Exponentially small eigenvalues of Witten Laplacians 2: Functions (p=0)

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Beijing 25/05/2017

Outline

- Assumptions and results
- Pairing of local minima with saddle points
- Singular values
- Reduced complex, quasimodes, final computation

Assumptions and results

(M, g) compact (oriented) manifold (or $\mathbb{R}^n$) .
f Morse function ($\liminf_{x\to\infty} |\nabla f(x)|^2 - |\Delta f(x)| = c > 0$ when $M = \mathbb{R}^n$) .

Generic Assumption: The critical values $f(U_i^{(p)})$, $p = 0, 1$ and $1 \leq i \leq m_p$, are distinct.
The differences $f(U_j^{(1)}) - f(U_k^{(0)})$, $1 \leq j \leq m_1$, $1 \leq k \leq m_0$, are distinct.

Witten Laplacian: $\Delta_{f,h} = (d_{f,h} + d_{f,h}^*)^2 = \bigoplus \Delta_{f,h}^{(p)}$.

Result :(REF Helffer-Klein-N.) The m_0 first eigenvalues of $\Delta_{f,h}^{(0)}$ satisfy $\lambda_1(h) = 0$ and

$$\lambda_{k\geq 2}(h) = \frac{h}{\pi}|\hat{\lambda}_1(U_{j(k)}^{(1)})| \sqrt{\frac{\left|\det(\mathrm{Hess}\, f(U_k^{(0)}))\right|}{\left|\det(\mathrm{Hess}\, f(U_{j(k)}^{(1)}))\right|}} (1 + c_k(h)) \exp\left(-\frac{2(f(U_{j(k)}^{(1)}) - f(U_k^{(0)}))}{h}\right)$$

with $c_k(h) \sim \sum_{\ell=1} c_{k,\ell} h^\ell$, $\hat{\lambda}_1(U)$ negative eigenvalue of $\mathrm{Hess} f(U)$.

$k \to j(k)$: specified below.

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Boundary Witten Laplacian (Neumann, Dirichlet): replace critical points by generalized critical point (∇f does not vanish on ∂M)

Result(Dirichlet):(REF Helffer-N.)

$$\lambda_k(h) = \frac{h}{\pi}|\widehat{\lambda}_1(U^{(1)}_{j(k)})|\sqrt{\frac{\left|\det(\mathrm{Hess}\, f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess}\, f(U^{(1)}_{j(k)}))\right|}}\,(1+hc^1_k(h))$$
$$\times \exp\left(-\frac{2\left(f(U^{(1)}_{j(k)})-f(U^{(0)}_k)\right)}{h}\right)\,,\quad \text{if } U^{(1)}_{j(k)}\notin \partial M\,,$$

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with $c^1_k(h)\sim \sum_{m=0}^{\infty} h^m c_{k,m}$.

Variation: Neumann B.C. (REF Le Peutrec) 4 cases. $k\to j(k)$: specified below.

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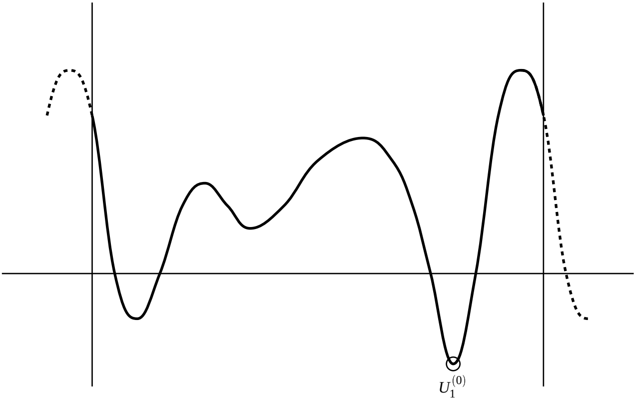
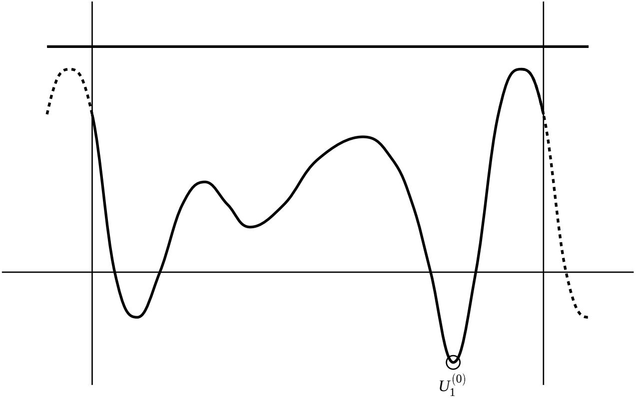
Pairing of minima with saddle points.



