On the cohomology of the ramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

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Abstract : *In this paper, we study the cohomology of the ramified PEL unitary Rapoport-Zink space of signature* $(1, n-1)$ *by using the Bruhat-Tits stratification on its special fiber. As such, we apply the same method that we developped for the unramified case in two previous papers. More precisely, we first investigate the cohomology of a given closed Bruhat-Tits stratum. It is isomorphic to a generalized Deligne-Lusztig variety which is in general not smooth, and is associated to a finite group of symplectic similitudes. We determine the weights of the Frobenius and most of the unipotent representations occuring in its cohomology. This computation involves the spectral sequence associated to a stratification by classical Deligne-Lusztig varieties, which are parabolically induced from Coxeter varieties of smaller symplectic groups. In particular, all the unipotent representations contribute to only two cuspidal series. Then, we introduce the analytical tubes of the closed Bruhat-Tits strata, which give an open cover of the generic fiber of the Rapoport-Zink space. Using the associated Čech spectral sequence, we prove that certain cohomology groups of the Rapoport-Zink space at hyperspecial level fail to be admissible if* n *is large enough. Eventually, when* $n = 2$ *in the split case, when* $n = 3$ *and when* $n = 4$ *in the non-split case, we give a complete description of the cohomology of the supersingular locus of the associated Shimura variety at hyperspecial level, in terms of automorphic representations.*

Contents

Introduction: Rapoport-Zink spaces are moduli spaces classifying the deformations of some p-divisible group $\mathbb{X}$ equipped with additional structures, called a framing object. It is a formal scheme $\mathcal{M} = \mathcal{M}_{\mathbb{X}}$ to which one can associate a projective system $\mathcal{M}_\infty = (\mathcal{M}_K)_K$ of analytic spaces called the Rapoport-Zink tower. It is equipped with compatible actions of two p-adic groups $G = G(\mathbb{Q}_p)$ and $J = J(\mathbb{Q}_p)$, with J being an inner form of some Levi complement of G. The cohomology of the tower $\mathcal{M}_\infty$ with coefficients in $\overline{\mathbb{Q}_\ell}$ for $\ell \neq p$ is naturally a representation of $G \times J \times W$ where W is the Weil group of the underlying reflex field E, which is a p-adic local field. These cohomology groups are believed to play a role in local Langlands correspondence. So far, relatively little is known on the cohomology of $\mathcal{M}_\infty$ in general. It has been computed entirely in the Lubin-Tate case in [Boy09], whose results have been used in [Dat07] to deduce the case of the Drinfeld space and compute the action of the monodrony. Both the Lubin-Tate and the Drinfeld cases correspond to Rapoport-Zink spaces of EL type, and their particular geometry allowed for explicit computations. Aside from this, the Kottwitz conjecture describes the part of the cohomology of $\mathcal{M}_\infty$ which is supercuspidal both for G and J. This has first been proved in the Lubin-Tate case in [Boy99] and [HT01]. It has been generalized to all unramified Rapoport-Zink spaces of EL type in [Far04] and [Shi12], and it has been proved more recently in the case of the unramified PEL unitary Rapoport-Zink space with signature $(r, n-r)$ with n odd in [Ngu19] and [BMN21].
One would like to obtain more information on the cohomology of general Rapoport-Zink spaces outside of the supercuspidal part, but this may remain out of reach unless we have a good understanding of the geometry of $\mathcal{M}$. In [GHN19] and [GHN22], the authors determined the complete list of all choices of the framing object $\mathbb{X}$ so that the resulting Rapoport-Zink space $\mathcal{M}$ exhibits a Bruhat-Tits stratification on its reduced special fiber $\mathcal{M}_{\text{red}}$. The resulting Bruhat-Tits strata are naturally isomorphic to Deligne-Lusztig varieties which, in the most favorable cases, are of Coxeter type. Classical Deligne-Lusztig theory is a field of mathematics giving a classification of all the irreducible complex representations of finite groups of Lie type. In

their foundational paper [DL76], the authors use the cohomology of Deligne-Lusztig varieties to define new induction and restriction functors, allowing them to build all such representations. The present paper is a contribution to a program aiming at making use of Deligne-Lusztig theory in order to access new information on the cohomology of the Rapoport-Zink space. In [Mul22b] and [Mul22a], we explored this idea with the unramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$. Historically, this space is the first for which the name "Bruhat-Tits stratification" was coined, and it has been studied in [Vol10] and [VW11]. In this case, the closed Bruhat-Tits strata are projective and smooth, and their cohomology was entirely computable. We used it to study the cohomology of the maximal level of the tower, ie. the cohomology of the Berkovich generic fiber $\mathcal{M}^{\mathrm{an}}$, as a representation of $J \times W$. We proved that some of these cohomology groups fail to be J-admissible for any $n \geqslant 3$, and we used our results to entirely compute the cohomology of the supersingular locus of the corresponding Shimura variety when $n = 3$ or 4. In the present paper, we now consider the ramified case for which the Bruhat-Tits stratification was described in [RTW14]. We reach very similar results despite new technical difficulties arising from the fact that the closed Bruhat-Tits strata are not smooth anymore, so that our method fails to encapsulate a certain part of its cohomology. Let us explain this in more details.

In the ramified case, the reflex field E is a quadratic ramified extension of $\mathbb{Q}_p$ with $p > 2$. When n is odd there is only one choice of framing object $\mathbb{X}$, but when n is even there are two choices $\mathbb{X}^+$ and $\mathbb{X}^-$. The group J is isomorphic to the group of unitary similitudes of a non-degenerate $E/\mathbb{Q}_p$-hermitian space C of dimension n, which is closely related to the Dieudonné isocristal of $\mathbb{X}$. If n is even, the hermitian space C is split if $\mathbb{X} = \mathbb{X}^+$ and non-split if $\mathbb{X} = \mathbb{X}^-$. The group J is quasi-split if and only if n is odd or n is even with C split. In [RTW14], the authors build the Bruhat-Tits stratification $\mathcal{M}_{\mathrm{red}} = \bigsqcup_{\Lambda} \mathcal{M}_{\Lambda}^{\circ}$ where Λ runs over the set of vertex lattices $\mathcal{L}$ in C and where $\mathcal{M}_{\Lambda}^{\circ}$ is isomorphic to the Coxeter variety associated to the finite group of symplectic similitudes $\mathrm{GSp}(2\theta, \mathbb{F}_p)$, where $0 \leqslant t(\Lambda) := 2\theta \leqslant n$ is the type of the vertex lattice Λ. The set of vertex lattices $\mathcal{L}$ forms a polysimplicial complex which is closely related to the Bruhat-Tits building of J, giving a good combinatorial description of the incidence relations of the strata. Let $\mathcal{M}_{\Lambda} := \overline{\mathcal{M}_{\Lambda}^{\circ}}$ denote the closure of a Bruhat-Tits stratum. Then $\mathcal{M}_{\Lambda}$ is a projective normal variety over $\mathbb{F}_p$ which is not smooth as soon as $\theta \geqslant 2$. It is isomorphic to the projective closure S_θ of a generalized Deligne-Lusztig variety for $\mathrm{GSp}(2\theta, \mathbb{F}_p)$, the kind of which does not fall under the scope of classical Deligne-Lusztig theory. However, the variety S_θ admits a stratification

$$S_\theta = \bigsqcup_{\theta'=0}^{\theta} X_{I_{\theta'}}(w_{\theta'}),$$

(notations of 1.1.5) such that the closure of a stratum $X_{I_{\theta'}}(w_{\theta'})$ is the union of all the smaller strata $X_{I_t}(w_t)$ for $0 \leqslant t \leqslant \theta'$. It turns out that each stratum $X_{I_{\theta'}}(w_{\theta'})$ is a classical Deligne-Lusztig variety, which is parabolically induced from the Coxeter variety attached to the smaller group of unitary similitudes $\mathrm{GSp}(2\theta', \mathbb{F}_p)$. In [Lus76], Lusztig has computed the cohomology of the Coxeter varieties for all classical groups. Using the combinatorical description of irreducible unipotent representations of symplectic groups in terms of Lusztig's notion of symbols, one may compute through parabolic induction the cohomology of any stratum $X_{I_{\theta'}}(w_{\theta'})$. The stratification then induces a $\mathrm{GSp}(2\theta, \mathbb{F}_p) \times \langle F \rangle$-equivariant spectral sequence

$$E_1^{a,b} = \mathrm{H}_c^{a+b}(X_{I_a}(w_a), \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(S_\theta, \overline{\mathbb{Q}_\ell}),$$

where F denotes the geometric Frobenius action. The study of the weights of the Frobenius on the terms $E_1^{a,b}$ shows that the sequence degenerates on the second page, and it offers substantial information on the cohomology of S_θ.

More precisely, S_θ has dimension θ, and for $0 \leqslant k \leqslant 2\theta$ the weights of the Frobenius on the cohomology group $\mathrm{H}_c^k(S_\theta, \overline{\mathbb{Q}_\ell})$ form a subset of $\{p^i, -p^{j+1}\}$ for $k - \min(k,\theta) \leqslant i \leqslant k - \lceil k/2 \rceil$ and for $k - \min(k,\theta) \leqslant j \leqslant k - \lceil k/2 \rceil - 1$. Among other things, if $i, j > k - \min(k,\theta)$ then we determine the eigenspaces of the Frobenius $\mathrm{H}_c^k(S_\theta, \overline{\mathbb{Q}_\ell})_{p^i}$ and $\mathrm{H}_c^k(S_\theta, \overline{\mathbb{Q}_\ell})_{-p^{j+1}}$ explicitely up to at most four irreducible representations of $\mathrm{GSp}(2\theta, \mathbb{F}_p)$. We refer to 1.4.2 and 1.4.3 for the detailed results, and to 1.2.3 Theorem for the notations regarding unipotent representations in terms of symbols, as it would be too long to fit this introduction.

In particular, we note that the cohomology of S_θ is entirely determined if $\theta \leqslant 1$ since S_0 is a point and $S_1 \simeq \mathbb{P}^1$. For $\theta \geqslant 2$ the variety S_θ has singularities and for $\theta \geqslant 3$ the action of the Frobenius on the cohomology is not pure (for $\theta = 2$ the non-purity is undetermined). All irreducible representations of $\mathrm{GSp}(2\theta, \mathbb{F}_p)$ occuring in an eigenspace of F for an eigenvalue of the form p^i belong to the unipotent principal series, whereas those corresponding to an eigenvalue of the form $-p^{j+1}$ belong to the cuspidal series determined by the unique cuspidal unipotent representation of $\mathrm{GSp}(4, \mathbb{F}_p)$.

We then introduce the analytical tube $U_\Lambda \subset \mathcal{M}^{\mathrm{an}}$ of any closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$. As we work at hyperspecial level, the associated integral model of the Shimura variety is smooth so that the nearby cycles are trivial. It allows us to identify the cohomology of U_Λ and of $\mathcal{M}_\Lambda$. Let $\mathcal{L}^{\max}$ denote the subset of vertex lattices Λ whose type $t(\Lambda)$ is maximal, ie. it is equal to

$$t_{\max} = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even and } C \text{ is split,} \\ n-2 & \text{if } n \text{ is even and } C \text{ is non-split.} \end{cases}$$

We also write $t_{\max} = 2\theta_{\max}$. Let $\{\Lambda_0, \ldots, \Lambda_{\theta_{\max}}\}$ be a maximal simplex in $\mathcal{L}$ such that $t(\Lambda_\theta) = 2\theta$ for all $0 \leqslant \theta \leqslant \theta_{\max}$. Let J_θ denote the fixator in J of the vertex lattice Λ_θ. Then the J_θ's are maximal compact subgroups of J, and any maximal compact subgroup of J is conjugate to one of the J_θ's. The collection $\{U_\Lambda\}_{\Lambda \in \mathcal{L}^{\max}}$ forms an open cover of the generic fiber $\mathcal{M}^{\mathrm{an}}$, from which we deduce the existence of a $(J \times W)$-equivariant spectral sequence (with the notations of 3.1.4 and 3.1.10)

$$E_1^{a,b} = \bigoplus_{\theta=0}^{\theta_{\max}} \mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \left(\mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}] \right) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

Since the non-zero terms $E_1^{a,b}$ are located in a finite range $0 \leqslant b \leqslant 2(n-1)$, this sequence eventually degenerates. Using this sequence, we are able to compute the cohomology group of

$\mathcal{M}^{\mathrm{an}}$ of highest degree, and when $\theta_{\max} = 1$, ie. $n = 2$ with C split, $n = 3$ or $n = 4$ with C non-split, using the combinatorics of the Bruhat-Tits building of J, we prove the vanishing of the cohomology group of degree $2(n-1)-1$. In the following statement, $J^\circ \subset J$ denotes the subgroup consisting of all unitary similitudes whose multiplier is a unit in $\mathbb{Z}_p$. It is also the subgroup of J generated by all its compact subgroups. We note that $J/J^\circ \simeq \mathbb{Z}$.

Theorem (3.1.13 and 3.2.3)**.** *There is an isomorphism*

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\,\mathbf{1},$$

where $\mathbf{1}$ *denotes the trivial representation, and where* Frob *acts like* $p^{n-1}\cdot\mathrm{id}$.
Assume that $\theta_{\max} = 1$*, ie.* $n = 2$ *with* C *split,* $n = 3$ *or* $n = 4$ *with* C *non-split. Then* $\mathrm{H}_c^{2(n-1)-1}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) = 0$.

In particular, when $n = 2$ with C split, the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ has dimension 1 and its cohomology is entirely computed. We sum in up in the following statement.

Corollary (3.2.3)**.** *Assume that* $n = 2$ *with* C *split. Then* $\mathrm{H}_c^0(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\,\mathbf{1}$ *with* τ *acting like* id, $\mathrm{H}_c^1(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) = 0$ *and* $\mathrm{H}_c^2(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\,\mathbf{1}$ *with* τ *acting like* $p\cdot\mathrm{id}$.

For general n, by looking carefully at the distribution of the Frobenius weights among the different terms, we are able to determine two terms which are left unchanged in the deeper pages of the sequence. It leads to the following statement, where $\tau := (\pi\mathrm{id}, \mathrm{Frob}) \in J \times W$ with π a uniformizer in E of trace zero and Frob the geometric Frobenius.

Theorem (3.1.8, 3.1.9 and 3.1.10)**.** *There is a* $J \times W$*-equivariant monomorphism*

$$\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J}\,\mathbf{1} \hookrightarrow \mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}}),$$

where on the left-hand side the inertia acts trivially and τ *acts like multiplication by the scalar* $p^{n-1-\theta_{\max}}$.
Assume that $n \geqslant 5$ *or that* $n = 4$ *with* C *split. There is a* $J \times W$*-equivariant monomorphism*

$$\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J}\,\rho_{\theta_{\max}} \hookrightarrow \mathrm{H}_c^{2(n-\theta_{\max})}(\mathcal{M}^{\mathrm{an}}),$$

where on the left-hand side the inertia acts trivially and τ *acts like multiplication by the scalar* $-p^{n-\theta_{\max}}$.

Here, $\mathbf{1}$ denotes the trivial representation and $\rho_{\theta_{\max}}$ is the inflation of a certain irreducible unipotent representation of the finite reductive quotient $J_{\theta_{\max}}/J_{\theta_{\max}}^{+}$ defined in 3.1.9. Using type theory, one may study the behaviour of such compactly induced representations to deduce the following proposition. Here, if V is any smooth representation of J and χ is any smooth character of the center $\mathrm{Z}(J) \simeq E^\times$, we denote by V_χ the largest quotient of V on which $\mathrm{Z}(J)$ acts through χ. For χ an unramified character of $\mathrm{Z}(J)$, when $\theta_{\max} = 0$ (ie. $n = 1$ or $n = 2$ with C non-split), we define an irreducible supercuspidal representation $\sigma_{0,\chi}$ of J in 3.1.11, and when $\theta_{\max} = 2$ (ie. $n = 4$ with C split, $n = 5$ or $n = 6$ with C non-split), we define another irreducible supercuspidal representation $\sigma_{2,\chi}$ of J in 3.1.12.

Proposition (3.1.11 and 3.1.9)**.** *Let χ be an unramified character of $\mathrm{Z}(J)$.*

(1) *If $n = 1$ or $n = 2$ with C non-split, all irreducible subquotients of $V := \mathrm{c} - \mathrm{Ind}_{J_0}^{J} \mathbf{1}$ are supercuspidal, and we have $V_\chi \simeq \sigma_{0,\chi}$.*

(2) *If $n \geqslant 3$ or if $n = 2$ with C split, then no irreducible subquotient of $V := \mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathbf{1}$ is supercuspidal. In this case, V_χ does not contain any non-zero admissible subrepresentation of J.*

(3) *If $n = 4$ with C split, if $n = 5$ or if $n = 6$ with C non-split, all irreducible subquotients of $V := \mathrm{c} - \mathrm{Ind}_{J_2}^{J} \rho_2$ are supercuspidal, and we have $V_\chi \simeq \sigma_{2,\chi}$.*

(4) *If $n \geqslant 7$ or if $n = 6$ with C split, then no irreducible subquotient of $V := \mathrm{c} - \mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{\theta_{\max}}$ is supercuspidal. In this case, V_χ does not contain any non-zero admissible subrepresentation of J.*

In particular, we obtain the following corollary.

Corollary. *Let χ be any unramified character of $\mathrm{Z}(J)$.*
If $n \geqslant 3$ or $n = 2$ with C split then $\mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}})_\chi$ is not J-admissible.
If $n \geqslant 7$ or $n = 6$ with C split then $\mathrm{H}_c^{2(n-\theta_{\max})}(\mathcal{M}^{\mathrm{an}})_\chi$ is not J-admissible.

This non-admissibility result was already observed in the unramified case, but it does not happen in the Lubin-Tate nor the Drinfeld cases.

Lastly, we introduce the PEL unitary Shimura variety S_{K^p} of signature $(1, n-1)$ at a ramified place, which is a smooth quasi-projective scheme over $\mathcal{O}_E$. It is given by a Shimura datum denoted $(\mathbb{G}, X)$. Let $\overline{S}^{\mathrm{ss}} := \varprojlim \overline{S}^{\mathrm{ss}}_{K^p}$ denote the supersingular locus in its special fiber. Via p-adic uniformization, the geometry of the supersingular locus is closely related to the geometry of $\mathcal{M}_{\mathrm{red}}$. At the level of cohomology, the following $(\mathbb{G}(\mathbb{A}_f^p) \times W)$-equivariant spectral sequence has been constructed in [Far04]

$$F_2^{a,b} = \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \implies \mathrm{H}_c^{a+b}(\overline{S}^{\mathrm{ss}}, \mathcal{L}_\xi),$$

where I is a certain inner form of $\mathbb{G}$ such that $I_{\mathbb{A}_f^p} = \mathbb{G}_{\mathbb{A}_f^p}$ and $I_{\mathbb{Q}_p} = J$, ξ is a finite dimensional irreducible algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbb{G}$ of weight $w(\xi) \in \mathbb{Z}$, $\mathcal{L}_\xi$ is the associated local system on the Shimura variety S_{K^p}, $\mathcal{A}_\xi(I)$ is the space of all automorphic representations of $I(\mathbb{A})$ of type ξ at infinity, and $\mathrm{H}_c^\bullet(\overline{S}^{\mathrm{ss}}, \mathcal{L}_\xi) := \varinjlim_{K^p} \mathrm{H}_c^\bullet(\overline{S}^{\mathrm{ss}}_{K^p}, \mathcal{L}_\xi)$. The semisimple rank of J is $\theta_{\max}$ so that $F_2^{a,b} = 0$ as soon as $a \geqslant \theta_{\max} + 1$. In particular, if $\theta_{\max} \leqslant 1$ then all the differentials are zero so that the spectral sequence already degenerates on the second page. Using our knowledge on the cohomology of the Rapoport-Zink space, one can compute all the non-zero terms $F_2^{a,b}$. We deduce the following automorphic description of the cohomology of the supersingular locus.
Let $X^{\mathrm{un}}(J)$ denote the set of unramified characters of J. Let St_J denote the Steinberg representation of J. Let us fix a square root π_ℓ of p in $\overline{\mathbb{Q}_\ell}$. If $\Pi \in \mathcal{A}_\xi(I)$, we define $\delta_{\Pi_p} :=$

$\omega_{\Pi_p}(\pi^{-1} \cdot \mathrm{id})\pi_\ell^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times$ where ω_{Π_p} is the central character of Π_p, and $\pi^{-1} \cdot \mathrm{id}$ lies in the center of J. For any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ we have $|\iota(\delta_{\Pi_p})| = 1$. Eventually, if $x \in \overline{\mathbb{Q}_\ell}^\times$, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of the Weil group W where the inertia acts trivially and Frob acts like multiplication by the scalar x.

Theorem (4.2.3)**.** *There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}_c^0(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p}\pi_\ell^{w(\xi)}],$$

$$\mathrm{H}_c^1(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p}\pi_\ell^{w(\xi)}],$$

$$\mathrm{H}_c^2(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p}\pi_\ell^{w(\xi)+2}].$$

This result is to be compared with the unramified case [Mul22a] Theorem 5.2.3. The cohomology of the supersingular locus both in the unramified and ramified cases are very similar, except that an additional term appears in H_c^1 in the unramified case, corresponding to automorphic representations Π such that Π_p is an unramified twist of a certain supercuspidal representation of J. The reason comes from the cohomology of a Bruhat-Tits stratum of maximal orbit type, where in the unramified case a cuspidal unipotent representation of the finite group of unitary similitudes in three variables $\mathrm{GU}(3, \mathbb{F}_p)$ occurs, however in the ramified case there is no such cuspidal unipotent representation of the finite group of simplectic similitudes in two variables $\mathrm{GSp}(2, \mathbb{F}_p)$.

Organisation of the paper: In the first section, we prove all the statements regarding the cohomology of the closed Bruhat-Tits strata by using Deligne-Lusztig theory only. Contrary to the introduction, we work over a general finite field $\mathbb{F}_q$ of characteristic p. However, only the case $q = p$ will be relevant in the context of the Rapoport-Zink space. Also, we work with the usual symplectic group Sp instead of the group of symplectic similitudes GSp, because the associated Deligne-Lusztig varieties are the same in virtue of 1.1.2 and 1.2.1. We recall the general definition of Deligne-Lusztig varieties, and we explain the combinatorics of symbols applied to the classification of unipotent representations of finite symplectic groups. We then translate Lusztig's computation of the cohomology of the Coxeter varieties in [Lus76] in terms of symbols, and we finally proceed to investigate the cohomology of a closed Bruhat-Tits stratum.

In section 2, we introduce the Rapoport-Zink space and recall the results from [RTW14] where the Bruhat-Tits stratification on the special fiber is built. We detail the combinatorics of vertex lattices, and we give a formula for the number of strata contained in or containing a fixed given stratum. Eventually, we also recall the p-adic uniformization of the supersingular locus of the associated Shimura variety.

In section 3, we move the Bruhat-Tits stratification to the generic fiber $\mathcal{M}^{\mathrm{an}}$ by considering the

analytical tubes U_Λ, and we study the associated Čech spectral sequence. This section contains all our results on the cohomology of the Rapoport-Zink space.
In the last section, we use the p-adic uniformization and our knowledge acquired so far on the cohomology of the Rapoport-Zink space, in order to compute the cohomology of the supersingular locus for small values of n.

Notations: Throughout the paper, we fix an integer $n \geqslant 1$ and an odd prime number p. If k is a perfect field of characteristic p, we denote by $W(k)$ the ring of Witt vectors and by $W(k)_{\mathbb{Q}}$ its fraction field, which is an unramified extension of $\mathbb{Q}_p$. We denote by $\sigma_k : x \mapsto x^p$ the Frobenius of $\mathrm{Aut}(k/\mathbb{F}_p)$, and we use the same notation for its lift to $\mathrm{Aut}(W(k)_{\mathbb{Q}}/\mathbb{Q}_p)$. If k'/k is a perfect field extension then $(\sigma_{k'})_{|k} = \sigma_k$, so we can remove the subscript and write σ unambiguously instead. If $q = p^e$ is a power of p, we write $\mathbb{F}_q$ for the field with q elements. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.
We fix $\epsilon \in \mathbb{Z}_p^\times$ such that $-\epsilon$ is not a square in $\mathbb{Z}_p$. We define $E_1 := \mathbb{Q}_p[\sqrt{-p}]$ and $E_2 := \mathbb{Q}_p[\sqrt{\epsilon p}]$. Any quadratic ramified extension of $\mathbb{Q}_p$ is isomorphic to either E_1 or E_2. We will denote by E either E_1 or E_2 with uniformizer π equal to $\sqrt{-p}$ or $\sqrt{\epsilon p}$ respectively. In both cases π^2 is a uniformizer in $\mathbb{Z}_p$. We write $\mathcal{O}_E$ for the ring of integers and $\kappa(E) = \mathbb{F}_p$ for the residue field. Let $\overline{\cdot} \in \mathrm{Gal}(E/\mathbb{Q}_p)$ be the non-trivial Galois involution, so that $\overline{\pi} = -\pi$.

