Semiclassical analysis and mean field dynamics.
Lecture summary

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1 Introduction.

This lecture is about the two problems :

1. The asymptotics as $h \to 0$ of the Schrödinger equation $ih\partial_t u = [-h^2\Delta + V(x)]u$, $x \in \mathbb{R}^d$ is related with the classical Hamiltonian dynamics $\dot{q} = 2p\,,\ \dot{p} = -\nabla_q V(q)\,,\ (q,p) \in \mathbb{R}^{2d} \sim \mathbb{C}^d\,.$

2. The mean field dynamics, that is the large N asymptotics, for

$$i\partial_t \Psi_N = \left[-\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N-1} \sum_{1\leq i<j\leq N} V(x_i - x_j)\right] \Psi_N\,,$$

$\Psi_N \in L^2(\mathbb{R}^{dN})$, is related with the nonlinear mean field dynamics

$$i\partial_t \psi = -\Delta\psi + (V * |\psi|^2)\psi \quad , \quad \psi \in L^2(\mathbb{R}^d;\mathbb{C})\,.$$

When dealing with bosonic particles, we shall see that 2) is exactly the infinite dimensional version of 1). After recalling why semiclassical (or Wigner) measures provide a very efficient and flexible tool to handle 1), we shall see that they are also very convenient to study the bosonic mean field dynamics.

Notations: We shall work (the contrary will be specified) with complex separable Hilbert spaces. The scalar product is left antilinear and right $\mathbb{C}$-linear. We shall simply write $L^2(M, d\mu)$ for $L^2(M, d\mu; \mathbb{C})$ and the scalar product will be

$$\langle u\,,\, v\rangle_{L^2(\mu)} = \int_M \overline{u(x)} v(x)\; d\mu(x)\,.$$

We shall use the bracket notation of physicist: $|v\rangle$ will be the vector v while $\langle u|$ is the form $v \to \langle u\,,\, v\rangle$. For a normalized vector u, $|u\rangle\langle u|$ is the orthogonal projection on $\mathbb{C}u$.
We shall work with a small parameter $h > 0$ or $\varepsilon > 0$. The rule of semiclassical analysis can be summarized as

- multiply any derivation by h, for any $\alpha \in \mathbb{N}^d$, $(h\partial_x)^\alpha = h^{|\alpha|}\partial_x^\alpha$ is an $\mathcal{O}(1)$ operator ;
- put a $\frac{1}{h}$ factor in any phase, $e^{i\frac{\varphi(x)}{h}}$ or $e^{\frac{\varphi}{h}}$;
- for integrations use the unit Lebesgue volume dx in the position variable and use $\frac{d\xi}{(2\pi h)^d}$ in the frequency of momentum variable.

The Fourier transform on $\mathbb{R}^d$ is normalized as

$$[F_h u](\xi) = \int_{\mathbb{R}^d} e^{-\frac{i\xi.x}{h}} u(x)\; dx \quad , \quad [F_h^{-1} v](x) = \int_{\mathbb{R}^d} e^{\frac{ix.\xi}{h}} v(\xi)\; \frac{d\xi}{(2\pi h)^d}\,.$$

(Even if there is no small paramater in the initial problem, putting a parameter $h > 0$ allows to switch easily from one to another normalization of the Fourier transform).
The operator D_x is defined by $D_x = \frac{1}{i}\partial_x$ and note the relations

$$F_h(u*v) = (F_h u)(F_h v) \quad , \quad F_h(uv) = (F_h u) \overset{2\pi h}{*} (F_h v) \quad , \quad F_h^{-1}(\xi\times)F_h = hD_x \,, .$$

The Schwartz space of rapidly decaying regular functions is denoted by $\mathcal{S}(\mathbb{R}^d)$ and its dual, the space of tempered distributions, by $\mathcal{S}'(\mathbb{R}^d)$.
Variables in the phase space are denoted by capital letters: X, Y, will be used for the variables $X = (x, \xi)$, $Y = (y, \eta)$ in $\mathbb{R}^{2d}$.
The symplectic form on $\mathbb{R}^{2d}$ will be denoted by

$$\sigma(X, Y) = \xi.y - x.\eta = \sum_{j=1}^{d} \xi_j y_j - x_j \eta_j \,.$$

2 Finite dimensional calculus

2.1 Phase-translations, coherent states and Weyl quantization

Set for $X_0 = (x_0, \xi_0) \in \mathbb{R}^{2d}$,

$$\varphi_0(x) = \frac{1}{(\pi h)^{d/4}} e^{-\frac{x^2}{2h}} \in L^2(\mathbb{R}^d)\,,$$
$$\forall u \in L^2(\mathbb{R}^d)\,, \quad [\tau^h_{X_0} u](x) = e^{i\frac{\xi_0\cdot(x - x_0/2)}{h}} u(x - x_0)\,,$$
$$\varphi_{X_0}(x) = \tau^h_{X_0}\varphi_0\,.$$

Definition 2.1. *For $X_0 = (x_0, \xi_0) \in \mathbb{R}^{2d}$, $\tau^h_{X_0}$ is called the phase translation of vector X_0.*
The function φ_{X_0} is the coherent state centered at X_0.

Properties:

- For any $X_0 \in \mathbb{R}^{2d}$, $\tau^h_{X_0}$ is a unitary operator in $L^2(\mathbb{R}^d)$ and $X_0 \to \tau^h_{X_0}$ is strongly continuous. For $X_1, X_2 \in \mathbb{R}^{2d}$ the Weyl relation

$$\tau^h_{X_1} \circ \tau^h_{X_2} = e^{i\frac{\sigma(X_1\,,\,X_2)}{2h}} \tau^h_{X_1+X_2}$$

  holds with $\sigma(X_1, X_2) = \xi_1.x_2 - x_1.\xi_2$.

- $$\tau^h_{X_0} = e^{i\frac{\xi_0.x - x_0.(hD_x)}{h}} = e^{i\frac{\sigma(X_0,(x,hD_x))}{h}}\,.$$

- If one denotes by $(\lambda, X) \in (\mathbb{R}/((2\pi h)\mathbb{Z})) \times \mathbb{R}^{2d}$ the element $e^{\frac{i}{h}\lambda}\tau_X^h$, it is a group with the law

$$(\lambda_1, X_1) \circ (\lambda_2, X_2) = (\lambda_1 + \lambda_2 + \frac{\sigma(X_1, X_2)}{2}, X_1 + X_2)\,,$$

(Heisenberg group).

- The Schwartz kernel of $\tau_{X_0}^h$ is given by

$$\tau_{X_0}^h(x,y) = \int_{\mathbb{R}^d} e^{i\frac{\xi.(x-y)}{h}} e^{i\frac{\xi_0\cdot(\frac{x+y}{2})-x_0\xi}{h}} \,\frac{d\xi}{(2\pi h)^d}$$

Definition 2.2. *For any $b \in \mathcal{S}'(\mathbb{R}^{2d}_{x,\xi})$, the Weyl quantized operator $b^W(x, hD_x) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is given by its kernel*

$$[b^W(x, hD_x)](x,y) = \int_{\mathbb{R}^d} e^{i\frac{\xi.(x-y)}{h}} b(\frac{x+y}{2}, \xi) \,\frac{d\xi}{(2\pi h)^d}\,.$$

Properties

- Any continuous operator $B : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ has a Weyl symbol, $B = b^W(x, hD_x)$ with

$$b(r,\xi) = \int e^{-i\frac{\xi.s}{h}} B(r + \frac{s}{2}, r - \frac{s}{2})\, ds\,.$$

- The formal adjoint of $b^W(x, hD_x)$ is $\overline{b}^W(x, hD_x)$.

- With the symplectic Fourier transform

$$\mathcal{F}_h b(P) = \int e^{i\frac{\sigma(P,X)}{h}} b(X) \frac{dX}{(2\pi h)^d}$$

the Weyl quantization is given by

$$b^W(x, hD_x) = \int_{\mathbb{R}^{2d}} \mathcal{F}_h b(P) \tau_P^h \,\frac{dP}{(2\pi h)^d}$$

for any $b \in \mathcal{F}_h L^1(\mathbb{R}^{2d})$.

- When $b \in L^2(\mathbb{R}^{2d})$, the operator $b^W(x, hD_x)$ is a Hilbert-Schmidt operator and

$$\mathrm{Tr}\,\left[b_1^W(x, hD_x)^* b_2^W(x, hD_x)\right] = \int_{\mathbb{R}^{2d}} \overline{b_1(x,\xi)} b_2(x,\xi)\, dx \frac{d\xi}{(2\pi h)^d}\,.$$

- The equality
$$\tau^h_{X_0} b^W(x,hD_x)\tau^h_{-X_0} = [b(.-X_0)]^W(x,hD_x)\,,$$
holds for any $b\in\mathcal{S}'(\mathbb{R}^{2d})$ and all $X_0\in\mathbb{R}^{2d}$.

- When $u,v\in\mathcal{S}(\mathbb{R}^d)$ the Weyl symbol of $|u\rangle\langle v|$ belongs to $\mathcal{S}(\mathbb{R}^{2d})$.

Example: The Weyl-symbol of $\Pi^h_{X_0}=|\varphi_{X_0}\rangle\langle\varphi_{X_0}|$ is $2^d e^{-\frac{(X-X_0)^2}{h}}$.
REF: [Fol][Rob][Mar][Per][Hep] [CoRo]

2.2 Anti-Wick quantization and Wick symbol

For $X_0\in\mathbb{R}^{2d}$ set $\Pi^h_{X_0}=|\varphi_{X_0}\rangle\langle\varphi_{X_0}|$.

Proposition 2.3. *For any $b\in\mathcal{S}'(\mathbb{R}^{2d})$ the operator*
$$b^{A-Wick}(x,hD_x)=\int_{\mathbb{R}^{2d}} b(X_0)\Pi^h_{X_0}\,\frac{dX_0}{(2\pi h)^d}\,,$$
is well defined and continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. The equality $b^{A-Wick}(x,hD_x)=c^W(x,hD_x)$ is equivalent to
$$c=b*\left(\frac{e^{-\frac{|\,|^2}{h}}}{(\pi h)^d}\right)\,.$$
The anti-Wick quantization is positive
$$(b\geq 0)\Rightarrow\left(b^{A-Wick}(x,hD_x)\geq 0\right)\,.$$

The equality $\mathrm{Id}_{L^2}=1^{A-Wick}(x,hD_x)$ says that the system of coherent states $(\varphi_{X_0})_{X_0\in\mathbb{R}^d}$ is overcomplete:
$$\mathrm{Id}_{L^2}=\int_{\mathbb{R}^{2d}}|\varphi_{X_0}\rangle\langle\varphi_{X_0}|\,\frac{dX_0}{(2\pi h)^d}$$
means
$$\forall u\in L^2(\mathbb{R}^d)\,,\quad u=\int_{\mathbb{R}^{2d}}\langle\varphi_{X_0}\,,\,u\rangle\varphi_{X_0}\,\frac{dX_0}{(2\pi h)^d}\,.$$

Proposition 2.4. *For any continuous operator $B:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ the symbol*
$$\sigma^{Wick}(B)(X_0)=\langle\varphi_{X_0}\,,\,B\varphi_{X_0}\rangle$$
is well-defined.
For $B=b^W(x,hD_x)$, one has
$$\sigma^{Wick}(b^W(x,hD_x))=b*\left(\frac{e^{-\frac{|\,|^2}{h}}}{(\pi h)^d}\right)\,.$$

Positivity:

$$(B \geq 0) \Rightarrow (\sigma^{Wick}(B) \geq 0)\,.$$

When B is trace class then $\sigma^{Wick}(B)$ belongs to $L^1(\mathbb{R}^{2d})$ and

$$\mathrm{Tr}\,[B] = \int_{\mathbb{R}^{2d}} \sigma^{Wick}(B)(X)\;\frac{dX}{(2\pi h)^d}\,.$$

The symbols which can be Wick quantized are the ones which belong to $\left(\frac{e^{-\frac{|\cdot|^2}{h}}}{(\pi h)^d}\right) * \mathcal{S}'(\mathbb{R}^{2d})$. It does not work for any symbol in $\mathcal{S}'(\mathbb{R}^{2d})$ but it works for example for polynomials. When it makes sense

$$\mathrm{Tr}\,\left[b^{A-Wick}C\right] = \int b(X)\sigma^{Wick}(C)(X)\;\frac{dX}{(2\pi h)^d}\,.$$

REF:[BeSh][Ler][UnUp]

2.3 Creation, annihilation, Wick quantization

Let $(e_1,\ldots,e_d)$ be the canonical orthonormal basis of $\mathbb{C}^d$ and set

$$a(e_j) = a_j = (h\partial_{x_j} + x_j) \quad , \quad a^*(e_j) = a_j^* = (-h\partial_{x_j} + x_j)\,,$$
$$\forall g = \sum_j g_j e_j \in \mathbb{C}^d\,, \quad a(g) = \sum_j \overline{g_j}a_j \quad , \quad a^*(g) = \sum_j g_j a_j^*\,.$$

Properties:

- Canonical Commutation relations:

$$\forall g,f \in \mathbb{C}^d\,, \qquad [a(g),a(f)] = [a^*(g),a^*(f)] = 0 \quad ,$$
$$[a(g),a^*(f)] = \varepsilon\langle g,f\rangle \quad , \quad \boxed{\varepsilon = 2h}$$

- Hermite functions: The vector $\varphi_0(x) = \frac{e^{-\frac{|x|^2}{2h}}}{(\pi h)^{d/4}}$ is also denoted by $|\Omega\rangle$ or ψ_0 and is called the vacuum. For $\alpha \in \mathbb{N}^d$, the α-th Hermite function is normalized as

$$\psi_\alpha = \frac{1}{\sqrt{\varepsilon^{|\alpha|}\alpha!}}(a^*)^\alpha|\Omega\rangle\,,$$

  and the family $(\psi_\alpha)_{\alpha\in\mathbb{N}^d}$ is a Hilbert basis of $L^2(\mathbb{R}^d)$. The multi-index notations $(\alpha^*)^\alpha = \prod_{j=1}^d a^*(e_j)^{\alpha_j}$ is allowed because the $a_j^* = a^*(e_j)$ commute. Remember also $\alpha! = \prod_{j=1}^d \alpha_j!$ and $|\alpha| = \sum_{j=1}^d \alpha_j$.

- Harmonic oscillator Hamiltonian: The operator $\mathbf{N} = (-h^2\Delta + x^2 - hd) = \sum_{j=1}^d a_j^* a_j$ is self-adjoint with eigenspaces $F_n = \oplus_{|\alpha|=n}\mathbb{C}\psi_\alpha$ and $\mathbf{N}\big|_{F_n} = \varepsilon n$.