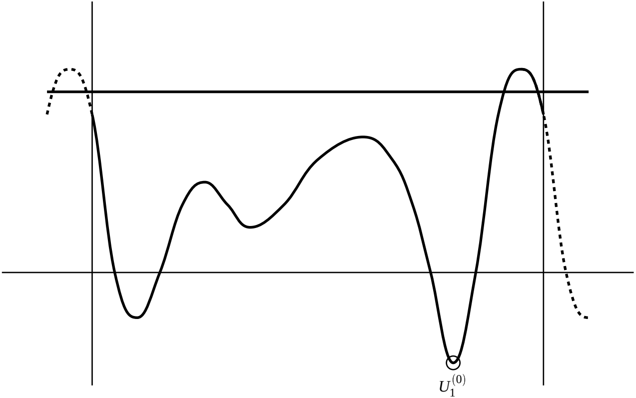
$U_1^{(0)}$ = Global minimum.
$M = \mathbb{S}^1$.

Pairing of minima with saddle points.



Consider the sublevel set $f^{\lambda} = \{x \in M, f(x) < \lambda\}$.

Pairing of minima with saddle points.



Decrease λ until the number of connected components f^{λ} is increased by $+1$.

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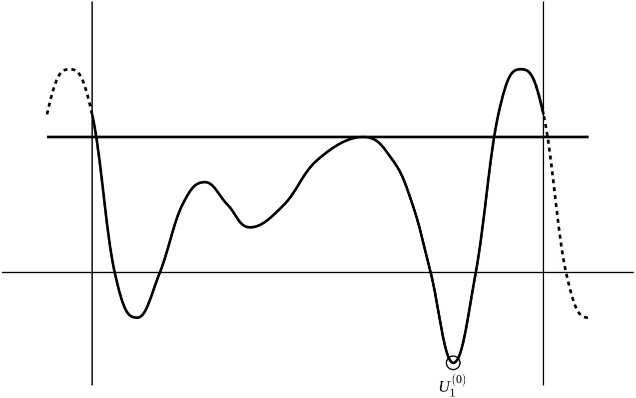

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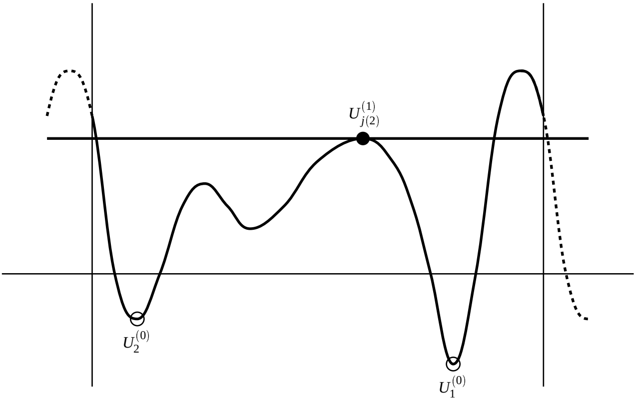
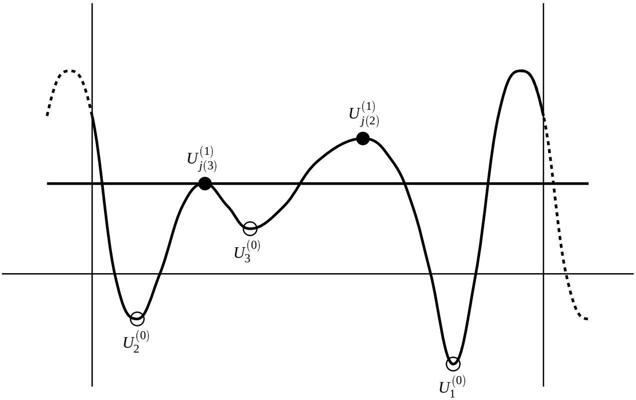


