - (x + T_n^+(f)) \in \ker(pr_{n+1}).$$

From condition 1) and Lemma 8.4-ii) we see that $\pi_{n-1} \circ (-T_n^-)(f)$ is an H-eigenvector of $R_1/\mathrm{Fil}^{r-1}(R_1) \cong \mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^r$ of associated eigencharacter $a^{-1}d^{r+1}$. We then deduce from Lemma 2.6 that

- if $r \neq 0$ the image of $\pi_{n-1} \circ (-T_n^-)(f)$ through the epimorphism

$$\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^r \stackrel{\pi}{\twoheadrightarrow} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2$$

is $\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix}$-invariant;

- if $r = 0$, then

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} \pi_{n-1} \circ (-T_n^-)(f) - \pi_{n-1} \circ (-T_n^-)(f) = c_1 \lambda \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, e]$$

inside $\mathrm{Ind}_{K_0(p)}^K 1$, for a suitable $c_1 \in \overline{\mathbf{F}}_p$.

It follows then from condition 2) and Lemma 8.4 that for any $\lambda \in \mathbf{F}_p$ the element

$$\sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p}(-\lambda)^{p-j} \sum_{\mu_0 \in \mathbf{F}_p} \mu_0^j \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] +$$
$$+\delta_{0,r} c_1 \lambda \sum_{\mu_0 \in \mathbf{F}_p} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] \in R_1/\mathrm{Fil}^{r-1}(R_1)$$

maps to zero via

$$\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^r \overset{\pi}{\twoheadrightarrow} \mathrm{Sym}^{\lceil r \rceil} \overline{\mathbf{F}}_p^2.$$

Thus, Lemma 2.10-*ii*) implies in particular that

$$\sum_{\mu_0 \in \mathbf{F}_p} \mu_0^{p-1} \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] \in \ker(\pi) \quad \text{for } r \neq 0$$
$$\sum_{\mu_0 \in \mathbf{F}_p} \mu_0 \begin{bmatrix} [\mu_0] & 1 \\ 1 & 0 \end{bmatrix} [1, Y^r] \in \ker(\pi) \quad \text{for } r = 0$$

giving an absurd for $r \neq 0$ and $r = 0$ respectively (cf. Remark 2.5). $\sharp$

Study of case B***.*** Let $f \in R_n$ be such that $pr_{n+1}(y + T_n^+(f)) \in \tau$. From the induced isomorphism $\tau \overset{\sim}{\to} \langle K \cdot y \rangle (\cong \det^{-1})$ and the behaviour of y in R_{n+1}/R_n we deduce the following necessary conditions:

1) for all $a, d \in \mathbf{F}_p^\times$ we have

$$\begin{bmatrix} [a] & 0 \\ 0 & [d] \end{bmatrix} (y + T_n^+(f)) - (ad)^{-1}(y + T_n^+(f)) \in \ker(pr_{n+1})$$

2) for all $\lambda \in \mathbf{F}_p$ we have

$$\begin{bmatrix} 1 & [\lambda] \\ 0 & 1 \end{bmatrix} (y + T_n^+(f)) - (y + T_n^+(f)) \in \ker(pr_{n+1}).$$

We then argue as in the previous case to get an absurd. The details are left to the reader. $\sharp$

This acheives the proof of Proposition 8.1 for $n \geqslant 3$, n odd, and we leave it to the reader to check (by the explicit description of T_1^-) that the same procedure applies also for $n = 1$. It is then obvious that the same proof applies to the case $m \in \mathbf{N}_>$ even and, with formal adjustments, to part *ii*) of Proposition 8.1 (as remarked after the proof of Lemma 8.3).

9. Conclusion

We are now ready to describe the socle filtration for the KZ-restriction of supersingular representations of $\mathrm{GL}_2(\mathbf{Q}_p)$: it will be a formal consequence of the explicit computations given in paragraphs 6, 7, 8.