- Weyl translations: Set for $g \in \mathbb{C}^d$, $\Phi(g) = \frac{1}{\sqrt{2}}(a(g)+a^*(g))$ is essentially self-adjoint on $\bigoplus_{n\in\mathbb{N}}^{alg} F_n$. The operator $W(f) = e^{i\Phi(f)}$ is nothing but

$$W(f) = \tau^h_{h\sqrt{2}X_0} \quad , \quad X_0 = (x_0, \xi_0)\,,\; f = \frac{1}{i}(x_0 + i\xi_0)\,.$$

They satisfy

$$\begin{aligned} &W(f_1)\circ W(f_2) = e^{-i\varepsilon\frac{\mathrm{Im}\,\langle f_1\,,\,f_2\rangle}{2}} W(f_1+f_2)\,,\\ &W(-f)a(g)W(f) = a(g) + \frac{i\varepsilon}{\sqrt{2}}\langle g\,,\,f\rangle\,,\\ &W(-f)a^*(g)W(f) = a^*(g) - \frac{i\varepsilon}{\sqrt{2}}\langle f\,,\,g\rangle\,,\\ &\langle \Omega\,, W(f)\Omega\rangle = e^{-\frac{\varepsilon|f|^2}{4}}\,. \end{aligned}$$

- Coherent states: For $z \in \mathbb{C}^d$, the function $W(\frac{\sqrt{2}}{i\varepsilon}z)|\Omega\rangle$ satisfies

$$\begin{aligned} &W(\frac{\sqrt{2}}{i\varepsilon}z)|\Omega\rangle = e^{\frac{a^*(z)-a(z)}{\varepsilon}}|\Omega\rangle = \tau^h_z\varphi_0 = \varphi_z\,, \quad z = z_R + iz_I = (z_R, z_I)\,,\\ &a(g)W(\frac{\sqrt{2}}{i\varepsilon}z)|\Omega\rangle = \langle g\,,\,z\rangle W(\frac{\sqrt{2}}{i\varepsilon}z)|\Omega\rangle\,. \end{aligned}$$

Proposition 2.5. *When $P(z,\overline{z}) = \sum_{|\alpha|+|\beta|\le n} c_{\alpha,\beta}\overline{z}^\alpha z^\beta$ is a polynomial on $\mathbb{R}^{2d}_{x,\xi}$ identified with $\mathbb{C}^d$ via $z = x + i\xi$, its Wick quantization is given*

$$P^{Wick}(x, hD_x) = \sum_{|\alpha|+|\beta|\le n} c_{\alpha,\beta}(a^*)^\alpha a^\beta\,,$$

i.e. by replacing z (resp. $\overline{z}$) by a (resp. a^*) and keeping the annihilation operators on the right-hand side.

2.4 Bargmann representation

The Bargmann transform is given by

$$[B_h u](z) = \frac{1}{(\pi h)^{d/4}} e^{\frac{z^2}{4h}} \int_{\mathbb{R}^d} e^{-\frac{(z-y)^2}{2h}} u(y)\;dy\,, \quad y \in \mathbb{R}^d,\; z \in \mathbb{C}^d\,.$$

It can also be written

$$[B_h u](z) = e^{\frac{|z|^2}{4h}}\langle \varphi_{\overline{z}}\,,\,u\rangle\,,$$

with the identification $\mathbb{C}^d \ni z = z_R + iz_I = (z_R, z_I) \in \mathbb{R}^{2d}$. The Lebesgue measure on $\mathbb{C}^d$ will be denoted by $L(dz)$.

Proposition 2.6. *The Bargmann transform B_h is an isometry from $L^2(\mathbb{R}^d, dx)$ into $L^2(\mathbb{C}^d, e^{-\frac{|z|^2}{2h}}\frac{L(dz)}{(2\pi h)^d})$, $B_h^* B_h = \mathrm{Id}$. Its range is the closed set of entire functions*

$$L^2(\mathbb{C}^d, e^{-\frac{|z|^2}{2h}}\frac{L(dz)}{(2\pi h)^d}) \cap \mathcal{H}(\mathbb{C}^d)\,.$$

The operator $B_h B_h^*$ is the orthogonal projection given by

$$[\Pi_h f](z) = \int_{\mathbb{C}^d} e^{\frac{z.\overline{z'}-|z'|^2}{2h}} f(z')\,\frac{L(dz')}{(2\pi h)^d}\,.$$

When $a_j = (h\partial_{x_j} + x_j)$ and $a_j^* = (-h\partial_{x_j} + x_j)$ one gets

$$B_h(a_j^*)B_h^* = z_j \times \quad , \quad B_h(a_j)B_h^* = \varepsilon\partial_{z_j} = \Pi_h(\overline{z_j}\times)\Pi_h \quad \boxed{\varepsilon = 2h}\,.$$

Additional properties:

- For any $\alpha \in \mathbb{N}^d$ the Hermite function ψ_α is transformed into $B_h\psi_\alpha = \frac{1}{\sqrt{\varepsilon^{|\alpha|}\alpha!}} z^\alpha$.
- For any $z_0 = z_{0,R} + iz_{0,I} = (z_{0,R}, z_{0,I})$ the coherent state φ_{z_0} is transformed into
$$B_h\varphi_{z_0} = e^{-\frac{|z_0|^2}{2\varepsilon}} e^{\frac{\langle \overline{z_0}\,,\, z\rangle}{\varepsilon}}\,.$$
- When $P(z,\overline{z}) = \sum_{|\alpha|+|\beta|\le n} c_{\alpha,\beta} z^\alpha \overline{z}^\beta$ is a polynomial on $\mathbb{R}^{2d}_{x,\xi}$ identified with $\mathbb{C}^d$ via $z = x + i\xi$, the anti-Wick quantization is given by
$$P^{A-Wick}(x, hD_x) = B_h^*(P\times)B_h \quad ,$$
and $B_h P^{A-Wick}(x, hD_x) B_h^*$ is nothing but the Toeplitz operator $\Pi_h(P\times)\Pi_h$.

Final remarks: Instead of holomorphic function one could consider a space of anti-holomorphic functions (simply replace z by $\overline{z}$ in the definition of B_h). Then, the anti-Wick quantization of $P(z,\overline{z}) = \sum_{|\alpha|+|\beta|\le n} c_{\alpha,\beta}\overline{z}^\alpha z^\beta$ is obtained by replacing z (resp. $\overline{z}$) by a (resp. by a^*) while keeping a on the left-hand side (Wick or anti-Wick refers to the order of a and a^* in the products, another not so widespread name is covariant or contravariant Berezin quantization).
Another consequence that we will develop further is that an element of $B_h L^2(\mathbb{R}^d)$ as an (anti)-entire function is equal to its Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!}\mathcal{D}^k_{\overline{z}} f(0).\overline{z}^{\otimes k} = \sum_{k:0}^{\infty}\frac{1}{k!}\langle z^{\otimes k}\,,\, \Sigma_k(f)\rangle_{(\mathbb{C}^d)^{\otimes k}}\,,$$

where the differential $\mathcal{D}^k_{\overline{z}} f(0)$ (or $\Sigma_k(f)$) is a symmetric k-tensor. Hence $L^2(\mathbb{R}^d)$ can also be viewed as a Hilbert sum of symmetric tensor powers of $\mathbb{C}^d$. We shall come back to this picture later.
REF:[Fol][Mar]

2.5 Weyl-Hörmander Calculus

We have already seen the Weyl quantization. Let us specify now sufficient conditions so that symbol classes become algebras.
Examples: Functions $b \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that

$$\forall \alpha, \beta \in \mathbb{N}^d\,, \quad |\partial_x^\alpha \partial_\xi^\beta b(X)| \leq C_{\alpha,\beta}$$

can be described as functions which fulfill the $N \in \mathbb{N}$-dependent estimates ($C_N > 0$ may depend on N): for any vector fields $T_1, \ldots T_N$ with $g(T_i) \leq 1$, $g = \sum_{j=1}^d dx_j^2 + d\xi_j^2$,

$$|T_1 \ldots T_N b(X)| \leq C_N\,.$$

The condition

$$|\partial_x^\alpha \partial_\xi^\beta b(X)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}\,,$$

which is fulfilled by polynomials in ξ, $\sum_{|\alpha| \leq n} c_\alpha(x) \xi^\alpha$, is equivalent to

$$|T_1 \ldots T_N b(X)| \leq C_N M(X) \quad g(T_i) \leq 1\,,$$

$$\text{with} \qquad g = dx^2 + \frac{d\xi^2}{\langle \xi \rangle^2} \quad \text{and} \quad M(X) = \langle \xi \rangle^m\,.$$

Other examples $g = \frac{dx^2}{\langle x \rangle^2} + d\xi^2$, $M(X) = \langle x \rangle^s$, $g = \frac{dx^2 + d\xi^2}{\langle X \rangle^2}$, $M(X) = \langle X \rangle^m$.

Definition 2.7. *Let g be a riemannian metric on $\mathbb{R}^{2d}$ and $M > 0$ be a continuous function. The space $S(M, g)$ is the subset of $\mathcal{C}^\infty(\mathbb{R}^{2d})$ made of functions b such that*

$$\forall N \in \mathbb{N}, \exists C_N > 0, \qquad \sup_{X \in \mathbb{R}^{2d}, g(T_i) \leq 1} \frac{|T_1 \ldots T_N b(X)|}{M(X)} \leq C_N\,.$$

Endowed with the seminorms $p_N(b) = \sup_{X \in \mathbb{R}^{2d}, g(T_i) \leq 1} \frac{|T_1 \ldots T_N b(X)|}{M(X)}$ it is a Fréchet-space.

Definition 2.8. *For a metric g on $\mathbb{R}^{2d}$, the dual metric g^σ is the dual metric for the symplectic form*

$$g_X^\sigma(T) = \max_{S \neq 0} \frac{|\sigma(T, S)|^2}{g_X(S)}$$

and the gain function is given by

$$\lambda(X)^2 = \min_{T \neq 0} \frac{g_X^\sigma(T)}{g_X(T)}\,.$$

Example: When $g = \sum_{j=1}^{d} \frac{dx_j^2}{a_j(X)^2} + \frac{d\xi_j^2}{b_j(X)^2}$ the dual metric is $g^\sigma = \sum_{j=1}^{d} b_j(X)^2 dx_j^2 + a_j(X)^2 d\xi_j^2$ and the gain function $\lambda(X) = \min_{j=\{1,\ldots,d\}} a_j(X) b_j(X)$.

Hörmander's conditions for the metric:

- Uncertainty principle: $\lambda \geq 1$.
- Slowness: There exists a constant $C_L > 0$ such that

$$\left(g_X(X-Y) \leq \frac{1}{C_L} \right) \Rightarrow \left(\left(\frac{g_X}{g_Y} \right)^{\pm 1} \leq C_L \right).$$

- Temperance: There exist two constant $C_T > 0$ and $N_T > 0$ such that

$$\left(\frac{g_X}{g_Y} \right)^{\pm 1} \leq C_T (1 + g_X^\sigma(X-Y))^{N_T}.$$

Hörmander's conditions for the weight:

- g-slowness:

$$\left(g_X(X-Y) \leq \frac{1}{C_{ML}} \right) \Rightarrow \left(\left(\frac{M(X)}{M(Y)} \right)^{\pm 1} \leq C_{ML} \right).$$

- g-temperance:

$$\left(\frac{M(X)}{M(Y)} \right)^{\pm 1} \leq C_{MT} (1 + g_X^\sigma(X-Y))^{N_{MT}}.$$

Proposition 2.9. *When g is a Hörmander metric and M a g-weight, for any $b \in S(M,g)$ and any $h \in (0, h_0)$, $b^W(x, hD_x)$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^d)$) into itself.*

When $b_j \in S(M_j, g)$, $j = 1, 2$, the operators $b_j^W(x, hD_x)$ can be composed.

Definition 2.10. *For $b_j \in S(M_j, g)$, g Hörmander metric, M_j g-weights, $j = 1, 2$, the Moyal product $b_1 \sharp^h b_2$ is the Weyl symbol of $b_1^W(x, hD_x) \circ b_2^W(x, hD_x)$.*

Theorem 2.11. *Assume that g is Hörmander metric, M_1 and M_2 are g-weights and assume $b_j \in S(M_j, g)$, $j = 1, 2$. Then the Moyal product $b_1 \sharp^h b_2$*

is given by

$$
\begin{aligned}
& b_1 \sharp^h b_2(X) = e^{i\frac{\sigma(hD_{X_1}, hD_{X_2})}{2h}} b(X_1)b(X_2)\big|_{X_1=X_2=X} \\
& = e^{\frac{ih}{2}\sigma(D_{X_1}, D_{X_2})} b(X_1)b(X_2)\big|_{X_1=X_2=X} \\
& = \sum_{k=0}^{K} \frac{h^k}{k!}\left(\frac{i\sigma(D_{X_1}, D_{X_2})}{2}\right)^k b(X_1)b(X_2)\big|_{X_1=X_2=X} \\
& + h^{K+1}\int_0^1 \frac{(1-t)^K}{K!} e^{\frac{ith}{2}\sigma(D_{X_1}, D_{X_2})}\, dt \left(\frac{i\sigma(D_{X_1}, D_{X_2})}{2}\right)^{K+1} a_1(X_1)a_2(X_2)\big|_{X_1=X_2=X} \\
& = \sum_{k=0}^{K} h^k T_k(b_1, b_2) + h^{K+1} R_{K+1}(b_1, b_2, h)\,,
\end{aligned}
$$

where the bilinear mapping

$$
\begin{aligned}
& T_k : S(M_1, g) \times S(M_2, g) \to S(M_1 M_2 \lambda^{-k}, g)\,, \\
\text{and} \quad & R_{K+1}(h) S(M_1, g) \times S(M_2, g) \to S(M_1 M_2 \lambda^{-K-1}, g)
\end{aligned}
$$

are bilinear continuous, uniformly w.r.t $h \in (0, h_0)$ (the seminorm estimates depend on the structural constants $C_L, C_T, C_{M_j,L}, C_{M_j,T}$ the dimension and the final seminorm).

Main terms:
At order 0:

$$
b_1 \sharp^h b_2 \sim b_1 b_2\,.
$$

At order 1:

$$
b_1 \sharp^h b_2 \sim b_1 b_2 + \frac{h}{2i}\{b_1, b_2\}
$$

and the commutator $\left[b_1^W(x, hD_x), b_2^W(x, hD_x)\right]$ has the Weyl symbol

$$
\frac{h}{i}\{b_1, b_2\} + \mathcal{O}(h^2)\,.
$$

Theorem 2.12. ***(Calderon-Vaillancourt)*** *When g is a Hörmander metric there exist $C, N > 0$ such that*

$$
\|b^W(x, hD_x)\|_{\mathcal{L}(L_2)} \leq C p_N(b)
$$

for all $b \in S(1, g)$ and all $h \in (0, h_0)$. When $g = dx^2 + d\xi^2$, the constants C and N depend on the dimension d (and h_0).