$U_2^{(0)}$: global minimum in the new connected component. $U_{j(2)}^{(1)}$: splitting point.

Pairing of minima with saddle points.



$f(U^{(1)}_{j(k)}) - f(U^{(0)}_{k})$ is strictly decreasing.

Pairing of local minima with saddle points: Dirichlet B.C.

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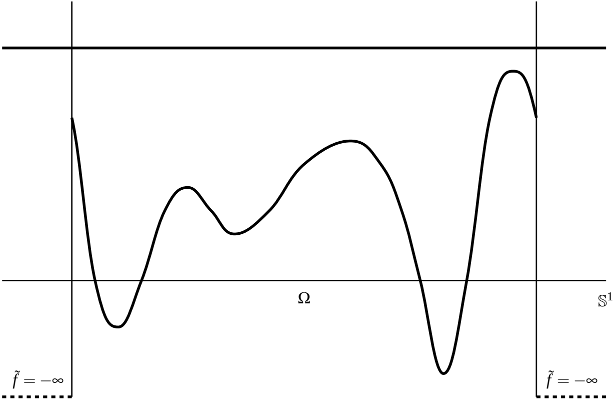
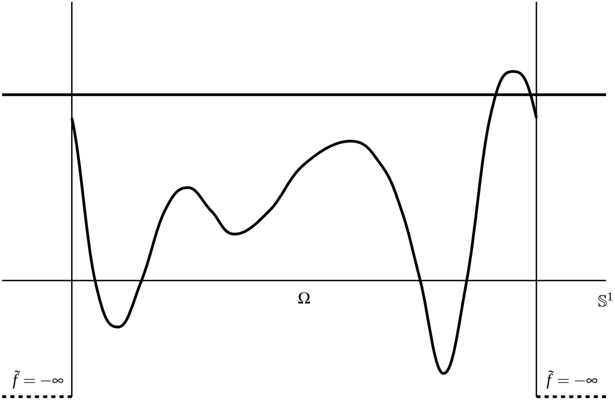
Dirichlet B.C.: $f = -\infty$ outside M
Neumann B.C. $f = +\infty$ outside M.

Pairing of local minima with saddle points: Dirichlet B.C.



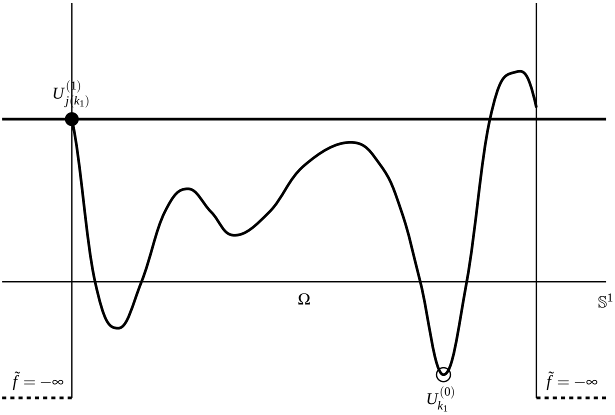
Consider the level set $\tilde{f}^\lambda = \left\{x \in M, \tilde{f}(x) < \lambda\right\}$.

