Cohomology of the Bruhat-Tits strata in the unramified unitary Rapoport-Zink space of signature $(1, n-1)$

J.Muller

October 1, 2021

Abstract : *In [Inventiones mathematicae, 184 (2011)], Vollaard and Wedhorn defined a stratification on the special fiber of the unitary unramified PEL Rapoport-Zink space with signature* $(1, n-1)$*. They constructed an isomorphism between the closure of a stratum, called a closed Bruhat-Tits stratum, and a Deligne-Lusztig variety which is not of classical type. In this paper, we describe the* ℓ*-adic cohomology groups over* $\overline{\mathbb{Q}_\ell}$ *of these Deligne-Lusztig varieties, where* $\ell \neq p$*. The computations involve the spectral sequence associated to the Ekedahl-Oort stratification of a closed Bruhat-Tits stratum, which translates into a stratification by Coxeter varieties whose cohomology is known. Eventually, we find out that the irreducible representations of the finite unitary group which appear inside the cohomology contribute to only two different unipotent Harish-Chandra series, one of them belonging to the principal series.*

Contents

1 The generalized Deligne-Lusztig variety $X_I(\mathrm{id})$ 4

2 Irreducible unipotent representations of the finite unitary group 9

3 Computing Harish-Chandra induction of unipotent representations in the finite unitary group 14

4 The cohomology of the Coxeter variety for the unitary group 19

5 The cohomology of the variety $X_I(\mathrm{id})$ 23

Bibliography 28

Rapoport-Zink spaces are geometric objects which can be seen as deformation spaces for a p-divisible group equipped with additional structures. They are formal schemes over the ring of integers of a p-adic field, and they are constructed by means of a moduli problem which grants them with commuting actions from some p-adic and Galois groups. Therefore, the étale cohomology of these spaces carries representations of these groups simultaneously, and it is expected to realize a local version of Langlands correspondance. Computing this cohomology is an arduous problem in general. So far it has only been entirely described in a few special cases such as the Lubin-Tate tower or the Drinfeld space ; in particular both of them correspond to Rapoport-Zink spaces of EL type.
The difficulty in studying the cohomology of the Rapoport-Zink spaces is maybe reflected by the lack of precise understanding of their geometry in general. However, for some specific choices of the set of data, the resulting moduli space may display some nice geometric features, giving hopes that their cohomology could be accessible. It is the case of the unitary unramified PEL Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$, whose special fiber $\mathcal{M}_{\mathrm{red}}$ is described by the Bruhat-Tits stratification constructed by Vollaard and Wedhorn in a series of two papers [Vol10] and [VW11]. This stratification $\{\mathcal{M}_\Lambda\}_\Lambda$ has two interesting features. On the one hand, the closed strata $\mathcal{M}_\Lambda$ are indexed by the set of vertices Λ of the Bruhat-Tits building of a p-adic group of unitary similitudes J defined by the PEL data. On the other hand, each individual closed stratum $\mathcal{M}_\Lambda$ is isomorphic to a Deligne-Lusztig variety. They usually show up in Deligne-Lusztig theory, whose aim is the classification of the irreducible representations of finite groups of Lie type. In particular, the cohomology of these varieties has been extensively studied in the past decades. In a series of two papers, we aim at exploiting these geometric observations in order to link the cohomology theories of Deligne-Lusztig varieties and of Rapoport-Zink spaces.
Our strategy in order to examine the cohomology of the Rapoport-Zink space $\mathcal{M}$ takes place in two steps: first we compute the cohomology groups $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda, \overline{\mathbb{Q}_\ell})$ (with $\ell \neq p$) of the closed strata; second we use the combinatorics of the Bruhat-Tits building to get information on the cohomology of $\mathcal{M}$. More precisely, in the second stage we introduce the analytical generic fiber $\mathcal{M}^{\mathrm{an}}$. It is covered with the analytical tubes U_Λ of the closed strata $\mathcal{M}_\Lambda$. These are open subdomains of $\mathcal{M}^{\mathrm{an}}$ whose cohomology coincides with the cohomology of the closed strata up to a suitable Tate twist and shift in degrees. Through the Čech spectral sequence associated to the open cover $\{U_\Lambda\}_\Lambda$, we prove the semisimplicity of the Frobenius action on $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$, and we determine the cuspidal supports of its irreducible subquotients as a smooth representation of the group of unitary similitudes J. It also turns out that these cohomology groups are not J-admissible in general.
Lastly, the p-adic uniformization theorem relates the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ with the basic stratum of an associated PEL unitary Shimura variety through a geometric isomorphism. It induces a Hochschild-Serre type spectral sequence on the cohomology, through which we compute the individual cohomology groups of the basic stratum in the case $n = 3$ and 4. In particular, we find out that some automorphic representations occur with a multiplicity depending on p which is a completely new phenomenon.
In the present paper, we carry out the first step of the strategy described above, namely we com-

pute the cohomology of the individual closed Bruhat-Tits strata by exploiting Deligne-Lusztig theory. The second step and the results stated above can be found in the sequel [Mul22].

Let q be a power of the prime number p. Let $\mathbf{G}$ be a connected reductive group over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. Assume that $\mathbf{G}$ is equipped with an $\mathbb{F}_q$-structure induced by a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$. For $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$, the associated generalized parabolic Deligne-Lusztig variety is defined by

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

Usually, parabolic Deligne-Lusztig varieties have been studied with the additional assumption that $\mathbf{P}$ contains a Levi complement $\mathbf{L}$ such that $F(\mathbf{L}) = \mathbf{L}$. Indeed, they are used to define the Deligne-Lusztig induction and restriction functors between the categories of representations of $\mathbf{L}^F$ and of $\mathbf{G}^F$, see for instance [DM14]. However, the closed Bruhat-Tits strata constructed by Vollaard and Wedhorn are isomorphic to Deligne-Lusztig varieties $X_{\mathbf{P}}$ associated to parabolic subgroups $\mathbf{P}$ which do not satisfy this assumption. We call them "generalized" and to our knowledge, their cohomology has not been studied so far.
The closed Bruhat-Tits strata $\mathcal{M}_\Lambda$ are isomorphic to generalized Deligne-Lusztig varieties $X_I(\mathrm{id})$ associated to finite unitary groups $\mathrm{U}_{2d+1}(p)$ in an odd number of variables (see 1.2 for the notations). Although only the case $q = p$ is relevant in the context of Vollaard and Wedhorn's paper [VW11], we will work in this paper with a general q. In loc. cit. the authors defined yet another stratification on each individual stratum. It is called the Ekedahl-Oort stratification and it gives a decomposition $X_I(\mathrm{id}) \simeq \bigsqcup_{0 \leqslant t \leqslant d} X_{I_t}(w_t)$ into locally closed subvarieties. It turns out that each Ekedahl-Oort stratum $X_{I_t}(w_t)$ is isomorphic to a Deligne-Lusztig variety which is not generalized. Moreover, they are closely related to Coxeter varieties whose cohomology is known thanks to the work of Lusztig in [Lus76]. The Ekedahl-Oort stratification on $X_I(\mathrm{id})$ induces a spectral sequence on the cohomology, through which we are able to entirely compute the individual cohomology groups in terms of representations of $\mathrm{U}_{2d+1}(q)$. The representations which occur are all unipotent, and these are classified by partitions of $2d+1$ or equivalently by Young diagrams, see [LS77] and [FS90]. Given a partition $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ of $2d+1$ with $\lambda_r > 0$, the associated irreducible unipotent representation of $\mathrm{U}_{2d+1}(q)$ is denoted by ρ_λ. We may now state our main result, whose proof covers the section 5 of the paper. In the statement, the prime number ℓ is different from p, the field $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_q$ and Frob is the geometric Frobenius relative to $\mathbb{F}_{q^2}$ acting on the cohomology groups.

Theorem. *The following statements hold.*

(1) *The cohomology group* $\mathrm{H}^i_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ *is zero unless* $0 \leqslant i \leqslant 2d$. *There is an isomorphism* $\mathrm{H}^i_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^{2d-i}_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})^\vee(d)$ *which is equivariant for the actions of* Frob *and of* $\mathrm{U}_{2d+1}(q)$.

(2) *The Frobenius element* Frob *acts like multiplication by* $(-q)^i$ *on* $\mathrm{H}^i_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$.

(3) *For* $0 \leqslant i \leqslant d$ *we have*

$$\mathrm{H}^{2i}_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(i,d-i)} \rho_{(2d+1-2s,2s)}.$$

(4) *For* $0 \leqslant i \leqslant d-1$ *we have*

$$\mathrm{H}^{2i+1}_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(i,d-1-i)} \rho_{(2d-2s,2s+1)}.$$

In particular, when the index is even all the representations in the cohomology groups contribute to the unipotent principal series, but when the index is odd the representations belong to the unipotent series determined by a minimal Levi complement of $\mathrm{U}_{2d+1}(q)$ which is not a torus.

Throughout the paper, we fix q a power of an odd prime number p. If k is a perfect field extension of $\mathbb{F}_q$, we denote by $\sigma : x \mapsto x^q$ the q-th power Frobenius of $\mathrm{Gal}(k/\mathbb{F}_q)$. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$.

Acknowledgement: This paper is part of a PhD thesis under the supervision of Pascal Boyer (Université Sorbonne Paris Nord) and Naoki Imai (University of Tokyo). I am grateful for their wise guidance throughout the research. I also wish to adress special thanks to Olivier Dudas (Université de Paris) who gave me precious support regarding Deligne-Lusztig theory. He taught me the subtleties of this field, and guided me through the vast literature with precise references in order to carry out the computations of the cohomology groups.

1 The generalized Deligne-Lusztig variety $X_I(\mathrm{id})$

1.1 Let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$. Let F be a Frobenius morphism defining an $\mathbb{F}_q$-structure on it. If $\mathbf{H}$ is an F-stable subgroup of $\mathbf{G}$, we denote by $H := \mathbf{H}^F \simeq \mathbf{H}(\mathbb{F}_q)$ its group of $\mathbb{F}_q$-rational points. We fix a pair $(\mathbf{T}, \mathbf{B})$ consisting of a maximal torus $\mathbf{T}$ contained in a Borel subgroup $\mathbf{B}$, both of them being F-stable. Such a pair always exists up to $G = \mathbf{G}^F$-conjugation. We obtain a Coxeter system $(\mathbf{W}, \mathbf{S})$ on which F acts, where $\mathbf{W} = \mathbf{W}(\mathbf{T})$ is the Weyl group attached to $\mathbf{T}$ and $\mathbf{S}$ is the set of simple reflexions. It can be identified with the Weyl group of $\mathbf{G}$ as defined in [DL76]. For $I \subset \mathbf{S}$, we write $\mathbf{P}_I, \mathbf{U}_I, \mathbf{L}_I$ respectively for the standard parabolic subgroup of type I, for its unipotent radical and for its unique Levi complement containing $\mathbf{T}$. We also write $\mathbf{W}_I$ for the parabolic subgroup of $\mathbf{W}$ generated by the simple reflexions in I.

1.2 We recall the definition of Deligne-Lusztig varieties from [BR06]. If $\mathbf{P}$ is any parabolic subgroup of $\mathbf{G}$, the associated generalized parabolic Deligne-Lusztig variety is

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

When these varieties were first introduced in [DL76] only the case of Borel subgroups was considered, hence the adjective "parabolic". Moreover, parabolic Deligne-Lusztig varieties have mostly been studied with the additional assumption that $\mathbf{P}$ contains an F-stable Levi complement, see for instance [DM14]. This is not required by the definition above, hence the adjective "generalized".

Using the Coxeter system as above, one may give an equivalent description of these varieties. For $I, I' \subset \mathbf{S}$ the generalized Bruhat decomposition is an isomorphism

$$\mathbf{P}_I\backslash\mathbf{G}/\mathbf{P}_{I'} = \bigsqcup_{w\in {}^I\mathbf{W}^{I'}} \mathbf{P}_I\backslash\mathbf{P}_I w\mathbf{P}_{I'}/\mathbf{P}_{I'} \simeq \mathbf{W}_I\backslash\mathbf{W}/\mathbf{W}_{I'}.$$

Here, we denote by ${}^I\mathbf{W}^{I'}$ the set of elements $w \in \mathbf{W}$ which are (I, I')-reduced, and it is identified with the set of double cosets $\mathbf{W}_I\backslash\mathbf{W}/\mathbf{W}_{I'}$. The generalized parabolic Deligne-Lusztig varieties is defined by

$$X_I(w) = \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}F(g) \in \mathbf{P}_I wF(\mathbf{P}_I)\}.$$

The families of varieties $X_{\mathbf{P}}$ and $X_I(w)$ are the same and [BR06] explains how to go from one description to the other. The case $I = \varnothing$ corresponds to usual Deligne-Lusztig varieties in $\mathbf{G}/\mathbf{B}$. Moreover, the additional assumption regarding the existence of a rational Levi complement translates into the equation

$$w^{-1}Iw = F(I), \tag{$*$}$$

which is a compatibility condition between the parameters w and I. The variety $X_I(w)$ is defined over $\mathbb{F}_{q^\iota}$, where ι is the least integer such that $F^\iota(I) = I$ and $F^\iota(w) = w$.