Proposition 9.1. *Assume p is odd; let r be an integer, with $0 \leqslant r \leqslant p-2$. The socle filtration for* $\varinjlim_{n \text{ odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ *is described by*

$$R_0 \text{—} \mathrm{SocFil}(R_2/R_1) \text{—} \ldots \text{—} \mathrm{SocFil}(R_{n+1}/R_n) \text{—} \ldots$$

while the socle filtration for $\varinjlim_{m\,\mathrm{even}} (R_1/R_0 \oplus_{R_2} \cdots \oplus_{R_m} R_{m+1})$ *is described by*

$$\mathrm{SocFil}(R_1/R_0)\text{—}\mathrm{SocFil}(R_3/R_2)\text{—}\ldots\text{—}\mathrm{SocFil}(R_{m+1}/R_m)\text{—}\ldots$$

Proof: The proof is by induction; we will treat the case n odd (the other is analogous). Fix an odd integer $n \in \mathbf{N}_{\geqslant 1}$ and let Q be a quotient coming from the socle filtration of R_{n-1}/R_{n-2}. Assume (by inductive hypothesis) we dispose of an inductive system

$$\{Q \oplus_{R_n} R_{n+1} \cdots \oplus_{R_m} R_{m+1}\}_{m \geqslant n-2,\, m\,\mathrm{odd}}$$

(with the convention $Q \oplus_{R_{n-2}} R_{n-1} \stackrel{\mathrm{def}}{=} Q$) and where the amalgamated sums are defined through the Hecke operators $T_j^{\pm}$ for $j \geqslant n$ as in §3.2, as well as natural exact sequences:

$$0 \to Q \oplus_{R_n} \cdots \oplus_{R_{m-2}} R_{m-1} \to Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1} \to R_{m+1}/R_m \to 0$$

for $m \geqslant n$, m odd. If we set

$$\tau \stackrel{\mathrm{def}}{=} \mathrm{soc}(Q)$$

we formally verify that for $\tau \neq Q$

$$Q/\tau \oplus_Q (Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1}) = \mathrm{coker}(\tau \to Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})$$

for any $m \geqslant n$, m odd, while, if $\tau = Q$,

$$R_{n+1}/R_n \oplus_{\tau \oplus_{R_n} R_{n+1}} (\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1}) = \mathrm{coker}(\tau \to Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})$$

for any $m > n$, m odd. We therefore get an inductive system:

$$\{Q/\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1}\}_{m \geqslant n-2,\, m\,\mathrm{odd}}$$

and natural exact sequences

$$0 \to Q/\tau \oplus_{R_n} \cdots \oplus_{R_{m-2}} R_{m-1} \to Q/\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1} \to R_{m+1}/R_m \to 0$$

for $m \geqslant n$, m odd (where we write R_{n+1} instead of $Q/\tau \oplus_{R_n} R_{n+1}$ in the case $\tau = Q$). As $\varinjlim$ is right exact, we deduce that

$$\mathrm{coker}(\tau \to \varinjlim_{m \geqslant n,\, m\,\mathrm{odd}} (Q \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})) = \varinjlim_{m \geqslant n,\, m\,\mathrm{odd}} (Q/\tau \oplus_{R_n} \cdots \oplus_{R_m} R_{m+1})$$

and the statement is now clear from Proposition 8.1 ♯

The socle filtration for $\pi(r, 0, 1)|_{KZ}$, with $0 \leqslant r \leqslant p-1$ and p odd is then immediate from Proposition 3.9 and from the isomorphism $\pi(0, 0, 1) \cong \pi(p-1, 0, 1)$.

We give now the idea of the socle filtration for $\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$:

$$\begin{aligned}&\mathrm{SocFil}(\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})) = \\ &= R_0\text{—}\mathrm{SocFil}(R_2/R_1)\text{—}\mathrm{SocFil}(R_4/R_3)\text{—}\ldots\end{aligned}$$

which gives, developing the socle filtration of the quotients R_{n+1}/R_n,

$$R_0\text{—}\mathrm{SocFil}(\mathrm{Fil}^0(R_2/R_1))\text{—}\mathrm{SocFil}(\mathrm{Fil}^1(R_2)/\mathrm{Fil}^0(R_2))\text{—}\mathrm{SocFil}(\mathrm{Fil}^2(R_2)/\mathrm{Fil}^1(R_2))\text{—}\ldots$$

and, using Proposition 7.1,

$$R_0\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+1})\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+2})\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^K \chi_r^s \mathfrak{a}^{r+3})\text{—}\ldots$$