Proposition 2.13. *When $\lim_{X\to\infty} M(X) = 0$ and $b \in S(M, g)$, $b^W(x, hD_x)$ is a compact operator on $L^2(\mathbb{R}^d)$.*

Proposition 2.14. *If $b \in S(1, dx^2 + d\xi^2)$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ satisfies $\chi \equiv 1$ in a neighborhood of 0, then*

$$\operatorname*{s-lim}_{n\to\infty} (b\chi(n^{-1}.))^W(x, hD_x) = b^W(x, hD_x)\,,$$

where the limit holds in the strong operator topology in $\mathcal{L}(L^2(\mathbb{R}^d))$. For $b \in S(1, dx^2 + d\xi^2)$

$$\|b^W(x, hD_x) - b^{A-Wick}(x, hD_x)\|_{\mathcal{L}(L^2)} = \mathcal{O}(h)\,.$$

REF: [Hor]-Chap XVIII, [Ler], [BoLe], [BoCh], [NaNi], [HeNi].

2.6 Semiclassical measures

We have already seen that when ϱ_h is a non negative trace class operator with $\mathrm{Tr}\,[\varrho_h] = 1$ (a normal state) then $\frac{1}{(2\pi h)^d}\sigma^{Wick}(\varrho_h)$ is a(n absolutely continuous) probability measure on $\mathbb{R}^{2d}$. Let $\mathcal{M}_b(\mathbb{R}^{2d})$ be the set of bounded Radon measures on $\mathbb{R}^d$. It is the dual of the separable space of $\mathcal{C}^0$ functions with limit 0 at infinity. Therefore bounded subsets are relatively sequentially compact for the weak-$*$ topology.

Definition 2.15. *For a family $(\varrho_h)_{h\in(0,h_0)}$ of normal states the semiclassical measure (or Wigner measures) are the weak-$*$ limit points of $\frac{1}{(2\pi h)^d}\sigma^{Wick}(\varrho_h)$ in $\mathcal{M}_b(\mathbb{R}^{2d})$.*
The set of semiclassical measures associated with $(\varrho_h)_{h\in(0,h_0)}$ is denoted by $\mathcal{M}(\varrho_h, h \in (0, h_0))$.
The family $(\varrho_h)_{h\in(0,h_0)}$ is said pure if $\mathcal{M}(\varrho_h, h \in (0, h_0))$ is reduced to a single element.
The same definitions can be used for general bounded family $(\varrho_h)_{h\in(0,h_0)}$ in $\mathcal{L}^1(L^2(\mathbb{R}^d))$ with possibly complex valued measures.

First properties:

- When $(\varrho_h)_{h\in(0,h_0)}$ is a family of states any $\mu \in \mathcal{M}(\varrho_h, h \in (0, h_0))$ satisfies
$$0 \le \mu \quad \text{and} \quad \int_{\mathbb{R}^{2d}} d\mu \le 1\,.$$
- The element μ of $\mathcal{M}(\varrho_h, h \in (0, h_0))$ are characterized by: There exists a sequence $(h_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} h_n = 0$ such that
$$\forall b \in \mathcal{D}\,, \quad \lim_{n\to\infty} \mathrm{Tr}\,[b^\bullet(x, h_nD_x)\varrho_{h_n}] = \int_{\mathbb{R}^{2d}} b(X)\; d\mu(X)\,, \tag{2.1}$$
where $\mathcal{D}$ is any dense subset of $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and $\bullet$ stands for W or $A-Wick$.

- For any $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ and any bounded family $(\varrho_h)_{h\in(0,h_0)}$ in $\mathcal{L}^1(L^2(\mathbb{R}^d))$ one has

$$\begin{aligned}\mathcal{M}\left(\chi^\bullet(x,hD_x)\varrho_h,\ h\in(0,h_0)\right) &= \mathcal{M}\left(\varrho_h\chi^\bullet(x,hD_x),\ h\in(0,h_0)\right)\\ &= \{\chi\mu\,,\ \mu\in\mathcal{M}(\varrho_h,\ h\in(0,h_0))\}\,.\end{aligned}$$

- When $(\varrho_h)_{h\in(0,h_0)}$ is a family of states such that

$$\forall h\in(0,h_0)\,,\quad \mathrm{Tr}\left[\chi_R^\bullet(x,hD_x)\varrho_h\right]\geq 1-\delta_R\,,$$

  for some $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^{2d})$, $0\leq\chi\leq 1$, $\chi_R(X)=\chi(R^{-1}X)$, with $\lim_{R\to\infty}\delta_R=0$, then any $\mu\in\mathcal{M}(\varrho_h,h\in(0,h_0))$ is a probability measure. Moreover $\mu\in\mathcal{M}(\varrho_h,\ h\in(0,h_0))$, is characterized by

$$\lim_{n\to\infty}\mathrm{Tr}\left[W(\frac{1}{i\sqrt{2}}X_0)\varrho_h\right]=\int_{\mathbb{R}^{2d}}e^{i\sigma(X_0,X)}\ d\mu(X_0)\,.$$

  A sufficient condition is $\mathrm{Tr}\left[\mathbf{N}^\nu\varrho_h\right]\leq C_\nu$ for some $\nu>0$.

- Assume that the family of states $(\varrho_h)_{h\in(0,h_0)}$ satisfies $\mathrm{Tr}\left[\mathbf{N}^\nu\varrho_h\right]\leq C_\nu$ for all $\nu>0$, then the convergence (2.1) holds for any $b\in S(\langle X\rangle^k,dX^2)$, in particular for any polynomial symbol b. Moreover in (2.1), $\bullet$ stands for W, $A-Wick$ or $Wick$.

- When $\varrho_h=\chi^W(x,hD_x)\varrho_h\chi^W(x,hD_x)$ for some compactly supported χ, then any $\mu\in\mathcal{M}(\varrho_h,\ h\in(0,h_0))$ has a compact support, $\mathrm{supp}\,\mu\subset\mathrm{supp}\,\chi$, and it is characterized by (2.1) when $\mathcal{D}$ is the set of polynomial functions on $\mathbb{R}^{2d}$ (Hamburger moment problem).

- Assume that $\varrho_h=\varrho_h^1\otimes\varrho_h^2$ in the decomposition $L^2(\mathbb{R}^d)=L^2(\mathbb{R}^{d_1})\otimes L^2(\mathbb{R}^{d_2})$, with $\mathrm{Tr}\left[\mathbf{N}^\delta\varrho_h\right]\leq C_\delta$ for some $\delta>0$, then

$$\mathcal{M}(\varrho_h,\ h\in(0,h_0))=\left\{\mu_1\otimes\mu_2\,,\ \mu_i\in\mathcal{M}(\varrho_h^i,\ h\in(0,h_0))\right\}\,.$$

Theorem 2.16. *Assume $V(x)\in S(1,dx^2;\mathbb{R})$ and consider the Hamiltonian $H^h=p(x,hD_x)=-h^2\Delta+V(x)$. Let $(\varrho_h)_{h\in(0,h_0)}$ be a family of states such that $\mathcal{M}(\varrho_h,h\in(0,h_0))=\{\mu_0\}$, then for any $t\in\mathbb{R}$ the family $(\varrho_h(t)=e^{-\frac{itH^h}{h}}\varrho_he^{\frac{it}{h}H_h},\ h\in(0,h_0))$ is pure with:*

$$\mathcal{M}(\varrho_h(t),\ h\in(0,h_0))=\{\mu_t\}\quad\textit{with}\quad\mu_t=\Phi(t)_*\mu_0\,,$$

where $\Phi(t)$ is the classical hamiltonian flow

$$(x(t),\xi(t))=\Phi(t)(x_0,\xi_0)\quad,\quad\left\{\begin{array}{l}\dot{x}=2\xi=\partial_\xi p\\ \dot{\xi}=-\partial_xV=-\partial_xp\,.\end{array}\right.$$

Remember that with the complex notation $z = x + i\xi$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_\xi)$, the Hamilton equations can be written

$$i\partial_t z = 2\partial_{\bar{z}} p\,.$$

Examples:

- When $\varrho_h = |\varphi_{X_0}\rangle\langle\varphi_{X_0}|$, then $\mathcal{M}(\varrho_h,\ h \in (0,h_0)) = \{\delta_{X_0}\}$.
- Any probability measure on $\mathbb{R}^{2d}$ is a semiclassical measure.
- When $d = 1$ and $\varrho_h = |\psi_\alpha\rangle\langle\psi_\alpha|$ with $\alpha = \left[\frac{c_1}{h}\right]$ then

$$\mu = \frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}\sqrt{c_1}}\, d\theta = \delta^{S^1}_{\sqrt{c_1}}\,.$$

- In dimension $d \geq 2$, with $\varrho_h = |\psi_\alpha\rangle\langle\psi_\alpha|$ with $\alpha = (\left[\frac{c_1}{h}\right], \left[\frac{c_2}{h}\right], 0, \ldots, 0)$ then

$$\mu = \delta^{S^1}_{c_1}(X_1) \times \delta^{S^1}_{c_2}(X_2) \times \delta_0(X_3, \ldots, X_d)\,.$$

Proposition 2.17. *Assume* $\|u_h^j\|_{L^2} = 1$ *for* $j = 1, 2$ *and* $h \in (0, h_0)$ *with*

$$\mathcal{M}(|u_j^h\rangle\langle u_j^h|,\ h \in (0,h_0)) = \{\mu_j\}$$

where $\mu_1 \perp \mu_2$. *Then the scalar product* $\langle u_1^h\,,\, u_2^h\rangle$ *tends to* 0 *as* $h \to 0$.

REF: [HMR][Ger][LiPa][Bur][GMMP][AmNi1].

2.7 Remarks

In this presentation, we focused on observables scaled as $a^\bullet(x, hD_x)$, which is the standard way for semiclassical analysis. A simple change of scale

$$b^\bullet(hx, D_x) = \mathrm{Dil}_h^{-1} \circ b^\bullet(x, hD_x) \circ \mathrm{Dil}_h$$
$$\text{with} \qquad (\mathrm{Dil}_t u)(x) = t^{-d/2} u(t^{-1}x)\,,$$

allows to translate the definition of semiclassical measures according to

$$\mathrm{Tr}\,\left[b^W(hx, D_x)\varrho_h\right] \overset{h' \to 0}{\to} \int b(X)\, d\mu'(X)\,.$$

In this scaling thing are measured at $\mathcal{O}(1)$ frequencies (or momentum) and at a macroscopic scale $x = \mathcal{O}(\frac{1}{h})$ for the position.
In order to keep a symmetric role of the position x and frequency ξ variables, another possible scaling is

$$b^\bullet(\sqrt{h}x, \sqrt{h}D_x) = \mathrm{Dil}_{\sqrt{h}}^{-1} \circ b^\bullet(x, hD_x) \circ \mathrm{Dil}_{\sqrt{h}}$$

with the definition of semiclassical measures as

$$\mathrm{Tr}\,\left[b^W(\sqrt{h}x,\sqrt{h}D_x)\varrho_h\right] \overset{h'\to 0}{\to} \int b(X)\; d\mu''(X)\,.$$

Note that with this last scaling, the creation and annihilation operators (and consequently the Wick calculus) are given by

$$\begin{aligned}
&\mathrm{Dil}_{\sqrt{h}}^{-1}\circ a(e_j)\circ \mathrm{Dil}_{\sqrt{h}} = \sqrt{2h}\frac{(\partial_{x_j}+x_j)}{\sqrt{2}} = a_\varepsilon(e_j) = \sqrt{\varepsilon}a_{\varepsilon=1}(e_j)\,,\\
&\mathrm{Dil}_{\sqrt{h}}^{-1}\circ a(e_j)\circ \mathrm{Dil}_{\sqrt{h}} = \sqrt{2h}\frac{(-\partial_{x_j}+x_j)}{\sqrt{2}} = a_\varepsilon^*(e_j) = \sqrt{\varepsilon}a^*_{\varepsilon=1}(e_j)\,,\\
&(\mathrm{Dil}_\lambda u)(x) = \lambda^{-d/2}u(\lambda^{-1}x)\,.
\end{aligned}$$

For the number operator $\mathbf{N}_\varepsilon = \varepsilon\mathbf{N}_{\varepsilon=1}$ and for a general homogeneous polynomial $P(z,\overline{z}) = \sum_{|\alpha|+|\beta|=m} c_{\alpha,\beta}z^\alpha\overline{z}^\beta$, $P_\varepsilon^{Wick} = \varepsilon^{\frac{m}{2}}P^{Wick}_{\varepsilon=1}$.

3 Infinite dimensional Wigner measures

3.1 Bosonic quantum field theory

3.1.1 Bosonic Fock space

Let $\mathcal{Z}$ be a separable complex Hilbert space with scalar product $\langle z_1\,,\,z_2\rangle$, the real Hilbert space structure associated with the real scalar product $S(z_1\,,\,z_2) = \mathrm{Re}\,\langle z_1\,,\,z_2\rangle$, and the natural symplectic form $\mathrm{Im}\,\langle z_1\,,\,z_2\rangle$. The norm of $z\in\mathcal{Z}$ with simply be denoted by $|z|$ (or possibly $|z|_{\mathcal{Z}}$) , with $|z|^2 = \langle z\,,\,z\rangle$.
Example: $\mathcal{Z} = \mathbb{C}^d$, $z = x+i\xi = \sum_{j=1}^d z^je_j = \sum_{j=1}^d x^je_j + i\sum_{j=1}^d \xi^je_j \in \mathbb{C}^d \sim \mathbb{R}^{2d}$,

$$\begin{aligned}
\langle z_1\,,\,z_2\rangle &= \sum_{j=1}^d \overline{z_1^j}z_2^j\,,\\
S(z_1\,,\,z_2) &= \sum_{j=1}^d x_1^jx_2^j + \xi_1^j\xi_2^j\\
\mathrm{Im}\,\langle z_1\,,\,z_2\rangle &= \sum_{j=1}^d -\xi_1^jx_2^j + x_1^j\xi_2^j = -\sigma(X_1\,,\,X_2)\,.
\end{aligned}$$

The symmetric tensor power of $\mathcal{Z}$ is denoted by $\bigvee^k\mathcal{Z}$ and spanned as a Hilbert space by the elements

$$z_1\vee\ldots\vee z_k = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} z_{\sigma(1)}\otimes\cdots\otimes z_{\sigma(k)} = \Pi_+(z_1\otimes\cdots\otimes z_k)\,,\quad z_j\in\mathcal{Z}\,.$$

Another possible definition is $\bigvee^k \mathcal{Z} = \Pi_+ Z^{\otimes k}$, where $\mathcal{Z}^{\otimes k}$ is the Hilbert tensor power of $\mathcal{Z}$ and Π_+ is the orthogonal projection given by the above formula.
Example: When $\mathcal{Z} = L^2(\mathbb{R}^d, dx; \mathbb{C})$, $\bigvee^k \mathcal{Z}$ is the set of symmetric L^2 functions on $\mathbb{R}^{dk}$.
The bosonic Fock space is given as the Hilbert direct sum

$$\Gamma_+(\mathcal{Z}) = \oplus_{k=0}^{\infty} \bigvee^k \mathcal{Z}\,.$$

Remark 3.1. *When it is not specified, direct sums and tensor products are considered as their Hilbert completion. For the algebraic direct sums or tensor product a* ^{alg} *exponent will be written. For example if* $\mathcal{H} = \Gamma_+(\mathcal{Z})$*, we shall often use the set* $\mathcal{H}_{fin} = \oplus_{k\in\mathbb{N}}^{alg} \bigvee^k \mathcal{Z}$.