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1 On the cohomology of a closed Bruhat-Tits stratum

1.1 The closed Deligne-Lusztig variety isomorphic to a closed Bruhat-Tits stratum

1.1.1 Let q be a power of p and let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$, together with a split $\mathbb{F}_q$-structure given by a geometric Frobenius morphism F. For $\mathbf{H}$ any F-stable subgroup of $\mathbf{G}$, we write $H := \mathbf{H}^F$ for its group of $\mathbb{F}_q$-rational points. Let $(\mathbf{T}, \mathbf{B})$ be a pair consisting of a maximal F-stable torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$. Let $(\mathbf{W}, \mathbf{S})$ be the associated Coxeter system, where $\mathbf{W} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Since the $\mathbb{F}_q$-structure on $\mathbf{G}$ is split, the Frobenius F acts trivially on $\mathbf{W}$. For $I \subset \mathbf{S}$, let $\mathbf{P}_I, \mathbf{U}_I, \mathbf{L}_I$ be respectively the standard parabolic subgroup of type I, its unipotent radical and its unique Levi complement containing $\mathbf{T}$. Let $\mathbf{W}_I$ be the subgroup of $\mathbf{W}$ generated by I.
For $\mathbf{P}$ any parabolic subgroup of $\mathbf{G}$, the associated **generalized parabolic Deligne-Lusztig variety** is

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

We say that the variety is **classical** (as opposed to generalized) when in addition the parabolic subgroup $\mathbf{P}$ contains an F-stable Levi complement. Note that $\mathbf{P}$ itself needs not be F-stable. We may give an equivalent definition using the Coxeter system $(\mathbf{W}, \mathbf{S})$. For $I \subset \mathbf{S}$, let ${}^I\mathbf{W}^I$ be the set of elements $w \in \mathbf{W}$ which are I-reduced-I. For $w \in {}^I\mathbf{W}^I$, the associated generalized parabolic Deligne-Lusztig variety is

$$X_I(w) := \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \,|\, g^{-1}F(g) \in \mathbf{P}_I w F(\mathbf{P}_I)\}.$$

The variety $X_I(w)$ is classical when $w^{-1}Iw = I$, and it is defined over $\mathbb{F}_q$. The dimension is given by $\dim X_I(w) = l(w)$ where $l(w)$ denotes the length of w with respect to $\mathbf{S}$.

1.1.2 Let $\mathbf{G}$ and $\mathbf{G}'$ be two reductive connected group over $\mathbb{F}$ both equipped with an $\mathbb{F}_q$-structure. We denote by F and F' the respective Frobenius morphisms. Let $f : \mathbf{G} \to \mathbf{G}'$ be an $\mathbb{F}_q$-isotypy, that is a homomorphism defined over $\mathbb{F}_q$ whose kernel is contained in the center of $\mathbf{G}$ and whose image contains the derived subgroup of $\mathbf{G}'$. Then, according to [DM14] proof of Proposition 11.3.8, we have $\mathbf{G}' = f(\mathbf{G})\mathrm{Z}(\mathbf{G}')^0$, where $\mathrm{Z}(\mathbf{G}')^0$ is the connected component of unity of the center of $\mathbf{G}'$. Thus intersecting with $f(\mathbf{G})$ defines a bijection between parabolic subgroups of $\mathbf{G}'$ and those of $f(\mathbf{G})$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and let $\mathbf{P}' = f(\mathbf{P})\mathrm{Z}(\mathbf{G}')^0$ be the corresponding parabolic of $\mathbf{G}'$. Then the map $g\mathbf{P} \mapsto f(g\mathbf{P})$ induces an isomorphism $f : X_{\mathbf{P}} \xrightarrow{\sim} X_{\mathbf{P}'}$ which is compatible with the actions of G and G' via f. Therefore $\mathbf{G}$ and $\mathbf{G}'$ generate the same Deligne-Lusztig varieties.

1.1.3 Let $\theta \geqslant 0$ and let V be a 2θ-dimensional $\mathbb{F}_q$-vector space equipped with a non-degenerate symplectic form $(\cdot,\cdot) : V \times V \to \mathbb{F}_q$. Fix a basis $(e_1, \ldots, e_{2\theta})$ in which $(\cdot,\cdot)$ is described by the matrix

$$\begin{pmatrix} 0 & A_\theta \\ -A_\theta & 0 \end{pmatrix},$$

where A_θ denotes the matrix having 1 on the anti-diagonal and 0 everywhere else. If k is a perfect field extension of $\mathbb{F}_q$, let $V_k := V \otimes_{\mathbb{F}_q} k$ denote the scalar extension to k equipped with its induced k-symplectic form $(\cdot,\cdot)$. Let $\tau : V_k \xrightarrow{\sim} V_k$ denote the map $\mathrm{id} \otimes \sigma$. If $U \subset V_k$, let $U^\perp$ denote its orthogonal.

We consider the finite symplectic group $\mathrm{Sp}(V, (\cdot,\cdot)) \simeq \mathrm{Sp}(2\theta, \mathbb{F}_q)$. It can be identified with $G = \mathbf{G}^F$ where $\mathbf{G}$ is the symplectic group $\mathrm{Sp}(V_{\mathbb{F}}, (\cdot,\cdot)) \simeq \mathrm{Sp}(2\theta, \mathbb{F})$ and F is the Frobenius raising the entries of a matrix to their q-th power. Let $\mathbf{T} \subset \mathbf{G}$ be the maximal torus of diagonal symplectic matrices and let $\mathbf{B} \subset \mathbf{G}$ be the Borel subgroup of upper-triangular symplectic matrices. The Weyl system of $(\mathbf{T}, \mathbf{B})$ is identified with $(W_\theta, \mathbf{S})$ where W_θ is the finite Coxeter group of type B_θ and $\mathbf{S} = \{s_1, \ldots, s_\theta\}$ is the set of simple reflexions. They satisfy the following relations

$$s_\theta s_{\theta-1} s_\theta s_{\theta-1} = s_{\theta-1} s_\theta s_{\theta-1} s_\theta, \qquad s_i s_{i-1} s_i = s_{i-1} s_i s_{i-1}, \qquad \forall\, 2 \leqslant i \leqslant \theta - 1,$$
$$s_i s_j = s_j s_i, \qquad \forall\, |i-j| \geqslant 2.$$

Concretely, the simple reflexion s_i acts on V by exchanging e_i and e_{i+1} as well as $e_{2\theta-i}$ and $e_{2\theta-i+1}$ for $1 \leqslant i \leqslant \theta - 1$, whereas s_θ exchanges e_θ and $e_{\theta+1}$. The Frobenius F acts trivially on W_θ.

1.1.4 We define the following subset of $\mathbf{S}$

$$I := \{s_1, \ldots, s_{\theta-1}\} = \mathbf{S} \backslash \{s_\theta\}.$$

We consider the generalized Deligne-Lusztig variety $X_I(s_\theta)$. Since $s_\theta s_{\theta-1} s_\theta \notin I$, it is not a classical Deligne-Lusztig variety. Let $S_\theta := \overline{X_I(s_\theta)}$ be its closure in $\mathbf{G}/\mathbf{P}_I$. This normal projective variety occurs as a closed Bruhat-Tits stratum in the special fiber of the ramified unitary PEL Rapoport-Zink space of signature $(1, n-1)$, as established in [RTW14]. In loc. cit. the authors describe the geometry of S_θ. We summarize their analysis.

Proposition ([RTW14] 5.3, 5.4)**.** *Let k be a perfect field extension of $\mathbb{F}_q$. The k-rational points of S_θ are given by*

$$S_\theta(k) \simeq \{U \subset V_k \,|\, U^\perp = U \text{ and } U \overset{\leqslant 1}{\subset} U + \tau(U)\},$$

where $\overset{\leqslant 1}{\subset}$ denotes an inclusion of subspaces with index at most 1. There is a decomposition

$$S_\theta = X_I(\mathrm{id}) \sqcup X_I(s_\theta),$$

where $X_I(\mathrm{id})$ is closed and of dimension 0, and $X_I(s_\theta)$ is open, dense of dimension θ. They correspond respectively to points U having $U = \tau(U)$ and $U \subsetneq U + \tau(U)$.
If $\theta \geqslant 2$ then S_θ is singular at the points of $X_I(\mathrm{id})$. When $\theta = 1$, we have $S_1 \simeq \mathbb{P}^1$.

1.1.5 For $0 \leqslant \theta' \leqslant \theta$, define

$$I_{\theta'} := \{s_1, \ldots, s_{\theta-\theta'-1}\},$$

and $w_{\theta'} := s_{\theta+1-\theta'} \ldots s_\theta$. In particular $I_0 = I$, $I_{\theta-1} = I_\theta = \varnothing$, $w_0 = \mathrm{id}$ and $w_1 = s_\theta$.

Proposition ([RTW14] 5.5)**.** *There is a stratification into locally closed subvarieties*

$$S_\theta = \bigsqcup_{\theta'=0}^{\theta} X_{I_{\theta'}}(w_{\theta'}).$$

The stratum $X_{I_{\theta'}}(w_{\theta'})$ corresponds to points U such that $\dim(U + \tau(U) + \ldots + \tau^{\theta'+1}(U)) = \theta + \theta'$. The closure in S_θ of a stratum $X_{I_{\theta'}}(w_{\theta'})$ is the union of all the strata $X_{I_t}(w_t)$ for $t \leqslant \theta'$. The stratum $X_{I_{\theta'}}(w_{\theta'})$ is of dimension θ', and $X_{I_\theta}(w_\theta)$ is open, dense and irreducible. In particular S_θ is irreducible.

Remark. This stratification plays the role of the Ekedahl-Oort stratification $\mathcal{M}_\Lambda = \bigsqcup_t \mathcal{M}_\Lambda(t)$ of the closed Bruhat-Tits strata in the unramified case, see [VW11].

1.1.6 It turns out that the strata $X_{I_{\theta'}}(w_{\theta'})$ are related to Coxeter varieties for symplectic groups of smaller sizes. For $0 \leqslant \theta' \leqslant \theta$, define

$$K_{\theta'} := \{s_1, \ldots s_{\theta-\theta'-1}, s_{\theta-\theta'+1}, \ldots, s_\theta\} = \mathbf{S} \backslash \{s_{\theta-\theta'}\}.$$

Note that $K_0 = I_0 = I$ and $K_\theta = \mathbf{S}$. We have $I_{\theta'} \subset K_{\theta'}$ with equality if and only if $\theta' = 0$.

Proposition. *There is an $\mathrm{Sp}(2\theta, \mathbb{F}_p)$-equivariant isomorphism*

$$X_{I_{\theta'}}(w_{\theta'}) \simeq \mathrm{Sp}(2\theta, \mathbb{F}_q)/U_{K_{\theta'}} \times_{L_{K_{\theta'}}} X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'}),$$

where $X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'})$ is a Deligne-Lusztig variety for $\mathbf{L}_{K_{\theta'}}$. The zero-dimensional variety $\mathrm{Sp}(2\theta, \mathbb{F}_q)/U_{K_{\theta'}}$ has a left action of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ and a right action of $L_{K_{\theta'}}$.

Proof. It is similar to [Mul22b] Proposition 8. □

1.1.7 The Levi complement $\mathbf{L}_{K_{\theta'}}$ is isomorphic to $\mathrm{GL}(\theta-\theta') \times \mathrm{Sp}(2\theta')$, and its Weyl group is isomorphic to $\mathfrak{S}_{\theta-\theta'} \times W_{\theta'}$. Via this decomposition, the permutation $w_{\theta'}$ corresponds to $\mathrm{id} \times w_{\theta'}$. The Deligne-Lusztig variety $X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'})$ decomposes as a product

$$X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'}) = X_{\mathbf{I}_{\theta'}}^{\mathrm{GL}(\theta-\theta')}(\mathrm{id}) \times X_{\varnothing}^{\mathrm{Sp}(2\theta')}(w_{\theta'}).$$

The variety $X_{\mathbf{I}_{\theta'}}^{\mathrm{GL}(\theta-\theta')}(\mathrm{id})$ is just a single point, but $X_{\varnothing}^{\mathrm{Sp}(2\theta')}(w_{\theta'})$ is the Coxeter variety for the symplectic group of size $2\theta'$. Indeed, $w_{\theta'}$ is a Coxeter element, ie. the product of all the simple reflexions of the Weyl group of $\mathrm{Sp}(2\theta')$.

1.2 Unipotent representations of the finite symplectic group

1.2.1 Recall that a (complex) irreducible representation of a finite group of Lie type $G = \mathbf{G}^F$ is said to be **unipotent**, if it occurs in the Deligne-Lusztig induction of the trivial representation of some maximal rational torus. Equivalently, it is unipotent if it occurs in the cohomology (with coefficient in $\overline{\mathbb{Q}_\ell}$ with $\ell \neq p$) of some Deligne-Lusztig variety of the form $X_{\mathbf{B}}$, with $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing a maximal rational torus.
Let $\mathbf{G}, \mathbf{G}'$ and let $f : \mathbf{G} \to \mathbf{G}'$ be an $\mathbb{F}_q$-isotypy as in 1.1.2. If $\mathbf{B}$ is such a Borel in $\mathbf{G}$, then $\mathbf{B}' := f(\mathbf{B})\mathrm{Z}(\mathbf{G}')^0$ is such a Borel in $\mathbf{B}'$, and f induces an isomorphism $X_{\mathbf{B}} \xrightarrow{\sim} X_{\mathbf{B}'}$ compatible with the actions. As a consequence, the map

$$\rho \mapsto f \circ \rho$$

defines a bijection between the sets of equivalence classes of unipotent representations of G' and of G. We will use this observation later in the case $\mathbf{G} = \mathrm{Sp}(2\theta)$ and $\mathbf{G}' = \mathrm{GSp}(2\theta)$, the symplectic group and the group of symplectic similitudes, the morphism f being the inclusion.

1.2.2 In this section, we recall the classification of the unipotent representations of the finite symplectic groups. The underlying combinatorics is described by Lusztig's notion of symbols. Our reference is [GM20] Section 4.4.

Definition. Let $\theta \geqslant 1$ and let d be an odd positive integer. The set of **symbols of rank** θ **and defect** d is

$$\mathcal{Y}^1_{d,\theta} := \left\{ S = (X,Y) \;\middle|\; \begin{array}{l} X = (x_1, \ldots, x_{r+d}) \\ Y = (y_1, \ldots, y_r) \end{array} \text{ with } x_i, y_j \in \mathbb{Z}_{\geqslant 0}, \begin{array}{l} x_{i+1} - x_i \geqslant 1, \\ y_{j+1} - y_j \geqslant 1, \end{array} \mathrm{rk}(S) = \theta \right\} \Big/ (\mathrm{shift}),$$

where the shift operation is defined by $\mathrm{shift}(X,Y) := (\{0\} \sqcup (X+1), \{0\} \sqcup (Y+1))$, and where the rank of S is given by

$$\mathrm{rk}(S) := \sum_{s \in S} s - \left\lfloor \frac{(\#S-1)^2}{4} \right\rfloor.$$

Note that the formula defining the rank is invariant under the shift operation, therefore it is well defined. By [Lus77], we have $\mathrm{rk}(S) \geqslant \left\lfloor \frac{d^2}{4} \right\rfloor$ so in particular $\mathcal{Y}^1_{d,\theta}$ is empty for d big enough. We write $\mathcal{Y}^1_\theta$ for the union of the $\mathcal{Y}^1_{d,\theta}$ with d odd, this is a finite set.

Example. In general, a symbol $S = (X, Y)$ will be written

$$S = \begin{pmatrix} x_1 & \dots & x_r & \dots & x_{r+d} \\ y_1 & \dots & y_r & & \end{pmatrix}.$$

We refer to X and Y as the first and second rows of S. The 6 elements of $\mathcal{Y}_2^1$ are given by

$$\begin{pmatrix} 2 \\ \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ & & \end{pmatrix}.$$

The last symbol has defect 3 whereas all the other symbols have defect 1.

1.2.3 The symbols can be used to classify the unipotent representations of the finite symplectic group.

Theorem ([Lus77] Theorem 8.2)**.** *There is a natural bijection between $\mathcal{Y}_\theta^1$ and the set of equivalence classes of unipotent representations of* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

If $S \in \mathcal{Y}_\theta^1$ we write ρ_S for the associated unipotent representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$. The classification is done so that the symbols

$$\begin{pmatrix} \theta \\ \end{pmatrix}, \qquad \begin{pmatrix} 0 & \dots & \theta-1 & \theta \\ 1 & \dots & \theta & \end{pmatrix},$$

correspond respectively to the trivial and the Steinberg representations.

1.2.4 Let $S = (X, Y)$ be a symbol and let $k \geqslant 1$. A k**-hook** h in S is an integer $z \geqslant k$ such that $z \in X, z-k \notin X$ or $z \in Y, z-k \notin Y$. A k**-cohook** c in S is an integer $z \geqslant k$ such that $z \in X, z-k \notin Y$ or $z \in Y, z-k \notin X$. The integer k is referred to as the **length** of the hook h or the cohook c, it is denoted $\ell(h)$ or $\ell(c)$. The **hook formula** gives an expression of $\dim(\rho_S)$ in terms of hooks and cohooks.

Proposition ([GM20] Proposition 4.4.17)**.** *We have*

$$\dim(\rho_S) = q^{a(S)} \frac{\prod_{i=1}^{\theta} (q^{2i} - 1)}{2^{b'(S)} \prod_h (q^{\ell(h)} - 1) \prod_c (q^{\ell(c)} + 1)},$$

where the products in the denominator run over all the hooks h and all the cohooks c in S, and the numbers $a(S)$ and $b'(S)$ are given by

$$a(S) = \sum_{\{s,t\} \subset S} \min(s, t) - \sum_{i \geqslant 1} \binom{\#S - 2i}{2}, \qquad b'(S) = \left\lfloor \frac{\#S - 1}{2} \right\rfloor - \#(X \cap Y).$$

1.2.5 For $\delta \geqslant 0$, we define the symbol

$$S_\delta := \begin{pmatrix} 0 & \dots & 2\delta \\ & & \end{pmatrix} \in \mathcal{Y}^1_{2\delta+1, \delta(\delta+1)}.$$

Definition. The **core** of a symbol $S \in \mathcal{Y}^1_{d,\theta}$ is defined by $\mathrm{core}(S) := S_\delta$ where $d = 2\delta + 1$. We say that S is **cuspidal** if $S = \mathrm{core}(S)$.

Remark. In general, we have $\mathrm{rk}(\mathrm{core}(S)) \leqslant \mathrm{rk}(S)$ with equality if and only if S is cuspidal.

The next theorem states that cuspidal unipotent representations correspond to cuspidal symbols.

Theorem ([GM20] Theorem 4.4.28)**.** *The group* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ *admits a cuspidal unipotent representation if and only if* $\theta = \delta(\delta+1)$ *for some* $\delta \geqslant 0$. *When this is the case, the cuspidal unipotent representation is unique and given by* ρ_{S_δ}.

1.2.6 The determination of the cuspidal unipotent representations leads to a description of the unipotent Harish-Chandra series.

Definition. Let $\delta \geqslant 0$ such that $\theta = \delta(\delta+1) + a$ for some $a \geqslant 0$. We write

$$L_\delta \simeq \mathrm{GL}(1, \mathbb{F}_q)^a \times \mathrm{Sp}(2\delta(\delta+1), \mathbb{F}_q)$$

for the block-diagonal Levi complement in $\mathrm{Sp}(2\theta, \mathbb{F}_q)$, with one middle block of size $2\delta(\delta+1)$ and other blocks of size 1. We write $\rho_\delta := (\mathbf{1})^a \boxtimes \rho_{S_\delta}$, which is a cuspidal representation of L_δ.

Proposition ([GM20] Proposition 4.4.29)**.** *Let* $S \in \mathcal{Y}^1_{\theta,d}$. *The cuspidal support of* ρ_S *is* (L_δ, ρ_δ) *where* $d = 2\delta + 1$.

In particular, the defect of the symbol S of rank θ classifies the unipotent Harish-Chandra series of $\mathrm{Sp}(2\theta, \mathbb{F}_p)$.

1.2.7 As it will be needed later, we explain how to compute a Harish-Chandra induction of the form

$$\mathrm{R}_L^G \, \mathbf{1} \boxtimes \rho_{S'},$$

where $G = \mathrm{Sp}(2\theta, \mathbb{F}_q)$, L is a block-diagonal Levi complement of the form $L \simeq \mathrm{GL}(a, \mathbb{F}_q) \times \mathrm{Sp}(2\theta', \mathbb{F}_q)$ and $S' \in \mathcal{Y}^1_{d,\theta'}$ is a symbol.

Definition. Let $S = (X, Y) \in \mathcal{Y}^1_{d,\theta}$ and let h be a k-hook of S given by some integer z. Assume that $z \in X$ and $z - k \notin X$ (resp. $z \in Y$ and $z - k \notin Y$). The **leg length** of h is given by the number of integers $s \in X$ (resp. Y) such that $z - k < s < z$.
Consider the symbol $S' = (X', Y')$ obtained by deleting z and replacing it with $z - k$ in the same row. We say that S' is obtained from S by **removing a** k**-hook**, or equivalently that S is obtained from S' by **adding a** k**-hook**.

Theorem ([FS90] Statement 4.B')**.** *Let* $S' = (X', Y') \in \mathcal{Y}^1_{d,\theta'}$. *We have*

$$\mathrm{R}_L^G \, \mathbf{1} \boxtimes \rho_{S'} = \sum_S \rho_S$$

where S *runs over all the symbols in* $\mathcal{Y}^1_{d,\theta}$ *such that, for some* $a_1, a_2 \geqslant 0$ *with* $a = a_1 + a_2$, S *is obtained from* S' *by adding an* a_1*-hook of leg length* 0 *to its first row and an* a_2*-hook of leg length* 0 *to its second row.*

This computation is a consequence of the Howlett-Lehrer comparison theorem [HL83] as well as the Pieri rule for Coxeter groups of type B, see [GP00] 6.1.9. We will use it in concrete examples in the following sections.

1.2.8 There is a similar rule to compute Harish-Chandra restrictions. Let $0 \leqslant \theta' \leqslant \theta$ and consider the embedding $G' \hookrightarrow L \hookrightarrow G$ where $G' = \mathrm{Sp}(2\theta', \mathbb{F}_q), G = \mathrm{Sp}(2\theta, \mathbb{F}_q)$ and L is the block diagonal Levi complement $\mathrm{GL}(a, \mathbb{F}_q) \times \mathrm{Sp}(2\theta', \mathbb{F}_q)$ where $a = \theta - \theta'$. We write ${}^*\mathrm{R}_{G'}^G$ for the composition of the Harish-Chandra restriction functor ${}^*\mathrm{R}_L^G$ with the usual restriction from L to G'.

Theorem. *Let $S = (X, Y) \in \mathcal{Y}^1_{d,\theta}$. We have*

$$ {}^*\mathrm{R}_{G'}^G\, \rho_S = \sum_{S'} \rho_{S'} $$

where S' runs over all the symbols in $\mathcal{Y}^1_{d,\theta'}$ such that, for some $a_1, a_2 \geqslant 0$ with $a = a_1 + a_2$, S' is obtained from S by removing an a_1-hook of leg length 0 to its first row and an a_2-hook of leg length 0 to its second row.

1.3 The cohomology of the Coxeter variety for the symplectic group

1.3.1 In this section we compute the cohomology of Coxeter varieties of finite symplectic groups, in terms of the classification of the unipotent characters that we recalled in 1.2.3.

Notation. We write $X^k := X_{\varnothing}(\mathrm{cox})$ for the Coxeter variety attached to the symplectic group $\mathrm{Sp}(2k, \mathbb{F}_q)$, and $\mathrm{H}^\bullet_c(X^k)$ instead of $\mathrm{H}^\bullet_c(X^k \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ where $\ell \neq p$.

We first recall known facts on the cohomology of X^k from Lusztig's work.

Theorem ([Lus76])**.** *The following statements hold.*

(1) *The variety X^k has dimension k and is affine. The cohomology group $\mathrm{H}^i_c(X^k)$ is zero unless $k \leqslant i \leqslant 2k$.*

(2) *The Frobenius F acts in a semisimple manner on the cohomology of X^k.*

(3) *The groups $\mathrm{H}^{2k-1}_c(X^k)$ and $\mathrm{H}^{2k}_c(X^k)$ are irreducible as $\mathrm{Sp}(2k, \mathbb{F}_q)$-representations, and the latter is the trivial representation. The Frobenius F acts with eigenvalues respectively q^{k-1} and q^k.*

(4) *The group $\mathrm{H}^{k+i}_c(X^k)$ for $0 \leqslant i \leqslant k-2$ is the direct sum of two eigenspaces of F, for the eigenvalues q^i and $-q^{i+1}$. Each eigenspace is an irreducible unipotent representation of $\mathrm{Sp}(2k, \mathbb{F}_q)$.*

(5) *The sum $\bigoplus_{i \geqslant 0} \mathrm{H}^i_c(X^k)$ is multiplicity-free as a representation of $\mathrm{Sp}(2k, \mathbb{F}_q)$.*

In other words, there exists a uniquely determined family of pairwise distinct symbols $S_0^k, \dots, S_k^k$ and $T_0^k, \dots, T_{k-2}^k$ in $\mathcal{Y}_k^1$ such that

$$\forall 0 \leqslant i \leqslant k-2, \qquad \mathrm{H}_c^{k+i}(X^k) \simeq \rho_{S_i^k} \oplus \rho_{T_i^k},$$
$$\forall k-1 \leqslant i \leqslant k, \qquad \mathrm{H}_c^{k+1}(X^k) \simeq \rho_{S_i^k}.$$

The representation $\rho_{S_i^k}$ (resp. $\rho_{T_i^k}$) corresponds to the eigenspace of the Frobenius F on $\bigoplus_{i\geqslant 0} \mathrm{H}_c^i(X^k)$ attached to p^i (resp. to $-p^{i+1}$). Moreover, we know that $\rho_{S_k^k}$ is the trivial representation, therefore

$$S_k^k = \begin{pmatrix} k \\ \end{pmatrix}.$$

Lusztig also gives a formula computing the dimension of the eigenspaces. Specializing to the case of the symplectic group, it reduces to the following statement.