1.3 Let d be a nonnegative integer and let V be a $(2d+1)$-dimensional $\mathbb{F}_{q^2}$-vector space. Let $(\cdot,\cdot) : V \times V \to \mathbb{F}_{q^2}$ be a non-degenerate hermitian form on V. This hermitian structure on V is unique up to isomorphism. In particular, we may once and for all a basis $\mathcal{B}$ of V in which $(\cdot,\cdot)$ is described by the square matrix $\dot{w}_0$ of size $2d+1$, having 1 on the anti-diagonal and 0 everywhere else. If k is a perfect field extension of $\mathbb{F}_{q^2}$, we may extend the pairing to $V_k := V \otimes_{\mathbb{F}_{q^2}} k$ by setting

$$(v \otimes x, w \otimes y) := xy^\sigma(v, w) \in k$$

for all $v, w \in V$ and $x, y \in k$. If U is a subspace of V_k we denote by $U^\perp$ its orthogonal, that is the subspace of all vectors $x \in V_k$ such that $(x, U) = 0$.

Let J denote the finite group of Lie type $\mathrm{U}(V, (\cdot,\cdot))$. It is defined as the group of F-fixed points of $\mathbf{J} := \mathrm{GL}(V)_{\mathbb{F}}$ with F a non-split Frobenius morphism. Using the basis $\mathcal{B}$, the group $\mathbf{J}$ is identified with GL_{2d+1} with $\mathbb{F}_q$-structure induced by the Frobenius morphism $F(M) := \dot{w}_0(M^{(q)})^{-t}\dot{w}_0$. Here, $M^{(q)}$ denotes the matrix M having all coefficients raised to the power q. We may then identify J with the usual finite unitary group $\mathrm{U}_{2d+1}(q)$.

The pair $(\mathbf{T}, \mathbf{B})$ consisting of the maximal torus of diagonal matrices and the Borel subgroup of upper-triangular matrices is F-stable. The Weyl system of $(\mathbf{T}, \mathbf{B})$ may be identified with $(\mathfrak{S}_{2d+1}, \mathbf{S})$ in the usual manner, where $\mathbf{S}$ is the set of simple transpositions $s_i := (i \quad i+1)$ for $1 \leqslant i \leqslant 2d$. Under this identification, the Frobenius acts on $\mathbf{W}$ as the conjugation by the element w_0, characterized for having the maximal length. It satisfies $w_0(i) = 2d + 2 - i$, and a natural representative of w_0 in the normalizer of $\mathbf{T}$ is no other than $\dot{w}_0$. Since w_0 has order 2, the action of the Frobenius on $\mathbf{W}$ is involutive. It also preserves the simple reflexions with the formula $F(s_i) = s_{2d+1-i}$.

1.4 We define the following subset of $\mathbf{S}$

$$I := \{s_1, \ldots, s_d, s_{d+2}, \ldots, s_{2d}\} = \mathbf{S}\backslash\{s_{d+1}\}.$$

We have $F(I) = \mathbf{S}\backslash\{s_d\} \neq I$. We consider the generalized Deligne-Lusztig variety $X_I(\mathrm{id})$. It corresponds to the variety denoted Y_Λ in [VW11] 4.5. It has dimension d and it does not satisfy the compatibility condition $(*)$.

Proposition ([VW11] 4.4)**.** *The variety $X_I(\mathrm{id})$ is defined over $\mathbb{F}_{q^2}$ and it is projective, smooth, geometrically irreducible of dimension d.*

Although the proposition in loc. cit. is only stated in the case $q = p$, the arguments carry over to general q. The geometric irreducibility is a consequence of the criterion proved in [BR06].

1.5 Rational points of Deligne-Lusztig varieties associated to the reductive group GL can be described in terms of vectorial flags, in a certain relative position with respect to their image by the Frobenius. Let k be a perfect field extension of $\mathbb{F}_{q^2}$. According to [Vol10] 2.12, the Frobenius acts on a flag $\mathcal{F}$ in V_k by sending it to its orthogonal flag $\mathcal{F}^\perp$. Explicitely, we have

$$\begin{aligned} \mathcal{F} &: \{0\} \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r \subset V_k, \\ \mathcal{F}^\perp &: \{0\} \subset \mathcal{F}_r^\perp \subset \ldots \subset \mathcal{F}_1^\perp \subset V_k. \end{aligned}$$

Here, given our choice of I, a k-rational point of $X_I(\mathrm{id})$ corresponds to a flag of the type

$$\mathcal{F} : \{0\} \subset U \subset V_k$$

with U having dimension $d+1$, and which is of relative position id with respect to $\mathcal{F}^\perp$. This precisely means that U must contain $U^\perp$.

Proposition. *The k-rational points of $X_I(\mathrm{id})$ are given by*

$$X_I(\mathrm{id})(k) \simeq \{U \subset V_k \,|\, \dim U = d+1 \text{ and } U^\perp \subset U\}.$$

1.6 In [VW11] 5.3, the authors defined the **Ekedahl-Oort stratification** on the Deligne-Lusztig variety $X_I(\mathrm{id})$. By loc. cit. Corollary 5.12, it turns out that each stratum is itself isomorphic to a parabolic Deligne-Lusztig variety which is not generalized. They are defined as follows.
For $0 \leqslant t \leqslant d$, we define the subset

$$I_t := \{s_1, \ldots, s_{d-t-1}, s_{d+t+2}, \ldots, s_{2d}\} \subset \mathbf{S}.$$

The subset I_t consists of all $2d$ simple reflexions in $\mathbf{S}$, except that we removed the $2t+2$ ones in the middle. Thus, it has cardinality $2(d-t-1)$. In particular, it is empty for $t = d$ or $d-1$. We also define the cycle $w_t := (d+t+1 \quad d+t \ldots d+1)$. Its decomposition into simple reflexions is $w_t = s_{d+1} \ldots s_{d+t}$. When $t = 0$, it is the identity. We note that even though $I_d = I_{d-1} = \varnothing$, we still have $w_d \neq w_{d-1}$.
One may check that $F(I_t) = I_t$ and that w_t belongs to ${}^{I_t}\mathbf{W}^{I_t}$. Moreover, the compatibility condition $(*)$ is satisfied for the pair (I_t, w_t). Indeed, the reduced decomposition for w_t does not use any simple reflexion that is adjacent to those in I_t.

Proposition ([VW11] 3.3 and 5.3)**.** *The Deligne-Lusztig variety $X_{I_t}(w_t)$ is defined over $\mathbb{F}_{q^2}$ and has dimension t. There is a natural immersion $X_{I_t}(w_t) \hookrightarrow X_I(\mathrm{id})$ inducing a stratification*

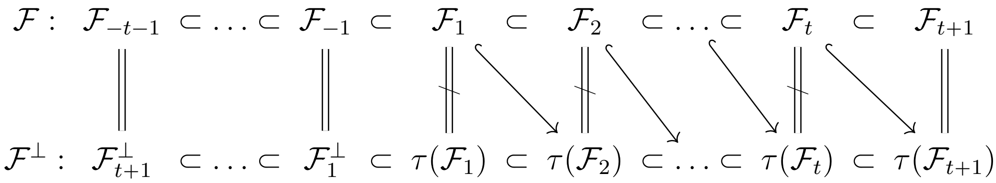
$$X_I(\mathrm{id}) = \bigsqcup_{0 \leqslant t \leqslant d} X_{I_t}(w_t).$$

The closure of the stratum $X_{I_t}(w_t)$ is the union of all the strata $X_{I_s}(w_s)$ for $s \leqslant t$.

1.7 Following the proof of Theorem 2.15 of [Vol10], we can describe the stratification at the level of rational points. Let k be a perfect field extension of $\mathbb{F}_{q^2}$. Because of the choice of I_t, a k-point of $X_{I_t}(w_t)$ is a flag

$$\mathcal{F} \;:\quad \{0\} \subset \mathcal{F}_{-t-1} \subset \ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{t+1} \subset V_k$$

with $\dim(\mathcal{F}_{-i}) = d+1-i$ and $\dim(\mathcal{F}_i) = d+i$ for $1 \leqslant i \leqslant t+1$, and which is in relative position w_t with respect to $\mathcal{F}^{\perp}$. It means that we have a diagram of the following type.



Here, $\tau := \sigma^2 \cdot \mathrm{id}$ is an $\mathbb{F}_{q^2}$-linear automorphism of V_k, and it satisfies $\tau(U) = (U^{\perp})^{\perp}$ for every subspace $U \subset (V_\Lambda)_k$. This diagram implies that $\tau(\mathcal{F}_i) = \mathcal{F}_{i-1} + \tau(\mathcal{F}_{i-1})$ for all $2 \leqslant i \leqslant t+1$. This rewrites as $\mathcal{F}_i = \mathcal{F}_{i-1} + \tau^{-1}(\mathcal{F}_{i-1})$. We deduce that

$$\mathcal{F}_i = \sum_{l=0}^{i-1} \tau^{-l}(\mathcal{F}_1)$$

for all $1 \leqslant i \leqslant t+1$. Thus, the whole flag is determined by the subspace $\mathcal{F}_1$, which has dimension $d+1$ and contains its orthogonal. The immersion $X_{I_t}(w_t) \hookrightarrow X_I(\mathrm{id})$ maps the flag $\mathcal{F}$ to $\mathcal{F}_1$.
Conversely, a k-point of $X_I(\mathrm{id})$ is given by a subspace $U \subset V_k$ of dimension $d+1$ containing its orthogonal. For $i \geqslant 1$ we define

$$\mathcal{F}_i := \sum_{l=0}^{i-1} \tau^{-1}(U) \subset V_k.$$

Then $(\mathcal{F}_i)_{i\geqslant 1}$ is a nondecreasing sequence of subspaces of V_k. Let t be the smallest integer such that $\mathcal{F}_{t+1} = \mathcal{F}_{t+2}$. It follows that $0 \leqslant t \leqslant d$ and that t is also the smallest integer such that $\mathcal{F}_{t+1} = \tau(\mathcal{F}_{t+1})$. Moreover the orthogonal $U^\perp$ has dimension d and we have $U^\perp \subset U$, so that $U^\perp \subset (U^\perp)^\perp = \tau(U)$. In particular, if $t > 0$ then $U \cap \tau(U) = U^\perp$. Thus, we have $\dim(\mathcal{F}_2) = d+2$. Similarly, we have $\dim(\mathcal{F}_i) = d+i$ for all $1 \leqslant i \leqslant t+1$. By setting $\mathcal{F}_{-i} := \mathcal{F}_i^\perp$, we obtain a flag $\mathcal{F}$ that is the k-rational point of $X_{I_t}(w_t)$ associated to U.

1.8 The Deligne-Lusztig varieties $X_{I_t}(w_t)$ are related to Coxeter varieties for smaller unitary groups as we now explain. We define

$$K_t := \{s_1, \ldots s_{d-t-1}, s_{d-t+1}, \ldots, s_{d+t}, s_{d+t+2}, \ldots, s_{2d}\} = \mathbf{S}\backslash\{s_{d-t}, s_{d+t+1}\}.$$

The set K_t is obtained from I_t by adding the $2t$ simple reflexions in the middle. It has cardinality $2d-2$ and satisfies $F(K_t) = K_t$. We have $I_t \subset K_t$ with equality if and only if $t = 0$.

Proposition. *There is a* $\mathrm{U}_{2d+1}(q)$*-equivariant isomorphism*

$$X_{I_t}(w_t) \simeq \mathrm{U}_{2d+1}(q)/U_{K_t} \times_{L_{K_t}} X_{I_t}^{\mathbf{L}_{K_t}}(w_t),$$

where $X_{I_t}^{\mathbf{L}_{K_t}}(w_t)$ *is a Deligne-Lusztig variety for* $\mathbf{L}_{K_t}$*. The zero-dimensional variety* $\mathrm{U}_{2d+1}(q)/U_{K_t}$ *has a left action of* $\mathrm{U}_{2d+1}(q)$ *and a right action of* L_{K_t}*.*

Proof. This is an application of [DM14] Proposition 7.19 which is the geometric identity behind the transitivity of the Deligne-Lusztig functors. It applies to the varieties $X_{I_t}(w_t)$ because they satisfy the compatibility condition $(*)$, and satisfies the following conditions: K_t contains I_t, it is stable by the Frobenius and w_t belongs to the parabolic subgroup $\mathbf{W}_{K_t} \simeq \mathfrak{S}_{d-t} \times \mathfrak{S}_{2t+1} \times \mathfrak{S}_{d-t} \subset \mathfrak{S}_{2d+1}$. □

1.9 The Levi complement $\mathbf{L}_{K_t}$ is isomorphic to the product $\mathrm{GL}_{d-t} \times \mathrm{GL}_{2t+1} \times \mathrm{GL}_{d-t}$ as a reductive group over $\mathbb{F}$. Given a matrix $M = \mathrm{diag}(A, C, B) \in \mathbf{L}_{K_t}$, we have $F(M) = \mathrm{diag}(F(B), F(C), F(A))$, where we still denote by F the Frobenius morphism for smaller linear groups. Writing $\mathbf{H}$ for the product of the two GL_{d-t} factors, we have $\mathbf{L}_{K_t} \simeq \mathbf{H} \times \mathrm{GL}_{2t+1}$ and both factors inherit an $\mathbb{F}_q$-structure by means of F. We have $L_{K_t} \simeq \mathrm{GL}_{d-t}(q^2) \times \mathrm{U}_{2t+1}(q)$, the first factor corresponding to H.
The Weyl group of $\mathbf{L}_{K_t}$ is isomorphic to $\mathbf{W}_\mathbf{H} \times \mathfrak{S}_{2t+1}$ where $\mathbf{W}_\mathbf{H} \simeq \mathfrak{S}_{d-t} \times \mathfrak{S}_{d-t}$ is the Weyl group of $\mathbf{H}$. Via this decomposition, the permutation w_t corresponds to $\mathrm{id} \times \widetilde{w}_t$, where $\widetilde{w}_t$ is the restriction of w_t to $\{d-t+1, \ldots, d+t+1\}$. Similarly, the set of simple reflexions $\mathbf{S}$ decomposes as $\mathbf{S}_\mathbf{H} \sqcup \widetilde{\mathbf{S}}$, the second term corresponding to the simple reflexions in $\mathfrak{S}_{2t+1}$. Then, we have $I_t = \mathbf{S}_\mathbf{H} \sqcup \varnothing$.
The Deligne-Lusztig variety for $\mathbf{L}_{K_t}$ decompose accordingly as the following product

$$X_{I_t}^{\mathbf{L}_{K_t}}(w_t) = X_{\mathbf{S}_\mathbf{H}}^{\mathbf{H}}(\mathrm{id}) \times X_{\varnothing}^{\mathrm{U}_{2t+1}(q)}(\widetilde{w}_t).$$

The variety $X_{\mathbf{S}_\mathbf{H}}^{\mathbf{H}}(\mathrm{id})$ is just a point, whereas $X_{\varnothing}^{\mathrm{U}_{2t+1}(q)}(\widetilde{w}_t)$ is a Deligne-Lusztig variety for the unitary group of size $2t+1$. We observe that the permutation $\widetilde{w}_t$ is a **Coxeter element** in $\mathfrak{S}_{2t+1}$, ie. the product of exactly one simple reflexion for each orbit of the Frobenius. Deligne-Lusztig varieties attached to Coxeter elements are called **Coxeter varieties**, and their cohomology with coefficients in $\overline{\mathbb{Q}_\ell}$ where ℓ is a prime number different from p are well understood thanks to the work of Lusztig in [Lus76]. Before stating the results of loc. cit. we recall parts of the representation theory of finite unitary groups.

