On the cohomology of the unramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

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Contents

Introduction

The Rapoport-Zink space $\mathcal{M}$

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

References

Introduction

Introduction

$p > 2$ prime number.
$\mathcal{D}$: a set of local EL or PEL datum (**P**olarization, **E**ndomorphism action, **L**evel structure).
The datum $\mathcal{D}$ determines

- a p-divisible group $\mathbb{X}$ with extra structures (the **framing object**),
- two p-adic groups $G(\mathbb{Q}_p)$ and $J(\mathbb{Q}_p) = \mathrm{Aut}(\mathbb{X})$,
- a p-adic field E (the **reflex field**), $\breve{E} := \widehat{E^{\mathrm{un}}}$.
- $K_0 \subset G(\mathbb{Q}_p)$ a parahoric subgroup.

Introduction

Rapoport-Zink space: the moduli space $\mathcal{M} = \mathcal{M}_{\mathcal{D}}$ over $\mathrm{Spf}(\mathcal{O}_{\breve{E}})$ classifying the deformations of $\mathbb{X}$ by quasi-isogenies. $J(\mathbb{Q}_p) \curvearrowright \mathcal{M}$ a natural action.

$\mathcal{M}^{\mathrm{an}}$: the Berkovich generic fiber of $\mathcal{M}$, an analytic space over $\breve{E}$.

$\forall K \subset K_0$ open compact, $\mathcal{M}_K \to \mathcal{M}^{\mathrm{an}}$ finite étale cover. In particular $\mathcal{M}_{K_0} = \mathcal{M}^{\mathrm{an}}$. Projective system $\mathcal{M}_\infty := (\mathcal{M}_K)_K$ called **the Rapoport-Zink tower**.

$G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \curvearrowright \mathcal{M}_\infty$ via Hecke correspondences.

$\ell \neq p$ prime number.
W : the Weil group of E.

Goal: study $\mathrm{H}_c^\bullet(\mathcal{M}_\infty \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ as a $(G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W)$-representation, expected to have applications to local Langlands program.

Remark: the W-action on the cohomology is given by **Rapoport-Zink's (non effective) descent datum** on $\mathcal{M}$.

Known results:

- $\mathrm{H}_c^\bullet(\mathcal{M}_\infty)$ entirely understood in the Lubin-Tate and Drinfeld cases by Dat (2006), Fargues-Genestier-Lafforgue (2006), Boyer (2009), etc. Both are EL type.
- Kottwitz's conjecture to describe the $(G(\mathbb{Q}_p) \times J(\mathbb{Q}_p))$-supercuspidal part. Known for
  - ✓ Lubin-Tate case by Boyer (1999) and Harris-Taylor (2001),
  - ✓ basic unramified RZ spaces of EL type by Fargues (2004) and Shin (2012),
  - ✓ basic unramified PEL unitary RZ space with signature $(r, n-r)$ and n odd by Nguyen (2019) and Bertoloni Meli-Nguyen (2021).

In this talk: consider the basic unramified unitary PEL datum $\mathcal{D}_{(1,n-1)}$ with signature $(1, n-1)$, and the associated RZ space $\mathcal{M} = \mathcal{M}_{\mathcal{D}_{(1,n-1)}}$. We study

$$\mathrm{H}_c^{\bullet}(\mathcal{M}^{\mathrm{an}}) = \mathrm{H}_c^{\bullet}(\mathcal{M}_{\infty})^{K_0}$$

as a $(J(\mathbb{Q}_p) \times W)$-representation, with K_0 hyperspecial.

Use the geometric description of the special fiber $\mathcal{M}_{\mathrm{red}}$ given by Vollaard (2010) and Vollaard-Wedhorn (2011).

The Rapoport-Zink space $\mathcal{M}$

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Notations:

- $p > 2$ prime number.
- $\mathbb{Z}_{p^2} := W(\mathbb{F}_{p^2})$ the ring of Witt vectors of $\mathbb{F}_{p^2}$.
- $E = \mathbb{Q}_{p^2} := \mathrm{Frac}(\mathbb{Z}_{p^2})$ the quadratic unramified extension of $\mathbb{Q}_p$.

