On the cohomology of the basic unramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

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Abstract : *In this paper, we study the cohomology of the unitary unramified PEL Rapoport-Zink space of signature $(1, n-1)$ at maximal level. Our method revolves around the spectral sequence associated to the open cover by the analytical tubes of the closed Bruhat-Tits strata in the special fiber, which were constructed by Vollaard and Wedhorn. The cohomology of these strata, which are isomorphic to generalized Deligne-Lusztig varieties, has been computed in [Mul21]. This spectral sequence allows us to prove the semisimplicity of the Frobenius action and to describe the inertial supports of the irreducible subquotients in the individual cohomology groups. In particular, when $n \geqslant 5$ no such subquotient is supercuspidal, but when $1 \leqslant n \leqslant 4$ supercuspidal subquotients do occur. Moreover, we also prove that the cohomology groups need not be admissible in general. Via p-adic uniformization, we relate the cohomology of the Rapoport-Zink space to the cohomology of the basic stratum of a Shimura variety with no level at p. In the case $n = 3$ or 4, we give a complete description of the cohomology of the basic stratum in terms of automorphic representations. In particular, some automorphic representations occur with multiplicities dependent on p.*

Contents

Introduction: By defining moduli problems classifying deformations of p-divisible groups with additional structures, Rapoport and Zink have constructed their eponymous spaces which consist in a projective system $(\mathcal{M}_{K_p})$ of non-archimedean analytic spaces. The set of data defining the moduli problem determines two p-adic groups $G(\mathbb{Q}_p)$ and J which both act on the tower. Its cohomology is therefore equipped with an action of $G(\mathbb{Q}_p) \times J \times W$ where W is the absolute Weyl group of a finite extension of $\mathbb{Q}_p$, called the local reflex field. This is expected to give a geometric incarnation of the local Langlands correspondance. So far, relatively little is known about the cohomology of Rapoport-Zink spaces in general. The Kottwitz conjecture describes the $G \times J(\mathbb{Q}_p)$-supercuspidal part of the cohomology but it is only known in a handful of cases. It was first proved for the Lubin-Tate tower in [Boy99] and in [HT01], from which the Drinfeld case follows by duality. The case of basic unramified EL Rapoport-Zink spaces has been treated in [Far04] and [Shi12]. As for the PEL case, it was proved for basic unramified unitary Rapoport-Zink spaces with signature $(1, n-1)$ with n odd in [Ngu19], and in [BMN21] for an arbitrary signature with an odd number of variables. Beyond the Kottwitz conjecture, one would like to understand the individual cohomology groups of the Rapoport-Zink spaces entirely. This has been done in [Boy09] for the Lubin-Tate case (and, dually, for the Drinfeld case as well) using a vanishing cycle approach. Boyer's results were later used in [Dat07] to recover the action of the monodrony and give an elegant form of geometric Jacquet-Langlands correspondance. However, this method relied heavily on the particuliar geometry of the Lubin-Tate tower, and we are faced with technical issues in other situations where we do not have a satisfactory understanding of the geometry of the Rapoport-Zink spaces.

In this paper, we aim at pursuing the goal of describing the individual cohomology groups of the Rapoport-Zink spaces in the basic PEL unramified unitary case with signature $(1, n-1)$. Here, $G(\mathbb{Q}_p)$ is an unramified group of unitary similitudes in n variables and J is an inner form of $G(\mathbb{Q}_p)$. In fact, J is isomorphic to $G(\mathbb{Q}_p)$ when n is odd and J is the non quasi-split inner form when n is even. Our approach is based on the geometric description of the reduced special fiber $\mathcal{M}_{\mathrm{red}}$ given in [Vol10] and [VW11]. In these papers, Vollaard and Wedhorn built the Bruhat-Tits stratification $\{\mathcal{M}_\Lambda\}$ on $\mathcal{M}_{\mathrm{red}}$ which is interesting for two reasons:

- the closed strata $\mathcal{M}_\Lambda$ are indexed by the vertices of the Bruhat-Tits building $\mathrm{BT}(J, \mathbb{Q}_p)$ of J. The combinatorics of the stratification can be read on the building.
- each individual stratum $\mathcal{M}_\Lambda$ is isomorphic to a generalized Deligne-Lusztig variety for a

finite group of Lie type of the form $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$, arising in the maximal reductive quotient of the maximal parahoric subgroup $J_\Lambda := \mathrm{Fix}_J(\Lambda)$.

In [Mul21], by exploiting the Ekedahl-Oort stratification on a given stratum $\mathcal{M}_\Lambda$, we computed its cohomology in terms of representations of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ with a Frobenius action. We consider the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}} := \mathcal{M}_{C_0}$ at maximal level, where $C_0 \subset G(\mathbb{Q}_p)$ is a hyperspecial maximal open compact subgroup. It admits an open cover by the analytical tubes U_Λ of the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$. This induces a $J \times W$-equivariant Čech spectral sequence computing the cohomology of $\mathcal{M}^{\mathrm{an}}$ (see 4.1.4 for the precise notations):

$$E_1^{a,b} : \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U_{\Lambda(\gamma)} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}).$$

Using Berkovich's comparison theorem, the cohomology of the tubes U_Λ can be identified, up to a shift in indices and a suitable Tate twist, with the cohomology of the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$. Let $\mathrm{Frob} \in W$ be a lift of the geometric Frobenius and let τ denote the action of the element $(p^{-1} \cdot \mathrm{id}, \mathrm{Frob}) \in J \times W$ on the cohomology. Then the action of τ on the cohomology of U_Λ is identified with the Frobenius action on the cohomology of $\mathcal{M}_\Lambda$. It follows in particular that τ acts in a semisimple manner on the cohomology of the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$.

Proposition (4.1.9)**.** *The spectral sequence degenerates on the second page E_2. For $0 \leqslant b \leqslant 2(n-1)$, the induced filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ splits, ie. we have an isomorphism*

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b \leqslant b' \leqslant 2(n-1)} E_2^{b-b',b'}.$$

The action of W on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ is trivial on the inertia subgroup and the action of the rational Frobenius element τ is semisimple. The subspace $E_2^{b-b',b'}$ is identified with the eigenspace of τ associated to the eigenvalue $(-p)^{b'}$.

Let $m := \lfloor \frac{n-1}{2} \rfloor$. In order to study the J-action, we rewrite the terms $E_1^{a,b}$ using compactly induced representations (see 4.1.10 for the precise notations)

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \left(\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell})\right)^{k_{-a+1,\theta}}.$$

The various J_θ's are maximal parahoric subgroups of J. Using type theory, we may describe the inertial supports of the irreducible subquotients of such compactly induced representations. An inertial class is a pair $[M, \tau]$ where M is a Levi complement of J and τ is a supercuspidal representation of M, up to conjugation and twist by an unramified character. Any smooth irreducible representation π of J determines a unique inertial class $\ell(\pi)$. If $\mathfrak{s}$ is an inertial class, let $\mathrm{Rep}^{\mathfrak{s}}(J)$ be the category of smooth representations of J all of whose irreducible subquotients π satisfy $\ell(\pi) = \mathfrak{s}$. In particular, we allow non-admissible representations in $\mathrm{Rep}^{\mathfrak{s}}(J)$. For $\mathfrak{S}$ a set of inertial classes, let $\mathrm{Rep}^{\mathfrak{S}}(J)$ be the direct product of the categories $\mathrm{Rep}^{\mathfrak{s}}(J)$ for $\mathfrak{s} \in \mathfrak{S}$.
Let $(\mathbf{V}, \{\cdot,\cdot\})$ be the n-dimensional $\mathbb{Q}_{p^2}$-hermitian space whose group of unitary similitudes is J, and let

$$\mathbf{V} = mH \oplus \mathbf{V}^{\mathrm{an}}$$

be a Witt decomposition, where H denotes the hyperbolic plane and where $\mathbf{V}^{\mathrm{an}}$ is anisotropic. Note that $\mathbf{V}^{\mathrm{an}}$ has dimension 1 or 2 depending on whether n is odd or even respectively. For $0 \leqslant f \leqslant m$, we define

$$M_f := \mathrm{G}\left(\mathrm{GU}(fH \oplus \mathbf{V}^{\mathrm{an}}) \times \mathrm{GU}_1(\mathbb{Q}_p)^{m-f}\right).$$

Then M_f can be seen as a Levi complement in J. In particular M_0 is a minimal Levi complement. Let τ_0 denote the trivial representation of M_0, and let τ_1 denote the representation of M_1 obtained by letting the GU_1-components act trivially, and the $\mathrm{GU}(H \oplus \mathbf{V}^{\mathrm{an}})$-component act through the compact induction of the inflation to a special maximal parahoric subgroup of the unique cuspidal unipotent representation of $\mathrm{GU}_3(\mathbb{F}_p)$. For $f = 0, 1$, the irreducible representation τ_f of M_f is supercuspidal.

Theorem (4.2.12)**.** *Let $\mathfrak{S} := \{[M_0, \tau_0], [M_1, \tau_1]\}$. As smooth representations of J, the cohomology groups $\mathrm{H}^k_c(\mathcal{M}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ are objects of $\mathrm{Rep}^{\mathfrak{S}}(J)$.*

Thus, the irreducible subquotients in the cohomology of $\mathcal{M}^{\mathrm{an}}$ may only contribute to two different inertial classes. In particular when $n \geqslant 5$, no such subquotient is supercuspidal. We note also that irreducible objects in $\mathrm{Rep}^{[M_0,\tau_0]}(J)$ belong to the principal series.
In general, the terms $E_2^{a,b}$ in the second page may be difficult to compute. However, the terms corresponding to $a = 0$ and $b \in \{2(n-1-m), 2(n-1-m)+1\}$ are not touched by any non-zero differential in the sequence, making their computations accessible. We note that $2(n-1-m)$ is equal to the middle degree when n is odd, and to one plus the middle degree when n is even. For λ a partition of $2m+1$, we denote by ρ_λ the associated irreducible unipotent representation of $\mathrm{GU}_{2m+1}(\mathbb{F}_p)$ via the classification of [LS77] which we recall in 2.6. We also write ρ_λ for its inflation to the maximal parahoric subgroup J_m.

Proposition (4.1.13)**.** *We have an isomorphism of J-representations*

$$E_2^{0,2(n-1-m)} \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J}\, \rho_{(2m+1)}.$$

If $n \geqslant 3$ then we also have an isomorphism

$$E_2^{0,2(n-1-m)+1} \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J}\, \rho_{(2m,1)}.$$

We note that the representation $\rho_{(2m+1)}$ is trivial. Let V be a smooth representation of J and let χ be a continuous character of the center $\mathrm{Z}(J)$. We denote by V_χ the maximal quotient of V on which the center acts like χ. Combining our previous results, we obtain the following proposition.

Proposition (4.2.13 and 4.2.14)**.** *Let χ be any unramified character of $\mathrm{Z}(J)$.*

- *Assume that $n \geqslant 3$. The representation $(E_2^{0,2(n-1-m)})_\chi$ contains no non-zero admissible subrepresentation, and it is not J-semisimple. If $n \geqslant 5$, then the same statement holds for $(E_2^{0,2(n-1-m)+1})_\chi$.*

– *For* $n = 1, 2, 3, 4$, *let* $b = 0, 2, 3, 5$ *respectively. Let* $f = 0$ *when* $n = 1, 2$ *and let* $f = 1$ *when* $n = 3, 4$. *Then* $(E_2^{0,b})_\chi \simeq \chi \otimes \tau_f$ *is an irreducible supercuspidal representation of* J.

By the first point in particular, we see that the representations $\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})_\chi$ need not be admissible nor J-semisimple in general. This seems to differ from the case of the Lubin-Tate tower.

Lastly, we introduce the unramified unitary PEL Shimura variety of signature $(1, n-1)$ with no structure level at p. It is defined over a quadratic extension E of $\mathbb{Q}$ in which the prime p is inert. The corresponding Shimura datum gives rise to a reductive group G over $\mathbb{Q}$, whose group of $\mathbb{Q}_p$-rational points is isomorphic to the group we denoted $G(\mathbb{Q}_p)$, and such that $G(\mathbb{R}) \simeq \mathrm{GU}(1, n-1)$. The Shimura varieties are indexed by the open compact subgroups $K^p \subset G(\mathbb{A}_f^p)$ which are small enough. Kottwitz constructed integral models at p of these Shimura varieties. Their special fibers are stratified by the Newton strata, and the unique closed stratum is called the basic stratum. We denote it $\overline{\mathrm{S}}_{K^p}(b_0)$. The p-adic uniformization theorem of [RZ96] is a geometric identity between the Rapoport-Zink space $\mathcal{M}$ and the basic stratum $\overline{\mathrm{S}}_{K^p}(b_0)$. In [Far04], Fargues constructed a Hochschild-Serre spectral sequence associated to this geometric identity, computing the cohomology of the basic stratum.

Let ξ be an irreducible algebraic finite dimensional representation of G, and let $\overline{\mathcal{L}_\xi}$ be the associated local system on the Shimura variety, restricted to the special fiber. It is a pure sheaf of some weight $w(\xi) \in \mathbb{Z}$. Let I be the inner form of G such that $I(\mathbb{A}_f) = J \times G(\mathbb{A}_f^p)$ and $I(\mathbb{R}) \simeq \mathrm{GU}(0, n)$. We denote by $\mathcal{A}_\xi(I)$ the set of automorphic representations of I of type $\check{\xi}$ at infinity, and counted with multiplicities. Fargues' spectral sequence is given in the second page by

$$F_2^{a,b} = \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^a_{J\text{-sm}}\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \implies \mathrm{H}_c^{a+b}(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}),$$

where $\overline{\mathrm{S}}(b_0) := \varprojlim_{K^p} \overline{\mathrm{S}}_{K^p}(b_0)$ and $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p$. It is $G(\mathbb{A}_f^p) \times W$-equivariant. When $n = 3$ or 4 this sequence degenerates on the second page, and our knowledge on the cohomology of the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ allows us to compute every term. We obtain a description of the cohomology of the basic stratum in terms of automorphic representations.

A smooth character of J is said to be unramified if it is trivial on all compact subgroups of J. Let $X^{\mathrm{un}}(J)$ denote the set of unramified characters of J. The subgroup of J generated by all its open compact subgroups is denoted by J°. It corresponds to all the elements of J whose factor of similitude is a unit. If $\Pi \in \mathcal{A}_\xi(I)$, we define $\delta_{\Pi_p} := \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id}) p^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times$ where ω_{Π_p} is the central character of Π_p, and $p^{-1} \cdot \mathrm{id}$ lies in the center of J. For any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ we have $|\iota(\delta_{\Pi_p})| = 1$. Eventually, if $x \in \overline{\mathbb{Q}_\ell}^\times$, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of the Weil group W where the inertia acts trivially and Frob acts like multiplication by the scalar x.

Theorem (5.2.6)**.** *There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

Moreover, there exists a $G(\mathbb{A}_f^p) \times W$-subspace $V \subset \mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ such that

$$V \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} d(\Pi_p) \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}],$$

and with quotient space isomorphic to

$$\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})/V \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0 \\ \dim(\Pi_p) > 1}} (\nu - 1 - d(\Pi_p)) \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} (\nu - d(\Pi_p)) \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

where $\nu = p$ if $n = 3$ and $\nu = p^3$ if $n = 4$, and $d(\Pi_p) := \dim \mathrm{Ext}^1_{J\text{-}sm}(\mathrm{c} - \mathrm{Ind}_{J^\circ}^J \mathbf{1}, \Pi_p)$.

For all $\Pi \in \mathcal{A}_\xi(I)$ the integer $d(\Pi_p) \geqslant 0$ is finite. If it is non-zero then $\Pi_p^{J_1} \neq 0$. In particular, if Π_p is supercuspidal or if the central character of Π_p is not unramified, then $d(\Pi_p) = 0$. Assume now that the Shimura variety is of Kottwitz-Harris-Taylor type, implying among other things that the reflex field E splits over a prime number p' different from p and ℓ. The cohomology of the whole Shimura variety has been computed in [Boy10]. In particular, it does not contain any multiplicity dependent on p such as ν, implying that such multiplicities should occur in other Newton strata as well. We may verify this directly in the case $n = 3$, where there is only one other Newton stratum which is the μ-ordinary locus of the Shimura variety. We denote it $\overline{\mathrm{S}}_{K^p}(b_1)$ and we also write $\overline{\mathrm{S}}(b_1) := \varprojlim_{K^p} \overline{\mathrm{S}}_{K^p}(b_1)$.

Proposition. *There is a $G(\mathbb{A}_f^p) \times W$-equivariant isomorphism*

$$\mathrm{H}_c^4(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+4}].$$

There is a $G(\mathbb{A}_f^p) \times W$-equivariant monomorphism

$$\mathrm{H}_c^3(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \hookrightarrow \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

There is a $G(\mathbb{A}_f^p) \times W$-equivariant monomorphism

$$\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \hookrightarrow \mathrm{H}_c^2(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

By the work of Mantovan in [Man05], the alternate sum of the cohomology of the Newton stratum $\overline{\mathrm{S}}(b_1)$ is closely related to the cohomology of an associated Igusa variety. In general, relatively little is known on the cohomology of Igusa varieties, but recent progress has been made in [KS21] and [BMN21] for instance. We hope that our results may offer some insight for research in this direction, as they suggest that multiplicities dependent on p may occur in the cohomology of Igusa varieties.

Throughout the paper, we fix an integer $n \geqslant 1$ and we write $m := \lfloor \frac{n-1}{2} \rfloor$ so that $n = 2m+1$ or $2(m+1)$ according to whether n is odd or even. We also fix an odd prime number p. If k is a perfect field of characteristic p, we denote by $W(k)$ the ring of Witt vectors and by $W(k)_{\mathbb{Q}}$ its fraction field, which is an unramified extension of $\mathbb{Q}_p$. We denote by $\sigma_k : x \mapsto x^p$ the Frobenius of $\mathrm{Gal}(k/\mathbb{F}_p)$, and we use the same notation for its (unique) lift to $\mathrm{Gal}(W(k)_{\mathbb{Q}}/\mathbb{Q}_p)$. If k'/k is a perfect field extension then $(\sigma_{k'})_{|k} = \sigma_k$, so we can remove the subscript and write σ unambiguously instead. If $q = p^e$ is a power of p, we write $\mathbb{F}_q$ for the field with q elements. In the special case where $q = p^2$, we also use the alternative notation $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$ and $\mathbb{Q}_{p^2} = W(\mathbb{F}_{p^2})_{\mathbb{Q}}$. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.

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1 The Bruhat-Tits stratification on the PEL unitary Rapoport-Zink space of signature $(1, n-1)$

1.1 The PEL unitary Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$

1.1.1 In [VW11], the authors introduce the PEL unitary Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$ as a moduli space, classifying the deformations of a given p-divisible group equipped with additional structures. We briefly recall the construction. Let Nilp denote the category of schemes over $\mathbb{Z}_{p^2}$ where p is locally nilpotent. For $S \in$ Nilp, a **unitary p-divisible group of signature** $(1, n-1)$ over S is a triple (X, ι_X, λ_X) where

- X is a p-divisible group over S.
- $\iota_X : \mathbb{Z}_{p^2} \to \mathrm{End}(X)$ is a $\mathbb{Z}_{p^2}$-action on X such that the induced action on its Lie algebra satisfies the **signature** $(1, n-1)$ **condition**: for every $a \in \mathbb{Z}_{p^2}$, the characteristic polynomial of $\iota_X(a)$ acting on $\mathrm{Lie}(X)$ is given by

$$(T-a)^1(T-\sigma(a))^{n-1} \in \mathbb{Z}_{p^2}[T] \subset \mathcal{O}_S[T].$$

- $\lambda_X : X \xrightarrow{\sim} {}^tX$ is a $\mathbb{Z}_{p^2}$-linear polarization where ${}^tX$ denotes the Serre dual of X.

The $\mathbb{Z}_{p^2}$-linearity of λ_X is with respect to the $\mathbb{Z}_{p^2}$-actions ι_X and the induced action $\iota_{{}^tX}$ on the dual. A specific example of unitary p-divisible group over $\mathbb{F}_{p^2}$ is given in [VW11] 2.4 by means of covariant Dieudonné theory. We denote it by $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and call it the **standard unitary p-divisible group**. The p-divisible group $\mathbb{X}$ is superspecial. The following set-valued functor $\mathcal{M}$ defines a moduli problem classifying deformations of $\mathbb{X}$ by quasi-isogenies. More precisely, for $S \in \mathrm{Nilp}$ the set $\mathcal{M}(S)$ consists of all isomorphism classes of tuples $(X, \iota_X, \lambda_X, \rho_X)$ such that

– (X, λ_X, ρ_X) is a unitary p-divisible group of signature $(1, n-1)$ over S.

– $\rho_X : X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}_{p^2}} \overline{S}$ is a $\mathbb{Z}_{p^2}$-linear quasi-isogeny compatible with the polarizations, in the sense that ${}^t\rho_X \circ \lambda_{\mathbb{X}} \circ \rho_X$ is a $\mathbb{Q}_p^\times$-multiple of λ_X.

In the second condition, $\overline{S}$ denotes the special fiber of S. By [RZ96] Corollary 3.40, this moduli problem is represented by a separated formal scheme $\mathcal{M}$ over $\mathrm{Spf}(\mathbb{Z}_{p^2})$, called a **Rapoport-Zink space**. It is formally locally of finite type, and because the associated PEL datum is unramified it is also formally smooth over $\mathbb{Z}_{p^2}$. The **reduced special fiber** of $\mathcal{M}$ is the reduced $\mathbb{F}_{p^2}$-scheme $\mathcal{M}_{\mathrm{red}}$ defined by the maximal ideal of definition. By loc. cit. Proposition 2.32, each irreducible component of $\mathcal{M}_{\mathrm{red}}$ is projective. The geometry of the special fiber has been thoroughly described in [Vol10] and [VW11], and we recall some of their constructions.

1.1.2 Rational points of $\mathcal{M}$ over a perfect field extension k of $\mathbb{F}_{p^2}$ can be understood in terms of semi-linear algebra by means of Dieudonné theory. We denote by $M(\mathbb{X})$ the Dieudonné module of $\mathbb{X}$, this is a free $\mathbb{Z}_{p^2}$-module of rank $2n$. We denote by $N(\mathbb{X}) := M(\mathbb{X}) \otimes \mathbb{Q}_{p^2}$ its isocrystal. By construction, the Frobenius and the Verschiebung agree on $N(\mathbb{X})$. In particular, we have $\mathbf{F}^2 = p \cdot \mathrm{id}$ on the isocrystal. The $\mathbb{Z}_{p^2}$-action $\iota_{\mathbb{X}}$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading $M(\mathbb{X}) = M(\mathbb{X})_0 \oplus M(\mathbb{X})_1$ as a sum of two free $\mathbb{Z}_{p^2}$-modules of rank n. The same goes for the isocrystal $N(\mathbb{X}) = N(\mathbb{X})_0 \oplus N(\mathbb{X})_1$ where $N(\mathbb{X})_i = M(\mathbb{X})_i \otimes \mathbb{Q}_{p^2}$ for $i = 0, 1$. The polarization $\lambda_{\mathbb{X}}$ induces a perfect σ-symplectic form on $N(\mathbb{X})$ which stabilizes the lattice $M(\mathbb{X})$ and for which $\mathbf{F}$ is self-adjoint. Compatibility with $\iota_{\mathbb{X}}$ implies that the pieces $N(\mathbb{X})_i$ are totally isotropic for $i = 0, 1$ and dual of each other. Moreover, the Frobenius $\mathbf{F}$ is then 1-homogeneous with respect to this grading. As in [VW11] 2.6, it is possible to modify the symplectic pairing so that it restricts to a non-degenerate $\mathbb{Q}_{p^2}$-valued σ-hermitian form $\{\cdot, \cdot\}$ on $N(\mathbb{X})_0$.

Notation. From now on, we will write $\mathbf{V} := N(\mathbb{X})_0$ and $\mathbf{M} := M(\mathbb{X})_0$.

Then $\mathbf{V}$ is a $\mathbb{Q}_{p^2}$-hermitian space of dimension n, and $\mathbf{M}$ is a given $\mathbb{Z}_{p^2}$-lattice, ie. a $\mathbb{Z}_{p^2}$-submodule containing a basis of $\mathbf{V}$. Given two lattices M_1 and M_2, the notation $M_1 \overset{d}{\subset} M_2$ means that $M_1 \subset M_2$ and the quotient module M_2/M_1 has length d. The integer d is called the **index** of M_1 in M_2, and is denoted $d = [M_2 : M_1]$. We have $0 \leqslant d \leqslant n$. Given a lattice $M \subset \mathbf{V}$, the dual lattice is denoted $M^\vee$. It consists of all the vectors $v \in \mathbf{V}$ such that $\{v, M\} \subset \mathbb{Z}_{p^2}$. Then, by construction the lattice $\mathbf{M}$ satisfies

$$p\mathbf{M}^\vee \overset{1}{\subset} \mathbf{M} \overset{n-1}{\subset} \mathbf{M}^\vee.$$

The existence of such a lattice $\mathbf{M}$ in $\mathbf{V}$ implies that the σ-hermitian structure on $\mathbf{V}$ is isomorphic to any one described by the following two matrices

$$T_{\text{odd}} := A_{2m+1}, \qquad T_{\text{even}} := \begin{pmatrix} & & & A_m \\ & 1 & 0 & \\ & 0 & p & \\ A_m & & & \end{pmatrix}.$$

Here, A_k denotes the $k \times k$ matrix with 1's in the antidiagonal and 0 everywhere else.

Proposition ([Vol10] 1.15)**.** *There exists a basis of* $\mathbf{V}$ *such that* $\{\cdot,\cdot\}$ *is represented by the matrix* T_{odd} *is* n *is odd and by* T_{even} *if* n *is even.*

1.1.3 A **Witt decomposition** on $\mathbf{V}$ is a set $\{L_i\}_{i\in I}$ of isotropic lines in $\mathbf{V}$ such that the following conditions are satisfied:

- For every $i \in I$, there is a unique $i' \in I$ such that $\{L_i, L_{i'}\} \neq 0$.
- The sum of the L_i's is direct.
- The orthogonal in $\mathbf{V}$ of the direct sum of the L_i's is an anisotropic subspace of $\mathbf{V}$.

Because each line L_i is isotropic, in the first condition one necessarily has $(i')' = i$ and $i \neq i'$. As a consequence, the cardinality of the index set I is an even number $\#I = 2w(\mathbf{V})$. The integer $w = w(\mathbf{V})$ is called the **Witt index** of $\mathbf{V}$ and it does not depend on the choice of a Witt decomposition. We write L^{an} for the orthogonal of the direct sum of the L_i's. The dimension of L^{an} is $n^{\text{an}} := n - 2w$, therefore it is also independent on the choice of the Witt decomposition.

Given any Witt decomposition, one may always find vectors $e_i \in L_i$ such that $\{e_i, e_j\} = \delta_{j,i'}$. Together with a choice of an orthogonal basis for L^{an}, these vectors define a basis of $\mathbf{V}$ which is said to be **adapted to the Witt decomposition**. For any $i \in I$, the direct sum $L_i \oplus L_{i'}$ is isometric to the hyperbolic plane $\mathbf{H}$. Therefore, we obtain a decomposition

$$\mathbf{V} = w\mathbf{H} \oplus L^{\text{an}}.$$

We may always rearrange the index set so that $I = \{-w, \ldots, -1, 1, \ldots, w\}$ and for every $i \in I$, we have $\{L_i, L_{-i}\} \neq 0$. Thus, the i' associated to i by the first condition is $-i$. Of course, this process is not unique as it relies on a choice of an ordering for the lines $\{L_i\}_{i\in I}$. In this context, we write L_0 instead of L^{an}.

1.1.4 We fix once and for all a basis e of $\mathbf{V}$ in which the hermitian form is represented by the matrix T_{odd} or T_{even}. In the case $n = 2m+1$ is odd, we will denote it

$$e = (e_{-m}, \ldots, e_{-1}, e_0^{\text{an}}, e_1, \ldots, e_m),$$

and in the case $n = 2(m+1)$ is even we will denote it

$$e = (e_{-m}, \ldots, e_{-1}, e_0^{\text{an}}, e_1^{\text{an}}, e_1, \ldots, e_m).$$

In this way, for every $1 \leqslant s \leqslant m$ the subspace generated by e_{-s} and e_s is isomorphic to the hyperbolic plane $\mathbf{H}$. Moreover, the vectors with a superscript $\cdot^{\mathrm{an}}$ generate an anisotropic subspace $\mathbf{V}^{\mathrm{an}}$ of $\mathbf{V}$. The choice of such a basis gives a Witt decomposition

$$\mathbf{V} = m\mathbf{H} \oplus \mathbf{V}^{\mathrm{an}}$$

consisting of an orthogonal sum of m copies of $\mathbf{H}$ and of the anisotropic subspace $\mathbf{V}^{\mathrm{an}}$. In particular, the Witt index of $\mathbf{V}$ is m and we have $n^{\mathrm{an}} = 1$ or 2 depending on whether n is odd or even respectively.

1.1.5 Given a perfect field extension k of $\mathbb{F}_{p^2}$, we denote by $\mathbf{V}_k$ the base change $\mathbf{V} \otimes_{\mathbb{Q}_{p^2}} W(k)_{\mathbb{Q}}$. The form may be extended to $\mathbf{V}_k$ by the formula

$$\{v \otimes x, w \otimes y\} := xy^{\sigma}\{v, w\} \in W(k)_{\mathbb{Q}}$$

for all $v, w \in \mathbf{V}$ and $x, y \in W(k)_{\mathbb{Q}}$. The notions of index and duality for $W(k)$-lattices can be extended as well. We have the following description of the rational points of the Rapoport-Zink space.

Proposition ([Vol10] 1.10)**.** *Let k be a perfect field extension of $\mathbb{F}_{p^2}$. There is a natural bijection between $\mathcal{M}(k) = \mathcal{M}_{\mathrm{red}}(k)$ and the set of lattices M in $\mathbf{V}_k$ such that for some integer $i \in \mathbb{Z}$, we have*

$$p^{i+1}M^{\vee} \overset{1}{\subset} M \overset{n-1}{\subset} p^{i}M^{\vee}.$$

1.1.6 There is a decomposition $\mathcal{M} = \bigsqcup_{i \in \mathbb{Z}} \mathcal{M}_i$ into formal connected subschemes which are open and closed. The rational points of $\mathcal{M}_i$ are those lattices M satisfying the relation above with the given integer i. Similarly, we have a decomposition into open and closed connected subschemes $\mathcal{M}_{\mathrm{red}} = \bigsqcup_{i \in \mathbb{Z}} \mathcal{M}_{i,\mathrm{red}}$. In particular, the lattice $\mathbf{M}$ defined in the previous paragraph is an element of $\mathcal{M}_0(\mathbb{F}_{p^2})$. Not all integers i can occur though, as a parity condition must be satisfied by the following lemma.