2 Irreducible unipotent representations of the finite unitary group

2.1 In this section, we recall the classification of the irreducible unipotent representations of the finite unitary group and we explain the underlying combinatorics.
We use the notations from 1.1. For $w \in \mathbf{W}$, let $\dot{w}$ be a representative of w in the normalizer $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$. By the Lang-Steinberg theorem, one can find $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$. Then ${}^g\mathbf{T} := g\mathbf{T}g^{-1}$ is another F-stable maximal torus, and $w \in \mathbf{W}$ is said to be the **type** of ${}^g\mathbf{T}$ with respect to $\mathbf{T}$. Every F-stable maximal torus arises in this manner. According to [DL76] Corollary 1.14, the G-conjugacy class of ${}^g\mathbf{T}$ only depends on the F-conjugacy class of the image w of the element $g^{-1}F(g) \in \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ in the Weyl group $\mathbf{W}$. Here, two elements w and w' in $\mathbf{W}$ are said to be F-conjugates if there exists some element $u \in \mathbf{W}$ such that $w = uw'F(u)^{-1}$.
For every $w \in \mathbf{W}$, we fix $\mathbf{T}_w$ an F-stable maximal torus of type w with respect to $\mathbf{T}$. The Deligne-Lusztig induction of the trivial representation of $\mathbf{T}_w$ is the virtual representation of G defined by the formula

$$R_w := \sum_{i \geqslant 0} (-1)^i \mathrm{H}^i_c(X_\varnothing(w))$$

where $X_\varnothing(w)$ is a Deligne-Lusztig variety for $\mathbf{G}$ as defined in 1.2. According to [DL76] Theorem 1.6, the virtual representation R_w only depends on the F-conjugacy class of w in $\mathbf{W}$. An irreducible representation of G is said to be **unipotent** if it occurs in R_w for some $w \in \mathbf{W}$. The set of isomorphism classes of unipotent representations of G is usually denoted $\mathcal{E}(G,1)$ following Lusztig's notations.

2.2 Assume that the Coxeter graph of the reductive group $\mathbf{G}$ is a union of subgraphs of type A_m (for various m). Let $\breve{\mathbf{W}}$ be the set of isomorphism classes of irreducible representations of its Weyl group $\mathbf{W}$. The action of the Frobenius F on $\mathbf{W}$ induces an action on $\breve{\mathbf{W}}$, and we consider the fixed point set $\breve{\mathbf{W}}^F$. Then, the following classification theorem is well known.

Theorem ([LS77] Theorem 2.2)**.** *There is a bijection between $\breve{\mathbf{W}}^F$ and the set of isomorphism classes of irreducible unipotent representations of $G = \mathbf{G}^F$.*

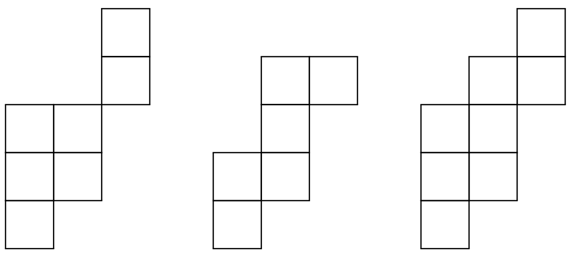
We recall how the bijection is constructed. If $V \in \breve{\mathbf{W}}^F$ is an irreducible F-stable representation of $\mathbf{W}$, according to loc. cit. there is a unique automorphism $\widetilde{F}$ of V of finite order such that

$$R(V) := \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \mathrm{Trace}(w \circ \widetilde{F} \,|\, V) R_w$$

is an irreducible representation of G. Then the map $V \mapsto R(V)$ is the desired bijection.
In the case $\mathbf{G} = \mathrm{GL}_n$ with the Frobenius morphism F being either standard or twisted (ie. $G = \mathrm{GL}_n(q)$ or $\mathrm{U}_n(q)$), we have an equality $\breve{\mathbf{W}}^F = \breve{\mathbf{W}}$. Moreover, the automorphism $\widetilde{F}$ is the identity in the former case and multiplication by w_0 on the latter, where w_0 is the element of maximal length in $\mathbf{W}$. Thus, in both cases the irreducible unipotent representations of G are classified by the irreducible representations of the Weyl group $\mathbf{W} \simeq \mathfrak{S}_n$, which in turn are classified by partitions of n or equivalently by Young diagrams. We now recall the underlying combinatorics behind the representation theory of the symmetric group. A general reference is [Jam84].

2.3 A partition of n is a tuple $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with $r \geqslant 1$ and the λ_i's are positive integers such that $\lambda_1 + \ldots + \lambda_r = n$. The integer n is called the length of the partition and it is also denoted by $|\lambda|$. If a partition has a series of repeating integers, it is common to write it shortly with an exponent. For instance, the partition $(3, 3, 2, 2, 1)$ of 11 will be denoted $(3^2, 2^2, 1)$. Partitions of n are naturally identified with Young diagrams of size n. The diagram attached to λ has r rows consisting successively of $\lambda_1, \ldots, \lambda_r$ boxes.
To any partition λ of n, one can naturally associate an irreducible representation χ_λ of the symmetric group $\mathfrak{S}_n$. An explicit construction is given, for instance, by the notion of Specht modules as explained in [Jam84] 7.1. In particular, the character $\chi_{(n)}$ is trivial while the character $\chi_{(1^n)}$ is the signature.

2.4 We recall the Murnaghan-Nakayama rule which gives a recursive formula to evaluate the characters χ_λ. We first need to introduce skew Young diagrams. Consider a pair λ and μ of two partitions respectively of integers $n+k$ and k. Assume that the Young diagram of μ is contained in the Young diagram of λ. By removing the boxes corresponding to μ from the diagram of λ, one finds a shape consisting of n boxes denoted by $\lambda \backslash \mu$. Any such shape is called a **skew Young diagram** of size n. It is said to be connected if one can go from a given box to any other by moving in a succession of adjacent boxes.
For example, consider the partition $\lambda = (3^2, 2^2, 1)$ and let us define the partitions $\mu_1 = (2^2)$, $\mu_2 = (3, 1^2)$ and $\mu_3 = (2, 1)$. The diagrams below correspond, from left to right, to the skew Young diagrams $\lambda \backslash \mu_i$ for $i = 1, 2, 3$.



The skew Young diagram $\lambda \backslash \mu_1$ is not connected, whereas the others are connected. A skew Young diagram is said to be a **border strip** if it is connected and if it does not contain any 2×2 square. The **height** of a border strip is defined as its number of rows minus 1. For instance, among the three skew Young diagrams above only $\lambda \backslash \mu_2$ is a border strip. Its size is

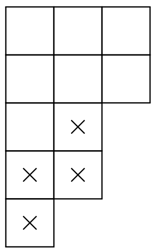
6 and its height is 3.
The characters χ_λ are class functions, so we only need to specify their values on conjugacy classes of the symmetric group $\mathfrak{S}_n$. These conjugacy classes are also naturally labelled by partitions of n. Indeed, up to ordering any permutation $\sigma \in \mathfrak{S}_n$ can be uniquely decomposed as a product of $r \geqslant 1$ cycles $c_1, \dots, c_r$ with disjoint supports. We denote by ν_i the cycle length of c_i and we order them so that $\nu_1 \geqslant \dots \geqslant \nu_r$. We allow cycles to have length 1, so that the union of the supports of all the c_i's is $\{1, \dots, n\}$. Thus, we obtain a partition $\nu = (\nu_1, \dots, \nu_r)$ of n which is called the **cycle type** of the permutation σ. Two permutations are conjugates in $\mathfrak{S}_n$ if and only if they share the same cycle type. We denote by $\chi_\lambda(\nu)$ the value of the character χ_λ on the conjugacy class labelled by ν.

Theorem (Murnaghan-Nakayama rule). *Let λ and ν be two partitions of n. We have*

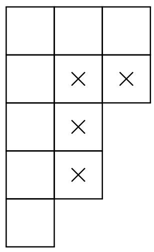
$$\chi_\lambda(\nu) = \sum_S (-1)^{\mathrm{ht}(S)} \chi_{\lambda \backslash S}(\nu \backslash \nu_1),$$

where S runs over the set of all border strips of size ν_1 in the Young diagram of λ, such that removing S from λ gives again a Young diagram. Here, the integer $\mathrm{ht}(S) \in \mathbb{Z}_{\geqslant 0}$ is the height of the border stip S, the Young diagram $\lambda \backslash S$ is the one obtained by removing S from λ, and $\nu \backslash \nu_1$ is the partition of $n - \nu_1$ obtained by removing ν_1 from ν.

Applying the Murnaghan-Nakayama rule in successions results in the value of $\chi_\lambda(\nu)$. We see in particular that $\chi_{(n)}$ is the trivial character whereas $\chi_{(1^n)}$ is the signature. We illustrate the computations with $\lambda = (3^2, 2^2, 1)$ and $\nu = (4^2, 3)$. There are only two elligible border strips of size 4 in the diagram of λ, as marked below.



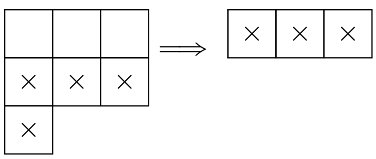
and

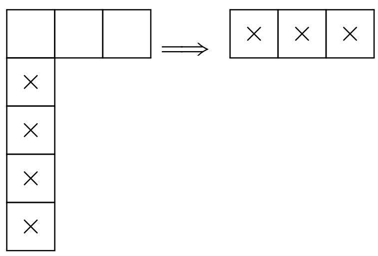


Both border strips have height 2. Thus, the formula gives

$$\chi_{(3^2,2^2,1)}(4^2, 3) = \chi_{(3^2,1)}(4, 3) + \chi_{(3,1^4)}(4, 3).$$

In each of the two Young diagrams obtained after removal of the border strips, there is only one elligible strip of size 4, and eventually the three last remaining boxes form the final border strip of size 3.





Taking the heights of the border strips into account, we find

$$\chi_{(3^2,1)}(4,3) = -\chi_{(3)}(3) = -\chi_{\varnothing} = -1, \qquad \chi_{(3,1^4)}(4,3) = -\chi_{(3)}(3) = -\chi_{\varnothing} = -1.$$

Here, $\varnothing$ denotes the empty partition. The computation finally gives $\chi_{(3^2,2^2,1)}(4^2,3) = -2$.

2.5 The irreducible unipotent representation of $\mathrm{U}_n(q)$ (resp. $\mathrm{GL}_n(q)$) associated to χ_λ by the bijection of 2.1 Theorem is denoted by $\rho_\lambda^{\mathrm{U}}$ (resp. $\rho_\lambda^{\mathrm{GL}}$). The partition (n) corresponds to the trivial representation and (1^n) to the Steinberg representation in both cases. We will omit the superscript when the group we are talking about is clear from the context.
The degrees of the representations $\rho_\lambda^{\mathrm{GL}}$ and $\rho_\lambda^{\mathrm{U}}$ are given by expressions known as **hook formula**. Given a box $\square$ in the Young diagram of λ, its **hook length** $h(\square)$ is 1 plus the number of boxes lying below it or on its right. For instance, in the following figure the hook length of every box of the Young diagram of $\lambda = (3^2, 2^2, 1)$ has been written inside it.

7	5	2
6	4	1
4	2	
3	1	
1		

Proposition ([GP00] Propositions 4.3.1 and 4.3.5)**.** *Let $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ be a partition of n. The degrees of the irreducible unipotent representations $\rho_\lambda^{\mathrm{GL}}$ and $\rho_\lambda^{\mathrm{U}}$, respectively of $\mathrm{GL}_n(q)$ and $\mathrm{U}_n(q)$, are given by the following formulas*

$$\deg(\rho_\lambda^{\mathrm{GL}}) = q^{a(\lambda)} \frac{\prod_{i=1}^n q^i - 1}{\prod_{\square \in \lambda} q^{h(\square)} - 1}, \qquad \deg(\rho_\lambda^{\mathrm{U}}) = q^{a(\lambda)} \frac{\prod_{i=1}^n q^i - (-1)^i}{\prod_{\square \in \lambda} q^{h(\square)} - (-1)^{h(\square)}},$$

where $a(\lambda) = \sum_{i=1}^r (i-1)\lambda_i$.