3.1.2 Basic operations

- Polarization identity

$$z_1 \vee \ldots \vee z_k = \frac{1}{2^k k!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \ldots \varepsilon_k \left(\sum_{j=1}^{k} \varepsilon_j z_j \right)^{\otimes k}\,.$$

  Consequence: $\bigvee^k \mathcal{Z} = \overline{\mathrm{Vect}(z^{\otimes k}\,,\ z \in \mathcal{Z})}$
  and $\mathcal{H} = \Gamma_+(\mathcal{Z}) = \overline{\mathrm{Vect}(z^{\otimes k}\,,\ z \in \mathcal{Z}\,,\ k \in \mathbb{N})}$.

- For any $f \in \mathcal{Z}$, the operator $a(f)$ is defined on $\mathcal{H}_{fin} = \bigoplus_{k\in\mathbb{N}}^{alg} \bigvee^k \mathcal{Z}$ by

$$a(f) z^{\otimes k} = \sqrt{\varepsilon k} \langle f, z \rangle z^{\otimes (k-1)}$$

  or

$$a(f) z_1 \vee \ldots \vee z_k = \frac{\sqrt{\varepsilon k}}{k!} \sum_{\sigma \in \mathfrak{S}_k} \langle f\,,\ z_{\sigma(1)} \rangle z_{\sigma(2)} \otimes \cdots \otimes z_{\sigma(k)}\,.$$

  Properties:

  - $[a(f_1), a(f_2)] = 0$;
  - $\|a(f)\big|_{\bigvee^k \mathcal{Z}}\| = \sqrt{\varepsilon k} |f|_{\mathcal{Z}}$;
  - $f \to a(f)$ is $\mathbb{C}$-antilinear;
  - when $\mathcal{Z} = L^2(\mathbb{R}^d, dx; \mathbb{C})$, one can define in the distributional sense $a(x)$ by

$$a(f) = \int_{\mathbb{R}^d} \overline{f}(x) a(x)\, dx\,, \quad \forall f \in \mathcal{S}(\mathbb{R}^d)$$

    meaning also

$$a(x) z^{\otimes k} = \sqrt{\varepsilon k} z(x) z^{\otimes (k-1)}\,, \quad \forall z \in \mathcal{S}(\mathbb{R}^d)\,.$$

- The formal adjoint of $a(f)$ for $f \in \mathcal{Z}$ is defined on $\mathcal{H}_{fin}$ by

$$a^*(f)z^{\otimes k} = \Pi_+ \left(\sqrt{\varepsilon(k+1)} f \otimes z^{\otimes k}\right) .$$

  Properties:

  - $[a^*(f_1), a^*(f_2)] = 0$;
  - $[a(g), a^*(f)] = \varepsilon\langle g\,,\, f\rangle$;
  - $\|a^*(f)\big|_{\bigvee^k \mathcal{Z}}\| = \sqrt{\varepsilon(k+1)}|z|$;
  - The mapping $f \to a^*(f)$ is $\mathbb{C}$-linear;
  - When $\mathcal{Z} = L^2(\mathbb{R}^d)$, $a^*(x)$ is defined by

$$a^*(f) = \int_{\mathbb{R}^d} f(x)a^*(x)\ dx\,,$$

    and one has $[a(x), a^*(x')] = \varepsilon\delta(x - x')$.

- Set $\bigvee^0 \mathcal{Z} = \mathbb{C}|\Omega\rangle$ where $|\Omega\rangle$ is called the vacuum. For any Hilbert basis $(e_j)_{j\in\mathbb{N}}$ the family $(\psi_\alpha)_{\alpha\in\mathbb{N}[X]}$ given by

$$\psi_\alpha = \frac{1}{\sqrt{\varepsilon^\alpha \alpha!}}(a^*)^\alpha|\Omega\rangle$$

  with $\alpha! = \prod_{j=0}^\infty \alpha_j!$ and $(a^*)^\alpha = \prod_{j=0}^\infty a^*(e_j)^{\alpha_j}$, is a Hilbert basis of $\mathcal{H}$.

- When $C \in \mathcal{L}(\mathcal{Z})$ is a contraction, $\|C\| \leq 1$, the operator $\Gamma_+(C)$ is the contraction of $\mathcal{H}$ defined by

$$\Gamma_+(C)(z_1 \vee \ldots \vee z_k) = (Cz_1) \vee \ldots \vee (Cz_k)$$
$$\text{or}\qquad \Gamma_+(C) = \Pi_+(C \otimes \cdots \otimes C)\Pi_+ = C \otimes \cdots \otimes C)\big|_{\Gamma_+(\mathcal{Z})} .$$

  When C is unitary, $\Gamma_+(C)$ is unitary.

- A particular case is when $C = e^{itA}$ where $(A, D(A))$ (resp. (A, D)) is self-adjoint (resp. essentially self-adjoint) in $\mathcal{Z}$. Then

$$\begin{aligned} d\Gamma_+(A) &= i\varepsilon\partial_t\Gamma_+(e^{-itA}) \\ &= \varepsilon\Pi_+ \left(A \otimes \mathrm{Id}_\mathcal{Z} \cdots \otimes \mathrm{Id}_\mathcal{Z} + \cdots + \mathrm{Id}_\mathcal{Z} \cdots \otimes \mathrm{Id}_\mathcal{Z} \otimes A\right) \Pi_+ \\ &= \varepsilon \left(A \otimes \mathrm{Id}_\mathcal{Z} \cdots \otimes \mathrm{Id}_\mathcal{Z} + \cdots + \mathrm{Id}_\mathcal{Z} \cdots \otimes \mathrm{Id}_\mathcal{Z} \otimes A\right) \big|_{\Gamma_+(\mathcal{Z})} \end{aligned}$$

  is essentially self-adjoint on $\oplus_{k\in\mathbb{N}}^{alg} \bigvee^{alg,k} D(A)$ (resp. $\oplus_{k\in\mathbb{N}}^{alg} \bigvee^{alg,k} D$).
  When $A = \mathrm{Id}_\mathcal{Z}$ one finds

$$\mathbf{N} = d\Gamma_+(\mathrm{Id}) \quad , \quad \mathbf{N}\psi_\alpha = \varepsilon|\alpha|\psi_\alpha\,,$$
$$\mathbf{N} = \sum_{j\in\mathbb{N}} a^*(e_j)a(e_j)\,,$$
$$\mathbf{N} = \int_{\mathbb{R}^d} a^*(x)a(x)\ dx \quad \text{when} \quad \mathcal{Z} = L^2(\mathbb{R}^d, dx; \mathbb{C})\,.$$

When $\mathcal{Z} = L^2(\mathbb{R}^d, dx; \mathbb{C})$ and the self-adjoint operator has the Schwartz kernel $A(x,y)$, the operator $d\Gamma(A)$ can be written

$$d\Gamma(A) = \int_{\mathbb{R}^{2d}} A(x,y) a^*(x) a(y)\ dxdy\,.$$

3.1.3 Separation of variables

Proposition 3.2.

$$\left(\mathcal{Z} = \mathcal{Z}_1 \overset{\perp}{\oplus} \mathcal{Z}_2\right) \Rightarrow \left(\Gamma_+(\mathcal{Z}) = \Gamma_+(\mathcal{Z}_1) \otimes \Gamma_+(\mathcal{Z}_2)\right)\,.$$

In this decomposition we have:

- $|\Omega\rangle = |\Omega_1\rangle \otimes |\Omega_2\rangle\,.$
- For $z_j \in \mathcal{Z}_j$, $j = 1, 2$,

$$\begin{aligned} &a(z_1) = a_1(z_1) \otimes \mathrm{Id}_{\Gamma_+(\mathcal{Z}_2)} \quad , \quad a^*(z_1) = a_1^*(z_1) \otimes \mathrm{Id}_{\Gamma_+(\mathcal{Z}_2)}\,,\\ &a(z_2) = \mathrm{Id}_{\Gamma_+(\mathcal{Z}_1)} \otimes a_2(z_2) \quad , \quad a^*(z_2) = \mathrm{Id}_{\Gamma_+(\mathcal{Z}_1)} \otimes a_2^*(z_2)\,,\\ &\mathbf{N} = \mathbf{N}_1 \otimes \mathrm{Id}_{\Gamma_+(\mathcal{Z}_2)} + \mathrm{Id}_{\Gamma_+(\mathcal{Z}_1)} \otimes \mathbf{N}_2\,. \end{aligned}$$

More generally when $(A_j, D(A_j))$, $j = 1, 2$, are self-adjoint operators in $\mathcal{Z}_j$,

$$d\Gamma(A_1 \oplus A_2) = d\Gamma_+(A_1) \otimes \mathrm{Id}_{\Gamma_+(\mathcal{Z}_2)} + \mathrm{Id}_{\Gamma_+(\mathcal{Z}_1)} \otimes d\Gamma_+(A_2)\,.$$

Examples:

- $\mathbb{C}^{d_1+d_2} = \mathbb{C}^{d_1} \overset{\perp}{\oplus} \mathbb{C}^{d_2}$ and $L^2(\mathbb{R}^{d_1+d_2}, dx) = L^2(\mathbb{R}^{d_1}, dx_1) \otimes L^2(\mathbb{R}^{d_2}, dx_2)$ (separation of variables).
- When $\Omega = \Omega_1 \sqcup \Omega_2$, $L^2(\Omega) = L^2(\Omega_1) \oplus L^2(\Omega_2)$ and when $(A_j, D(A_j))$, $j = 1, 2$, are self-adjoint operators with Schwartz kernels $A_j(x,y)$, one gets

$$\begin{aligned} d\Gamma(A) = \int_{\Omega_1\times\Omega_1} & A_1(x_1,y_1) a_1^*(x_1) a(y_1)\ dx_1 dy_1\\ &+ \int_{\Omega_1\times\Omega_2} A_2(x_2,y_2) a_2^*(x_2) a(y_2)\ dx_2 dy_2\,. \end{aligned}$$

- When $\mathcal{Z}$ is a Hilbert space and p is a finite rank orthogonal projection

$$\Gamma_+(\mathcal{Z}) = \Gamma_+(p\mathcal{Z}) \otimes \Gamma_+((1-p)\mathcal{Z})$$

and $p\mathcal{Z}$ (resp. $\Gamma_+(p\mathcal{Z})$) is isomorphic to $\mathbb{C}^{\mathrm{rk}\ p}$ (resp. to $L^2(\mathbb{R}^{\mathrm{rk}\ p}, dx)$).

Definition 3.3. *The set of finite rank orthogonal projections in $\mathcal{Z}$ will be denoted by $\mathbb{P}$.*

Remember the non commutative Fubini rule: When $\varrho_j \in \mathcal{L}^1(\mathcal{H}_j)$, $j = 1, 2$, then $\varrho_1 \otimes \varrho_2 \in \mathcal{L}^1(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with

$$\mathrm{Tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2}[\varrho_1 \otimes \varrho_2] = \mathrm{Tr}_{\mathcal{H}_1}[\varrho_1] \times \mathrm{Tr}_{\mathcal{H}_2}[\varrho_2] \,,$$

and that reciprocally when $\varrho \in \mathcal{L}^1(\mathcal{H})$ the partial traces define trace class operators

$$\begin{aligned} & \mathrm{Tr}_{\mathcal{H}_1}[\varrho] \in \mathcal{L}^1(\mathcal{H}_2) \quad , \quad \mathrm{Tr}_{\mathcal{H}_2}[\varrho] \in \mathcal{L}^1(\mathcal{H}_1)\,, \\ \text{with} \quad & \mathrm{Tr}\,[\varrho] = \mathrm{Tr}_{\mathcal{H}_2}[\mathrm{Tr}_{\mathcal{H}_1}[\varrho]] = \mathrm{Tr}_{\mathcal{H}_1}[\mathrm{Tr}_{\mathcal{H}_2}[\varrho]] \,. \end{aligned}$$

REF: [ReSi][Ber][BSZ]

3.2 Weyl translation, coherent states

Definition 3.4. *For $z \in \mathcal{Z}$, the field operator is defined by*

$$\Phi(z) = \frac{1}{\sqrt{2}}(a(z) + a^*(z))\,,$$

the Weyl translations by

$$W(z) = e^{i\Phi(z)}\,,$$

and the coherent state by

$$E(z) = W(\frac{\sqrt{2}}{i\varepsilon}z)|\Omega\rangle\,.$$

Properties:

- When $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ then for any $z = z_1 \oplus z_2$,

$$W_{\mathcal{Z}}(z) = W_{\mathcal{Z}_1}(z_1) \otimes W_{\mathcal{Z}_2}(z_2)\,.$$

- All the formulas of the finite dimensional case carry over to $\mathcal{Z}$ (e.g. for $W(-z_2)a(z_1)W(z_2)$ write $\mathcal{Z} = p\mathcal{Z} \oplus (p\mathcal{Z})^\perp$ with $z_1, z_2 \in p\mathcal{Z}$ and use the result in $p\mathcal{Z}$). Note that $a_j = \sqrt{h}(\partial_{x_j} + x_j) = \sqrt{\varepsilon}\frac{(\partial_{x_j}+x_j)}{\sqrt{2}}$ according to Section 2.7.