Pairing of local minima with saddle points: Dirichlet B.C.



Decrease λ until the number of connected components $\tilde{f}^{\lambda}$ is increased by $+1$.

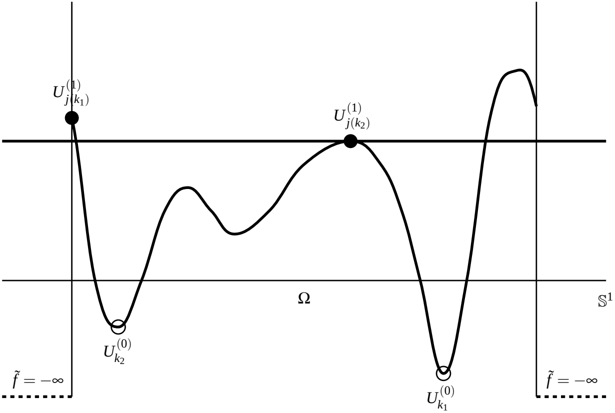
Pairing of local minima with saddle points: Dirichlet B.C.



$U^{(0)}_{k_1}$: highest "global" minimum of $\tilde{f}\big|_{f^\lambda)}$.

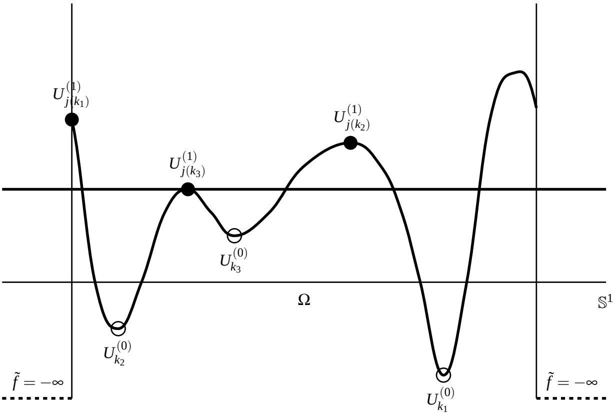
$U^{(1)}_{j(k_1)}$: splitting point.

Pairing of local minima with saddle points: Dirichlet B.C.



$U^{(0)}_{k_2}$: highest "global" minimum.
$U^{(1)}_{j(k_2)}$: splitting point.

Pairing of local minima with saddle points: Dirichlet B.C.



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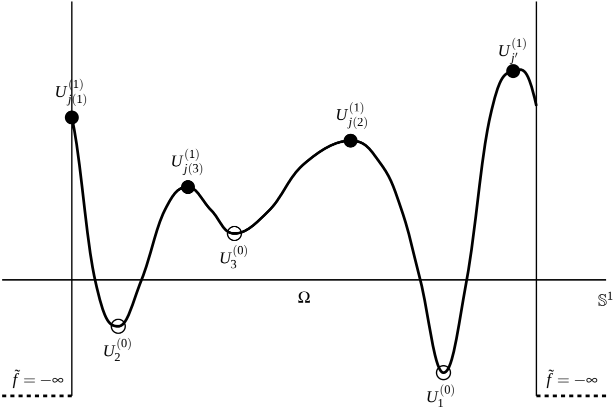

Reorder the k_i according to the decreasing order of $f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)$.