$\mathbb{X}$ is a **superspecial** p-divisible group over $\mathbb{F}_{p^2}$ with polarization $\lambda_{\mathbb{X}} : \mathbb{X} \xrightarrow{\sim} \mathbb{X}^\vee$ and action $\iota_{\mathbb{X}} : \mathbb{Z}_{p^2} \to \mathrm{End}(\mathbb{X})$ satisfying the signature $(1, n-1)$ condition.

The Rapoport-Zink space $\mathcal{M}$

Nilp : the category of $\mathbb{Z}_{p^2}$-schemes S where p is locally nilpotent.

Definition

Let $S \in \mathrm{Nilp}$ and $\overline{S} := S \times \mathbb{F}_{p^2}$. Define $\mathcal{M}(S) = \{(X, \iota_X, \lambda_X, \rho_X)\}/\simeq$ where

- (X, ι_X, λ_X) is a p-divisible group over S with polarization λ_X and $\mathbb{Z}_{p^2}$-action ι_X of signature $(1, n-1)$,
- $\rho_X : X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}_{p^2}} \overline{S}$ is a quasi-isogeny compatible with the extra structures.

The Rapoport-Zink space $\mathcal{M}$

Theorem (Rapoport, Zink, 1996)

$\mathcal{M}$ is a formal scheme over $\mathrm{Spf}(\mathbb{Z}_{p^2})$ formally smooth and locally formally of finite type, called the **basic unramified PEL unitary Rapoport-Zink space with signature** $(1, n-1)$.

Remark: $\mathcal{M}$ can be defined over $\mathbb{Z}_{p^2}$ in this case.

$\mathcal{M}_{\mathrm{red}}$: the reduced special fiber of $\mathcal{M}$, a scheme over $\mathrm{Spec}(\mathbb{F}_{p^2})$.

The geometry of $\mathcal{M}_{\mathrm{red}}$ has been described by Vollaard and Wedhorn (2010, 2011).

The Rapoport-Zink space $\mathcal{M}$

Here, $G(\mathbb{Q}_p) \simeq \mathrm{GU}_n(\mathbb{Q}_p)$ quasi-split group of unitary similitudes in n variables, and

$$J(\mathbb{Q}_p) \simeq \begin{cases} G(\mathbb{Q}_p) & \text{if } n \text{ is odd,} \\ \text{the non quasi-split inner form of } G(\mathbb{Q}_p) & \text{if } n \text{ is even.} \end{cases}$$

$\mathrm{BT}(J)$: set of vertices of the **Bruhat-Tits building** of $J(\mathbb{Q}_p)$.

The Rapoport-Zink space $\mathcal{M}$

Vollaard-Wedhorn's results (2010,2011):

The **Bruhat-Tits stratification** $\mathcal{M}_{\mathrm{red}} = \bigsqcup_{\Lambda} \mathcal{M}_{\Lambda}^{\circ}$ where $\Lambda \in \mathrm{BT}(J)$ and $\mathcal{M}_{\Lambda}^{\circ} \hookrightarrow \mathcal{M}_{\mathrm{red}}$ locally closed subscheme.
$\mathcal{M}_{\Lambda} := \overline{\mathcal{M}_{\Lambda}^{\circ}}$ a **closed Bruhat-Tits stratum**.

Two main features:
1) The incidence relations of the $\mathcal{M}_{\Lambda}$'s are described by the combinatorics of $\mathrm{BT}(J)$.

Action $J(\mathbb{Q}_p) \curvearrowright \mathcal{M}$ compatible with BT stratification: for $g \in J(\mathbb{Q}_p)$,

$$g : \mathcal{M}_{\Lambda} \xrightarrow{\sim} \mathcal{M}_{g\cdot\Lambda}.$$

The Rapoport-Zink space $\mathcal{M}$

$J_\Lambda := \mathrm{Fix}_J(\Lambda)$ maximal parahoric subgroup of $J(\mathbb{Q}_p)$.
J_Λ^+ : pro-unipotent radical.
$\mathcal{J}_\Lambda := J_\Lambda / J_\Lambda^+$ maximal reductive quotient.
We have $\mathcal{J}_\Lambda \simeq \mathrm{G}\left(\mathrm{GU}_{2\theta+1}(\mathbb{F}_p) \times \mathrm{GU}_{n-2\theta-1}(\mathbb{F}_p)\right)$.
$t(\Lambda) := 2\theta + 1$ is the **type** of Λ. We have $1 \leq t(\Lambda) \leq n$.