To be even more explicit, if we suppose $1 \leqslant r \leqslant p-6$ the beginning of the socle filtration for $\varinjlim_{n,\,\mathrm{odd}} (R_0 \oplus_{R_1} \cdots \oplus_{R_n} R_{n+1})$ looks as follows:

$$\mathrm{Sym}^r\overline{\mathbf{F}}_p^2 — \mathrm{Sym}^{p-3-r}\overline{\mathbf{F}}_p^2 \otimes \det{}^{r+1} — \mathrm{Sym}^{r+2}\overline{\mathbf{F}}_p^2 \otimes \det{}^{p-2} — \mathrm{Sym}^{p-5-r}\overline{\mathbf{F}}_p^2 \otimes \det{}^{r+2} — \dots$$

10. The principal series and the Steinberg

In this section we want to describe the socle filtration for the K-restriction of principal series and the Steinberg representation for $\mathrm{GL}_2(\mathbf{Q}_p)$. The techniques are very close to those of §6 and therefore will be mainly left to the reader. If $\lambda \in \overline{\mathbf{F}}_p^\times$ and $r \in \{0, \dots, p-1\}$ we recall the parabolic induction

$$\mathrm{Ind}_B^G(\mathrm{un}_\lambda \otimes \omega^r \mathrm{un}_{\lambda^{-1}}). \tag{6}$$

If $V_{\lambda,r}$ is the underlying vector space associated to the B-representation $\mathrm{un}_\lambda \otimes \omega^r \mathrm{un}_{\lambda^{-1}}$, the induction (6) is the $\overline{\mathbf{F}}_p$-vector space of locally constant functions $f : G \to V_{\lambda,r}$ such that $f(bg) = b \cdot f(g)$ for any $b \in B$, $g \in G$; the left G-action defined by right translation of functions gives (6) a structure of smooth G-representation.

We recall also that, for $(\lambda, r) \notin \{(0, \pm 1), (p-1, \pm 1)\}$, the representations (6) are irreducible (referred to as principal series), otherwise they fit into a short exact sequence

$$0 \to 1 \to \mathrm{Ind}_B^G 1 \to \mathrm{St} \to 0$$

and the quotient St is referred to as the “Steinberg” representation.

We turn our attention to the K-restriction of the inductions given by (6).

Lemma 10.1. *For any $\lambda \in \overline{\mathbf{F}}_p^\times$ and $r \in \{0, \dots, p-1\}$ we have a K-equivariant isomorphism*

$$(\mathrm{Ind}_B^G(\mathrm{un}_\lambda \otimes \omega^r \mathrm{un}_{\lambda^{-1}}))|_K \cong \mathrm{Ind}_{K\cap B}^K \chi_r^s$$

where χ_r^s, which is a character of $B(\mathbf{F}_p)$, is seen as a smooth character of $B \cap K$ by inflation.

Proof: It is an immediate consequence of Mackey theorem and the Iwasawa decomposition $G = KB$.♯

We have a natural homeomorphism

$$K/K\cap B \xrightarrow{\sim} \mathbf{P}^1_{\mathbf{Z}_p}$$

(coming from the natural left action of K on $[1:0] \in \mathbf{P}^1_{\mathbf{Z}_p}$) and the decomposition of corollary 3.2 let us deduce an open disjoint covering of $\mathbf{P}^1_{\mathbf{Z}_p}$ with balls of radius $(\frac{1}{p})^n$ (for the normalised norm on $\mathbf{Z}_p$: $|p| \overset{\mathrm{def}}{=} \frac{1}{p}$). The following result is then clear