Singular values

$$A_0(h) = B(h)^* B(h) \,,$$
$$B(h) \; : F^{(0)} \to F^{(1)} \quad \dim F^{(i)} = m_i$$
$$B(h) \underbrace{A_0(h)}_{\geq 0} = \underbrace{A_1(h)}_{\geq 0} B(h) \,;$$
$$\psi_k^{(0)} = \psi_k^{(0)}(\varepsilon, h), \qquad \langle \psi_k^{(0)} \,|\, \psi_{k'}^{(0)} \rangle = \delta_{k,k'} + O(e^{-\frac{\alpha}{h}}) \,,$$
$$\psi_j^{(1)} = \psi_j^{(1)}(\varepsilon, h), \qquad \langle \psi_j^{(1)} \,|\, \psi_{j'}^{(1)} \rangle = \delta_{j,j'} + O(e^{-\frac{\alpha}{h}}) \,;$$
$$h \log |\langle \psi_{j(k)}^{(0)} \,|\, B(h)\psi_k^{(0)} \rangle| \overset{h\to 0}{\sim} -\alpha_k \qquad |\langle \psi_{j'}^{(0)} \,|\, B(h)\psi_k^{(0)} \rangle| \leq e^{-\frac{\alpha_k + \alpha}{h}} \;;$$
$$\alpha_1 > \ldots > \alpha_k > \alpha_{k+1} > \ldots > \alpha_{m_0} > 0 \,.$$

The eigenvalues of $A_0(h)$ satisfy

$$\lambda_k(h) = \left| \langle \psi_{j(k)}^{(1)} \,|\, B(h)\psi_k^{(0)} \rangle \right|^2 (1 + O_\eta(e^{-\frac{\eta}{h}})) \quad (\eta > 0).$$

$$(\, \lambda_1(h) < \ldots < \lambda_{m_0}(h) \,.)$$

Stable after change of two bases: with almost orthogonal transformation, $[C^{(i)}]^* C^{(i)} = \mathrm{Id}_{F^{(i)}} + \mathcal{O}(e^{-\frac{\alpha}{h}}))$ (particular case of Fan inequalities).

$$
\begin{aligned}
&A_0(h) = B(h)^* B(h)\ ,\\
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Reduced complex

$$d_{f,h}^{(p)}\chi(\Delta_{f,h}^{(p)}) = \chi(\Delta_{f,h}^{(p+1)})d_{f,h}^{(p)}. \tag{0.1}$$

Witten complex : Set $F^{(p)} = \mathrm{Ran}\ 1_{[0,h^{3/2})}(\Delta_{f,h}^{(p)})$ and $\beta_{f,h}^{(p)} = d_{f,h}^{(p)}\big|_{F^{(p)}}$.

$$\dim F^{(p)} = m_p$$

$$\begin{array}{l} 0 \to F^{(0)} \overset{\beta_{f,h}^{(0)}}{\to} F^{(1)} \overset{\beta_{f,h}^{(1)}}{\to} \dots \overset{\beta_{f,h}^{(n-1)}}{\to} F^{(n)} \to 0 \\ 0 \leftarrow F^{(0)} \overset{\beta_{f,h}^{(0),*}}{\leftarrow} F^{(1)} \overset{\beta_{f,h}^{(1),*}}{\leftarrow} \dots \overset{\beta_{f,h}^{(n-1),*}}{\leftarrow} F^{(n)} \leftarrow 0 \end{array} \quad \text{exact.}$$

Take

$$A_0(h) = \Delta_{f,h}^{(0)}\big|_{F^{(0)}}$$

$$A_1(h) = \Delta_{f,h}^{(1)}\big|_{F^{(1)}}$$

$$B(h) = \beta_{f,h}^{(0)} = d_{f,h}^{(0)}\big|_{F^{(0)}}\ .$$

Works as well for Dirichlet or Neumann realization ((0.1) still holds).
$F^{(p)} = 1_{[0,h^{6/5}]}(F^{(p)})$, m_p number of generalized critical points.

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$$B(h) = \beta^{(0)}_{f,h} = d^{(0)}_{f,h}\big|_{F^{(0)}} .$$

Works as well for Dirichlet or Neumann realization ((0.1) still holds).
$F^{(p)} = 1_{[0,h^{6/5}]}(F^{(p)})$, m_p number of generalized critical points.