Proposition ([Lus76]). *For $0 \leqslant i \leqslant k$ we have*

$$\deg(\rho_{S_i^k}) = q^{(k-i)^2} \prod_{s=1}^{k-i} \frac{q^{s+i}-1}{q^s-1} \prod_{s=0}^{k-i-1} \frac{q^{s+i}+1}{q^s+1}.$$

For $0 \leqslant j \leqslant k-2$ we have

$$\deg(\rho_{T_j^k}) = q^{(k-j-1)^2} \frac{(q^{k-1}-1)(q^k-1)}{2(q+1)} \prod_{s=1}^{k-j-2} \frac{q^{s+j}-1}{q^s-1} \prod_{s=2}^{k-j-1} \frac{q^{s+j}+1}{q^s+1}.$$

1.3.2 Our goal in this section is to determine the symbols S_i^k and T_j^k explicitly. This is done in the following proposition.

Proposition. *For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant k-2$, we have*

$$S_i^k = \begin{pmatrix} 0 \dots k-i-1 & k \\ 1 \dots \quad k-i & \end{pmatrix}, \qquad T_j^k = \begin{pmatrix} 0 & \dots & k-j-3 & k-j-2 & k-j-1 & k \\ 1 & \dots & k-j-2 & & & \end{pmatrix}.$$

We note that the statement is coherent with the two dimension formulae that we provided earlier. That is, the degree of $\rho_{S_i^k}$ (resp. of $\rho_{T_j^k}$) computed with the hook formula 1.2.4, agrees with the dimension of the eigenspace of p^i (resp. of $-p^{j+1}$) in the cohomology of X^k as given in the previous paragraph.

Proof. We use induction on $k \geqslant 0$. Since we already know that S_k^k is the symbol corresponding to the trivial representation, the proposition is proved for $k = 0$. Thus we may assume $k \geqslant 1$. We consider the block diagonal Levi complement $L \simeq \mathrm{GL}(1, \mathbb{F}_q) \times \mathrm{Sp}(2(k-1), \mathbb{F}_q)$, and we write ${}^*\mathrm{R}_{k-1}^k$ for the composition of the Harish-Chandra restriction from $\mathrm{Sp}(2k, \mathbb{F}_q)$ to L, with the usual restriction from L to $\mathrm{Sp}(2(k-1), \mathbb{F}_q)$. As in the proof of [Mul22b] Proposition 19, for all $0 \leqslant i \leqslant k$ we have an $\mathrm{Sp}(2(k-1), \mathbb{F}_q) \times \langle F \rangle$-equivariant isomorphism

$${}^*\mathrm{R}_{k-1}^k \left(\mathrm{H}_c^{k+i}(X^k)\right) \simeq \mathrm{H}_c^{k-1+i}(X^{k-1}) \oplus \mathrm{H}_c^{k-1+(i-1)}(X^{k-1})(1). \qquad (*)$$

Here, (1) denotes the Tate twist. This recursive formula is established by Lusztig in [Lus76] Corollary 2.10. The right-hand side is known by induction hypothesis whereas the left-hand side can be computed using 1.2.8 Theorem. We establish the proposition by comparing the different eigenspaces of F on both sides.
If $S \in \mathcal{Y}^1_{d,k}$ is any symbol, the restriction ${}^*\mathrm{R}^k_{k-1}\,\rho_S$ is the sum of all the representations $\rho_{S'}$ where S' is obtained from S by removing a 1-hook from any of its rows.
We distinguish different cases depending on the values of k and i.

- **Case k = 1**. We only need to determine S^1_0. For $i = 0$, the right-hand side of $(*)$ is $\rho_{S^0_0}$ with eigenvalue 1. Thus, the symbol $S^1_0 \in \mathcal{Y}^1_1$ has defect 1 and admits only one 1-hook. If we remove this hook we obtain S^0_0. Therefore, S^1_0 must be one of the two following symbols

$$\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ \ \end{pmatrix}.$$

  By 1.3.1, we know that $\rho_{S^1_0}$ has degree q, thus S^1_0 must be equal to the former symbol.

From now, we assume $k \geqslant 2$ and we determine S^k_i for $0 \leqslant i < k$.

- **Case k = 2 and i = 0**. The eigenspace attached to 1 on the right-hand side of $(*)$ is $\rho_{S^1_0}$. Thus, the symbol $S^2_0 \in \mathcal{Y}^1_k$ has defect 1 and admits only one 1-hook. If we remove this hook we obtain S^1_0. Therefore, S^2_0 must be one of the two following symbols

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}.$$

  By 1.3.1, we know that $\rho_{S^2_0}$ has degree q^4, thus S^2_0 must be equal to the former symbol.
- **Case k > 2 and i = 0**. The eigenspace attached to 1 on the right-hand side of $(*)$ is $\rho_{S^{k-1}_0}$. Thus, the symbol $S^k_0 \in \mathcal{Y}^1_k$ has defect 1 and admits only one 1-hook. If we remove this hook we obtain S^{k-1}_0. The only such symbol is

$$S^k_0 = \begin{pmatrix} 0 & \dots & k-1 & k \\ 1 & \dots & k & \end{pmatrix}.$$

- **Case $1 \leqslant \mathbf{i} \leqslant \mathbf{k}-1$**. The eigenspace attached to p^i on the right-hand side of $(*)$ is $\rho_{S^{k-1}_i} \oplus \rho_{S^{k-1}_{i-1}}$. Thus, the symbol $S^k_i \in \mathcal{Y}^1_k$ has defect 1 and admits only two 1-hooks. If we remove one of these hooks we obtain either S^{k-1}_i or S^{k-1}_{i-1}. The only such symbol is

$$S^k_i = \begin{pmatrix} 0 & \dots & k-i-1 & k \\ 1 & \dots & k-i & \end{pmatrix}.$$

It remains to determine T^k_j for $0 \leqslant j \leqslant k-2$.

- **Case k = 2**. The eigenspace attached to $-p$ on the right-hand side of $(*)$ is 0. Thus, the symbol $T^2_0 \in \mathcal{Y}^1_2$ has no hook at all, implying that it is cuspidal in the sense of 1.2.5. Since $\mathrm{Sp}(4, \mathbb{F}_q)$ admits only 1 unipotent cuspidal representation, we deduce that

$$T^2_0 = \begin{pmatrix} 0 & 1 & 2 \\ & & \end{pmatrix}.$$

– **Case k = 3**. First when $j = 0$, the eigenspace attached to $-p$ on the right-hand side of $(*)$ is $\rho_{T_0^2}$. Thus, the symbol $T_0^3 \in \mathcal{Y}_3^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^2. Therefore, T_0^3 must be one of the two following symbols

$$\begin{pmatrix} 0\ 1\ 2\ 3 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 3 \\ & & \end{pmatrix}.$$

By 1.3.1, we know that $\rho_{T_0^3}$ has degree $q^4 \frac{(q^2-1)(q^3-1)}{2(q+1)}$, thus T_0^3 must be equal to the former symbol.
Then when $j = 1$, the eigenspace attached to $-p^2$ on the right-hand side of $(*)$ is $\rho_{T_0^2}$. Thus, the symbol $T_1^3 \in \mathcal{Y}_3^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^2. Thus T_1^3 is also one of the two symbols above. We can deduce that it is equal to the latter by comparing the dimensions or by using the fact that the symbols T_j^k are pairwise distinct.

From now, we assume $k \geqslant 4$ and we determine T_j^k for $0 \leqslant j \leqslant k-2$.

– **Case k = 4 and j = 0**. The eigenspace attached to $-p$ on the right-hand side of $(*)$ is $\rho_{T_0^3}$. Thus, the symbol $T_0^4 \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^3. Therefore, T_0^4 must be one of the two following symbols

$$\begin{pmatrix} 0\ 1\ 2\ 3\ 4 \\ 1\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & & & \end{pmatrix}.$$

By 1.3.1, we know that $\rho_{T_0^4}$ has degree $q^9 \frac{(q^3-1)(q^4-1)}{2(q+1)}$, thus T_0^4 must be equal to the former symbol.

– **Case k > 4 and j = 0**. The eigenspace attached to $-p$ on the right-hand side of $(*)$ is $\rho_{T_0^{k-1}}$. Thus, the symbol $T_0^k \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^{k-1}. The only such symbol is

$$T_0^k = \begin{pmatrix} 0 \ \dots \ k-3 \ k-2 \ k-1 \ k \\ 1 \ \dots \ k-2 \end{pmatrix}.$$

– **Case k = 4 and j = k − 2**. The eigenspace attached to $-p^3$ on the right-hand side of $(*)$ is $\rho_{T_1^3}$. Thus, the symbol $T_2^4 \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_1^3. Therefore, T_2^4 must be one of the two following symbols

$$\begin{pmatrix} 0\ 1\ 4 \\ \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 & 3 \\ & & \end{pmatrix}.$$

By 1.3.1, we know that $\rho_{T_2^4}$ has degree $q\frac{(q^3-1)(q^4-1)}{2(q+1)}$, thus T_2^4 must be equal to the former symbol.

– **Case k > 4 and j = k − 2**. The eigenspace attached to $-p^{k-1}$ on the right-hand side of $(*)$ is $\rho_{T_{k-3}^{k-1}}$. Thus, the symbol $T_{k-2}^k \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_{k-3}^{k-1}. The only such symbol is

$$T_{k-2}^k = \begin{pmatrix} 0\ 1\ k \\ \end{pmatrix}.$$

– **Case $\mathbf{1 \leqslant j \leqslant k-3}$**. The eigenspace attached to $-p^{j+1}$ on the right-hand side of $(*)$ is $\rho_{T_j^{k-1}} \oplus \rho_{T_{j-1}^{k-1}}$. Thus, the symbol $T_j^k \in \mathcal{Y}_k^1$ has defect 3 and admits only two 1-hooks. If we remove one of these hooks we obtain either T_j^{k-1} or T_{j-1}^{k-1}. The only such symbol is

$$T_j^k = \begin{pmatrix} 0 & \ldots & k-j-3 & k-j-2 & k-j-1 & k \\ 1 & \ldots & k-j-2 & & & \end{pmatrix}.$$

□

1.4 On the cohomology of a closed Bruhat-Tits stratum

1.4.1 Recall from 1.1.4 the θ-dimensional normal projective variety $S_\theta := \overline{X_I(s_\theta)}$ defined over $\mathbb{F}_q$. It is equipped with an action of the finite symplectic group $\mathrm{Sp}(2\theta, \mathbb{F}_q)$. We use the stratification of 1.1.5 Proposition to study its cohomology over $\overline{\mathbb{Q}_\ell}$. If λ is a scalar, we write $\mathrm{H}_c^\bullet(S_\theta)_\lambda$ to denote the eigenspace of the Frobenius F associated to λ (we do not in principle assume the eigenspace to be non zero). We give a series of statements before proving all of them at once in the remaining of this section.

Proposition. *The Frobenius F acts semi-simply on $\mathrm{H}_c^\bullet(S_\theta)$. Its eigenvalues form a subset of*

$$\{q^i \,|\, 0 \leqslant i \leqslant \theta\} \cup \{-q^{j+1} \,|\, 0 \leqslant j \leqslant \theta - 2\}.$$

1.4.2 In a first statement, we give our results regarding the eigenspaces attached to a scalar of the form q^i for some i. Recall from 1.2.6 the cuspidal supports (L_δ, ρ_δ) for the finite symplectic group $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

Theorem. *Let $0 \leqslant i \leqslant \theta$ and $\theta' \in \mathbb{Z}$.*

(1) *The eigenspace $\mathrm{H}_c^{\theta'+i}(S_\theta)_{q^i}$ is zero when $\theta' < i$ or $\theta' > \theta$.*

We now assume that $0 \leqslant i \leqslant \theta' \leqslant \theta$.

(2) *All the irreducible representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ in the eigenspace $\mathrm{H}_c^{\theta'+i}(S_\theta)_{q^i}$ belong to the unipotent principal series, ie. they have cuspidal support (L_0, ρ_0).*

(3) *We have*

$$\mathrm{H}_c^0(S_\theta) = \mathrm{H}_c^0(S_\theta)_1 \simeq \rho_{\binom{\theta}{}}, \qquad \mathrm{H}_c^{2\theta}(S_\theta) = \mathrm{H}_c^{2\theta}(S_\theta)_{q^\theta} \simeq \rho_{\binom{\theta}{}}.$$

(4) *If $i+2 \leqslant \theta'$ then*

$$\bigoplus_{0 \leqslant d \leqslant \theta-\theta'-1} \rho_{\begin{pmatrix} 0 & \ldots & \theta'-i-2 & \theta'-i-1 & \theta'+d \\ 1 & \ldots & \theta'-i-1 & \theta-i-d & \end{pmatrix}} \oplus \bigoplus_{\substack{1 \leqslant d \leqslant \\ \min(i, \theta-\theta'-1)}} \rho_{\begin{pmatrix} 0 & \ldots & \theta'-i-2 & \theta'-i-1+d & \theta' \\ 1 & \ldots & \theta'-i-1 & \theta-i-d & \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{\theta'+i}(S_\theta)_{q^i}.$$

The cokernel of this map consists of at most 4 irreducible representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

(5) *When* $i = \theta' \neq \theta$, *we have*

$$\rho_{\begin{pmatrix}\theta\\ \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{2i}(S_\theta)_{q^i} \text{ if } 2i < \theta, \qquad \rho_{\begin{pmatrix}\theta\\ \end{pmatrix}} \oplus \rho_{\begin{pmatrix}\theta - i & i+1\\ 0 & \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{2i}(S_\theta)_{q^i} \text{ if } 2i \geqslant \theta.$$

(6) *When* $\theta' = \theta$ *we have*

$$\mathrm{H}_c^{\theta+i}(S_\theta)_{q^i} \simeq 0 \text{ or } \rho_{\begin{pmatrix}0 & \ldots & \theta-i-1 & \theta\\ 1 & \ldots & \theta-i & \end{pmatrix}}.$$

(7) *When* $\theta' = 1$ *and* $i = 0$, *we have*

$$\mathrm{H}_c^1(S_1) = 0, \qquad \mathrm{H}_c^1(S_\theta) = \mathrm{H}_c^1(S_\theta)_1 \simeq 0 \text{ or } \rho_{\begin{pmatrix}0 & 1 & \theta\\ 1 & 2 & \end{pmatrix}} \text{ when } \theta \geqslant 2.$$

We note that when $\theta' = \theta$, the formula of (4) does not say anything about the eigenspace $\mathrm{H}_c^{\theta+i}(S_\theta)_{q^i}$ since the sums are empty. However, by (6) we understand that this eigenspace is either 0 either irreducible.
We note also that the theorem does not give any information in the case $i + 1 = \theta'$, except when $\theta' = 1$ and $i = 0$ which corresponds to (7).

1.4.3 In a second statement, we give our results regarding the eigenspaces attached to a scalar of the form $-q^{j+1}$ for some j.

Theorem. *Let* $0 \leqslant j \leqslant \theta - 2$ *and* $\theta' \in \mathbb{Z}$.

(1) *The eigenspace* $\mathrm{H}_c^{\theta'+j}(S_\theta)_{-q^{j+1}}$ *is zero when* $\theta' < j + 2$ *or* $\theta' > \theta$.

We now assume that $2 \leqslant j + 2 \leqslant \theta' \leqslant \theta$.

(2) *All the irreducible representations of* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ *in the eigenspace* $\mathrm{H}_c^{\theta'+j}(S_\theta)_{-q^{j+1}}$ *are unipotent with cuspidal support* (L_1, ρ_1).

(3) *We have*

$$\mathrm{H}_c^{2\theta-2}(S_\theta)_{-q^{\theta-1}} \simeq \rho_{\begin{pmatrix}0 & 1 & \theta\\ \end{pmatrix}}.$$

(4) *If* $j + 4 \leqslant \theta' \leqslant \theta$ *then*

$$\bigoplus_{0 \leqslant d \leqslant \theta-\theta'-1} \rho_{\begin{pmatrix}0 & \ldots & \theta'-i-4 & \theta'-i-3 & \theta'-j-2 & \theta'-j-1 & \theta'+d\\ 1 & \ldots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix}} \oplus$$
$$\bigoplus_{\substack{1 \leqslant d \leqslant \\ \min(i, \theta-\theta'-1)}} \rho_{\begin{pmatrix}0 & \ldots & \theta'-i-4 & \theta'-i-3 & \theta'-j-2 & \theta'-j-1+d & \theta'\\ 1 & \ldots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{\theta'+j}(S_\theta)_{-q^{j+1}}.$$

The cokernel of this map consists of at most 4 *irreducible representations of* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

(5) *When* $j + 2 = \theta' \neq \theta$, *we have*

$$\rho_{\begin{pmatrix}0 & 1 & \theta\\ \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{2(j+1)}(S_\theta)_{-q^{j+1}} \qquad \text{if } 2(j+1) < \theta,$$
$$\rho_{\begin{pmatrix}0 & 1 & \theta\\ \end{pmatrix}} \oplus \rho_{\begin{pmatrix}0 & \theta-i-1 & i+2\\ \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{2(j+1)}(S_\theta)_{-q^{j+1}} \qquad \text{if } 2(j+1) \geqslant \theta.$$

(6) *When* $\theta' = \theta$ *we have*

$$\mathrm{H}_c^{\theta+j}(S_\theta)_{-q^{j+1}} \simeq 0 \text{ or } \rho_{\begin{pmatrix} 0 & \dots & \theta-j-3 & \theta-j-2 & \theta-j-1 & \theta \\ 1 & \dots & \theta-j-2 & & & \end{pmatrix}}.$$

We note that when $\theta' = \theta$, the formula of (4) does not say anything about the eigenspace $\mathrm{H}_c^{\theta+j}(S_\theta)_{-q^{j+1}}$ since the sums are empty. However, by (6) we understand that this eigenspace is either 0 either irreducible.
We note also that the theorem does not give any information in the case $j + 3 = \theta'$.

Remark. A cuspidal representation occurs in the cohomology of S_θ only in the cases $\theta = 0$ and $\theta = 2$. When $\theta = 0$ it corresponds to $\mathrm{H}_c^0(S_0)$ which is trivial. When $\theta = 2$ it corresponds to $\mathrm{H}_c^2(S_2)_{-q}$ as described by (3) in the theorem above.

1.4.4 The remaining of this section is dedicated to proving the theorems stated above. Recall from 1.1.5 that we have a stratification $S_\theta = \bigsqcup_{\theta'=0}^{\theta} X_{I_{\theta'}}(w_{\theta'})$. It induces a spectral sequence on the cohomology whose first page is given by

$$E_1^{a,b} = \mathrm{H}_c^{a+b}(X_{I_a}(w_a)) \implies \mathrm{H}_c^{a+b}(S_\theta). \tag{E}$$

Now, recall that the strata $X_{I_{\theta'}}(w_{\theta'})$ are related to Coxeter varieties for the finite symplectic group $\mathrm{Sp}(2\theta', \mathbb{F}_q)$. Using 1.1.7, the geometric isomorphism given in 1.1.6 Proposition induces an isomorphism on the cohomology

$$\mathrm{H}_c^\bullet(X_{I_{\theta'}}(w_{\theta'})) \simeq \mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathbf{1} \boxtimes \mathrm{H}_c^\bullet(X^{\mathrm{Sp}(2\theta')}(w_{\theta'})), \tag{$**$}$$

where $L_{K_{\theta'}}$ denotes the block-diagonal Levi complement isomorphic to $\mathrm{GL}(\theta-\theta', \mathbb{F}_q) \times \mathrm{Sp}(2\theta', \mathbb{F}_q)$. The variety $X^{\mathrm{Sp}(2\theta')}(w_{\theta'})$ is nothing but the Coxeter variety that we denoted by $X^{k'}$ in 1.3.1, and whose cohomology we have described. For $0 \leqslant i \leqslant \theta'$ and $0 \leqslant j \leqslant \theta' - 2$, recall from 1.3.2 the symbols $S_i^{\theta'}$ and $T_j^{\theta'}$. We define

$$\mathrm{R}_{i,\theta'}^S := \mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{S_i^{\theta'}}, \qquad \mathrm{R}_{j,\theta'}^T := \mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{T_j^{\theta'}}.$$

Then by ($**$), we have

$$\mathrm{H}_c^{\theta'+i}(X_{I_{\theta'}}(w_{\theta'})) \simeq \mathrm{R}_{i,\theta'}^S \oplus \mathrm{R}_{i,\theta'}^T \qquad \forall 0 \leqslant i \leqslant \theta' - 2,$$
$$\mathrm{H}_c^{\theta'+i}(X_{I_{\theta'}}(w_{\theta'})) \simeq \mathrm{R}_{i,\theta'}^S \qquad \forall \theta' - 1 \leqslant i \leqslant \theta'.$$

The cohomology groups of other degrees vanish. The representation $\mathrm{R}_{i,\theta'}^S$ corresponds to the eigenvalue q^i of F, whereas $\mathrm{R}_{j,\theta'}^T$ corresponds to $-q^{j+1}$.

Lemma. *Let* $0 \leqslant \theta' \leqslant \theta$, $0 \leqslant i \leqslant \theta'$ *and* $0 \leqslant j \leqslant \theta' - 2$.

– *If* $i < \theta'$, *the representation* $\mathrm{R}_{i,\theta'}^S$ *is the multiplicity-free sum of the unipotent representations* ρ_S *where* $S \in \mathcal{Y}_{1,\theta}^1$ *runs over the following* 4 *distinct families of symbols*

(S1) $\begin{pmatrix} 0 & \dots & \theta'-i-2 & \theta'-i-1 & \theta'+d \\ 1 & \dots & \theta'-i-1 & \theta-i-d & \end{pmatrix}$ $\forall 0 \leqslant d \leqslant \theta-\theta'$,

(S2) $\begin{pmatrix} 0 & \dots & \theta'-i-2 & \theta'-i-1+d & \theta' \\ 1 & \dots & \theta'-i-1 & \theta-i-d & \end{pmatrix}$ $\forall 1 \leqslant d \leqslant \min(i, \theta-\theta')$,

(S Exc 1) $\begin{pmatrix} 0 & \dots & \theta'-i-1 & \theta'-i & \theta \\ 1 & \dots & \theta'-i & \theta'-i+1 & \end{pmatrix}$ *if* $\theta' \neq \theta$,

(S Exc 2) $\begin{pmatrix} 0 & \dots & \theta'-i-1 & \theta-i-1 & \theta'+1 \\ 1 & \dots & \theta'-i & \theta'-i+1 & \end{pmatrix}$ *if* $\theta' \neq \theta, \theta-1$ *and* $\theta \leqslant \theta'+i+1$.

– *The representation* $\mathrm{R}^{S}_{\theta',\theta'}$ *is the multiplicity-free sum of the unipotent representations* ρ_S *where* $S \in \mathcal{Y}^1_{1,\theta}$ *runs over the following* 2 *distinct families of symbols*

(S1') $\begin{pmatrix} 0 & \theta'+1+d \\ \theta-\theta'-d & \end{pmatrix}$ $\forall 0 \leqslant d \leqslant \theta-\theta'$,

(S2') $\begin{pmatrix} d & \theta'+1 \\ \theta-\theta'-d & \end{pmatrix}$ $\forall 1 \leqslant d \leqslant \min(\theta', \theta-\theta')$.

– *If* $j+2 < \theta'$, *the representation* $\mathrm{R}^{T}_{j,\theta'}$ *is the multiplicity-free sum of the unipotent representations* ρ_T *where* $T \in \mathcal{Y}^1_{3,\theta}$ *runs over the following* 4 *distinct families of symbols*

(T1) $\begin{pmatrix} 0 & \dots & \theta'-j-4 & \theta'-j-3 & \theta'-j-2 & \theta'-j-1 & \theta'+d \\ 1 & \dots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix}$ $\forall 0 \leqslant d \leqslant \theta-\theta'$,

(T2) $\begin{pmatrix} 0 & \dots & \theta'-j-4 & \theta'-j-3 & \theta'-j-2 & \theta'-j-1+d & \theta' \\ 1 & \dots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix}$ $\forall 1 \leqslant d \leqslant \min(j, \theta-\theta')$,

(T Exc 1) $\begin{pmatrix} 0 & \dots & \theta'-j-2 & \theta'-j-1 & \theta'-j & \theta \\ 1 & \dots & \theta'-j-1 & & & \end{pmatrix}$ *if* $\theta' \neq \theta$,

(T Exc 2) $\begin{pmatrix} 0 & \dots & \theta'-j-2 & \theta'-j-1 & \theta-j-1 & \theta'+1 \\ 1 & \dots & \theta'-j-1 & & & \end{pmatrix}$ *if* $\theta' \neq \theta, \theta-1$ *and* $\theta \leqslant \theta'+j+1$.