Lemma ([Vol10] 1.7)**.** *The formal scheme $\mathcal{M}_i$ is empty if ni is odd.*

1.1.7 Let $J = \mathrm{GU}(\mathbf{V})$ be the group of unitary similitudes attached to $\mathbf{V}$. It consists of all linear transformations g which preserve the hermitian form up to a unit $c(g) \in \mathbb{Q}_p^{\times}$, called the **multiplier**. One may think of J as the group of $\mathbb{Q}_p$-rational point of a reductive algebraic group. The space $\mathcal{M}$ is endowed with a natural action of J. At the level of points, the element g acts by sending a lattice M to $g(M)$.
By [Vol10] 1.16, the action of $g \in J$ induces, for every integer i, an isomorphism $\mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_{i+\alpha(g)}$ where $\alpha(g)$ is the p-adic valuation of the multiplier $c(g)$. This defines a continous homomorphism

$$\alpha : J \to \mathbb{Z}$$

where $\mathbb{Z}$ is given the discrete topology. According to 1.17 in loc. cit. the image of α is $\mathbb{Z}$ if n is even, and it is $2\mathbb{Z}$ if n is odd. The center $\mathrm{Z}(J)$ of J consists of all the multiple of the identity. Therefore it can be identified with $\mathbb{Q}_{p^2}^\times$. If $\lambda \in \mathbb{Q}_{p^2}^\times$, then $c(\lambda \cdot \mathrm{id}) = \lambda\sigma(\lambda) = \mathrm{Norm}(\lambda) \in \mathbb{Q}_p^\times$, where Norm is the norm map relative to the quadratic extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$. In particular, $\alpha(\mathrm{Z}(J)) = 2\mathbb{Z}$. Thus, the restriction of α to the center of J is surjective onto the image of α only when n is odd. When n is even, we define the following element

$$g_0 := \begin{pmatrix} & & & I_m \\ & 0 & p & \\ & 1 & 0 & \\ pI_m & & & \end{pmatrix}$$

where I_m denotes the $m \times m$ identity matrix. Then $g_0 \in J$ and $c(g_0) = p$ so that $\alpha(g_0) = 1$. Moreover $g_0^2 = p \cdot \mathrm{id}$ belongs to $\mathrm{Z}(J)$.
Let i and i' be two integers such that ni and ni' are even. Following [Vol10] Proposition 1.18, we define a morphism $\psi_{i,i'} : \mathcal{M}_i \to \mathcal{M}_{i'}$ by sending, for any perfect field extension $k/\mathbb{F}_{p^2}$, a point $M \in \mathcal{M}_i$ to

$$\psi_{i,i'}(M) = \begin{cases} p^{\frac{i'-i}{2}} \cdot M & \text{if } i \equiv i' \mod 2. \\ p^{\frac{i'-i-1}{2}} g_0 \cdot M & \text{if } i \not\equiv i' \mod 2. \end{cases}$$

This is well defined as the second case may only happen when n is even. We obtain the following proposition.

Proposition ([Vol10] 1.18)**.** *The map $\psi_{i,i'}$ is an isomorphism between $\mathcal{M}_i$ and $\mathcal{M}_{i'}$. Moreover they are compatible with each other in the sense that if i, i' and i'' are three integers such that ni, ni' and ni'' are even, then we have $\psi_{i',i''} \circ \psi_{i,i'} = \psi_{i,i''}$.*

The same statement also holds for the special fiber $\mathcal{M}_{\mathrm{red}}$. In particular, we have $\mathcal{M}_i \neq \emptyset$ if and only if ni is even.

1.2 The Bruhat-Tits stratification of the special fiber $\mathcal{M}_{\mathrm{red}}$

1.2.1 We now recall the construction of the Bruhat-Tits stratification on $\mathcal{M}_{\mathrm{red}}$ as in [VW11]. Let i be an integer such that ni is even. We define

$$\mathcal{L}_i := \{\Lambda \subset \mathbf{V} \text{ a lattice} \,|\, p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i\Lambda^\vee\}.$$

If $\Lambda \in \mathcal{L}_i$, we define its **orbit type** $t(\Lambda) := [\Lambda : p^{i+1}\Lambda^\vee]$. We also call it the type of Λ. In particular, the lattices in $\mathcal{L}_i$ of type 1 are precisely the $\mathbb{F}_{p^2}$-rational points of $\mathcal{M}_{i,\mathrm{red}}$. By sending Λ to $g(\Lambda)$, an element $g \in J$ defines a map $\mathcal{L}_i \to \mathcal{L}_{i+\alpha(g)}$.

Proposition ([Vol10] Remark 2.3 and [VW11] Remark 4.1)**.** *Let i be an integer such that ni is even and let $\Lambda \in \mathcal{L}_i$.*

- *The map $\mathcal{L}_i \to \mathcal{L}_{i+\alpha(g)}$ induced by an element $g \in J$ is an inclusion preserving, type preserving bijection.*
- *We have $1 \leqslant t(\Lambda) \leqslant n$. Furthermore $t(\Lambda)$ is odd.*

– *The sets $\mathcal{L}_i$'s for various i's are pairwise disjoint.*

Moreover, two lattices $\Lambda, \Lambda' \in \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{L}_i$ are in the same orbit under the action of J if and only if $t(\Lambda) = t(\Lambda')$.

Proof. The first three points are proved in [Vol10]. Thus, we only explain the last statement. If Λ and Λ' are in the same J-orbit, because the action of J preserves the type we have $t(\Lambda) = t(\Lambda')$.
For the converse, assume that Λ and Λ' have the same type. Let i and i' be the integers such that $\Lambda \in \mathcal{L}_i$ and $\Lambda' \in \mathcal{L}_{i'}$. According to 1.1.7, we can always find $g \in J$ such that $\alpha(g) = i - i'$. Hence, replacing Λ' by $g \cdot \Lambda'$ we may assume that $i = i'$. Then the statement follows from [VW11] Remark 4.1. □

We write $\mathcal{L} := \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{L}_i$. For any integer i such that ni is even and any odd number t between 1 and n, there exists a lattice $\Lambda \in \mathcal{L}_i$ of orbit type t. Indeed, by fixing a bijection $\mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_0$ it is enough to find such a lattice for $i = 0$. Then, examples of lattices in $\mathcal{L}_0$ of any type are given in 1.2.6 below.

1.2.2 Write $t_{\max} := 2m + 1$, so that the orbit type t of any lattice in $\mathcal{L}$ satisfies $1 \leqslant t \leqslant t_{\max}$. The following lemma will be useful later.

Lemma. *Let $i \in \mathbb{Z}$ such that ni is even, and let $\Lambda \in \mathcal{L}$. We have $\Lambda^\vee \in \mathcal{L}$ if and only if either n is even, either n is odd and $t(\Lambda) = t_{\max}$.*
If this condition is satisfied and n is even, then $\Lambda^\vee \in \mathcal{L}_{-i-1}$ and $t(\Lambda^\vee) = n - t(\Lambda)$. If on the contrary n is odd, then $\Lambda^\vee \in \mathcal{L}_{-i}$ and $t(\Lambda^\vee) = t(\Lambda)$.

Proof. First we prove the converse. We have the following chain of inclusions

$$p^{-i}\Lambda \overset{n-t(\Lambda)}{\subset} \Lambda^\vee \overset{t(\Lambda)}{\subset} p^{-i-1}\Lambda.$$

If n is even, then $-n(i+1)$ is also even and $n - t(\Lambda) \neq 0$. Since $(\Lambda^\vee)^\vee = \Lambda$, we deduce that $\Lambda^\vee \in \mathcal{L}_{-i-1}$ with orbit type $n - t(\Lambda)$. Assume now that n is odd and that $t(\Lambda) = t_{\max} = n$. Then $\Lambda^\vee = p^{-i}\Lambda \in \mathcal{L}_{-i}$.
Let us now assume that $\Lambda^\vee \in \mathcal{L}$ and that n is odd. Let $i' \in 2\mathbb{Z}$ such that $\Lambda^\vee \in \mathcal{L}_{i'}$. We have

$$\Lambda^\vee \overset{n-t(\Lambda^\vee)}{\subset} p^{i'}\Lambda \overset{n-t(\Lambda)}{\subset} p^{i'+i}\Lambda^\vee, \qquad \Lambda^\vee \overset{t(\Lambda)}{\subset} p^{-i-1}\Lambda \overset{t(\Lambda^\vee)}{\subset} p^{-i-i'-2}\Lambda^\vee,$$

therefore $-2 \leqslant i+i' \leqslant 0$. Since $i+i'$ is even it is either -2 or 0. If it were -2, then we would have $t(\Lambda) = t(\Lambda^\vee) = 0$ which is absurd. Therefore $i+i' = 0$, and we have $n-t(\Lambda) = n-t(\Lambda^\vee) = 0$. □

1.2.3 With the help of $\mathcal{L}_i$, one may construct an abstract simplicial complex $\mathcal{B}_i$. For $s \geqslant 0$, an s-simplex of $\mathcal{B}_i$ is a subset $S \subset \mathcal{L}_i$ of cardinality $s+1$ such that for some ordering $\Lambda_0, \ldots, \Lambda_s$ of its elements, we have a chain of inclusions $p^{i+1}\Lambda_s^\vee \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \ldots \subsetneq \Lambda_s$. We must have $0 \leqslant s \leqslant m$ for such a simplex to exist.
We introduce $\tilde{J} = \mathrm{SU}(\mathbf{V})$, the derived group of J. We consider the abstract simplicial complex

$\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ of the Bruhat-Tits building of $\tilde{J}$ over $\mathbb{Q}_p$. A concrete description of this complex is given in [Vol10], while proving the following theorem.

Theorem ([Vol10] 3.5)**.** *The abstract simplicial complex* $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ *of the Bruhat-Tits building of* $\tilde{J}$ *is naturally identified with* $\mathcal{B}_i$ *for any fixed integer* i *such that* ni *is even. There is in particular an identification of* $\mathcal{L}_i$ *with the set of vertices of* $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$*. The identification is* $\tilde{J}$*-equivariant.*

Apartments in the Bruhat-Tits building $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ are in 1 to 1 correspondence with Witt decompositions of $\mathbf{V}$. Let $L = \{L_j\}_{i\in I}$ be a Witt decomposition of $\mathbf{V}$ and let $f = (f_i)_{i\in I} \sqcup B^{\mathrm{an}}$ be a basis of $\mathbf{V}$ adapted to the decomposition, where B^{an} is an orthogonal basis of L^{an}. Under the identification of $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ with $\mathcal{B}_i$, the vertices inside the apartment associated to L correspond to the lattices $\Lambda \in \mathcal{L}_i$ which are equal to the direct sum of $\Lambda \cap L^{\mathrm{an}}$ and of the modules $p^{r_i}\mathbb{Z}_{p^2} f_i$ for some integers $(r_i)_{i\in I}$. The subset of $\mathcal{L}_i$ consisting of all such lattices will be denoted $\mathcal{A}_i^L$ or, with an abuse of notations, $\mathcal{A}_i^f$. We call such a set $\mathcal{A}_i^L$ the **apartment associated to L in $\mathcal{L}_i$.**

Remark. The set of vertices of the Bruhat-Tits building of $J = \mathrm{GU}(\mathbf{V})$ may then be identified with the disjoint union $\mathcal{L}$ of the $\mathcal{L}_i$'s. The subsets of lattices in a common apartment correspond to the sets $\mathcal{A}^L := \bigsqcup_{ni\in 2\mathbb{Z}} \mathcal{A}_i^L$ where L is some Witt decomposition of $\mathbf{V}$. The set $\mathcal{A}^L$ will be called the apartment associated to L.

We recall a general result regarding Bruhat-Tits buildings.

Proposition. *Let* i *be an integer such that* ni *is even. Any two lattices* Λ *and* Λ' *in* $\mathcal{L}_i$ *(resp.* $\mathcal{L}$*) lie inside a common apartment* $\mathcal{A}_i^L$ *(resp.* $\mathcal{A}^L$*) for some Witt decomposition* L*.*
Moreover, the action of the group $\tilde{J}$ *sends apartments to apartments. It acts transitively on the set* $\{\mathcal{A}_i^L\}_L$*. The same is true for* J *acting on the set* $\{\mathcal{A}^L\}_L$*.*

1.2.4 Recall the basis e of $\mathbf{V}$ that we fixed in **1.4**. We will denote by

$$\Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$$

the $\mathbb{Z}_{p^2}$-lattice generated by the vectors $p^{r_j}e_j$ for all $j = \pm 1, \ldots, \pm m$, by $p^{s_0}e_0^{\mathrm{an}}$ and if n is even, by $p^{s_1}e_1^{\mathrm{an}}$ too. Here, the r_j's are integers and s denotes either the integer s_0 if n is odd or the pair of integers (s_0, s_1) if n is even.

Proposition. *Let* i *be an integer such that* ni *is even. Let* (r_j, s) *be a family of integers as above. The corresponding lattice* $\Lambda = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$ *belongs to* $\mathcal{L}_i$ *if and only if the following conditions are satisfied*

- *for all* $1 \leqslant j \leqslant m$*, we have* $r_{-j} + r_j \in \{i, i+1\}$*,*
- $s_0 = \lfloor \frac{i+1}{2} \rfloor$*,*
- *if* n *is even, then* $s_1 = \lfloor \frac{i}{2} \rfloor$*.*

Moreover, when that is the case the type of Λ *is given by*

$$t(\Lambda) = 1 + 2\#\{1 \leqslant j \leqslant m \,|\, r_{-j} + r_j = i\}.$$

Proof. The lattice Λ belongs to $\mathcal{L}_i$ if and only if the following chain of inclusions holds:

$$p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i\Lambda^\vee.$$

The dual lattice $\Lambda^\vee$ is equal to the lattice $\Lambda(-r_m, \ldots, -r_1, s', -r_{-1}, \ldots, -r_{-m})$, where $s' = -s_0$ when n is odd, and $s' = (-s_0, -s_1 - 1)$ when n is even. Thus, the inclusions above are equivalent to the following inequalities:

$$i - r_{-j} \leqslant r_j \leqslant i + 1 - r_{-j}, \qquad i - s_0 \leqslant s_0 \leqslant i + 1 - s_0,$$
$$i - 1 - s_1 \leqslant s_1 \leqslant i - s_1 \text{ (if } n \text{ is even)}.$$

This proves the desired condition on the integers r_j's and on s.
Let us now assume that $\Lambda \in \mathcal{L}_i$. Its orbit type is equal to the index $[\Lambda, p^{i+1}\Lambda^\vee]$. This corresponds to the number of times equality occurs with the left-hand side in all the inequalities above. Of course, if the equality $i - r_{-j} = r_j$ occurs for some j, then it occurs also for $-j$. Moreover, if i is even then the equality $i - s_0 = s_0$ occurs whereas $i - 1 - s_1 \neq s_1$. On the contrary if i is odd, then the equality $i - 1 - s_1 = s_1$ occurs whereas $i - s_0 \neq s_0$. Thus in all cases, only one of s_0 and s_1 contributes to the index. Putting things together, we deduce the desired formula. □

1.2.5 We deduce the following corollary.

Corollary. *The apartment A_i^e (resp. A^e) consists of all the lattices of the form*

$$\Lambda = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$$

which belong to $\mathcal{L}_i$ (resp. to $\mathcal{L}$).

Proof. According to the previous proposition, it is clear that all lattices which belong to $\mathcal{L}_i$ and are of the form $\Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$ are elements of $\mathcal{A}_i^e$. We shall prove the converse. Let $\Lambda \in \mathcal{A}_i^e$. By definition, there exists integers (r_j) such that

$$\Lambda = \Lambda \cap \mathbf{V}^{\mathrm{an}} \oplus \bigoplus_{1 \leqslant j \leqslant m} \left(p^{r_{-j}}\mathbb{Z}_{p^2}e_{-j} \oplus p^{r_j}\mathbb{Z}_{p^2}e_j\right).$$

Write $\Lambda' = \Lambda \cap \mathbf{V}^{\mathrm{an}}$. This is a lattice in $\mathbf{V}^{\mathrm{an}}$ which satisfies the chain of inclusions

$$p^{i+1}\Lambda'^{\,\vee} \subset \Lambda' \subset p^i\Lambda'^{\,\vee},$$

where the duals are taken with respect to the restriction of $\{\cdot, \cdot\}$ to $\mathbf{V}^{\mathrm{an}}$. Since $\mathbf{V}^{\mathrm{an}}$ is anisotropic, there is only a single lattice satisfying the chain of inclusions above. If we write $a := \lfloor \frac{i+1}{2} \rfloor$ and $b := \lfloor \frac{i}{2} \rfloor$, it is given by $p^a\mathbb{Z}_{p^2}e_0^{\mathrm{an}}$ if n is odd, and by $p^a\mathbb{Z}_{p^2}e_0^{\mathrm{an}} \oplus p^b\mathbb{Z}_{p^2}e_1^{\mathrm{an}}$ if n is even. Thus, it must be equal to Λ' and it concludes the proof. □

1.2.6 We fix a maximal simplex in $\mathcal{L}_0$ lying inside the apartment $\mathcal{A}_0^e$. For $0 \leqslant \theta \leqslant m$ we define

$$\Lambda_\theta := \Lambda(\underbrace{0, \ldots, 0}_{m}, 0, \underbrace{0, \ldots, 0}_{\theta}, \underbrace{1, \ldots, 1}_{m-\theta}).$$

Here, the 0 in the middle stands for $(0,0)$ in case n is even. The lattice Λ_θ belongs to $\mathcal{L}_0$, its orbit type is $2\theta+1$ and together they fit inside the following chain of inclusions

$$p\Lambda_0^\vee \subsetneq \Lambda_0 \subset \ldots \subset \Lambda_m.$$

Thus, they form an m-simplex in $\mathcal{L}_0$.

1.2.7 Given a lattice $\Lambda \in \mathcal{L}_i$, the authors of [VW11] define a subfunctor $\mathcal{M}_\Lambda$ of $\mathcal{M}_{i,\mathrm{red}}$ classifying those p-divisible groups for which a certain quasi-isogeny, depending on Λ, is in fact an actual isogeny. In Lemma 4.2, they prove that it is representable by a projective scheme over $\mathbb{F}_{p^2}$, and that the natural morphism $\mathcal{M}_\Lambda \hookrightarrow \mathcal{M}_{i,\mathrm{red}}$ is a closed immersion. The schemes $\mathcal{M}_\Lambda$ are called the **closed Bruhat-Tits strata of $\mathcal{M}$**. Their rational points are described as follows.

Proposition ([VW11] Lemma 4.3)**.** *Let k be a perfect field extension of $\mathbb{F}_{p^2}$, and let $M \in \mathcal{M}_{i,\mathrm{red}}(k)$. Then we have the equivalence*

$$M \in \mathcal{M}_\Lambda(k) \iff M \subset \Lambda_k := \Lambda \otimes_{\mathbb{Z}_{p^2}} W(k).$$

The set of lattices satisfying the condition above was conjectured in [Vol10] to be the set of points of a subscheme of $\mathcal{M}_{i,\mathrm{red}}$, and it was proved in the special cases $n = 2,3$. In [VW11], the general argument is given by the construction of $\mathcal{M}_\Lambda$. The action of an element $g \in J$ on $\mathcal{M}_{\mathrm{red}}$ induces an isomorphism $\mathcal{M}_\Lambda \xrightarrow{\sim} \mathcal{M}_{g\cdot\Lambda}$.

1.2.8 Let $\Lambda \in \mathcal{L}$, we denote by J_Λ the fixator of Λ under the action of J. If $\Lambda = \Lambda_\theta$ for some $0 \leqslant \theta \leqslant m$, we will write J_θ instead. These are **maximal parahoric subgroups** of J. In unramified unitary similitude groups, maximal parahoric subgroups and maximal compact subgroups are the same. A general **parahoric subgroup** is an intersection $J_{\Lambda_1} \cap \ldots \cap J_{\Lambda_s}$ where $\{\Lambda_1, \ldots, \Lambda_s\}$ is an s-simplex in $\mathcal{L}_i$ for some i. Any parahoric subgroup is compact and open in J.
Let i be the integer such that $\Lambda \in \mathcal{L}_i$. We define $V_\Lambda^0 := \Lambda/p^{i+1}\Lambda^\vee$ and $V_\Lambda^1 := p^i\Lambda^\vee/\Lambda$. Because $p\Lambda \subset p\cdot p^i\Lambda^\vee$ and $p\cdot p^i\Lambda^\vee \subset \Lambda$, these are both $\mathbb{F}_{p^2}$-vector space of dimensions respectively $t(\Lambda)$ and $n - t(\Lambda)$. Both spaces come together with a non-degenerate σ-hermitian form $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ with values in $\mathbb{F}_{p^2}$, respectively induced by $p^{-i}\{\cdot,\cdot\}$ and by $p^{-i+1}\{\cdot,\cdot\}$. If k is a perfect field extension of $\mathbb{F}_{p^2}$ and if $\epsilon \in \{0,1\}$, we may extend the pairings to $(V_\Lambda^\epsilon)_k = V_\Lambda^\epsilon \otimes_{\mathbb{F}_{p^2}} k$ by setting

$$(v \otimes x, w \otimes y)_\epsilon := xy^\sigma (v,w)_\epsilon \in k$$

for all $v, w \in V_\Lambda^\epsilon$ and $x, y \in k$. If U is a subspace of $(V_\Lambda^\epsilon)_k$ we denote by $U^\perp$ its orthogonal, that is the subspace of all vectors $x \in (V_\Lambda^\epsilon)_k$ such that $(x, U)_\epsilon = 0$.
Denote by J_Λ^+ the pro-unipotent radical of J_Λ and write $\mathcal{J}_\Lambda := J_\Lambda/J_\Lambda^+$. This is a finite group of Lie type, called the **maximal reductive quotient** of J_Λ. We have an identification $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{GU}(V_\Lambda^0)\times\mathrm{GU}(V_\Lambda^1))$, that is the group of pairs (g_0, g_1) where for $\epsilon \in \{0,1\}$ we have $g_\epsilon \in \mathrm{GU}(V_\Lambda^\epsilon)$ and $c(g_0) = c(g_1)$. Here, $c(g_\epsilon) \in \mathbb{F}_p^\times$ denotes the multiplier of g_ϵ.

For $0 \leqslant \theta \leqslant m$ and $\epsilon \in \{0,1\}$, we will write V_θ^ϵ and $\mathcal{J}_\theta$ instead of $V_{\Lambda_\theta}^\epsilon$ and $\mathcal{J}_{\Lambda_\theta}$. A basis of V_θ^0 is given by the images of the $2\theta+1$ vectors $e_{-\theta} \dots, e_{-1}, e_0^{\mathrm{an}}, e_1, \dots, e_\theta$. As for V_θ^1, a basis is given by the images of the $n-2\theta-1$ vectors $p^{-1}e_{-m}, \dots, p^{-1}e_{-\theta-1}, e_{\theta+1}, \dots, e_m$ when n is odd, and in case n is even one must add the image of $p^{-1}e_1^{\mathrm{an}}$ to the basis.

1.2.9 Let $\Lambda \in \mathcal{L}_i$ where ni is even. We write $t(\Lambda) = 2\theta+1$. Let k be a perfect field extension of $\mathbb{F}_{p^2}$. Let T be any $W(k)$-lattice in $\mathbf{V}_k$ such that

$$p^{i+1}T^\vee \overset{2\theta'+1}{\subset} T \subset \Lambda_k$$

where $0 \leqslant \theta' \leqslant \theta$. Then T must contain $p^{i+1}\Lambda_k^\vee$ and $[\Lambda_k : T] = \theta - \theta'$. We may consider $\overline{T} := T/p^{i+1}\Lambda_k^\vee$ the image of T in $V_\Lambda^{(0)}$. Then $\overline{T}$ is an $\mathbb{F}_{p^2}$-subspace of dimension $\theta+\theta'+1$. Moreover, one may check that $\overline{p^{i+1}T^\vee} = \overline{T}^\perp$, therefore the subspace $\overline{T}$ contains its orthogonal. These observations lead to the following proposition.

Proposition ([Vol10] 2.7)**.** *The mapping $T \mapsto \overline{T}$ defines a bijection between the set of $W(k)$-lattices T in $\mathbf{V}_k$ such that $p^{i+1}T^\vee \overset{2\theta'+1}{\subset} T \subset \Lambda_k$ and the set*

$$\{U \subset (V_\Lambda^0)_k \mid \dim U = \theta + \theta' + 1 \textit{ and } U^\perp \subset U\}.$$

In particular taking $\theta' = 0$, this set is in bijection with $\mathcal{M}_\Lambda(k)$.

Remark. Similarly, the set of $W(k)$-lattices T such that $\Lambda_k \subset T \overset{2\theta'+1}{\subset} p^iT^\vee$ for some $\theta \leqslant \theta' \leqslant m$ is in bijection with

$$\{U \subset (V_\Lambda^1)_k \mid \dim U = n - \theta' - \theta - 1 \text{ and } U^\perp \subset U\}.$$

The bijection is given by $T \mapsto \overline{T}^\perp$ where $\overline{T} := T/\Lambda_k \subset V_k^{(1)}$. These sets can be seen as the k-rational points of some flag variety for $\mathrm{GU}(V_\Lambda^{(0)})$ and $\mathrm{GU}(V_\Lambda^{(1)})$, which are special instances of Deligne-Lusztig varieties. This is accounted for in the next paragraph.

1.2.10 Let $\Lambda \in \mathcal{L}$. The action of J on the Rapoport-Zink space $\mathcal{M}$ restricts to an action of the parahoric subgroup J_Λ on the closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$. This action factors through the maximal reductive quotient $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{GU}(V_\Lambda^0) \times \mathrm{GU}(V_\Lambda^1))$. This action is trivial on the normal subgroup $\{\mathrm{id}\} \times \mathrm{U}(V_\Lambda^1) \subset \mathcal{J}_\Lambda$, thus it factors again through the quotient which is isomorphic to $\mathrm{GU}(V_\Lambda^0)$.

Theorem ([VW11] Theorem 4.8)**.** *There is an isomorphism between $\mathcal{M}_\Lambda$ and a certain "generalized" parabolic Deligne-Lusztig variety for the finite group of Lie type $\mathrm{GU}(V_\Lambda^0)$, compatible with the actions. In particular, if $t(\Lambda) = 2\theta+1$ then the scheme $\mathcal{M}_\Lambda$ is projective, smooth, geometrically irreducible of dimension θ.*

We refer to [Mul21] Section 2 for the definition of Deligne-Lusztig varieties. In particular, the adjective "generalized" is understood according to loc. cit. The Deligne-Lusztig variety isomorphic to $\mathcal{M}_\Lambda$ is introduced in [VW11] 4.5, and it is denoted by Y_Λ there.

1.2.11 We now explain how the different closed Bruhat-Tits strata behave together.

Theorem ([VW11] Theorem 5.1). *Let $i \in \mathbb{Z}$ such that ni is even. Consider Λ and Λ' two lattices in $\mathcal{L}_i$. The following statements hold.*

(1) *The inclusion $\Lambda \subset \Lambda'$ is equivalent to the scheme-theoretic inclusion $\mathcal{M}_\Lambda \subset \mathcal{M}_{\Lambda'}$. It also implies $t(\Lambda) \leqslant t(\Lambda')$ and there is equality if and only if $\Lambda = \Lambda'$.*

(2) *The three following assertions are equivalent.*

(i) $\Lambda \cap \Lambda' \in \mathcal{L}_i$. (ii) $\Lambda \cap \Lambda'$ *contains a lattice of* $\mathcal{L}_i$. (iii) $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} \neq \emptyset$.

If these conditions are satisfied, then $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} = \mathcal{M}_{\Lambda \cap \Lambda'}$, where we understand the left hand side as the scheme theoretic intersection inside $\mathcal{M}_{i,\mathrm{red}}$.

(3) *The three following assertions are equivalent*

(i) $\Lambda + \Lambda' \in \mathcal{L}_i$. (ii) $\Lambda + \Lambda'$ *is contained in a lattice of* $\mathcal{L}_i$.

(iii) $\mathcal{M}_\Lambda, \mathcal{M}_{\Lambda'} \subset \mathcal{M}_{\widetilde{\Lambda}}$ *for some* $\widetilde{\Lambda}$ *in* $\mathcal{L}_i$.

If these conditions are satisfied, then $\mathcal{M}_{\Lambda+\Lambda'}$ is the smallest subscheme of the form $\mathcal{M}_{\widetilde{\Lambda}}$ containing both $\mathcal{M}_\Lambda$ and $\mathcal{M}_{\Lambda'}$.

(4) *If k is a perfect field field extension of $\mathbb{F}_{p^2}$ then $\mathcal{M}_i(k) = \bigcup_{\Lambda \in \mathcal{L}_i} \mathcal{M}_\Lambda(k)$.*

In essence, the previous statements explain how the stratification given by the $\mathcal{M}_\Lambda$ mimics the combinatorics of the Bruhat-Tits building of $\tilde{J}$, hence the name.

1.3 On the maximal parahoric subgroups of J

1.3.1 In this section we give a few results that will be useful later regarding the maximal parahoric subgroups J_Λ. First, we study their conjugacy classes. It starts with the following lemma.

Lemma. *Let $\Lambda, \Lambda' \in \mathcal{L}$.*

(i) The parahoric subgroup J_Λ acts transitively on the set of apartments containing Λ.

(ii) We have $J_\Lambda = J_{\Lambda'}$ if and only if there exists $k \in \mathbb{Z}$ such that $\Lambda = p^k\Lambda'$ or $\Lambda = p^k{\Lambda'}^\vee$.

Proof. The first point is a general fact from the theory of Bruhat-Tits buildings.
For the second point, the converse is clear. Indeed, if $x \in \mathbb{Q}_{p^2}^\times$ then $J_{x\Lambda} = J_\Lambda$, and an element $g \in J$ fixes a lattice Λ if and only if it fixes its dual $\Lambda^\vee$.
Now, let $\Lambda, \Lambda' \in \mathcal{L}$ such that $J_\Lambda = J_{\Lambda'}$. Up to replacing Λ' by an appropriate lattice $g \cdot \Lambda'$, it is enough to treat the case $\Lambda' = \Lambda_\theta$ for some $0 \leqslant \theta \leqslant m$. By 1.2.3 Proposition, we can find an apartment $\mathcal{A}^L$ containing both Λ_θ and Λ. By the first point, we can find $g \in J_\theta = J_\Lambda$ which sends $\mathcal{A}^L$ to $\mathcal{A}^e$. Therefore $g \cdot \Lambda = \Lambda$ belongs to $\mathcal{A}^e$. According to 1.2.5, we may write

$$\Lambda = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$$

for some integers (r_j, s). Let i be the integer such that $\Lambda \in \mathcal{L}_i$. Then according to 1.2.4 we have

- $\forall 1 \leqslant j \leqslant m, r_{-j} + r_j \in \{i, i+1\}$.
- $s_0 = \lfloor \frac{i+1}{2} \rfloor$.
- if n is even then $s_1 = \lfloor \frac{i}{2} \rfloor$.