- Application: By writing $\mathcal{Z} = (\mathbb{C}z) \overset{\perp}{\oplus} (\mathbb{C}z)^\perp$ and using $a(z)E(z) = |z|^2E(z)$ in dimension 1 one gets

$$E(z) = e^{-\frac{|z|^2}{2\varepsilon}} \sum_{k=0}^{\infty} \frac{1}{\sqrt{\varepsilon^k k!}} z^{\otimes k}\,.$$

From the above decomposition we can Weyl-quantize some symbols.

Definition 3.5. *A function f on $\mathcal{Z}$ is said to be cylindrical if there exists $p \in \mathbb{P}$ and a function g on $p\mathcal{Z}$ such that*

$$\forall z \in \mathcal{Z}\,, \quad f(z) = g(pz)\,.$$

For a finite dimensional functional space usually denoted by $\mathcal{A}$ we shall use the notation $\mathcal{A}_{cyl}(\mathcal{Z})$ for the set of its cylindrical version. For example, $f \in \mathcal{S}_{cyl}(\mathcal{Z})$ means that there exists $p \in \mathbb{P}$ and $g \in \mathcal{S}(p\mathcal{Z})$ such that $f = g \circ p$.
If $f = g \circ p$, one says that f is based on $p\mathcal{Z}$.
The Lebesgue measure on $p\mathcal{Z}$, associated with the norm $|z|_{p\mathcal{Z}} = |z|_{\mathcal{Z}}$ for $z \in p\mathcal{Z}$, is denoted by $L_{p\mathcal{Z}}(dz)$.

Remark: In general $\mathcal{A}_{cyl}(\mathcal{Z})$ is not an algebra nor a vector space. It is the case if one takes $\mathcal{A} = \mathcal{C}^0_b$ the set of bounded continuous functions. With many examples of algebras, the Stone-Weierstrass theorem, says that $\mathcal{A}_{cyl}$ is dense in the set of continuous function on the ball $\{z \in \mathcal{Z}, |z| \leq R\})$ endowed with weak topology of $\mathcal{Z}$. The measurable version of Stone-Weierstrass theorem is also useful (see e.g. [Cou]) and provides a way to identify a bounded Borel probability measure on $\mathcal{Z}$ carried by a ball $\{|z| \leq R\}$ by testing with a "small" set of test functions. A convenient subalgebra of $\mathcal{C}^0_{b,cyl}(\mathcal{Z})$ is $\cup_{p\in\mathbb{P}}\mathcal{F}^{-1}(\mathcal{M}_b(p\mathcal{Z}))$, which can be denoted by $(\mathcal{F}^{-1}\mathcal{M}_b)_{cyl}$. Symbol classes are associated with metrics on finite dimensional phase-spaces, e.g.:

$$S^s_\nu(p\mathcal{Z}) = S(\langle z\rangle^s_{p\mathcal{Z}}, \frac{d|z|^2_{p\mathcal{Z}}}{\langle z\rangle^{2\nu}_{p\mathcal{Z}}}) \quad s \in \mathbb{R},\, \nu \in [0,1]\,, \quad \langle z\rangle^2_{p\mathcal{Z}} = 1 + |z|^2_{p\mathcal{Z}}\,.$$

One defines as above $S^s_{\nu,cyl} = \cup_{p\in\mathbb{P}} S^s_\nu(p\mathcal{Z})$ and the case $s = 0$, $\nu = 0$, provides another example of algebra.

Definition 3.6. *When $b \in \mathcal{S}_{cyl}(\mathcal{Z})$ is based on $p\mathcal{Z}$, one defines*

$$(\mathcal{F}b)(z) = \int_{p\mathcal{Z}} b(\xi) e^{-2i\pi S(z,\xi)}\; L_{p\mathcal{Z}}(d\xi)\,,$$

$$\text{and} \qquad b^W = \int_{p\mathcal{Z}} (\mathcal{F}b)(z) W(\sqrt{2}\pi z)\; L_{p\mathcal{Z}}(dz)\,.$$

This definition can be extended to $b \in (\mathcal{F}^{-1}\mathcal{M}_b)_{cyl}$ or to $b \in S^s_{\nu,cyl}$ with $s \in \mathbb{R}$, $\nu \in [0,1]$.

REF:[Ber][BSZ][BrRo][Cou][AmNi1]

3.3 Wick calculus

Let us specify first what is a polynomial function of $z, \overline{z}$ with $z \in \mathcal{Z}$.

Definition 3.7. *For $p, q \in \mathbb{N}$, $\mathcal{P}_{p,q}(\mathcal{Z})$ denotes the set of (p,q)-homogeneous polynomial functions on $\mathcal{Z}$ which fulfill :*

$$b(z) = \left\langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p} \right\rangle \quad \text{with} \quad \tilde{b} \in \mathcal{L}(\bigvee{}^{p}\mathcal{Z}, \bigvee{}^{q}\mathcal{Z})\,.$$

The subspace of $\mathcal{P}_{p,q}(\mathcal{Z})$ made of polynomials b such that $\tilde{b}$ is a compact operator $\tilde{b} \in \mathcal{L}^{\infty}(\bigvee^{p}\mathcal{Z}, \bigvee^{q}\mathcal{Z})$ (resp. $b \in \mathcal{L}^{r}(\bigvee^{p}\mathcal{Z}, \bigvee^{q}\mathcal{Z})$), is denoted by $\mathcal{P}^{\infty}_{p,q}(\mathcal{Z})$ (resp. $\mathcal{P}^{r}_{p,q}(\mathcal{Z})$).
On those spaces, the natural norms are

$$|b|_{\mathcal{P}_{p,q}} = \|\tilde{b}\|_{\mathcal{L}(\bigvee^{p}\mathcal{Z}, \bigvee^{q}\mathcal{Z})} \quad \text{and} \quad |b|_{\mathcal{P}^{r}_{p,q}} = \|\tilde{b}\|_{\mathcal{L}^{r}(\bigvee^{p}\mathcal{Z}, \bigvee^{q}\mathcal{Z})}\,, \quad 1 \leq r\,.$$

The set of non homogeneous polynomials, the algebraic direct sum $\oplus^{alg}_{p,q\in\mathbb{N}}$ $\mathcal{P}_{p,q}(\mathcal{Z})$ (resp. $\oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}^{r}_{p,q}(\mathcal{Z})$ with $1 \leq r \leq \infty$), will be denoted by $\mathcal{P}_{alg}(\mathcal{Z})$ (resp. $\mathcal{P}^{r}_{alg}(\mathcal{Z})$).

Owing to the condition $\tilde{b} \in \mathcal{L}(\bigvee^{p}\mathcal{Z}, \bigvee^{q}\mathcal{Z})$ for $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, this definition implies that any Gâteaux differential $\partial^{j}_{\overline{z}}\partial^{k}_{z}b(z)$ at the point $z \in \mathcal{Z}$ belongs to $\mathcal{L}(\bigvee{}^{k}\mathcal{Z}, \bigvee{}^{j}\mathcal{Z})$ with

$$\langle \varphi, \partial^{j}_{\overline{z}}\partial^{k}_{z}b(z)\psi\rangle = \frac{p!}{(p-k)!}\frac{q!}{(q-j)!}\langle z^{\otimes q-j} \vee \varphi, \tilde{b}\, z^{\otimes p-k} \vee \psi\rangle\,.$$

In particular $\tilde{b}$ equals $\frac{1}{p!}\frac{1}{q!}\partial^{p}_{z}\partial^{q}_{\overline{z}}b(z)$.
With any "symbol" $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, a linear operator b^{Wick} called Wick monomial can be associated, according to:

$$\begin{aligned} & b^{Wick} : \mathcal{H}_{fin} \to \mathcal{H}_{fin}, \\ & b^{Wick}_{\bigvee^{n}} = 0 \quad \text{for } n < p \\ b^{Wick}_{|\bigvee^{n+p}\mathcal{Z}} = {} & \frac{\sqrt{(n+p)!(n+q)!}}{n!}\, \varepsilon^{\frac{p+q}{2}} \underbrace{\left(\tilde{b} \bigvee I_{\bigvee^{n}\mathcal{Z}}\right)}_{\in\mathcal{L}(\bigvee^{n+p}\mathcal{Z}, \bigvee^{n+q}\mathcal{Z})}\,, \end{aligned} \tag{3.1}$$

with $\tilde{b} = (p!)^{-1}(q!)^{-1}\partial^{p}_{z}\partial^{q}_{\overline{z}}b(z)$.
The basic symbol-operator correspondence is:

$$\begin{array}{cccccccccc} \langle z, \xi\rangle & \longleftrightarrow & a^{*}(\xi) & \sqrt{2}S(\xi, z) & \longleftrightarrow & \Phi(\xi) & \langle z, Az\rangle & \longleftrightarrow & d\Gamma(A) \\ \langle \xi, z\rangle & \longleftrightarrow & a(\xi) & \sqrt{2}\sigma(\xi, z) & \longleftrightarrow & \Pi(\xi) & |z|^{2} & \longleftrightarrow & \mathbf{N}\,, \end{array}$$

and more generally

$$\left(\prod_{i=1}^{p}\langle z, \eta_{i}\rangle \times \prod_{j=1}^{q}\langle \xi_{j}, z\rangle\right)^{Wick} = a^{*}(\eta_{1})\cdots a^{*}(\eta_{p})a(\xi_{1})\cdots a(\xi_{q}).$$

We have the following properties.

Proposition 3.8. *The following identities hold true on $\mathcal{H}_{fin}$ for every $b \in \mathcal{P}_{p,q}(\mathcal{Z})$:*
(i) $\left(b^{Wick}\right)^* = \bar{b}^{Wick}$.
(ii) $\left(C(z)b(z)A(z)\right)^{Wick} = C^{Wick}b^{Wick}A^{Wick}$, *if* $A \in \mathcal{P}_{\alpha,0}(\mathcal{Z})$, $C \in \mathcal{P}_{0,\beta}(\mathcal{Z})$.
(iii) $e^{i\frac{t}{\varepsilon}d\Gamma(A)}b^{Wick}e^{-i\frac{t}{\varepsilon}d\Gamma(A)} = \left(b(e^{-itA}z)\right)^{Wick}$, *if A is a self-adjoint operator on* $\mathcal{Z}$.

A consequence of i) says that b^{Wick} is symmetric when $q = p$ and $\tilde{b}^* = \tilde{b}$. Moreover the definition (3.1) gives

$$\left(q = p \quad \text{and} \quad \tilde{b} \geq 0\right) \Rightarrow \left(b^{Wick} \geq 0 \text{ on } \mathcal{H}_{fin}\right) , \tag{3.2}$$

which is false for general non negative polynomial symbols[1]. For an increasing net of non negative operators $(\tilde{b}_\alpha)_\alpha$, $\tilde{b}_\alpha \in \mathcal{L}(\bigvee^p \mathcal{Z})$ (again $q = p$), it also gives

$$\left(\tilde{b} = \sup_\alpha\ \tilde{b}_\alpha \ \text{ in } \ \mathcal{L}(\bigvee{}^p\mathcal{Z})\right) \ \Rightarrow \tag{3.3}$$
$$\left(\forall \varphi \in \mathcal{H}_{fin}\,, \quad \langle \varphi\,,\, b^{Wick}\varphi\rangle = \sup_\alpha \langle \varphi\,,\, b_\alpha^{Wick}\varphi\rangle\right) .$$

When $\mathcal{Z} = L^2(\mathbb{R}^d, dx)$, the general formula for b^{Wick} with $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ is simply

$$b^{Wick} = \int_{\mathbb{R}^{d(p+q)}} \tilde{b}(y_1, \ldots, y_q, x_1, \ldots, x_p)\, a^*(y_1) \ldots a^*(y_q)$$
$$a(x_1) \ldots a(x_p)\ dx_1 \cdots dx_p\, dy_1 \cdots dy_q\,,$$

where $\tilde{b}(y, x)$ is the Schwartz kernel of $\tilde{b}$ which has to be understood as

$$\langle z_1^{n+q}\,,\, b^{Wick} z_2^{n+p}\rangle = \sqrt{\frac{(n+q)!(n+p)!}{n!}}\langle z_1\,,\, z_2\rangle^n \varepsilon^{\frac{p+q}{2}}$$
$$\times \int_{\mathbb{R}^{d(p+q)}} \overline{z_1^{\otimes q}(y)} z_2^{\otimes p}(x) b(y, x)\ dxdy$$

for $z_1, z_2 \in \mathcal{S}(\mathbb{R}^d)$. Such a weak formulation makes sense for $b \in \mathcal{S}'$ only for the Wick order.

Proposition 3.9. *For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the following number estimate holds*

$$\left\| \langle \mathbf{N}\rangle^{-\frac{q}{2}}\, b^{Wick}\, \langle \mathbf{N}\rangle^{-\frac{p}{2}} \right\|_{\mathcal{L}(\mathcal{H})} \leq |b|_{\mathcal{P}_{p,q}} . \tag{3.4}$$

[1] This property should not be confused with the positivity of the finite dimensional Anti-Wick quantization which associates a non negative operator to any non negative symbol.

An important property of our class of Wick polynomials is that a composition of $b_1^{Wick} \circ b_2^{Wick}$ with $b_1, b_2 \in \mathcal{P}_{alg}(\mathcal{Z})$ is a Wick polynomial with symbol in $\mathcal{P}_{alg}(\mathcal{Z})$. For $b_1 \in \mathcal{P}_{p_1,q_1}(\mathcal{Z})$, $b_2 \in \mathcal{P}_{p_2,q_2}(\mathcal{Z})$, $k \in \mathbb{N}$ and any fixed $z \in \mathcal{Z}$, $\partial_z^k b_1(z) \in \mathcal{L}(\bigvee^k \mathcal{Z}; \mathbb{C})$ while $\partial_{\bar{z}}^k b_2(z) \in \bigvee^k \mathcal{Z}$. The $\mathbb{C}$-bilinear duality product $\partial_z^k b_1(z).\partial_{\bar{z}}^k b_2(z)$ defines a function of $z \in \mathcal{Z}$ simply denoted by $\partial_z^k b_1.\partial_{\bar{z}}^k b_2$. We also use the following notation for multiple Poisson brackets:

$$\begin{aligned} &\{b_1, b_2\}^{(k)} = \partial_z^k b_1.\partial_{\bar{z}}^k b_2 - \partial_z^k b_2.\partial_{\bar{z}}^k b_1, \quad k \in \mathbb{N}, \\ &\{b_1, b_2\} = \{b_1, b_2\}^{(1)}. \end{aligned}$$

Proposition 3.10. *Let* $b_1 \in \mathcal{P}_{p_1,q_1}(\mathcal{Z})$ *and* $b_2 \in \mathcal{P}_{p_2,q_2}(\mathcal{Z})$ *.*
For any $k \in \{0, \ldots, \min\{p_1, q_2\}\}$, $\partial_z^k b_1.\partial_{\bar{z}}^k b_2$ *belongs to* $\mathcal{P}_{p_1+p_2-k,q_1+q_2-k}(\mathcal{Z})$ *with the estimate*

$$|\partial_z^k b_1.\partial_{\bar{z}}^k b_2|_{\mathcal{P}_{p_1+p_2,q_1+q_2}} \leq \frac{p_1!}{(p_1-k)!}\frac{q_2!}{(q_2-k)!}|b_1|_{\mathcal{P}_{p_1,q_1}}|b_2|_{\mathcal{P}_{p_2,q_2}}\,.$$

The formulas

$$\begin{aligned} (i) \quad b_1^{Wick} \circ b_2^{Wick} &= \left(\sum_{k=0}^{\min\{p_1,q_2\}} \frac{\varepsilon^k}{k!} \; \partial_z^k b_1.\partial_{\bar{z}}^k b_2\right)^{Wick} \\ &= \left(e^{\varepsilon\langle\partial_z,\partial_{\bar{\omega}}\rangle} b_1(z) b_2(\omega)\,|_{z=\omega}\right)^{Wick}, \\ (ii) \quad [b_1^{Wick}, b_2^{Wick}] &= \left(\sum_{k=1}^{\max\{\min\{p_1,q_2\}\,,\min\{p_2,q_1\}\}} \frac{\varepsilon^k}{k!} \; \{b_1, b_2\}^{(k)}\right)^{Wick}, \end{aligned}$$

hold as identities on $\mathcal{H}_{fin}$.