Then $J_\Lambda \curvearrowright \mathcal{M}_\Lambda$ factors through $\mathcal{J}_\Lambda$, and then to an action of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$.

2) $\mathcal{M}_\Lambda$ is isomorphic to a **generalized Deligne-Lusztig variety** for $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$.

Our strategy:

1. Compute $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ the cohomology of one stratum using Deligne-Lusztig theory.
2. Use the Bruhat-Tits stratification and its combinatorics to study $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$.

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

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q : a power of p.

$\mathbb{H}$: connected reductive group over $\overline{\mathbb{F}_p}$ with an $\mathbb{F}_q$-structure.

$F : \mathbb{H} \to \mathbb{H}$ the associated geometric Frobenius.

$H := \mathbb{H}(\mathbb{F}_q) \simeq \mathbb{H}^F$ **finite group of Lie type**.

$\mathbb{P} \subset \mathbb{H}$ any parabolic subgroup.

Definition

The associated **generalized Deligne-Lusztig variety** is

$$X_{\mathbb{P}} := \left\{ h\mathbb{P} \in \mathbb{H}/\mathbb{P} \,|\, h^{-1}F(h) \in \mathbb{P}F(\mathbb{P}) \right\}.$$

Defined over $\mathbb{F}_{q^\delta}$ where $\delta \geq 1$ smallest integer such that $F^\delta(\mathbb{P}) = \mathbb{P}$.
We have $H \curvearrowright X_{\mathbb{P}}$ by left translations.

Remark: The variety $X_{\mathbb{P}}$ is **classical** if in addition

$$\text{“}\exists \mathbb{L} \subset \mathbb{P} \text{ a Levi complement such that } F(\mathbb{L}) = \mathbb{L}.\text{”} \qquad (*)$$

Then we have $H \curvearrowright X_{\mathbb{P}} \curvearrowleft L := \mathbb{L}^F$.

The cohomology $\mathrm{H}^\bullet_c(X_{\mathbb{P}} \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ gives the **Deligne-Lusztig's induction and restriction functors** R_L^H and ${}^*\mathrm{R}_L^H$ between the categories of representations of L and of H.

$\Longrightarrow$ Classification of irreducible representations of finite groups of Lie type.

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

Fix $\Lambda \in \mathrm{BT}(J)$, write $t(\Lambda) = 2\theta + 1$.
Consider $\mathbb{H} = \mathrm{GL}_{2\theta+1} \times \mathrm{GL}_1$.
Let $F : \mathbb{H} \to \mathbb{H}$ the *twisted* Frobenius.
Then $H = \mathbb{H}^F = \mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$.
Define

$$\mathbb{P} := \left\{ \left(\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}_{\underbrace{}_{\theta+1}\underbrace{}_{\theta}}, * \right) \in \mathrm{GL}_{2\theta+1} \times \mathrm{GL}_1 \right\}.$$

Remark: Condition $(*)$ is not satisfied for $X_{\mathbb{P}}$, thus it is **not classical**.

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

Theorem (Vollaard, Wedhorn, 2011)

There is a $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$-equivariant isomorphism

$$\mathcal{M}_\Lambda \xrightarrow{\sim} X_{\mathbb{P}}.$$

In particular $\mathcal{M}_\Lambda$ is smooth, irreducible, projective of dimension θ.

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

An irreducible representation ρ of a finite group of Lie type $H = \mathbb{H}^F$ is **unipotent** if it occurs in $\mathrm{R}_T^H 1$ for some maximal rational torus $T \subset H$ (Deligne-Lusztig induction).

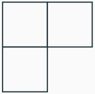
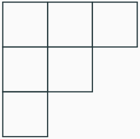
Theorem (Lusztig, Srinivasan, 1977)

The unipotent irreducible representations ρ_λ of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ are classified by partitions λ of $2\theta+1$ (or Young diagrams of size $2\theta+1$).

Example: $\rho_{(2\theta+1)} = \rho_{\square\square\cdots\square} = 1$ and $\rho_{(1^{2\theta+1})} = \rho_{\begin{smallmatrix}\square\\ \square\\ \vdots\\ \square\end{smallmatrix}} = \mathrm{St}.$

Remark: ρ_Λ is cuspidal iff $2\theta + 1 = \frac{t(t+1)}{2}$ for some $t \geq 1$ and $\lambda = \Delta_t = (t, t-1, \ldots, 1)$.