Lemma 10.2. *Let $n \in \mathbf{N}$, $r \in \{0, \dots, p-2\}$; we fix a basis $\{e\}$ of the underlying vector space of χ_r^s. We have K-equivariant monomorphisms*

$$\mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \overset{\iota_{n+1}}{\hookrightarrow} \mathrm{Ind}_{K\cap B}^K \chi_r^s, \qquad \mathrm{Ind}_{K_0(p^{n+1})}^K \chi_r^s \overset{\iota_{n+1,n+2}}{\hookrightarrow} \mathrm{Ind}_{K_0(p^{n+2})}^K \chi_r^s$$

characterzed by

i) $\iota_{n+1}([1,e])$ *is the unique function* $f \in \mathrm{Ind}_{K\cap B}^K \chi_r^s$ *such that* $\mathrm{Supp}(f) = K_0(p^{n+1})$ *and* $f(1) = e$;

ii)

$$\iota_{n+1,n+2}([1,e]) = \sum_{\mu_{n+1}\in\mathbf{F}_p} \begin{bmatrix} 1 & 0 \\ p^{n+1}[\mu_{n+1}] & 1 \end{bmatrix} [1,e]$$

Proof: It is a standard verification that the conditions in *i)* and *ii)* define K-equivariant morphisms ι_{n+1}, $\iota_{n+1,n+2}$. Such morphisms are then injective by support reasons. $\sharp$

From the monomorphisms defined in Lemma 10.2 we deduce then a natural monomorphism:

$$\varinjlim_{n\in\mathbf{N}}(\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s)\hookrightarrow \mathrm{Ind}_{K\cap B}^K\chi_r^s. \tag{7}$$

As K is compact and all functions $f\in \mathrm{Ind}_{K\cap B}^K\chi_r^s$ are locally constant, we conclude that (7) is actually an isomorphism. Moreover:

LEMMA 10.3. *Let $n\in\mathbf{N}$, $r\in\{0,\dots,p-2\}$. Then*

$$\mathrm{coker}(\iota_{n+1,n+2})=Q_{0,\dots,0,1}^{(0,n+2)}.$$

Proof: From the definitions of $Q_{0,\dots,0,1}^{(0,n+2)}$ and $\iota_{n+1,n+2}$ we deduce a natural epimorphism $\mathrm{coker}(\iota_{n+1,n+2})\twoheadrightarrow Q_{0,\dots,0,1}^{(0,n+2)}$. The result follows, as the two spaces have the same dimension.$\sharp$

We dispose now of K-equivariant exact sequences, where $n\in\mathbf{N}$:

$$0\to \mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s\to \mathrm{Ind}_{K_0(p^{n+2})}^K\chi_r^s\to Q_{0,\dots,0,1}^{(0,n+2)}\to 0.$$

Thanks to the explicit description of $\mathrm{soc}(Q^{0,n+2}_{(0,\dots,0,1)})$ we deduce, with arguments which are very similar to those of Proposition 8.1, the following result

PROPOSITION 10.4. *Let $n\in\mathbf{N}$, $r\in\{0,\dots,p-2\}$. Then*

$$\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s)=\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+2})}^K\chi_r^s).$$

More generally, if $Q\leqslant \mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s$ is a K-subrepresentation coming from the socle filtration of $\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s$, we have

$$\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+1})}^K\chi_r^s/Q)=\mathrm{soc}(\mathrm{Ind}_{K_0(p^{n+2})}^K\chi_r^s/\iota_{n+1,n+2}(Q)).$$

Proof: It suffices to use the same arguments of the proof of Proposition 8.1, and similar explicit computations. The details are left to the reader. $\sharp$

Once again, we can use Proposition 10.4 to describe the behaviour of the socle filtration for $\mathrm{Ind}_{K\cap B}^K\chi_r^s$. The graded pieces of such a filtration look as follows:

$$\mathrm{SocFil}(\mathrm{Ind}_{K\cap B}^K\chi_r^s)=\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^K\chi_r^s)\text{—}\mathrm{SocFil}(Q_{0,1}^{(0,2)})\text{—}\mathrm{SocFil}(Q_{0,0,1}^{(0,3)})\text{—}\dots$$

and, developing the socle filtration of $Q_{0,\dots,0,1}^{(0,n+2)}$,

$$\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^K\chi_r^s\mathfrak{a})\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^2)\text{—}\mathrm{SocFil}(\mathrm{Ind}_{K_0(p)}^K\chi_r^s\mathfrak{a}^3)\text{—}\dots$$

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Stefano Morra stefano.morra@math.uvsq.fr
Université de Versailles, 45 Avenue des Etats-Unis, 78035 Versailles, France