REF:[ReSi][DeGe][AmNi1][AmNi3][FGS][FKP] [KrRa][Las]

3.4 Wigner measures

The Wigner measures are defined after the next result proved in [AmNi1, Theorem 6.2].

Theorem 3.11. *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ *be a family of normal states on* $\mathcal{H}$ *parametrized by* ε*. Assume* $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\delta] \leq C_\delta$ *uniformly w.r.t.* $\varepsilon \in (0, \bar{\varepsilon})$ *for some fixed* $\delta > 0$ *and* $C_\delta \in (0, +\infty)$*. Then for every sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *with* $\lim_{n\to\infty} \varepsilon_n = 0$ *there exists a subsequence* $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ *and a Borel probability measure* μ *on* $\mathcal{Z}$ *such that*

$$\lim_{k\to\infty} \mathrm{Tr}\,[\varrho_{\varepsilon_{n_k}} b^W] = \int_{\mathcal{Z}} b(z)\; d\mu(z)\,,$$

for all $b \in (\mathcal{F}^{-1}\mathcal{M}_b)_{cyl}$.

Moreover this probability measure μ *satisfies* $\displaystyle\int_{\mathcal{Z}} |z|^{2\delta}\, d\mu(z) < \infty.$

Definition 3.12.

The set of Wigner measures associated with a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ (resp. a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$) which satisfies the assumptions of Theorem 3.11 is denoted by

$$\mathcal{M}(\varrho_\varepsilon, \varepsilon\in(0,\bar\varepsilon))\,, \quad (\textit{resp. } \mathcal{M}(\varrho_{\varepsilon_n}, n\in\mathbb{N}))\,.$$

When $\mathcal{M}(\varrho_\varepsilon, \varepsilon\in(0,\bar\varepsilon)) = \{\mu\}$, the family $(\varrho_\varrho)_{\varepsilon\in(0,\bar\varepsilon)}$ is said pure.
By linearity, this definition can be extended to any family of trace-class operators $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ such that the uniform estimate $\|\langle\mathbf{N}\rangle^\delta\varrho_\varepsilon\langle\mathbf{N}\rangle^\delta\|_{\mathcal{L}^1(\mathcal{H})} \leq C_\delta$ holds for some $\delta>0$.

Examples:

- When $\varrho_\varepsilon = \Pi_0 = |\Omega\rangle\langle\Omega|$ for all $\varepsilon\in(0,\bar\varepsilon)$, then $\mathcal{M}(\varrho_\varepsilon, \varepsilon\in(0,\bar\varepsilon)) = \{\delta_0\}$.
- When $\varrho_\varepsilon = \varrho_{\varepsilon,1}\otimes\varrho_{\varepsilon,2}$ in the decomposition $\Gamma_+(\mathcal{Z}_1 \overset{\perp}{\oplus} \mathcal{Z}_2) = \Gamma_1(\mathcal{Z}_1)\otimes\Gamma_+(\mathcal{Z}_2)$ with

$$\begin{aligned} &\mathrm{Tr}\left[\mathbf{N}_j^\delta\varrho_{\varepsilon,j}\right] \leq C_\delta\,,\\ \text{and}\quad &\mathcal{M}(\varrho_{\varepsilon,j}, \varepsilon\in(0,\bar\varepsilon)) = \{\mu_j\} \quad \text{for } j=1,2, \end{aligned}$$

  then $\mathcal{M}(\varrho_\varepsilon, \varepsilon\in(0,\bar\varepsilon)) = \{\mu_1\otimes\mu_2\}$.
- For $\varrho_\varepsilon = \Pi_{z_0} = |E(z_0)\rangle\langle E(z_0)| = \Pi_{z_0}\otimes\Pi_0$ in the decomposition $\mathcal{Z} = (\mathbb{C}z_0) \overset{\perp}{\oplus} (\mathbb{C}z_0)^\perp$ and the finite dimensional result gives

$$\mathcal{M}(\Pi_{z_0}\,,\ \varepsilon\in(0,\bar\varepsilon)) = \{\delta_{z_0}\}\,.$$

  The same argument for the Hermite state $|\psi(z_0,n)\rangle\langle\psi(z_0,n)|$ with $\psi(z_0,n) = \frac{1}{\sqrt{\varepsilon^n n!}}[a^*(z_0)]^n|\Omega\rangle$ $\lim_{\varepsilon\to0} n\varepsilon = 1$ and now $|z_0|=1$, gives

$$\mathcal{M}(|\psi_{z_0,n}\rangle\langle\psi_{z_0,n}|,\ \varepsilon\in(0,\bar\varepsilon)) = \delta_{z_0}^{S^1} = \frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}z_0}\,d\theta\,.$$

  This can be tensorized according to $\psi(z_1,z_2,n_1,n_2)$, $\langle z_1\,,\ z_2\rangle = 0$, $|z_1| = |z_2| = 1$ and $\lim_{\varepsilon\to0} n_j\varepsilon = \frac{1}{2}$:

$$\mathcal{M}(|\psi(z_1,z_2,n_1,n_2)\rangle\langle\psi(z_1,z_2,n_1,n_2), \varepsilon\in(0,\bar\varepsilon)) = \delta^{S^1}_{\frac{\sqrt2}{2}z_1}\otimes\delta^{S^1}_{\frac{\sqrt2}{2}z_2}\,.$$

- Any Borel probability measure on $\mathcal{Z}$ can be realized as a semiclassical measure: Take $\varrho_\varepsilon = \int_{\mathcal{Z}} |E(z)\rangle\langle E(z)|\ d\mu(z)$.
- Let $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $\mathcal{Z}$ and consider $\varrho_\varepsilon = |E(e_n)\rangle\langle E(e_n)|$ with $n = [\varepsilon^{-1}]$ then

$$\mathcal{M}(\varrho_\varepsilon,\ \varepsilon\in(0,\bar\varepsilon)) = \{\delta_0\}$$

- The free Bose gas on a torus: Take $\mathcal{Z} = L^2(\mathbb{R}^d/\mathbb{Z}^d)$ and consider the asymptotic behaviour as $\varepsilon \to 0$ of the Gibbs state
$$\varrho_\varepsilon = \frac{1}{\mathrm{Tr}\,\left[\Gamma(e^{-\beta(-\varepsilon^{2/d}\Delta - \mu_\varepsilon)})\right]}\Gamma(e^{-\beta(-\varepsilon^{2/d}\Delta-\mu_\varepsilon)})\,.$$
The Bose-Einstein condensation occurs when $d \geq 3$ and when the chemical potential satisfies $\mu_\varepsilon = \varepsilon\mu$ with $\mu < \mu(\beta)$. Depending on the case:

  $\mu > \mu(\beta)$**:** $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\delta_0\}$.

  $\mu < \mu(\beta)$**:** $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\gamma_\mu \otimes \delta_0\}$ in the decomposition $\mathcal{Z} = (\mathbb{C}1) \oplus (\mathbb{C}1)^\perp$ with $\gamma_\nu(z_1) = \frac{e^{-\frac{|z_1|^2}{c\mu}}}{(\pi c_\mu)} L(dz_1)$.

REF: [Sko] [AmNi1]

3.5 Wigner measures and BBGKY hierarchy

We consider a family of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ such that
$$\forall k \in \mathbb{N}\,,\ \exists C_k > 0\,,\ \forall \varepsilon \in (0,\bar{\varepsilon})\,, \quad \mathrm{Tr}\,\left[\mathbf{N}^k \varrho_\varepsilon\right] \leq C_k\,, \tag{3.5}$$
$$\mathcal{M}\left(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\right) = \{\mu\}\,. \tag{3.6}$$

Then the probability measure μ satisfies
$$\forall k \in \mathbb{N},\, \forall p \in \mathbb{P},\ \int_{\mathcal{Z}} \langle pz\rangle^{2k}\, d\mu(z) \leq \int_{\mathcal{Z}} \langle z\rangle^{2k}\, d\mu(z) \leq \liminf_{\varepsilon\to 0} \mathrm{Tr}\,\left[\langle \mathbf{N}\rangle^k \varrho_\varepsilon\right] \leq C'_k\,,$$
and the convergence
$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,\left[b^W \varrho_\varepsilon\right] = \int_{\mathcal{Z}} b(z)\ d\mu(z)$$
holds for any cylindrical $b(z) = g(pz)$ with $g \in \mathcal{S}(\langle z\rangle^n_{p\mathcal{Z}}, \frac{dz^2}{\langle z\rangle^2_{p\mathcal{Z}}})$ and therefore when g is a polynomial function on $p\mathcal{Z}$, $p \in \mathbb{P}$. Moreover the finite dimensional case says that b^W can be replaced by b^{Wick}. A monomial $b(z) = \langle z^{\otimes q}\,, \tilde{b}z^{\otimes p}\rangle$ is cylindrical when $\tilde{b} \in \mathcal{L}(\bigvee^p \mathcal{Z}; \bigvee^q \mathcal{Z})$ has a finite rank. The number estimates (3.4) and (3.5) lead to the uniform estimate to
$$\left|\mathrm{Tr}\,\left[(b-b')^{Wick}\varrho_\varepsilon\right]\right| \leq C_{p,q}|b-b'|_{\mathbb{P}_{p,q}} = C_{p,q}\|\tilde{b}-\tilde{b}'\|\,,$$
so that the convergence can be extended to any $\tilde{b}$ compact.

Proposition 3.13. *Under the conditions* (3.5)(3.6)*, the convergence*
$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,\left[b^{Wick}\varrho_\varepsilon\right] = \int_{\mathcal{Z}} b(z)\ d\mu(z)\,,$$
holds for any $b \in \mathcal{P}_\infty(\mathcal{Z})$.

Then, the question is whether it holds for any $b \in \mathcal{P}(\mathcal{Z})$ without the compactness of $\tilde{b}$.
The answer is: **NO**. In [AmNi1], we called the corresponding phenomenon the infinite dimensional defect of compactness (because it is due to the infinite dimension of the phase-space and may occur while remaining in finite ball of the phase-space).
Examples:

- When $\varrho_\varepsilon = |E(e_n)\rangle\langle E(e_n)|$ with $n = [\varepsilon^{-1}]$ and $(e_n)_{n\in\mathbb{N}}$ is a Hilbert basis of $\mathcal{Z}$, take $b(z) = |z|^2$, $\tilde{b} = \mathrm{Id}$ and computed

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,[\mathbf{N}\varrho_\varepsilon] = \lim_{n\to\infty} |e_n|^2 = 1 \neq 0 = \int_{\mathcal{Z}} |z|^2\ \delta_0(z)\,.$$

- The free Bose gas presented above provides another example.

In [AmNi2] we introduced the condition

$$(PI)\quad \forall k \in \mathbb{N},\ \lim_{\varepsilon\to 0} \mathrm{Tr}\,\left[\mathbf{N}^k \varrho_\varepsilon\right] = \int_{\mathcal{Z}} |z|^{2k}\ d\mu(z)$$

and proved

Proposition 3.14. *When the conditions* (3.6) *and* (PI) *are satisfied the convergence*

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,\left[b^{Wick}\varrho_\varepsilon\right] = \int_{\mathcal{Z}} b(z)\ d\mu(z)\,,$$

holds for any $b \in \mathcal{P}(\mathcal{Z})$ *.*

When $\mathcal{Z} = L^2(\mathbb{R}^d, dx_1)$ and $\varrho_\varepsilon = |\psi_N\rangle\langle\psi_N|$ with $\psi_N \in \bigvee^N \mathcal{Z}$, $N \sim \frac{1}{\varepsilon}$, the reduced density matrix of order $k \in [0, N]$ is defined by

$$\gamma_\varepsilon^{(k)}(x_1,\ldots,x_k,y_1,\ldots,y_k) = \int_{\mathbb{R}^{d(N-k)}} \psi_N(x_1,\ldots,x_k,X)\overline{\psi(y_1,\ldots,y_k,X)}\ dX\,,$$

or $\quad \forall b \in \mathcal{P}_{kk}(\mathcal{Z})\,,\quad \varepsilon^k \dfrac{N!}{(N-k)!}\mathrm{Tr}\,\left[\tilde{b}\gamma_\varepsilon^k\right] = \mathrm{Tr}\,\left[b^{Wick}\varrho_\varepsilon\right]\,.$

After noticing $\varepsilon^k \frac{N!}{(N-k)!} = \mathrm{Tr}\,\left[(|z|^{2k})^{Wick} varrho_\varepsilon\right]$, with

$$\left(\frac{(|z|^{2k})^{Wick}}{\mathbf{N}^k}\right)^{\pm 1} = 1 + \mathcal{O}(\varepsilon)\quad \text{in } \mathcal{L}(\mathcal{H})\,,$$

the general definition is the following.