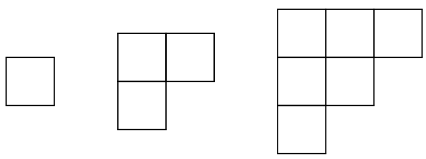
2.6 We recall from [GM20] 3.1 and 3.2 some definitions on classical Harish-Chandra theory. A parabolic subgroup of G is a subgroup $P \subset G$ such that there exists an F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with $P = \mathbf{P}^F$. A Levi complement of G is a subgroup $L \subset G$ such that there exists an F-stable Levi complement $\mathbf{L}$ of $\mathbf{G}$, contained inside some F-stable parabolic subgroup, such that $L = \mathbf{L}^F$. Any parabolic subgroup P of G has a Levi complement L.
Let $L = \mathbf{L}^F$ be a Levi complement of G inside a parabolic subgroup $P = \mathbf{P}^F$. Let $U = \mathbf{U}^F$ be the F-fixed points of the unipotent radical $\mathbf{U}$ of $\mathbf{P}$. The **Harish-Chandra induction and restriction functors** are defined by the following formulas.

$$\mathrm{R}_{L\subset P}^G : \mathrm{Rep}(L) \to \mathrm{Rep}(G) \qquad\qquad {}^*\mathrm{R}_{L\subset P}^G : \mathrm{Rep}(G) \to \mathrm{Rep}(L)$$
$$\sigma \mapsto \mathbb{C}[G/U] \otimes_{\mathbb{C}[L]} \sigma \qquad\qquad \rho \mapsto \mathrm{Hom}_G(\mathbb{C}[G/U], \rho)$$

Here, $\mathrm{Rep}(G)$ is the category of complex representations of G, and similarly for $\mathrm{Rep}(L)$. These two functors are adjoint, and up to isomorphism they do not depend on the choice of the

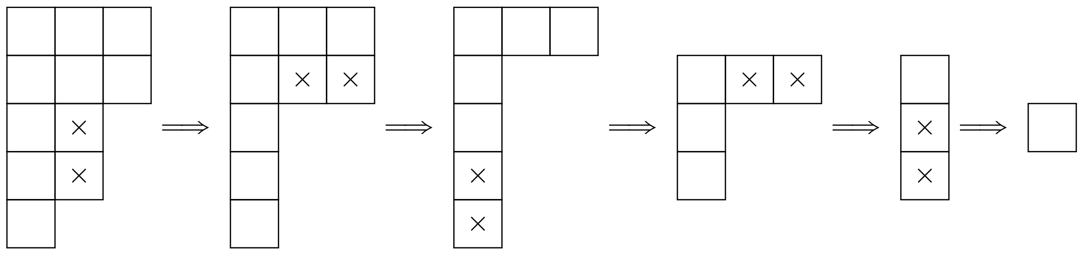
parabolic subgroup P containing the Levi complement L. For this reason, we will denote the functors R_L^G and ${}^*\mathrm{R}_L^G$ instead.
An irreducible representation of G is called **cuspidal** if its Harish-Chandra restriction to any proper Levi complement is zero. We consider pairs (L, X) where L is a Levi complement of G and X is an irreducible representation of L. We define an order on the set of such pairs by setting $(L, X) \leqslant (M, Y)$ if $L \subset M$ and if X occurs in the Harish-Chandra restriction of Y to L. A pair is said to be **cuspidal** if it is minimal with respect to this order, in which case X is a cuspidal representation of L. If (L, X) is a cuspidal pair, we will denote by $[L, X]$ its conjugacy class under G.
Given a cuspidal pair (L, X) of G, its associated **Harish-Chandra series** $\mathcal{E}(G, (L, X))$ is defined as the set of isomorphism classes of irreducible constituents in the induction of X to G. Each series is non empty. Two of them are either disjoint or equal, the latter occuring if and only if the two cuspidal pairs are conjugates in G. Thus, the series are indexed by the conjugacy classes of cuspidal pairs $[L, X]$. Moreover, the isomorphism class of any irreducible representation of G belongs to some Harish-Chandra series. Thus, Harish-Chandra series form a partition of the set of isomorphism classes of irreducible representations of G. If ρ is an irreducible representation of G, the conjugacy class $[L, X]$ corresponding to the series to which ρ belongs is called the **cuspidal support** of ρ. If T denotes a maximal torus in G, then the series $\mathcal{E}(G, (T, 1))$ is called the **unipotent principal series** of G.

2.7 For the general linear group $\mathrm{GL}_n(q)$, there is no unipotent cuspidal representation unless $n = 1$, in which case the trivial representation is cuspidal. Moreover, the unipotent representations all belong to the principal series. The situation for the unitary group is very different. First, by [Lus77] 9.2 and 9.4 there exists an irreducible unipotent cuspidal representation of $\mathrm{U}_n(q)$ if and only if n is an integer of the form $n = \frac{x(x+1)}{2}$ for some $x \geqslant 0$, and when that is the case it is the one associated to the partition $\Delta_x := (x, x-1, \ldots, 1)$, whose Young diagram has the distinctive shape of a reversed staircase. Here, as a convention $\mathrm{U}_0(q)$ denotes the trivial group.
For example, here are the Young diagrams of Δ_1, Δ_2 and Δ_3. Of course, the one of Δ_0 the empty diagram.



Furthermore the unipotent representations decompose non trivially into various Harish-Chandra series, as we recall from [GM20] 4.3.
We consider an integer $x \geqslant 0$ such that n decomposes as $n = 2a + \frac{x(x+1)}{2}$ for some $a \geqslant 0$. We also consider the standard Levi complement $L_x \simeq \mathrm{GL}_1(q^2)^a \times \mathrm{U}_{\frac{x(x+1)}{2}}(q)$ which corresponds to the choice of simple reflexions $s_{a+1}, \ldots, s_{n-a-1}$. We write ρ_x for the inflation of $\rho^{\mathrm{U}}_{\Delta_x}$ to an irreducible representation of L_x. Then $\mathcal{E}(\mathrm{U}_n(q), 1)$ decomposes as the disjoint union of all the Harish-Chandra series $\mathcal{E}(\mathrm{U}_n(q), (L_x, \rho_x))$ for all possible choices of x. With these notations, the principal unipotent series corresponds to $x = 0$ if n is even and to $x = 1$ if n is odd.

2.8 Given an irreducible unipotent representation ρ_λ of $\mathrm{U}_n(q)$, there is a combinatorical way of determining the Harish-Chandra series to which it belongs. We consider the Young diagram of λ. We call **domino** any pair of adjacent boxes in the diagram. It may be either vertical or horizontal. We remove dominoes from the rim of the diagram of λ so that the resulting shape is again a Young diagram, until one can not proceed further. This process results in the Young diagram of the partition Δ_x for some $x \geqslant 0$, and it is called the **2-core** of λ. It does not depend on the successive choices for the dominoes. Then, the representation ρ_λ belongs to the series $\mathcal{E}(\mathrm{U}_n(q), (L_x, \rho_x))$ if and only if λ has 2-core Δ_x.
For instance, the diagram $\lambda = (3^2, 2^2, 1)$ has 2-core Δ_1, as it can be determined by the following steps. We put crosses inside the successive dominoes that we remove from the diagram. Thus, the unipotent representation ρ_λ of $\mathrm{U}_{11}(q)$ belongs to the unipotent principal series $\mathcal{E}(\mathrm{U}_{11}(q), (L_1, \rho_1))$.



3 Computing Harish-Chandra induction of unipotent representations in the finite unitary group

3.1 In this paragraph, we recover the notations from 1.1. We recall from [GM20] 3.2 how Harish-Chandra induction of unipotent representations can be explicitely computed. Let $W = \mathbf{W}^F$ be the Weyl group of G. It is still a Coxeter group, whose set of simple reflexions S is identified with the set of F-orbits on $\mathbf{S}$. Let (L, X) be a cuspidal pair of G. The **relative Weyl group of** L is given by $W_G(L) := N_{\mathbf{G}}(\mathbf{L})^F/L \subset W$. The relative Weyl group of the pair (L, X), also called **the ramification group of** X in [HL83], is the subgroup $W_G(L, X)$ of $W_G(L)$ consisting of elements w such that $wX \simeq X$, where wX denotes the representation $wX(g) := X(wgw^{-1})$ of L. It is yet again a Coxeter group if $\mathbf{G}$ has a connected center or if X is unipotent.
Theorem 3.2.5 of [GM20] establishes an isomorphism between the endomorphism algebra of the induced representation $\mathrm{R}_L^G(X)$ and the complex group ring of the ramification group $W_G(L, X)$. In particular, this gives an bijection between the Harish-Chandra series $\mathcal{E}(G, (L, X))$ and the set $\mathrm{Irr}(W_G(L, X))$ of isomorphism classes of irreducible complex characters of $W_G(L, X)$. These bijections for G and for various Levi complements in G can be chosen to be compatible with Harish-Chandra induction. This is known as Howlett and Lehrer's comparison theorem which was proved in [HL83].

Theorem ([GM20] Comparison Theorem 3.2.7)**.** *Let (L, X) be a cuspidal pair for the finite group of Lie type G. For every Levi complement M in G containing L, the bijection between*

$\mathrm{Irr}(W_M(L,X))$ *and* $\mathcal{E}(M,(L,X))$ *can be taken so that the diagrams*

$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}(G,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_G(L,X)) \\ \uparrow {\scriptstyle \mathrm{R}_M^G} & & \uparrow {\scriptstyle \mathrm{Ind}} \\ \mathbb{Z}\mathcal{E}(M,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_M(L,X)) \end{array} \qquad \begin{array}{ccc} \mathbb{Z}\mathcal{E}(G,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_G(L,X)) \\ \downarrow {\scriptstyle {}^*\mathrm{R}_M^G} & & \downarrow {\scriptstyle \mathrm{Res}} \\ \mathbb{Z}\mathcal{E}(M,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_M(L,X)) \end{array}$$

are commutative. Here, Ind *and* Res *on the right-hand side of the diagrams are the classical induction and restriction functors for representations of finite groups.*

In other words, computing Harish-Chandra induction and restrictions of representations in G can be entirely done at the level of the associated Coxeter groups. In order to use this statement for unitary groups, we need to make the horizontal arrows explicit and to understand the combinatorics behind induction and restriction of the irreducible representations of the relevant Coxeter groups. This has been explained consistently in [FS90] for classical groups.

3.2 We focus on the case of the unitary group. Let $x \geqslant 0$ such that $n = 2a + \frac{x(x+1)}{2}$ for some $a \geqslant 0$. We consider the cuspidal pair (L_x, ρ_x) as in 2.7, with $L_x = \mathrm{GL}_1(q^2)^a \times \mathrm{U}_{\frac{x(x+1)}{2}}(q)$. The relative Weyl group $W_{\mathrm{U}_n(q)}(L_x)$ is isomorphic to the Coxeter group of type B_a, which is usually denoted by W_a. Indeed, the Weyl group $W_{\mathrm{U}_n(q)}(L_x)$ admits a presentation by elements $\sigma_1, \ldots, \sigma_{a-1}$ and θ of order 2 satisfying the relations

$$\begin{aligned} &\theta\sigma_1\theta\sigma_1 = \sigma_1\theta\sigma_1\theta, && \theta\sigma_i = \sigma_i\theta, && \forall\, 2 \leqslant i \leqslant m-1. \\ &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, && \sigma_i\sigma_j = \sigma_j\sigma_i, && \forall\, |i-j| \geqslant 2. \end{aligned}$$

Explicitely, the element σ_i is represented by the permutation matrix of the double transposition $(i \quad i+1)(n-i \quad n-i+1)$ and the element θ by the matrix of the transposition $(1 \quad n)$, all of which belong to $N_{\mathrm{U}_n(q)}(L_x)$. This presentation coincide with the Coxeter group W_a of type B_a, see in [GP00] 1.4.1. Moreover, the ramification group $W_{\mathrm{U}_n(q)}(L_x, \rho_x)$ is equal to the whole of $W_{\mathrm{U}_n(q)}(L_x) \simeq W_a$. The identification between the ramification group and the Coxeter group W_a is naturally induced by the isomorphism between the absolute Weyl group $\mathbf{W}$ and the symmetric group $\mathfrak{S}_n$. In order to proceed further, we need to explain the representation theory of the group W_a.