– *The representation* $\mathrm{R}^{T}_{\theta'-2,\theta'}$ *is the multiplicity-free sum of the unipotent representations* ρ_T *where* $T \in \mathcal{Y}^1_{3,\theta}$ *runs over the following* 2 *distinct families of symbols*

(T1') $\begin{pmatrix} 0 & 1 & 2 & \theta'+1+d \\ \theta-\theta'-d & & & \end{pmatrix}$ $\forall 0 \leqslant d \leqslant \theta-\theta'$,

(T2') $\begin{pmatrix} 0 & 1 & 2+d & \theta'+1 \\ \theta-\theta'-d & & & \end{pmatrix}$ $\forall 1 \leqslant d \leqslant \min(\theta'-2, \theta-\theta')$.

This lemma results directly from the computational rule explained in 1.2.7. In concrete terms, an induction of the form

$$\mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{S'}$$

is the sum of all the representations ρ_S where S is obtained from S' by adding a hook of leg length 0 to both rows, whose lengths sum to $\theta - \theta'$. We illustrate the arguments by looking at a concrete example.

With $\theta = 6, \theta' = 3$ and $i = 2$ let us explain the computation of

$$\mathrm{R}_{2,3}^{S} = \mathrm{R}_{L_{K_3}}^{\mathrm{Sp}(12,\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{S_2^3}.$$

Recall that

$$S_2^3 = \begin{pmatrix} 0 & 3 \\ 1 & \end{pmatrix}.$$

For $0 \leqslant d \leqslant \theta - \theta' = 3$, we add a d-hook of leg length 0 to the first row of S_2^3, and a $(3-d)$-hook of leg length 0 to its second row.

We may always add the hooks to the last entries of each row. By doing so we obtain the representations corresponding to the family of symbols (S1):

$$\begin{pmatrix} 0 & 3 \\ 4 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 4 \\ 3 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 5 \\ 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 6 \\ 1 & \end{pmatrix}.$$

When $d \leqslant \min(\theta - \theta', i) = \min(3, 2) = 2$, we may also add the first hook to the penultimate entry of the first row. Note that since $i < \theta'$, the first row of $S_i^{\theta'}$ has at least 2 entries. By doing so, we obtain the representations corresponding to the family of symbols (S2):

$$\begin{pmatrix} 1 & 3 \\ 3 & \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 \\ 2 & \end{pmatrix}.$$

Now, recall that symbols are equal up to shifts. Therefore, one may rewrite S_2^3 as

$$S_2^3 = \mathrm{shift}(S_2^3) = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 2 & \end{pmatrix}.$$

Written this way, we notice that a 1-hook can be added to the first entry of the second row, which is a 0. Then one must add to the first row a hook of length $d = \theta - \theta' - 1 = 2$. One may always add it to the last entry, which results in the first "exceptional" representation (S Exc 1). Moreover if $d \leqslant i$, which is the case here, one may also add this hook to the penultimate entry of the first row, which leads to the second "exceptional" representation (S Exc 2):

$$\begin{pmatrix} 0 & 1 & 6 \\ 1 & 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 2 & \end{pmatrix}.$$

The sum of the representations attached to all the 8 symbols written above is isomorphic to $\mathrm{R}_{2,3}^{S}$.

We also explain in detail the special case $i = \theta$. Thus we compute

$$\mathrm{R}^S_{\theta,\theta} = \mathrm{R}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}_{L_{K_{\theta'}}} \mathbf{1} \boxtimes \rho_{S^{\theta'}_{\theta'}}.$$

Recall that

$$S^{\theta'}_{\theta'} = \begin{pmatrix} \theta' \\ \; \end{pmatrix}$$

corresponds to the trivial representation of $\mathrm{Sp}(2\theta', \mathbb{F}_q)$. In order to compute this induction, we shift the symbol $S^{\theta'}_{\theta'}$ first:

$$S^{\theta'}_{\theta'} = \begin{pmatrix} 0 & \theta'+1 \\ 0 & \end{pmatrix}.$$

For $0 \leqslant d \leqslant \theta - \theta'$, we add a d-hook of leg length 0 to the first row and a $(\theta - \theta' - d)$-hook of leg length 0 to the second row. We may always add the hooks to the last entries of each row. By doing so, we obtain the representations corresponding to the family of symbols $(S1')$. Moreover when $d \leqslant \min(\theta', \theta - \theta')$, we may also add the first hook to the 0 in the first row. It leads to the representations corresponding to the family of symbols $(S2')$.
In particular, we notice that the symbol of $(S1')$ with $d = \theta - \theta'$ corresponds to the trivial representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

1.4.5 Now, we have an explicit description of the terms $E_1^{a,b}$ in the first page of the spectral sequence (E). In the Figure 1, we draw the shape of the first page.

$$\begin{array}{cccccccccc}
 & & & & & & & & & \mathrm{R}^S_{\theta,\theta} \\
 & & & & & & & \mathrm{R}^S_{\theta-1,\theta-1} & \longrightarrow & \mathrm{R}^S_{\theta-1,\theta} \\
 & & & & & \mathrm{R}^S_{\theta-2,\theta-2} & \longrightarrow & \mathrm{R}^S_{\theta-2,\theta-1} & \longrightarrow & \mathrm{R}^S_{\theta-2,\theta} \oplus \mathrm{R}^T_{\theta-2,\theta} \\
 & & & & \iddots & & & & & \vdots \\
 & & \mathrm{R}^S_{2,2} & \longrightarrow & \dots \longrightarrow & \mathrm{R}^S_{2,\theta-2} \oplus \mathrm{R}^T_{2,\theta-2} & \longrightarrow & \mathrm{R}^S_{2,\theta-1} \oplus \mathrm{R}^T_{2,\theta-1} & \longrightarrow & \mathrm{R}^S_{2,\theta} \oplus \mathrm{R}^T_{2,\theta} \\
 & \mathrm{R}^S_{1,1} \longrightarrow & \mathrm{R}^S_{1,2} & \longrightarrow & \dots \longrightarrow & \mathrm{R}^S_{1,\theta-2} \oplus \mathrm{R}^T_{1,\theta-2} & \longrightarrow & \mathrm{R}^S_{1,\theta-1} \oplus \mathrm{R}^T_{1,\theta-1} & \longrightarrow & \mathrm{R}^S_{1,\theta} \oplus \mathrm{R}^T_{1,\theta} \\
\mathrm{R}^S_{0,0} \longrightarrow & \mathrm{R}^S_{0,1} \longrightarrow & \mathrm{R}^S_{0,2} \oplus \mathrm{R}^T_{0,2} & \longrightarrow & \dots \longrightarrow & \mathrm{R}^S_{0,\theta-2} \oplus \mathrm{R}^T_{0,\theta-2} & \longrightarrow & \mathrm{R}^S_{0,\theta-1} \oplus \mathrm{R}^T_{0,\theta-1} & \longrightarrow & \mathrm{R}^S_{0,\theta} \oplus \mathrm{R}^T_{0,\theta}
\end{array}$$

Figure 1: The first page of the spectral sequence.

First, since the Frobenius F acts with the eigenvalue q^i (resp. $-q^{j+1}$) on the representations $\mathrm{R}^S_{i,\theta'}$ (resp. $\mathrm{R}^T_{j,\theta'}$), 1.4.1 Proposition as well as point (1) of 1.4.2 and 1.4.3 Theorems follow from the triangular shape of the spectral sequence. Point (2) also follows from 1.4.4 Lemma.
Next, we notice that on the b-th row of the first page E_1, the eigenvalues of F which occur are

q^b and $-q^{b+1}$. In particular, the eigenvalues on different rows are all distinct. It follows that all the arrows in the deeper pages of the sequence are zero, therefore it degenerates on the second page. Moreover, the filtration induced by the spectral sequence on the abutment splits, so that $\mathrm{H}_c^k(S_\theta)$ is isomorphic to the direct sum of the terms $E_2^{k-b,b}$ on the k-th diagonal of the second page.

We prove point (3) of 1.4.2 and 1.4.3 Theorems. By the shape of the spectral sequence, we see that

$$\mathrm{H}_c^{2\theta}(S_\theta) = \mathrm{H}_c^{2\theta}(S_\theta)_{q^\theta} \simeq \mathrm{R}^S_{\theta,\theta} \simeq \rho_{\begin{pmatrix} \theta \\ \end{pmatrix}}, \qquad \mathrm{H}_c^{2\theta-2}(S_\theta)_{-q^{\theta-1}} \simeq \mathrm{R}^T_{\theta-2,\theta} \simeq \rho_{\begin{pmatrix} 0 & 1 & \theta \\ \end{pmatrix}}.$$

Moreover, by the spectral sequence we know that $\mathrm{H}_c^0(S_\theta)$ is a subspace of $\mathrm{R}^S_{0,0}$, thus the Frobenius F acts like the identity. Since S_θ is projective and irreducible, the cohomology group $\mathrm{H}_c^0(S_\theta) = \mathrm{H}^0(S_\theta)$ is trivial.

We now prove point (4) of 1.4.2 and 1.4.3 Theorems. Let $2 \leqslant i+2 \leqslant \theta' \leqslant \theta - 1$. By extracting the eigenvalue q^i in the spectral sequence, we have a chain

$$\ldots \longrightarrow \mathrm{R}^S_{i,\theta'-1} \xrightarrow{u} \mathrm{R}^S_{i,\theta'} \xrightarrow{v} \mathrm{R}^S_{i,\theta'+1} \longrightarrow \ldots$$

The quotient $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is isomorphic to the eigenspace $\mathrm{H}_c^{\theta'+i}(S_\theta)_{q^i}$.
The middle term $\mathrm{R}^S_{i,\theta'}$ is the sum of the representations ρ_S where S runs over the families of symbols (S1), (S2), (S Exc 1) and (S Exc 2) as in 1.4.4 Lemma. All these symbols are written in their "reduced" form, meaning that they can not be written as the shift of another symbol. Let us look at the length of the second row of these symbols. If S belongs to (S1) or (S2), then the second row has length $\theta' - i$. If S belongs to (S Exc 1) or (S Exc 2), then the second row has length $\theta' - i + 1$.
We may do a similar analysis for the left term (resp. the right term) by replacing θ' with $\theta' - 1$ (resp. $\theta' + 1$). In the left term $\mathrm{R}^S_{i,\theta'-1}$, all the representations corresponding to the families (S1) and (S2) have second row of length $\theta' - i - 1$. No such representation occurs in the middle term, therefore they all automatically lie in the $\mathrm{Ker}(u)$. Then, in the left term the representation corresponding to (S Exc 1) occurs since $\theta' - 1 \neq \theta$. We observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'}$ with S in the family (S1) and $d = \theta - \theta'$. Further, assume that $\theta \leqslant \theta' + i$ so that the representation corresponding to (S Exc 2) occurs in $\mathrm{R}^S_{i,\theta'-1}$. Then we observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'}$ with S in the family (S2) and $d = \theta - \theta' = \min(i, \theta - \theta')$. Hence, it follows that $\mathrm{Im}(u)$ consists of at most 2 irreducible subrepresentations of $\mathrm{R}^S_{i,\theta'}$, and they correspond to the symbols of (S1) and (S2) with $d = \theta - \theta'$.
Next, all the subrepresentations ρ_S of $\mathrm{R}^S_{i,\theta'}$ with S in (S1) or (S2) belong to $\mathrm{Ker}(v)$, since no component of $\mathrm{R}^S_{i,\theta'+1}$ correspond to a symbol whose second row has length $\theta' - i$. Since $\theta' \neq \theta$, the represensation corresponding to (S Exc 1) occurs in $\mathrm{R}^S_{i,\theta'}$. We observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'+1}$ with S in the family (S1) and $d = \theta - \theta' - 1$. Assume

that $\theta' \leqslant \theta - 2$ and $\theta \leqslant \theta' + i + 1$, so that the representation corresponding to (S Exc 2) occurs in $\mathrm{R}^S_{i,\theta'}$. Then we observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'+1}$ with S in the family (S2) and $d = \theta - \theta' - 1 = \min(i, \theta - \theta' - 1)$. Therefore, it is not possible to tell whether the components of $\mathrm{R}^S_{i,\theta'}$ corresponding to (S Exc 1) and (S Exc 2) are in $\mathrm{Ker}(v)$ or not. In all cases, we conclude that $\mathrm{Ker}(v)/\mathrm{Im}(u)$ contains at least all the representations corresponding to the symbols S in (S1) and (S2) with $d < \theta - \theta'$. With this description we miss up to four irreducible representations, which correspond to (S1) and (S2) with $d = \theta - \theta'$, (S Exc 1) and (S Exc 2). This proves point (4) of 1.4.2 Theorem.
The point (4) of 1.4.3 Theorem is proved by identical arguments.

We now prove point (5) of 1.4.2 and 1.4.3 Theorems. We consider $i = \theta' \neq \theta$. By extracting the eigenvalue q^i in the spectral sequence, we have a chain

$$\mathrm{R}^S_{i,i} \xrightarrow{u} \mathrm{R}^S_{i,i+1} \longrightarrow \dots$$

The kernel $\mathrm{Ker}(u)$ is isomorphic to the eigenspace $\mathrm{H}^{2i}_c(S_\theta)_{q^i}$. The left term $\mathrm{R}^S_{i,i}$ is the sum of the representations $\rho_{S'}$ where S' runs over the families of symbols (S1') and (S2'). We observe that the representation $\rho_{S'}$ with S' in (S1') corresponding to some $0 \leqslant d' \leqslant \theta - i - 1$ is equivalent to the component ρ_S of $\mathrm{R}^S_{i,i+1}$ with S in (S1) corresponding to $d = d'$. Similarly, we observe that the representation $\rho_{S'}$ with S' in (S2') corresponding to some $1 \leqslant d' \leqslant \min(i, \theta - i - 1)$ is equivalent to the component ρ_S of $\mathrm{R}^S_{i,i+1}$ with S in (S2) corresponding to $d = d'$.
Therefore, the representation ρ_S corresponding to S in (S1') with $d' = \theta - i$ belongs to $\mathrm{Ker}(u)$. This is no other than the trivial representation. Moreover, if $\min(i, \theta - i - 1) \neq \min(i, \theta - i)$, ie. if $2i \geqslant \theta$, then the representation ρ_S corresponding to S in (S2') with $d' = \theta - i$ also belongs to $\mathrm{Ker}(u)$. This proves point (5) of 1.4.2 Theorem.
The point (5) of 1.4.3 Theorem is proved by identical arguments.

Points (6) of 1.4.2 and 1.4.3 Theorems follows easily from the shape of the spectral sequence. Indeed, it suffices to notice that all the terms $\mathrm{R}^S_{i,\theta}$ and $\mathrm{R}^T_{j,\theta}$ in the rightmost column of the sequence are irreducible. Thus, they may either vanish, either remain the same in the second page.

Lastly we prove point (7) of 1.4.2. Assume first that $\theta = 1$. The 0-th row of the spectral sequence is given by

$$\rho_{\begin{pmatrix} 1 \\ \end{pmatrix}} \oplus \rho_{\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}} \xrightarrow{u} \rho_{\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}}$$

We have $\mathrm{H}^1_c(S_1) \simeq \mathrm{Coker}(u)$. Since we already know that $\mathrm{H}^0_c(S_1) \simeq \mathrm{Ker}(u)$ is the trivial representation of $\mathrm{Sp}(2, \mathbb{F}_q)$, we see that u must be surjective. Therefore $\mathrm{H}^1_c(S_1) = 0$.

Remark. The vanishing of $\mathrm{H}^1_c(S_1)$ also follows directly from the fact that $S_1 \simeq \mathbb{P}^1$.

Let us now assume $\theta \geqslant 2$. The first terms of the 0-th row of the spectral sequence are

$$\mathrm{R}^S_{0,0} \xrightarrow{u} \mathrm{R}^S_{0,1} \xrightarrow{v} \mathrm{R}^S_{0,2} \longrightarrow \dots$$

We have $\mathrm{H}^1_c(S_\theta) = \mathrm{H}^1_c(S_\theta)_1 \simeq \mathrm{Ker}(v)/\mathrm{Im}(u)$. The middle term $\mathrm{R}^S_{0,1}$ is the sum of all the representations corresponding to the following symbols

$$\begin{pmatrix} 0 & 1 & \theta \\ 1 & 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & & 1+d \\ \theta - d & & \end{pmatrix}, \qquad \forall 0 \leqslant d \leqslant \theta - 1.$$

On the other hand, the left term $\mathrm{R}^S_{0,0}$ is the sum of all the representations corresponding to the following symbols

$$\begin{pmatrix} 0 & & 1+d \\ \theta - d & & \end{pmatrix}, \qquad \forall 0 \leqslant d \leqslant \theta.$$

Since we already know that $\mathrm{H}^0_c(S_\theta) \simeq \mathrm{Ker}(u)$ is the trivial representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$, we see that $\mathrm{Im}(u)$ contains all the components of $\mathrm{R}^S_{0,1}$ associated to a symbol whose second row has length 1. Therefore, $\mathrm{H}^1_c(S_\theta)$ is either 0 either irreducible, depending on whether the remaining component

$$\begin{pmatrix} 0 & 1 & \theta \\ 1 & 2 & \end{pmatrix}$$

is in $\mathrm{Ker}(v)$ or not. This proves point (7) and concludes the proof of 1.4.2 and 1.4.3 Theorems.

2 The geometry of the ramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

2.1 The Bruhat-Tits stratification

2.1.1 Recall that $E = \mathbb{Q}_p[\pi]$ is a quadratic ramified extension of $\mathbb{Q}_p$ with $\pi = \sqrt{-p}$ (case $E = E_1$) or $\pi = \sqrt{\epsilon p}$ (case $E = E_2$). If k is any perfect field over $\mathbb{F}_p$, we define $E_k := E \otimes_{\mathbb{Q}_p} W(k)_{\mathbb{Q}}$ with the embedding $E \hookrightarrow E_k, x \mapsto x \otimes 1$. We still write $\overline{\cdot}$ and σ for $\overline{\cdot} \otimes \mathrm{id}$ and $\mathrm{id} \otimes \sigma$ respectively on E_k. We define

$$E' := \begin{cases} E & \text{if } E = E_1, \\ E_{\mathbb{F}_{p^2}} & \text{if } E = E_2. \end{cases}$$

Eventually we write $\breve{E} := E_{\mathbb{F}}$, where $\mathbb{F} := \overline{\mathbb{F}_p}$. In [RTW14], the authors introduce the ramified PEL unitary Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$ as a moduli space which classifies the deformations of a given p-divisible group $\mathbb{X}$ equipped with additional structures, called the **framing object**. The latter is defined over $\mathbb{F}$ and the Rapoport-Zink space $\mathcal{M}$ is defined over $\mathcal{O}_{\breve{E}}$. For our purpose, it will be convenient to define this space over $\mathcal{O}_{E_k}$ where k is the smallest possible perfect extension of $\mathbb{F}_p$. Therefore we start by defining the framing object over a finite field. Denote by A_k the Dieudonné ring over k, that is the p-adic completion of the associative ring $W(k)\langle F, V\rangle$ with two indeterminates satisfying the relations $FV = VF = p$, $F\lambda = \lambda^\sigma F$ and $V\lambda^\sigma = \lambda V$ for all $\lambda \in W(k)$. We denote by $\mathbb{D}(\cdot)$ the *covariant* Dieudonné module functor from the category of p-divisible groups over k, to the category of A_k-modules that are free of finite rank over $W(k)$. A p-divisible group X over k is called **superspecial** if $\mathbb{D}(X) \otimes_{W(k)} W(L) \simeq A_{1,1}^{\oplus g} \otimes_{W(k)} W(L)$ for some $g \geqslant 1$ and some algebraically closed extension L/k, where $A_{1,1} := A_k/A_k(F-V)$ seen as a quotient of left A_k-modules. In particular, if X is superspecial then $2g = \text{height}(X) = 2\dim(X)$. Eventually, we define non-negative integers m, m^+ and m^- via the formula

$$n = \begin{cases} 2m+1 & \text{if } n \text{ is odd,} \\ 2m^+ = 2(m^-+1) & \text{if } n \text{ is even.} \end{cases}$$

2.1.2 Assume first that $E = E_1$, in which case $\mathbb{X}$ can be defined over $\mathbb{F}_p$. According to [LO98] §1.2, there exists an elliptic curve $\mathcal{E}$ over $\mathbb{F}_p$ whose relative Frobenius $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ satisfies $\mathcal{F}^2 + [p] = 0$. Its Dieudonné module is isomorphic to $\mathbb{D}(\mathcal{E}) \simeq A_{\mathbb{F}_p}/A_{\mathbb{F}_p}(F+V)$. If L is any extension of k containing $\mathbb{F}_{p^4}$ then $\mathbb{D}(\mathcal{E}) \otimes W(L) \simeq A_{1,1} \otimes W(L)$ so that $\mathcal{E}$ is supersingular. By [Tat66] Theorem 2, the endomorphism algebra $\text{End}^\circ(\mathcal{E})$ is commutative and isomorphic to $\mathbb{Q}[\mathcal{F}]$. Thus we have an action of $\mathcal{O}_E$ on the p-divisible group $\mathcal{E}[p^\infty]$ via the choice of an embedding

$$\iota_{\mathcal{E}} : E \xrightarrow{\sim} \text{End}^\circ(\mathcal{E}) \otimes \mathbb{Q}_p = \text{End}^\circ(\mathcal{E}[p^\infty]).$$

Eventually we have a canonical principal polarization $\lambda_{\mathcal{E}} : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^\vee$. Next, as in [RTW14] we define $\mathbb{X}_2^+ := \mathcal{E}[p^\infty] \times \mathcal{E}[p^\infty]$ with diagonal $\mathcal{O}_E$-action and polarization induced by the 2×2 matrix having 1's on the anti-diagonal and 0's on the diagonal. In the same manner, define $\mathbb{X}_2^-$ but with polarization induced by a 2×2 diagonal matrix having coefficients $u_1, u_2 \in \mathbb{Z}_p^\times$ such that $-u_1u_2$ is not a norm of E. The framing object $\mathbb{X}$ is given by any of the three following cases

$(\mathbb{X}_2^+)^m \times \mathcal{E}[p^\infty]$	when n is odd,
$(\mathbb{X}_2^+)^{m^+}$	when n is even (split case),
$(\mathbb{X}_2^+)^{m^-} \times \mathbb{X}_2^-$	when n is even (non-split case),

with diagonal $\mathcal{O}_E$-action $\iota_{\mathbb{X}}$ and polarization $\lambda_{\mathbb{X}}$.

Assume now that $E = E_2$. Since there is no supersingular elliptic curve over $\mathbb{F}_p$ whose endomorphism algebra at p contains E, the framing object $\mathbb{X}$ may only be defined over $\mathbb{F}_{p^2}$ in this case. There exists an elliptic curve $\mathcal{E}'$ over $\mathbb{F}_{p^2}$ whose Dieudonné module is isomorphic to $A_{1,1}$. The endomorphism algebra $\text{End}^\circ(\mathcal{E})$ is a central simple algebra over $\mathbb{Q}$ of degree 4 which ramifies only at p and infinity. At p, it is a quaternion algebra over $\mathbb{Q}_p$ generated by elements i, j such that $i^2 = -\epsilon$, $j^2 = p$ and $ij = -ji$. By fixing an embedding of E, we obtain an $\mathcal{O}_E$-action on $\mathcal{E}'[p^\infty]$ and we equip it with its natural polarization. We may then proceed with defining $\mathbb{X}_2^+$, $\mathbb{X}_2^-$ and $\mathbb{X}$ exactly as in the previous paragraph, except that we use $\mathcal{E}'$ instead of $\mathcal{E}$.

2.1.3 Let Nilp denote the category of $\mathcal{O}_{E'}$-schemes where π is locally nilpotent. For $S \in$ Nilp, a **unitary p-divisible group of signature** $(1, n-1)$ over S is a triple (X, ι_X, λ_X) where

- X is a p-divisible group over S.
- $\iota_X : \mathcal{O}_E \to \mathrm{End}(X)$ is a $\mathcal{O}_E$-action on X such that the induced action on its Lie algebra satisfies the Kottwitz and the Pappas conditions:

$$\forall a \in \mathcal{O}_E, \qquad \mathrm{char}(\iota(a) \,|\, \mathrm{Lie}(X)) = (T-a)^1(T-\overline{a})^{n-1},$$
$$\forall n \geqslant 3, \qquad \bigwedge^n(\iota(\pi) - \pi \,|\, \mathrm{Lie}(X)) = 0 \text{ and } \bigwedge^2(\iota(\pi) + \pi \,|\, \mathrm{Lie}(X)) = 0.$$

- $\lambda_X : X \xrightarrow{\sim} {}^tX$ is a principal polarization, where ${}^tX$ denotes the Serre dual of X. We assume that the associated Rosati involution induces $\overline{\cdot}$ on $\mathcal{O}_E$.