Definition 3.15. *For* $\varrho_\varepsilon \in \mathcal{L}_1(\mathcal{H})$*,* $\varrho_\varepsilon \geq 0$*,* $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$*, the reduced density matrix* γ_ε^k *of order* $k \in \mathbb{N}^*$ *is defined by*

$$\forall b \in \mathcal{P}_{kk}(\mathcal{Z})\,,\quad \mathrm{Tr}\,\left[\tilde{b}\gamma_\varepsilon^k\right] = \begin{cases} \frac{\mathrm{Tr}\,\left[b^{Wick}\varrho_\varepsilon\right]}{\mathrm{Tr}\,\left[(|z|^{2k})^{Wick}\varrho_\varepsilon\right]} & \textit{if } \mathrm{Tr}\,\left[(|z|^{2k})^{Wick}\varrho_\varepsilon\right] \neq 0\,, \\ 0 & \textit{else}\,. \end{cases}$$

This allows to compute the asymptotic values of the reduced density matrices under condition (PI).

Proposition 3.16. *Assume the conditions* (3.6) *and* (PI) *with* $\int_{\mathcal{Z}} |z|^{2k}\, d\mu \neq 0$ *then the convergence*

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^k = \frac{1}{\int_{\mathcal{Z}} |z|^{2k}\ d\mu(z)} \int_{\mathcal{Z}} |z^{\otimes k}\rangle\langle z^{\otimes k}|\ d\mu(z)\,,$$

holds in the norm topology of $\mathcal{L}^1(\bigvee^k \mathcal{Z})$.

REF: [Spo][BGM] [ESY][KlMa][AmNi2][AmNi3]

4 Mean field dynamics and Wigner measures

4.1 The mean field problem

Consider $\mathcal{Z} = L^2(\mathbb{R}^d;\mathbb{C})$ as the set of 1-particle wave functions and $\bigvee^N \mathcal{Z} = L^2_s(\mathbb{R}^{dN};\mathbb{C})$ the set of symmetric (bosonic) N-particles wave functions. The N-body Hamiltonian is for example

$$H_N = \underbrace{\sum_{j=1}^N -\Delta_{x_j}}_{\text{kinetic energy}} + \underbrace{\frac{1}{2(N-1)}\sum_{i\neq j} V(x_i - x_j)}_{\text{pair interaction potential}}\,, \quad V(-x) = V(x)\,,$$

and the problem of mean field dynamics consists in describing by a one particle flow in the limit $N\to\infty$, the solution to

$$i\partial_t \Psi_N = H_N\Psi_N \quad , \quad \Psi_N(t=0) = \Psi_{N,0} \in \bigvee^N \mathcal{Z}\,. \tag{4.1}$$

The N-particles Hamiltonian H_N preserves the symmetry and the number of particles. By setting

$$\varepsilon = \frac{1}{N} \quad \text{and} \quad \varrho_\varepsilon(t) = |\Psi_N(t)\rangle\langle\Psi_N(t)| \quad , \quad \varrho_\varepsilon = |\Psi_{N,0}\rangle\langle\Psi_{N,0}|\,,$$

and dividing (4.1) by N, the problem becomes

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}(\frac{1}{N}H_N)}\varrho_\varepsilon e^{-i\frac{t}{\varepsilon}(\frac{1}{N}H_N)}\,.$$

Up to the constant $\frac{1}{2N}V(0)$, which vanishes in the above expression (we shall consider also singular potentials), the Hamiltonian $\frac{1}{N}H_N$ is nothing but the restriction to $\bigvee^N \mathcal{Z}$ of

$$H_\varepsilon = d\Gamma(-\Delta) + \frac{1}{2}\langle z^{\otimes 2}\,,\, V(x-y)z^{\otimes 2}\rangle^{Wick} = h(z)^{Wick}$$

with

$$\begin{aligned} h(z) &= \langle z, (-\Delta)z\rangle + \frac{1}{2}\langle z^{\otimes 2}\,,\, V(x-y)z^{\otimes 2}\rangle \\ &= \int_{\mathbb{R}^d} |\nabla_x z(x)|^2\ dx + \frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y)|z(x)|^2|z(y)|^2\ dxdy\,. \end{aligned}$$

and the previous normalization

$$[a(g), a^*(f)] = \varepsilon\langle g\,,\, f\rangle\,.$$

The mean field flow, $\Phi(t)$, is given by the nonlinear Hamiltonian dynamics, formally written:

$$i\partial_t z = \partial_{\bar{z}} h(z) = -\Delta z + (V * |z|^2)z\,.$$

The option followed in [AmNi1][AmNi2][AmNi3] consists in proving that

$$\mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})) = \{\mu_t\} \quad \text{with} \quad \mu_t = \Phi(t-s)_*\mu_s\,,$$

and possibly deduce the evolution of the reduced density matrices. While doing this, we have been especially careful with the invariance w.r.t the quantum and mean field flows of the assumptions.
More standard approaches are

- Hepp method (see [Hep][GiVe1][GiVe2]): It consists in analyzing the evolution of a coherent state by proving $\Psi_\varepsilon(t) \sim E(\varphi_t)$ when

$$i\varepsilon\partial_t\Psi_t = H_\varepsilon\Psi_t \quad , \quad \Psi_{t=0} \sim E(\varphi_0) = W(\frac{\sqrt{2}}{i\varepsilon}z_0)|\Omega\rangle\,.$$

- BBGKY approach (see e.g. [Spo][BGM][FKP][ESY]): It consists in studying directly the evolution of the BBGKY hierarchy by assuming $\gamma_\varepsilon^{(1)}(0) \to |\varphi\rangle\langle\varphi|$.

With these two methods, the assumptions are not always dynamically invariant but those more specific assumptions sometimes allow more refined results.
We shall show in the last section how the Wigner measure point of view allows to understand the dynamics of non trivial mean field correlations.

4.2 Two kinds of results

Two results were obtained in [AmNi3] and [AmNi4] with different assumptions and different approaches.

4.2.1 Case of bounded interactions

Consider the general Hamiltonian

$$H_\varepsilon = d\Gamma(A) + Q^{Wick} = h^{Wick} \quad , \quad h(z) = \langle z\,,\, Az\rangle + Q(z)\,,$$

with $(A, D(A))$ self-ajoint in $\mathcal{Z}$ and $Q \in \oplus^{alg}_{p\in\mathbb{N}}\mathcal{P}_{p,p}(\mathcal{Z})$, Q real-valued,

$$Q = \sum_{j=0}^{m} Q_j \quad , \quad Q_j(z) = \langle z^{\otimes j}\,,\, \tilde{Q}_j z^{\otimes j}\rangle \quad , \quad \tilde{Q}_j = \tilde{Q}_j^* \in \mathcal{L}(\bigvee^{j} \mathcal{Z})\,.$$

Under those assumptions the equation

$$i\partial_t z = \partial_{\bar{z}} h(z) \tag{4.2}$$

defines a Hamiltonian flow on the phase-space $(\mathcal{Z}, \operatorname{Im}\langle\ ,\ \rangle)$.

Theorem 4.1. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\mathcal{H}$ with a single Wigner measure μ_0 and such that*

$$(PI) \quad \forall \alpha \in \mathbb{N}, \quad \lim_{\varepsilon\to 0} \operatorname{Tr}[\varrho_\varepsilon \mathbf{N}^\alpha] = \int_{\mathcal{Z}} |z|^{2\alpha}\ d\mu_0(z) < +\infty\,. \tag{4.3}$$

Then for all $t \in \mathbb{R}$, the family $(\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar{\varepsilon})}$ has a unique Wigner measure $\mu_t = (\Phi_t)_*\mu_0$, which is the initial measure μ_0 pushed forward by the flow associated with (4.2).
Moreover the convergence

$$\lim_{\varepsilon\to 0} \operatorname{Tr}\left[\varrho_\varepsilon(t) b^{Wick}\right] = \int_{\mathcal{Z}} b\circ\Phi_t(z)\ d\mu_0(z)$$

holds for any $b \in \mathcal{P}_{alg}(\mathcal{Z}) = \oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z})$.
Finally when $\mu_0 \neq \delta_0$, the convergence of the reduced density matrices

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^{(p)}(t) = \frac{1}{\int_{\mathcal{Z}} |z|^{2p}\ d\mu_t(z)} \int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\ d\mu_t(z) =: \gamma_0^{(p)}(t)\,,$$

holds in the $\mathcal{L}^1(\bigvee^p \mathcal{Z})$-norm for all $p \in \mathbb{N}$.

REF:[AmNi3]

4.2.2 Case of unbounded interactions

The unbounded case considers the more specific Hamiltonian

$$H_\varepsilon = d\Gamma(-\Delta) + \frac{1}{2}\langle z^{\otimes 2}\,,\, V(x-y) z^{\otimes^2}\rangle$$

with the assumptions

$$\begin{aligned} & V(-x) = V(x) \in \mathbb{R}\,, \\ & V(1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))\,, \\ \text{and} \qquad & (1-\Delta)^{-1/2} V (1-\Delta)^{-1/2} \in \mathcal{L}^\infty(L^2(\mathbb{R}^d))\,. \end{aligned}$$

We shall use the notation

$$H_\varepsilon^0 = d\Gamma(-\Delta)\,.$$

Those assumptions include the attractive or repulsive Coulombic case in dimension $d = 3$. They also ensure that the Hartree equation

$$i\partial_t z = -\Delta z + (V * |z|^2) z$$

defined a flow on $H^1(\mathbb{R}^d)$. Since the flow is not well defined on $L^2(\mathbb{R}^d; \mathbb{C})$, it is necessary to consider to the two phase-spaces

$$\mathcal{Z}_0 = L^2(\mathbb{R}^d; \mathbb{C}) \quad \text{and} \quad \mathcal{Z}_1 = H^1(\mathbb{R}^d)\,,$$

endowed with the symplectic form $\mathrm{Im}\,\langle\ ,\ \rangle_{\mathcal{Z}_0}$.

Theorem 4.2. *Let $(\varrho_\varepsilon)_{\varepsilon \in (0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ with a single Wigner measure μ_0 such that the bound*

$$\mathrm{Tr}[(\mathbf{N} + H_\varepsilon^0)^\delta \varrho_\varepsilon] \le C_\delta < +\infty\,, \tag{4.4}$$

holds uniformly w.r.t $\varepsilon \in (0, \bar\varepsilon)$ for some $\delta > 0$.
Then for all $t \in \mathbb{R}$, the family $(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar\varepsilon)}$ has a unique Wigner measure μ_t which is a Borel measure on $\mathcal{Z}_1 = H^1(\mathbb{R}^d)$. This measure $\mu_t = \Phi(t)_*\mu_0$ is the push forward of the initial measure μ_0 by the Hartree flow, well defined on $\mathcal{Z}_1$.

Proposition 4.3. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$, satisfying the hypothesis of Theorem 4.2, with a single Wigner measure μ_0 such that*

$$\forall \alpha \in \mathbb{N}, \quad \lim_{\varepsilon\to 0} \mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\alpha] = \int_{\mathcal{Z}_0} |z|^{2\alpha}\, d\mu_0(z) < +\infty\,. \tag{4.5}$$

Then for all $t \in \mathbb{R}$, the convergence

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\,\left[\varrho_\varepsilon(t) b^{Wick}\right] = \int_{\mathcal{Z}_0} b(\Phi(t)z)\, d\mu_0(z) = \int_{\mathcal{Z}_0} b(z)\, d\mu_t(z)$$

holds for any $b \in \mathcal{P}_{alg}(\mathcal{Z}_0) = \oplus^{alg}_{p,q\in\mathbb{N}} \mathcal{P}_{p,q}(\mathcal{Z}_0)$, with $\mu_t = \Phi(t)_*\mu_0$.
Finally when $\mu_0 \neq \delta_0$, the convergence of the reduced density matrices

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^{(p)}(t) = \frac{1}{\int_{\mathcal{Z}_0} |z|^{2p}\, d\mu_t(z)} \int_{\mathcal{Z}_0} |z^{\otimes p}\rangle\langle z^{\otimes p}|\, d\mu_t(z)\,,$$

holds in the $\mathcal{L}^1(L^2_s(\mathbb{R}^{dp}))$-norm for all $p \in \mathbb{N}$.

REF:[AmNi4]

4.3 Strategy of the proof(s)

Contrary to the finite dimensional case, the nonlinear flow does not preserve the algebra of observables, ε-quantized symbols. Wigner measures are now identified after measuring the state with Weyl-quantized cylindrical functions or possibly Wick-quantized polynomial functions. In general, neither cylindrical nor polynomial functions keep their property after composing with a nonlinear flow. The propagation of Wigner measures has to be tackled with a different approach, which can be summarized in two steps:

- Ascoli type argument: Even when $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\mu_0\}$, it is a priori neither known whether the family $(\varrho_\varepsilon(t))_{\varepsilon\in(0,\bar{\varepsilon})}$ has a single measure μ_t for $t \neq 0$, nor whether the extraction of a subsequence involved in the definition of Wigner measures can be performed simultaneously for all $t \in [-T,T]$. Actually this requires some equicontinuity estimate w.r.t $t \in [-T,T]$ in order to apply some diagonal extraction process. It is more convenient to work with $\tilde{\varrho}_\varepsilon(t) = e^{i\frac{t}{\varepsilon}d\Gamma(A)}\varrho_\varepsilon(t)e^{-i\frac{t}{\varepsilon}d\Gamma(A)}$ with $A = -\Delta$ in the singular case.
  In the polynomial bounded case, it is provided by
  $$|\mathrm{Tr}\ [[\tilde{\varrho}_\varepsilon(t) - \tilde{\varrho}_\varepsilon(s)]\, W(\xi)]| \geq C|t-s|(1+|\xi|_{\mathcal{Z}})^{2m}\,,$$
  when $\mathrm{Tr}\ [\mathbf{N^m}\varrho_\varepsilon] \leq C$. In the singular case, it is provided by
  $$|\mathrm{Tr}\ [[\tilde{\varrho}_\varepsilon(t) - \tilde{\varrho}_\varepsilon(s)]\, W(\xi)]| \geq C|t-s|(1+|\xi|_{\mathcal{Z}_1})^{4}\,,$$
  when $\mathrm{Tr}\ [(1+H_\varepsilon+\mathbf{N^3})\varrho_\varepsilon] \leq C$. This last additional condition is then removed by a uniform approximation of ϱ_ε , leading to uniform errors for the Wigner measures.