3.3 Let W_a be a Coxeter group of type B_a given with a presentation by elements $\sigma_1, \ldots, \sigma_{a-1}$ and θ satisfying equations as in 3.2. For $1 \leqslant i \leqslant a-1$, we define $\theta_i = \sigma_i \ldots \sigma_1\theta\sigma_1 \ldots \sigma_i$. In particular $\theta_0 = \theta$. Following [GP00] 3.4.2, we define **signed blocks** to be elements of the following form. Given $k \geqslant 0$ and $e \geqslant 1$ such that $k + e \leqslant a$, the positive (resp. negative) block of length e starting at k is

$$b^+_{k,e} := \sigma_{k+1}\sigma_{k+2}\ldots\sigma_{k+e-1}, \qquad b^-_{k,e} := \theta_k\sigma_{k+1}\sigma_{k+2}\ldots\sigma_{k+e-1}.$$

A **bipartition** of a is an ordered pair (α, β) where α is a partition of some integer $0 \leqslant j \leqslant a$ and β is a partition of $a - j$. Given a bipartition (α, β) of a and writing $\alpha = (\alpha_1, \ldots, \alpha_r)$ and

$\beta = (\beta_1, \ldots, \beta_s)$, we define the element

$$w_{\alpha,\beta} := b^-_{k_1,\beta_1} \ldots b^-_{k_s,\beta_s} b^+_{k_{s+1},\alpha_1} \ldots b^+_{k_{s+r},\alpha_r}$$

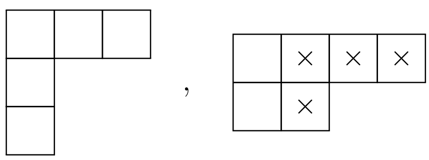
where $k_1 = 0$, $k_{i+1} = k_i + \beta_i$ if $1 \leqslant i \leqslant s$ and $k_{i+1} = k_i + \alpha_{i-s}$ if $s+1 \leqslant i \leqslant s+r-1$. In particular, we have $k_{r+s} + \alpha_r = a$. According to [GP00] Proposition 3.4.7, the conjugacy classes in W_a are labelled by bipartitions of a, and a representative of minimal length of the conjugacy class corresponding to the bipartition (α, β) is given by $w_{\alpha,\beta}$. Thus, the irreducible representations of W_a can be labelled by bipartitions of a as well. An explicit construction of these irreducible representations is given in [GP00] 5.5. We will not recall it, however we may again give a method to compute the character values, similar to the Murnaghan-Nakayama formula. The character of the irreducible representation of W_a associated in loc. cit. to the bipartition (α, β) of a will be denoted $\chi_{\alpha,\beta}$. If (γ, δ) is another bipartition of a, we denote by $\chi_{\alpha,\beta}(\gamma, \delta)$ the value of the character $\chi_{\alpha,\beta}$ on the conjugacy class of W_a labelled by (γ, δ).
One can think of a bipartition (α, β) of a as an ordered pair of two Young diagrams of combined size a. A **border strip** of a bipartition (α, β) is a border strip either of the partition α or of β. The height of a border strip is defined in the same way.

Theorem ([GP00] Theorem 10.3.1)**.** *Let (α, β) and (γ, δ) be two bipartitions of a. If $\gamma \neq \varnothing$, let $\epsilon = 1$ and let x be the last integer in the partition γ. If $\gamma = \varnothing$, let $\epsilon = -1$ and let x be the last integer of the partition δ. We have*

$$\chi_{\alpha,\beta}(\gamma, \delta) = \sum_S (-1)^{\mathrm{ht}(S)} \epsilon^{f_S} \chi_{(\alpha,\beta)\backslash S}((\gamma, \delta)\backslash x),$$

where S runs over the set of all border strips of size x in the bipartition (α, β), such that removing S from (α, β) gives again a pair of Young diagrams. Here, the pair of Young diagrams $(\alpha, \beta)\backslash S$ is the one obtained after removing S, and $(\gamma, \delta)\backslash x$ is the bipartition obtained by removing x from (γ, δ). Eventually, the integer f_S is 0 if S is a border strip of α, and it is 1 if S is a border strip of β.

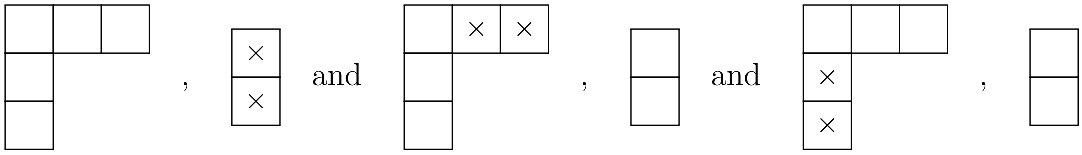
Applying this formula in successions results in the value of $\chi_{(\alpha,\beta)}(\gamma, \delta)$. In particular, one sees that $\chi_{(a),\varnothing}$ is the trivial character and $\chi_{\varnothing,(1^a)}$ is the signature character of W_a. We illustrate the computations with $(\alpha, \beta) = ((3, 1^2), (4, 2))$ and $(\gamma, d) = ((4), (5, 2))$. There is only elligible border strip of size 4 in the pair of diagrams (α, β), as marked below.



This border strip S has height 1. It was taken in the diagram of β so $f_S = 1$. Since $\gamma \neq \varnothing$ we have $\epsilon = 1$. Applying the formula, we obtain

$$\chi_{(3,1^2),(4,2)}((4), (5, 2)) = -\chi_{(3,1^2),(1^2)}(\varnothing, (5, 2)).$$

We are now looking for border strips of size 2 in the pair of diagrams of the bipartition $(3, 1^2), (1^2)$. Three of them are eligible, as marked below.



These three border strips have respective heights $1, 0$ and 1. The corresponding values of f_S are respectively 1, 0 and 0. Moreover, the partition γ is now empty so $\epsilon = -1$. The formula gives

$$\chi_{(3,1^2),(1^2)}(\varnothing,(5,2)) = \chi_{(3,1^2),\varnothing}(\varnothing,(5)) + \chi_{(1^3),(1^2)}(\varnothing,(5)) - \chi_{(3),(1^2)}(\varnothing,(5)).$$

In the bipartitions $((1^3),(1^2))$ and $((3),(1^2))$ there is no border strip of size 5 at all. Thus, the formula tells us that the corresponding character values are 0. On the other hand, the bipartition $((3,1^2),\varnothing)$ consists of a single border strip of size 5 and height 2. The formula gives

$$\chi_{(3,1^2),\varnothing}(\varnothing,(5)) = \chi_{\varnothing} = 1.$$

Putting things together, we deduce that $\chi_{(3,1^2),(4,2)}((4),(5,2)) = -1$.

3.4 We may now describe the horizontal arrows in 3.1 Theorem for the unitary group. To do this, we need an alternate labelling of the irreducible unipotent representations of the unitary group. We refer to [FS90] for the details.
The new labelling of the irreducible unipotent representations of $\mathrm{U}_n(q)$ involves triples of the form $(\Delta_x, \alpha, \beta)$ where x is a nonnegative integer such that $n = 2a + \frac{x(x+1)}{2}$ for some integer $a \geqslant 0$, and where (α, β) is a bipartition of a. The corresponding representation will be denoted $\rho_{\Delta_x,\alpha,\beta}$. With this labelling, the unipotent Harish-Chandra series $\mathcal{E}(\mathrm{U}_n(q), (L_x, \rho_x))$ consists precisely of all the representations $\rho_{\Delta_x,\alpha,\beta}$ with (α, β) varying over all bipartitions of a. The bijection $\mathbb{Z}\mathcal{E}(\mathrm{U}_n(q), (L_x, \rho_x)) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(W_{\mathrm{U}_n(q)}(L_x, \rho_x))$ involved in the Comparison theorem simply sends $\rho_{\Delta_x,\alpha,\beta}$ to $\chi_{\alpha,\beta}$. Here, we made use of the identification $W_{\mathrm{U}_n(q)}(L_x, \rho_x) \simeq W_a$ as in 3.2.
More generally, if M is a standard Levi complement in $\mathrm{U}_n(q)$ containing L_x, we may write $M \simeq \mathrm{U}_b(q) \times \mathrm{GL}_{a_1}(q^2) \times \ldots \times \mathrm{GL}_{a_r}(q^2)$ where $n = 2(a_1 + \ldots + a_r) + b$ and $b \geqslant \frac{x(x+1)}{2}$. The irreducible unipotent representations of M in the Harish-Chandra series $\mathcal{E}(M, (L_x, \rho_x))$ are those of the form $\rho_{\Delta_x,\alpha,\beta} \boxtimes \rho^{\mathrm{GL}}_{\lambda_1} \boxtimes \ldots \boxtimes \rho^{\mathrm{GL}}_{\lambda_r}$ where λ_i is a partition of a_i for $1 \leqslant i \leqslant r$ and (α, β) is a bipartition of the integer $c := \frac{1}{2}\left(b - \frac{x(x+1)}{2}\right)$. On the other hand, the relative Weyl group $W_M(L_x, \rho_x)$ can be identified with the subgroup of $W_{\mathrm{U}_n(q)}(L_x, \rho_x) \simeq W_a$ isomorphic to the product $W_c \times \mathfrak{S}_{a_1} \times \ldots \times \mathfrak{S}_{a_r}$ (note that $c + a_1 + \ldots + a_r = a$). With the notations of 3.2, the W_c-component is generated by the elements $\theta, \sigma_1, \ldots \sigma_{c-1}$, the $\mathfrak{S}_{a_1}$-component by the elements $\sigma_{c+1} \ldots, \sigma_{c+a_1-1}$, and so on. Irreducible characters of $W_M(L_x, \rho_x)$ have the shape $\chi_{\alpha,\beta} \boxtimes \chi_{\lambda_1} \boxtimes \ldots \boxtimes \chi_{\lambda_r}$ where (α, β) is a bipartition of c and λ_i is a partition of a_i for $1 \leqslant i \leqslant r$. Then, according to [FS90] (4.2), the bijection $\mathbb{Z}\mathcal{E}(\mathrm{M}, (L_x, \rho_x)) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(W_M(L_x, \rho_x))$ involved in the Comparison theorem in 3.1 sends $\rho_{\Delta_x,\alpha,\beta} \boxtimes \rho^{\mathrm{GL}}_{\lambda_1} \boxtimes \ldots \boxtimes \rho^{\mathrm{GL}}_{\lambda_r}$ to $\chi_{\alpha,\beta} \boxtimes \chi_{\lambda_1} \boxtimes \ldots \boxtimes \chi_{\lambda_r}$.

3.5 We explain how the two different labellings of the irreducible unipotent representations of $\mathrm{U}_n(q)$ are related. To do this, one needs the notion of 2-quotient. For the following definitions, we allow partitions to have 0 terms at the end. Thus, let us write $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with

$\lambda_r \geqslant 0$. The **β-set** of λ is the sequence of decreasing nonnegative integers $\beta_i := \lambda_i + r - i$ for $1 \leqslant i \leqslant r$. Mapping a partition λ to its β-set gives a bijection between the set of partitions having r terms and the set of decreasing sequences of nonnegative integers of length r. The inverse mapping sends a sequence $(\beta_1 > \ldots > \beta_r \geqslant 0)$ to the partition λ given by $\lambda_i = \beta_i + i - r$. Let λ be a partition of n as above, and let β be its β-set. We let β_{even} (resp. β_{odd}) be the subsequence consisting of all even (resp. odd) integers of β. Then, we define the following sequences.

$$\beta^0 := \left(\frac{\beta_i}{2} \,\middle|\, \beta_i \in \beta_{\text{even}} \right) \qquad\qquad \beta^1 := \left(\frac{\beta_i - 1}{2} \,\middle|\, \beta_i \in \beta_{\text{odd}} \right)$$

The sequences β^0 and β^1 are the β-sets of two partitions, which we call μ^0 and μ^1 respectively. Then, the **2-quotient** of λ is the bipartition (μ^0, μ^1) if r is odd, and (μ^1, μ^0) if r is even. We note that the ordering of μ^0 and μ^1 in the 2-quotient may vary in the literature. Here, we followed the conventions of [FS90] section 1. A different ordering is used in [Jam84] 2.7.29. In loc. cit. Theorem 2.7.37, another construction of the 2-quotient using Young diagrams is proposed.

Let λ' be another partition which differs from λ only by 0 terms at the end. While the β-sets of λ and λ' are not the same, the resulting 2-quotients are equal up to 0 terms at the end of the partitions. Thus, from now on we identify all partitions differing only from 0 terms by removing all of them. The 2-quotient of a partition is then well-defined.

Theorem ([Jam84] Theorem 2.7.30)**.** *A partition λ is uniquely characterized by the data of its 2-core Δ_x and its 2-quotient (λ^0, λ^1). Moreover, the lengths of these partitions are related by the equation*

$$|\lambda| = |\Delta_x| + 2(|\lambda^0| + |\lambda^1|)$$

and $|\Delta_x| = \frac{x(x+1)}{2}$.

For instance, the 2-quotient of the partition $\lambda = (3^2, 2^2, 1)$ is $(2^2, 1)$. Recall that the 2-core of λ is Δ_1. Thus, the equation on the lengths of the partitions is satisfied, as we have $11 = 1 + 2(4+1)$. We may now relate the two labellings $\{\rho_\lambda^{\mathrm{U}}\}$ and $\{\rho_{\Delta_x,\alpha,\beta}\}$ of the irreducible unipotent representations of $\mathrm{U}_n(q)$ together.

Proposition ([FS90] Appendix)**.** *Let λ be a partition of n. Denote by Δ_y its 2-core and by (λ^0, λ^1) its 2-quotient. On the other hand, let $x \geqslant 0$ be such that $n = 2a + \frac{x(x+1)}{2}$ for some $a \geqslant 0$ and let (α, β) be a bipartition of a. Then the irreducible representations ρ_λ^U and $\rho_{\Delta_x,\alpha,\beta}$ are equivalent if and only if $x = y$ and $(\lambda^0, \lambda^1) = (\alpha, \beta)$ if x is even or $(\lambda^0, \lambda^1) = (\beta, \alpha)$ if x is odd.*

For instance, for $\lambda = (3^2, 2^2, 1)$ the representation $\rho_\lambda^{\mathrm{U}}$ is equivalent to $\rho_{\Delta_1,(1),(2^2)}$.

3.6 In order to apply the comparison theorem 3.1 for unitary groups, it remains to understand how to compute inductions in Coxeter groups of type B. Such computations are carried out in [GP00] Section 6.1. It turns out that we will only need one specific case of such inductions, and the corresponding method is known as the Pieri rule for groups of type B.