Note that $\mathrm{char}(\iota(a) \,|\, \mathrm{Lie}(X))$ is a polynomial with coefficients in $\mathcal{O}_S$. The Kottwitz condition compares it with a polynomial with coefficients in $\mathcal{O}_E \subset \mathcal{O}_{E'}$ via the structure morphism $S \to \mathcal{O}_{E'}$. For instance, the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ defined in the previous paragraph is an exemple of unitary p-divisible group of signature $(1, n-1)$ over $\kappa(E')$.
The following set-valued functor $\mathcal{M}$ defines a moduli problem classifying deformations of $\mathbb{X}$ by quasi-isogenies. More precisely, for $S \in$ Nilp the set $\mathcal{M}(S)$ consists of all isomorphism classes of tuples $(X, \iota_X, \lambda_X, \rho_X)$ such that

- (X, ι_X, λ_X) is a unitary p-divisible group of signature $(1, n-1)$ over S,
- $\rho_X : X \times_S \overline{S} \to \mathbb{X} \times_{\kappa(E')} \overline{S}$ is an $\mathcal{O}_E$-linear quasi-isogeny compatible with the polarizations, in the sense that ${}^t\rho_X \circ \lambda_{\mathbb{X}} \circ \rho_X$ is a $\mathbb{Q}_p^\times$-multiple of λ_X.

In the second condition, $\overline{S}$ denotes the special fiber of S. By [RZ96] Corollary 3.40, this moduli problem is represented by a separated formal scheme $\mathcal{M}$ over $\mathrm{Spf}(\mathcal{O}_{E'})$ called a **Rapoport-Zink space**. It is formally locally of finite type and flat over $\mathcal{O}_{E'}$. Let $\mathcal{M}_{\mathrm{red}}$ denote the reduced special fiber of $\mathcal{M}$, which is a scheme locally of finite type over $\mathrm{Spec}(\kappa(E'))$.

Remark. In [RZ96], Corollary 3.40 is stated under the assumption that the residue field $\kappa(E')$ contains $\mathbb{F}_{p^s}$, where $s > 0$ is an exponent appearing in a decency condition for the isocristal of $\mathbb{X}$. In general, we say that an isocristal N with Frobenius F is **decent** if it is generated by elements $n \in N$ such that $F^s n = p^r n$ for some integers $r \geqslant 0$ and $s > 0$ (loc. cit. Definition 2.13). By construction, the isocristal of $\mathbb{X}$ is decent. If $E = E_2$, then we have $F^2 = p\mathrm{id}$ on $\mathbb{D}(\mathbb{X})_{\mathbb{Q}}$, so that $s = 2$ and $\kappa(E')$ contains $\mathbb{F}_{p^2}$. However, if $E = E_1$ then $F^2 = -p\mathrm{id}$ on $\mathbb{D}(\mathbb{X})$, so that a decency equation is given by $F^4 = p^2\mathrm{id}$. In this case $s = 4$ and $\kappa(E') = \mathbb{F}_p$ does not contain $\mathbb{F}_{p^4}$.
Nonetheless, to our understanding, the condition that $\kappa(E')$ contains $\mathbb{F}_{p^s}$ can be relaxed. It seems to be used only in loc. cit. Lemma 3.37 in order to scale a bilinear form by a suitable unit so that it corresponds to a polarization of isocristals. In our case, since the isocristal $\mathbb{D}(\mathbb{X})$ already comes from a polarized p-divisible group, this lemma does not seems necessary.

2.1.4 We have a decomposition

$$\mathcal{M} = \bigsqcup_{i \in \mathbb{Z}} \mathcal{M}_i$$

into a disjoint union of open and closed formal connected subschemes, where the points of $\mathcal{M}_i$ correspond to those tuples $(X, \iota_X, \lambda_X, \rho_X)$ such that ${}^t\rho_X \circ \lambda_{\mathbb{X}} \circ \rho_X = c\lambda_X$ with $c \in \mathbb{Q}_p^\times$ having p-adic valuation i.

2.1.5 Dieudonné theory can be used to describe the rational points of $\mathcal{M}$ over a perfect field extension k of $\kappa(E')$. Let $N := \mathbb{D}(\mathbb{X})_{\mathbb{Q}}$ denote the Dieudonné isocristal of $\mathbb{X}$. The $\mathcal{O}_E$-action $\iota_{\mathbb{X}}$ induces an E'-vector space structure on N of dimension n. The polarization $\lambda_{\mathbb{X}}$ induces a $W(\kappa(E'))_{\mathbb{Q}}$-bilinear skew-symmetric form $\langle\cdot,\cdot\rangle$ on N such that

$$\forall x, y \in N, \qquad \langle \mathbf{F}x, y\rangle = \langle x, \mathbf{V}y\rangle,$$
$$\forall a \in E, \qquad \langle a\,\cdot,\cdot\rangle = \langle\cdot,\overline{a}\,\cdot\rangle,$$

where $\mathbf{F}$ and $\mathbf{V}$ denote respectively the Frobenius and the Verschiebung on N. Let $\tau := \eta\pi\mathbf{V}^{-1} : N \xrightarrow{\sim} N$ where $\eta \in W(\kappa(E'))_{\mathbb{Q}}$ is 1 if $E = E_1$ and a square root of $-\epsilon^{-1}$ if $E = E_2$. Notice that we have $(\eta\pi)^2 = -p$ in both cases. Let $C := N^\tau$ be the subset of vectors in N which are fixed by τ. It is naturally an E-vector space of dimension n and the natural map $C \otimes_E E' \xrightarrow{\sim} N$ is an isomorphism under which τ corresponds to $\mathrm{id} \otimes \sigma$. If $x, y \in C$ then

$$\langle x, y\rangle = \langle\tau(x), \tau(y)\rangle = \langle \eta\pi\mathbf{V}^{-1}x, \eta\pi\mathbf{V}^{-1}y\rangle = -p^{-1}(\pi\eta)^2\langle x, y\rangle^\sigma = \langle x, y\rangle^\sigma.$$

Therefore the restriction of $\langle\cdot,\cdot\rangle$ to C takes value in $\mathbb{Q}_p$. We define an E-hermitian form $(\cdot,\cdot)$ on C by the formula

$$\forall x, y \in C, \qquad (x, y) := \langle \pi x, y\rangle + \langle x, y\rangle\pi \in E.$$

Let k be any perfect field extension of $\mathbb{F}_p$. We extend $(\cdot,\cdot)$ to an E_k-hermitian form on $C_k := C \otimes_E E_k$ by the formula

$$\forall x, y \in C, \forall a, b \in E_k, \qquad (x \otimes a, y \otimes b) := a\overline{b}(x, y).$$

We still denote by τ the map $\mathrm{id} \otimes \sigma$ on C_k. For M an $\mathcal{O}_{E_k}$-lattice in C_k, we define its dual lattice $M^\sharp := \{x \in C_k \mid (x, M) \in \mathcal{O}_{E_k}\}$.

2.1.6 Let k be a perfect extension of $\kappa(E')$. The k-rational points of $\mathcal{M}$ are classified by the following proposition.

Proposition ([RTW14] Proposition 2.4)**.** *There is a bijection*

$$\mathcal{M}_i(k) \simeq \left\{ M \subset C_k \text{ an } \mathcal{O}_{E_k}\text{-lattice} \,\middle|\, M = p^i M^\sharp, \pi\tau(M) \subset M \subset \pi^{-1}\tau(M), M \overset{\leq 1}{\subset} M + \tau(M) \right\}.$$

The notation $\overset{\leq 1}{\subset}$ denotes an inclusion of $\mathcal{O}_{E_k}$-lattices with index at most 1.

2.1.7 Recall the integers m, m^+ and m^- that we defined depending on the parity of n. For $k \geqslant 0$, let A_k denote the $k \times k$ matrix with 1 on the antidiagonal and 0 everywhere else. We fix some scalars $u_1, u_2 \in \mathbb{Z}_p^\times$ such that $-u_1u_2 \notin \mathrm{Norm}_{E/\mathbb{Q}_p}(E^\times)$. We then define the three matrices

$$T_{\mathrm{odd}} := A_{2m+1}, \qquad T_{\mathrm{even}}^+ := A_{2m^+}, \qquad T_{\mathrm{even}}^- := \begin{pmatrix} & & & A_{m^-} \\ & u_1 & 0 & \\ & 0 & u_2 & \\ A_{m^-} & & & \end{pmatrix}.$$

By construction, C has a basis in which $(\cdot,\cdot)$ is given by $T_{\mathrm{odd}}, T_{\mathrm{even}}^+$ or T_{even}^- when n is odd, when n is even and $\mathbb{X} = (\mathbb{X}_2^+)^{m^+}$ (split case) or when n is even and $\mathbb{X} = (\mathbb{X}_2^+)^{m^-} \times \mathbb{X}_2^-$ (non-split case) respectively. We denote such a basis by $e = (e_{-j}, e_0^{\mathrm{an}}, e_j)_{1 \leqslant j \leqslant m}$ when n is odd, and if n is even by $e = (e_{-j}, e_j)_{1 \leqslant j \leqslant m^+}$ in the split case and by $e = (e_{-j}, e_0^{\mathrm{an}}, e_1^{\mathrm{an}}, e_j)_{1 \leqslant j \leqslant m^-}$ in the non-split case.

Remark. The integers m, m^+ and m^- correspond to the Witt index of C in each of the three cases.

2.1.8 Let $J = \mathrm{Aut}(\mathbb{X})$ be the group of automorphisms of $\mathbb{X}$ compatible with the additional structures. By [RTW14] Lemma 2.3, we have an isomorphism $J \simeq \mathrm{GU}(C, (\cdot,\cdot))$. As a reductive group over $\mathbb{Q}_p$, J is quasi-split if and only if n is odd or n is even and C is split. Let

$$c : J \mapsto \mathbb{Q}_p^\times$$

denote the multiplier character. For instance, $\pi^k \mathrm{id} \in J$ has multiplier $\pi^{2k} \in \mathbb{Q}_p^\times$. We define a surjective morphism $\alpha : J \mapsto \mathbb{Z}$ by $\alpha(g) := v_p(c(g))$ where v_p is the p-adic valuation. We denote by J° the kernel of α. Then J° is the subgroup generated by all compact subgroups of J.
The group J acts on $\mathcal{M}$ via

$$g \cdot (X, \iota_X, \lambda_X, \rho_X) := (X, \iota_X, \lambda_X, g \circ \rho_X).$$

An element $g \in J$ induces an isomorphism $g : \mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_{i+\alpha(g)}$.

2.1.9 For $i \in \mathbb{Z}$ we define

$$\mathcal{L}_i := \left\{\Lambda \subset C \text{ an } \mathcal{O}_E\text{-lattice} \,|\, p^i\Lambda^\sharp \subset \Lambda \subset \pi^{-1}p^i\Lambda^\sharp\right\}.$$

We also write $\mathcal{L}$ for the (disjoint) union of the $\mathcal{L}_i$'s. Elements of $\mathcal{L}$ are called **vertex lattices**. If Λ is a vertex lattice, its **orbit type** $t(\Lambda)$ is the lattice index $[\Lambda : p^i\Lambda^\sharp]$. According to [RTW14], $t(\Lambda)$ is an even integer between 0 and n.
The group J acts on $\mathcal{L}$ via $g \cdot \Lambda := g(\Lambda)$. An element $g \in J$ defines a type preserving, inclusion preserving bijection $g : \mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_{i+\alpha(g)}$. With arguments similar to those used in the unramified case in [Vol10], one may prove the following proposition.

Proposition. *Two vertex lattices $\Lambda, \Lambda' \in \mathcal{L}$ are in the same J-orbit if and only if $t(\Lambda) = t(\Lambda')$.*

2.1.10 Recall the basis e of C that we fixed in 2.1.7. For a family of integers (r_j, s) where

- $1 \leqslant j \leqslant m$ and $s \in \mathbb{Z}$ if n is odd,
- $1 \leqslant j \leqslant m^+$ and $s = \varnothing$ if n is even and C is split,
- $1 \leqslant j \leqslant m^-$ and $s \in \mathbb{Z}^2$ if n is even and C is non-split,

we denote by

$$\Lambda(r_{-j}\,;\,s\,;\,r_j)$$

the $\mathcal{O}_E$-lattice generated by the vectors $p^{r_{\pm j}} e_{\pm j}$, and by $p^{s_0} e_0^{\mathrm{an}}$ and $p^{s_1} e_1^{\mathrm{an}}$ when it makes sense.

Proposition. *A lattice $\Lambda = \Lambda(r_{-j}\,;\,s\,;\,r_j)$ is a vertex lattice if and only if for some $i \in \mathbb{Z}$, $r_{-j} + r_j \in \{2i-1, 2i\}$ for all j, and s is respectively given by $i, \varnothing$ or (i,i) depending on whether n is odd or even with C split or not. When Λ is a vertex lattice, its orbit type is given by*

$$t(\Lambda) = 2\#\{j \,|\, r_{-j} + r_j = 2i-1\}.$$

This is proved in the same way as [Mul22a] 1.2.4 Proposition. In particular, when n is even and C is non-split there is no vertex lattice of orbit type n. Let $t_{\max}$ denote the maximal type of a vertex lattice. We have

$$t_{\max} = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even and } C \text{ is split,} \\ n-2 & \text{if } n \text{ is even and } C \text{ is non-split.} \end{cases}$$

We also write $t_{\max} = 2\theta_{\max}$, so that $\theta_{\max} = m, m^+$ or m^- depending on whether n is odd, n is even with C split or n is even with C non-split respectively.

2.1.11 The set $\mathcal{L}$ of vertex lattices can be given the structure of a polysimplicial complex, by declaring that an s-simplex in $\mathcal{L}$ is a subset $\{\Lambda_0, \ldots, \Lambda_s\} \subset \mathcal{L}_i$ for some $i \in \mathbb{Z}$ such that, up to reordering, we have

$$\Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_s.$$

Depending on whether n is odd or even with C split or not, for an s-simplex to exist we must have s between 0 and $\theta_{\max}$. We fix a specific maximal simplex in each case.

If n is odd, for $0 \leqslant \theta \leqslant \theta_{\max}$ we define

$$\Lambda_\theta := \Lambda(0^{\theta_{\max}}\,;\,0\,;\,0^{\theta_{\max}-\theta}, -1^{\theta}).$$

If n is even and C is split, for $0 \leqslant \theta \leqslant \theta_{\max}$ we define

$$\Lambda_\theta := \Lambda(0^{\theta_{\max}}\,;\,0^{\theta_{\max}-\theta}, -1^{\theta}).$$

If n is even and C is non-split, for $0 \leqslant \theta \leqslant \theta_{\max}$ we define

$$\Lambda_\theta := \Lambda(0^{\theta_{\max}}\,;\,0,0\,;\,0^{\theta_{\max}-\theta}, -1^{\theta}).$$

In each case we have $\Lambda_\theta \in \mathcal{L}_0$ and $\Lambda_\theta \subset \Lambda_{\theta+1}$. Moreover the orbit type of Λ_θ is 2θ.

2.1.12 For $\Lambda \in \mathcal{L}$, let J_Λ denote the fixator of Λ in J. Let J_Λ^+ be its pro-unipotent radical, and write $\mathcal{J}_\Lambda := J_\Lambda/J_\Lambda^+$ for the **finite reductive quotient**. It is a finite group of Lie type over $\mathbb{F}_p$.
We define also the quotients

$$V_\Lambda^0 := \Lambda/p^i\Lambda^\sharp, \qquad\qquad V_\Lambda^1 := \pi^{-1}p^i\Lambda^\sharp/\Lambda.$$

They are both $\mathcal{O}_E/\pi\mathcal{O}_E \simeq \mathbb{F}_p$-vector spaces of dimension respectively $t(\Lambda)$ and $n - t(\Lambda)$. Both spaces inherit a perfect $\mathbb{F}_p$-bilinear form, which we denote by the same notation $\{\cdot,\cdot\}$, induced respectively by $\pi p^{-i}(\cdot,\cdot)$ and by $p^{1-i}(\cdot,\cdot)$. Then $\{\cdot,\cdot\}$ is symplectic on V_Λ^0 whereas it is symmetric on V_Λ^1. If k is a perfect field extension of $\mathbb{F}_p$, we denote by $V_{\Lambda,k}^0$ and $V_{\Lambda,k}^1$ the scalar extensions to k, equipped with their perfect k-bilinear forms $\{\cdot,\cdot\}$, and we denote by τ the map $\mathrm{id}\otimes\sigma$ on both spaces. If U is a subspace, we denote by $U^\perp$ its orthogonal.
We denote by $\mathrm{GSp}(\cdot)$ and $\mathrm{GO}(\cdot)$ the associated groups of symplectic or orthogonal similitudes. Then we have a natural isomorphism

$$\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{Sp}(V_\Lambda^0) \times \mathrm{O}(V_\Lambda^1)),$$

where the right-hand side is the subgroup of $\mathrm{GSp}(V_\Lambda^0) \times \mathrm{GO}(V_\Lambda^1)$ with both factors sharing the same multiplier in $\mathbb{F}_p^\times$. Let $\mathcal{J}_\Lambda^\circ$ be the connected component of unity and let J_Λ° be its preimage in J_Λ. We recall some known facts about parahoric subgroups of J, see for instance [LS20] section 2 for a complete summary.

Proposition. *Let $\Lambda \in \mathcal{L}$.*

- *The fixator J_Λ is a maximal compact subgroup of J. All maximal compact subgroups arise this way.*
- *The subgroup J_Λ° is a parahoric subgroup of J. It is a maximal parahoric subgroup unless n is even, C is split and $t(\Lambda) = n-2$. All maximal parahoric subgroups of J arise this way.*
- *The parahoric subgroup J_Λ° consists of all the elements $g \in J_\Lambda$ such that the induced orthogonal similitude on V_Λ^1 has determinant 1.*
- *If $t(\Lambda) \neq n$ then J_Λ° has index 2 in J_Λ. If $t(\Lambda) = n$ then $J_\Lambda^\circ = J_\Lambda$.*

We note that the condition $t(\Lambda) = n$ can only occur when n is even and C is split. Besides, in this case any vertex lattice $\Lambda \in \mathcal{L}_i$ of orbit type $n-2$ is contained in precisely two different vertex lattices $\Lambda_1, \Lambda_2 \in \mathcal{L}_i$ of orbit type n (see 2.2). Then the parahoric subgroup J_Λ° is the intersection of the two maximal parahoric subgroups $J_{\Lambda_1}^\circ = J_{\Lambda_1}$ and $J_{\Lambda_2}^\circ = J_{\Lambda_2}$.

Notation. If Λ is one of the Λ_θ's, we write J_θ, V_θ^0 and V_θ^1 instead of J_{Λ_θ}, $V_{\Lambda_\theta}^0$ and $V_{\Lambda_\theta}^1$ respectively.

2.1.13 In this paragraph, we compute the normalizer of the maximal compact subgroups J_Λ and their attached parahoric subgroup J_Λ°.

Lemma. *Let $\Lambda, \Lambda' \in \mathcal{L}$. The following statements are equivalent.*

(1) $J_\Lambda = J_{\Lambda'}$,
(2) $J_\Lambda^\circ = J_{\Lambda'}^\circ$,
(3) $\Lambda' = \pi^k\Lambda$ *for some* $k \in \mathbb{Z}$.

Proof. The implications (3) $\implies$ (1), (2) are clear. Given two vertex lattices, there exists a Witt decomposition of C in which both lattices split. Thus, for the converse implications it is enough to treat the case $\Lambda = \Lambda_\theta$ where $t(\Lambda) = 2\theta$, and Λ' is a vertex lattice of the form

$$\Lambda' = \Lambda(r_{-j}; s; r_j) \in \mathcal{L}_i,$$

for some $i \in \mathbb{Z}$. By 2.1.10 Proposition we have $r_j + r_{-j} \in \{2i-1, 2i\}$ for all j, and $s = i, \varnothing$ or (i,i) depending on whether n is odd or even with C split or not respectively. A basis of the vector space V_θ^1 is given by the images of the vectors e_{-j}, e_j for $1 \leqslant j \leqslant \theta_{\max} - \theta$ and the vectors $e_0^{\mathrm{an}}, e_1^{\mathrm{an}}$ when they exist. We now assume that we have $J_\theta = J_{\Lambda'}$ or $J_\theta^\circ = J_{\Lambda'}^\circ$. In both cases we have $J_\theta^\circ \subset J_{\Lambda'}$, and it is all we need to prove that $\Lambda' = \pi^i\Lambda_\theta$.
First, let us assume that n is odd or that n is even and C is non-split. Thus, the vector e_0^{an} exists. For $1 \leqslant j \leqslant \theta$, consider $g \in \mathrm{GL}(C)$ swapping e_{-j} and e_j, sending e_0^{an} to $-e_0^{\mathrm{an}}$ and fixing all the other vectors in the basis e. Then g is a unitary similitude of multiplier equal to 1, and we have $g \cdot \Lambda_\theta = \Lambda_\theta$. Moreover g induces an orthogonal isometry on V_θ^1 of determinant 1. Thus $g \in J_\theta^\circ \subset J_{\Lambda'}$. It follows that $r_{-j} = r_j$, hence $r_j = 2i$. On the other hand, for $\theta + 1 \leqslant j \leqslant \theta_{\max}$, consider $g \in \mathrm{GL}(C)$ sending e_j to πe_{-j} and e_{-j} to $-\pi^{-1}e_j$ while fixing all the other vectors of e. Then g defines an element of J of multiplier equal to 1, which fixes Λ_θ and induces identity on V_θ^1. Thus $g \in J_\theta^\circ \subset J_{\Lambda'}$. We deduce that $r_{-j} = r_j + 1$, so that $r_j = i - 1$. In other words, we have $\Lambda' = \pi^i\Lambda_\theta$.
Let us now assume that n is even and C is split. Consider $g \in \mathrm{GL}(C)$ which is defined by $e_{-j} \mapsto -e_j, e_j \mapsto e_{-j}$ for $1 \leqslant j \leqslant \theta$, and by $e_j \mapsto \pi e_{-j}, e_{-j} \mapsto \pi^{-1}e_j$ for $\theta + 1 \leqslant j \leqslant \theta_{\max}$. Then g is a unitary similitudes of multiplier equal to -1, which fixes Λ_θ and induces an orthogonal similitude of determinant 1 on V_θ^1. Thus, $g \in J_\Lambda^\circ \subset J_{\Lambda'}$. We deduce that $r_{-j} = r_j$ for all $1 \leqslant j \leqslant \theta$ and that $r_{-j} = r_j + 1$ for all $\theta + 1 \leqslant j \leqslant \theta_{\max}$. Thus we have $\Lambda' = \pi^i\Lambda_\theta$ as above. ☐

Proposition. *Let* $\Lambda \in \mathcal{L}$ *be a vertex lattice. We have* $\mathrm{N}_J(J_\Lambda) = \mathrm{N}_J(J_\Lambda^\circ) = \mathrm{Z}(J)J_\theta$.

Proof. It is clear that $\mathrm{Z}(J)J_\Lambda$ is contained in both $\mathrm{N}_J(J_\Lambda)$ and $\mathrm{N}_J(J_\Lambda^\circ)$. Moreover, if g belongs to $\mathrm{N}_J(J_\Lambda)$ (resp. to $\mathrm{N}_J(J_\Lambda^\circ)$), then we have $J_{g\cdot\Lambda} = J_\Lambda$ (resp. $J_{g\cdot\Lambda}^\circ = J_\Lambda^\circ$). By the previous Lemma, we deduce that $g \cdot \Lambda = \pi^k\Lambda$ for some $k \in \mathbb{Z}$. Thus $\pi^{-k}g \in J_\Lambda$, hence $g \in \pi^kJ_\Lambda \subset \mathrm{Z}(J)J_\Lambda$. ☐

2.1.14 In [RTW14] section 6, the authors attach to any vertex lattice $\Lambda \in \mathcal{L}_i$ a closed projective subscheme $\mathcal{M}_\Lambda \hookrightarrow \mathcal{M}_{i,\mathrm{red}}$, which is called a **closed Bruhat-Tits stratum**. Its rational points are described by the following proposition.

Proposition ([RTW14] Corollary 6.3)**.** *Let k be a perfect field extension of $\kappa(E')$ and let $\Lambda \in \mathcal{L}_i$. We have a natural bijection*

$$\mathcal{M}_\Lambda(k) \simeq \{M \in \mathcal{M}_i(k) \,|\, M \subset \Lambda_k := \Lambda \otimes_{\mathcal{O}_E} \mathcal{O}_{E_k}\}.$$

By mapping $M \in \mathcal{M}_\Lambda(k)$ to its image $\overline{M} := M/p^i\Lambda_k^\sharp$ in $V^0_{\Lambda,k}$, one obtains a bijection between $\mathcal{M}_\Lambda(k)$ and the set

$$\{U \subset V^0_{\Lambda,k} \,|\, U^\perp = U \text{ and } U \overset{\leqslant 1}{\subset} U + \tau(U)\}.$$

The action of J on $\mathcal{M}$ restricts to an action of J_Λ on $\mathcal{M}_\Lambda$. This action factors through the finite reductive quotient $\mathcal{J}_\Lambda$, and the $\mathrm{GO}(V^1_\Lambda)$-component acts trivially. Therefore we obtain an action of $\mathrm{GSp}(V^0_\Lambda) \simeq \mathrm{GSp}(2\theta, \mathbb{F}_p)$ where $t(\Lambda) = 2\theta$ on $\mathcal{M}_\Lambda$. The main theorem of loc. cit. is the construction of a natural isomorphism between the closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$ and the closed Deligne-Lusztig variety S_θ that we introduced in 1.1.4.