- Writing and solving a transport equation for μ_t: At this level the two cases differ.
  In the bounded polynomial case, writing and solving the transport equation is done in the same time: One uses a truncated Dyson expansion in the spirit of [FGS][FKP] in order to write
  $$e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}d\Gamma(A)}b^{Wick}e^{i\frac{t}{\varepsilon}d\Gamma(A)}e^{-i\frac{t}{\varepsilon}H_\varepsilon} \simeq \sum_{k=0}^{K} b_k(t)^{Wick} + R_k(t),\,,$$
  where the b_k's are time-dependent polynomials (with some explicit expression) and the estimate $\mathrm{Tr}\ [R_k(t)\varrho_\varepsilon] = o(1)$ as $K\to\infty$ uniformly w.r.t $\varepsilon \in (0,\bar{\varepsilon})$. By using the existence of μ_t or $\tilde{\mu}_t = (e^{itA})_*\mu_t$ for $\tilde{\varrho}_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}d\Gamma(A)}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}d\Gamma(A)}$ and the condition (PI) for ϱ_ε (the $\tilde{b}_k$ are not compact) , one obtains
  $$\int b(z)\ d\tilde{\mu}_t(z) = \sum_{k=0}^{K}\int b_k(z,t)\ d\mu_0 + \lim_{\varepsilon\to 0}\mathrm{Tr}\ [R_k(t)\varrho_\varepsilon]\ ,$$

for $b \in \mathcal{P}^\infty(\mathcal{Z})$. As a second step the uniform estimate of Tr $[R_K(t)\varrho_\varepsilon]$ allows to take the limit as $K \to \infty$. The explicit expressions of the b_k's allows to identify the series $\sum_{k=0}^\infty b_k(e^{itA}z) = b(\Phi(t)z)$ and $\mu_t = \Phi(t)_*\mu_0$. The actual proof is a bit more involved and we refer the reader to [AmNi3].
In the singular case, the assumed compactness of the operator $(1-\Delta)^{-1/2}V(1-\Delta)^{-1/2}$, allows to take the limit in the equation

$$i\partial_t \mathrm{Tr}\ [\tilde{\varrho}_\varepsilon(t)W(\xi)] - \frac{1}{\varepsilon}\mathrm{Tr}\ \left[[V_t^{Wick}, \tilde{\varrho}_\varepsilon(t)]W(\xi)\right]$$

with $V_t(z) = \langle (e^{it\Delta}z)^{\otimes 2}, V(x-y)(e^{it\Delta}z)^{\otimes 2}\rangle$ defined for $z \in \mathcal{Z}_1$. By assuming

$$\forall \alpha \in \mathbb{N},\ \mathrm{Tr}\ \left[(1 + H_\varepsilon^0 + \mathbf{N}^3)\varrho_\varepsilon(1 + H_\varepsilon^0 + \mathbf{N}^3)\mathbf{N}^\alpha\right] \le C_\alpha\,,$$

one can prove that the measure $\tilde{\mu}_t$ satisfies

$$\forall t \in [-T, T], \quad \int_{\mathcal{Z}_0} |z|_{\mathcal{Z}_1}^4 |z|_{\mathcal{Z}_0}^2\ d\mu(z) \le C_T$$

and that for any cylindrical function $f \in \mathcal{C}_{0,cyl}^\infty(\mathcal{Z}_1 \times \mathbb{R})$

$$\int_{\mathbb{R}}\int_{\mathcal{Z}_1} (\partial_t f + i(\partial_z V_t.\partial_{\bar{z}} f - \partial_z f.\partial_{\bar{z}} V_t))\ d\mu_t(z)dt = 0\,.$$

The above equation is a weak form of the time-dependent Liouville equation

$$\partial_t \mu + i\,\{V_t, \mu\} = 0\,.$$

Another property coming from taking the limit as $\varepsilon \to 0$ of the original quantities, says that the measure $\tilde{\mu}_t$ is Lipschitz-continuous for the Wasserstein distance

$$W(\mu^1, \mu^2) = \inf_{\nu \in \Gamma(\mu^1, \mu^2)} \int_{\mathcal{Z}_1^2} |z_1 - z_2|^2\ d\nu(z_1, z_2)$$

where $\Gamma(\mu^1, \mu^2)$ is the set of Borel probability measures on $\mathcal{Z}_1^2$ with marginals μ^1 and μ^2. We conclude by an adaptation of a result of [AGS], which can be seen as an extension of the characteristic method for transport equations in the infinite dimensional setting. Again details may be found in [AmNi4].

REF: [FGS][FKP][AGS][AmNi1][AmNi2][AmNi3][AmNi4]

4.4 A example of the dynamics of mean field correlations

For two elements $\psi_1, \psi_2 \in \mathcal{Z}_1 \subset \mathcal{Z}_0$ such that $|\psi_1| = |\psi_2| = 1$ and $\langle \psi_1\,,\,\psi_2\rangle = 0$, the space $\mathcal{Z}_0$ can be decomposed into

$$\mathcal{Z}_0 = \mathbb{C}\psi_1 \overset{\perp}{\oplus} \mathbb{C}\psi_2 \overset{\perp}{\oplus} \psi^\perp\,.$$

This decomposition is second-quantized into the Hilbert tensor product

$$\mathcal{H} = \Gamma_s(\mathcal{Z}_0) = \Gamma_s(\mathbb{C}\psi_1) \otimes \Gamma_s(\mathbb{C}\psi_2) \otimes \Gamma_s(\psi^\perp)\,,$$

which allows an analysis by separating the variables. The number observable is now

$$\mathbf{N} = (\mathbf{N}_1 \otimes \mathrm{Id} \otimes \mathrm{Id}) \oplus (\mathrm{Id} \otimes \mathbf{N}_2 \otimes \mathrm{Id}) \oplus (\mathrm{Id} \otimes \mathrm{Id} \otimes \mathbf{N}')\,,$$

simply written as $\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}'$ and where $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}'$ are respectively the number operators on $\Gamma_s(\mathbb{C}\psi_1)$, $\Gamma_s(\mathbb{C}\psi_2)$ and $\Gamma_s(\psi^\perp)$. Consider in this decomposition, the state

$$\varrho_\varepsilon = \varrho_\varepsilon^1 \otimes \varrho_\varepsilon^2 \otimes (|\Omega'\rangle\langle\Omega'|)$$

where $|\Omega'\rangle$ is the vacuum state of $\Gamma_s(\psi^\perp)$ and

$$\varrho_\varepsilon^1 = |\psi_1^{\otimes n_1}\rangle\langle\psi_1^{\otimes n_1}| \quad , \quad \varrho_\varepsilon^2 = |\psi_2^{\otimes n_2}\rangle\langle\psi_2^{\otimes n_2}|\,,$$
$$\text{with} \quad \lim_{\varepsilon\to 0} \varepsilon n_1 = \lim_{\varepsilon\to 0} \varepsilon n_2 = \frac{1}{2}\,.$$

In $\mathcal{H} = \Gamma_s(\mathcal{Z}_0)$, this state is explicitly written (see [AmNi3]) as

$$\varrho_\varepsilon = |\psi^{\vee(n_1,n_2)}\rangle\langle\psi^{\vee(n_1,n_2)}| \tag{4.6}$$

$$\text{with} \quad \psi^{\vee(n_1,n_2)} = \frac{1}{\sqrt{\varepsilon^{n_1+n_2} n_1! n_2!}} \overbrace{a^*(\psi_1)\dots a^*(\psi_1)}^{n_1\ \text{times}} \overbrace{a^*(\psi_2)\dots a^*(\psi_2)}^{n_2\ \text{times}} |\Omega\rangle \tag{4.7}$$

The state satisfies

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\mathbf{N}^k \varrho_\varepsilon\right] = (\frac{1}{2} + \frac{1}{2})^k = 1\,,$$

owing to $\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}'$. Moreover, with (4.6)(4.7), $\mathbf{N} + H_\varepsilon^0 = d\Gamma(1-\Delta)$ and the help of Wick calculus, it also fulfills

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[(\mathbf{N} + H_\varepsilon^0)\varrho_\varepsilon\right] = \frac{|\psi_1|^2_{\mathcal{Z}_1} + |\psi_2|^2_{\mathcal{Z}_1}}{2}\,.$$

Meanwhile the separation of variables allows to compute explicitly the (it is unique) Wigner measure of $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$

$$\mu_0 = \delta^{S^1}_{\frac{\sqrt{2}}{2}\psi_1}\otimes\delta^{S^1}_{\frac{\sqrt{2}}{2}\psi_2}\otimes\delta_0 \quad \text{on} \quad \mathcal{Z}_1=(\mathbb{C}\psi_1)\times(\mathbb{C}\psi_2)\times\psi^\perp\,,$$
$$\text{with} \qquad \delta^{S^1}_u = \frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}u}\,d\theta\,.$$

We get

$$\int_{\mathcal{Z}_1}|z|^{2k}\,d\mu_0(z)=\int_{\mathcal{Z}_1}\left(|z_1|^2+|z_2|^2+|z'|^2\right)^k\,d\mu_0(z)=1=\lim_{\varepsilon\to\infty}\mathrm{Tr}\,\left[\mathbf{N}^k\varrho_\varepsilon\right]\,.$$

Hence all the assumptions of Theorem 4.2 and Proposition 4.3 are fulfilled. This measure is carried by a torus in $\mathcal{Z}_1$ better described by using an other orthonormal basis of $\mathbb{C}\psi_1\oplus\mathbb{C}\psi_2$:

$$\begin{aligned}
&\psi_0=\frac{\sqrt{2}}{2}(\psi_1+\psi_2)\quad,\quad \psi_{\frac{\pi}{2}}=i\frac{\sqrt{2}}{2}(\psi_1-\psi_2)\,,\\
&\psi_\varphi=\cos(\varphi)\psi_0+\sin(\varphi)\psi_{\frac{\pi}{2}}\,,\\
&\frac{\sqrt{2}}{2}(e^{i\theta}\psi_1+e^{i\theta'}\psi_2)=e^{i\frac{\theta+\theta'}{2}}\psi_{\frac{\theta-\theta'}{2}}\,,\\
&\mu_0=\frac{1}{2\pi}\int_0^{2\pi}\delta^{S^1}_{\psi_\varphi}\,d\varphi\,.
\end{aligned}$$

Two elements $e^{i\theta}\psi_\varphi$ and $e^{i\theta'}\psi_{\varphi'}$ in the support of μ_0 are equal when

$$(\theta'=\theta \text{ and } \varphi'=\varphi)\quad \text{or}\quad (\theta'=\theta+\pi \text{ and } \varphi'=\varphi+\pi)\,.$$

Hence a one to one parametrization of the torus can be done by $\varphi\in[0,2\pi)$ and $\theta\in[\varphi,\varphi+\pi)$.

Let $\psi_\varphi(t)=\Phi(t,0)\psi_\varphi$, be the solution to the Hartree equation

$$\left\{\begin{array}{l} i\partial_t\psi_\varphi(t)=-\Delta\psi_\varphi(t)+(V*|\psi_\varphi(t)|^2)\psi_\varphi(t)\\ \psi_\varphi(t=0)=\psi_\varphi=e^{i\frac{\pi}{4}}\cos(\varphi)\psi_1+e^{-i\frac{\pi}{4}}\sin(\varphi)\psi_2\end{array}\right.\,,$$

The gauge invariance of the equation says that for any $\theta\in[0,2\pi]$, $e^{i\theta}\psi_\varphi(t)=\Phi(t,0)\left[e^{i\theta}\psi_\varphi\right]$. By applying the result of Theorem 4.2 and Proposition 4.3 we get

$$\begin{aligned}
&\mu_t=\frac{1}{2\pi}\int_0^{2\pi}\delta^{S^1}_{\psi_\varphi(t)}\,d\varphi=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}\psi_\varphi(t)}\,d\varphi d\theta\\
&\forall p\in\mathbb{N},\quad \lim_{\varepsilon\to0}\gamma_\varepsilon^{(p)}(t)=\frac{1}{2\pi}\int_0^{2\pi}|[\psi_\varphi(t)]^{\otimes p}\rangle\langle[\psi_\varphi(t)]^{\otimes p}|\,d\varphi\,.
\end{aligned}$$

Since the Hartree flow is non linear, the complete hierarchy of reduced density matrices have to be taken into account if one wants to write an evolution equation for them. More simply, they can be computed after solving an autonomous equation for the Wigner measure. Due to the nonlinear term the dynamics of correlations is by far nontrivial. This can also be thought geometrically: The initial measure is initially carried by a torus which lies in a 2-dimensional complex vector space (think of the circle in the plane $\mathbb{R}\psi_0 \oplus \mathbb{R}\psi_{\frac{\pi}{2}}$); along the time evolution, the measure μ_t is still carried by a torus in $\mathcal{Z}_1$, which nevertheless, is a priori not embedded in any finite dimensional subspace.

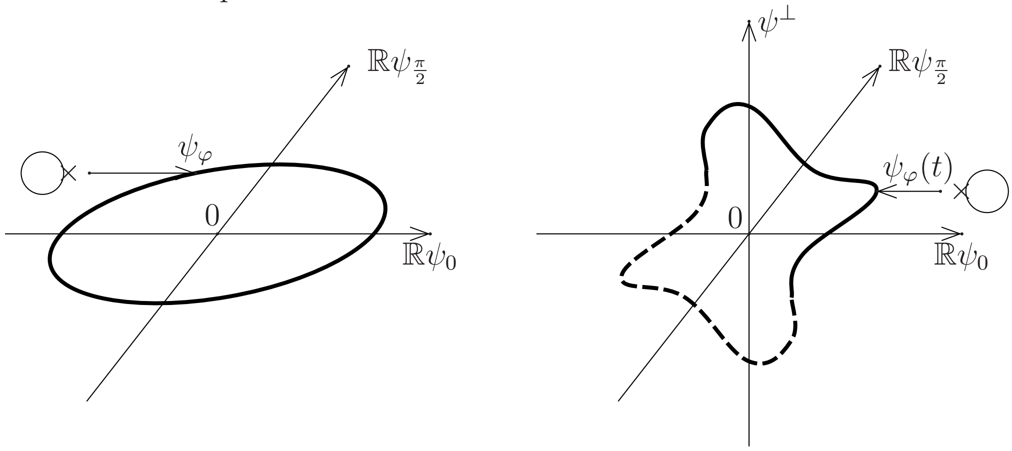

Fig.1: Evolution of the measure initially carried by a torus in $\mathbb{C}\psi_0 \oplus \mathbb{C}\psi_{\frac{\pi}{2}}$.
The complex gauge parameter $e^{i\theta}$ is represented by the small circle.

In Figure 1, the deformed torus for time $t \neq 0$, has to be imagined in the infinite-dimensional phase-space $\mathcal{Z}_1 \subset \mathcal{Z}_0$. Contrary to the picture, there might be no intersection with the real plane $\mathbb{R}\psi_0 \oplus \mathbb{R}\psi_{\frac{\pi}{2}}$.
This discussion can also be extended to higher dimensional tori after taking a finite (or countable) orthonormal family $(\psi_n)_{1\leq n\leq N}$ for building the initial states ϱ_ε with a measure $\prod_{j=1}^N \delta^{S^1}_{\lambda_j\psi_j}$ (see [AmNi3]).

REF:[AmNi4]

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