Quasimodes

Quasimodes for $\Delta^{(0)}_{f,h}$ Take $\psi_k(\varepsilon, h) = C(k,h)\chi_{k,\varepsilon}(x)e^{-\frac{f(x)-f(U^{(0)}_k)}{h}}$.
The normalisation factor $C(k,h)$: Laplace method.

$$d^{(0)}_{f,h}\psi_{k,\varepsilon} = C_{k,h}e^{-\frac{f(x)-f(U^{(0)}_k)}{h}}\left(hd\chi_{k,\varepsilon}\right).$$

The cut-off $\chi_{k,\varepsilon}$ is modelled on the connected component of $U^{(0)}_k$ in $\left\{f < f(U^{(1)}_{j(k)})\right\}$ with

$$\operatorname{supp}\nabla\chi_{k,\varepsilon} \subset \left\{\left|f(x) - f(U^{(1)}_{j(k)})\right| \leq \varepsilon\right\}.$$

(see picture).

Quasimodes

Quasimodes for $\Delta^{(1)}_{f,h}$

In a ball $B(U_j^{(1)}, 2\varepsilon_1)$, $\varepsilon_1 > 0$ independent of ε, consider the Dirichlet realization $\Delta^{D,(1)}_{f,h}$ and its first normalized eigenvector $u_j^{(1)}$.

$$\psi_j^{(1)}(h) = \chi_j(x) u_j^{(1)}$$

$$\text{with} \qquad \chi_j \in \mathcal{C}_0^\infty(B(U_j^{(1)}, 2\varepsilon_1)), \quad \chi_j \equiv 1 \quad \text{in } B(U_j^{(1)}, \frac{3}{2}\varepsilon_1).$$

$$\text{In } B(U_j, \varepsilon_1) \qquad |\partial_x^\alpha \psi_j^{(1)}(x)| = \tilde{O}(e^{-\frac{\varphi(x)}{h}})$$

$$\left|\partial_x^\alpha(\psi_j^{(1)} - u_j^{wkb})(x)\right| = O(h^\infty e^{-\frac{\varphi(x)}{h}}).$$

$$\varphi(x) = d_{Agmon}(x, U_j^{(1)}) \geq \left|f(x) - f(U_j^{(1)})\right|$$

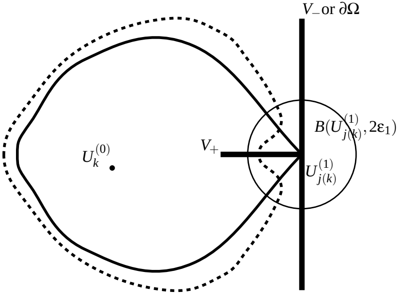
with equality only along the stable manifold V_+ (1D) and, depending on the case,

the unstable manifold V_- of ∇f;

or the boundary $\partial M \cap B(U_j^{(1)}, 2\epsilon_1)$ (Dirichlet BC).

This part relies on Helffer-Sjstrand (86), see also Helffer (Lect. Notes in Math. 1336)

Quasimodes



The support of $d\chi_{k,\varepsilon}$ is localized around the dashed curve.

Final computation

$(j \neq j(k)) \Rightarrow (\langle \psi_j^{(1)}(h) \mid d_{f,h}\psi_k^{(0)}(\varepsilon, h)\rangle = 0)$.
$j = j(k)$: For a good choice of $\chi_{k,\varepsilon}$,

$$\langle \psi_j^{(1)}(h) \mid d_{f,h}\psi_k^{(0)}(\varepsilon, h)\rangle =$$

$$C_{k,h} \int_{B(U^1_{j(k)}),\varepsilon_1)} \langle \psi_j^{(1)}(h) \mid hd\chi_{k,\varepsilon}\rangle(x) e^{-\frac{f(x)-f(U_k^{(0)})}{h}}\, dx + O(e^{-c\varepsilon/h}).$$

Replace $\psi_j^{(1)}$ by u_j^{wkb}.
Reduce the integration domain to a neigborhood of V_+.
Integration along V_+ : Stokes Formula.
Integration transverse to V_+ : Laplace method.