Theorem ([RTW14] Proposition 6.7)**.** *Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta$ for its orbit type. There is a natural isomorphism*

$$\mathcal{M}_\Lambda \xrightarrow{\sim} S_\theta \otimes_{\mathbb{F}_p} \kappa(E')$$

which is $\mathrm{GSp}(2\theta, \mathbb{F}_p)$*-equivariant.*

In particular, the variety $\mathcal{M}_\Lambda$ is always defined over $\mathbb{F}_p$. We note also that the $\mathrm{GSp}(2\theta, \mathbb{F}_p)$-action on S_θ is induced from 1.1.2. Identifying the unipotent representations of $\mathrm{Sp}(2\theta, \mathbb{F}_p)$ and of $\mathrm{GSp}(2\theta, \mathbb{F}_p)$ as in 1.2.1, the theorems 1.4.2 and 1.4.3 give us a certain knowledge of the cohomology of the closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$.

2.1.15 The closed subschemes $\mathcal{M}_\Lambda$ form the **Bruhat-Tits stratification** of the reduced special fiber $\mathcal{M}_{\mathrm{red}}$, whose incidence relations mimic the combinatorics of vertex lattices.

Theorem ([RTW14] Theorem 6.10)**.** *Let $i \in \mathbb{Z}$ and $\Lambda, \Lambda' \in \mathcal{L}_i$.*

(1) *The inclusion $\Lambda \subset \Lambda'$ is equivalent to the scheme-theoretic inclusion $\mathcal{M}_\Lambda \subset \mathcal{M}_{\Lambda'}$. It implies $t(\Lambda) \leqslant t(\Lambda')$ with equality if and only if $\Lambda = \Lambda'$.*

(2) *The three following assertions are equivalent.*

(i) $\Lambda \cap \Lambda' \in \mathcal{L}_i$. (ii) $\Lambda \cap \Lambda'$ *contains a lattice of* $\mathcal{L}_i$. (iii) $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} \neq \varnothing$.

If these conditions are satisfied, then $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} = \mathcal{M}_{\Lambda \cap \Lambda'}$ scheme-theoretically.

(3) *If k is a perfect field field extension of $\kappa(E')$ then $\mathcal{M}_i(k) = \bigcup_{\Lambda \in \mathcal{L}_i} \mathcal{M}_\Lambda(k)$.*

It follows in particular that $\mathcal{M}_{\mathrm{red}}$ has pure dimension $\theta_{\max}$.

2.2 Counting the Bruhat-Tits strata

In this short section we give a formula for the number of closed Bruhat-Tits strata of a certain dimension, which are included in or which contain a fixed stratum. Let $d \geqslant 0$ and let V be a d-dimensional $\mathbb{F}_p$-vector space equipped with a non-degenerate symmetric or symplectic form $\{\cdot,\cdot\} : V \times V \to \mathbb{F}_p$. Define the integer δ by

$$d = \begin{cases} 2\delta & \text{if } \{\cdot,\cdot\} \text{ is symplectic, or if it is symmetric, } d \text{ is even and } V \text{ is split,} \\ 2(\delta+1) & \text{if } \{\cdot,\cdot\} \text{ is symmetric, } d \text{ is even and } V \text{ is not split,} \\ 2\delta+1 & \text{if } \{\cdot,\cdot\} \text{ is symmetric, } d \text{ is odd.} \end{cases}$$

Thus δ corresponds to the Witt index of V. For $0 \leqslant r \leqslant \delta$ we define

$$N(r, V) := \{U \subset V \mid \dim U = r \text{ and } U \subset U^\perp\}.$$

Let $i \in \mathbb{Z}$ and let $\Lambda \in \mathcal{L}_i$ be a vertex lattice. Write $t(\Lambda) = 2\theta$ so that $0 \leqslant \theta \leqslant \theta_{\max}$.

Proposition. (1) *The set of vertex lattices $\Lambda' \in \mathcal{L}_i$ of orbit type $t(\Lambda') = 2\theta'$ such that $\Lambda' \subset \Lambda$ is in bijection with $N(\theta - \theta', V_\Lambda^0)$.*

(2) *The set of vertex lattices $\Lambda' \in \mathcal{L}_i$ of orbit type $t(\Lambda') = 2\theta'$ such that $\Lambda \subset \Lambda'$ is in bijection with $N(\theta' - \theta, V_\Lambda^1)$.*

The bijection is established by mapping Λ' to the image of $p^i(\Lambda')^\sharp$ in V_Λ^0 in case (1), and to its own image in V_Λ^1 in case (2). The following statement gives the cardinality of $N(r, V)$.

Proposition. *Let δ be the Witt index of V and let $0 \leqslant r \leqslant \delta$.*

- *If $\{\cdot,\cdot\}$ is symplectic, or if it is symmetric and d is odd, then*

$$\# N(r, V) = \prod_{i=1}^{r} \frac{p^{2(i+\delta-r)} - 1}{p^i - 1}.$$

- *If $\{\cdot,\cdot\}$ is symmetric, d is even and V is not split then*

$$\# N(r, V) = \frac{p^{\delta+1} + 1}{p^{\delta+1-r} + 1} \prod_{i=1}^{r} \frac{p^{2(i+\delta-r)} - 1}{p^i - 1}.$$

- *If $\{\cdot,\cdot\}$ is symmetric, d is even and V is split then*

$$\# N(r, V) = \frac{p^{\delta-r} + 1}{p^{\delta} + 1} \prod_{i=1}^{r} \frac{p^{2(i+\delta-r)} - 1}{p^i - 1}.$$

The proof of this proposition is very similar to [Mul22a] 1.4.2 Proposition, therefore we omit it.

Remark. Assume that n is even and that C is split. Let $\Lambda \in \mathcal{L}_i$ with orbit type $n-2 = 2(\theta_{\max}-1)$. The set of vertices $\Lambda' \in \mathcal{L}_i$ of maximal orbit type $n = 2\theta_{\max}$ which contain Λ is in bijection with $N(1, V_\Lambda^1)$. The space V_Λ^1 has dimension 2 with a symmetric form and is split. According to the formula above, the number of such lattices Λ' is

$$\frac{p^{1-1} + 1}{p^1 + 1} \prod_{i=1}^{1} \frac{p^{2i} - 1}{p^i - 1} = 2.$$

We recover the fact stated in [RTW14] proof of Proposition 3.4 that in the even split case, vertex lattices of orbit type $n-2$ correspond in fact to an edge in the Bruhat-Tits building of J.

2.3 Shimura variety and p-adic uniformization of the basic stratum

2.3.1 In this section, we introduce the integral model of a Shimura variety whose supersingular locus is uniformized by the Rapoport-Zink space $\mathcal{M}$. We follow the construction of [RTW14] Section 7. Let $\mathbb{E}$ be an imaginary quadratic field in which $p > 2$ ramifies, and let $\overline{\cdot}$ denote the non-trivial element of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q})$. Let $\mathbb{V}$ be an n-dimensional hermitian $\mathbb{E}$-vector space of signature $(1, n-1)$ at infinity. Let $\mathbb{G}$ denote the group of unitary similitudes of $\mathbb{V}$ as a reductive group over $\mathbb{Q}$.
First, we give the moduli description of the canonical model of the Shimura variety associated to the data above. Let $K \subset \mathbb{G}(\mathbb{A}_f)$ be an open compact subgroup. For a locally noetherian $\mathbb{E}$-scheme S, let $\mathrm{Sh}_K(S)$ denote the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta})$ where

- A is an abelian scheme over S.
- $\lambda : A \to \widehat{A}$ is a polarization.
- $\iota : \mathbb{E} \to \mathrm{End}(A) \otimes \mathbb{Q}$ is a $\mathbb{E}$-action on A such that $\iota(\overline{x}) = \iota(x)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ, and such that the Kottwitz determinant condition is satisfied:

$$\forall x \in \mathbb{E},\ \det(T - \iota(x) \,|\, \mathrm{Lie}(A)) = (T - x)^1 (T - \overline{x})^{n-1} \in \mathbb{E}[T].$$

- $\overline{\eta}$ is a K-level structure, that is a K-orbit of isomorphisms of $\mathbb{E} \otimes \mathbb{A}_f$-modules $\mathrm{H}_1(A, \mathbb{A}_f) \xrightarrow{\sim} \mathbb{V} \otimes \mathbb{A}_f$ that is compatible with the other data.

According to [KR14] Proposition 4.3, when K is small enough the functor Sh_K is represented by a smooth quasi-projective scheme over $\mathbb{E}$. As the level K varies, the Shimura varieties sit together in a projective system $(\mathrm{Sh}_K)_K$ on which $\mathbb{G}(\mathbb{A}_f)$ acts by Hecke correspondences.

2.3.2 In order to define integral models for these Shimura varieties, let us assume that there exists a self-dual $\mathcal{O}_{\mathbb{E}}$-lattice Γ in $\mathbb{V}$. Let $K \subset \mathbb{G}(\mathbb{A}_f)$ denote the stabilizer of Γ. Let $K^p \subset K \cap \mathbb{G}(\mathbb{A}_f^p)$ be an open compact subgroup. For an $\mathcal{O}_{\mathbb{E},(p)}$-scheme S, let $\mathrm{S}_{K^p}(S)$ denote the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta}^p)$ where

- A is an abelian scheme over S.
- $\lambda : A \to \widehat{A}$ is a polarization of order prime to p.
- $\iota : \mathcal{O}_{\mathbb{E}} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is an $\mathcal{O}_{\mathbb{E}}$-action on A such that $\iota(\overline{x}) = \iota(x)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ, and such that the Kottwitz determinant condition is satisfied:

$$\forall x \in \mathcal{O}_{\mathbb{E}},\ \det(T - \iota(x) \,|\, \mathrm{Lie}(A)) = (T - x)^1 (T - \overline{x})^{n-1} \in \mathcal{O}_{\mathbb{E}}[T].$$

- $\overline{\eta}^p$ is a K^p-level structure, that is a K^p-orbit of isomorphisms of $\mathbb{E} \otimes \mathbb{A}_f^p$-modules $\mathrm{H}_1(A, \mathbb{A}_f^p) \xrightarrow{\sim} \mathbb{V} \otimes \mathbb{A}_f^p$ that is compatible with the other data.

By [RTW14] Section 7, when K^p is small enough the functor S_{K^p} is represented by a smooth quasi-projective $\mathcal{O}_{\mathbb{E},(p)}$-scheme. As the level K^p varies, these integral models form a projective system on which $\mathbb{G}(\mathbb{A}_f^p)$ acts by Hecke correspondences. We have natural isomorphisms

$$\mathrm{Sh}_{K_pK^p} \simeq \mathrm{S}_{K^p} \otimes_{\mathcal{O}_{\mathbb{E},(p)}} \mathbb{E},$$

which are compatible as K^p varies, where K_p is the stabilizer of $\Gamma \otimes \mathbb{Z}_p$ in $\mathbb{V}_{\mathbb{Q}_p}$.

Remark. From now on, the notation S_{K^p} will be used to denote the $O_{\mathbb{E}_p}$-scheme obtained by base change.

2.3.3 Let $\overline{S}_{K^p}$ denote the special fiber of the Shimura variety over the residue field $\kappa(\mathbb{E}_p)$. Let $\overline{S}_{K^p}^{\mathrm{ss}}$ denote the **supersingular locus**, it is a closed projective subscheme of $\overline{S}_{K^p}$. Let $\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss}}$ denote the formal completion of S_{K^p} along the supersingular locus. Eventually, let $\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss,an}}$ denote the Berkovich generic fiber of the formal scheme $\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss}}$, a smooth analytic space over $\mathbb{E}_p$.
From now on, we write $E := \mathbb{E}_p$ and $\kappa(\mathbb{E}_p) = \kappa(E) = \mathbb{F}_p$. Let $(A^0, \lambda^0, \iota^0, \overline{\eta}^0)$ be the $\kappa(E')$-rational point of $\overline{S}_{K^p}^{\mathrm{ss}}$ given by the product of elliptic curves used to define the framing object $\mathbb{X}$ as in 2.1.2. In particular the p-divisible group $A^0[p^\infty]$ is identified with $\mathbb{X}$ and we may consider the associated Rapoport-Zink space $\mathcal{M}$ over $\mathrm{Spf}(\mathcal{O}_{E'})$. As observed in [RTW14] Remark 7.1, when n is even the discriminants of the hermitian spaces C and $\mathbb{V} \otimes E$ are different, so that one space is split precisely when the other is non-split. Let I denote the group of quasi-isogenies of A^0 which respect all additional structures. Since A^0 is in the basic stratum, I can be seen as an inner-form of $\mathbb{G}$ such that $I_{\mathbb{A}_f^p} \simeq \mathbb{G}_{\mathbb{A}_f^p}$ and $I_{\mathbb{Q}_p} \simeq J$. One may therefore think of $I(\mathbb{Q})$ as a subgroup both of $\mathbb{G}(\mathbb{A}_f^p)$ and of J at the same time. The p-adic uniformization theorem gives a geometric link between the Rapoport-Zink space and the supersingular locus of the Shimura variety.

Theorem ([RTW14])**.** *There is an isomorphism of formal schemes over* $\mathrm{Spf}(\mathcal{O}_{E'})$

$$\Theta_{K^p} : I(\mathbb{Q})\backslash \left(\mathcal{M} \times \mathbb{G}(\mathbb{A}_f^p)/K^p\right) \xrightarrow{\sim} \widehat{\mathrm{S}}_{K^p}^{\mathrm{ss}} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$$

which is compatible with the $G(\mathbb{A}_f^p)$-action as the level K^p varies.

As in [Mul22a] 3.6, one also obtains uniformization isomorphisms $(\Theta_{K^p})_s$ and $\Theta_{K^p}^{\mathrm{an}}$ for the special and the generic fibers respectively.

2.3.4 Let $g_1, \ldots, g_s \in \mathbb{G}(\mathbb{A}_f^p)$ be a system of representatives for the double coset space $I(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_f^p)/K^p$, and let $\Gamma_k := I(\mathbb{Q}) \cap g_k K^p g_k^{-1}$ for $1 \leqslant k \leqslant s$. These are subgroups of J which are discrete and cocompact modulo the center. The uniformization theorem for the special fiber may be written as

$$(\Theta_{K^p})_s : \bigsqcup_{k=1}^{s} \Gamma_k\backslash\mathcal{M}_{\mathrm{red}} \xrightarrow{\sim} \overline{S}_{K^p}^{\mathrm{ss}} \otimes_{\mathbb{F}_p} \kappa(E').$$

Let $\Phi_{K^p}^k$ be the composition $\mathcal{M}_{\mathrm{red}} \to \Gamma_k\backslash\mathcal{M}_{\mathrm{red}} \to \overline{S}_{K^p}^{\mathrm{ss}}$, and let Φ_{K^p} be the union of the $\Phi_{K^p}^k$. By the same arguments as [VW11] Section 6.4, the surjection Φ_{K^p} is a local isomorphism. Moreover the restriction of $\Phi^k_{K^p}$ to any closed Bruhat-Tits stratum $\mathcal{M}_\Lambda \subset \mathcal{M}_{\mathrm{red}}$ is an isomorphism onto its image. We will denote this scheme-theoretic image by $\overline{\mathrm{S}}_{K^p,\Lambda,k}$. For varying Λ and k, these subschemes constitute the closed strata of the **Bruhat-Tits stratification** of the supersingular locus of the Shimura variety.

3 On the cohomology of the Rapoport-Zink space

3.1 The spectral sequence associated to the Bruhat-Tits open cover of $\mathcal{M}^{\mathrm{an}}$

3.1.1 Let $\mathcal{M}^{\mathrm{an}}$ denote the Berkovich generic fiber of the Rapoport-Zink space. It is a smooth analytic space over E' of dimension $n-1$. Let $\mathrm{red} : \mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$ denote the reduction map. Write $\mathcal{M}^{\mathrm{an}}_i := \mathrm{red}^{-1}(\mathcal{M}_{\mathrm{red},i})$ so that

$$\mathcal{M}^{\mathrm{an}} = \bigsqcup_{i\in\mathbb{Z}} \mathcal{M}^{\mathrm{an}}_i,$$

each $\mathcal{M}^{\mathrm{an}}_i$ being connected. For a vertex lattice $\Lambda \in \mathcal{L}_i$, let

$$U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda) \subset \mathcal{M}^{\mathrm{an}}_i$$

denote the analytical tube of the closed Bruhat-Tits stratum indexed by Λ. Since the map red is anticontinuous, it is an open subspace of the generic fiber. The group J acts on $\mathcal{M}^{\mathrm{an}}$ and the map red is J-equivariant. The action restricts to an action of the maximal compact subgroup J_Λ on U_Λ.

3.1.2 We fix a prime number $\ell \neq p$ and we consider the cohomology groups

$$\mathrm{H}^\bullet_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) := \varinjlim_{U} \varprojlim_{k} \mathrm{H}^\bullet_c(U \widehat{\otimes} \mathbb{C}_p, \mathbb{Z}/\ell^k\mathbb{Z}) \otimes \overline{\mathbb{Q}_\ell},$$

where U runs over all relatively compact open subspaces of $\mathcal{M}_{\mathrm{an}}$. These cohomology groups are representations of $J \times W$ where W is the absolute Weil group of E. The W-action on the cohomology group is defined in the following specific way. The inertia $I \subset W$ acts on the coefficients $\mathbb{C}_p$, whereas the action of the Frobenius is given by **Rapoport and Zink's descent datum** on $\mathcal{M} \widehat{\otimes} \mathcal{O}_{\breve{E}}$. As we recalled in [Mul22a] 4.1.2, this descent datum is an isomorphism $\alpha_{\mathrm{RZ}} : \mathcal{M} \widehat{\otimes} \mathcal{O}_{\breve{E}} \xrightarrow{\sim} \sigma^*(\mathcal{M} \widehat{\otimes} \mathcal{O}_{\breve{E}})$, where $\sigma \in \mathrm{Gal}(\breve{E}/E) \simeq \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ is the arithmetic Frobenius. The right-hand side can be identified with the Rapoport-Zink space for $(\mathbb{X} \otimes \mathbb{F})^{(p)}$. This isomorphism is induced by the relative Frobenius $\mathcal{F}_{\mathbb{X}} : \mathbb{X} \otimes \mathbb{F} \to (\mathbb{X} \otimes \mathbb{F})^{(p)}$, via $(X, \iota, \lambda, \rho) \mapsto (X, \iota, \lambda, \mathcal{F}_{\mathbb{X}} \circ \rho)$. We fix a lift $\mathrm{Frob} \in W$ of the *geometric* Frobenius. Then the action of Frob on $\mathrm{H}^\bullet_c(\mathcal{M}^{\mathrm{an}})$ is induced by $\alpha_{\mathrm{RZ}}^{-1}$.
Via covariant Dieudonné theory, the relative Frobenius $\mathcal{F}_{\mathbb{X}}$ corresponds to the Verschiebung morphism $\mathbf{V}$ on $C_{\mathbb{F}} = C \otimes_E \breve{E}$. If k is any perfect extension of $\kappa(\breve{E})$, the Verschiebung sends a k-rational point $M \in \mathcal{M}(k)$ to $\mathbf{V}M = \eta^{\sigma^{-1}}\pi\tau^{-1}(M) = \pi\tau^{-1}(M)$ (since η is a scalar unit). Hence, the descent datum α_{RZ} sends a k-point M to $\pi\tau^{-1}(M)$.

Remark. The descent datum α_{RZ} is not effective. The rational structure of $\mathcal{M}$ over $\mathcal{O}_{E'}$ is induced by $\pi\alpha_{\mathrm{RZ}}^{-1}$ if $E = E_1$, and by $(\pi\alpha_{\mathrm{RZ}}^{-1})^2$ if $E = E_2$. It maps a k-point M to $\tau(M)$ in the first case, and to $\tau^2(M)$ in the second case.
In any case, we will denote by τ the action on the cohomology induced by $\pi\alpha_{\mathrm{RZ}}^{-1}$, and we refer to it as the **rational Frobenius**. We have $\tau = (\pi^{-1} \cdot \mathrm{id}, \mathrm{Frob}) \in J \times W$, the π^{-1} coming from contravariance of cohomology with compact support.

3.1.3 One also defines in a similar way the cohomology of the connected components $\mathcal{M}_i^{\mathrm{an}}$. Any element $g \in J$ induces an isomorphism

$$g : \mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^\bullet(\mathcal{M}_{i+\alpha(g)}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

Besides, Frob induces an isomorphism between the cohomology of $\mathcal{M}_i^{\mathrm{an}}$ and that of $\mathcal{M}_{i+1}^{\mathrm{an}}$. Let $(J \times W)^\circ$ be the subgroup of elements $(g, u\mathrm{Frob}^j)$ where $u \in I$ and $\alpha(g) = -j$. In fact we have $(J \times W)^\circ = (J^\circ \times I)\tau^{\mathbb{Z}}$. Then each cohomology group $\mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is a $(J \times W)^\circ$-representation and we have

$$\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c} - \mathrm{Ind}_{(J\times W)^\circ}^{J\times W} \mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

3.1.4 We write $\mathcal{L}_i^{\max}$ for the subset of $\mathcal{L}_i$ consisting only of lattices of orbit type $t_{\max}$, and we write $\mathcal{L}^{\max}$ for the disjoint union of the $\mathcal{L}_i^{\max}$. The collection $\{U_\Lambda\}_{\Lambda \in \mathcal{L}^{\max}}$ forms a locally finite open cover of $\mathcal{M}^{\mathrm{an}}$. By [Far04] Proposition 4.2.2, we obtain a spectral sequence concentrated in degrees $a \leqslant 0$ and $0 \leqslant b \leqslant 2(n-1)$,

$$E_1^{a,b} = \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

Here for $s \geqslant 1$ the index set is given by

$$I_s := \left\{ \gamma = (\Lambda^1, \ldots, \Lambda^s) \,\middle|\, \forall 1 \leqslant j \leqslant s, \Lambda^j \in \mathcal{L}^{\max} \text{ and } U(\gamma) := \bigcap_{j=1}^{s} U_{\Lambda^j} \neq \varnothing \right\}.$$

We note that if $\gamma = (\Lambda^1, \ldots, \Lambda^s) \in I_s$ then there exists $i \in \mathbb{Z}$ such that $\Lambda^j \in \mathcal{L}_i^{\max}$ for all j, and $U(\gamma) = U_{\Lambda(\gamma)}$ where $\Lambda(\gamma) := \bigcap_{j=1}^{s} \Lambda^j \in \mathcal{L}_i$.
For $\Lambda, \Lambda' \in \mathcal{L}_i$ with $\Lambda' \subset \Lambda$, let $f_{\Lambda',\Lambda}^b : \mathrm{H}_c^b(U_{\Lambda'}, \overline{\mathbb{Q}_\ell}) \to \mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$ denote the natural map induced by the inclusion of the open subspace $U_{\Lambda'} \subset U_\Lambda$. For $\gamma = (\Lambda^1, \ldots, \Lambda^s) \in I_s$, let $\gamma_j := (\Lambda^1, \ldots, \widehat{\Lambda^j}, \ldots, \Lambda^s) \in I_{s-1}$ denote tuple obtained by removing the j-th component from γ. For $a \leqslant -1$, the differential $\varphi_{-a}^b : E_1^{a,b} \to E_1^{a+1,b}$ is the direct sum over all $\gamma \in I_{-a+1}$ of the maps

$$\mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) \to \bigoplus_{\delta \in \{\gamma_1, \ldots \gamma_{-a+1}\}} \mathrm{H}_c^b(U(\delta), \overline{\mathbb{Q}_\ell})$$
$$v \mapsto \sum_{j=1}^{-a+1} \gamma_j \cdot (-1)^{j+1} f_{\Lambda(\gamma),\Lambda(\gamma_j)}^b(v).$$

The notation $\gamma_j \cdot (-1)^{j+1} f_{\Lambda(\gamma),\Lambda(\gamma_j)}^b(v)$ means that we consider the vector $(-1)^{j+1} f_{\Lambda(\gamma),\Lambda(\gamma_j)}^b(v)$ inside the summand $\mathrm{H}_c^b(U(\delta), \overline{\mathbb{Q}_\ell})$ corresponding to $\delta = \gamma_j$.

There is a natural action of J on I_s, and $g^{-1} \in J$ induces an isomorphism between the cohomology of $U(\gamma)$ and that of $U(g \cdot \gamma)$. This induces a J-action on $E_1^{a,b}$ for which the spectral sequence is equivariant.

Remark. The descent datum $\pi\alpha_{\mathrm{RZ}}^{-1}$, mapping a k-point M to $\tau(M)$, induces the $\mathbb{F}_p$-rational structure on $\mathcal{M}_\Lambda \otimes \mathbb{F}$. It induces an action of τ on the cohomology of U_Λ. Since for any $\gamma \in I_s$ we have $\pi \cdot \gamma \in I_s$, each term $E_1^{a,b}$ carries a W-representation. The spectral sequence is then $J \times W$-equivariant.

3.1.5 For $\Lambda \in \mathcal{L}$, the cohomology groups $\mathrm{H}_c^\bullet(U_\Lambda, \overline{\mathbb{Q}_\ell})$ are representations of the subgroup $(J_\Lambda \times I) \cdot \tau^{\mathbb{Z}} \subset (J \times W)^\circ$. They are related to the cohomology of the special fiber $\mathcal{M}_\Lambda$ by the following proposition.

Proposition. *Let $\Lambda \in \mathcal{L}$. There is a natural $(J_\Lambda \times I) \cdot \tau^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^{2(n-1)-b}(\mathcal{M}_\Lambda, \overline{\mathbb{Q}_\ell})^\vee(n-1),$$

where on the right-hand side the inertia I acts trivially and the rational Frobenius τ acts like the Frobenius F.