For $1 \leqslant j \leqslant \theta$, let g_j be the automorphism of $\mathbf{V}$ which exchanges e_{-j} and e_j while fixing all the other vectors in the basis e. Then, from the definition of Λ_θ we have $g_j \in J_\theta$. Therefore g_j must fix Λ too, which implies that $r_{-j} = r_j$. And for $\theta + 1 \leqslant j \leqslant m$, let g_j be the automorphism sending e_j to $p^{-1}e_{-j}$ and e_{-j} to pe_j while fixing all the other vectors in the basis e. Then again we have $g_j \in J_\theta = J_\Lambda$ which implies that $r_{-j} = r_j - 1$.
Assume first that $i = 2i'$ is even. Combining the previous observations, we have $r_j = i'$ for all $1 \leqslant j \leqslant \theta$ and $r_j = i' + 1$ for all $\theta + 1 \leqslant j \leqslant m$. Moreover we have $s_0 = i'$ and if n is even, we have $s_1 = i'$. In other words, we have $\Lambda = p^{i'}\Lambda_\theta$.
Assume now that $i = 2i' + 1$ is odd. This implies that n is even. Combining the previous observations, we have $r_j = i' + 1$ for all $1 \leqslant j \leqslant m$. Moreover we have $s_0 = i' + 1$ and if n is even, we have $s_1 = i'$. In other words, we have $\Lambda = p^{i'+1}\Lambda_\theta^\vee$. □

1.3.2 We may now describe the conjugacy classes of these maximal parahoric subgroups.

Corollary. *Let $\Lambda, \Lambda' \in \mathcal{L}$.*

(i) If n is odd, then $t(\Lambda) = t(\Lambda')$ if and only if the associated maximal parahoric subgroups J_Λ and $J_{\Lambda'}$ are conjugate in J. Each such subgroup is conjugate to J_θ for a unique $0 \leqslant \theta \leqslant m$.

(ii) If n is even, then $t(\Lambda) \in \{t(\Lambda'), n - t(\Lambda')\}$ if and only if the associated maximal parahoric subgroups J_Λ and $J_{\Lambda'}$ are conjugate in J. Each such subgroup is conjugate to J_θ for a unique $0 \leqslant \theta \leqslant \lfloor \frac{m}{2} \rfloor$.

Thus, there are $m+1$ conjugacy classes of maximal parahoric subgroups when n is odd, and only $\lfloor \frac{m}{2} \rfloor + 1$ when n is even. If n is odd the subgroups J_θ are pairwise non conjugate, whereas J_θ is conjugate to $J_{m-\theta}$ when n is even.

Remark. The special maximal compact subgroups are the conjugates of J_0 and of J_m. When n is odd, the conjugates of J_m are hyperspecial.

Proof. For the first point, assume that $t(\Lambda) = t(\Lambda')$. By 1.2.1 Proposition, we can find $g \in J$ such that $g \cdot \Lambda = \Lambda'$. Therefore $J_{\Lambda'} = J_{g\cdot\Lambda} = {}^gJ_\Lambda$, the two parahoric subgroups are conjugate. For the converse, assume that $J_{\Lambda'} = {}^gJ_\Lambda$ for some $g \in J$. Then $J_{\Lambda'} = J_{g\cdot\Lambda}$. By 1.3.1 there is some $k \in \mathbb{Z}$ such that $\Lambda' = p^k g \cdot \Lambda$ or $(\Lambda')^\vee = p^k g \cdot \Lambda$. This implies that $t(\Lambda) = t(\Lambda')$. Indeed, it is clear in the first case, and in the second case we have in particular $(\Lambda')^\vee \in \mathcal{L}$. Since n is odd, by 1.2.2 we have $t(\Lambda') = t((\Lambda')^\vee)$, so that we are done.
For the second point, if $t(\Lambda') = t(\Lambda)$ then we reason the same way as above. If $t(\Lambda') = n - t(\Lambda)$ then Λ' and $\Lambda^\vee$ have the same type. By the first case, we know that $J_{\Lambda'}$ and $J_{\Lambda^\vee} = J_\Lambda$ are conjugate. The converse goes the same way as above, except that the case $(\Lambda')^\vee = p^k g \cdot \Lambda$ now implies that $t(\Lambda') = n - t(\Lambda)$ therefore we are done. □

1.3.3 As another corollary of 1.3.1 we may also describe the normalizers of the maximal parahoric subgroups.

Corollary. *Let $\Lambda \in \mathcal{L}$. If $t(\Lambda) \neq n - t(\Lambda)$ then the normalizer of J_Λ in J is $\mathrm{N}_J(J_\Lambda) = \mathrm{Z}(J)J_\Lambda$. Otherwise, n is even and there exists an element $h_0 \in J$ such that $h_0^2 = p \cdot \mathrm{id}$ and $N_J(J_\lambda)$ is the subgroup generated by J_Λ and h_0. In particular, $\mathrm{Z}(J)J_\Lambda$ is a subgroup of index 2 in $N_J(J_\Lambda)$.*

Remark. The condition $t(\Lambda) \neq n - t(\Lambda)$ is automatically satisfied if n is odd. If n is even, it is satisfied when $t(\Lambda) \neq m + 1$, this is the case in particular when m is odd.

Proof. It is clear that $\mathrm{Z}(J)J_\Lambda \subset \mathrm{N}_J(J_\Lambda)$. Conversely, let $g \in N_J(J_\Lambda)$, so that we have $J_\Lambda = {}^gJ_\Lambda = J_{g\cdot\Lambda}$. We apply 1.3.1 to deduce the existence of $k \in \mathbb{Z}$ such that $g \cdot \Lambda = p^k\Lambda$ (case 1) or $g \cdot \Lambda = p^k\Lambda^\vee$ (case 2). If we are in case 1, then $g \in p^kJ_\Lambda \subset \mathrm{Z}(J)J_\Lambda$ and we are done. If n is even, the assumption that $t(\Lambda) \neq n - t(\Lambda)$ makes the case 2 impossible. If n is odd and we are in case 2, then in particular $\Lambda^\vee \in \mathcal{L}$. By 1.2.2, we must have $\Lambda = p^i\Lambda^\vee$ for some even $i \in \mathbb{Z}$. In particular, we are also in case 1. Therefore, no matter the parity of n, we are always in case 1. Assume now that $t(\Lambda) = n - t(\Lambda)$, in particular n and m are both even. We write $m = 2m'$ so that $t(\Lambda) = 2m' + 1$ and we solve the case $\Lambda = \Lambda_{m'}$ first. Recall the element g_0 that was defined in 1.1.7. By direct computation, we see that $g_0 \cdot \Lambda_{m'} = p\Lambda_{m'}^\vee$. Therefore ${}^{g_0}J_{m'} = J_{p\Lambda_{m'}^\vee} = J_{m'}$ so that $g_0 \in \mathrm{N}_J(J_{m'})$. Now let g be any element normalizing J_m, so that $J_{m'} = {}^gJ_{m'} = J_{g\cdot\Lambda_{m'}}$. According to 1.3.1 there exists $k \in \mathbb{Z}$ such that $g\cdot\Lambda_{m'} = p^k\Lambda_{m'}$ or $g\cdot\Lambda_{m'} = p^k\Lambda_{m'}^\vee = p^{k-1}g_0\cdot\Lambda_{m'}$. In the first case we have $g \in p^kJ_{m'}$ and in the second case we have $g \in p^{k-1}g_0J_{m'}$. Because $g_0^2 = p \cdot \mathrm{id}$, the claim is proved with $h_0 = g_0$.
In the general case, we have $t(\Lambda) = 2m' + 1 = t(\Lambda_{m'})$. By 1.2.1 there exists some $g \in J$ such that $\Lambda = g \cdot \Lambda_{m'}$. Then $\mathrm{N}_J(\Lambda) = {}^g\mathrm{N}_J(\Lambda_{m'})$ so that the claim follows with $h_0 := gg_0g^{-1}$. □

1.3.4 Let J° be the kernel of $\alpha : J \to \mathbb{Z}$. In other words, J° is the subgroup of J consisting of all $g \in J$ whose multiplier $c(g)$ is a unit in $\mathbb{Z}_p^\times$. We have an isomorphism $J/J^\circ \simeq \mathbb{Z}$ induced by α when n is even, and by $\frac{1}{2}\alpha$ when n is odd. Note that J° contains all the compact subgroups of J, in particular $J_\Lambda \subset J^\circ$ for every $\Lambda \in \mathcal{L}$. Let K be the subgroup generated by all the J_Λ for $\Lambda \in \mathcal{L}$ having maximal orbit type $t(\Lambda) = 2m + 1$. We will prove the following result.

Proposition. *We have $K = J^\circ$.*

The proof requires the following lemma.

Lemma. *Let $i \in \mathbb{Z}$ such that ni is even and let $\Lambda \in \mathcal{L}_i$ be a lattice of maximal orbit type. Let $\Lambda', \Lambda'' \in \mathcal{L}_i$ such that $\Lambda' \cap \Lambda$ and $\Lambda'' \cap \Lambda$ belong to $\mathcal{L}_i$. There exists $g \in J_\Lambda$ such that $g \cdot \Lambda' = \Lambda''$ if and only if $t(\Lambda') = t(\Lambda'')$ and $t(\Lambda' \cap \Lambda) = t(\Lambda'' \cap \Lambda)$.*

Proof. The forward direction is clear because the action of J preserves the types of the lattices. We prove the converse. Since J acts transitively on $\mathcal{L}$ while preserving types and inclusions, it is enough to look at the case $i = 0$ and $\Lambda = \Lambda_m = \Lambda(0, \ldots, 0)$. Let $0 \leqslant \theta_- \leqslant \theta_+ \leqslant m$. We fix a certain $\Lambda' \in \mathcal{L}_0$ such that $t(\Lambda') = 2\theta_+ + 1$ and $t(\Lambda' \cap \Lambda) = 2\theta_- + 1$, and we prove that any $\Lambda'' \in \mathcal{L}_0$ satisfying the hypotheses of the lemma is in the J_m-orbit of Λ'. We define

$$\Lambda' = \Lambda(0^{\theta_-}, 1^{\theta_+ - \theta_-}, 1^{m-\theta_+}, 0, 0^{m-\theta_+}, -1^{\theta_+ - \theta_-}, 0^{\theta_-})$$

where the 0 in the middle stands for 0 when n is odd and the pair $(0, 0)$ when n is even. Then, we have

$$\Lambda' \cap \Lambda = \Lambda(0^{\theta_-}, 1^{m-\theta_-}, 0, 0^{m-\theta_-}, 0^{\theta_-})$$

so that Λ' satisfies the required conditions. Let Λ'' be as in the lemma. Let L be a Witt decomposition of $\mathbf{V}$ such that the corresponding apartment $\mathcal{A}^L$ contains both Λ and Λ''. Since J_m acts transitively on the set of apartments containing Λ_m, we can find some $g \in J_m$ such that $g \cdot \mathcal{A}^L = \mathcal{A}^e$. Up to replacing Λ'' by $g \cdot \Lambda''$, we may then assume that $\Lambda'' \in \mathcal{A}^e$. Therefore, there exists integers $r_{-m}, \ldots, r_m, s$ such that

$$\Lambda'' = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m).$$

Since $\Lambda'' \in \mathcal{L}_0$, by 1.2.4 we have $s = 0$ and $r_j + r_{-j} \in \{0, 1\}$ for all $1 \leqslant j \leqslant m$. Let us write $r_{-j} = r_j + \epsilon_j$ where $\epsilon_j \in \{0, 1\}$. Since $t(\Lambda'') = 2\theta_+ + 1$, there are θ_+ indices $1 \leqslant j_1 \leqslant \ldots \leqslant j_{\theta_+} \leqslant m$ such that $\epsilon_j = 0$ if and only if j is one of the j_k's. Moreover, we have

$$\Lambda'' \cap \Lambda = \Lambda\left(\max(-r_m + \epsilon_m, 0), \ldots, \max(-r_1 + \epsilon_1, 0), 0, \max(r_1, 0), \ldots, \max(r_m, 0)\right).$$

This lattice is in $\mathcal{L}_0$, thus for every $1 \leqslant j \leqslant m$ we have $0 \leqslant \max(-r_j + \epsilon_j, 0) + \max(r_j, 0) \leqslant 1$. Hence, if $j = j_k$ for some k then $\epsilon_j = 0$ and

$$\max(-r_j + \epsilon_j, 0) + \max(r_j, 0) = \max(-r_j, 0) + \max(r_j, 0) = |r_j|.$$

Thus, $|r_j| = 0$ or 1. If $j \neq j_k$ for all k, then $\epsilon_j = 1$ and

$$\max(-r_j + \epsilon_j, 0) + \max(r_j, 0) = \max(-r_j + 1, 0) + \max(r_j, 0) = \frac{1}{2} + \frac{|r_j| + |r_j - 1|}{2}.$$

This sum is a positive integer between 0 and 1, therefore it is always 1. It means that $|r_j| + |r_j - 1| = 1$ and as a consequence, $r_j = 0$ or 1.
Lastly, we have $t(\Lambda'' \cap \Lambda) = 2\theta_- + 1$ so there are exactly θ_- indices j for which the sum $\max(-r_j + \epsilon_j, 0) + \max(r_j, 0)$ is zero. As we have just seen, this may only happen when j is one of the j_k's. Thus, among the indices $j = j_1, \ldots, j_{\theta_+}$, there are exactly θ_- of them for which $(r_{-j}, r_j) = (0, 0)$, and for the others we have $(r_{-j}, r_j) = (1, -1)$ or $(-1, 1)$. If j is not one of the j_k's, we have $(r_{-j}, r_j) = (0, 1)$ or $(1, 0)$. In other words, the pairs of indices (r_{-j}, r_j) are, up to shifts and ordering, the same as the corresponding pairs of indices defining Λ'. By considering appropriate permutation matrices, we may change a pair (r_{-j}, r_j) into (r_j, r_{-j}) and we may change the order so that Λ'' is sent to Λ'. This transformation defines an element of J which stabilizes $\Lambda = \Lambda(0, \ldots, 0)$. □

1.3.5 We may now prove the proposition.

Proof. It is clear that $K \subset \mathrm{Ker}(\alpha)$, so we prove the reverse inclusion. Let $g^0 \in J^\circ$. We will write g^0 as a product of elements in J, each of which fixes some lattice of maximal orbit type in the Bruhat-Tits building. We write $\Lambda := \Lambda_m = \Lambda(0,\dots,0)$ and $\Lambda^0 := g^0 \cdot \Lambda$. Since $g^0 \in J^\circ$, we have $\Lambda^0 \in \mathcal{L}_0$. We would like to send Λ^0 back to Λ by using elements of K only. Let L be some Witt decomposition of $\mathbf{V}$ such that the corresponding apartment $\mathcal{A}^L$ contains both Λ and Λ^0. We can find some $g_1 \in J_\Lambda$ which sends $\mathcal{A}^L$ to $\mathcal{A}^e$. We define $g^1 := g_1 g^0$ and $\Lambda^1 := g^1 \cdot \Lambda$. Then $\Lambda^1 \in \mathcal{L}_0$ and it belongs to the apartment $\mathcal{A}^e$. Therefore, there exists integers $r_{-m},\dots,r_m,s$ such that

$$\Lambda^1 = \Lambda(r_{-m},\dots,r_{-1},s,r_1,\dots,r_m).$$

Since $\Lambda^1 \in \mathcal{L}_0$ and its orbit type is maximal, we have $s = 0$ and $r_{-j} = -r_j$ for all $1 \leqslant j \leqslant m$. Let $1 \leqslant j_1 < \ldots < j_a \leqslant m$ be the indices j for which r_j is odd. We have $0 \leqslant a \leqslant m$. For $1 \leqslant j \leqslant m$ we write $r_j = 2r'_j + 1$ if j is some of the $j'_k s$ and $r_j = 2r'_j$ otherwise. We also write $r'_{-j} = -r'_j$, so that we have $r_{-j} = 2r'_{-j} - 1$ if j is some of the j_k's and $r_{-j} = 2r'_{-j}$ otherwise. We define g_2 the endomorphism of $\mathbf{V}$ sending e_{-j} to $p^{2r'_j} e_j$ for $-m \leqslant j \leqslant m$ and $j \neq 0$, and which acts like identity on $\mathbf{V}^{\mathrm{an}}$. Then g_2 is an element of J with multiplier equal to 1. Moreover, g_2 stabilizes the lattice $\Lambda(r'_{-m},\dots,r'_{-1},0,r'_1,\dots,r'_m) \in \mathcal{L}_0$ whose orbit type is maximal, therefore $g_2 \in K$. We define $g^2 := g_2 g^1$ and $\Lambda^2 := g^2 \cdot \Lambda \in \mathcal{L}_0$. Concretely, the lattice Λ^2 still lies in the apartment $\mathcal{A}^e$ and its coefficients are obtained from those of Λ^1 by replacing each pair (r_{-j_k}, r_{j_k}) by $(1,-1)$ and the other pairs (r_{-j}, r_j) by $(0,0)$. Let us note that if $a = 0$ then we already have $\Lambda^2 = \Lambda$.

Let us now assume that $a > 0$. The intersection of the lattices Λ^2 and Λ has the following shape.

$$\Lambda^2 \cap \Lambda = \Lambda(\underbrace{0 \text{ or } 1,\dots,0 \text{ or } 1}_{a \text{ times } 1 \text{ and } m-a \text{ times } 0}, 0, 0^m).$$

The coefficient takes the value 1 if and only if its index is one of the $-j_k$'s. This is a lattice in $\mathcal{L}_0$ of orbit type $2(m-a)+1$. We will use 1.3.4 Lemma in order to send Λ^2 to Λ while fixing some lattice of maximal orbit type. In order to find this lattice, we need to leave the apartment $\mathcal{A}^e$. Let $\delta \in \mathbb{Z}_{p^2}^\times$ such that $\sigma(\delta) = -\delta$. We define the following vectors

$$f_j = \begin{cases} e_j & \text{if } j \text{ is not one of the } \pm j_k\text{'s.} \\ pe_{-j_k} & \text{if } j = -j_k. \\ p^{-1}e_{j_k} + \delta e_{-j_k} & \text{if } j = j_k. \end{cases}$$

We also define $f_i^{\mathrm{an}} = e_i^{\mathrm{an}}$ for $i \in \{0,1\}$ (the case $i = 1$ only occurs if n is even). All together, these vectors form a basis f of $\mathbf{V}$. We write Λ_f for the $\mathbb{Z}_{p^2}$-lattice generated by the basis f. One may check that $\langle f_j, f_{j'} \rangle = \delta_{j',-j}$ for every j and j'. It follows that $\Lambda_f \in \mathcal{L}_0$ and it has maximal orbit type. It turns out that both intersections $\Lambda^2 \cap \Lambda_f$ and $\Lambda \cap \Lambda_f$ are equal to $\Lambda^2 \cap \Lambda$, as we prove in the following two points.

- $\underline{\Lambda^2 \cap \Lambda_f}$: The lattice $\Lambda^2 \cap \Lambda_f$ contains all the vectors e_j where j is not of the $\pm j_k$'s. It also contains the vectors pe_{-j_k} and $p \cdot (p^{-1}e_{j_k} + \delta e_{-j_k}) = e_{j_k} + \delta p e_{-j_k}$ for all $1 \leqslant k \leqslant a$. Therefore, it must contain the vectors e_{j_k}'s as well. This gives the inclusion $\Lambda^2 \cap \Lambda \subset \Lambda^2 \cap \Lambda_f$. For the converse, if $x \in \Lambda_f$ then we may write

$$\begin{aligned} x &= \sum_{j \neq \pm j_k} \mu_j e_j + \sum_{k=1}^{s} \lambda_k p e_{-j_k} + \lambda'_k (p^{-1} e_{j_k} + \delta e_{-j_k}) \\ &= \sum_{j \neq \pm j_k} \mu_j e_j + \sum_{k=1}^{s} (\lambda_k p + \lambda'_k \delta) e_{-j_k} + \lambda'_k p^{-1} e_{j_k} \end{aligned}$$

with the scalars μ_j, λ_k and λ'_k in $\mathbb{Z}_{p^2}$. If moreover $x \in \Lambda^2$ then in the last formula, we must have $\lambda_k p + \lambda'_k \delta \in p\mathbb{Z}_{p^2}$. It follows that the scalars λ'_k belong to $p\mathbb{Z}_{p^2}$ and thus $x \in \Lambda^2 \cap \Lambda$.

– $\underline{\Lambda \cap \Lambda_f}$: By the same arguments as above, we prove that $\Lambda^2 \cap \Lambda \subset \Lambda \cap \Lambda_f$. For the converse, let $x \in \Lambda_f$ as above. If moreover $x \in \Lambda$ then the scalars λ'_k are elements of $p\mathbb{Z}_{p^2}$. It implies that $\lambda_k p + \lambda'_k \delta \in p\mathbb{Z}_{p^2}$, whence $x \in \Lambda^2 \cap \Lambda$.

Eventually we may apply 1.3.4 Lemma to the lattices Λ_f, Λ^2 and Λ. It gives the existence of an element $g_3 \in J$ which stabilizes Λ_f and sends Λ^2 to Λ. We write $g^3 := g_3 g^2$. It follows that $g^3 \cdot \Lambda = \Lambda$, therefore $g^3 \in J_\Lambda \subset K$. But $g^3 = g_3 g_2 g_1 g^0$ and each of the elements g_1, g_2 and g_3 also lies in K. Therefore $g^0 \in K$ as well. □

1.4 Counting the closed Bruhat-Tits strata

1.4.1 In this section we count the number of closed Bruhat-Tits strata which contain or which are contained in another given one. Let $d \geqslant 0$ and consider V a d-dimensional $\mathbb{F}_{p^2}$-vector space equipped with a non degenerate hermitian form. This structure is uniquely determined up to isomorphism as we are working over a finite field. As in [VW11], for $\lceil \frac{d}{2} \rceil \leqslant r \leqslant d$, we define

$$\begin{aligned} N(r, V) &:= \{U \,|\, U \text{ is an } r\text{-dimensional subspace of } V \text{ such that } U^\perp \subset U\}, \\ \nu(r, d) &:= \#N(r, V), \end{aligned}$$

where $U^\perp$ denotes the orthogonal of U with respect to the hermitian form on V. As remarked in [VW11], the set $N(r, V)$ can be seen as the set of rational points of a certain flag variety for the unitary group of V.

Proposition ([VW11] Corollary 5.7). *Let $\Lambda \in \mathcal{L}$. Write $t(\Lambda) = 2\theta + 1$ for some $0 \leqslant \theta \leqslant m$.*

- *Let θ' be an integer such that $0 \leqslant \theta' \leqslant \theta$. The number of closed Bruhat-Tits strata of dimension θ' contained in $\mathcal{M}_\Lambda$ is $\nu(\theta + \theta' + 1, 2\theta + 1)$.*
- *Let θ' be an integer such that $\theta \leqslant \theta' \leqslant m$. The number of closed Bruhat-Tits strata of dimension θ' containing $\mathcal{M}_\Lambda$ is $\nu(n - \theta - \theta' - 1, n - 2\theta - 1)$.*

These follows from 1.2.9 Proposition and Remark. Another way to formulate the proposition is to say that $\nu(\theta + \theta' + 1, 2\theta + 1)$ (resp. $\nu(n - \theta - \theta' - 1, n - 2\theta - 1)$) is the number of vertices of type $2\theta' + 1$ in the Bruhat-Tits building of $\tilde{J}$ which are neighbors of a given vertex of type $2\theta + 1$ for $\theta' \leqslant \theta$ (resp. $\theta' \geqslant \theta$).

1.4.2 In [VW11], an explicit formula is given for $\nu(d-1,d)$. The next proposition gives a formula to compute $\nu(r,d)$ for general r and d.

Proposition. *Let $d \geqslant 0$ and let $\lceil \frac{d}{2} \rceil \leqslant r \leqslant d$. We have*

$$\nu(r,d) = \frac{\prod_{j=1}^{2(d-r)} \left(p^{2r-d+j} - (-1)^{2r-d+j}\right)}{\prod_{j=1}^{d-r} (p^{2j}-1)}$$

Proof. Recall that for any integer k, we denote by A_k the $k \times k$ matrix having 1 in the antidiagonal and 0 everywhere else. We fix a basis $(e_1, \ldots, e_d)$ of V in which the hermitian form is represented by the matrix A_d. We denote by U_0 the subspace generated by the vectors $e_1, \ldots, e_r$. Then the orthogonal of U_0 is generated by $e_1, \ldots, e_{d-r}$. Since r is an integer between $\lceil \frac{d}{2} \rceil$ and d, we have $0 \leqslant d-r \leqslant r$ and therefore U_0 contains its orthogonal. Thus, U_0 defines an element of $N(r,V)$. The unitary group $\mathrm{U}(V) \simeq \mathrm{U}_d(\mathbb{F}_p)$ acts on the set $N(r,V)$: an element $g \in \mathrm{U}(V)$ sends the subspace U to $g(U)$. This action is transitive. Indeed, any $U \in N(r,V)$ can be sent to U_0 by using an equivalent of the Gram-Schmidt orthogonalization process over $\mathbb{F}_{p^2}$ (note that $p \neq 2$). The stabilizer of U_0 in $\mathrm{U}_d(\mathbb{F}_p)$ is the standard parabolic subgroup

$$P_0 := \left\{ \begin{pmatrix} B & * & * \\ 0 & M & * \\ 0 & 0 & F(B) \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; B \in \mathrm{GL}_{d-r}(\mathbb{F}_{p^2}), M \in \mathrm{U}_{2r-d}(\mathbb{F}_p) \right\}.$$

Here, $F(B) = A_{d-r}(B^{(p)})^{-T} A_{d-r}$ where $B^{(p)}$ is the matrix B with all coefficients raised to the power p. Therefore, the set $N(r,V)$ is in bijection with the quotient $\mathrm{U}_d(\mathbb{F}_p)/P_0$. The order of $\mathrm{U}_d(\mathbb{F}_p)$ is well known and given by the formula

$$\#\mathrm{U}_d(\mathbb{F}_p) = p^{\frac{d(d-1)}{2}} \prod_{j=1}^{d} \left(p^j - (-1)^j\right).$$

It remains to compute the order of P_0. We have a Levi decomposition $P_0 = L_0 N_0$ with $L_0 \cap N_0 = \{1\}$ where

$$L_0 := \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & F(B) \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; B \in \mathrm{GL}_{d-r}(\mathbb{F}_{p^2}), M \in \mathrm{U}_{2r-d}(\mathbb{F}_p) \right\},$$

$$N_0 := \left\{ \begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; X \in \mathrm{M}_{d-r,2r-d}(\mathbb{F}_{p^2}), Y \in \mathrm{M}_{2r-d,d-r}(\mathbb{F}_{p^2}), Z \in \mathrm{M}_{d-r}(\mathbb{F}_{p^2}) \right\}.$$

The order of L_0 is given by

$$\#L_0 = \#\mathrm{GL}_{d-r}(\mathbb{F}_{p^2})\#\mathrm{U}_{2r-d}(\mathbb{F}_p) = p^{(d-r)(d-r-1)+\frac{(2r-d)(2r-d-1)}{2}} \prod_{j=1}^{d-r} \left(p^{2j}-1\right) \prod_{j=1}^{2r-d} \left(p^j - (-1)^j\right).$$

As for N_0, we need some more conditions on the matrices X, Y and Z. By direct computations, one checks that such a matrix belongs to $\mathrm{U}_d(\mathbb{F}_p)$ if and only if

$$Y = -A_{2r-d}(X^{(p)})^T A_{d-r}, \qquad Z + A_{d-r}(Z^{(p)})^T A_{d-r} = XY \in \mathrm{M}_{d-r}(\mathbb{F}_{p^2}).$$

Thus, X is any matrix of size $(d-r) \times (2d-r)$ and Y is determined by X. Let us look at the second equation. The matrix $A_{d-r}(Z^{(p)})^T A_{d-r}$ is the reflexion of $Z^{(p)}$ with respect to the antidiagonal. The equation implies that the coefficients below the antidiagonal of Z determine those above the antidiagonal. Furthermore, if z is a coefficient in the antidiagonal then the equation determines the value of $\mathrm{Tr}(z) = z + z^p$, where $\mathrm{Tr} : \mathbb{F}_{p^2} \to \mathbb{F}_p$ is the trace relative to the extension $\mathbb{F}_{p^2}/\mathbb{F}_p$. The trace is surjective and its kernel has order p. Thus, there are only p possibilities for each antidiagonal coefficient. Putting things together, the order of N_0 is given by

$$\#N_0 = p^{2(d-r)(2r-d)} \cdot p^{2\frac{(d-r)(d-r-1)}{2}} \cdot p^{d-r} = p^{(d-r)(3r-d)}$$

where the three terms take account respectively of the choice of X, the choice of the coefficients below the antidiagonal of Z and the choice of the coefficients in the antidiagonal of Z.
Hence the order of P_0 is given by

$$\#P_0 = \#L_0 \#N_0 = p^{\frac{d(d-1)}{2}} \prod_{j=1}^{d-r} \left(p^{2j} - 1\right) \prod_{j=1}^{2r-d} \left(p^j - (-1)^j\right).$$

Upon taking the quotient $\nu(r, d) = \#\mathrm{U}_d(\mathbb{F}_p)/\#P_0$, the result follows. □

In particular with $r = d - 1$, we obtain

$$\nu(d-1, d) = \frac{(p^{d-1} - (-1)^{d-1})(p^d - (-1)^d)}{p^2 - 1}.$$

If $d = 2\delta$ is even, it is equal to $(p^{d-1} + 1) \sum_{j=0}^{\delta-1} p^{2j}$, and if $d = 2\delta + 1$ is odd, it is equal to $(p^d + 1) \sum_{j=0}^{\delta-1} p^{2j}$. This coincides with the formula given in [VW11] Example 5.6.

2 The cohomology of a closed Bruhat-Tits stratum

2.1 In [Mul21], we computed the cohomology groups $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ of the closed Bruhat-Tits strata (recall that $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_p$). The computation relies on the Ekedahl-Oort stratification on $\mathcal{M}_\Lambda$ which, in the language of Deligne-Lusztig varieties, translates into a stratification by Coxeter varieties for unitary groups of smaller sizes. The cohomology of Coxeter varieties is well known thanks to the work of Lusztig in [Lus76]. In order to state our results, we recall the classification of unipotent representations of the finite unitary group over $\overline{\mathbb{Q}_\ell}$.