Proposition ([GP00] 6.1.9). *Let $a \geqslant 1$ and consider $r, s \geqslant 0$ such that $r + s = a$. We think of the group $W_r \times \mathfrak{S}_s$ as a subgroup of W_a as in 3.4.*

- *Let (α, β) be a bipartition of r. Then the induced character*

$$\mathrm{Ind}_{W_r \times \mathfrak{S}_s}^{W_a} \left(\chi_{(\alpha,\beta)} \boxtimes \chi_{(s)}\right)$$

is the multiplicity-free sum of all the characters $\chi_{\gamma,\delta}$ such that for some $0 \leqslant k \leqslant s$, the Young diagram of γ (resp. δ) can be obtained from that of α (resp. β) by adding k boxes (resp. $s-k$ boxes) so that no two of them lie in the same column.

- *Let (γ, δ) be a bipartition of a. The restricted character*

$$\mathrm{Res}_{W_r}^{W_a} \left(\chi_{\gamma,\delta}\right)$$

is the multiplicity-free sum of all the characters $\chi_{(\alpha,\beta)}$ such that for some $0 \leqslant k \leqslant s$, the Young diagram of α (resp. β) can be obtained from that of γ (resp. δ) by deleting k boxes (resp. $s-k$ boxes) so that no two of them lie in the same column.

We will use this rule on concrete examples in the sections that follow.

4 The cohomology of the Coxeter variety for the unitary group

4.1 In this section, we describe the cohomology of the Coxeter varieties for the unitary groups in odd dimension in terms of the classification of unipotent representations that we recalled in the previous section. The cohomology groups are entirely understood by the work of Lusztig in [Lus76].
Let $t \geqslant 0$. The **Coxeter variety** for $\mathrm{U}_{2t+1}(q)$ is the Deligne-Lusztig variety $X_{\varnothing}(\mathrm{cox})$, where cox is any Coxeter element of the Weyl group $\mathbf{W} \simeq \mathfrak{S}_{2t+1}$. Recall that a Coxeter element is a permutation which can be written as the product, in any order, of exactly one simple reflexion for each F-orbit on $\mathbf{S}$. The variety $X_{\varnothing}(\mathrm{cox})$ does not depend on the choice of the Coxeter element. It is defined over $\mathbb{F}_{q^2}$ and is equipped with commuting actions of both $\mathrm{U}_{2t+1}(q)$ and F^2.

Notation. We write $X^t = X_{\varnothing}(\mathrm{cox})$ for the Coxeter variety attached to the unitary group $\mathrm{U}_{2t+1}(q)$. We also write $\mathrm{H}_c^{\bullet}(X^t)$ instead of $\mathrm{H}_c^{\bullet}(X_{\varnothing}(\mathrm{cox}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$, where $\ell \nmid p$.

We first recall known facts on the cohomology of X^t from Lusztig's work.

Theorem ([Lus76]). *The following statements hold.*

(1) *The variety X^t has dimension t and is affine. The cohomology group $\mathrm{H}_c^{t+i}(X^t)$ is zero unless $0 \leqslant i \leqslant t$.*

(2) *The Frobenius F^2 acts in a semisimple manner on the cohomology of X^t.*

(3) *The group* $\mathrm{H}_c^{2t}(X^t)$ *is* 1*-dimensional, the unitary group* $\mathrm{U}_{2t+1}(q)$ *acts trivially whereas* F^2 *has a single eigenvalue* q^{2t}.

(4) *The group* $\mathrm{H}_c^{t+i}(X^t)$ *for* $0 \leqslant i < t$ *is the direct sum of two eigenspaces of* F^2*, for the eigenvalues* q^{2i} *and* $-q^{2i+1}$*. Each eigenspace is an irreducible unipotent representation of* $\mathrm{U}_{2t+1}(q)$.

(5) *If* $0 \leqslant a \leqslant 2t$*, the dimension of the eigenspace of* $(-q)^a$ *inside the sum* $\sum_{i\geqslant 0} \mathrm{H}_c^{t+i}(X^t)$ *is given by the formula*

$$q^{\frac{(2t-a)(2t+1-a)}{2}} \prod_{j=1}^{2t-a} \frac{q^{a+j}-(-1)^{a+j}}{q^j-(-1)^j}.$$

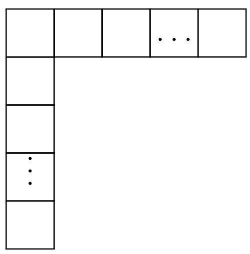
(6) *The sum* $\sum_{i\geqslant 0} \mathrm{H}_c^{t+i}(X^t)$ *is multiplicity-free as a representation of* $\mathrm{U}_{2t+1}(q)$.

4.2 We wish to identify these unipotent representations of $\mathrm{U}_{2t+1}(q)$ occuring in the cohomology of X^t. To this purpose, we start by defining the following partitions. If $0 \leqslant a \leqslant 2t$, we put $\lambda_a^t := (1+a, 1^{2t-a})$. Note that $\lambda_0^t = (1^{2t+1})$ and $\lambda_{2t}^t = (2t+1)$.

Lemma. *For* $0 \leqslant i \leqslant t$*, the* 2*-core of* λ_{2i}^t *is* Δ_1 *and its* 2*-quotient is* $((1^{t-i}), (i))$.
For $0 \leqslant i < t$*, then the* 2*-core of* λ_{2i+1}^t *is* Δ_2 *and its* 2*-quotient is* $((i), (1^{t-i-1}))$.

In particular, according to 3.5 the irreducible unipotent representation $\rho_{\lambda_{2i}^t}$ of $\mathrm{U}_{2t+1}(q)$ is equivalent to the representation $\rho_{\Delta_1,(i),(1^{t-i})}$, and $\rho_{\lambda_{2i+1}^t}$ to $\rho_{\Delta_2,(i),(1^{t-i-1})}$.

Proof. The Young diagram of the partition λ_a^t has the following shape.



The first row has an odd number of boxes when a is even, and an even number of boxes when a is odd. To compute the 2-core, one removes horizontal dominoes from the first row, right to left, and vertical dominoes from the first column, bottom to top. The process results in Δ_1 when a is even and Δ_2 when a is odd.
The partition λ_a^t has $2t+1-a$ non zero terms. Its β-set is given by the sequence

$$\beta = (2t+1, 2t-a, 2t-a-1, \ldots, 1).$$

Assume that $a = 2i$ is even. Then the sequences β^0 and β^1 are given by

$$\beta^0 = (t-i, t-i-1, \ldots 1), \qquad \beta^1 = (t, t-i-1, t-i-2, \ldots, 0).$$

The sequence β^0 has length $t-i$ while β^1 has length $t-i+1$. The associated permutations are then respectively $\mu_0 = (1^{t-i})$ and $\mu_1 = (i)$. Since $2t+1-a$ is odd, the 2-quotient is given

by (μ_0, μ_1) as claimed.
Assume now that $a = 2i+1$ is odd. Then the sequences β^0 and β^1 are given by

$$\beta^0 = (t-i-1, t-i-2, \dots 1), \qquad \beta^1 = (t, t-i-1, t-i-2, \dots, 0).$$

The sequence β^0 has length $t-i-1$ while β^1 has length $t-i+1$. The associated permutations are then respectively $\mu_0 = (1^{t-i-1})$ and $\mu_1 = (i)$. Since $2t+1-a$ is even, the 2-quotient is given by (μ_1, μ_0) as claimed. □

4.3 We may now identify the irreducible unipotent representations occuring in the cohomology of the Coxeter variety X^k.

Proposition. *For $0 \leqslant i < t$, the cohomology group of the Coxeter variety for the finite unitary group $\mathrm{U}_{2t+1}(q)$ is given by*

$$\mathrm{H}_c^{t+i}(X^t) = \rho_{\lambda^t_{2i}} \oplus \rho_{\lambda^t_{2i+1}}$$

with the first summand corresponding to the eigenvalue q^{2i} of F^2 and the second to $-q^{2i+1}$. Moreover, $\mathrm{H}_c^{2t}(X^t) = \rho_{\lambda^t_{2t}}$ with eigenvalue q^{2t}.

Before going to the proof, one may notice that the statement is consistent with the dimensions. Indeed, the formula given in 4.1 Theorem (5) coincides with the hook formula for the degree of the representation $\rho^{\mathrm{U}}_{\lambda^t_a}$ given in 2.5 Proposition.

Proof. First, the statement on the highest cohomology group $\mathrm{H}_c^{2t}(X^t)$ follows from 4.1 Theorem (3). It is the only cohomology group in the case $t = 0$. We will prove the formula by induction on t. Let us now assume $t \geqslant 1$ and that the proposition is known for $t-1$. If $0 \leqslant i \leqslant t-1$, we know that $\mathrm{H}_c^{t+i}(X^t)$ is the sum of two irreducible unipotent representations. So let us write

$$\mathrm{H}_c^{t+i}(X^t) = \rho_{\mu_i} \oplus \rho_{\nu_i}$$

where μ_i and ν_i are two partitions of $2t+1$, and so that ρ_{μ_i} corresponds to the eigenvalue q^{2i} of F^2 whereas ρ_{ν_i} corresponds to $-q^{2i+1}$.
We consider the standard Levi complement $L \simeq \mathrm{GL}_1(q^2) \times \mathrm{U}_{2t-1}(q) \subset \mathrm{U}_{2t+1}(q)$. Let V denote the unipotent radical of the standard parabolic subgroup containing L. According to [Lus76] Corollary 2.10, one can build a geometric isomorphism between the quotient variety X^t/V and the product of the Coxeter variety for L and of a copy of $\mathbb{G}_m$. Even though this geometric isomorphism is not L-equivariant, Lusztig proves that the induced map on cohomology is L-equivariant. By a discussion similar to that in 1.9, the Coxeter variety for L is isomorphic to the Coxeter variety X^{t-1} for $\mathrm{U}_{2t-1}(q)$. We write ${}^*\mathrm{R}^t_{t-1}$ for the composition of the Harish-Chandra restriction from $\mathrm{U}_{2t+1}(q)$ to L, with the usual restriction from L to the subgroup $\mathrm{U}_{2t-1}(q)$. For any nonnegative integer i, the $\mathrm{U}_{2t-1}(q), F^2$-equivariant induced map on the cohomology is an isomorphism

$$ {}^*\mathrm{R}^t_{t-1}\left(\mathrm{H}_c^{t+i}(X^t)\right) \simeq \mathrm{H}_c^{t-1+i}(X^{t-1}) \oplus \mathrm{H}_c^{t-1+(i-1)}(X^{t-1})(1). \qquad (**)$$

Here, (1) denotes the Tate twist (the action of F^2 on a twist $M(n)$ is obtained from the action on the space M by multiplication with q^{2n}). The right-hand side of this identity is given by

the induction hypothesis. Let us look at the left-hand side.
We fix $0 \leqslant i \leqslant t-1$ and we denote by $(\Delta_x, \alpha, \beta)$ and by $(\Delta_y, \gamma, \delta)$ the alternative labelling of the representations ρ_{μ_i} and ρ_{ν_i} respectively as introduced in 3.4 and 3.5. By the Howlett-Lehrer comparison theorem for restriction in 3.1 and by the Pieri rule in 3.6, we know that the restriction ${}^*\mathrm{R}^t_{t-1}(\rho_{\Delta_x,\alpha,\beta})$ is the multiplicity-free sum of all the representations $\rho_{\Delta_x,\alpha',\beta'}$ where the bipartition (α', β') can be obtained from (α, β) by removing exactly one box, of either α or β. The similar description also holds for ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right)$.
By using 4.2 Lemma and the induction hypothesis, we may write down the identity $(**)$ explicitely. Moreover, as it is F^2-equivariant we can identify the components corresponding to the same eigenvalues on both sides. We distinguish 4 different cases depending on the values of t and i.

- **Case t = 1**. We only need to consider $i = 0$. On the right-hand side of $(**)$, the second term is 0 because $t-1+(i-1) = -1 < 0$. On the other hand, the first term is $\rho_{\lambda^0_0} \simeq \rho_{\Delta_1,\varnothing,\varnothing}$ and it corresponds to the eigenvalue $(-q)^0 = 1$. By identifying the eigenspaces, we have ${}^*\mathrm{R}^1_0(\rho_{\Delta_x,\alpha,\beta}) \simeq \rho_{\Delta_1,\varnothing,\varnothing}$ and ${}^*\mathrm{R}^1_0\left(\rho_{\Delta_y,\gamma,\delta}\right) = 0$. The second equation implies that there is no box to remove from γ nor from δ. Thus, $\gamma = \delta = \varnothing$. The value of y is given by the relation $2t+1 = 3 = 2(0+0) + \frac{y(y+1)}{2}$, that is $y = 2$. This corresponds to the partition $\nu_0 = \lambda^1_1$. We notice in passing that the representation ρ_{ν_0} is the unique unipotent cuspidal representation of $\mathrm{U}_3(q)$.
  As for μ_0, the equation ${}^*\mathrm{R}^1_0(\rho_{\Delta_x,\alpha,\beta}) \simeq \rho_{\Delta_1,\varnothing,\varnothing}$ tells us that there is only one removable box from (α, β). After removal of this box, both partitions are empty. Thus, we deduce that $x = 1$ and $(\alpha, \beta) = (1, \varnothing)$ or $(\varnothing, 1)$. This corresponds respectively to $\mu_0 = \lambda^1_2$ or $\mu_0 = \lambda^1_0$. That is, ρ_{μ_0} is either the trivial or the Steinberg representation of $\mathrm{U}_3(q)$. We can deduce which one it is by comparing the degree of the representations with the formula of 4.1 Theorem (5). According to this formula, the dimension of the eigenspace for $(-q)^0$ is q^3. This is precisely the degree of the Steinberg representation $\rho_{\lambda^1_0}$ as given by the hook formula in 2.5 Proposition, and it excludes the possibility of ρ_{μ_0} being trivial. Thus, we have $\mu_0 = \lambda^1_0$ as claimed.