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

$\Delta_1 =$  $\Delta_2 =$  $\Delta_3 =$ 

Δ_t has the shape of a staircase.

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

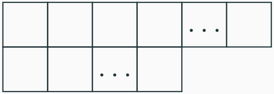
Theorem (M.)

Let $\Lambda \in \mathrm{BT}(J)$ and write $t(\Lambda) = 2\theta + 1$.

1. $\mathrm{H}^i_c(\mathcal{M}_\Lambda) \neq 0$ iff $0 \leq i \leq 2\theta$.
2. The Frobenius F^2 acts like $(-p)^i \cdot \mathrm{id}$ on $\mathrm{H}^i_c(\mathcal{M}_\Lambda)$.
3. For $0 \leq i \leq \theta$ we have
$$\mathrm{H}^{2i}_c(\mathcal{M}_\Lambda) \simeq \bigoplus_{s=0}^{\min(i,\theta-i)} \rho_{(2\theta+1-2s,2s)}.$$
4. For $0 \leq i \leq \theta - 1$ we have
$$\mathrm{H}^{2i+1}_c(\mathcal{M}_\Lambda) \simeq \bigoplus_{s=0}^{\min(i,\theta-1-i)} \rho_{(2\theta-2s,2s+1)}.$$

Remarks:

- All representations associated to a Young diagram λ with at most 2 rows appear in $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda)$.

$\lambda =$ 

- In $\mathrm{H}_c^{2i}(\mathcal{M}_\Lambda)$ the representations belong to the unipotent principal series.
- In $\mathrm{H}_c^{2i+1}(\mathcal{M}_\Lambda)$ belong to the cuspidal series given by ρ_{Δ_2}, cuspidal representation of $\mathrm{GU}_3(\mathbb{F}_p)$.

Step 1: the cohomology of a BT stratum $\mathcal{M}_\Lambda$

Idea of proof: Ekedahl-Oort stratification

$$\mathcal{M}_\Lambda = \bigsqcup_{0 \leq \theta' \leq \theta} \mathcal{M}_\Lambda(\theta').$$

The EO stratum $\mathcal{M}_\Lambda(\theta')$ is related to a classical Deligne-Lusztig variety **of Coxeter type** for $\mathrm{GU}_{2\theta'+1}(\mathbb{F}_p)$.

$\implies$ Compute $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda(\theta'))$ using work of Lusztig (1976), then use spectral sequence

$$E_1^{a,b} = \mathrm{H}_c^{a+b}(\mathcal{M}_\Lambda(a)) \implies \mathrm{H}_c^{a+b}(\mathcal{M}_\Lambda).$$

□

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

$\mathrm{red} : \mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$ anticontinuous map.
$U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda)$ the analytical tube of $\mathcal{M}_\Lambda$. It is open in $\mathcal{M}^{\mathrm{an}}$, smooth analytical space over $\mathbb{Q}_{p^2}$ of dimension $n-1$.

Recall: $1 \leq t(\Lambda) \leq n$ is odd. Write

$$n = \begin{cases} 2m+1 & \text{if } n \text{ is odd,} \\ 2(m+1) & \text{if } n \text{ is even.} \end{cases}$$

Then $t_{\max} := 2m+1$.
$\mathrm{BT}(J)^m := \{\Lambda \in \mathrm{BT}(J) \mid t(\Lambda) = t_{\max}\}$.

$\implies \{U_\Lambda\}_{\Lambda \in \mathrm{BT}(J)^m}$ is open cover of $\mathcal{M}^{\mathrm{an}}$.

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

The open cover induces a Čech spectral sequence on cohomology

$$E_1^{a,b} = \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U(\gamma)\widehat{\otimes}\,\mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}\widehat{\otimes}\,\mathbb{C}_p, \overline{\mathbb{Q}_\ell}),$$

where for $a \leq 0$

$$I_{-a+1} := \left\{ \gamma \subset \mathrm{BT}(J)^m \,\middle|\, \#\gamma = -a+1 \text{ and } U(\gamma) := \bigcap_{\Lambda \in \gamma} U_\Lambda \neq \emptyset \right\}.$$

Note that $\exists \Lambda(\gamma) \in \mathrm{BT}(J)$ such that $U(\gamma) = U_{\Lambda(\gamma)}$.