2.2 Let q be a power of prime number p, and let $\mathbf{G}$ be a reductive connected group over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Assume that $\mathbf{G}$ is equipped with an $\mathbb{F}_q$-structure induced by a Frobenius morphism F. Let $G = \mathbf{G}^F$ be the associated finite group of Lie type. Let $(\mathbf{T}, \mathbf{B})$ be a pair consisting of an F-stable maximal torus $\mathbf{T}$ and an F-stable Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\mathbf{W} = \mathbf{W}(\mathbf{T})$ denote the Weyl group of $\mathbf{G}$. The Frobenius F induces an action on $\mathbf{W}$. For $w \in \mathbf{W}$, let $\dot{w}$ be a representative of w in the normalizer $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$. By the Lang-Steinberg theorem, one can find $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$. Then ${}^g\mathbf{T} := g\mathbf{T}g^{-1}$ is another F-stable maximal torus, and $w \in \mathbf{W}$ is said to be the **type** of ${}^g\mathbf{T}$ with respect to $\mathbf{T}$. Every F-stable maximal torus arises in this manner. According to [DL76] Corollary 1.14, the G-conjugacy class of ${}^g\mathbf{T}$ only depends on the F-conjugacy class of w in the Weyl group $\mathbf{W}$. Here, two elements w and w' in $\mathbf{W}$ are said to be F-conjugates if there exists some element $\tau \in \mathbf{W}$ such that $w = \tau w' F(\tau)^{-1}$. For every $w \in \mathbf{W}$, we fix $\mathbf{T}_w$ an F-stable maximal torus of type w with respect to $\mathbf{T}$. The Deligne-Lusztig induction of the trivial representation of $\mathbf{T}_w$ is the virtual representation of G defined by the formula

$$R_w := \sum_{i \geqslant 0} (-1)^i \mathrm{H}^i_c(X_\varnothing(w) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}),$$

where $X_\varnothing(w)$ is the Deligne-Lusztig variety for $\mathbf{G}$ given by

$$X_\varnothing(w) := \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\}.$$

According to [DL76] Theorem 1.6, the virtual representation R_w only depends on the F-conjugacy class of w in $\mathbf{W}$. An irreducible representation of G is said to be **unipotent** if it occurs in R_w for some $w \in \mathbf{W}$. The set of isomorphism classes of unipotent representations of G is usually denoted $\mathcal{E}(G, 1)$ following Lusztig's notations.

Remark. Since the center $\mathrm{Z}(G)$ acts trivially on the variety $X_\varnothing(w)$, any irreducible unipotent representation of G has trivial central character.

2.3 Let $\mathbf{G}$ and $\mathbf{G}'$ be two reductive connected group over $\mathbb{F}$ both equipped with an $\mathbb{F}_q$-structure. We denote by F and F' the respective Frobenius morphisms. Let $f : \mathbf{G} \to \mathbf{G}'$ be an $\mathbb{F}_q$-isotypy, that is a homomorphism defined over $\mathbb{F}_q$ whose kernel is contained in the center of $\mathbf{G}$ and whose image contains the derived subgroup of $\mathbf{G}'$. Then, according to [DM20] Proposition 11.3.8, we have an equality

$$\mathcal{E}(G, 1) = \{\rho \circ f \mid \rho \in \mathcal{E}(G', 1)\}.$$

Thus, the irreducible unipotent representations of G and of G' can be identified. We will use this observation in the case $G = \mathrm{U}_k(\mathbb{F}_q)$ and $G' = \mathrm{GU}_k(\mathbb{F}_q)$. The corresponding reductive groups are $\mathbf{G} = \mathrm{GL}_k$ and $\mathbf{G}' = \mathrm{GL}_k \times \mathrm{GL}_1$. The Frobenius morphisms can be defined as

$$F(M) = \dot{w}_0 (M^{(q)})^{-T} \dot{w}_0, \qquad\qquad F'(M, c) = (c^q \dot{w}_0 (M^{(q)})^{-T} \dot{w}_0, c^q).$$

Here, $\dot{w}_0$ is the $k \times k$ matrix with only 1's in the antidiagonal and $M^{(q)}$ is the matrix M whose entries are all raised to the power q. The isotypy $f : \mathbf{G} \to \mathbf{G}'$ is defined by $f(M) = (M, 1)$. It satisfies $F' \circ f = f \circ F$, it is injective and its image contains the derived subgroup $\mathrm{SL}_n \times \{1\} \subset \mathbf{G}'$. Hence, we obtain the following result.

Proposition. *The irreducible unipotent representations of the finite groups of Lie type $\mathrm{U}_k(\mathbb{F}_q)$ and $\mathrm{GU}_k(\mathbb{F}_q)$ can be naturally identified.*

2.4 Assume that the Coxeter graph of the reductive group $\mathbf{G}$ is a union of subgraphs of type A_m (for various m). Let $\breve{\mathbf{W}}$ be the set of isomorphism classes of irreducible representations of its Weyl group $\mathbf{W}$. The action of the Frobenius F on $\mathbf{W}$ induces an action on $\breve{\mathbf{W}}$, and we consider the fixed point set $\breve{\mathbf{W}}^F$. The following theorem classifies the irreducible unipotent representations of G.

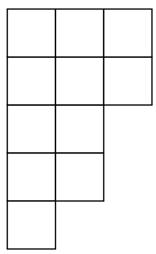
Theorem ([LS77] Theorem 2.2)**.** *There is a bijection between $\breve{\mathbf{W}}^F$ and the set of isomorphism classes of irreducible unipotent representations of G.*

We recall how the bijection is constructed. According to loc. cit. if $V \in \breve{\mathbf{W}}^F$ there is a unique automorphism $\widetilde{F}$ of V of finite order such that

$$R(V) := \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \mathrm{Trace}(w \circ \widetilde{F} \,|\, V) R_w$$

is an irreducible representation of G. Then the map $V \mapsto R(V)$ is the desired bijection. In the case of $\mathrm{U}_k(\mathbb{F}_q)$ or $\mathrm{GU}_k(\mathbb{F}_q)$, the Weyl group $\mathbf{W}$ is identified with the symmetric group $\mathfrak{S}_k$ and we have an equality $\breve{\mathbf{W}}^F = \breve{\mathbf{W}}$. Moreover, the automorphism $\widetilde{F}$ is the multiplication by w_0, where w_0 is the element of maximal length in $\mathbf{W}$. Thus, in both cases the irreducible unipotent representations of G are classified by the irreducible representations of the Weyl group $\mathbf{W} \simeq \mathfrak{S}_k$, which in turn are classified by partitions of k or equivalently by Young diagrams, as we briefly recall in the next paragraph.

2.5 A partition of k is a tuple $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with $r \geqslant 1$ and each λ_i is a positive integer, such that $\lambda_1 + \ldots + \lambda_r = k$. The integer k is called the length of the partition, and it is denoted by $|\lambda|$. A Young diagram of size k is a top left justified collection of k boxes, arranged in rows and columns. There is a correspondance between Young diagrams of size k and partitions of k, by associating to a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ the Young diagram having r rows consisting successively of $\lambda_1, \ldots, \lambda_r$ boxes. We will often identify a partition with its Young diagram, and conversely. For example, the Young diagram associated to $\lambda = (3^2, 2^2, 1)$ is the following one.



To any partition λ of k, one can naturally associate an irreducible character χ_λ of the symmetric group $\mathfrak{S}_k$. An explicit construction is given, for instance, by the notion of Specht modules as explained in [Jam84] 7.1. We will not recall their definition.

2.6 The irreducible unipotent representation of $\mathrm{U}_k(\mathbb{F}_q)$ (resp. $\mathrm{GU}_k(\mathbb{F}_q)$) associated to χ_λ by the bijection of 2.4 is denoted by $\rho_\lambda^{\mathrm{U}}$ (resp. $\rho_\lambda^{\mathrm{GU}}$). In virtue of 2.3, for every λ we have $\rho_\lambda^{\mathrm{U}} = \rho_\lambda^{\mathrm{GU}} \circ f$, where $f : \mathrm{U}_k(\mathbb{F}_q) \to \mathrm{GU}_k(\mathbb{F}_q)$ is the inclusion. Thus, it is harmless to identify $\rho_\lambda^{\mathrm{U}}$ and $\rho_\lambda^{\mathrm{GU}}$ so that from now on, we will omit the superscript. The partition (k) corresponds to the trivial representation and (1^k) to the Steinberg representation. The degree of the representations ρ_λ is given by expressions known as **hook formula**. Given a box $\square$ in the Young diagram of λ, its **hook length** $h(\square)$ is 1 plus the number of boxes lying below it or on its right. For instance, in the following figure the hook length of every box of the Young diagram of $\lambda = (3^2, 2^2, 1)$ has been written inside it.

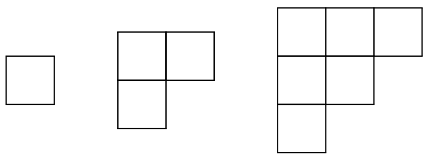
7	5	2
6	4	1
4	2	
3	1	
1		

Proposition ([GP00] Propositions 4.3.5)**.** *Let $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ be a partition of n. The degree of the irreducible unipotent representation ρ_λ is given by the following formula*

$$\deg(\rho_\lambda) = q^{a(\lambda)} \frac{\prod_{i=1}^{k} q^i - (-1)^i}{\prod_{\square \in \lambda} q^{h(\square)} - (-1)^{h(\square)}}$$

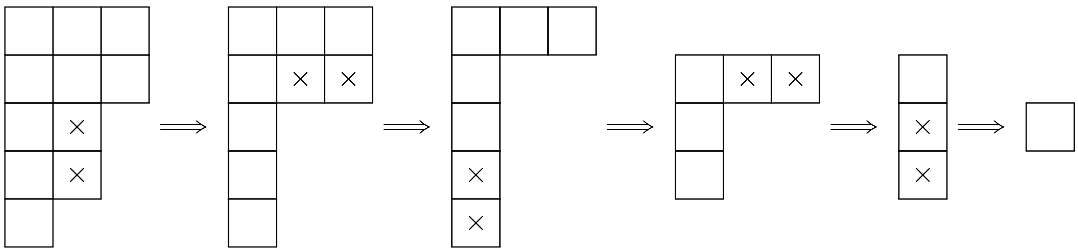
where $a(\lambda) = \sum_{i=1}^{r} (i-1)\lambda_i$.

2.7 We may describe the cuspidal support of the unipotent representations ρ_λ. According to [Lus77] 9.2 and 9.4 there exists an irreducible unipotent cuspidal character of $\mathrm{U}_k(\mathbb{F}_q)$ (or $\mathrm{GU}_k(\mathbb{F}_q)$) if and only if k is an integer of the form $k = \frac{t(t+1)}{2}$ for some $t \geqslant 0$, and when that is the case it is the one associated to the partition $\Delta_t := (t, t-1, \ldots, 1)$, whose Young diagram has the distinctive shape of a reversed staircase. Here, as a convention $U_0(\mathbb{F}_q) = \mathrm{GU}_0(\mathbb{F}_q)$ denotes the trivial group. For example, here are the Young diagrams of Δ_1, Δ_2 and Δ_3. Of course, the one of Δ_0 the empty diagram.



We consider an integer $t \geqslant 0$ such that k decomposes as $k = 2e + \frac{t(t+1)}{2}$ for some $e \geqslant 0$. Let G denote $\mathrm{U}_k(\mathbb{F}_q)$ or $\mathrm{GU}_k(\mathbb{F}_q)$, and consider L_t the subgroup consisting of block-diagonal matrices having one middle block of size $\frac{t(t+1)}{2}$ and all other blocks of size 1. This is a standard Levi subgroup of G. For $\mathrm{U}_k(\mathbb{F}_q)$, it is isomorphic to $\mathrm{GL}_1(\mathbb{F}_{q^2})^e \times \mathrm{U}_{\frac{t(t+1)}{2}}(\mathbb{F}_q)$ whereas in the case of $\mathrm{GU}_k(\mathbb{F}_q)$ it is isomorphic to $\mathrm{G}\left(\mathrm{GU}_1(\mathbb{F}_q)^e \times \mathrm{GU}_{\frac{t(t+1)}{2}}(\mathbb{F}_q)\right)$. In both cases, L_t admits a quotient which is isomorphic to a group of the same type as G but of size $\frac{t(t+1)}{2}$. We write ρ_t for the inflation to L_t of the unipotent cuspidal representation ρ_{Δ_t} of this quotient. If λ is a partition of k, the cuspidal support of the representation ρ_λ is given by exactly one of the pair (L_t, ρ_t) up to conjugacy, where $t \geqslant 0$ is an integer such that for some $e \geqslant 0$ we have $k = 2e + \frac{t(t+1)}{2}$. Note that in particular k and $\frac{t(t+1)}{2}$ must have the same parity. With these notations, the irreducible unipotent representations belonging to the principal series are those with cuspidal support (L_0, ρ_0) if k is even and (L_1, ρ_1) is k is odd.

2.8 Given an irreducible unipotent representation ρ_λ, there is a combinatorical way to determine the Harish-Chandra series to which it belongs, as we recalled in [Mul21] 4.6. We consider the Young diagram of λ. We call **domino** any pair of adjacent boxes in the diagram. It may be either vertical or horizontal. We remove dominoes from the diagram of λ so that the resulting shape is again a Young diagram, until one can not proceed further. This process results in the Young diagram of the partition Δ_t for some $t \geqslant 0$, and it is called the **2-core** of λ. It does not depend on the successive choices for the dominoes. Then, the representation ρ_λ has cuspidal support (L_t, ρ_t) if and only if λ has 2-core Δ_t. For instance, the diagram $\lambda = (3^2, 2^2, 1)$ given in 2.5 has 2-core Δ_1, as it can be determined by the following steps. We put crosses inside the successive dominoes that we remove from the diagram. Thus, the unipotent representation ρ_λ of $\mathrm{U}_{11}(\mathbb{F}_q)$ or $\mathrm{GU}_{11}(\mathbb{F}_q)$ has cuspidal support (L_1, ρ_1), so in particular it is a principal series representation.



2.9 From now on, we take $q = p$. We consider the ℓ-adic cohomology with compact support of a closed Bruhat-Tits stratum $\mathcal{M}_\Lambda \otimes \mathbb{F}$, where ℓ is a prime number different from p and $\Lambda \in \mathcal{L}$ has orbit type $t(\Lambda) = 2\theta + 1$, $0 \leqslant \theta \leqslant m$. Recall from 1.2.10 that the stratum $\mathcal{M}_\Lambda$ is equipped with an action of the finite group of Lie type $\mathrm{GU}(V_\Lambda^0) \simeq \mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$, and as such it is isomorphic to a Deligne-Lusztig variety. Let F be the Frobenius morphism of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ as defined in 2.3. Then F^2 induces a geometric Frobenius morphism $\mathcal{M}_\Lambda \otimes \mathbb{F} \to \mathcal{M}_\Lambda \otimes \mathbb{F}$ relative to the $\mathbb{F}_{p^2}$-structure of $\mathcal{M}_\Lambda$. Because it is a finite morphism, it induces a linear endomorphism on the cohomology groups, and it is in fact an automorphism. In [Mul21], we computed these cohomology groups in terms of a $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p) \times \langle F^2 \rangle$-representation.

Theorem. *Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$ for some $0 \leqslant \theta \leqslant m$.*

(1) *The cohomology group $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ is zero unless $0 \leqslant j \leqslant 2\theta$. There is an isomorphism*

$$\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}_c^{2\theta-j}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})^\vee(\theta)$$

which is equivariant for the action of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p) \times \langle F^2 \rangle$.

(2) *The Frobenius F^2 acts like multiplication by $(-p)^j$ on $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$.*

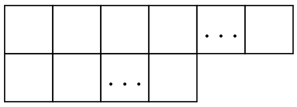
(3) *For $0 \leqslant j \leqslant \theta$ we have*

$$\mathrm{H}_c^{2j}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-j)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant j \leqslant \theta - 1$ we have

$$\mathrm{H}_c^{2j+1}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-1-j)} \rho_{(2\theta-2s,2s+1)}.$$

Thus, in the cohomology of $\mathcal{M}_\Lambda$ all the representations associated to a Young diagram with at most 2 rows occur, and there is no other. Such a diagram has the following general shape.



Remarks. Let us make a few comments.

- Part (1) of the theorem follows from general theory of etale cohomology given that the variety $\mathcal{M}_\Lambda$ is smooth and projective over $\mathbb{F}_{p^2}$. The identity is a consequence of Poincaré duality. The notation (θ) is a Tate twist, it modifies the action of F^2 by multiplying it with $p^{2\theta}$.
- The cohomology groups of index 0 and 2θ are the trivial representation of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$.
- All irreducible representations in the cohomology groups of even index belong to the unipotent principal series, whereas all the ones in the groups of odd index have cuspidal support (L_2, ρ_2).
- The cohomology group $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ contains no cuspidal representation of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ unless $\theta = j = 0$ or $\theta = j = 1$. If $\theta = 0$ then H_c^0 is the trivial representation of $\mathrm{GU}_1(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times$, and if $\theta = 1$ then H_c^1 is the representation ρ_{Δ_2} of $\mathrm{GU}_3(\mathbb{F}_p)$. Both of them are cuspidal.

3 Shimura variety and p-adic uniformization of the basic stratum

3.1 In this section, we introduce the PEL unitary Shimura variety with signature $(1, n-1)$ as in [VW11] 6.1 and 6.2, and we recall the p-adic uniformization theorem of its basic (or supersingular) locus. The Shimura variety can be defined as a moduli problem classifying abelian varieties with additional structures, as follows. Let E be a quadratic imaginary extension of $\mathbb{Q}$ in which p **is inert**. Let B/F be a simple central algebra of degree $d \geqslant 1$ which splits over p and at infinity. Let $*$ be a positive involution of the second kind on B, and let $\mathbb{V}$ be a non-zero finitely generated left B-module equipped with a non-degenerate $*$-alternating form $\langle \cdot, \cdot \rangle$ taking values in $\mathbb{Q}$. Assume also that $\dim_E(\mathbb{V}) = nd$. Let G be the connected reductive group over $\mathbb{Q}$ whose points over a $\mathbb{Q}$-algebra R are given by

$$G(R) := \{g \in \mathrm{GL}_{E \otimes R}(\mathbb{V} \otimes R) \,|\, \exists c \in R^\times \text{ such that for all } v, w \in \mathbb{V} \otimes R, \langle gv, gw \rangle = c\langle v, w \rangle\}.$$

We denote by $c : G \to \mathbb{G}_m$ the **multiplier** character. The base change $G_\mathbb{R}$ is isomorphic to a group of unitary similitudes $\mathrm{GU}(r, s)$ of a hermitian space with signature (r, s) where $r + s = n$. We assume that $r = 1$ and $s = n - 1$. We consider a Shimura datum of the form (G, X), where X denotes the unique $G(\mathbb{R})$-conjugacy class of homorphisms $h : \mathbb{C}^\times \to G_\mathbb{R}$ such that for all $z \in \mathbb{C}^\times$ we have $\langle h(z)\cdot, \cdot \rangle = \langle \cdot, h(\overline{z})\cdot \rangle$, and such that the $\mathbb{R}$-pairing $\langle \cdot, h(i)\cdot \rangle$ is positive definite. Such a homomorphism h induces a decomposition $\mathbb{V} \otimes \mathbb{C} = \mathbb{V}_1 \oplus \mathbb{V}_2$. Concretely, $\mathbb{V}_1$ (resp. $\mathbb{V}_2$) is the subspace where $h(z)$ acts like z (resp. like $\overline{z}$). The reflex field associated to this PEL data, that is the field of definition of $\mathbb{V}_1$ as a complex representation of B, is E unless $n = 2$ in which case it is $\mathbb{Q}$. Nonetheless, for simplicity we will consider the associated Shimura varieties over E even in the case $n = 2$.

Remark. As remarked in [Vol10] Section 6, the group G satisfies the Hasse principle, ie. $\ker^1(\mathbb{Q}, G)$ is a singleton. Therefore, the Shimura variety associated to the Shimura datum (G, X) coincides with the moduli space of abelian varieties that we are going to define.

3.2 Let $\mathbb{A}_f$ denote the ring of finite adèles over $\mathbb{Q}$ and let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup. We define a functor Sh_K by associating to an E-scheme S the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta})$ where

- A is an abelian scheme over S.
- $\lambda : A \to \widehat{A}$ is a polarization.
- $\iota : B \to \mathrm{End}(A) \otimes \mathbb{Q}$ is a morphism of algebras such that $\iota(b^*) = \iota(b)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ, and such that the Kottwitz determinant condition is satisfied:
$$\forall b \in B,\ \det(\iota(b)) = \det(b \,|\, \mathbb{V}_1).$$
- $\overline{\eta}$ is a K-level structure, that is a K-orbit of isomorphisms of $B\otimes\mathbb{A}_f$-modules $\mathrm{H}_1(A, \mathbb{A}_f) \xrightarrow{\sim} \mathbb{V} \otimes \mathbb{A}_f$ that is compatible with the other data.

The Kottwitz condition in the third point is independent on the choice of $h \in X$. If K is sufficiently small, this moduli problem is represented by a smooth quasi-projective scheme Sh_K over E. When the level K varies, the Shimura varieties form a projective system $(\mathrm{Sh}_K)_K$ equipped with an action of $G(\mathbb{A}_f)$ by Hecke correspondences.

3.3 We assume the existence of a $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ in B, stable under the involution $*$, such that its p-adic completion is a maximal order in $B_{\mathbb{Q}_p}$. We also assume that there is a $\mathbb{Z}_p$-lattice Γ in $\mathbb{V} \otimes \mathbb{Q}_p$, invariant under $\mathcal{O}_B$ and self-dual for $\langle \cdot, \cdot \rangle$. We may fix isomorphisms $E_p \simeq \mathbb{Q}_{p^2}$ and $B_{\mathbb{Q}_p} \simeq \mathrm{M}_d(\mathbb{Q}_{p^2})$ such that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is identified with $\mathrm{M}_d(\mathbb{Z}_{p^2})$.
As a consequence of the existence of Γ, the group $G_{\mathbb{Q}_p}$ is unramified. Let $K_0 := \mathrm{Fix}(\Gamma)$ be the subgroup of $G(\mathbb{Q}_p)$ consisting of all g such that $g \cdot \Gamma = \Gamma$. It is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. We will consider levels of the form $K = K_0 K^p$ where K^p is an open compact subgroup of $G(\mathbb{A}_f^p)$. Note that K is sufficiently small as soon as K^p is sufficiently small. By the work of Kottwitz in [Kot92], the Shimura varieties $\mathrm{Sh}_{K_0 K^p}$ admit integral models over $\mathcal{O}_{E,(p)}$ which have the following moduli interpretation. We define a functor S_{K^p} by associating to an $\mathcal{O}_{E,(p)}$-scheme S the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta}^p)$ where

- A is an abelian scheme over S.
- $\lambda : A \to \widehat{A}$ is a polarization whose order is prime to p.
- $\iota : \mathcal{O}_B \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is a morphism of algebras such that $\iota(b^*) = \iota(b)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ, and such that the Kottwitz determinant condition is satisfied:
$$\forall b \in \mathcal{O}_B,\ \det(\iota(b)) = \det(b \,|\, \mathbb{V}_1).$$

– $\overline{\eta}^p$ is a K^p-level structure, that is a K^p-orbit of isomorphisms of $B\otimes\mathbb{A}_f^p$-modules $\mathrm{H}_1(A,\mathbb{A}_f^p) \xrightarrow{\sim} \mathbb{V}\otimes\mathbb{A}_f^p$ that is compatible with the other data.

If K^p is sufficiently small, this moduli problem is also representable by a smooth quasi-projective scheme over $\mathcal{O}_{E,(p)}$. When the level K^p varies, these integral Shimura varieties form a projective system $(\mathrm{S}_{K^p})_{K^p}$ equipped with an action of $G(\mathbb{A}_f^p)$ by Hecke correspondences. We have a family of isomorphisms

$$\mathrm{Sh}_{K_0K^p} \simeq \mathrm{S}_{K^p} \otimes_{\mathcal{O}_{E,(p)}} E$$

which are compatible as the level K^p varies.

Notation. Unless explicitly mentioned, from now on the notation S_{K^p} will refer to the smooth quasi-projective $\mathbb{Z}_{p^2}$-scheme $\mathrm{S}_{K^p} \otimes_{\mathcal{O}_{E,(p)}} \mathbb{Z}_{p^2}$. Here, we implicitly use the identification of E_p with $\mathbb{Q}_{p^2}$.

Therefore, with this convention we have isomorphisms $\mathrm{Sh}_{K_0K^p}\otimes_E\mathbb{Q}_{p^2} \simeq \mathrm{S}_{\mathrm{K^p}}\otimes_{\mathbb{Z}_{p^2}}\mathbb{Q}_{p^2}$ compatible as the level K^p varies.

3.4 Let $\overline{\mathrm{S}}_{K^p} := \mathrm{S}_{K^p} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ denote the special fiber of the Shimura variety. It is a smooth quasi-projective variety over $\mathbb{F}_{p^2}$. Its geometry can be described in terms of the Newton stratification as follows. Recall the Shimura datum introduced in 3.1. To any homomorphism $h \in X$, we can associate the cocharacter

$$\mu_h : \mathbb{C}^\times \to G_{\mathbb{C}} = \bigsqcup_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} G_{\mathbb{R}}$$

which is given by $h : \mathbb{C}^\times \to G_{\mathbb{R}}$ into the summand corresponding to the identity in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The conjugacy class μ of μ_h is well-determined by X. The field of definition of μ is by definition the reflex field of the Shimura datum, that is E when $n \neq 2$ and $\mathbb{Q}$ otherwise. We fix an algebraic closure $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}_p}$) containing E (resp. $\mathbb{Q}_{p^2}$). We also fix an embedding $\nu : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ compatible with the identification $E_p \simeq \mathbb{Q}_{p^2}$. We may then consider the local datum $(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$ where $\mu_{\overline{\mathbb{Q}_p}}$ is the conjugacy class of cocharacters $\overline{\mathbb{Q}_p}^\times \to G_{\overline{\mathbb{Q}_p}}$ induced by μ and ν. Let $B(G_{\mathbb{Q}_p})$ denote the set of σ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$ where $\breve{\mathbb{Q}}_p := \widehat{W(\mathbb{F})_{\mathbb{Q}}}$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$. As in [Kot97], we may associate the Kottwitz set $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}}) \subset B(G_{\mathbb{Q}_p})$. It is a finite set equipped with a partial order. An element $b \in B(G_{\mathbb{Q}_p})$ is said to be $\mu_{\overline{\mathbb{Q}_p}}$**–admissible** when it belongs to $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$. The set $B(G_{\mathbb{Q}_p})$ (resp. $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$) canonically classifies the isomorphism classes of isocrystals with a $G_{\mathbb{Q}_p}$-structure (resp. compatible $\mu_{\overline{\mathbb{Q}_p}}, G_{\mathbb{Q}_p}$-structures).
Let $\mathcal{A}_{K^p}$ denote the universal abelian scheme over S_{K^p}, and let $\overline{\mathcal{A}}_{K^p}$ denote its reduction modulo p. The associated p-divisible group $\mathcal{A}_{K^p}[p^\infty]$ is denoted by X_{K^p}. For any geometric point $x \in \overline{\mathrm{S}}_{K^p}$, the p-divisible group $(X_{K^p})_x$ is equipped with compatible $\mu_{\overline{\mathbb{Q}_p}}, G_{\mathbb{Q}_p}$-structures therefore it determines an element $b_x \in B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$. For $b \in B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$, the set

$$\overline{\mathrm{S}}_{K^p}(b) := \{x \in \overline{\mathrm{S}}_{K^p} \mid b_x = b\}$$

is locally closed in $\overline{\mathrm{S}}_{K^p}$. It is the underlying topological space of a reduced subscheme which we still denote by $\overline{\mathrm{S}}_{K^p}(b)$. They are called the **Newton strata** of the special fiber of the Shimura variety. For a fixed b, as the level K^p varies the strata form a projective tower $(\overline{\mathrm{S}}_{K^p}(b))_{K^p}$ equipped with an action of $G(\mathbb{A}_f^p)$ by Hecke correspondences.

3.5 In [BW05], the combinatorics of the Newton stratification is described in the case of a PEL unitary Shimura variety of signature $(1, n-1)$. The set $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$ contains $\lfloor \frac{n}{2} \rfloor + 1$ elements $b_0 < b_1 < \ldots < b_{\lfloor \frac{n}{2} \rfloor}$ and we have

$$\overline{\mathrm{S}}_{K^p} = \bigsqcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} \overline{\mathrm{S}}_{K^p}(b_i).$$

The stratification is linear, that is the closure of a stratum $\overline{\mathrm{S}}_{K^p}(b_i)$ is the union of all the strata $\overline{\mathrm{S}}_{K^p}(b_j)$ for $j \leqslant i$. The stratum corresponding to b_i has dimension $m + i$. The element $b_{\lfloor \frac{n}{2} \rfloor}$ is μ-ordinary, and the corresponding stratum $\overline{\mathrm{S}}_{K^p}(b_{\lfloor \frac{n}{2} \rfloor})$ is called the **μ-ordinary locus**. It is open and dense in $\overline{\mathrm{S}}_{K^p}$. The unique basic element is b_0, and the corresponding stratum $\overline{\mathrm{S}}_{K^p}(b_0)$ is called the **basic stratum**. It coincides with the **supersingular locus**. It is a closed subscheme of $\overline{\mathrm{S}}_{K^p}$.

3.6 The geometry of the basic stratum can be described using the Rapoport-Zink space $\mathcal{M}$ in a process called p-adic uniformization, see [RZ96] and [Far04]. Let x be a geometric point of $\overline{\mathrm{S}}_{K^p}(b_0)$. Since G satisfies the Hasse principle, according to [Far04] Proposition 3.1.8 the isogeny class of the triple $(\mathcal{A}_x, \lambda, \iota)$, consisting of the abelian variety $\mathcal{A}_x$ together with its additional structures, does not depend on the choice of x. We define

$$I := \mathrm{Aut}(\mathcal{A}_x, \lambda, \iota).$$

It is a reductive group over $\mathbb{Q}$. In fact, since we are considering the basic stratum, according to loc. cit. the group I is the inner form of G such that $I(\mathbb{A}_f) = J \times G(\mathbb{A}_f^p)$ and $I(\mathbb{R}) \simeq \mathrm{GU}(0, n)$, that is the unique inner form of $G(\mathbb{R})$ which is compact modulo center. In particular, one can think of $I(\mathbb{Q})$ as a subgroup both of J and of $G(\mathbb{A}_f^p)$. Let $(\widehat{\mathrm{S}}_{K^p})_{|b_0}$ denote the formal completion of S_{K^p} along the basic stratum.

Theorem ([RZ96] Theorem 6.24)**.** *There is an isomorphism of formal schemes over* $\mathrm{Spf}(\mathbb{Z}_{p^2})$

$$\Theta_{K^p} : I(\mathbb{Q}) \backslash \left(\mathcal{M} \times G(\mathbb{A}_f^p)/K^p \right) \xrightarrow{\sim} (\widehat{\mathrm{S}}_{K^p})_{|b_0}$$

which is compatible with the $G(\mathbb{A}_f^p)$-action by Hecke correspondences as the level K^p varies.