Proof. By the same arguments as in [Mul22a] 4.1.5 Proposition, we have an isomorphism

$$\mathrm{H}^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^b(\mathcal{M}_\Lambda, \overline{\mathbb{Q}_\ell}).$$

This requires the fact that the integral model of the Shimura variety S_{K^p} with hyperspecial level at p is smooth, so that the nearby cycles sheaf is trivial. Since $\mathcal{M}_\Lambda$ is projective, the right-hand side coincide with the cohomology with compact support. On the other hand, we apply Poincaré duality on the left-hand side to obtain

$$\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^{2(n-1)-b}(U_\Lambda, \overline{\mathbb{Q}_\ell})^\vee(n-1) \simeq \mathrm{H}_c^{2(n-1)-b}(\mathcal{M}_\Lambda, \overline{\mathbb{Q}_\ell})^\vee(n-1).$$

□

Remark. The cohomology groups $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda)$ decompose as a sum of irreducible unipotent representations of $\mathrm{GSp}(2\theta, \mathbb{F}_p)$, inflated to J_Λ. The smallest field of definition of unipotent representations of classical groups is $\mathbb{Q}$ by [Lus02], therefore they are autodual. Thus, we have a $\mathrm{GSp}(2\theta, \mathbb{F}_p)$-equivariant isomorphism $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda)^\vee \simeq \mathrm{H}_c^\bullet(\mathcal{M}_\Lambda)$, but it is not equivariant for the action of the Frobenius F.

The situation is less favorable than in the unramified case since the Frobenius action on the cohomology of $\mathcal{M}_\Lambda$, and consequently of U_Λ as well, is not pure (at least when $t(\Lambda) \geqslant 6$). Therefore, [Mul22a] 4.1.7 Corollary does not seem to hold in general, ie. one may not deduce from the previous proposition that the spectral sequence degenerates on the second page, splits and that τ acts semi-simply on the abutment. However, the spectral sequence does eventually degenerate in deeper pages since the non-zero terms $E_1^{a,b}$ are concentrated in a finite range for b. In particular, the inertia acts trivially on $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}})$.

3.1.6 The non-zero $E_1^{a,b}$ terms extend indefinitely in the range $a \leqslant 0$, since the index set I_{-a+1} allows for tuples of the form $\gamma = (\Lambda, \ldots, \Lambda)$ for $\Lambda \in \mathcal{L}^{\max}$ for instance. As in [Mul22a] 4.1.8, we may introduce the alternating version of the Čech spectral sequence in order to fix this issue. If $v \in E_1^{a,b}$, let $v_\gamma \in \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ denote the component of v in the summand of $E_1^{a,b}$ indexed by γ. For all a, b we define

$$E_{1,\mathrm{alt}}^{a,b} := \{v \in E_1^{a,b} \,|\, \forall \gamma \in I_{-a+1}, \forall \sigma \in \mathfrak{S}_{-a+1}, v_{\sigma(\gamma)} = \mathrm{sgn}(\sigma) v_\gamma\},$$

where $\sigma(\gamma)$ denotes the tuple obtained after permuting the components of γ via the permutation σ.

The subspace $E_{1,\mathrm{alt}}^{a,b} \subset E_1^{a,b}$ is stable under the action of $J \times W$, and is compatible with the differentials φ_{-a}^b. As stated in [Mul22a] 4.1.8 Proposition, we have a natural isomorphism $E_{2,\mathrm{alt}}^{a,b} \simeq E_2^{a,b}$.

By definition we have $E_{1,\mathrm{alt}}^{0,b} = E_1^{0,b}$. Moreover, as explained in [Mul22a] 4.1.9, we have the following statement. It holds because if $\Lambda, \Lambda' \in \mathcal{L}_i^{\max}$ are two distinct vertex lattices such that $\Lambda \cap \Lambda' \in \mathcal{L}_i$, then $t(\Lambda \cap \Lambda') < t_{\max}$.

Proposition. *We have $E_2^{0,2(n-1-\theta_{\max})} \simeq E_1^{0,2(n-1-\theta_{\max})}$. If moreover $\theta_{\max} \geqslant 1$ (ie. $n \geqslant 2$ with C split or $n \geqslant 3$), then we have $E_2^{0,2(n-1-\theta_{\max})+1} \simeq E_1^{0,2(n-1-\theta_{\max})+1}$ as well.*

3.1.7 Similarily, let us assume $\theta_{\max} \geqslant 2$, ie. $n = 4$ with C split or $n \geqslant 5$. Consider the case $b = 2(n - \theta_{\max})$. If $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$ with $a \leqslant -1$, and if there exists $\Lambda^j \neq \Lambda^{j'}$, then $t(\Lambda(\gamma)) < 2\theta_{\max}$. Thus, according to 3.1.5 Proposition and 1.4.3 Theorem, there is no eigenvalue of the form $-p^{j+1}$ occuring in $\mathrm{H}_c^{2(n-\theta_{\max})}(U(\gamma), \overline{\mathbb{Q}_\ell})$. It follows that the image of the differential $\varphi_1^{2(n-\theta_{\max})} : E_{1,\mathrm{alt}}^{-1,2(n-\theta_{\max})} \to E_{1,\mathrm{alt}}^{0,2(n-\theta_{\max})}$ intersects trivially with the eigenspace of $E_{1,\mathrm{alt}}^{0,2(n-\theta_{\max})}$ attached to $-p^{n-\theta_{\max}}$. This observation gives the following Proposition.

Proposition. *We have an isomorphism $(E_2^{0,2(n-\theta_{\max})})_{-p^{n-\theta_{\max}}} \simeq (E_1^{0,2(n-\theta_{\max})})_{-p^{n-\theta_{\max}}}$ between the two eigenspaces of τ associated to the eigenvalue $-p^{n-\theta_{\max}}$.*

3.1.8 Let us focus on the term $E_1^{0,2(n-1-\theta_{\max})}$. It is the direct sum of all the cohomology groups $\mathrm{H}_c^{2(n-1-\theta_{\max})}(U_\Lambda, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}_c^{2\theta_{\max}}(\mathcal{M}_\Lambda, \overline{\mathbb{Q}_\ell})^\vee(n-1)$ for $\Lambda \in \mathcal{L}^{\max}$. By 1.4.2 Theorem, we have $\mathrm{H}_c^{2(n-1-\theta_{\max})}(U_\Lambda, \overline{\mathbb{Q}_\ell}) \simeq \mathbf{1}$ and the rational Frobenius τ acts like multiplication by $p^{n-1-\theta_{\max}}$. The same eigenvalue occurs in all the non-zero terms of the row $b = 2(n-1-\theta_{\max})$, but nowhere else. Indeed, let us consider another non-zero term $E_1^{a,b}$ in the spectral sequence with $b > 2(n-1-\theta_{\max})$. There must be some vertex lattice $\Lambda = \Lambda(\gamma) \in \mathcal{L}$ with $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$ such that $\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$ is not zero. If all the Λ^j's are equal, then $t(\Lambda) = 2\theta_{\max}$ and the eigenvalue $p^{n-1-\theta_{\max}}$ does not occur in $E_1^{a,b}$ since, according to 1.4.2 Theorem, the eigenvalue $p^{\theta_{\max}}$ only occurs in the cohomology group of $\mathcal{M}_\Lambda$ of highest degree. If there exists $\Lambda^j \neq \Lambda^{j'}$, then $t(\Lambda) = 2\theta$ with $\theta < \theta_{\max}$ and by 3.1.5 Proposition, we have that $0 \leqslant 2(n-1) - b \leqslant 2\theta$. The possible Frobenius eigenvalues on $\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}_\Lambda)$ have the form $-p^{j+1}$ for some $j \geqslant 0$ and p^i for some $0 \leqslant i \leqslant n-1-b+\lceil b/2 \rceil$ according to 1.4.2 Theorem. After taking dual and Tate twist by $n-1$, it follows that the possible Frobenius eigenvalues on $H_c^b(U_\Lambda)$ have the form $-p^{j+1}$ for some j and p^i for some $b - \lceil b/2 \rceil \leqslant i \leqslant n-1$. Since $b \geqslant 2(n-1-\theta)$ we have $b - \lceil b/2 \rceil \geqslant n-1-\theta$.

In particular, we conclude that $i > n-1-\theta_{\max}$ so that the eigenvalue $p^{n-1-\theta_{\max}}$ does not appear in $E_1^{a,b}$, and so neither in $E_k^{a,b}$ in the deeper pages.
To sum up, we have observed that the eigenvalue $p^{n-1-\theta_{\max}}$ of τ only appears in the row $b = 2(n-1-\theta_{\max})$. The Frobenius equivariance of the spectral sequence forces all the differentials connected to this term in the deeper pages E_k for $k \geqslant 2$ to be zero. Combining this with 3.1.6 Proposition, we conclude that for all $k \geqslant 1$, we have $E_k^{0,2(n-1-\theta_{\max})} \simeq E_1^{0,2(n-1-\theta_{\max})}$. Since this term is the last non zero term of its diagonal, it contributes to a subspace of $\mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}})$. Thus, we have obtained the following statement.

Theorem. *There is a $J \times W$-equivariant monomorphism*

$$E_1^{0,2(n-1-\theta_{\max})} \hookrightarrow \mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}}).$$

3.1.9 We may repeat the exact same arguments by looking this time at the smallest eigenvalue of the form $-p^{j+1}$ in the spectral sequence. Assume that $n = 4$ with C split or that $n \geqslant 5$, so that $\theta_{\max} \geqslant 2$. We observe that the eigenvalue $-p^{n-\theta_{\max}}$ only appears in the row $b = 2(n-\theta_{\max})$ of the spectral sequence. Combining this with 3.1.7 Proposition, and since the term $E_1^{0,2(n-\theta_{\max})}$ is the last non zero of its diagonal, we obtain the following statement.

Theorem. *Assume that $n = 4$ with C split or that $n \geqslant 5$. There is a $J \times W$-equivariant monomorphism*

$$(E_1^{0,2(n-\theta_{\max})})_{-p^{n-\theta_{\max}}} \hookrightarrow \mathrm{H}_c^{2(n-\theta_{\max})}(\mathcal{M}^{\mathrm{an}})_{-p^{n-\theta_{\max}}}.$$

3.1.10 In order to analyze the J-action on $E_1^{a,b}$, we rewrite the direct sum by making compactly induced representations appear. For $s \geqslant 1$ we define

$$I_s^{(\theta)} := \{\gamma \in I_s \,|\, t(\Lambda(\gamma)) = 2\theta\}.$$

We denote by $N(\Lambda_\theta)$ the set $N(\theta_{\max} - \theta, V_\theta^1)$ that we defined in 2.2. It corresponds to the set of lattices $\Lambda \in \mathcal{L}_0$ of orbit type $t_{\max}$ containing Λ_θ. We then define

$$K_s^{(\theta)} := \{\gamma \in I_s^{(\theta)} \,|\, \Lambda(\gamma) = \Lambda_\theta\}.$$

The action of J on $I_s^{(\theta)}$ restricts to an action of J_θ on the finite set $K_s^{(\theta)}$. If $\gamma \in I_s^{(\theta)}$ then there exists some $g \in J$ such that $g \cdot \Lambda(\gamma) = \Lambda_\theta$. Therefore $g \cdot \gamma \in K_s^{(\theta)}$, and the coset $J_\theta \cdot g$ is uniquely determined. This induces a bijection of orbit sets

$$J \backslash I_s^{(\theta)} \xrightarrow{\sim} J_\theta \backslash K_s^{(\theta)}.$$

The terms of the spectral sequence can be rewritten in the following way.

Proposition. *We have an isomorphism*

$$\begin{aligned} E_1^{a,b} &\simeq \bigoplus_{\theta=0}^{\theta_{\max}} \bigoplus_{[\gamma] \in J_\theta \backslash K_{-a+1}^{(\theta)}} \mathrm{c}-\mathrm{Ind}_{\mathrm{Fix}(\gamma)}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\gamma)} \\ &\simeq \bigoplus_{\theta=0}^{\theta_{\max}} \mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \left(\mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]\right), \end{aligned}$$

where $\overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]$ is the permutation representation associated to the action of J_θ on the finite set $K_{-a+1}^{(\theta)}$.

The proof is strictly identic to [Mul22a] 4.1.10 Proposition. In particular, when $a = 0$ we have

$$E_1^{0,b} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathrm{H}_c^b(U_{\Lambda_{\theta_{\max}}}, \overline{\mathbb{Q}_\ell}).$$

In order to alleviate the notations, for any integer $\theta \geqslant 2$ we will write

$$\rho_\theta := \rho_{\begin{pmatrix} 0 & 1 & \theta \end{pmatrix}}.$$

Combining with the results of the two previous paragraphs, we have the following statement.

Corollary. *We have*

$$E_1^{0,2(n-1-\theta_{\max})} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathbf{1} \hookrightarrow H_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}}).$$

Assume that $n = 4$ with C split or that $n \geqslant 5$. We have

$$(E_1^{0,2(n-\theta_{\max})})_{-p^{n-\theta_{\max}}} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{\theta_{\max}} \hookrightarrow \mathrm{H}_c^{2(n-\theta_{\max})}(\mathcal{M}^{\mathrm{an}})_{-p^{n-\theta_{\max}}}.$$

3.1.11 In [Mul22a] 4.2, we made a summary of a general analysis of compactly induced representation in regards to type theory, using results of [BK98], [Bus90] and [Mor99]. It allows us to describe the irreducible subquotients of the representation $\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathbf{1}$. For $n = 1$ and for $n = 2$ with C non-split, we have $\theta_{\max} = 0$ and the trivial representation $\mathbf{1}$ is cuspidal for the maximal reductive quotient $\mathcal{J}_0 \simeq \mathrm{G}(\mathrm{Sp}(0, \mathbb{F}_p) \times \mathrm{O}(V_0^1))$. Here, we note that V_0^1 has dimension n, and if $n = 2$ with C non-split then $\mathrm{GO}(V_0^1) \simeq \mathrm{GO}^-(2, \mathbb{F}_p)$ is the non-split finite orthogonal group in two variables. Let χ be an unramified character of $\mathrm{Z}(J) \simeq E^\times$. We then define

$$\sigma_{0,\chi} := \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_0)}^{J} \mathbf{1} \otimes \chi,$$

where, as in 2.1.13 Proposition, $\mathrm{N}_J(J_0) = \mathrm{Z}(J)J_0$ is the normalizer of J_0 in J. According to 2.1.12 Proposition, the parahoric subgroup $J_0^\circ \subset J_0$ is maximal. Besides, we have $\mathrm{N}_J(J_0) = \mathrm{N}_J(J_0^\circ)$. Thus, according to [Mor99] 4.1 Proposition, $\sigma_{0,\chi}$ is an irreducible supercuspidal representation of J. If $n \geqslant 3$ or if $n = 2$ with C split, then the trivial representation is not cuspidal for $\mathcal{J}_{\theta_{\max}}$.
Eventually, if V is any smooth representation of J and if χ is any smooth character of $\mathrm{Z}(J) \simeq E^\times$, we denote by V_χ the largest quotient of V on which $\mathrm{Z}(J)$ acts through χ.

Proposition. *Let χ be an unramified character of $E^\times$.*

1. *If $n = 1$ or $n = 2$ with C non-split, all irreducible subquotients of $V := \mathrm{c}-\mathrm{Ind}_{J_0}^{J} \mathbf{1}$ are supercuspidal, and we have $V_\chi \simeq \sigma_{0,\chi}$.*
2. *If $n \geqslant 3$ or if $n = 2$ with C split, then no irreducible subquotient of $V := \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathbf{1}$ is supercuspidal. In this case, V_χ does not contain any non-zero admissible subrepresentation of J.*

Combining this proposition with 3.1.10 Corollary, we deduce the following statement.

Corollary. *If $n \geqslant 3$ or $n = 2$ with C split, and if χ is any unramified character of $E^\times$, then $\mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}})_\chi$ is not J-admissible.*

3.1.12 We repeat the same argument to analyze the irreducible subquotients of $\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J}\rho_{\theta_{\max}}$. If $n=4$ with C split, if $n=5$ or if $n=6$ with C non-split, we have $\theta_{\max}=2$ and ρ_2 is a cuspidal representation of $\mathcal{J}_2 \simeq \mathrm{G}(\mathrm{Sp}(4,\mathbb{F}_p)\times \mathrm{O}(V_2^1))$ (we note that V_2^1 has dimension $n-4$). Let χ be an unramified character of $\mathrm{Z}(J)\simeq E^\times$. We define

$$\sigma_{2,\chi} := \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_2)}^{J}\rho_2\otimes\chi,$$

where $\mathrm{N}_J(J_2)=\mathrm{N}_J(J_2^\circ)=\mathrm{Z}(J)J_2$. Then $\sigma_{2,\chi}$ is an irreducible supercuspidal representation of J. If $n\geqslant 7$ or if $n=6$ with C split, then $\rho_{\theta_{\max}}$ is not a cuspidal representation of $\mathcal{J}_{\theta_{\max}}$. We deduce the following consequences.

Proposition. *Let χ be an unramified character of $E^\times$.*

(1) *If $n=4$ with C split, if $n=5$ or if $n=6$ with C non-split, all irreducible subquotients of $V:=\mathrm{c}-\mathrm{Ind}_{J_2}^{J}\rho_2$ are supercuspidal, and we have $V_\chi\simeq\sigma_{2,\chi}$.*

(2) *If $n\geqslant 7$ or if $n=6$ with C split, then no irreducible subquotient of $V:=\mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J}\rho_{\theta_{\max}}$ is supercuspidal. In this case, V_χ does not contain any non-zero admissible subrepresentation of J.*

Combining this proposition with 3.1.10 Corollary, we deduce the following statement.

Corollary. *If $n\geqslant 7$ or $n=6$ with C split, and if χ is any unramified character of $E^\times$, then $\mathrm{H}_c^{2(n-\theta_{\max})}(\mathcal{M}^{\mathrm{an}})_\chi$ is not J-admissible.*

3.1.13 We finish this section with the following observation regarding the cohomology group (with compact support) of highest degree.

Proposition. *There is an isomorphism*

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}},\overline{\mathbb{Q}_\ell})\simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\mathbf{1},$$

where $\mathbf{1}$ *denotes the trivial representation, and where* Frob *acts like* $p^{n-1}\cdot\mathrm{id}$.

The proof is identic to [Mul22a] 4.1.12 Proposition.

3.2 The spectral sequence for small values of n

3.2.1 If $\theta_{\max}=0$, ie. if $n=1$ or $n=2$ and C is non-split, then C is anisotropic and $\mathcal{L}_i$ is a singleton. There is only one non-zero term in the alternate version $E_{1,\mathrm{alt}}$ of the spectral sequence, equal to $\mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\mathbf{1}$, and it computes the cohomology group $\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}})$ as we already checked in 3.1.13. Thus, there is not much to say in this case.

3.2.2 We assume now that $\theta_{\max} = 1$, ie. $n = 2$ and C is split, $n = 3$ or $n = 4$ and C is non-split. In this case, the orbit type 2θ of any vertex lattice is 0 or 2, so that $\theta = 0$ or 1. Recall from 3.1.10 that $N(\Lambda_0)$ denotes the set of vertex lattices $\Lambda \in \mathcal{L}_0$ of orbit type $t_{\max} = 2$ containing Λ_0. According to 2.2 Proposition, we have

$$\#N(\Lambda_0) = \begin{cases} 2 & \text{if } n = 2 \text{ and } C \text{ is split,} \\ p+1 & \text{if } n = 3, \\ p^2+1 & \text{if } n = 4 \text{ and } C \text{ is non-split.} \end{cases}$$

We may precisely locate the non-zero terms in the alternate version of the spectral sequence.

$$E_{1,\mathrm{alt}}^{a,b} \neq 0 \iff \begin{cases} (a,b) \in \{(0,0);(0,2);(-1,2)\} & \text{if } n = 2 \text{ and } C \text{ is split,} \\ (a,b) \in \{(0,2);(-k,4)\,|\,0 \leqslant k \leqslant p\} & \text{if } n = 3, \\ (a,b) \in \{(0,4);(-k,6)\,|\,0 \leqslant k \leqslant p^2\} & \text{if } n = 4 \text{ and } C \text{ is non-split.} \end{cases}$$

In Figure 2 and 3, we draw the first page of the alternate version of the Čech spectral sequence respectively when $n = 2$ and C is split, and when $n = 3$. In brackets we have written the scalar by which τ acts on each term. The spectral sequence in the case $n = 4$ with C non-split is similar to Figure 3, except that two more 0 rows must be added at the bottom, and all eigenvalues of τ are multiplied by p. In order to alleviate the notations, we write φ_s for the differentials in the top row. Given the shape of these sequences and taking into account the Frobenius weights of each term, we observe that they degenerate on the second page.

$$E_{1,\mathrm{alt}}^{-1,2}[p] \xrightarrow{\varphi_1} \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}[p]$$

$$0$$

$$\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}[1]$$

Figure 2: The first page $E_{1,\mathrm{alt}}$ when $n = 2$ and C is split.

3.2.3 When $\theta_{\max} = 1$, the simplicial complex $\mathcal{L}_0$ is actually a tree. In this case, we have the following proposition.

Proposition. *Let $b = 2, 4$ or 6 respectively if $n = 2$ with C split, $n = 3$ or $n = 4$ with C non-split. We have $E_2^{-1,b} = 0$.*

The proof is strictly identic to [Mul22a] 4.3.2 Proposition, so that we omit it.
In particular, we obtain the following statement.

Theorem. *Let $b = 1, 3$ or 5 respectively if $n = 2$ with C split, $n = 3$ or $n = 4$ with C non-split. We have $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) = 0$.*

$$\ldots \xrightarrow{\varphi_4} E_{1,\mathrm{alt}}^{-3,2}[p^2] \xrightarrow{\varphi_3} E_{1,\mathrm{alt}}^{-2,2}[p^2] \xrightarrow{\varphi_2} E_{1,\mathrm{alt}}^{-1,2}[p^2] \xrightarrow{\varphi_1} \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}[p^2]$$

$$0$$

$$\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}[p]$$

$$0$$

$$0$$

Figure 3: The first page $E_{1,\mathrm{alt}}$ when $n = 3$.

We observe that in the case $n = 2$ with C split, the cohomology of $\mathcal{M}^{\mathrm{an}}$ is now entirely understood. Namely, $\mathcal{M}^{\mathrm{an}}$ has dimension 1, and we have $\mathrm{H}_c^0(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}$ with τ acting like id, $\mathrm{H}_c^1(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) = 0$ and $\mathrm{H}_c^2(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\mathbf{1}$ with τ acting like $p \cdot \mathrm{id}$.

4 The cohomology of the basic stratum of the Shimura variety for small values of n

4.1 The Hochschild-Serre spectral sequence induced by p-adic uniformization

4.1.1 Related to the p-adic uniformization theorem in 2.3.3, Fargues has built in [Far04] a spectral sequence relating the cohomology of $\mathcal{M}^{\mathrm{an}}$ to that of $\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss,an}}$. Even though the construction of loc. cit. is done in the context of unramified Rapoport-Zink spaces, it works in greater generality as mentioned in the last paragraph of 4.5.2.1.
Recall the notations of section 2.3. Let $\xi : \mathbb{G} \to W_\xi$ be a finite-dimensional irreducible algebraic representation of $\mathbb{G}$ over $\overline{\mathbb{Q}_\ell}$. In [Mul22a] 5.1.1, we recalled the classification of all such representations ξ following [HT01] III.2. Let $\mathbb{V}_0$ denote the dual of $\mathbb{V} \otimes \overline{\mathbb{Q}_\ell}$ on which $\mathbb{G}$ acts. There exists uniquely defined integers $t(\xi), m(\xi) \geqslant 0$ and an idempotent $\epsilon(\xi) \in \mathrm{End}(\mathbb{V}_0^{\otimes m(\xi)})$ such that

$$W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)}),$$

where c denotes the similitude factor. The weight of ξ is defined by

$$w(\xi) := m(\xi) - 2t(\xi).$$

One can associate to ξ a local system $\mathcal{L}_\xi$ on the tower $(\mathrm{S}_{K^p})_{K^p}$ of Shimura varieties. Let $\mathcal{A}_{K^p}$ be the universal abelian scheme over S_{K^p}. We write $\pi_{K^p}^m : \mathcal{A}_{K^p}^m \to \mathrm{S}_{K^p}$ for the structure morphism of the m-fold product of $\mathcal{A}_{K^p}$ with itself over S_{K^p}. Then

$$\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)}\left(\mathrm{R}^{m(\xi)}(\pi_{K^p}^{m(\xi)})_*\overline{\mathbb{Q}_\ell}(t(\xi))\right),$$

where $\epsilon_{m(\xi)}$ is some idempotent. We denote by $\overline{\mathcal{L}_\xi}$ its restriction to the special fiber $\overline{\mathrm{S}}_{K^p}$.

4.1.2 Let $\mathcal{A}_\xi$ be the space of **automorphic forms of I of type ξ at infinity**. Explicitly, it is given by

$$\mathcal{A}_\xi = \{f : I(\mathbb{A}_f) \to W_\xi \,|\, f \text{ is } I(\mathbb{A}_f)\text{-smooth by right translations and } \forall \gamma \in I(\mathbb{Q}), f(\gamma \cdot) = \xi(\gamma) f(\cdot)\}\,.$$

We denote by $\mathcal{L}_\xi^{\mathrm{an}}$ the analytification of $\mathcal{L}_\xi$.