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

Recall: $W = W_{\mathbb{Q}_{p^2}}$ the Weyl group.
$\mathrm{Frob} \in W$ geometric Frobenius. $\tau := (p \cdot \mathrm{id}, \mathrm{Frob}) \in J(\mathbb{Q}_p) \times W$.

Proposition

Let $\Lambda \in \mathrm{BT}(J)$ with $t(\Lambda) = 2\theta + 1$ and $0 \leq b \leq 2(n-1)$. There is an isomorphism

$$\mathrm{H}_c^b(U_\Lambda) \simeq \mathrm{H}_c^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda)(n-1-\theta)$$

compatible with the J_Λ and W actions. The action of τ on the LHS agrees with the action of F^2 on the RHS.

Proof: Hyperspecial level so smooth integral model. The vanishing cycles are trivial. Apply Poincaré duality. □

In particular, τ acts like $(-p)^b \cdot \mathrm{id}$ on $E_1^{a,b}$.

Corollary

The spectral sequence degenerates on E_2 and splits, ie.

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}) \simeq \bigoplus_{b \leq b' \leq 2(n-1)} E_2^{b-b',b'}.$$

Then $E_2^{b-b',b'}$ (may be 0) is the eigenspace of τ attached to the eigenvalue $(-p)^{b'}$.

Remark: The inertia acts trivially.

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

Fix $\{\Lambda_0, \dots, \Lambda_m\}$ an alcôve (ie. maximal simplex) in $\mathrm{BT}(J)$. Let $J_\theta := J_{\Lambda_\theta}$ maximal parahoric.

Proposition

There exists $k_{-a+1,\theta} \in \mathbb{Z}_{\geq 0}$ such that

$$E_1^{a,b} \simeq \bigoplus_{\theta=0}^{m} \left(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta})\right)^{k_{-a+1,\theta}}.$$

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

Example: When $n = 3$ so $m = 1$.

$$\ldots \longrightarrow \left(\mathrm{c}-\mathrm{Ind}_{J_0}^{J} 1\right)^{k_{3,0}} \longrightarrow \left(\mathrm{c}-\mathrm{Ind}_{J_0}^{J} 1\right)^{k_{2,0}} \longrightarrow \mathrm{c}-\mathrm{Ind}_{J_1}^{J} 1$$

$$\mathrm{c}-\mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2}$$

$$\mathrm{c}-\mathrm{Ind}_{J_1}^{J} 1$$

$$0$$

$$0$$

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

Proposition

We have an isomorphism of J-representations

$$E_2^{0,2(n-1-m)} \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J}\,\rho_{(2m+1)}.$$

If $n \geq 3$ then we also have an isomorphism

$$E_2^{0,2(n-1-m)+1} \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J}\,\rho_{(2m,1)}.$$

For $V \in \mathrm{Rep}(J(\mathbb{Q}_p))$ and χ a character of $\mathrm{Z}(J(\mathbb{Q}_p)) \simeq \mathbb{Q}_{p^2}^{\times}$, write V_χ for **the largest quotient of V where $\mathrm{Z}(J(\mathbb{Q}_p))$ acts like χ.**

Step 2: on the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}}$

Using type theory, we prove the following.

Corollary (M.)

Let χ be any unramified character of $\mathrm{Z}(J(\mathbb{Q}_p))$.

1. Let $n \geq 3$. The representation $(E_2^{0,2(n-1-m)})_\chi$ contains no non-zero admissible subrepresentation, and is not J-semisimple. If $n \geq 5$, the same holds for $(E_2^{0,2(n-1-m)+1})_\chi$.
2. For $n = 1, 2, 3, 4$, let $b = 0, 2, 3, 5$ respectively. Then $(E_2^{0,b})_\chi$ is an irreducible supercuspidal representation of $J(\mathbb{Q}_p)$.

In particular, $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}})_\chi$ **needs not be admissible** as a $J(\mathbb{Q}_p)$-representation.

$\implies$ Different from the Lubin-Tate and Drinfeld cases!

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

$(\mathbb{G}, X)$: Shimura datum inducing the local PEL datum at p.
$\implies \mathbb{G}_{\mathbb{R}} \simeq \mathrm{GU}(1, n-1)$ and $\mathbb{G}_{\mathbb{Q}_p} \simeq G$.