This isomorphism is known as the **p-adic uniformization** of the basic stratum. The induced map on the special fiber is an isomorphism

$$(\Theta_{K^p})_s : I(\mathbb{Q}) \backslash \left(\mathcal{M}_{\mathrm{red}} \times G(\mathbb{A}_f^p)/K^p \right) \xrightarrow{\sim} \overline{\mathrm{S}}_{K^p}(b_0)$$

of schemes over $\mathrm{Spec}(\mathbb{F}_{p^2})$. We denote by $\mathcal{M}^{\mathrm{an}}$ (resp. $(\widehat{\mathrm{S}}_{K^p})_{|b_0}^{\mathrm{an}}$) the smooth analytic space over $\mathbb{Q}_{p^2}$ associated to the formal scheme $\mathcal{M}$ (resp. $(\widehat{\mathrm{S}}_{K^p})_{|b_0}$) by the Berkovich functor as defined in [Ber96]. Note that both formal schemes are special in the sense of loc. cit. so that we may use Berkovich's constructions. These analytic spaces play the role of the generic fibers of the formal schemes over $\mathrm{Spf}(\mathbb{Z}_{p^2})$. By [Far04] Théorème 3.2.6, p-adic uniformization induces an isomorphism

$$\Theta_{K^p}^{\mathrm{an}} : I(\mathbb{Q})\backslash \left(\mathcal{M}^{\mathrm{an}} \times G(\mathbb{A}_f^p)/K^p\right) \xrightarrow{\sim} (\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}$$

of analytic spaces over $\mathbb{Q}_{p^2}$. We denote by red the reduction map from the generic fiber to the special fiber. It is an anticontinuous map of topological spaces, which means that the preimage of an open subset is closed and the preimage of a closed subet is open. Then, the uniformization on the generic and special fibers are compatible in the sense that the diagram

$$\begin{array}{ccc} I(\mathbb{Q})\backslash \left(\mathcal{M}^{\mathrm{an}} \times G(\mathbb{A}_f^p)/K^p\right) & \xrightarrow{\Theta_{K^p}^{\mathrm{an}}} & (\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0} \\ \downarrow{\scriptstyle \mathrm{red}} & & \downarrow{\scriptstyle \mathrm{red}} \\ I(\mathbb{Q})\backslash \left(\mathcal{M}_{\mathrm{red}} \times G(\mathbb{A}_f^p)/K^p\right) & \xrightarrow{(\Theta_{K^p})_s} & \overline{\mathrm{S}}_{K^p}(b_0) \end{array}$$

is commutative.

3.7 The double coset space $I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)/K^p$ is finite, so that we may fix a system of representatives $g_1, \ldots, g_s \in G(\mathbb{A}_f^p)$. For every $1 \leqslant k \leqslant s$, we define $\Gamma_k := I(\mathbb{Q}) \cap g_k K^p g_k^{-1}$, which we see as a discrete subgroup of J that is cocompact modulo the center. The left hand side of the p-adic uniformization theorem is isomorphic to the disjoint union of the various quotients of $\mathcal{M}$ (or $\mathcal{M}_{\mathrm{red}}$ or $\mathcal{M}^{\mathrm{an}}$) by the subgroups $\Gamma_k \subset J$. In particular for the special fiber, it is an isomorphism

$$(\Theta_{K^p})_s : \bigsqcup_{k=1}^{s} \Gamma_k\backslash\mathcal{M}_{\mathrm{red}} \xrightarrow{\sim} \overline{\mathrm{S}}_{K^p}(b_0).$$

Let $\Phi_{K^p}^k$ be the composition $\mathcal{M}_{\mathrm{red}} \to \Gamma_k\backslash\mathcal{M}_{\mathrm{red}} \to \overline{\mathrm{Sh}}_{C^p}^{\mathrm{ss}}$ and let Φ_{K^p} be the disjoint union of the $\Phi_{K^p}^k$. The map Φ_{K^p} is surjective onto $\overline{\mathrm{S}}_{K^p}(b_0)$. According to [VW11] Section 6.4, it is a local isomorphism which can be used in order to transport the Bruhat-Tits stratification from $\mathcal{M}_{\mathrm{red}}$ to $\overline{\mathrm{S}}_{K^p}(b_0)$. Recall the notations of 1.2.3.

Proposition ([VW11] Proof of Proposition 6.5)**.** *Let $\Lambda \in \mathcal{L}$. For any $1 \leqslant k \leqslant s$, the restriction of $\Phi_{K^p}^k$ to $\mathcal{M}_\Lambda$ is an isomorphism onto its image.*

We will denote by $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ the scheme theoretic image of $\mathcal{M}_\Lambda$ through Φ^k. A subscheme of the form $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is called a **closed Bruhat-Tits stratum** of the Shimura variety. Together, they form the Bruhat-Tits stratification of the basic stratum, whose combinatorics is described by the union of the complexes $\Gamma_k\backslash\mathcal{L}$.

4 The cohomology of the Rapoport-Zink space at maximal level

4.1 The spectral sequence associated to an open cover of $\mathcal{M}^{\mathrm{an}}$

4.1.1 As in 3.6, we consider the generic fiber $\mathcal{M}^{\mathrm{an}}$ of the Rapoport-Zink space as a smooth Berkovich analytic space over $\mathbb{Q}_{p^2}$. Let $\mathrm{red} : \mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$ be the reduction map. If Z is a locally closed subset of $\mathcal{M}_{\mathrm{red}}$, then the preimage $\mathrm{red}^{-1}(Z)$ is called the **analytical tube over** Z. It is an analytic domain in $\mathcal{M}^{\mathrm{an}}$ and it coincides with the generic fiber of the formal completion of $\mathcal{M}_{\mathrm{red}}$ along Z. If $i \in \mathbb{Z}$ such that ni is even, then the tube $\mathrm{red}^{-1}(\mathcal{M}_i) = \mathcal{M}_i^{\mathrm{an}}$ is open and closed in $\mathcal{M}^{\mathrm{an}}$ and we have

$$\mathcal{M}^{\mathrm{an}} = \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{M}_i^{\mathrm{an}}.$$

If $\Lambda \in \mathcal{L}$, we define

$$U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda)$$

the tube over $\mathcal{M}_\Lambda$. The action of J on $\mathcal{M}$ induces an action on the generic fiber $\mathcal{M}^{\mathrm{an}}$ such that red is J-equivariant. By restriction it induces an action of J_Λ on U_Λ. The analytic space $\mathcal{M}^{\mathrm{an}}$ and each of the open subspaces U_Λ have dimension $n-1$.

4.1.2 We fix a prime number $\ell \neq p$. In [Ber93], Berkovich developped a theory of étale cohomology for his analytic spaces. Using it we may define the cohomology of the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ by the formula

$$\begin{aligned} \mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) &:= \varinjlim_U \mathrm{H}_c^\bullet(U \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \\ &= \varinjlim_U \varprojlim_n \mathrm{H}_c^\bullet(U \widehat{\otimes}\, \mathbb{C}_p, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \overline{\mathbb{Q}_\ell} \end{aligned}$$

where U goes over all relatively compact open of $\mathcal{M}^{\mathrm{an}}$. These cohomology groups are equipped with commuting actions of J and of W, the Weyl group of $\mathbb{Q}_{p^2}$. The J-action causes no problem of interpretation, but the W-action needs explanations. Let $\tau := \sigma^2$ be the Frobenius relative to $\mathbb{F}_{p^2}$. We fix a lift $\mathrm{Frob} \in W$ of the geometric Frobenius $\tau^{-1} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^2})$. The inertia subgroup $I \subset W$ acts on $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ via the coefficients $\mathbb{C}_p$, whereas Frob acts via the **Weil descent datum** defined by Rapoport and Zink in [RZ96] 3.48, as we explain now.
Recall the standard unitary p-divisible group $\mathbb{X}$ introduced in 1.1.1. Let

$$\mathcal{F}_{\mathbb{X}} : \mathbb{X} \otimes \mathbb{F} \to \tau^*(\mathbb{X} \otimes \mathbb{F})$$

denote the Frobenius morphism relative to $\mathbb{F}_{p^2}$. Let $(\mathcal{M} \widehat{\otimes}\, \mathcal{O}_{\breve{\mathbb{Q}}_p})^\tau$ be the functor defined by

$$(\mathcal{M} \widehat{\otimes}\, \mathcal{O}_{\breve{\mathbb{Q}}_p})^\tau(S) := \mathcal{M}(S_\tau)$$

for all $\mathcal{O}_{\breve{\mathbb{Q}}_p}$-scheme S where p is locally nilpotent. Here, S_τ denotes the scheme S but with structure morphism the composition $S \to \mathrm{Spec}(\mathcal{O}_{\breve{\mathbb{Q}}_p}) \xrightarrow{\tau} \mathrm{Spec}(\mathcal{O}_{\breve{\mathbb{Q}}_p})$. The Weil descent datum is the isomorphism $\alpha_{\mathrm{RZ}} : \mathcal{M}\widehat{\otimes}\mathcal{O}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} (\mathcal{M}\widehat{\otimes}\mathcal{O}_{\breve{\mathbb{Q}}_p})^{\tau}$ given by $(X,\iota,\lambda,\rho) \in \mathcal{M}(S) \mapsto (X,\iota,\lambda,\mathcal{F}_{\mathbb{X}}\circ\rho)$. We may describe this in terms of k-rational points, where k is a perfect field extension of $\mathbb{F}$. Since we use covariant Dieudonné theory, the relative Frobenius $\mathcal{F}_{\mathbb{X}}$ corresponds to the Verschiebung $\mathbf{V}^2$ in the Dieudonné module. By construction of $\mathbb{X}$, we have $\mathbf{V}^2 = p\tau^{-1}$. Therefore, if $S = \mathrm{Spec}(k)$ with $k/\mathbb{F}_{p^2}$ perfect, then α_{RZ} sends a Dieudonné module $M \in \mathcal{M}(k)$ to $p\tau^{-1}(M)$. Since Frob $\in W$ is a *geometric* Frobenius element, its action on the cohomology of $\mathcal{M}^{\mathrm{an}}$ is induced by the inverse $\alpha_{\mathrm{RZ}}^{-1}$.

Remark. The Rapoport-Zink space is defined over $\mathbb{Z}_{p^2}$ and this rational structure is induced by the effective descent datum $p\alpha_{\mathrm{RZ}}^{-1}$, with $p = p\cdot\mathrm{id}$ seen as an element of the center of J. It sends a point M to $\tau(M)$. Consequently, in the following we will write $\tau := (p^{-1}\cdot\mathrm{id}, \mathrm{Frob}) \in J\times W$, and we refer to it as the *rational Frobenius*. We note that $p^{-1}\cdot\mathrm{id}$ comes from contravariance of cohomology with compact support: the action of $g \in J$ on the cohomology of $\mathcal{M}^{\mathrm{an}}$ is induced by the action of g^{-1} on the space $\mathcal{M}^{\mathrm{an}}$.

Notation. In order to shorten the notations, we will omit the coefficients $\mathbb{C}_p$. Thefore we write $\mathrm{H}_c^{\bullet}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ and similarly for subspaces of $\mathcal{M}^{\mathrm{an}}$.

4.1.3 The cohomology groups $\mathrm{H}_c^{\bullet}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ are concentrated in degrees 0 to $2\dim(\mathcal{M}^{\mathrm{an}}) = 2(n-1)$. According to [Far04] Corollaire 4.4.7, these groups are smooth for the J-action and continous for the I-action. In a similar way as for $\mathcal{M}^{\mathrm{an}}$, we can also define the cohomology groups $\mathrm{H}_c^{\bullet}(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ for every $i \in \mathbb{Z}$ such that ni is even. The action of an element $g \in J$ induces an isomorphism

$$g : \mathrm{H}_c^{\bullet}(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^{\bullet}(\mathcal{M}_{i+\alpha(g)}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

In particular, the action of Frob gives an isomorphism from the cohomology of $\mathcal{M}_i^{\mathrm{an}}$ to that of $\mathcal{M}_{i+2}^{\mathrm{an}}$. Let $(J\times W)^{\circ}$ be the subgroup of $J\times W$ consisting of all elements of the form $(g, u\mathrm{Frob}^j)$ with $u \in I$ and $\alpha(g) = -2j$. In fact, we have $(J\times W)^{\circ} = (J^{\circ}\times I)\tau^{\mathbb{Z}}$. Each group $\mathrm{H}_c^{\bullet}(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is a $(J\times W)^{\circ}$-representation, and we have an isomorphism

$$\mathrm{H}_c^{\bullet}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{(J\times W)^{\circ}}^{J\times W}\, \mathrm{H}_c^{\bullet}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

In particular, when $\mathrm{H}_c^{k}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is non-zero it is infinite dimensional. However, by loc. cit. Proposition 4.4.13, these cohomology groups are always of finite type as J-modules.

4.1.4 In order to obtain information on the cohomology of $\mathcal{M}^{\mathrm{an}}$, we study the spectral sequence associated to the covering by the open subspaces U_Λ for $\Lambda \in \mathcal{L}$. The spaces U_Λ satisfy the same incidence relations as the $\mathcal{M}_\Lambda$, as described in 1.2.11 Theorem (1), (2) and (3). As a consequence, the open covering of $\mathcal{M}^{\mathrm{an}}$ by the $\{U_\Lambda\}$ is locally finite. For $i \in \mathbb{Z}$ such that ni is even and for $0 \leqslant \theta \leqslant m$, we denote by $\mathcal{L}_i^{(\theta)}$ the subset of $\mathcal{L}_i$ whose elements are those lattices of orbit type $2\theta+1$. We also write $\mathcal{L}^{(\theta)}$ for the union of the $\mathcal{L}_i^{(\theta)}$. Then $\{U_\Lambda\}_{\Lambda\in\mathcal{L}^{(m)}}$ is an open cover of $\mathcal{M}^{\mathrm{an}}$. We may apply [Far04] Proposition 4.2.2 to deduce the existence of the following spectral sequence computing the cohomology of the Rapoport-Zink space.

$$E_1^{a,b} : \bigoplus_{\gamma\in I_{-a+1}} \mathrm{H}_c^{b}(U(\gamma), \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

Here, for $s \geqslant 1$ the set I_s is defined by

$$I_s := \left\{ \gamma \subset \mathcal{L}^{(m)} \;\middle|\; \#\gamma = s \text{ and } U(\gamma) := \bigcap_{\Lambda \in \gamma} U_\Lambda \neq \varnothing \right\}.$$

Necessarily, if $\gamma \in I_s$ then there exists a unique i such that ni is even and $\gamma \subset \mathcal{L}_i^{(m)}$. For $\gamma \in I_s$ we also write

$$\Lambda(\gamma) := \bigcap_{\Lambda \in \gamma} \Lambda.$$

Then $\Lambda(\gamma) \in \mathcal{L}_i$ and $U(\gamma) = U_{\Lambda(\gamma)}$. In particular, the open subspace $U(\gamma)$ depends only on the intersection $\Lambda(\gamma)$ of the elements in γ. It follows from the definition and from 1.4.1 that if $s > \nu(m+1, 2m+1)$ then $I_s = \varnothing$. In particular, the spectral sequence is concentrated in the finite strip defined by the degrees $1 - \nu(m+1, 2m+1) \leqslant a \leqslant 0$ and $0 \leqslant b \leqslant 2(n-1)$. An element $g \in J$ acts on the set I_s by sending an element γ to $g \cdot \gamma = \{g \cdot \Lambda \,|\, \Lambda \in \gamma\}$. It induces an isomorphism

$$g : \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^b(U(g \cdot \gamma), \overline{\mathbb{Q}_\ell}).$$

The spectral sequence E is J-equivariant.

Remark. The map $p\alpha_{\mathrm{RZ}}^{-1}$ defines a Weil descent datum on $\mathcal{M}_\Lambda \otimes \mathbb{F}$ which is effective, and coincides with the natural $\mathbb{F}_{p^2}$-structure. Hence, the same holds for the analytical tube $U_\Lambda \widehat{\otimes} \mathbb{C}_p$. The descent datum $p\alpha_{\mathrm{RZ}}^{-1}$ induces the action of τ on the cohomology of U_Λ. If $\gamma \in I_{-a+1}$ then $p \cdot \gamma \in I_{-a+1}$. It follows that each term $E_1^{a,b}$ is equipped with an action of W. The spectral sequence E is in fact $J \times W$-equivariant.

4.1.5 An alternative is to restrict to the 0-th connected component $\mathcal{M}_0^{\mathrm{an}}$. Considering its open cover $\{U_\Lambda\}$ for $\Lambda \in \mathcal{L}_0^{(m)}$, we have a J°-equivariant spectral sequence

$$E_1'^{a,b} : \bigoplus_{\gamma \in I'_{-a+1}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}),$$

where I'_{-a+1} is defined as I_{-a+1} except that we only consider subsets γ of $\mathcal{L}_0^{(m)}$. As above, each term $E_1'^{a,b}$ is in fact a $(J \times W)^\circ$-representation, and the spectral sequence E' is $(J \times W)^\circ$-equivariant.

The compact induction operation $\mathrm{c} - \mathrm{Ind}_{J^\circ}^J$ defines an additive exact functor between the categories of smooth representations of J° and of J. The spectral sequences E and E' are related by

$$E = \mathrm{c} - \mathrm{Ind}_{J^\circ}^J E',$$

in the sense that all terms and differentials in the sequence E are the compact induction of the terms and differentials of E'. This observation will play a crucial role later in 5.2.5. We note that if we take the Galois action into consideration, then E is actually obtained from E' by applying $\mathrm{c} - \mathrm{Ind}_{(J \times W)^\circ}^{J \times W}$.

4.1.6 First we relate the cohomology of a tube U_Λ to the cohomology of the corresponding closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$. We observe that $\mathrm{H}_c^\bullet(U_\Lambda, \overline{\mathbb{Q}_\ell})$ is naturally a representation of the subgroup $(J_\Lambda \times I)\tau^{\mathbb{Z}} \subset J \times W$.

Proposition. *Let $\Lambda \in \mathcal{L}$ and let $0 \leqslant b \leqslant 2(n-1)$. There is a $(J_\Lambda \times I)\tau^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}^b(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$$

where, on the left-hand side, the inertia I acts trivially and τ acts like the geometric Frobenius F^2.

In particular, the inertia acts trivially on the cohomology of U_Λ.

Proof. Recall the notations of 3.7 regarding the Bruhat-Tits stratification on the Shimura variety $\overline{\mathrm{S}}_{K^p}$, where K^p is any open compact subgroup of $G(\mathbb{A}_f^p)$ that is small enough. Fix an integer $1 \leqslant k \leqslant s$ and consider the closed Bruhat-Tits stratum $\overline{\mathrm{S}}_{K^p,\Lambda,k}$, that is the isomorphic image of $\mathcal{M}_\Lambda$ through $\Phi_{K^p}^k$. Let $\mathrm{Sh}_{K^p,\Lambda,k}$ be the analytic tube of $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ inside $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}$. By compatibility of the p-adic uniformization, the tube $\mathrm{Sh}_{K^p,\Lambda,k}$ is the isomorphic image of U_Λ through $(\Phi_{K^p}^k)^{\mathrm{an}}$, which is the composition $\mathcal{M}^{\mathrm{an}} \to \Gamma_k \backslash \mathcal{M}^{\mathrm{an}} \to (\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}$. Thus, the following diagram is commutative.

$$\begin{array}{ccc} U_\Lambda & \xrightarrow{\sim} & \mathrm{Sh}_{K^p,\Lambda,k} \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ \mathcal{M}_\Lambda & \xrightarrow{\sim} & \overline{\mathrm{S}}_{K^p,\Lambda,k} \end{array}$$

Berkovich's comparison theorem gives the desired isomorphism. More precisely, let $\widehat{\mathrm{S}}_{K^p}$ denote the formal completion of the Shimura variety S_{K^p} along its special fiber. Since it is a smooth formal scheme over $\mathrm{Spf}(\mathbb{Z}_{p^2})$, we may apply [Ber96] Corollary 3.7 to deduce the existence of a natural isomorphism

$$\mathrm{H}^b(\overline{\mathrm{S}}_{K^p,\Lambda,k} \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^b(\mathrm{Sh}_{K^p,\Lambda,k}, \overline{\mathbb{Q}_\ell}).$$

This isomorphism is equivariant for the action of $(J_\Lambda \times I)\tau^{\mathbb{Z}}$, with the rational Frobenius τ on the right-hand side corresponding to F^2 on the left-hand side. □

Remark. It is a priori not possible to use Berkovich's result directly on the Rapoport-Zink space because $\mathcal{M}$ is not a smooth formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^2)$. In fact, it is not adic unless $n = 1$ or 2, see [Far04] Remarque 2.3.5. It is the reason why we have to introduce the Shimura variety in the proof.

Corollary. *Let $\Lambda \in \mathcal{L}$ and let $0 \leqslant b \leqslant 2(n-1)$. There is a $(J_\Lambda \times I)\tau^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})(n-1-\theta)$$

where $t(\Lambda) = 2\theta + 1$.

Proof. This is a consequence of algebraic and analytic Poincaré duality, respectively for U_Λ and for $\mathcal{M}_\Lambda$. Indeed, we have

$$\begin{aligned}\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) &\simeq \mathrm{H}^{2(n-1)-b}(U_\Lambda, \overline{\mathbb{Q}_\ell})^\vee(n-1)\\ &\simeq \mathrm{H}^{2(n-1)-b}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})^\vee(n-1)\\ &\simeq \mathrm{H}_c^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})(n-1-\theta).\end{aligned}$$

□

4.1.7 Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$. If λ is a partition of $2\theta + 1$, recall the unipotent irreducible representation ρ_λ of $\mathrm{GU}(V_\Lambda^{(0)}) \simeq \mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ that we introduced in 2.6. It can be inflated to the maximal reductive quotient $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$, and then to the maximal parahoric subgroup J_Λ. With an abuse of notation, we still denote this inflated representation by ρ_λ. In virtue of 2.9, the isomorphism in the last paragraph translates into the following result.

Proposition. *Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$. The following statements hold.*

(1) *The cohomology group $\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$ is zero unless $2(n-1-\theta) \leqslant b \leqslant 2(n-1)$.*
(2) *The action of J_Λ on the cohomology factors through an action of the finite group of Lie type $\mathrm{GU}(V_\Lambda^0)$. The rational Frobenius τ acts like multiplication by $(-p)^b$ on $\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$.*
(3) *For $0 \leqslant b \leqslant \theta$ we have*

$$\mathrm{H}_c^{2b+2(n-1-\theta)}(U_\Lambda, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-j)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant b \leqslant \theta - 1$ we have

$$\mathrm{H}_c^{2b+1+2(n-1-\theta)}(U_\Lambda, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-1-j)} \rho_{(2\delta-2s,2s+1)}.$$

In particular, all the terms $E_1^{a,b}$ in the first page of the spectral sequence are entirely understood.

4.1.8 We may now precisely locate the non-zero terms.

Corollary. *There is an equivalence*

$$E_1^{a,b} \text{ is non-zero } \iff 2(n-1-m) \leqslant b \leqslant 2(n-1) \text{ and } 1-\nu\left(\left\lfloor\frac{b}{2}\right\rfloor - m, 2\left\lfloor\frac{b}{2}\right\rfloor - (n-1)\right) \leqslant a \leqslant 0.$$

Proof. First we prove the reverse implication. We fix a lattice $\Lambda \in \mathcal{L}$ such that $t(\Lambda) = 2\theta + 1$ with $\theta = n - 1 - \left\lfloor\frac{b}{2}\right\rfloor$. This is possible since the condition on b implies $0 \leqslant \theta \leqslant m$. We have $1 \leqslant 1 - a \leqslant \nu\left(\left\lfloor\frac{b}{2}\right\rfloor - m, 2\left\lfloor\frac{b}{2}\right\rfloor - (n-1)\right)$. According to 1.4.1, the right-hand side is the number of lattices Λ' containing Λ and having maximal orbit type $2m + 1$. Therefore, we can consider any collection γ of $1 - a$ such lattices. Their intersection $\Lambda(\gamma)$ contains Λ, therefore $\Lambda(\gamma) \in \mathcal{L}$. It follows that $\gamma \in I_{-a+1}$, and the orbit type of $\Lambda(\gamma)$ is $2\theta' + 1$ with $\theta' \geqslant \theta = n - 1 - \left\lfloor\frac{b}{2}\right\rfloor$. Thus,

by the previous proposition the cohomology group $\mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ is non-zero, and it follows that $E_1^{a,b} \neq 0$.
We now prove the direct implication. By the previous proposition, since $E_1^{a,b} \neq 0$ there must be some $\gamma \in I_{-a+1}$ such that $\Lambda(\gamma)$ has orbit type $2\theta + 1$ with $2(n-1-\theta) \leqslant b \leqslant 2(n-1)$. Since $\theta \leqslant m$ the condition on b is satisfied, and because $I_{-a+1} \neq \varnothing$ we have $a \leqslant 0$. The $-a+1$ elements of γ are distinct lattices Λ' of maximal orbit type $2m+1$, all of which contains $\Lambda(\gamma)$. According to 1.4.1, there are $\nu(n-\theta-m-1, n-2\theta-1)$ such lattices in total. Therefore, we have $a \geqslant 1 - \nu(n-\theta-m-1, n-2\theta-1)$. By 1.4.2 we have

$$\nu(n-\theta-m-1, n-2\theta-1) = \frac{\prod_{j=1}^{2(m-\theta)} \left(p^{n-2m-1+j} - (-1)^{n-2m-1+j}\right)}{\prod_{j=1}^{m-\theta} \left(p^{2j}-1\right)}.$$

This fraction simplifies to $\prod_{j'=0}^{m-\theta-1} \left(p^{2j'+1}+1\right)$ when n is odd, and to $\prod_{j'=1}^{m-\theta} \left(p^{2j'+1}+1\right)$ when n is even. In particular, this expression is decreasing with θ. Since $\theta \geqslant n-1-\left\lfloor \frac{b}{2} \right\rfloor$, we have $\nu(n-\theta-m-1, n-2\theta-1) \leqslant \nu\left(\left\lfloor \frac{b}{2} \right\rfloor - m, 2\left\lfloor \frac{b}{2} \right\rfloor - (n-1)\right)$ and the result follows. □

Remark. In particular, the sequence is in general much "longer" than it is "high": the range of the b's is $2m+1$ whereas the range of the a's is at most $\nu(n-m-1, n-1)$, which is a rational fraction in p.

4.1.9 The description of the rational Frobenius action yields the following result.

Corollary. *The spectral sequence degenerates on the second page E_2. For $0 \leqslant b \leqslant 2(n-1)$, the induced filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ splits, ie. we have an isomorphism*

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b \leqslant b' \leqslant 2(n-1)} E_2^{b-b',b'}.$$

The action of W on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is trivial on the inertia subgroup and the action of the rational Frobenius element τ is semisimple. The subspace $E_2^{b-b',b'}$ is identified with the eigenspace of τ associated to the eigenvalue $(-p)^{b'}$.

Remark. In the previous statement, the terms $E_2^{b-b',b'}$ may be zero.

Proof. The (a,b)-term in the first page of the spectral sequence is the direct sum of the cohomology groups $H_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ for all $\gamma \in I_{-a+1}$. On each of these cohomology groups, the rational Frobenius τ acts like multiplication by $(-p)^b$. This action is in particular independant of γ and of a. Thus, on the b-th row of the first page of the sequence, the Frobenius acts everywhere as multiplication by $(-p)^b$. Starting from the second page, the differentials in the sequence connect two terms lying in different rows. Since the differentials are equivariant for the τ-action, they must all be zero. Thus, the sequence degenerates on the second page. By the machinery of spectral sequences, there is a filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ whose graded factors are given by the terms $E_2^{b-b',b'}$ of the second page. Only a finite number of these terms are non-zero, and since they all lie on different rows, the Frobenius τ acts like multiplication by a different scalar on each graded factor of the filtration. It follows that the filtration splits, ie. the abutment is the direct sum of the graded pieces of the filtration, as they correspond to the eigenspaces of τ. Consequently, its action is semisimple. □

Since the sequence computes the cohomology of the Rapoport-Zink space and it degenerates on the second page, the non-zero terms $E_2^{a,b}$ must be in the locus delimited by $0 \leqslant a+b \leqslant 2(n-1)$. Hence, even though the first page is very "long", most of the terms must vanish in the second page.

4.1.10 In order to study the action of J we need to rearrange the terms $E_1^{a,b}$ of the spectral sequence. To do this, let us introduce a few more notations. For $0 \leqslant \theta \leqslant m$ and $s \geqslant 1$, we define

$$I_s^{(\theta)} := \{\gamma \in I_s \,|\, t(\Lambda(\gamma)) = 2\theta + 1\}.$$

The subset $I_s^{(\theta)} \subset I_s$ is stable under the action of J, and it may be empty if s is too big with respect to θ. We denote by $N(\Lambda_\theta)$ the set $N(n-\theta-m-1, V_\theta^1)$ as defined in paragraph 1.4.1. It corresponds to the set of lattices $\Lambda \in \mathcal{L}_0$ of maximal orbit type $t(\Lambda) = 2m+1$ containing Λ_θ. It is finite of cardinality $\nu(n-\theta-m-1, n-2\theta-1)$. For $s \geqslant 1$ we define

$$K_s^{(\theta)} := \{\delta \subset N(\Lambda_\theta) \,|\, |\delta| = s \text{ and } \Lambda(\delta) = \Lambda_\theta\}.$$

Then $K_s^{(\theta)}$ is a finite subset of $I_s^{(\theta)}$ and it is stable under the action of J_θ. We also write

$$k_{s,\theta} := \# K_s^{(\theta)}.$$

If $\gamma \in I_s^{(\theta)}$, there exists some $g \in J$ such that $g \cdot \Lambda(\gamma) = \Lambda_\theta$ because both lattices share the same orbit type. Moreover, the coset $J_\theta \cdot g$ is uniquely determined, and $g \cdot \gamma$ is an element of $K_s^{(\theta)}$. This mapping results in a natural bijection between the orbit sets

$$J \backslash I_s^{(\theta)} \xrightarrow{\sim} J_\theta \backslash K_s^{(\theta)}.$$

The bijection sends the orbit $J \cdot \alpha$ to the orbit $J_\theta \cdot (g \cdot \alpha)$ where g is chosen as above. The inverse sends an orbit $J_\theta \cdot \beta$ to $J \cdot \beta$. We note that both orbit sets are finite.

4.1.11 We may now rearrange the terms in the spectral sequence.

Proposition. *We have an equality*

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \left(\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})\right)^{k_{-a+1,\theta}}.$$

Remark. For a given b, the cohomology group $\mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})$ is zero as soon as $\theta < n-1-\frac{b}{2}$. Thus, the direct sum can start at $\theta = c$ where $c := n-1-\lfloor \frac{b}{2} \rfloor$ instead.