Notation. We write $\mathrm{H}^\bullet(\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}})$ for the cohomology of $\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss,an}} \widehat{\otimes} \mathbb{C}_p$ with coefficients in $\mathcal{L}_\xi^{\mathrm{an}}$.

Theorem ([Far04] 4.5.12)**.** *There is a W-equivariant spectral sequence*

$$F_2^{a,b}(K^p) : \mathrm{Ext}_J^a\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \mathcal{A}_\xi^{K^p}\right) \implies \mathrm{H}^{a+b}(\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}}).$$

These spectral sequences are compatible as the open compact subgroup K^p varies in $\mathbb{G}(\mathbb{A}_f^p)$.

We may take the limit $\varinjlim_{K^p}$ for all terms and obtain a $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant spectral sequence. According to [Far04] Lemme 4.4.12, we have $F_2^{a,b} = 0$ when $a > \theta_{\max}$ since $\theta_{\max}$ is also the semisimple rank of J. Since the Shimura variety S_{K^p} is smooth, the comparison theorem [Ber96] Corollary 3.7 of Berkovich gives an isomorphism

$$\mathrm{H}_c^{a+b}(\overline{S}_{K^p}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) = \mathrm{H}^{a+b}(\overline{S}_{K^p}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \xrightarrow{\sim} \mathrm{H}^{a+b}(\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}}),$$

where first equality follows from the supersingular locus being a proper variety. Since $\dim \overline{S}_{K^p}^{\mathrm{ss}} = \theta_{\max}$ by [RTW14] Theorem 7.2, the cohomology $\mathrm{H}^\bullet(\widehat{\mathrm{S}}_{K^p}^{\mathrm{ss,an}}, \mathcal{L}_\xi^{\mathrm{an}})$ is concentrated in degrees 0 to $2\theta_{\max}$.

4.1.3 Let $\mathcal{A}(I)$ denote the set of all automorphic representations of I counted with multiplicities, and let $\check{\xi}$ be the contragredient of ξ. We also define

$$\mathcal{A}_\xi(I) := \{\Pi \in \mathcal{A}(I) \,|\, \Pi_\infty = \check{\xi}\}.$$

According to [Far04] 4.6, we have an identification

$$\mathcal{A}_\xi^{K_p} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \Pi_p \otimes (\Pi^p)^{K_p}.$$

We deduce that

$$F_2^{a,b} := \varinjlim_{K^p} F_2^{a,b}(K^p) \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p.$$

The spectral sequence defined by the terms $F_2^{a,b}$ computes the cohomology of $\overline{S}^{\mathrm{ss}} := \varinjlim_{K^p} \overline{S}_{K^p}^{\mathrm{ss}}$.

4.1.4 Let us focus on the Frobenius action on the Ext groups occuring in the spectral sequence. To this effect, we need the following lemma. For $\Pi \in \mathcal{A}_\xi(I)$, let ω_Π denote the central character. For any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, we define $|\cdot|_\iota := |\iota(\cdot)|$.

Lemma. *We have* $|\omega_{\Pi_p}(\pi^{-1}\cdot \mathrm{id})|_\iota = p^{w(\xi)/2}$.

Proof. We have

$$|\omega_{\Pi_p}(\pi^{-1}\cdot \mathrm{id})|_\iota^2 = |\omega_{\Pi_p}(p^{-1}\cdot \mathrm{id})|_\iota.$$

Indeed, π^2 is equal to p up to a unit in $\mathbb{Z}_p^\times$. The value of the central character ω_{Π_p} at this unit has complex modulus 1 under any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$. Since I is the group of unitary similitudes of some $\mathbb{E}/\mathbb{Q}$-hermitian space, its center is isomorphic to $\mathbb{E}^\times \cdot \mathrm{id}$. In particular, the element $p^{-1}\cdot \mathrm{id}$ in $\mathrm{Z}(J)$ can be seen as the image of $p^{-1}\cdot\mathrm{id}$ in $\mathrm{Z}(\mathrm{I}(\mathbb{Q}))$. We have $\omega_\Pi(p^{-1}\cdot\mathrm{id}) = 1$, and at every finite place q different from p we have $|\omega_{\Pi_q}(p^{-1}\cdot\mathrm{id})|_\iota = 1$, since $p^{-1}\cdot\mathrm{id}$ lies in the maximal compact subgroup of $\mathrm{Z}(I(\mathbb{Q}_q))$. Eventually, the fact that $\Pi_\infty = \check{\xi}$ implies that

$$|\omega_{\Pi_p}(p^{-1}\cdot \mathrm{id})|_\iota = |\omega_{\check{\xi}}(p^{-1}\cdot \mathrm{id})|_\iota^{-1} = |\omega_{\xi}(p^{-1}\cdot \mathrm{id})|_\iota = p^{w(\xi)},$$

the last equality being a consequence of $W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)})$ (see 4.1.1). □

Let us fix a square root π_ℓ of p in $\overline{\mathbb{Q}_\ell}$. We define

$$\delta_{\Pi_p} := \omega_{\Pi_p}(\pi^{-1}\cdot\mathrm{id})\pi_\ell^{-w(\xi)}.$$

The lemma implies that $|\delta_{\Pi_p}|_\iota = 1$ for any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$.
By convention, the action of Frob on a space $\mathrm{Ext}^a_{J\text{-sm}}(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p)$ is given by functoriality of Ext applied to Frob^{-1} acting on the cohomology of $\mathcal{M}^{\mathrm{an}}$. Recall that the action of Frob on the cohomology is the composition of τ and of $\pi\cdot\mathrm{id} \in J$. Let $P_\bullet$ be a projective resolution of $\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n)$ in the category of smooth representations of J. Let $\mathcal{T} : P_\bullet \to P_\bullet$ be a lift of τ^{-1} as a morphism of chain complexes. For $a \geqslant 0$, the action of Frob on an element of $\mathrm{Ext}^a_{J\text{-sm}}(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p)$ represented by a function $f : P_a \to \Pi_p$, is given by

$$\mathrm{Frob}^* f(v) = f((\pi^{-1}\cdot\mathrm{id})\mathcal{T}_a v) = \omega_{\Pi_p}(\pi^{-1}\cdot\mathrm{id})f(\mathcal{T}_a v) = \delta_{\Pi_p}\pi_\ell^{w(\xi)} f(\mathcal{T}_a v).$$

In particular, if τ acts like $x\cdot\mathrm{id}$ on the cohomology of $\mathcal{M}^{\mathrm{an}}$ for some $x \in \overline{\mathbb{Q}_\ell}^\times$, then Frob acts by multiplication by $\delta_{\Pi_p}\pi_\ell^{w(\xi)}x^{-1}p^{n-1}$ on the corresponding Ext groups.

4.1.5 If $x \in \overline{\mathbb{Q}_\ell}^\times$, let $\overline{\mathbb{Q}_\ell}[x]$ denote the 1-dimensional representation of W where the inertia acts trivially and Frob acts like multiplication by the scalar x. Let $X^{\mathrm{un}}(J)$ denote the set of unramified characters of J. Looking at the diagonal $a + b = 0$ in the spectral sequence, we obtain the following result.

Proposition. *There is a* $(G(\mathbb{A}_f^p)\times W)$*-equivariant isomorphism*

$$\mathrm{H}_c^0(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi\in\mathcal{A}_\xi(I)\\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p}\pi_\ell^{w(\xi)}].$$

The proof is identic to [Mul22a] 5.1.5 Proposition. It follows from the following facts. First, $F_2^{0,0}$ is the only non-zero term on the diagonal $a+b=0$ and all differentials connected to $F_k^{0,0}$ are zero for $k \geqslant 2$. Then, by 3.1.13 we have $\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}) \simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J} \mathbf{1}$ with τ acting by multiplication by p^{n-1}. Eventually, since J° is normal in J, $J/J^\circ \simeq \mathbb{Z}$ and J° is generated by all the compact open subgroups of J, any smooth irreducible representation of J having some non-zero J°-fixed vectors is actually an unramified character of J.

4.2 The cohomology when $n=2$ with C split, when $n=3$ and when $n=4$ with C non-split

4.2.1 From now on we assume that $\theta_{\max}=1$ so that $n=2$ with C split, $n=3$ or $n=4$ with C non-split. We will use our knowledge of the spectral sequence given by $E_1^{a,b}$ as detailed in section 3.2 in order to compute all the terms $F_2^{a,b}$, and as a consequence we obtain a description of the cohomology of the supersingular locus of the Shimura variety. All the arguments used here are exactly the same as in [Mul22a] Section 5.2.
Since $\theta_{\max}=1$, we have $F_2^{a,b}=0$ for all $a \geqslant 2$. As a consequence, all differentials in the second and deeper pages of the sequence are zero, so that it already degenerates on the second page. Moreover, the supersingular locus $\overline{S}^{\mathrm{ss}}$ has dimension one, thus $F_2^{0,b}=0$ for $b \geqslant 3$ and $F_2^{1,b}=0$ for $b \geqslant 2$.
In Figure 4, we draw the second page F_2 and we write between brackets the *complex modulus* of the possible eigenvalues of Frob on each term. as computed in 4.1.4.
Remark. The fact that $F_2^{0,1}=F_2^{1,1}=0$ follows from 3.2.3 Theorem.

$F_2^{0,2}[p^{1+w(\xi)/2}, p^{w(\xi)/2}]$	0
0	0
$F_2^{0,0}[p^{w(\xi)/2}]$	$F_2^{1,0}[p^{w(\xi)/2}]$

Figure 4: The second page F_2 with the complex modulus of possible eigenvalues of Frob on each term.

Proposition. *The eigenspaces of* Frob *on* $F_2^{0,2}$ *attached to any eigenvalue of complex modulus* $p^{w(\xi)/2}$ *are zero.*

Proof. By the machinery of spectral sequences, we have a $\mathbb{G}(\mathbb{A}_f^p) \times W$-equivariant isomorphism $\mathrm{H}_c^2(\overline{S}_{K^p}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq F_2^{0,2}$. We prove that no eigenvalue of Frob on this H_c^2 cohomology group has complex modulus $p^{w(\xi)/2}$, and the result readily follows.
Let $K^p \subset \mathbb{G}(\mathbb{A}_f^p)$ be small enough. Recall from 2.3.4 the definition of the Bruhat-Tits strata $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ inside the supersingular locus $\overline{S}_{K^p}^{\mathrm{ss}}$. Each stratum $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is isomorphic to $\mathcal{M}_\Lambda$. For $\Lambda \in \mathcal{L}_i$, define

$$\mathcal{M}_\Lambda^\circ := \mathcal{M}_\Lambda \backslash \bigcup_{\Lambda' \subsetneq \Lambda} \mathcal{M}_{\Lambda'},$$

where Λ' runs over all vertex lattices of $\mathcal{L}_i$ strictly contained in Λ. By [RTW14] Theorem 6.10, each $\mathcal{M}_\Lambda^\circ$ is open dense in $\mathcal{M}_\Lambda$. Via the isomorphism $\mathcal{M}_\Lambda \xrightarrow{\sim} S_\theta$ where $t(\Lambda) = 2\theta$, we have $\mathcal{M}_\Lambda^\circ \xrightarrow{\sim} X_{I_\theta}(w_\theta)$ with the notations of 1.1.5. In particular, $\mathcal{M}_\Lambda^\circ$ is isomorphic to the Coxeter variety for $\mathrm{Sp}(2\theta, \mathbb{F}_p)$. Let $\overline{\mathrm{S}}^\circ_{K^p,\Lambda,k} \subset \overline{\mathrm{S}}_{K^p,\Lambda,k}$ be the scheme theoric image of $\mathcal{M}_\Lambda^\circ$ in the k-th copy of $\mathcal{M}_{\mathrm{red}}$ via the p-adic uniformization $(\Theta_{K^p})_s$ of 2.3.4. We have a stratification

$$\overline{S}^{\mathrm{ss}}_{K^p} = \overline{S}^{\mathrm{ss}}_{K^p}[0] \sqcup \overline{S}^{\mathrm{ss}}_{K^p}[1],$$

where for $i = 0, 1$ the stratum $\overline{S}^{\mathrm{ss}}_{K^p}[i]$ is the finite disjoint union of the $\overline{\mathrm{S}}^\circ_{K^p,\Lambda,k}$ for various k and Λ of orbit type $t(\Lambda) = 2i$. The stratum $\overline{S}^{\mathrm{ss}}_{K^p}[0]$ is closed of dimension 0, and the stratum $\overline{S}^{\mathrm{ss}}_{K^p}[1]$ is open dense of dimension 1. Therefore, we have an isomorphism between the highest degree cohomology groups

$$\mathrm{H}^2_c(\overline{S}^{\mathrm{ss}}_{K^p}, \overline{\mathcal{L}_\xi}) \simeq \mathrm{H}^2_c(\overline{S}^{\mathrm{ss}}_{K^p}[1], \overline{\mathcal{L}_\xi}).$$

Since $\overline{S}^{\mathrm{ss}}_{K^p}[1] = \bigsqcup_{t(\Lambda)=2,k} \overline{\mathrm{S}}^\circ_{K^p,\Lambda,k}$ and each $\overline{\mathrm{S}}^\circ_{K^p,\Lambda,k}$ is open and closed in $\overline{S}^{\mathrm{ss}}_{K^p}[1]$, we have

$$\mathrm{H}^2_c(\overline{S}^{\mathrm{ss}}_{K^p}[1], \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{t(\Lambda)=2,k} \mathrm{H}^2_c(\overline{\mathrm{S}}^\circ_{K^p,\Lambda,k}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{t(\Lambda)=2,k} \mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p,\Lambda,k}, \overline{\mathcal{L}_\xi}),$$

where the last isomorphism follows from the stratification

$$\overline{\mathrm{S}}_{K^p,\Lambda,k} = \overline{\mathrm{S}}^\circ_{K^p,\Lambda,k} \sqcup \bigsqcup_{\Lambda' \subsetneq \Lambda} \overline{\mathrm{S}}^\circ_{K^p,\Lambda',k},$$

with the first term being open dense of dimension 1 and the second term being closed of dimension 0. Since $\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)}\left(\mathrm{R}^{m(\xi)}(\pi^{m(\xi)}_{K^p})_*\overline{\mathbb{Q}_\ell}(t(\xi))\right)$, the local system $\overline{\mathcal{L}_\xi}$ is pure of weight $w(\xi)$. Moreover the variety $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is projective and smooth for $\theta = 1$ (it is actually isomorphic to $\mathbb{P}^1$ by 1.1.4 Proposition). Hence, all eigenvalues of the Frobenius on $H^2_c(\overline{\mathrm{S}}_{K^p,\Lambda,k}, \overline{\mathcal{L}_\xi})$, and therefore on $\mathrm{H}^2_c(\overline{S}^{\mathrm{ss}}_{K^p}, \overline{\mathcal{L}_\xi})$ as well, must have complex modulus $p^{1+t(\xi)/2}$. □

4.2.2 In this paragraph, we compute the term

$$\begin{aligned} F_2^{1,0} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^1_J\left(\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^1_J\left(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1}(1-n), \Pi_p\right) \otimes \Pi^p. \end{aligned}$$

Let St_J denote the Steinberg representation of J, and recall that $X^{\mathrm{un}}(J)$ denotes the set of unramified characters of J.

Proposition. *Let π be an irreducible smooth representation of J. Then*

$$\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1}, \pi) = \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } \exists \chi \in X^{\mathrm{un}}(J), \pi \simeq \chi \cdot \mathrm{St}_J, \\ 0 & \text{otherwise.} \end{cases}$$

The proof follows the same lines as [Mul22a] 5.2.2 Proposition. We abbreviate the similar arguments, but we focus on the main difference caused by the ramified case. Let $J^1 := \mathrm{U}(C) \subset J^\circ \subset J$ denote the unitary group of the $E/\mathbb{Q}_p$-hermitian space C. For the following lemma only, n can be any positive integer.

Lemma. *If n is odd then $\mathrm{Z}(J^\circ)J^1 = J^\circ$. If n is even then $\mathrm{Z}(J^\circ)J^1$ has index 2 in J°.*

Proof. The subgroup $Z(J^\circ)J^1$ consists of all $g \in J^\circ$ such that the multiplier $c(g)$ is a norm, ie. belongs to the subgroup $\mathrm{Norm}_{E/\mathbb{Q}_p}(\mathcal{O}_E^\times) \subset \mathbb{Z}_p^\times$ of index 2 (since $E/\mathbb{Q}_p$ is a ramified quadratic extension). Let $g \in J$ and let M be the matrix of g in the basis e. Let Ω denote the matrix $T_{\text{odd}}, T_{\text{even}}^+$ or T_{even}^- depending on whether n is odd or even with C split or not, as defined in 2.1.7. We have the relation

$$M\Omega\overline{M}^T = c(g)\Omega,$$

where $\overline{M} = (\overline{M_{ij}})_{i,j}$. Taking the determinant, we have $\det(M)\overline{\det(M)} = c(g)^n$. Thus, $c(g)^n$ is a norm, and if n is odd it follows that $c(g)$ is a norm too.
Assume now that n is even and let $\epsilon \in \mathbb{Z}_p^\times$ be a unit which is not a norm. It is enough to exhibit an element $g_0 \in J^\circ$ such that $c(g_0) = \epsilon$. We distinguish three cases.

- **Case C split:** we can take g_0 given by $e_{-j} \mapsto \epsilon e_{-j}$ and $e_j \mapsto e_j$ for all $1 \leqslant j \leqslant \theta_{\max}$.
- **Case C non-split and $p = 1 \mod 4$:** then -1 is a square in $\mathbb{Z}_p^\times$, so in particular it is a norm. Recall that $\Omega = T_{\text{even}}^-$ is defined with two scalars $u_1, u_2 \in \mathbb{Z}_p^\times$ such that $-u_1u_2$ is not a norm. It follows that $u_1u_2^{-1}\epsilon$ is a norm. Let us write $u_1u_2^{-1}\epsilon = \lambda\overline{\lambda}$ for some $\lambda \in \mathcal{O}_E^\times$, and define g_0 by
$$\forall 1 \leqslant j \leqslant \theta_{\max}, \quad e_{-j} \mapsto \epsilon e_{-j}, \qquad e_j \mapsto e_j,$$
$$e_0^{\text{an}} \mapsto \lambda e_1^{\text{an}}, \qquad e_1^{\text{an}} \mapsto \frac{\epsilon}{\lambda} e_0^{\text{an}}.$$
Then one may check that $g_0 \in J^\circ$ with $c(g_0) = \epsilon$.
- **Case C non-split and $p = 3 \mod 4$:** since -1 is not a norm, we may assume that $u_1 = u_2 = 1$. Let $\underline{\epsilon} \in \mathcal{O}_E/\pi\mathcal{O}_E = \mathbb{F}_p$ denote the π-adic residue of ϵ. The polynomial $f(X,Y) = X^2 + Y^2 - \underline{\epsilon}$ has a root in $\mathbb{F}_p^2$, thus by a multivariate version of Hensel's lemma, there exists $\alpha, \beta \in \mathbb{Z}_p^\times$ such that $\alpha^2 + \beta^2 = \epsilon$. We define g_0 by
$$\forall 1 \leqslant j \leqslant \theta_{\max}, \quad e_{-j} \mapsto \epsilon e_{-j}, \qquad e_j \mapsto e_j,$$
$$e_0^{\text{an}} \mapsto \alpha e_0^{\text{an}} + \beta e_1^{\text{an}}, \qquad e_1^{\text{an}} \mapsto \beta e_0^{\text{an}} - \alpha e_1^{\text{an}}.$$
Then one may check that $g_0 \in J^\circ$ with $c(g_0) = \epsilon$.

□

From now, we assume again that $\theta_{\max} = 1$, so that $n = 2$ with C split, $n = 3$ or $n = 4$ with C non-split. In the unramified case [Mul22a] 5.2.2, we have $\mathrm{Z}(J^\circ)J^1 = J^\circ$, thus the proof of the Proposition when n is odd works as verbatim. Let us assume that n is even. Let π be an irreducible smooth representation of J. As explained in loc. cit. we have

$$\mathrm{Ext}_J^1(\mathrm{c} - \mathrm{Ind}_{J^\circ}^J \mathbf{1}, \pi) \simeq \mathrm{Ext}_{J^1}^1(\mathbf{1}, \pi_{|J^1})^{J^\circ/J^1}.$$

In particular, $\mathrm{Ext}_J^1(\mathrm{c} - \mathrm{Ind}_{J^\circ}^J \mathbf{1}, \pi) = 0$ if the central character of π is not unramified. If σ is a smooth irreducible representation of J^1, then

$$\mathrm{Ext}_{J^1}^1(\mathbf{1}, \sigma) = \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } \sigma = \mathrm{St}_{J^1}, \\ 0 & \text{otherwise.} \end{cases}$$

Here St_{J^1} denotes the Steinberg representation of J^1. Since $J = \mathrm{Z}(J)J^{\circ}$, we observe that $\mathrm{Z}(J)J^1$ also has index 2 in J. Since the Steinberg representation St_{J^1} is isomorphic to its conjuguates by elements of J, and since $(\mathrm{St}_J)_{|J^1} = \mathrm{St}_{J^1}$, any smooth irreducible representation π of J which contains St_{J^1} is of the form $\pi \simeq \chi \cdot \mathrm{St}_J$ for some character χ of J which is trivial on J^1. Let χ_0 be the unique non-trivial character of J which is trivial on $\mathrm{Z}(J)J^1$. Then the smooth irreducible representations π of J with unramified central character and whose restriction to J^1 is isomorphic to St_{J^1} are precisely the representations $\chi \cdot \mathrm{St}$ and $\chi\chi_0 \cdot \mathrm{St}_J$, for $\chi \in X^{\mathrm{un}}(J)$. Consider the action of J°/J^1 on the 1-dimensional vector spaces $\mathrm{Ext}^1_{J^1}(\mathbf{1}, (\mathrm{St}_J)_{|J^1})$ and $\mathrm{Ext}^1_{J^1}(\mathbf{1}, (\chi_0 \cdot \mathrm{St}_J)_{|J^1})$. The action is trivial on $\mathrm{Z}(J^{\circ})$, so that it factors through an action of $J^{\circ}/\mathrm{Z}(J^{\circ})J^1 \simeq \mathbb{Z}/2\mathbb{Z}$. The non trivial element of $J^{\circ}/\mathrm{Z}(J^{\circ})J^1$ must act like id on one space, and like $-$id on the other. Taking the invariants, it follows that exactly one of $\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^{\circ}}^J \mathbf{1}, \mathrm{St}_J)$ and $\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^{\circ}}^J \mathbf{1}, \chi_0 \cdot \mathrm{St}_J)$ is $\overline{\mathbb{Q}_\ell}$, the other is 0. Consider P_0 a minimal parabolic subgroup of J and δ_{P_0} the associated modulus character. Let $\iota_{P_0}^J$ denote the normalized parabolic induction functor. By definition, $\iota_{P_0}^J \delta_{P_0}^{-1/2}$ defines a non-trivial extension of St_J by the trivial representation. Since $\mathrm{Z}(J)J^{\circ} = J$, its restriction to J° defines a non-trivial extension of $(\mathrm{St}_J)_{|J^{\circ}}$ by the trivial representation, ie. a non-zero element of $\mathrm{Ext}^1_{J^{\circ}}(\mathbf{1}, (\mathrm{St}_J)_{|J^{\circ}}) \simeq \mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^{\circ}}^J \mathbf{1}, \mathrm{St}_J)$. Thus, it is this Ext group which is non zero, and it completes the proof of the Proposition in the case n even.

4.2.3 As a summary of the analysis detailed in the previous paragraph, we may state our main result. We use the same notations as 4.1.5.

Theorem. *There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}^0_c(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} \pi_\ell^{w(\xi)}],$$

$$\mathrm{H}^1_c(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} \pi_\ell^{w(\xi)}],$$

$$\mathrm{H}^2_c(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} \pi_\ell^{w(\xi)+2}].$$

We note that the statement regarding H^0_c has already been proved in 4.1.5, and the statement regarding H^1_c follows directly from 4.2.1 Figure 4 and 4.2.2 Proposition. Thus, it only remains to justify the formula for H^2_c. We have

$$\mathrm{H}^2_c(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq F_2^{0,2} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J \left(E_2^{0,b}(1-n), \Pi_p\right) \otimes \Pi^p,$$

where $b = 0$ if $n = 2$ with C split, $b = 2$ if $n = 3$ and $b = 4$ if $n = 4$ with C non-split. In all cases we have $E_2^{0,b} \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^J \mathbf{1}$. Using Frobenius reciprocity we have

$$F_2^{0,2} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J(\mathrm{c}-\mathrm{Ind}_{J_1}^J \mathbf{1}(1-n), \Pi_p) \otimes \Pi^p \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J_1}(\mathbf{1}(1-n), (\Pi_p)_{|J_1}) \otimes \Pi^p.$$

Since J_1 is a special maximal compact subgroup of J, we have $\dim(\pi^{J_1}) = 1$ for all smooth irreducible representations of J such that $\pi^{J_1} \neq 0$. Therefore, we have

$$\mathrm{H}^2_c(\overline{S}^{\mathrm{ss}}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} \pi_\ell^{w(\xi)+2}]$$

as claimed, using 4.1.4 for the action of Frob.

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