$K^p \subset \mathbb{G}(\mathbb{A}_f^p)$ small enough open compact.
S_{K^p}: integral model of the Shimura variety, smooth quasi-projective over $\mathrm{Spec}(\mathbb{Z}_{p^2})$.
$\overline{\mathrm{S}}_{K^p}$: special fiber.
$\overline{\mathrm{S}}_{K^p}(b_0)$: the basic locus.
$\widehat{\mathrm{S}}_{K^p}(b_0)^{\mathrm{an}}$: the analytical tube of $\overline{\mathrm{S}}_{K^p}(b_0)$.

I: inner form of $\mathbb{G}$ such that $I_{\mathbb{A}_f^p} = \mathbb{G}_{\mathbb{A}_f^p}$, $I_{\mathbb{Q}_p} = J$ and $I_{\mathbb{R}} \simeq \mathrm{GU}(0, n)$.

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

p-adic uniformization theorem (Rapoport, Zink, 1996)

There is a natural isomorphism

$$I(\mathbb{Q})\backslash \left(\mathcal{M}^{\mathrm{an}} \times \mathbb{G}(\mathbb{A}_f^p)/K^p\right) \xrightarrow{\simeq} \widehat{\mathrm{S}}_{K^p}(b_0)^{\mathrm{an}}.$$

ξ: finite dimensional irreducible algebraic representation over $\overline{\mathbb{Q}_\ell}$.

$t(\xi) \in \mathbb{Z}_{\geq 0}$ the weight of ξ.

$\mathcal{L}_\xi$: the associated local system on the Shimura variety.

$\mathcal{A}(I)$: space of automorphic representations of I counted with multiplicities.

$\mathcal{A}_\xi(I) := \{\Pi \in \mathcal{A}(I) \mid \Pi_\infty = \check{\xi}\}$.

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

Theorem (Fargues, 2004)

There is a $W \times \mathbb{G}(\mathbb{A}_f^p)$-equivariant spectral sequence

$$F_2^{a,b} = \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}})(1-n), \Pi_p) \otimes \Pi^p \Longrightarrow \mathrm{H}_c^{a+b}(\overline{\mathrm{S}}(b_0), \mathcal{L}_\xi),$$

where $\overline{\mathrm{S}}(b_0) := \varprojlim_{K^p} \overline{\mathrm{S}}_{K^p}(b_0)$.

From now assume $m = 1$, ie. $n = 3$ or 4. Then $\dim(\overline{\mathrm{S}}(b_0)) = 1$.
Let $\sigma := \mathrm{c} - \mathrm{Ind}_{\mathrm{N}_J(J_1)}^J \rho_{\Delta_2}$. It is an irreducible supercuspidal representation of $J(\mathbb{Q}_p)$.

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

Theorem (M.)

There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms

$$\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0), \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[p^{t(\xi)}],$$

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0), \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[p^{t(\xi)+2}],$$

where $\overline{\mathbb{Q}_\ell}[x]$ is the 1-dimensional representation of W with I acting trivially and Frob acts like $x \cdot \mathrm{id}$, and $X^{\mathrm{un}}(J)$ is the set of unramified characters of $J(\mathbb{Q}_p)$.

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

Theorem (M.)

$$\mathrm{H}^1_c(\overline{\mathrm{S}}(b_0), \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0 \\ \dim(\Pi_p) > 1}} (\nu - 1)\Pi^p \otimes \overline{\mathbb{Q}_\ell}[p^{t(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \nu \Pi^p \otimes \overline{\mathbb{Q}_\ell}[p^{t(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \sigma}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-p^{t(\xi)+1}],$$

where $\nu \in \mathbb{Z}_{\geq 0}$ is a multiplicity given by $\nu = p$ if $n = 3$, and $\nu = p^3$ if $n = 4$.

Cohomology of the basic locus of the $\mathrm{GU}(1, n-1)$ Shimura variety for $n = 3, 4$

Remark: The cohomology of the whole Shimura variety $\overline{\mathrm{S}}$ has been computed by Boyer (2010) when it is of Kottwitz-Harris-Taylor type.
In particular, no multiplicity such as ν occurs.

$\implies$ These multiplicities must also occur in the cohomology of the non-basic Newton strata. Expected connections with cohomology of Igusa varieties.

Thank you for your attention.

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