Proof. First, by decomposing I_{-a+1} as the disjoint union of the $I_{-a+1}^{(\theta)}$ for $0 \leqslant \theta \leqslant m$, we may write

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \bigoplus_{\gamma \in I_{-a+1}^{(\theta)}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}).$$

For each orbit $X \in J\backslash I^{(\theta)}_{-a+1}$, we fix a representative δ_X which lie in $K^{(\theta)}_{-a+1}$. Thus, δ_X is also a representative of the J_θ-orbit that is the image of X via $J\backslash I^{(\theta)}_{-a+1} \xrightarrow{\sim} J_\theta\backslash K^{(\theta)}_{-a+1}$. We may write

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \bigoplus_{X\in J\backslash I^{(\theta)}_{-a+1}} \bigoplus_{\gamma\in X} \mathrm{H}^b_c(U(\gamma), \overline{\mathbb{Q}_\ell}) = \bigoplus_{\theta=0}^{m} \bigoplus_{X\in J\backslash I^{(\theta)}_{-a+1}} \bigoplus_{g\in J/\mathrm{Fix}(\delta_X)} g\cdot \mathrm{H}^b_c(U(\delta_X), \overline{\mathbb{Q}_\ell})$$

Here, $\mathrm{Fix}(\delta_X)$ denotes the fixator of δ_X in J, that is the subgroup of all $g \in J$ such that $g\cdot\delta_X = \delta_X$. We have the inclusions $\mathrm{Fix}(\delta_X) \subset J_\theta \subset J$. Indeed, since $\delta_X \in K^{(\theta)}_{-a+1}$ we have $\Lambda(\delta_X) = \Lambda_\theta$. Any $g \in \mathrm{Fix}(\delta_X)$ fixes the intersection $\Lambda(\delta_X)$ of the elements in δ_X, thus it fixes Λ_θ. Hence, we may see the quotient $J_\theta/\mathrm{Fix}(\delta_X)$ as a subset of $J/\mathrm{Fix}(\delta_X)$. Since $U(\delta_X) = U_{\Lambda_\theta}$, if $g \in J_\theta/\mathrm{Fix}(\delta_X)$ then $g\cdot \mathrm{H}^b_c(U(\delta_X), \overline{\mathbb{Q}_\ell}) = \mathrm{H}^b_c(U(\delta_X), \overline{\mathbb{Q}_\ell})$. Thus, we may take the sum over the quotient

$$(J/\mathrm{Fix}(\delta_X))\,/\,(J_\theta/\mathrm{Fix}(\delta_X)) \simeq J/J_\theta$$

without forgetting the multiplicity $k_X := \#\left(J_\theta/\mathrm{Fix}(\delta_X)\right)$. Thus, we have

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \bigoplus_{X\in J_\theta\backslash K^{(\theta)}_{-a+1}} \bigoplus_{g\in J/J_\theta} \left(g\cdot \mathrm{H}^b_c(U(\delta_X), \overline{\mathbb{Q}_\ell})\right)^{k_X}.$$

The third sum can be interpreted as a compact induction from J_θ to J. Using the fact that $U(\delta_X) = U_{\Lambda_\theta}$ does not depend on the orbit X, we have

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \bigoplus_{X\in J_\theta\backslash K^{(\theta)}_{-a+1}} \left(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})\right)^{k_X}.$$

Eventually, since there is no dependance in X the second sum only contributes to the multiplicity. We notice that

$$\sum_{X\in J_\theta\backslash K^{(\theta)}_{-a+1}} k_X = \# K^{(\theta)}_{-a+1} = k_{-a+1,\theta}$$

and hence, we obtain the desired formula

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \left(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})\right)^{k_{-a+1,\theta}}.$$

□

4.1.12 We do not have an explicit formula computing the integers $k_{s,\theta}$. However, they satisfy the following properties.

Proposition. *We write $\nu_\theta = \nu(n-\theta-m-1, n-2\theta-1)$. The following statements hold.*

- *We have $k_{s,\theta} = 0$ if $s \leqslant 0$, or if $s > \nu_\theta$, or if $s = 1$ and $\theta \neq m$.*
- *We have $k_{1,m} = 1$ and $k_{\nu_\theta,0} = 1$.*
- *We have $k_{s,\theta} \leqslant \binom{\nu_\theta}{s}$.*

Proof. It follows directly from the definition of the set $K^{(\theta)}_s$. □

We may give an interpretation of the integer $k_{s,\theta}$ in terms of linear algebra. The set $N(\Lambda_\theta)$ is identified with the set of subspaces U of V_θ^1 of dimension $n-\theta-m-1$ containing their orthogonal. We note that the space V_θ^1 has dimension $n-2\theta-1$, that is $2(m-\theta)$ if n is odd and $2(m-\theta)+1$ if n is even. Then, the subspace U has half dimension $m-\theta$ if n is odd and dimension $m-\theta+1$ if n is even. In particular, if n is odd the condition requires that U is equal to its own orthogonal. The set $K_s^{(\theta)}$ corresponds to the set of collections of s such subspaces whose intersection is $\{0\}$.

4.1.13 In general, the first page of the spectral sequence contains many non-zero arrows so that it is difficult to make out what $E_2^{a,b}$ is. If $m \geqslant 1$, that is $n \geqslant 3$, we notice that there are exactly two terms $E_1^{a,b}$ which are not the start nor the target of any non-zero arrow. They correspond to $a=0$ and $b \in \{2(n-1-m), 2(n-1-m)+1\}$. Note that

$$2(n-1-m) = \begin{cases} n-1 & \text{when } n \text{ is odd,} \\ n & \text{when } n \text{ is even.} \end{cases}$$

Thus $2(n-1-m)$ is the middle degree of the cohomology of $\mathcal{M}^{\mathrm{an}}$ when n is odd, and it is one plus the middle degree when n is even.
In the case $m=0$, the spectral sequence E_1 consists of a single non-zero term and the same observation can be made. With the notations of 4.1.7, we deduce the following statement.

Proposition. *We have an isomorphism of J-representations*

$$E_2^{0,2(n-1-m)} \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \rho_{(2m+1)}.$$

If $m \geqslant 1$ then we also have an isomorphism

$$E_2^{0,2(n-1-m)+1} \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \rho_{(2m,1)}.$$

Remark. The representation $\rho_{(2m+1)}$ is the trivial representation of J_m.

4.1.14 Lastly, we remark that the cohomology group of highest degree $\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ can be computed as a J-representation without the use of the spectral sequence.

Proposition. *There is an isomorphism*

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J} \mathbf{1}$$

where $\mathbf{1}$ denotes the trivial representation of J°.

Proof. When $n=1$ or 2, we have $J^\circ = J_m$ therefore the statement follows from the last paragraph. In general, we have an isomorphism of J-representations

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J^\circ}^{J} \mathrm{H}_c^{2(n-1)}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

Since the space $\mathcal{M}_0^{\mathrm{an}}$ is smooth, Poincaré duality gives a J-equivariant isomorphism

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^0(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})^\vee.$$

Since $\mathcal{M}_0^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p$ is connected, this group is the trivial representation of J. □

If we take the W-action into account, the isomorphism can be rewritten as $\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{(J\times W)^\circ}^{J\times W} \mathbf{1}$ where an element $(g, u\mathrm{Frob}^j) \in (J\times W)^\circ$ with $u \in I$ acts on $\mathbf{1}$ like multiplication by $p^{2(n-1)j}$.

4.2 Compactly induced representations and type theory

4.2.1 Let $\mathrm{Rep}(J)$ denote the category of smooth $\overline{\mathbb{Q}_\ell}$-representations of G. Let χ be a continuous character of the center $\mathrm{Z}(J) \simeq \mathbb{Q}_{p^2}$ and let $V \in \mathrm{Rep}(J)$. We define **the maximal quotient of V on which the center acts like** χ as follows. Let us consider the set

$$\Omega := \{W \mid W \text{ is a subrepresentation of } V \text{ and } \mathrm{Z}(J) \text{ acts like } \chi \text{ on } V/W\}.$$

The set Ω is stable under arbitrary intersection, so that $W_\circ := \bigcap_{W\in\Omega} W \in \Omega$. The maximal quotient is defined by

$$V_\chi := V/W_\circ.$$

It satisfies the following universal property.

Proposition. *Let χ be a continuous character of $\mathrm{Z}(J)$ and let $V, V' \in \mathrm{Rep}(J)$. Assume that $\mathrm{Z}(J)$ acts like χ on V'. Then any morphism $V \to V'$ factors through V_χ.*

Proof. Let $f : V \to V'$ be a morphism of J-representations. Since $V/\mathrm{Ker}(f) \simeq \mathrm{Im}(f) \subset V'$, the center $\mathrm{Z}(J)$ acts like χ on the quotient $V/\mathrm{Ker}(f)$. Therefore $\mathrm{Ker}(f) \in \Omega$. It follows that $\mathrm{Ker}(f)$ contains $W_\circ$ and as a consequence, f factors through V_χ. □

4.2.2 As representations of J, the terms $E_1^{a,b}$ of the spectral sequence 4.1.4 consist of representations of the form

$$\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \rho,$$

where ρ is the inflation to J_θ of a representation of the finite group of Lie type $\mathcal{J}_\theta$. We note that such a compactly induced representation does not contain any smooth irreducible subrepresentation of J. Indeed, the center $\mathrm{Z}(J) \simeq \mathbb{Q}_{p^2}^\times$ does not fix any finite dimensional subspace. In order to rectify this, it is customary to fix a continuous character χ of $\mathrm{Z}(J)$ which agrees with the central character of ρ on $\mathrm{Z}(J) \cap J_\theta \simeq \mathbb{Z}_{p^2}^\times$, and to describe the space $(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \rho)_\chi$ instead.

Lemma. *The representation $(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \rho)_\chi$ of J is isomorphic to the compactly induced representation $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J} \chi\otimes\rho$.*

Proof. By Frobenius reciprocity, the identity map on $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J} \chi\otimes\rho$ gives a morphism $\chi\otimes\rho \to \left(\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J} \chi\otimes\rho\right)_{|\mathrm{Z}(J)J_\theta}$ of $\mathrm{Z}(J)J_\theta$-representations. Restricting further to J_θ, we obtain a morphism $\rho \to \left(\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J} \chi\otimes\rho\right)_{|J_\theta}$. By Frobenius reciprocity, this corresponds to a morphism $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \rho \to \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J} \chi\otimes\rho$ of J-representations. Because $\mathrm{Z}(J)$ acts via the character χ on the target space, this morphism factors through a map $(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \rho)_\chi \to$

$\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho$. In order to prove that this is an isomorphism, we build its inverse. The quotient morphism $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi$ corresponds, via Frobenius reciprocity, to a morphism $\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_{\chi\,|J_\theta}$ of J_θ-representations. Because $\mathrm{Z}(J)$ acts via the character χ on the target space, this arrow may be extended to a morphism $\chi\otimes\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_{\chi\,|\mathrm{Z}(J)J_\theta}$ of $\mathrm{Z}(J)J_\theta$-representations. By Frobenius reciprocity, this corresponds to a morphism $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi$, and this is our desired inverse. □

4.2.3 We recall a general theorem from [Bus90] describing certain compactly induced representations. In this paragraph only, let G be any p-adic group, and let L be an open subgroup of G which contains the center $\mathrm{Z}(G)$ and which is compact modulo $\mathrm{Z}(G)$.

Theorem ([Bus90] Theorem 2 (supp)). *Let (σ, V) be an irreducible smooth representation of L. There is a canonical decomposition*

$$\mathrm{c}-\mathrm{Ind}_{L}^{G}\sigma \simeq V_0 \oplus V_\infty$$

where V_0 is the sum of all supercuspidal subrepresentations of $\mathrm{c}-\mathrm{Ind}_{L}^{G}\sigma$, and where V_∞ contains no non-zero admissible subrepresentation. Moreover, V_0 is a finite sum of irreducible supercuspidal subrepresentations of G.

The spaces V_0 or V_∞ could be zero. Note also that since G is p-adic, any irreducible representation is admissible. So in particular, V_∞ does not contain any irreducible subrepresentation. However, it may have many irreducible quotients and subquotients. Thus, the space V_∞ is in general not G-semisimple. Hence, the structure of the compactly induced representation $\mathrm{c}-\mathrm{Ind}_{L}^{G}\sigma$ heavily depends on the supercuspidal supports of its irreducible subquotients.
We go back to our previous notations. Let $0 \leqslant \theta \leqslant m$, let ρ be a smooth irreducible representation of J_θ and let χ be a character of $\mathrm{Z}(J)$ agreeing with the central character of ρ on $\mathrm{Z}(J) \cap J_\theta$. Since the group $\mathrm{Z}(J)J_\theta$ contains the center and is compact modulo the center, we have a canonical decomposition

$$(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi \simeq V_{\rho,\chi,0} \oplus V_{\rho,\chi,\infty}.$$

In order to describe the spaces $V_{\rho,\chi,0}$ and $V_{\rho,\chi,\infty}$, we determine the supercuspidal supports of the irreducible subquotients of $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho$ through type theory, with the assumption that ρ is inflated from $\mathcal{J}_\theta$.

4.2.4 In the following paragraphs, we recall a few general facts from type theory. For more details, we refer to [BK98] and [Mor99]. Let G be the group of F-rational points of a reductive connected group $\mathbf{G}$ over a p-adic field F. A parabolic subgroup P (resp. Levi complement L) of G is defined as the group of F-rational points of an F-rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ (resp. an F-rational Levi complement $\mathbf{L} \subset \mathbf{G}$). Every parabolic subgroup P admits a Levi decomposition $P = LU$ where U is the unipotent radical of P. We denote by $X_F(G)$ the set of F-rational $\overline{\mathbb{Q}_\ell}$-characters of $\mathbf{G}$, and by $X^{\mathrm{un}}(G)$ the set of **unramified characters** of G, ie. the continuous characters of G which are trivial on all compact subgroups. We consider pairs (L,τ) where L is a Levi complement of G and τ is a supercuspidal representation of L. Two pairs (L,τ) and (L',τ') are said to be **inertially equivalent** if for some $g \in G$ and $\chi \in X^{\mathrm{un}}(G)$ we have $L' = L^g$ and $\tau' \simeq \tau^g \otimes \chi$ where τ^g is the representation of L^g defined by $\tau^g(l) := \tau(g^{-1}lg)$. This is an equivalence relation, and we denote by $[L,\tau]_G$ or $[L,\tau]$ the inertial equivalence class of (L,τ) in G. The set of all inertial equivalence classes is denoted $\mathrm{IC}(G)$. If P is a parabolic subgroup of G, we write ι_P^G for the normalised parabolic induction functor. Any smooth irreducible representation π of G is isomorphic to a subquotient of some parabolically induced representation $\iota_P^G(\tau)$ where $P = LU$ for some Levi complement L and τ is a supercuspidal representation of L. We denote by $\ell(\pi) \in \mathrm{IC}(G)$ the inertial equivalence class $[L,\tau]$. This is uniquely determined by π and it is called the **inertial support** of π.

4.2.5 Let $\mathfrak{s} \in \mathrm{IC}(G)$. We denote by $\mathrm{Rep}^{\mathfrak{s}}(G)$ the full subcategory of $\mathrm{Rep}(G)$ whose objects are the smooth representations of G all of whose irreducible subquotients have inertial support $\mathfrak{s}$. This definition corresponds to the one given in [BD84] 2.8. If $\mathfrak{S} \subset \mathrm{IC}(G)$, we write $\mathrm{Rep}^{\mathfrak{S}}(G)$ for the direct product of the categories $\mathrm{Rep}^{\mathfrak{s}}(G)$ where $\mathfrak{s}$ runs over $\mathfrak{S}$. We recall the main results from loc. cit.

Theorem ([BD84] 2.8 and 2.10)**.** *The category* $\mathrm{Rep}(G)$ *decomposes as the direct product of the subcategories* $\mathrm{Rep}^{\mathfrak{s}}(G)$ *where* $\mathfrak{s}$ *runs over* $\mathrm{IC}(G)$*. Moreover, if* $\mathfrak{S} \subset \mathrm{IC}(G)$ *then the category* $\mathrm{Rep}^{\mathfrak{S}}(G)$ *is stable under direct sums and subquotients.*

Type theory was then introduced in [BK98] in order to describe the categories $\mathrm{Rep}^{\mathfrak{s}}(G)$ which are called the **Bernstein blocks**.

4.2.6 Let $\mathfrak{S}$ be a subset of $\mathrm{IC}(G)$. A **$\mathfrak{S}$-type** in G is a pair (K,ρ) where K is an open compact subgroup of G and ρ is a smooth irreducible representation of K, such that for every smooth irreducible representation π of G we have

$$\pi_{|K} \text{ contains } \rho \iff \ell(\pi) \in \mathfrak{S}.$$

When $\mathfrak{S}$ is a singleton $\{\mathfrak{s}\}$, we call it an $\mathfrak{s}$-type instead.

Remark. By Frobenius reciprocity, the condition that $\pi_{|K}$ contains ρ is equivalent to π being isomorphic to an irreducible quotient of $\mathrm{c}-\mathrm{Ind}_K^G\,\rho$. In fact, we can say a little bit more. Let K be an open compact subgroup of G and let ρ be an irreducible smooth representation of K. Let $\mathrm{Rep}_\rho(G)$ denote the full subcategory of $\mathrm{Rep}(G)$ whose objects are those representations which are generated by their ρ-isotypic component. If (K,ρ) is an $\mathfrak{S}$-type, then [BK98] Theorem 4.3 establishes the equality of categories $\mathrm{Rep}_\rho(G) = \mathrm{Rep}^{\mathfrak{S}}(G)$. By definition of compact induction, the representation $\mathrm{c}-\mathrm{Ind}_K^G\,\rho$ is generated by its ρ-isotypic vectors. Therefore any irreducible subquotient of $\mathrm{c}-\mathrm{Ind}_K^G\,\rho$ has inertial support in $\mathfrak{S}$.

4.2.7 An important class of types are those of depth zero, and they are the only ones we shall encounter. First, we recall the following result. If K is a parahoric subgroup of G, we denote by $\mathcal{K}$ its maximal reductive quotient. It is a finite group of Lie type over the residue field of F.

Proposition ([Mor99] 4.1)**.** *Let K be a maximal parahoric subgroup of G and let ρ be an irreducible cuspidal representation of $\mathcal{K}$. We see ρ as a representation of K by inflation. Let π be an irreducible smooth representation of G and assume that $\pi_{|K}$ contains ρ. Then π is supercuspidal and there exists an irreducible smooth representation $\tilde{\rho}$ of the normalizer $\mathrm{N}_G(K)$ such that $\tilde{\rho}_{|K}$ contains ρ and $\pi \simeq \mathrm{c}-\mathrm{Ind}_{N_G(K)}^G\tilde{\rho}$.*

Such representations π are called **depth-0 supercupidal representations** of G. More generally, a smooth irreducible representation π of G is said to be of **depth-0** if it contains a non-zero vector that is fixed by the pro-unipotent radical of some parahoric subgroup of G. A **depth-0 type** in G is a pair (K, ρ) where K is a parahoric subgroup of G and ρ is an irreducible cuspidal representation of $\mathcal{K}$, inflated to K. The name is justified by the following theorem.

Theorem ([Mor99] 4.8)**.** *Let (K, ρ) be a depth-0 type. Then there exists a (unique) finite set $\mathfrak{S} \subset \mathrm{IC}(G)$ such that (K, ρ) is an $\mathfrak{S}$-type of G.*

In loc. cit. it is also proved that any depth-0 supercuspidal representation of G contains a unique conjugacy class of depth-0 types. Let K be a parahoric subgroup of G. Using the Bruhat-Tits building of G, one may canonically associate a Levi complement L of G such that $K_L := L \cap K$ is a maximal parahoric subgroup of L, whose maximal reductive quotient $\mathcal{K}_L$ is naturally identified with $\mathcal{K}$. This is precisely described in [Mor99] 2.1. Moreover, we have $L = G$ if and only if K is a maximal parahoric subgroup of G. Now, let (K, ρ) be a depth-0 type of G and denote by $\mathfrak{S}$ the finite subset of $\mathrm{IC}(G)$ such that it is an $\mathfrak{S}$-type of G. Since ρ is a cuspidal representation of $\mathcal{K} \simeq \mathcal{K}_L$, we may inflate it to K_L. Then, the pair (K_L, ρ) is a depth-0 type of L. We say that (K, ρ) is a G**-cover** of (K_L, ρ). By the previous theorem, there is a finite set $\mathfrak{S}_L \subset \mathrm{IC}(L)$ such that (K_L, ρ) is an $\mathfrak{S}_L$-type of L. Then the proof of Theorem 4.8 in [Mor99] shows that we have the relation

$$\mathfrak{S} = \left\{[M, \tau]_G \,\middle|\, [M, \tau]_L \in \mathfrak{S}_L\right\}.$$

In this set, M is some Levi complement of L, therefore it may also be seen as a Levi complement in G. Thus, an inertial equivalence class $[M, \tau]_L$ in L gives rise to a class $[M, \tau]_G$ in G. Because K_L is maximal in L, in virtue of the proposition above any element of $\mathfrak{S}_L$ has the form $[L, \pi]_L$ for some supercuspidal representation π of L. In particular, every smooth irreducible representation of G containing the type (K, ρ) has a conjugate of L as cuspidal support. We deduce the following corollary.

Corollary. *Let (K, ρ) be a depth-0 type in G and assume that K is not a maximal parahoric subgroup. Then no smooth irreducible representation π of G containing the type (K, ρ) is supercuspidal.*

4.2.8 Thus, up to replacing G with a Levi complement, the study of any depth-0 type (K, ρ) can be reduced to the case where K is a maximal parahoric subgroup. Let us assume that it is the case, and let $\mathfrak{S}$ be the associated finite subset of $\mathrm{IC}(G)$. While $\mathfrak{S}$ is in general not a singleton, it becomes one once we modify the pair (K, ρ) a little bit. Let $\widehat{K}$ be the maximal open compact subgroup of $\mathrm{N}_G(K)$. We have $K \subset \widehat{K}$ but in general this inclusion may be strict. Let $\widehat{\rho}$ be an irreducible component of the restriction $\tilde{\rho}_{|\widehat{K}}$.

Theorem ([Mor99] Variant 4.7)**.** *The pair* $(\widehat{K}, \widehat{\rho})$ *is a* $[G, \pi]$*-type.*

The conclusion does not depend on the choice of $\widehat{\rho}$ as an irreducible component of $\tilde{\rho}_{|\widehat{K}}$. Any one of them affords a type for the same singleton $\mathfrak{s} = [G, \pi]$.

4.2.9 We go back to the context of the unitary similitudes group J. Let $0 \leqslant \theta \leqslant m$ and let λ be a partition of $2\theta + 1$. Let ρ_λ be the irreducible unipotent representation of $\mathrm{GU}(V_\theta^0)$ attached to λ as in 2.6. We still denote by ρ_λ its inflation to the maximal reductive quotient $\mathcal{J}_\theta = \mathrm{G}(\mathrm{U}(V_\theta^0) \times \mathrm{U}(V_\theta^1))$. Let Δ_t be the 2-core of the partition λ so that we may write $2\theta + 1 = 2e + \frac{t(t+1)}{2}$ for some $e \geqslant 0$. In particular the integer $\frac{t(t+1)}{2}$ is odd, so it can be written as $2f + 1$ for some $f \geqslant 0$. Hence, we have $\theta = e + f$.
Following the notations of 2.7, the cuspidal support of ρ_λ is given by the pair $(\mathrm{G}(L_t \times L_0), \rho_t \otimes \rho_0)$ where L_t (resp. L_0) is seen as a Levi complement of $\mathrm{GU}(V_\theta^0) \simeq \mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ (resp. of $\mathrm{GU}(V_\theta^1) \simeq \mathrm{GU}_{n-2\theta-1}(\mathbb{F}_p)$), and $\mathrm{G}(L_t \times L_0)$ is the subgroup of $\mathcal{J}_\theta$ consisting of the pairs $(g, h) \in L_t \times L_0$ such that $c(g) = c(h)$. Then with a slight abuse of notations, the representation $\rho_t \otimes \rho_0$ denotes its restriction from $L_t \times L_0$ to $\mathrm{G}(L_t \times L_0)$. Note that L_0 is the maximal torus of diagonal elements in $\mathrm{GU}_{n-2\theta-1}(\mathbb{F}_p)$ and ρ_0 is the trivial representation. Therefore, $\rho_t \otimes \rho_0$ may also be seen as the inflation of ρ_t from L_t to $\mathrm{G}(L_t \times L_0)$.

Lemma. *The preimage via the quotient map* $J_\theta \twoheadrightarrow \mathcal{J}_\theta$ *of the standard parabolic subgroup of* $\mathcal{J}_\theta$ *containing the Levi complement* $\mathrm{G}(L_t \times L_0)$ *is the parahoric subgroup*

$$J_{f,\dots,m} := J_f \cap J_{f+1} \cap \ldots \cap J_m.$$

Proof. The standard parabolic subgroup decomposes as a product $\mathrm{G}(P_t \times P_0)$ where $L_t \subset P_t \subset \mathrm{GU}(V_\theta^0)$ and $L_0 \subset P_0 \subset \mathrm{GU}(V_\theta^1)$. The group P_t (resp. P_0) is the standard parabolic subgroup of $\mathrm{GU}(V_\theta^0)$ (resp. of $\mathrm{GU}(V_\theta^1)$) containing L_t (resp. L_0). It consists of the block upper-triangular matrices whose diagonal blocks have sizes prescribed by the Levi complement L_t or L_0 respectively. Recall the bases of V_θ^ϵ that we fixed in 1.2.8 for $\epsilon = 0, 1$. For $0 \leqslant j \leqslant \theta$, we denote by U_j^0 the subspace of V_θ^0 generated by the images of the vectors $e_{-\theta}, \ldots, e_{-1}, e_0^{\mathrm{an}}, e_1, \ldots, e_j$. Let $W_j^0 := (U_j^0)^\perp$ be the orthogonal. It is generated by the images of $e_{-\theta}, \ldots, e_{-j-1}$. Then P_t is the stabilizer of the flag

$$\{0\} = W_\theta^0 \subset W_{\theta-1}^0 \subset \ldots \subset W_f^0 \subset U_f^0 \subset \ldots \subset U_{\theta-1}^0 \subset U_\theta^0 = V_\theta^0.$$

Similarly, for $0 \leqslant j \leqslant m - \theta$ we write U_j^1 for the subspace of V_θ^1 generated by the images of $p^{-1}e_{-m}, \ldots, p^{-1}e_{-(m-j+1)}, e_{\theta+1}, \ldots, e_m$, and in case n is even we also add the image of $p^{-1}e_1^{\mathrm{an}}$. Let $W_j^1 := (U_j^1)^\perp$ be the orthogonal. It is generated by the images of the vectors $e_{\theta+1}, \ldots e_{m-j}$. Then P_0 is the stabilizer of the (full) flag

$$\{0\} = W_{m-\theta}^1 \subset W_{m-\theta-1}^1 \subset \ldots \subset W_0^1 \subset U_0^1 \subset \ldots \subset U_{m-\theta-1}^1 \subset U_{m-\theta}^1 = V_\theta^1.$$

In other words, P_0 is the Borel subgroup of upper-triangular matrices in $\mathrm{GU}(V_\theta^1) \simeq \mathrm{GU}_{n-2\theta-1}(\mathbb{F}_p)$. Now, we consider $0 \leqslant \theta' \leqslant m$. If $\theta' \leqslant \theta$ we have $\Lambda_{\theta'} \subset \Lambda_\theta$ and the image of $\Lambda_{\theta'}$ inside V_θ^0 is $U_{\theta'}^0$. If $\theta' \geqslant \theta$ we have $\Lambda_\theta \subset \Lambda_{\theta'}$ and the image of $\Lambda_{\theta'}$ inside V_θ^1 is $W_{m-\theta'}^1$. Thus, an element $g \in J_\theta$ has image in $\mathrm{G}(P_t \times P_0) \subset \mathcal{J}_\theta$ if and only if it fixes the subspaces $U_f^0, \ldots, U_\theta^0, W_0^1, \ldots, W_{m-\theta}^1$. This happens if and only if g fixes the lattices $\Lambda_f, \ldots, \Lambda_m$, ie. if and only $g \in J_{f,\ldots,m}$. □

4.2.10 Let $\epsilon = 0$ when n is odd and $\epsilon = 1$ when n is even. The maximal reductive quotient of the parahoric subgroup $J_{f,\ldots,m}$ is given by $\mathrm{G}(\mathrm{GU}_1^{m-f+e}(\mathbb{F}_p) \times \mathrm{GU}_{2f+1}(\mathbb{F}_p))$, which can be identified with $\mathrm{G}(L_t \times L_0)$. By inflating the cuspidal representation $\rho_t \otimes \rho_0$ to $J_{f,\ldots,m}$, we obtain a depth-0 J-type $(J_{f,\ldots,m}, \rho_t \otimes \rho_0)$. By 4.2.7 Theorem, there exists a unique finite set $\mathfrak{S} \subset \mathrm{IC}(J)$ such that it is an $\mathfrak{S}$-type. We may describe the set $\mathfrak{S}$ as follows.
Let M_f be the Levi complement of J that is associated to the parahoric subgroup $J_{f,\ldots,m}$. Let $\mathbf{V}^f$ be the subspace of $\mathbf{V}$ generated by $\mathbf{V}^{\mathrm{an}}$ and by the vectors $e_{\pm 1}, \ldots, e_{\pm f}$. We have $\dim \mathbf{V}^f = 2f + 1 + \epsilon$. The space $\mathbf{V}^f$ is equipped with the restriction of the hermitian form on $\mathbf{V}$. As a group, the Levi complement M_f is isomorphic to $\mathrm{G}(\mathrm{GU}(\mathbf{V}^f) \times \mathrm{GU}_1(\mathbb{Q}_p)^{m-f})$. We write J^f for the unitary similitude group $\mathrm{GU}(\mathbf{V}^f)$. The group J^f is of the same type of J but of smaller size. The intersection $M_f \cap J_{f,\ldots,m}$ is a maximal parahoric subgroup of M_f. As a group it is isomorphic to $\mathrm{G}(\mathrm{GU}(\Lambda^f) \times \mathrm{GU}_1(\mathbb{Z}_p)^{m-f})$ where Λ^f is the $\mathbb{Z}_{p^2}$-lattice of $\mathbf{V}^f$ generated by the vectors $e_{\pm 1}, \ldots, e_{\pm f}, e_0^{\mathrm{an}}$ and e_1^{an} when n is even. As a lattice in $\mathbf{V}^f$, it has maximal type. Therefore the summand $\mathrm{GU}(\Lambda^f)$ is a special maximal parahoric subgroup of J^f. The maximal reductive quotient of $M_f \cap J_{f,\ldots,m}$ is identified with the one of $J_{f,\ldots,m}$, that is $\mathrm{G}(L_t \times L_0)$. Thus, we may inflate the representation $\rho_t \otimes \rho_0$ of $\mathrm{G}(L_t \times L_0)$ to the parahoric subgroup $M_f \cap J_{f,\ldots,m}$ of M_f. We will still denote this inflation by $\rho_t \otimes \rho_0$.
The normalizer $\mathrm{N}_{M_f}(M_f \cap J_{f,\ldots,m})$ is isomorphic to $\mathrm{G}(\mathrm{N}_{J^f}(\mathrm{GU}(\Lambda^f)) \times \mathrm{GU}_1(\mathbb{Q}_p)^{m-f})$. According to the description of normalizers of maximal parahoric subgroups given in 1.3.3, the maximal open compact subgroup of $\mathrm{N}_{M_f}(M_f \cap J_{f,\ldots,m})$ is no other that $M_f \cap J_{f,\ldots,m}$ itself. Moreover, the normalizer $\mathrm{N}_{M_f}(M_f \cap J_{f,\ldots,m})$ has a quotient isomorphic to $M_f \cap J_{f,\ldots,m}$. Thus, we may inflate the representation $\rho_t \otimes \rho_0$ to an irreducible representation $\widetilde{\rho_t \otimes \rho_0}$ of $\mathrm{N}_{M_f}(M_f \cap J_{f,\ldots,m})$. We define

$$\tau_f := \mathrm{c} - \mathrm{Ind}_{\mathrm{N}_{M_f}(M_f \cap J_{f,\ldots,m})}^{M_f} \widetilde{\rho_t \otimes \rho_0}.$$

According to 4.2.7 Proposition, τ_f is an irreducible supercuspidal representation of M_f.