From now, we assume $t \geqslant 2$.

- **Case i = 0**. On the right-hand side of $(**)$, the second term is 0 because $t-1+(i-1) = t-2 < t-1$. The first term is $\rho_{\lambda^{t-1}_0} \oplus \rho_{\lambda^{t-1}_1} \simeq \rho_{\Delta_1,\varnothing,(1^{t-1})} \oplus \rho_{\Delta_2,\varnothing,(1^{t-2})}$. Identifying the eigenspaces, we have ${}^*\mathrm{R}^t_{t-1}(\rho_{\Delta_x,\alpha,\beta}) \simeq \rho_{\Delta_1,\varnothing,(1^{t-1})}$ and ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right) \simeq \rho_{\Delta_2,\varnothing,(1^{t-2})}$. We deduce that $x = 1$ and $y = 2$. Moreover, it also follows that there is only one removable box in (α, β) and in (γ, δ). After removal, we should obtain respectively $(\varnothing, (1^{t-1}))$ and $(\varnothing, (1^{t-2}))$. The only possibility is that $(\alpha, \beta) = (\varnothing, (1^t))$ and $(\gamma, \delta) = (\varnothing, (1^{t-1}))$. This corresponds to $\mu_0 = \lambda^t_0$ and $\nu_0 = \lambda^t_1$ as claimed.
- **Case i = t − 1**. On the right-hand side of $(**)$, the first term is $\rho_{\lambda^{t-1}_{2(t-1)}} \simeq \rho_{\Delta_1,(t-1),\varnothing}$ and the second term is $\rho_{\lambda^{t-1}_{2(t-2)}} \oplus \rho_{\lambda^{t-1}_{2(t-2)+1}} \simeq \rho_{\Delta_1,(t-2),(1)} \oplus \rho_{\Delta_2,(t-2),\varnothing}$. Identifying the eigenspaces while taking the Tate twist into account, we have ${}^*\mathrm{R}^t_{t-1}(\rho_{\Delta_x,\alpha,\beta}) \simeq \rho_{\Delta_1,(t-1),\varnothing} \oplus \rho_{\Delta_1,(t-2),(1)}$ and ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right) \simeq \rho_{\Delta_2,(t-2),\varnothing}$. We deduce that $x = 1$ and $y = 2$. Moreover, there are

two removable boxes in (α,β) and only one removable box in (γ,δ). After removal of one of the two boxes in (α,β), we can get either $((t-1),\varnothing)$ or $((t-2),(1))$; and after removal of the box in (γ,δ) we obtain $((t-2),\varnothing)$. The only possibility is that $(\alpha,\beta) = ((t-1),(1))$ and $(\gamma,\delta) = ((t-1),\varnothing)$. This corresponds to $\mu_{t-1} = \lambda^t_{2(t-1)}$ and $\nu_{t-1} = \lambda^t_{2(t-1)+1}$ as claimed.

– **Case $\mathbf{1 \leqslant i \leqslant t-2}$**. On the right-hand side of (**), the first term is $\rho_{\lambda^{t-1}_{2i}} \oplus \rho_{\lambda^{t-1}_{2i+1}} \simeq \rho_{\Delta_1,(i),(1^{t-1-i})} \oplus \rho_{\Delta_2,(i),(1^{t-2-i})}$. The second term is $\rho_{\lambda^{t-1}_{2(i-1)}} \oplus \rho_{\lambda^{t-1}_{2(i-1)+1}} \simeq \rho_{\Delta_1,(i-1),(1^{t-i})} \oplus \rho_{\Delta_2,(i-1),(1^{t-1-i})}$. Identifying the eigenspaces while taking the Tate twist into account, we have ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_x,\alpha,\beta}\right) \simeq \rho_{\Delta_1,(i),(1^{t-1-i})} \oplus \rho_{\Delta_1,(i-1),(1^{t-i})}$ and ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right) \simeq \rho_{\Delta_2,(i),(1^{t-2-i})} \oplus \rho_{\Delta_2,(i-1),(1^{t-1-i})}$. We deduce that $x = 1$ and $y = 2$. Moreover, there are exactly two removable boxes from (α,β) and from (γ,δ). After removal of one of the two boxes in (α,β), we can get either $((i),(1^{t-1-i}))$ or $((i-1),(1^{t-i}))$; and after removal of one of the two boxes in (γ,δ), we can get either $((i),(1^{t-2-i}))$ or $((i-1),(1^{t-1-i}))$. The only possibility is that $(\alpha,\beta) = ((i),(1^{t-i}))$ and $(\gamma,\delta) = ((i),(1^{t-1-i}))$. This corresponds to $\mu_i = \lambda^t_{2i}$ and $\nu_i = \lambda^t_{2i+1}$ as claimed.

□

5 The cohomology of the variety $X_I(\mathrm{id})$

5.1 We go on with the computation of the cohomology of the variety $X_I(\mathrm{id})$. We use the same notations as in section 1. We first compute the cohomology of each Ekedahl-Oort stratum $X_{I_t}(w_t)$, before using the spectral sequence associated to the stratification to conclude.
Recall that $X_I(\mathrm{id})$ has dimension d, is defined over $\mathbb{F}_{q^2}$ and is equipped with an action of $J \simeq \mathrm{U}_{2d+1}(q)$. As before, we will write $\mathrm{H}^\bullet_c(X_I(\mathrm{id}))$ as a shortcut for $\mathrm{H}^\bullet_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$.

Theorem. *The following statements hold.*

(1) *The cohomology group $\mathrm{H}^i_c(X_I(\mathrm{id}))$ is zero unless $0 \leqslant i \leqslant 2d$. There is an isomorphism $\mathrm{H}^i_c(X_I(\mathrm{id})) \simeq \mathrm{H}^{2d-i}_c(X_I(\mathrm{id}))^\vee(d)$ which is equivariant for the actions of F^2 and of $\mathrm{U}_{2d+1}(q)$.*

(2) *The Frobenius F^2 acts like multiplication by $(-q)^i$ on $\mathrm{H}^i_c(X_I(\mathrm{id}))$.*

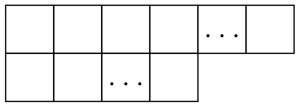
(3) *For $0 \leqslant i \leqslant d$ we have*

$$\mathrm{H}^{2i}_c(X_I(\mathrm{id})) = \bigoplus_{s=0}^{\min(i,d-i)} \rho_{(2d+1-2s,2s)}.$$

For $0 \leqslant i \leqslant d-1$ we have

$$\mathrm{H}^{2i+1}_c(X_I(\mathrm{id})) = \bigoplus_{s=0}^{\min(i,d-1-i)} \rho_{(2d-2s,2s+1)}.$$

Thus, in the cohomology of $X_I(\mathrm{id})$ all the representations associated to a Young diagram with at most 2 rows occur, and there is no other. Such a diagram has the following general shape.



We may rephrase the result by using the alternative labelling of the irreducible unipotent representations as in 3.5. The partition $(2d+1-2s, 2s)$ has 2-core Δ_1 and 2-quotient $(\varnothing, (d-s, s))$; whereas the partition $(2d-2s, 2s+1)$ has 2-core Δ_2 and 2-quotient $((d-1-s, s), \varnothing)$. Thus, according to 3.5 Proposition, we have

$$\rho_{(2d+1-2s,2s)} \simeq \rho_{\Delta_1,(d-s,s),\varnothing}, \qquad \rho_{(2d-2s,2s+1)} \simeq \rho_{\Delta_2,(d-1-s,s),\varnothing}.$$

In particular, all irreducible representations in the cohomology groups of even index belong to the unipotent principal series $\mathcal{E}(\mathrm{U}_{2d+1}(q), (L_1, \rho_1))$, whereas all the ones in the groups of odd index belong to the Harish-Chandra series $\mathcal{E}(\mathrm{U}_{2d+1}(q), (L_2, \rho_2))$.

Proof. Point (1) of the statement follows from a general property of the cohomology groups, namely Poincaré duality. It is due to the fact that $X_I(\mathrm{id})$ is projective and smooth. It also implies the purity of the Frobenius F^2 on the cohomology : we know at this stage that all eigenvalues of F^2 on $\mathrm{H}^i_c(X_I(\mathrm{id}))$ have complex modulus q^i under any choice of an isomorphism $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$.
We prove the points (2) and (3) by explicit computations. As in 4.2, we denote by λ^t_a the partition $(1+a, 1^{2t-a})$ of $2t+1$. Let $0 \leqslant t \leqslant d$. For $0 \leqslant a \leqslant 2t$ we will write

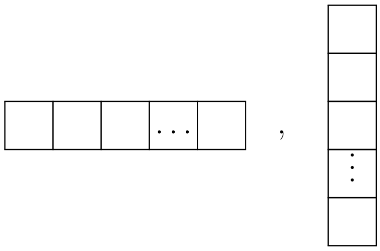
$$\mathrm{R}^t_a := \mathrm{R}^{\mathrm{U}_{2d+1}(q)}_{L_{K_t}} \left(\rho^{\mathrm{GL}}_{(d-t)} \boxtimes \rho^{\mathrm{U}}_{\lambda^t_a}\right).$$

Recall that 1.8 Proposition gives an isomorphism between the Ekedahl-Oort stratum $X_{I_t}(w_t)$ and the variety $\mathrm{U}_{2d+1}(q)/U_{K_t} \times_{L_{K_t}} X^{\mathbf{L}_{K_t}}_{I_t}(w_t)$. It implies that the cohomology of the Ekedahl-Oort stratum is the Harish-Chandra induction of the cohomology of the Deligne-Lusztig variety $X^{\mathbf{L}_{K_t}}_{I_t}(w_t)$. According to 1.9, this cohomology is related to that of the Coxeter variety for $\mathrm{U}_{2t+1}(q)$. Combining with the formula of 4.3 Proposition, for $0 \leqslant i \leqslant t-1$ it follows that

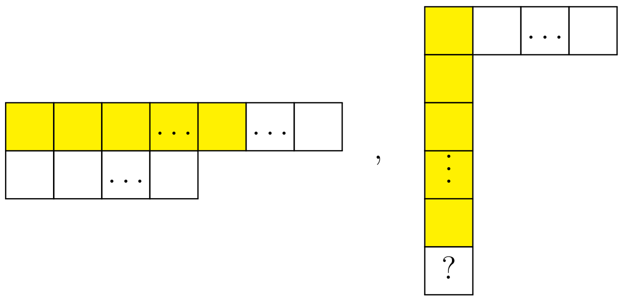
$$\mathrm{H}^{t+i}_c(X_{I_t}(w_t)) = \mathrm{R}^t_{2i} \oplus \mathrm{R}^t_{2i+1}, \qquad \mathrm{H}^{2t}_c(X_{I_t}(w_t)) = \mathrm{R}^t_{2t}.$$

The representation R^t_a in this formula is associated to the eigenvalue $(-q)^a$ of F^2.

We first compute R^t_a explicitely. By the combination of the Howhlett-Lehrer comparison theorem in 3.1 and the Pieri rule for groups of type B as in 3.6, one can compute the Harish-Chandra induction R^t_a by adding $d-t$ boxes to the bipartition corresponding to the representation $\rho^{\mathrm{U}}_{\lambda^t_a}$ with no two added boxes in the same column. Recall from 4.2 Lemma that the representation $\rho_{\lambda^t_{2i}}$ of $\mathrm{U}_{2t+1}(q)$ is equivalent to the representation $\rho_{\Delta_1,(i),(1^{t-i})}$, and that $\rho_{\lambda^t_{2i+1}}$ is equivalent to $\rho_{\Delta_2,(i),(1^{t-1-i})}$.
In order to illustrate the argument, let us say that we want to add N boxes to a bipartition of the shape as in the figure below, so that no two added boxes lie in the same column.



We will add N_1 boxes to the first diagram and N_2 to the second, where $N = N_1 + N_2$. In the first diagram, the only places where we can add boxes are in the second row from left to right, and at the end of the first row. Because no two added boxes must be in the same column, the number of boxes we add on the second row must be at most the number of boxes already lying in the first row. Of course, it must also be at most N_1.
In the second diagram, the only places where we can add boxes are at the bottom of the first column and at the end of the first row. Because no two added boxes must be in the same column, we can only put up to one box at the bottom of the first column and all the remaining ones will align at the end of the first row.
At the end of the process, we will obtain a bipartition of the following general shape.



We colored in yellow the boxes that were already there before we added new ones. The box with a question mark may or may not be placed there.
We now make the result more precise, and write down exactly what the irreducible components of R^t_a are depending on the parity of a.