Proposition. *The pair $(J_{f,\ldots,m}, \rho_t \otimes \rho_0)$ is an $[M_f, \tau_f]$-type.*

This is a direct consequence of 4.2.8. Thus, the set $\mathfrak{S} = \{[M_f, \tau_f]\}$ is a singleton.

4.2.11 Let us go back to the representation ρ_λ of $\mathcal{J}_\theta \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$. If Δ_t is the 2-core of the partition λ, then $(\mathrm{G}(L_t \times L_0), \rho_t \otimes \rho_0)$ is the cuspidal support of ρ_λ. Thus, ρ_λ occurs as a subrepresentation of the Harish-Chandra induction of $\rho_t \otimes \rho_0$. Recall that Harish-Chandra induction is the process of extending $\rho_t \otimes \rho_0$ to a representation of the parabolic subgroup $\mathrm{G}(P_t \times P_0)$ by letting the unipotent radical act trivially, then applying usual induction for finite groups from $\mathrm{G}(P_t \times P_0)$ to $\mathcal{J}_\theta$.

We may interpret this process at the level of parahoric subgroups. We now think of $\rho_t \otimes \rho_0$ as its inflation to the parahoric subgroup $J_{f,\dots,m}$. We may extend it to a representation of $J_{f,\dots,m}J_\theta^+$ by letting J_θ^+ act trivially, and we consider the representation $\mathrm{Ind}_{J_{f,\dots,m}J_\theta^+}^{J_\theta} \rho_t \otimes \rho_0$ via smooth induction. Then the inflation of ρ_λ to J_θ is an irreducible subrepresentation of this induction. Since J_θ is already compact, smooth induction with or without compact support coincide. Thus in other words, we have an injection

$$\rho_\lambda \hookrightarrow \mathrm{c}-\mathrm{Ind}_{J_{f,\dots,m}J_\theta^+}^{J_\theta} \rho_t \otimes \rho_0$$

as smooth representations of the parahoric subgroup J_θ. Moreover, by Frobenius reciprocity there is an injection

$$\mathrm{c}-\mathrm{Ind}_{J_{f,\dots,m}J_\theta^+}^{J_\theta} \rho_t \otimes \rho_0 \hookrightarrow \mathrm{c}-\mathrm{Ind}_{J_{f,\dots,m}}^{J_\theta} \rho_t \otimes \rho_0.$$

Therefore, by transitivity and exacteness of compact induction it follows that we have a natural injection

$$\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \rho_\lambda \hookrightarrow \mathrm{c}-\mathrm{Ind}_{J_{f,\dots,m}}^{J} \rho_t \otimes \rho_0$$

as smooth representations of J.

Corollary. *Let $0 \leqslant \theta \leqslant m$ and let λ be a partition of $2\theta+1$. Let Δ_t be the 2-core of λ and write $\frac{t(t+1)}{2} = 2f+1$. The representation $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \rho_\lambda$ is an object of $\mathrm{Rep}^{[M_f,\tau_f]}(J)$.*

Proof. This is a combination of 4.2.6 Remark and 4.2.10 Proposition. ∎

4.2.12 We may apply this discussion to the representations $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})$ appearing in the spectral sequence. By a combination of 4.1.7 and 4.2.5, these representations are objects of the category $\mathrm{Rep}^{[M_f,\tau_f]}(J)$ where $f = 0$ when b is even and $f = 1$ when b is odd. It follows that the whole term $E_1^{a,b}$ is again an object of $\mathrm{Rep}^{[M_f,\tau_f]}(J)$ for the same $f \in \{0,1\}$.

Theorem. *Let $\mathfrak{S} := \{[M_0,\tau_0],[M_1,\tau_1]\}$. As smooth representations of J, the cohomology groups $\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ are objects of $\mathrm{Rep}^{\mathfrak{S}}(J)$.*

Proof. Recall from 4.1.9 that the filtration on the cohomology of the Rapoport-Zink space splits, that is we have

$$\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{k \leqslant b \leqslant 2(n-1)} E_2^{k-b,b}.$$

The term $E_2^{k-b,b}$ is a subquotient of $E_1^{k-b,b}$. By 4.2.5 it is again an object of $\mathrm{Rep}^{[M_f,\tau_f]}(J)$ where $f = 0$ if b is even and $f = 1$ if b is odd, so the result follows. ∎

Corollary. *For $n \geqslant 5$, the representation $\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ has no irreducible supercuspidal subquotient.*

Proof. We know that the cohomology group $\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is an object of the category $\mathrm{Rep}^{\mathfrak{S}}(J)$ where $\mathfrak{S} := \{[M_0,\tau_0],[M_1,\tau_1]\}$. According to 4.2.5, any subquotient of $\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is again an object of the category $\mathrm{Rep}^{\mathfrak{S}}(J)$. We observe that for $n \geqslant 5$, the Levi complements M_0 and M_1 are both proper in J. Therefore, no irreducible object in $\mathrm{Rep}^{\mathfrak{S}}(J)$ is supercuspidal. ∎

4.2.13 Let us consider a summand $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho_\lambda$ appearing in a non-zero term $E_1^{a,b}$ of the spectral sequence. It is an object of the category $\mathrm{Rep}^{[M_f,\tau_f]}(J)$ for some $f \in \{0,1\}$ depending on the parity of b. Assume that M_f is a proper parabolic subgroup of J, which implies $n \geqslant 3$. If $n = 3$ (resp. $n = 4$), we have $M_1 = J$ so the condition is equivalent to $b \neq 3$ (resp. $b \neq 5$). Eventually, if $n \geqslant 5$ then M_f is always proper. In this case, no irreducible subquotient of $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho_\lambda$ is supercuspidal. Let χ be an unramified character of $\mathrm{Z}(J) \simeq \mathbb{Q}_{p^2}$, ie. it is trivial on $J_\theta \cap \mathrm{Z}(J) \simeq \mathbb{Z}_{p^2}$. Recall the representation $(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho_\lambda)_\chi$ from 4.2.2, as well as the decomposition

$$(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho_\lambda)_\chi = V_{\rho_\lambda,\chi,0} \oplus V_{\rho_\lambda,\chi,\infty}$$

from 4.2.3. Then, since $V_{\rho_\lambda,\chi,0}$ is a direct sum of supercuspidal representations of J, it must be zero. It follows that $(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho_\lambda)_\chi = V_{\rho_\lambda,\chi,\infty}$ contains no non-zero admissible subrepresentation, and in particular no irreducible subrepresentation. Combining with the observations of 4.1.13, we deduce the following fact.

Proposition. *Assume that $n \geqslant 3$. The representation $(E_2^{0,2(n-1-m)})_\chi$ contains no non-zero admissible subrepresentation, and it is not J-semisimple. If $n \geqslant 5$, then the same statement holds for $(E_2^{0,2(n-1-m)+1})_\chi$.*

4.2.14 With the same notations, assume now that $M_f = J$. It means that ρ_λ is a cuspidal representation of the reductive quotient $\mathcal{J}_\theta$. It implies that $1 \leqslant n \leqslant 4$ and $\theta = m$. Such a summand $\mathrm{c}-\mathrm{Ind}_{J_m}^{J}\rho_\lambda$ occurs only in a single term $E_1^{a,b}$ for each n. For $n = 1,2,3,4$, the corresponding pair (a,b) is respectively given by $(0,0),(0,2),(0,3),(0,5)$. If $n = 1$ or 2, then we have $\lambda = \Delta_1$ and $f = 0$; if $n = 3$ or 4, then we have $\lambda = \Delta_2$ and $f = 1$.
The pair (J_m, ρ_λ) is a depth-0 J-type, thus evey irreducible subquotient of $\mathrm{c}-\mathrm{Ind}_{J_m}^{J}\rho_\lambda$ is supercuspidal. The corresponding set $\mathfrak{S}$ is the singleton $\{[J,\tau_f]\}$. Note that by definition of τ_f, its central character is trivial. Let χ be an unramified character of $\mathrm{Z}(J)$. Then, we have $V_{\rho_\lambda,\chi,\infty} = 0$ and $(\mathrm{c}-\mathrm{Ind}_{J_m}^{J}\rho_\lambda)_\chi = V_{\rho_\lambda,\chi,0} \simeq \chi \otimes \tau_f$. In particular, it is an irreducible supercuspidal representation of J. Combining with the observations of 4.1.13, we deduce the following.

Proposition. *For $n = 1,2,3,4$, let $b = 0,2,3,5$ respectively. Let $f = 0$ when $n = 1,2$ and let $f = 1$ when $n = 3,4$. Let χ be an unramified character of $\mathrm{Z}(J)$. Then $(E_2^{0,b})_\chi \simeq \chi \otimes \tau_f$ is an irreducible supercuspidal representation of J.*

5 The cohomology of the basic stratum of the Shimura variety for $n = 3, 4$

5.1 The Hochschild-Serre spectral sequence induced by p-adic uniformization

5.1.1 In this section, we still assume that n is any integer $\geqslant 1$. We recover the notations of Part 3 regarding Shimura varieties. As we have seen in 3.6, p-adic uniformization is a

geometric identity relating the Rapoport-Zink space $\mathcal{M}$ with the basic stratum $\overline{\mathrm{S}}_{K^p}(b_0)$. In [Far04], Fargues constructed a Hochschild-Serre spectral sequence using the uniformization theorem on the generic fibers, which we introduce in the following paragraphs.
Recall the PEL datum introduced in 3.1. Let $\xi : G \to W_\xi$ be a finite-dimensional irreducible algebraic $\overline{\mathbb{Q}_\ell}$-representation of G. Such representations have been classified in [HT01] III.2. We look at $\mathbb{V}_{\overline{\mathbb{Q}_\ell}} := \mathbb{V} \otimes \overline{\mathbb{Q}_\ell}$ as a representation of G, whose dual is denoted by $\mathbb{V}_0$. Using the alternating form $\langle \cdot, \cdot \rangle$, we have an isomorphism $\mathbb{V}_0 \simeq \mathbb{V}_{\overline{\mathbb{Q}_\ell}} \otimes c^{-1}$, where c is the multiplier character of G.

Proposition ([HT01] III.2)**.** *There exists unique integers $t(\xi), m(\xi) \geqslant 0$ and an idempotent $\epsilon(\xi) \in \mathrm{End}(\mathbb{V}_0^{\otimes m(\xi)})$ such that*

$$W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)}).$$

The weight $w(\xi)$ is defined by

$$w(\xi) := m(\xi) - 2t(\xi).$$

To any ξ as above, we can associate a local system $\mathcal{L}_\xi$ which is defined on the tower $(\mathrm{S}_{K^p})_{K^p}$ of Shimura varieties. We still write $\mathcal{L}_\xi$ for its restriction to the generic fiber $\mathrm{Sh}_{K_0K^p} \otimes_E \mathbb{Z}_{p^2}$, and we denote by $\overline{\mathcal{L}_\xi}$ its restriction to the special fiber $\overline{\mathrm{S}}_{K^p}$. Let $\mathcal{A}_{K^p}$ be the universal abelian scheme over S_{K^p}. We write $\pi^m_{K^p} : \mathcal{A}^m_{K^p} \to \mathrm{S}_{K^p}$ for the structure morphism of the m-fold product of $\mathcal{A}_{K^p}$ with itself over S_{K^p}. If $m = 0$ it is just the identity on S_{K^p}. According to [HT01] III.2, we have an isomorphism

$$\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)}\left(\mathrm{R}^{m(\xi)}(\pi^{m(\xi)}_{K^p})_*\overline{\mathbb{Q}_\ell}(t(\xi))\right),$$

where $\epsilon_{m(\xi)}$ is some idempotent. In particular, if ξ is the trivial representation of G then $\mathcal{L}_\xi = \overline{\mathbb{Q}_\ell}$.

5.1.2 We fix an irreducible algebraic representation $\xi : G \to W_\xi$ as above. We associate the space $\mathcal{A}_\xi$ of **automorphic forms of I of type ξ at infinity**. Explicitly, it is given by

$$\mathcal{A}_\xi = \{f : I(\mathbb{A}_f) \to W_\xi \,|\, f \text{ is } I(\mathbb{A}_f)\text{-smooth by right translations and } \forall \gamma \in I(\mathbb{Q}), f(\gamma \cdot) = \xi(\gamma)f(\cdot)\}.$$

We denote by $\mathcal{L}_\xi^{\mathrm{an}}$ the analytification of $\mathcal{L}_\xi$ to $\mathrm{Sh}^{\mathrm{an}}_{K_0K^p}$, as well as for its restriction to any open subspace.

Notation. We write $\mathrm{H}^\bullet((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}})$ for the cohomology of $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0} \widehat{\otimes} \mathbb{C}_p$ with coefficients in $\mathcal{L}_\xi^{\mathrm{an}}$.

Theorem ([Far04] 4.5.12)**.** *There is a W-equivariant spectral sequence*

$$F_2^{a,b}(K^p) : \mathrm{Ext}^a_{J\text{-}sm}\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \mathcal{A}_\xi^{K^p}\right) \implies \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}}).$$

These spectral sequences are compatible as the open compact subgroup K^p varies in $G(\mathbb{A}_f^p)$.

The W-action on $F_2^{a,b}(K^p)$ is inherited from the cohomology group $\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n)$. By the compatibility with K^p, we may take the limit $\varinjlim_{K^p}$ for all terms and obtain a $G(\mathbb{A}_f^p)\times W$-equivariant spectral sequence. Since m is the semisimple rank of J, the terms $F_2^{a,b}(K^p)$ are zero for $a > m$ according to [Far04] Lemme 4.4.12. Therefore, the non-zero terms $F_2^{a,b}$ are located in the finite strip delimited by $0 \leqslant a \leqslant m$ and $0 \leqslant b \leqslant 2(n-1)$.
Let us look at the abutment of the sequence. Since the formal completion $\widehat{\mathrm{S}}_{K^p}$ of S_{K^p} along its special fiber is a smooth formal scheme, Berkovich's comparison theorem ([Ber96] Corollary 3.7) gives an isomorphism

$$\mathrm{H}_c^{a+b}(\overline{\mathrm{S}}_{K^p}(b_0)\otimes\mathbb{F}, \overline{\mathcal{L}_\xi}) = \mathrm{H}^{a+b}(\overline{\mathrm{S}}_{K^p}(b_0)\otimes\mathbb{F}, \overline{\mathcal{L}_\xi}) \xrightarrow{\sim} \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}}).$$

The first equality follows from $\overline{\mathrm{S}}_{K^p}(b_0)$ being a proper variety. Since this variety has dimension m, the cohomology $\mathrm{H}^\bullet((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}})$ is concentrated in degrees 0 to $2m$.

5.1.3 Let $\mathcal{A}(I)$ denote the set of all automorphic representations of I counted with multiplicities. We write $\check{\xi}$ for the dual of ξ. We also define

$$\mathcal{A}_\xi(I) := \{\Pi \in \mathcal{A}(I) \,|\, \Pi_\infty = \check{\xi}\}.$$

According to [Far04] 4.6, we have an identification

$$\mathcal{A}_\xi^{K_p} \simeq \bigoplus_{\Pi\in\mathcal{A}_\xi(I)} \Pi_p \otimes (\Pi^p)^{K_p}.$$

It yields, for every a and b, an isomorphism

$$F_2^{a,b}(K^p) \simeq \bigoplus_{\Pi\in\mathcal{A}_\xi(I)} \mathrm{Ext}^a_{J\text{-sm}}\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes (\Pi^p)^{K_p}.$$

Taking the limit over K^p, we deduce that

$$F_2^{a,b} := \varinjlim_{K^p} F_2^{a,b}(K^p) \simeq \bigoplus_{\Pi\in\mathcal{A}_\xi(I)} \mathrm{Ext}^a_{J\text{-sm}}\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p.$$

The spectral sequence defined by the terms $F_2^{a,b}$ computes $\mathrm{H}^{a+b}(\widehat{\mathrm{S}}^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}}) := \varinjlim_{K^p} \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}})$. It is isomorphic to $\mathrm{H}_c^{a+b}(\overline{\mathrm{S}}(b_0)\otimes\mathbb{F}, \overline{\mathcal{L}_\xi}) := \varinjlim_{K^p} \mathrm{H}_c^{a+b}(\overline{\mathrm{S}}_{K^p}(b_0)\otimes\mathbb{F}, \overline{\mathcal{L}_\xi})$.

5.1.4 Recall from 4.1.9 that we have a decomposition

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b\leqslant b'\leqslant 2(n-1)} E_2^{b-b',b'},$$

and $E_2^{b-b',b'}$ corresponds to the eigenspace of τ associated to the eigenvalue $(-p)^b$. Accordingly, we have a decomposition

$$F_2^{a,b} \simeq \bigoplus_{\substack{2(n-1)-b\leqslant \\ b'\leqslant 2(n-1)}} \bigoplus_{\Pi\in\mathcal{A}_\xi(I)} \mathrm{Ext}^a_{J\text{-sm}}\left(E_2^{2(n-1)-b-b',b'}(1-n), \Pi_p\right) \otimes \Pi^p.$$

For $\Pi \in \mathcal{A}_\xi(I)$, we denote by ω_Π the central character. We define

$$\delta_{\Pi_p} := \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id}) p^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times.$$

Let ι be any isomorphism $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, and write $|\cdot|_\iota := |\iota(\cdot)|$. Since I is a group of unitary similitudes of an $E/\mathbb{Q}$-hermitian space, its center is $E^\times \cdot \mathrm{id}$. The element $p^{-1} \cdot \mathrm{id} \in \mathrm{Z}(J)$ can be seen as the image of $p^{-1} \cdot \mathrm{id} \in \mathrm{Z}(I(\mathbb{Q}))$. We have $\omega_\Pi(p^{-1} \cdot \mathrm{id}) = 1$. Moreover, for any finite place $q \neq p$, the element $p^{-1} \cdot \mathrm{id}$ lies inside the maximal compact subgroup of $\mathrm{Z}(I(\mathbb{Q}_q))$, so $|\omega_{\Pi_q}(p^{-1}\mathrm{id})|_\iota = 1$. Besides $\Pi_\infty = \check{\xi}$, so we have

$$|\omega_{\Pi_p}(p^{-1} \cdot \mathrm{id})|_\iota = |\omega_{\check{\xi}}(p^{-1} \cdot \mathrm{id})|_\iota^{-1} = |\omega_\xi(p^{-1} \cdot \mathrm{id})|_\iota = |p^{w(\xi)}|_\iota = p^{w(\xi)}.$$

The last equality comes from the isomorphism $W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)})$, see 5.1.1. In particular $|\delta_{\Pi_p}|_\iota = 1$ for any isomorphism ι.

Proposition. *The W-action on $\mathrm{Ext}^a_{J\text{-}sm}(E_2^{2(n-1)-b-b',b'}(1-n), \Pi_p)$ is trivial on the inertia I, and the Frobenius element* Frob *acts like multiplication by $(-1)^{-b'} \delta_{\Pi_p} p^{-b'+2(n-1)+w(\xi)}$.*

Proof. Let us write $X := E_2^{2(n-1)-b-b',b'}(1-n)$. By convention, the action of Frob on a space $\mathrm{Ext}^a_{J\text{-sm}}(X, \Pi_p)$ is induced by functoriality of Ext applied to $\mathrm{Frob}^{-1} : X \to X$. Let us consider a projective resolution of X in the category of smooth representations of J

$$\ldots \xrightarrow{u_3} P_2 \xrightarrow{u_2} P_1 \xrightarrow{u_1} P_0 \xrightarrow{u_0} X \longrightarrow 0.$$

Since Frob^{-1} commutes with the action of J, we can choose a lift $\mathcal{F} = (\mathcal{F}_i)_{i \geq 0}$ of Frob^{-1} to a morphism of chain complexes.

$$\begin{array}{ccccccccccc}
\ldots & \xrightarrow{u_3} & P_2 & \xrightarrow{u_2} & P_1 & \xrightarrow{u_1} & P_0 & \xrightarrow{u_0} & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{F}_2} & & \downarrow{\scriptstyle \mathcal{F}_1} & & \downarrow{\scriptstyle \mathcal{F}_0} & & \downarrow{\scriptstyle \mathrm{Frob}^{-1}} & & \\
\ldots & \xrightarrow{u_3} & P_2 & \xrightarrow{u_2} & P_1 & \xrightarrow{u_1} & P_0 & \xrightarrow{u_0} & X & \longrightarrow & 0
\end{array}$$

After applying $\mathrm{Hom}_J(\cdot, \Pi_p)$ and forgetting about the first term, we obtain a morphism $\mathcal{F}^*$ of chain complexes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_J(P_0, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_1, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_2, \Pi_p) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \mathcal{F}_0^*} & & \downarrow{\scriptstyle \mathcal{F}_1^*} & & \downarrow{\scriptstyle \mathcal{F}_2^*} & & \\
0 & \longrightarrow & \mathrm{Hom}_J(P_0, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_1, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_2, \Pi_p) & \longrightarrow & \ldots
\end{array}$$

Here $\mathcal{F}_i^* f(v) := f(\mathcal{F}_i(v))$. It induces morphisms on the cohomology

$$\mathcal{F}_i^* : \mathrm{Ext}^i_J(X, \Pi_p) \to \mathrm{Ext}^i_J(X, \Pi_p),$$

which do not depend on the choice of the lift $\mathcal{F}$. Recall that Frob is the composition of τ and $p \cdot \mathrm{id} \in J$. Since τ is multiplication by the scalar $(-1)^{b'} p^{b'-2(n-1)}$ on X, we may choose the lift $\mathcal{F}_i := (-1)^{-b'} p^{-b'+2(n-1)} (p^{-1} \cdot \mathrm{id})$ for all i.

Consider an element of $\mathrm{Ext}^i_J(X, \Pi_p)$ represented by a morphism $f : P_i \to \Pi_p$. For any $v \in P_i$ we have

$$\mathcal{F}_i^* f(v) = f(\mathcal{F}_i(v)) = (-1)^{-b'} p^{-b'+2(n-1)} f((p^{-1} \cdot \mathrm{id}) \cdot v) = (-1)^{-b'} p^{-b'+2(n-1)} \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id}) f(v).$$

It follows that Frob acts on $\mathrm{Ext}^i_J(X, \Pi_p)$ via multiplication by the scalar $(-1)^{-b'} \delta_{\Pi_p} p^{-b'+2(n-1)+w(\xi)}$.

□

5.1.5 In general, the Hochschild-Serre spectral sequence has many differentials between non-zero terms. However, focusing on the diagonal defined by $a + b = 0$, it is possible to compute $\mathrm{H}^0_c(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$. Recall that $X^{\mathrm{un}}(J)$ denotes the set of unramified characters of J. If $x \in \overline{\mathbb{Q}_\ell}^\times$ is any non-zero scalar, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of W where the inertia I acts trivially and the geometric Frobenius Frob acts like $x \cdot \mathrm{id}$.

Proposition. *We have an isomorphism of $G(\mathbb{A}_f^p) \times W$-representations*

$$\mathrm{H}^0_c(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}].$$

Proof. The only non-zero term $F_2^{a,b}$ on the diagonal defined by $a + b = 0$ is $F_2^{0,0}$. Since there is no non-zero arrow pointing at nor coming from this term, it is untouched in all the successive pages of the sequence. Therefore we have an isomorphism

$$F_2^{0,0} \simeq \mathrm{H}^0_c(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

Using 4.1.14, we also have isomorphisms

$$\begin{aligned} F_2^{0,0} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J\text{-sm}} \left(\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p \right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J\text{-sm}} \left((\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1})(1-n), \Pi_p \right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J^\circ} \left(\mathbf{1}(1-n), \Pi_{p|J^\circ} \right) \otimes \Pi^p. \end{aligned}$$

Thus, only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ with $\Pi_p^{J^\circ} \neq 0$ contribute to the sum. Consider such a Π. The irreducible representation Π_p is generated by a J°-invariant vector. Since J° is normal in J, the whole representation Π_p is trivial on J°. Thus, it is an irreducible representation of $J/J^\circ \simeq \mathbb{Z}$. Therefore, it is one-dimensional. Since J° is generated by all compact subgroups of J, it follows that $\Pi_p^{J^\circ} \neq 0 \iff \Pi_p \in X^{\mathrm{un}}(J)$. When it is satisfied, the W-representation $V_\Pi^0 := \mathrm{Hom}_{J^\circ}(\mathbf{1}(1-n), \Pi_p)$ has dimension one and the Frobenius action was described in 5.1.4. □

5.2 The case $n = 3$ or 4

5.2.1 In this section, we assume that $n = 3$ or 4, ie. we have $m = 1$. In this case, the Hochschild-Serre spectral sequence 5.1.2 has no non-zero arrow, so that it degenerates on the second page. Using the spectral sequence 4.1.4 for the cohomology of the Rapoport-Zink space, it makes it possible to entirely compute the cohomology of the basic stratum.

5.2.2 First we describe the spectral sequence $E_1^{a,b}$ when $n = 3$ or 4. They have similar shapes in both cases, but the indices and multiplicities vary a little bit. According to 4.1.8 and 1.4.2, we have

$$E_1^{a,b} \neq 0 \iff \begin{cases} (a,b) \in \{(0,2); (0,3); (-k,4) \,|\, 0 \leqslant k \leqslant p\} & \text{if } n = 3, \\ (a,b) \in \{(0,4); (0,5); (-k,6) \,|\, 0 \leqslant k \leqslant p^3\} & \text{if } n = 4. \end{cases}$$

In both cases $n = 3$ or 4, it is possible to compute the multiplicities $k_{s,\theta} = \#K_s^{(\theta)}$ that were introduced in 4.1.10. Here $0 \leqslant \theta \leqslant 1$. When $\theta = 1$ we already know by 4.1.12 that $k_{s,1} = 1$ if $s = 1$ and 0 otherwise. Thus, we only need to compute the $k_{s,0}$. Recall the definition of the set

$$K_s^{(0)} = \{\delta \subset N(\Lambda_0) \,|\, \#\delta = s \text{ and } \Lambda(\delta) = \Lambda_0\}.$$

Here, $N(\Lambda_0)$ is the set of lattices $\Lambda \in \mathcal{L}$ such that $\Lambda_0 \subset \Lambda$ and $t(\Lambda) = 2m + 1 = 3$ is maximal. The condition $\Lambda(\delta) = \Lambda_0$ is automatically satisfied as soon as $s \geqslant 2$. Indeed, the intersection of at least two lattices in $N(\Lambda_0)$ must contain Λ_0, and must be of orbit type strictly less than 3. Therefore, it has orbit type $1 = t(\Lambda_0)$ so that it is equal to Λ_0. The set $N(\Lambda_0)$ has cardinality $\nu(n-2, n-1)$ which is $p + 1$ if $n = 3$ and $p^3 + 1$ if $n = 4$. Hence, for $s \geqslant 2$ we have

$$k_{s,0} = \begin{cases} \binom{p+1}{s} & \text{if } s \geqslant 2 \text{ and } n = 3 \\ \binom{p^3+1}{s} & \text{if } s \geqslant 2 \text{ and } n = 4 \end{cases}$$

and of course, $k_{1,0} = 0$ in both cases.

5.2.3 Using the notations from 4.1.7, we draw the first page of the spectral sequence $E_1^{a,b}$ for $n = 3$ in Figure 1. The bottom right 0 corresponds to the $(0,0)$-coordinates. The first page when $n = 4$ looks precisely the same, except that two more 0 rows should be added at the bottom. The arrows on the top row are denoted φ_i for $1 \leqslant i \leqslant p$ when $n = 3$, and $1 \leqslant i \leqslant p^3$ when $n = 4$.

$$\begin{array}{ccccccccr} \ldots & \xrightarrow{\varphi_4} & \left(\mathrm{c-Ind}_{J_0}^J \mathbf{1}\right)^{k_{4,0}} & \xrightarrow{\varphi_3} & \left(\mathrm{c-Ind}_{J_0}^J \mathbf{1}\right)^{k_{3,0}} & \xrightarrow{\varphi_2} & \left(\mathrm{c-Ind}_{J_0}^J \mathbf{1}\right)^{k_{2,0}} & \xrightarrow{\varphi_1} & \mathrm{c-Ind}_{J_1}^J \mathbf{1} \\ & & & & & & & & \mathrm{c-Ind}_{J_1}^J \rho_{\Delta_2} \\ & & & & & & & & \mathrm{c-Ind}_{J_1}^J \mathbf{1} \\ & & & & & & & & 0 \\ & & & & & & & & 0 \end{array}$$

Figure 1: The first page E_1 when $n = 3$.

5.2.4 Our focus lies on the top row, and more precisely what happens on the second page. Since the spectral sequence computes the cohomology of the Rapoport-Zink space, we know that $E_2^{a,4}$ is 0 for $a \leqslant -5$ when $n = 3$, and that $E_2^{a,6}$ is 0 for $a \leqslant -7$ when $n = 4$. However, it is unclear what these terms look like for other values of a ; except for the case $a = 0$ which is done in 4.1.14. In order to obtain more information, we introduce the Hochschild-Serre spectral sequence $F_2^{a,b}$ computing the cohomology of the basic stratum of the Shimura variety. Let ξ be an irreducible finite dimensional algebraic representation of G as in 5.1.1. When $n = 3$ or 4, the semisimple rank of J is $m = 1$, therefore the terms $F_2^{a,b}$ are zero for $a > 1$. In particular, the spectral sequence degenerates on the second page. Since it computes the cohomology of the basic locus $\overline{\mathrm{S}}(b_0)$ which is 1-dimensional, we also have $F_2^{0,b} = 0$ for $b \geqslant 3$, and $F_2^{1,b} = 0$ for $b \geqslant 2$. In Figure 2, we draw the second page F_2 and we write between brackets the *complex modulus* of the possible eigenvalues of Frob on each term under any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, as computed in 5.1.4.

$F_2^{0,2}[p^{w(\xi)+2}, p^{w(\xi)}]$	0
$F_2^{0,1}[p^{w(\xi)+1}, p^{w(\xi)}]$	$F_2^{1,1}[p^{w(\xi)+1}, p^{w(\xi)}]$
$F_2^{0,0}[p^{w(\xi)}]$	$F_2^{1,0}[p^{w(\xi)}]$

Figure 2: The second page F_2 with the complex modulus of possible eigenvalues of Frob on each term.

Proposition. *We have $F_2^{1,1} = 0$ and the eigenspaces of* Frob *on $F_2^{0,2}$ attached to any eigenvalue of complex modulus $p^{w(\xi)}$ are zero.*

Proof. By the machinery of spectral sequences, there is a $G(\mathbb{A}_f^p) \times W$-subspace of $\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ isomorphic to $F_2^{1,1}$, and the quotient by this subspace is isomorphic to $F_2^{0,2}$. We prove that all eigenvalues of Frob on $\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ have complex modulus $p^{w(\xi)+2}$. The proposition then readily follows.
We need the Ekedahl-Oort stratification on the basic stratum of the Shimura variety. Let $K^p \subset G(\mathbb{A}_f^p)$ be small enough. In [VW11] 3.3 and 6.3, the authors define the Ekedahl-Oort stratification on $\mathcal{M}_{\mathrm{red}}$ and on $\overline{\mathrm{S}}_{K^p}(b_0)$ respectively, and they are compatible via the p-adic uniformization isomorphism. For $n = 3$ or 4, the stratification on the basic stratum take the following form

$$\overline{\mathrm{S}}_{K^p}(b_0) = \overline{\mathrm{S}}_{K^p}[1] \sqcup \overline{\mathrm{S}}_{K^p}[3].$$

The stratum $\overline{\mathrm{S}}_{K^p}[1]$ is closed and 0-dimensional, whereas the other stratum $\overline{\mathrm{S}}_{K^p}[3]$ is open, dense and 1-dimensional. In particular, we have a Frobenius equivariant isomorphism between the cohomology groups of highest degree

$$\mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}[3] \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

According the [VW11] 5.3, the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$ and $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ also admit an Ekedahl-Oort stratification of a similar form, and we have a decomposition

$$\overline{\mathrm{S}}_{K^p}[3] = \bigsqcup_{\Lambda,k} \overline{\mathrm{S}}_{K^p,\Lambda,k}[3]$$

into a finite disjoint union of open and closed subvarieties. As a consequence, we have the following Frobenius equivariant isomorphisms

$$\mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}[3] \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\Lambda,k} \mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p,\Lambda,k}[3] \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\Lambda,k} \mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p,\Lambda,k} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$$

where the last isomorphism between cohomology groups of highest degree follows from the stratification on the closed Bruhat-Tits strata $\overline{\mathrm{S}}_{K^p,\Lambda,k}$. Now, recall from 5.1.1 that the local system $\mathcal{L}_\xi$ is given by

$$\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)} \left(\mathrm{R}^{m(\xi)}(\pi_{K^p}^{m(\xi)})_* \overline{\mathbb{Q}_\ell}(t(\xi))\right).$$

It implies that $\overline{\mathcal{L}_\xi}$ is pure of weight $w(\xi)$. Since the variety $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is smooth and projective, it follows that all eigenvalues of Frob on the cohomology group $\mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p,\Lambda,k} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ must have complex modulus $p^{w(\xi)+2}$ under any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$. The result follows by taking the limit over K^p. □

5.2.5 We will need the following lemma.

Lemma. *For all $1 \leqslant i \leqslant p$ when $n = 3$, and for all $1 \leqslant i \leqslant p^3$ when $n = 4$, we have $\mathrm{Ext}^1_{J\text{-}sm}(\mathrm{Ker}(\varphi_i), \pi) = \mathrm{Ext}^1_{J\text{-}sm}(\mathrm{Im}(\varphi_i), \pi) = 0$ for any smooth representation π of J.*

Proof. Recall that for any a and b we have

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \left(\mathrm{c}-\mathrm{Ind}_{J_\theta}^J \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})\right)^{k_{-a+1,\theta}}.$$

By Frobenius reciprocity, if K is any compact open subgroup of J and ρ is any smooth representation of K, the compactly induced representation $\mathrm{c}-\mathrm{Ind}_K^J \rho$ is projective in the category of smooth representations of J. Therefore all the terms $E_1^{a,b}$ are projective. It follows that, if π is any smooth representation,

$$\mathrm{Ext}^1_{J\text{-sm}}(\mathrm{Ker}(\varphi_i), \pi) \simeq \mathrm{Ext}^2_{J\text{-sm}}(\mathrm{Im}(\varphi_i), \pi) \quad \text{and} \quad \mathrm{Ext}^1_{J\text{-sm}}(\mathrm{Im}(\varphi_i), \pi) = \mathrm{Ext}^2_{J\text{-sm}}(\mathrm{Coker}(\varphi_i), \pi).$$

Recall from 4.1.5 that the spectral sequence E is obtained by applying the compact induction functor $\mathrm{c}-\mathrm{Ind}_{J^\circ}^J$ to E'. Let us denote by φ_i' the differential $E_1'^{-i,b} \to E_1'^{-i+1,b}$ where $b = 4$ if $n = 3$ and $b = 6$ if $n = 4$. Since compact induction is exact, we have $\mathrm{Coker}(\varphi_i) = \mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathrm{Coker}(\varphi_i')$ and $\mathrm{Im}(\varphi_i) = \mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathrm{Im}(\varphi_i')$. The Lemme 4.4.12 in [Far04] proves that the Ext groups of degree greater than the semisimple rank of J, from any smooth representation compactly induced from J° to any smooth representation of J, vanish. Since the semisimple rank of J is equal to $m = 1$, the Ext^2 groups above are zero. □

5.2.6 We may now compute the cohomology of the basic stratum. Recall the supercuspidal representation τ_1 of the Levi complement $M_1 \subset J$ that we defined in 4.2.10. When $n = 3$ or 4, we actually have $M_1 = J$ and

$$\tau_1 = \mathrm{c} - \mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J} \widetilde{\rho_{\Delta_2}}$$

is a supercuspidal representation of J, where $\mathrm{N}_J(J_1) = \mathrm{Z}(J)J_1$ (see 1.3.3) and $\widetilde{\rho_{\Delta_2}}$ is the inflation of ρ_{Δ_2} to $\mathrm{N}_J(J_1) = \mathrm{Z}(J)J_1$ (see 1.3.3) obtained by letting the center act trivially. We use the same notations as in 5.1.5.

Theorem. *There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

Moreover, there exists a $G(\mathbb{A}_f^p) \times W$-subspace $V \subset \mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ such that

$$V \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} d(\Pi_p)\Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}],$$

and with quotient space isomorphic to

$$\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})/V \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0 \\ \dim(\Pi_p) > 1}} (\nu - 1 - d(\Pi_p))\Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} (\nu - d(\Pi_p))\Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

where $\nu = p$ if $n = 3$ and $\nu = p^3$ if $n = 4$, and $d(\Pi_p) := \dim \mathrm{Ext}^1_{J\text{-sm}}(\mathrm{c} - \mathrm{Ind}_{J^\circ}^J \mathbf{1}, \Pi_p)$.

The integers $d(\Pi_p)$ are finite, and all the multiplicities $\nu - 1 - d(\Pi_p)$ and $\nu - d(\Pi_p)$ occuring in the formula are non negative. For all $\Pi \in \mathcal{A}_\xi(I)$ we have $d(\Pi_p) \neq 0 \implies \Pi_p^{J_0} \neq 0$ and $\Pi_p^{J_1} \neq 0$. In particular, if Π_p is supercuspidal or if the central character of Π_p is not unramified, then $d(\Pi_p) = 0$. All of these facts are byproducts of the proof below, but may also be proved by direct computation. We are not aware of a formula computing $d(\Pi_p)$ in general.

Proof. The statement regarding $\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ was already proved in 5.1.5.
Let us prove the statement regarding $\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ first. By 5.2.4, we have

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq F_2^{0,2} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J\text{-sm}}\left(E_2^{0,b}(1-n), \Pi_p\right) \otimes \Pi^p,$$

where $b = 2$ if $n = 3$ and $b = 4$ if $n = 4$. The term $E_2^{0,b}$ is isomorphic to $\mathrm{c} - \mathrm{Ind}_{J_1}^J \mathbf{1}$. Therefore, by Frobenius reciprocity we have

$$\mathrm{Hom}_{J\text{-sm}}\left(E_2^{0,b}(1-n), \Pi_p\right) \simeq \mathrm{Hom}_{J_1\text{-sm}}\left(\mathbf{1}(1-n), \Pi_p\right).$$

Hence, only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ with $\Pi_p^{J_1} \neq 0$ contribute to $F_2^{0,2}$. Such a representation Π_p is said to be J_1**-spherical**. Since J_1 is a special maximal compact subgroup of J, according to [Mın11] 2.1, we have $\dim(\pi^{J_1}) = 1$ for every smooth irreducible J_1-spherical representation π of J. The result follows using 5.1.4 to describe the eigenvalues of Frob.

We now prove the statement regarding $\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$. By the spectral sequence, there exists a $G(\mathbb{A}_f^p) \times W$-subspace V' of this cohomology group such that

$$V' \simeq F_2^{1,0} \quad \text{and} \quad \mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})/V' \simeq F_2^{0,1}.$$

We have

$$\begin{aligned} F_2^{1,0} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^1_{J\text{-sm}}\left(\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^1_{J\text{-sm}}\left(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1}(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} d(\Pi_p)\Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}], \end{aligned}$$

where the eigenvalues of Frob are given by 5.1.4.
On the other hand, we have

$$F_2^{0,1} \simeq \underbrace{\bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J\text{-sm}}\left(E_2^{0,2(n-1)-1}(1-n), \Pi_p\right) \otimes \Pi^p}_{=:V_0} \oplus \underbrace{\bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J\text{-sm}}\left(E_2^{-1,2(n-1)}(1-n), \Pi_p\right) \otimes \Pi^p}_{=:V_1}.$$

By 5.1.4, Frob acts on a summand of V_0 by the scalar $-\delta_{\Pi_p} p^{w(\xi)+1}$, and on a summand of V_1 by the scalar $\delta_{\Pi_p} p^{w(\xi)}$. Since $\mathrm{Frob}_{|V'}$ has no eigenvalue of complex modulus $p^{w(\xi)+1}$, the V_0-part of the quotient by V' actually splits, so that V_0 is naturally a subspace of $\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$. Defining $V := V' \oplus V_0$, we have $\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})/V \simeq V_1$. Let us now compute V_0.
Since $2(n-1) - 1 = 2(n-1-m) + 1$ when $m = 1$, by 4.1.13 we have

$$E_2^{0,2(n-1)-1} \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^J \rho_{\Delta_2},$$

with τ acting like multiplication by $-p^3$ when $n = 3$ and by $-p^5$ when $n = 4$, and $\Delta_2 = (2,1)$ is the partition of $2m + 1 = 3$ defined in 2.7. Hence, we have an isomorphism

$$\begin{aligned} V_0 &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J\text{-sm}}\left(\mathrm{c}-\mathrm{Ind}_{J_1}^J \rho_{\Delta_2}(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J_1\text{-sm}}\left(\rho_{\Delta_2}(1-n), \Pi_{p|J_1}\right) \otimes \Pi^p. \end{aligned}$$

It follows that only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ whose p-component Π_p contains the supercuspidal representation ρ_{Δ_2} when restricted to J_1, contribute to the sum. According to 4.2.7, such Π_p are precisely those of the form $\chi \cdot \tau_1$ for some $\chi \in X^{\mathrm{un}}(J)$. By the Mackey formula we have

$$\begin{aligned}\mathrm{Hom}_{J\text{-sm}}\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\rho_{\Delta_2},\chi\cdot\tau_1\right) &\simeq \mathrm{Hom}_{J_1\text{-sm}}\left(\rho_{\Delta_2},\tau_{1|J_1}\right)\\ &\simeq \mathrm{Hom}_{J_1\text{-sm}}\left(\rho_{\Delta_2},(\mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J}\widetilde{\rho_{\Delta_2}})_{|J_1}\right)\\ &\simeq \bigoplus_{h\in J_1\backslash J/\mathrm{N}_J(J_1)} \mathrm{Hom}_{J_1\cap{}^h\mathrm{N}_J(J_1)\text{-sm}}(\rho_{\Delta_2},{}^h\widetilde{\rho_{\Delta_2}}),\end{aligned}$$

where in the last formula we omitted to write the restrictions to $J_1\cap{}^h\mathrm{N}_J(J_1)$. We used the fact that $\chi_{|J_1}$ is trivial. Since $\widetilde{\rho_{\Delta_2}}$ is just the inflation of ρ_{Δ_2} from J_1 to $\mathrm{N}_J(J_1)=\mathrm{Z}(J)J_1$ obtained by letting $\mathrm{Z}(J)$ act trivially, we have a bijection

$$\mathrm{Hom}_{J_1\cap{}^h\mathrm{N}_J(J_1)\text{-sm}}(\rho_{\Delta_2},{}^h\widetilde{\rho_{\Delta_2}}) \simeq \mathrm{Hom}_{\mathrm{N}_J(J_1)\cap{}^h\mathrm{N}_J(J_1)\text{-sm}}(\widetilde{\rho_{\Delta_2}},{}^h\widetilde{\rho_{\Delta_2}}).$$

Now, $\mathrm{N}_J(J_1)$ contains the center, is compact modulo the center, and $\tau_1=\mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J}\widetilde{\rho_{\Delta_2}}$ is supercuspidal. It follows that an element $h\in J$ intertwines $\widetilde{\rho_{\Delta_2}}$ if and only if $h\in\mathrm{N}_J(J_1)$ (see for instance [BH06] 11.4 Theorem along with Remarks 1 and 2). Therefore, only the trivial double coset contributes to the sum and we have

$$\mathrm{Hom}_{J\text{-sm}}\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\rho_{\Delta_2},\chi\cdot\tau_1\right)\simeq \mathrm{Hom}_{J_1\text{-sm}}(\rho_{\Delta_2},\rho_{\Delta_2})\simeq\overline{\mathbb{Q}_\ell}.$$

To sum up, we have

$$V_0\simeq\bigoplus_{\substack{\Pi\in\mathcal{A}_\xi(I)\\ \exists\chi\in X^{\mathrm{un}}(J),\\ \Pi_p=\chi\cdot\tau_1}}\Pi^p\otimes\overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p}p^{w(\xi)+1}].$$

Let us now compute V_1. Let $b=2(n-1)$ which is 4 if $n=3$ and 6 if $n=4$. We must compute $\mathrm{Hom}_{J\text{-sm}}(E_2^{-1,b},\Pi_p)$ for $\Pi\in\mathcal{A}_\xi(I)$. Consider the short exact sequence

$$0\to\mathrm{Im}(\varphi_1)\to E_1^{0,b}\to E_2^{0,b}\to 0,$$

with $E_1^{0,b}\simeq\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}$ and $E_2^{0,b}\simeq\mathrm{c}-\mathrm{Ind}_{J^\circ}^{J}\mathbf{1}$ by 4.1.14. Since $E_1^{0,b}$ is projective in the category of smooth representations of J, we obtain a short exact sequence

$$0\to\mathrm{Hom}_{J\text{-sm}}(E_2^{0,b},\Pi_p)\to\mathrm{Hom}_{J\text{-sm}}(E_1^{0,b},\Pi_p)\to\mathrm{Hom}_{J\text{-sm}}(\mathrm{Im}(\varphi_1),\Pi_p)\to\mathrm{Ext}^1_{J\text{-sm}}(E_2^{0,b},\Pi_p)\to 0.$$

By Frobenius reciprocity, the left-hand side is $\overline{\mathbb{Q}_\ell}$ if $\Pi_p^{J^\circ}\neq 0$ and 0 otherwise. As we remarked in the proof of 5.1.5, the existence of a non-zero J°-fixed vector is equivalent to $\Pi_p\in X^{\mathrm{un}}(J)$. Moreover, the second term from the left is $\overline{\mathbb{Q}_\ell}$ if $\Pi_p^{J_1}\neq 0$ and 0 otherwise. We deduce that

$$\mathrm{Hom}_{J\text{-sm}}\left(\mathrm{Im}(\varphi_1),\Pi_p\right)\simeq\begin{cases}\overline{\mathbb{Q}_\ell}^{d(\Pi_p)} & \text{if } \Pi_p\in X^{\mathrm{un}}(J)\text{ or }\Pi_p^{J_1}=0,\\ \overline{\mathbb{Q}_\ell}^{d(\Pi_p)+1} & \text{if } \Pi_p\notin X^{\mathrm{un}}(J)\text{ and }\Pi_p^{J_1}\neq 0,\end{cases}$$

where $d(\Pi_p)=\dim\mathrm{Ext}^1_{J\text{-sm}}(E_2^{0,b},\Pi_p)$. Next, we consider the short exact sequence

$$0\to\mathrm{Ker}(\varphi_1)\to E_1^{-1,b}\to\mathrm{Im}(\varphi_1)\to 0.$$

By 5.2.5, we know that $\mathrm{Ext}^1_{J\text{-sm}}(\mathrm{Im}(\varphi_1), \Pi_p) = 0$. Therefore it induces the following exact sequence

$$0 \to \mathrm{Hom}_{J\text{-sm}}(\mathrm{Im}(\varphi_1), \Pi_p) \to \mathrm{Hom}_{J\text{-sm}}(E_1^{-1,b}, \Pi_p) \to \mathrm{Hom}_{J\text{-sm}}(\mathrm{Ker}(\varphi_1), \Pi_p) \to 0.$$

Since $E_1^{-1,b} \simeq (\mathrm{c}-\mathrm{Ind}_{J_0}^{J}\mathbf{1})^{k_{2,0}}$, by Frobenius reciprocity the middle term is $(\Pi_p^{J_0})^{k_{2,0}}$. Since J_0 is also a special maximal compact subgroup of J, we know that $\Pi_p^{J_0}$ is zero or one dimensional. We observe that the injectivity of the left arrow implies that $d(\Pi_p) > 0 \implies \Pi_p^{J_0} \neq 0$, and that $(\Pi_p \notin X^{\mathrm{un}}(J)$ and $\Pi_p^{J_1} \neq 0) \implies \Pi_p^{J_0} \neq 0$. Of course, if Π_p is an unramified character then it has some non-zero J_0-invariants. Therefore, we really have $\forall \Pi \in \mathcal{A}_\xi(I), \Pi_p^{J_1} \neq 0 \implies \Pi_p^{J_0} \neq 0$. Before pursuing, let us make a short digression to prove that this implication is in fact an equivalence.

We need to recall from [Mın11] how spherical representations are classified. Let B be a minimal parabolic subgroup of J with Levi complement T. Let ι_B^J denote the normalized parabolic induction. If $\chi \in X^{\mathrm{un}}(T)$ then the representation $\iota_B^J\chi$ has composition length at most 2. Since J_0 and J_1 are special, there is a unique composition factor of $\iota_B^J\chi$ which is J_0-spherical (resp. J_1-spherical). Every spherical representation arises in this way. Moreover, the space of J_0-invariants (resp. J_1-invariants) has dimension 1.
Now, assume towards a contradiction that $\Pi_p^{J_0} \neq 0$ and $\Pi_p^{J_1} = 0$. Let $\chi \in X^{\mathrm{un}}(T)$ be an unramified character such that Π_p is a composition factor of $\iota_B^J\chi$. Since Π_p has no J_1-invariants, this induced representation must have composition length 2. The other composition factor is the Aubert-Zelevinski dual $D(\Pi_p)$. There exists an automorphic representation $\Pi' \in \mathcal{A}_\xi(I)$ such that $\Pi'_p = D(\Pi_p)$, and so $(\Pi'_p)^{J_1} \neq 0$ since one composition factor of $\iota_B^J\chi$ is J_1-spherical. By the direct implication we have $(\Pi'_p)^{J_0} \neq 0$, and by unicity of the J_0-spherical composition factor, we deduce that $D(\Pi_p) = \Pi'_p = \Pi_p$ which is a contradiction.

In particular, if $d(\Pi_p) > 0$ then $\Pi_p^{J_1} \neq 0$. From the previous exact sequence, we deduce that

$$\mathrm{Hom}_{J\text{-sm}}(\mathrm{Ker}(\varphi_1), \Pi_p) \simeq \begin{cases} 0 & \Pi_p^{J_1} = 0, \\ \overline{\mathbb{Q}_\ell}^{k_{2,0}-d(\Pi_p)} & \text{if } \Pi_p \in X^{\mathrm{un}}(J), \\ \overline{\mathbb{Q}_\ell}^{k_{2,0}-d(\Pi_p)-1} & \text{if } \Pi_p \notin X^{\mathrm{un}}(J) \text{ and } \Pi_p^{J_1} \neq 0. \end{cases}$$

It remains to compute $\mathrm{Hom}_{J\text{-sm}}(\mathrm{Im}(\varphi_2), \Pi_p)$. We may do it by induction. For all $2 \leqslant i \leqslant p-1$ if $n = 3$ and $2 \leqslant i \leqslant p^3 - 1$ if $n = 4$, let $d_i(\Pi_p) := \dim \mathrm{Hom}_{J\text{-sm}}(\mathrm{Im}(\varphi_i), \Pi_p)$. Consider the short exact sequence

$$0 \to \mathrm{Im}(\varphi_{i+1}) \to \mathrm{Ker}(\varphi_i) \to E_2^{-i,b} \to 0.$$

Since $F_2^{1,i} = 0$ (see Figure 2), we know that $\mathrm{Ext}^1_{J\text{-sm}}(E_2^{-i,b}, \Pi_p) = 0$. Moreover, if $i \geqslant 3$ we have $F_2^{0,i} = 0$ as well so that $\mathrm{Hom}_{J\text{-sm}}(E_2^{-i,b}, \Pi_p) = 0$. If $i = 2$ this Hom is also 0 by 5.2.4. Thus we deduce an isomorphism

$$\mathrm{Hom}_{J\text{-sm}}(\mathrm{Ker}(\varphi_i), \Pi_p) \simeq \mathrm{Hom}_{J\text{-sm}}(\mathrm{Im}(\varphi_{i+1}), \Pi_p).$$

Next, we consider the short exact sequence

$$0 \to \mathrm{Ker}(\varphi_i) \to E_1^{-i,b} \to \mathrm{Im}(\varphi_i) \to 0.$$

By 5.2.5 we have $\mathrm{Ext}^1_{J\text{-sm}}(\mathrm{Im}(\varphi_i), \Pi_p) = 0$. Thus we obtain the following exact sequence

$$0 \to \mathrm{Hom}_{J\text{-sm}}(\mathrm{Im}(\varphi_i), \Pi_p) \to \mathrm{Hom}_{J\text{-sm}}(E_1^{-i,b}, \Pi_p) \to \mathrm{Hom}_{J\text{-sm}}(\mathrm{Ker}(\varphi_i), \Pi_p) \to 0.$$

Since $E_1^{-i,b} \simeq (\mathrm{c}-\mathrm{Ind}_{J_0}^{J}\mathbf{1})^{k_{i+1,0}}$, by Frobenius reciprocity the middle term is $\overline{\mathbb{Q}_\ell}^{k_{i+1,0}}$ if $\Pi_p^{J_0} \neq 0$ (or equivalently $\Pi_p^{J_1} \neq 0$), and 0 otherwise. Therefore, we deduce that

$$d_i(\Pi_p) = \begin{cases} 0 & \text{if } \Pi_p^{J_1} = 0, \\ k_{i+1,0} - d_{i+1}(\Pi_p) & \text{if } \Pi_p^{J_1} \neq 0. \end{cases}$$

Eventually for $\Pi_p^{J_1} \neq 0$, if $n = 3$ (resp. if $n = 4$), we have $d_p(\Pi_p) = k_{p+1,0}$ (resp. $d_{p^3}(\Pi_p) = k_{p^3+1,0}$). It follows by induction that $d_2(\Pi_p)$ is the sum of the $(-1)^{i-1}k_{i,0}$ for i ranging from 3 to $p+1$ when $n = 3$, and to p^3+1 when $n = 4$.
Lastly, consider the short exact sequence

$$0 \to \mathrm{Im}(\varphi_2) \to \mathrm{Ker}(\varphi_1) \to E_2^{-1,b} \to 0.$$

By 5.2.4 we know that $F_2^{1,1} = 0$ therefore $\mathrm{Ext}^1_{J\text{-sm}}(E_2^{-1,b}, \Pi_p) = 0$. We deduce the following exact sequence

$$0 \to \mathrm{Hom}_{J\text{-sm}}(E_2^{-1,b}, \Pi_p) \to \mathrm{Hom}_{J\text{-sm}}(\mathrm{Ker}(\varphi_1), \Pi_p) \to \mathrm{Hom}_{J\text{-sm}}(\mathrm{Im}(\varphi_2), \Pi_p) \to 0.$$

We deduce that

$$\mathrm{Hom}_{J\text{-sm}}(E_2^{-1,b}, \Pi_p) \simeq \begin{cases} 0 & \text{if } \Pi_p^{J_1} = 0, \\ \overline{\mathbb{Q}_\ell}^{k_{2,0}-d_2(\Pi_p)-d(\Pi_p)} & \text{if } \Pi_p \in X^{\mathrm{un}}(J), \\ \overline{\mathbb{Q}_\ell}^{k_{2,0}-d_2(\Pi_p)-d(\Pi_p)-1} & \text{if } \Pi_p \notin X^{\mathrm{un}}(J) \text{ and } \Pi_p^{J_1} \neq 0. \end{cases}$$

For $\Pi_p^{J_1} \neq 0$ it remains to compute

$$k_{2,0} - d_2(\Pi_p) = \begin{cases} \sum_{i=2}^{p+1}(-1)^i\binom{p+1}{i} = p & \text{if } n = 3, \\ \sum_{i=2}^{p^3+1}(-1)^i\binom{p^3+1}{i} = p^3 & \text{if } n = 4. \end{cases}$$

Eventually, we observe that $\forall \Pi \in \mathcal{A}_\xi(I), (\Pi_p \notin X^{\mathrm{un}}(J) \text{ and } \Pi_p^{J_1} \neq 0) \iff (\dim(\Pi_p) > 1 \text{ and } \Pi_p^{J_1} \neq 0)$. Indeed, the reverse implication is obvious. The direct implication follows given that if $\Pi_p^{J_1} \neq 0$ and Π_p is a character, Π_p is trivial on all the conjugates of J_1. By 1.3.4 it follows that Π_p is trivial on J° hence it is an unramified character. One may reach the same conclusion without referring to 1.3.4 by using the fact that Π_p also has J_0-invariants, so it is in fact trivial on the subgroup generated by all conjugates of J_0 and of J_1. By general theory, this group is no other than J°. Hence, we have proved that

$$V_1 \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0 \\ \dim(\Pi_p) > 1}} (\nu - 1 - d(\Pi_p))\Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} (\nu - d(\Pi_p))\Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

where $\nu = p$ if $n = 3$ and $\nu = p^3$ if $n = 4$. This concludes the proof. □

5.3 On the cohomology of the ordinary locus when $n = 3$

5.3.1 In this section, we assume that the Shimura variety is of Kottwitz-Harris-Taylor type. According to [HT01] I.7, it amounts to assuming that the algebra B from 3.1 is a division algebra satisfying a few additional conditions. In particular, B_v is either split either a division algebra for every place v of $\mathbb{Q}$, and there must be at least one prime number p' (different from p) which splits in E and such that B splits over p'. In this situation, the Shimura variety is compact.
According to 3.5, when $n = 3$ there is a single Newton stratum other than the basic one. It is the μ-ordinary locus $\overline{\mathrm{S}}_{K^p}(b_1)$, and it is an open dense subscheme of the special fiber of the Shimura variety. Moreover, since the Shimura variety is compact, the ordinary locus is also an affine scheme according to [GN17] and [KW18]. By using the spectral sequence associated to the stratification

$$\overline{\mathrm{S}}_{K^p} = \overline{\mathrm{S}}_{K^p}(b_0) \sqcup \overline{\mathrm{S}}_{K^p}(b_1),$$

we may deduce information on the cohomology of the ordinary locus. The spectral sequence is given by

$$G_1^{a,b} : \mathrm{H}_c^b(\overline{\mathrm{S}}_{K^p}(b_a) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathrm{S}_{K^p} \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}).$$

In figure 3, we draw the first page of this sequence.

$$\begin{array}{ccc}
 & & \mathrm{H}_c^4(\overline{\mathrm{S}}_{K^p}(b_1) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \\
\mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) & \xrightarrow{\phi} & \mathrm{H}_c^3(\overline{\mathrm{S}}_{K^p}(b_1) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \\
\mathrm{H}_c^1(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) & \xrightarrow{\psi} & \mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}(b_1) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \\
\mathrm{H}_c^0(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) & &
\end{array}$$

Figure 3: The first page G_1.

5.3.2 Let v be a place of E above p'. The cohomology of the Shimura variety $\mathrm{Sh}_{C_0K^p} \otimes_E E_v$ has been entirely computed in [Boy10]. Note that as $G(\mathbb{A}_f^p)$-representations, the cohomology of $\mathrm{Sh}_{C_0K^p} \otimes_E E_v$ is isomorphic to the cohomology of $\mathrm{Sh}_{C_0K^p} \otimes_E \mathbb{Q}_{p^2}$, which in turn is isomorphic to the cohomology of the special fiber $\overline{\mathrm{S}}_{K^p}$ using nearby cycles. In particular, we understand perfectly the abutment of the spectral sequence $G_1^{a,b}$. Since $\overline{\mathrm{S}}_{K^p}$ is smooth and projective, its cohomology admits a symmetry with respect to the middle degree 2. Moreover, by the results of loc. cit. the groups of degree 1 and 3 are zero. It follows that ϕ is surjective and ψ is injective. Combining with our computations, we deduce the following proposition.

Proposition. *There is a $G(\mathbb{A}_f^p) \times W$-equivariant isomorphism*

$$\mathrm{H}_c^4(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+4}].$$

There is a $G(\mathbb{A}_f^p) \times W$-equivariant monomorphism

$$\mathrm{H}_c^3(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \hookrightarrow \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

There is a $G(\mathbb{A}_f^p) \times W$-equivariant monomorphism

$$\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \hookrightarrow \mathrm{H}_c^2(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

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