- For $0 \leqslant i \leqslant t$, the representation R^t_{2i} is the multiplicity-free sum of all the representations $\rho_{\Delta_1,\alpha,\beta}$ where the bipartition (α, β) satisfies, for some $0 \leqslant x \leqslant d-t$,

$$\begin{cases} \alpha = (i+x-s, s) \text{ for some } 0 \leqslant s \leqslant \min(x, i), \\ \beta = (d-t-x, 1^{t-i}) \text{ or } (d-t-x+1, 1^{t-i-1}). \end{cases}$$

- For $0 \leqslant i \leqslant t-1$, the representation R^t_{2i+1} is the multiplicity-free sum of all representations $\rho_{\Delta_2,\alpha,\beta}$ where the bipartition (α, β) satisfies, for some $0 \leqslant x \leqslant d-t$,

$$\begin{cases} \alpha = (i+x-s, s) \text{ for some } 0 \leqslant s \leqslant \min(x, i), \\ \beta = (d-t+1-x, 1^{t-1-i}) \text{ or } (d-t+2-x, 1^{t-2-i}). \end{cases}$$

In our notations, we used the convention that the partitions (0) and (1^0) are the empty partition $\varnothing$. The integer x corresponds to the number of boxes we add to the first partition. We notice that if i takes the maximal value, there is only one possibility for β that is respectively $(d-t-x)$

in the first case and $(d-t+1-x)$ in the second case.
Recall from 1.6 that the variety $X_I(\mathrm{id})$ is the union of the Ekedahl-Oort strata $X_{I_t}(w_t)$ for $0 \leqslant t \leqslant d$ and the closure of the stratum for t is the union of all strata $X_{I_s}(w_s)$ for $s \leqslant t$. At the level of cohomology, it translates into the following $F^2, \mathrm{U}_{2d+1}(q)$-equivariant spectral sequence

$$\mathrm{E}_1^{t,i} : \mathrm{H}_c^{t+i}(X_{I_t}(w_t)) \implies \mathrm{H}_c^{t+i}(X_I(\mathrm{id})).$$

The first page of the sequence is drawn in the Figure 1, it has a triangular shape.

$$\begin{array}{ccccccc}
 & & & & & & \mathrm{R}^d_{2d} \\
 & & & & & \mathrm{R}^{d-1}_{2d-2} \longrightarrow & \mathrm{R}^d_{2d-2} \oplus \mathrm{R}^d_{2d-1} \\
 & & & \iddots & & & \vdots \\
 & & \mathrm{R}^2_4 \longrightarrow & \dots & \longrightarrow \mathrm{R}^{d-1}_4 \oplus \mathrm{R}^{d-1}_5 \longrightarrow & & \mathrm{R}^d_4 \oplus \mathrm{R}^d_5 \\
 & \mathrm{R}^1_2 \longrightarrow & \mathrm{R}^2_2 \oplus \mathrm{R}^2_3 \longrightarrow & \dots & \longrightarrow \mathrm{R}^{d-1}_2 \oplus \mathrm{R}^{d-1}_3 \longrightarrow & & \mathrm{R}^d_2 \oplus \mathrm{R}^d_3 \\
\mathrm{R}^0_0 \longrightarrow & \mathrm{R}^1_0 \oplus \mathrm{R}^1_1 \longrightarrow & \mathrm{R}^2_0 \oplus \mathrm{R}^2_1 \longrightarrow & \dots & \longrightarrow \mathrm{R}^{d-1}_0 \oplus \mathrm{R}^{d-1}_1 \longrightarrow & & \mathrm{R}^d_0 \oplus \mathrm{R}^d_1
\end{array}$$

Figure 1: The first page of the spectral sequence.

The representation R^t_a corresponds to the eigenvalue $(-p)^a$ of F^2 as before. The only eigenvalues of F^2 on the i-th row of the spectral sequence are p^{2i} and $-p^{2i+1}$. In particular, the eigenvalues on two distinct rows are different. Since the differentials in deeper pages of the sequence map terms from different rows, their F^2-equivariance implies that they vanish. Therefore, the sequence degenerates on the second page.
Moreover, by the machinery of spectral sequences, for $0 \leqslant k \leqslant 2d$ there exists a filtration by $\mathrm{U}_{2d+1}(q) \times \langle F^2 \rangle$-modules on $\mathrm{H}_c^k(X_I(\mathrm{id}))$ whose graded components are the terms of the second page lying on the anti-diagonal $t+i=k$. Since the group algebra $\overline{\mathbb{Q}_\ell}[\mathrm{U}_{2d+1}(q)]$ is semi-simple, the filtration splits, meaning that $\mathrm{H}_c^k(X_I(\mathrm{id}))$ is actually the direct sum of the graded components. The purity of $\mathrm{H}_c^k(X_I(\mathrm{id}))$ then implies that all the terms of the second page lying on the anti-diagonal $t+i=k$, which are associated to an eigenvalue whose modulus is not equal to q^k, must be zero. Therefore, the second page has the shape described in Figure 2. The Frobenius F^2 acts via q^{2i} on the term $\mathrm{E}_2^{i,i}$, and via $-q^{2i+1}$ on the term $\mathrm{E}_2^{i+1,i}$. Point (2) of the Theorem readily follows.

By the previous computations, we understand precisely all the terms in the first page of the spectral sequence. The key observation to compute the second page is that two terms on the

$$\begin{array}{cccccc}
 & & & & & \mathrm{E}_2^{d,d} \\
 & & & & \mathrm{E}_2^{d-1,d-1} & \mathrm{E}_2^{d,d-1} \\
 & & & \iddots & & \vdots \\
 & \mathrm{E}_2^{1,1} & \mathrm{E}_2^{2,1} & 0 & \cdots & 0 \\
\mathrm{E}_2^{0,0} & \mathrm{E}_2^{1,0} & 0 & 0 & \cdots & 0
\end{array}$$

Figure 2: The second page of the spectral sequence.

first page which lie on the same row, but are separated by at least 2 arrows, do not have any irreducible component in common. We make the argument more precise in the following two paragraphs, distinguishing the cohomology groups of even and odd index.

We first compute the cohomology group $\mathrm{H}_c^{2t}(X_I(\mathrm{id}))$ for $0 \leqslant t \leqslant d$. We look at the following portion of the first page

$$\mathrm{R}_{2t}^{t} \longrightarrow \mathrm{R}_{2t}^{t+1} \oplus \mathrm{R}_{2t+1}^{t+1} \longrightarrow \mathrm{R}_{2t}^{t+2} \oplus \mathrm{R}_{2t+1}^{t+2} .$$

By extracting the eigenspaces corresponding to q^{2t}, we actually have the following sequence

$$\mathrm{R}_{2t}^{t} \xrightarrow{u} \mathrm{R}_{2t}^{t+1} \xrightarrow{v} \mathrm{R}_{2t}^{t+2} .$$

The representation R_{2t}^{t} is the sum of all the representations $\rho_{\Delta_1,\alpha,\beta}$ where for some $0 \leqslant x \leqslant d-t$ and for some $0 \leqslant s \leqslant \min(x,t)$, we have $\alpha = (t+x-s, s)$ and $\beta = (d-t-x)$.
The representation R_{2t}^{t+1} is the sum of all the representations $\rho_{\Delta_1,\alpha',\beta'}$ where for some $0 \leqslant x' \leqslant d-t-1$ and for some $0 \leqslant s \leqslant \min(x',t)$, we have $\alpha' = (t+x'-s, s)$ and $\beta' = (d-t-x')$ or $(d-t-x'-1, 1)$.
The quotient space $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is isomorphic to the eigenspace of q^{2t} in $E_2^{t+1,t}$, which is zero. Besides, in the representation R_{2t}^{t+2} all the irreducible components have the shape $\rho_{\Delta_1,\alpha'',\beta''}$ with β'' a partition of length 2 or 3. In particular, all the representations $\rho_{\Delta_1,\alpha',\beta'}$ of R_{2t}^{t+1} with β' a partition of length 1 automatically lie inside $\mathrm{Ker}(v) = \mathrm{Im}(u)$. Such representations correspond to all the irreducible components $\rho_{\Delta_1,\alpha,\beta}$ of R_{2t}^{t} having $x \neq d-t$. Thus, none of them lies in $\mathrm{Ker}(u) \simeq E_2^{t,t}$.
The remaining components of R_{2t}^{t} are those having $x = d-t$, and they do not occur in the codomain of u so that they lie in $\mathrm{Ker}(u)$. By the previous argument, they must form the whole of $\mathrm{Ker}(u)$.

Thus, we have proved that

$$E_2^{t,t} \simeq \mathrm{H}_c^{2t}(X_I(\mathrm{id})) \simeq \mathrm{Ker}(u) = \bigoplus_{s=0}^{\min(t,d-t)} \rho_{\Delta_1,(t-s,s),\varnothing}$$

and it coincides with the formula of point (3).

We now compute the cohomology group $\mathrm{H}_c^{2t+1}(X_I(\mathrm{id}))$ for $0 \leqslant t \leqslant d-1$. We look at the following portion of the first page

$$\mathrm{R}_{2t}^{t} \longrightarrow \mathrm{R}_{2t}^{t+1} \oplus \mathrm{R}_{2t+1}^{t+1} \longrightarrow \mathrm{R}_{2t}^{t+2} \oplus \mathrm{R}_{2t+1}^{t+2} \longrightarrow \mathrm{R}_{2t}^{t+3} \oplus \mathrm{R}_{2t+1}^{t+3} .$$

By extracting the eigenspaces corresponding to $-q^{2t+1}$, we actually have the following sequence

$$0 \longrightarrow \mathrm{R}_{2t+1}^{t+1} \xrightarrow{u} \mathrm{R}_{2t+1}^{t+2} \xrightarrow{v} \mathrm{R}_{2t+1}^{t+3} .$$

The representation R_{2t+1}^{t+1} is the sum of all the representations $\rho_{\Delta_2,\alpha,\beta}$ where for some $0 \leqslant x \leqslant d-t-1$ and for some $0 \leqslant s \leqslant \min(x,t)$, we have $\alpha = (t+x-s,s)$ and $\beta = (d-t-x)$.
The representation R_{2t+1}^{t+2} is the sum of all the representations $\rho_{\Delta_2,\alpha',\beta'}$ where for some $0 \leqslant x' \leqslant d-t-2$ and for some $0 \leqslant s \leqslant \min(x',t)$, we have $\alpha' = (t+x'-s,s)$ and $\beta' = (d-t-1-x',1)$ or $(d-t-x')$.
The quotient space $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is isomorphic to the eigenspace of $-q^{2t+1}$ in $E_2^{t+2,t}$, which is zero. Besides, in the representation R_{2t+1}^{t+3} all the irreducible components have the shape $\rho_{\Delta_2,\alpha'',\beta''}$ with β'' a partition of length 2 or 3. In particular, all the representations $\rho_{\Delta_2,\alpha',\beta'}$ of R_{2t+1}^{t+2} with β' a partition of length 1 automatically lie inside $\mathrm{Ker}(v) \simeq \mathrm{Im}(u)$. Such representations correspond to all the irreducible components $\rho_{\Delta_2,\alpha,\beta}$ of R_{2t+1}^{t+1} having $x \neq d-t-1$. Thus, none of them lies in $\mathrm{Ker}(u) \simeq E_2^{t+1,t}$.
The remaining components of R_{2t+1}^{t+1} are those having $x = d-t-1$, and they do not occur in the codomain of u so that they lie in $\mathrm{Ker}(u)$. By the argument above, they must form the whole of $\mathrm{Ker}(u)$.
Thus, we have proved that

$$E_2^{t+1,t} \simeq \mathrm{H}_c^{2t+1}(X_I(\mathrm{id})) \simeq \mathrm{Ker}(u) = \bigoplus_{s=0}^{\min(d-t-1,t)} \rho_{\Delta_2,(t-1-s,s),\varnothing}$$

and one may check that it coincides with the formula of point (3). □

Bibliography

[BR06] C. Bonnafé and R. Rouquier. “On the irreducibility of Deligne–Lusztig varieties”. In: *Comptes Rendus Mathematique* 343.1 (2006). DOI: 10.1016/j.crma.2006.04.014.

[DL76] P. Deligne and G. Lusztig. “Representations of Reductive Groups Over Finite Fields”. In: *Annals of Mathematics* 103 (1976). DOI: 10.2307/1971021.

[DM14] F. Digne and J. Michel. “Parabolic Deligne–Lusztig varieties”. In: *Advances in Mathematics* 257 (2014). DOI: 10.1016/j.aim.2014.02.023.

[FS90] P. Fong and B. Srinivasan. "Brauer trees in classical groups". en. In: *Journal of Algebra* 131.1 (1990). DOI: 10.1016/0021-8693(90)90172-K.

[GM20] M. Geck and G. Malle. *The Character Theory of Finite Groups of Lie Type: A Guided Tour*. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2020. ISBN: 978-1-108-48962-1. DOI: 10.1017/9781108779081.

[GP00] M. Geck and G. Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. London Mathematical Society Monographs. Oxford University Press, 2000. ISBN: 978-0-19-850250-0.

[HL83] R. B. Howlett and G. I. Lehrer. "Representations of Generic Algebras and Finite Groups of Lie Type". In: *Transactions of the American Mathematical Society* 280.2 (1983). DOI: 10.2307/1999645.

[Jam84] G. James. *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1984. ISBN: 978-0-521-10412-8.

[LS77] G. Lusztig and B. Srinivasan. "The characters of the finite unitary groups". In: *Journal of Algebra* 49.1 (1977). DOI: 10.1016/0021-8693(77)90277-0.

[Lus76] G. Lusztig. "Coxeter orbits and eigenspaces of Frobenius". In: *Inventiones mathematicae* 38.2 (1976). DOI: 10.1007/BF01408569.

[Lus77] G. Lusztig. "Irreducible representations of finite classical groups". In: *Inventiones mathematicae* 43.2 (1977). DOI: 10.1007/BF01390002.

[Mul22] J. Muller. "Cohomology of the basic unramified PEL unitary Rapoport-Zink space of signature (1,n-1)". In: (2022). arXiv: 2201.10229.

[Vol10] I. Vollaard. "The Supersingular Locus of the Shimura Variety for GU(1, s)". In: *Canadian Journal of Mathematics* 62.3 (2010). DOI: 10.4153/CJM-2010-031-2.

[VW11] I. Vollaard and T. Wedhorn. "The supersingular locus of the Shimura variety of GU(1,n−1) II". In: *Inventiones mathematicae* 184.3 (2011). DOI: 10.1007/s00222-010-0299-y.