Exponentially small eigenvalues of Witten Laplacians 3: Morse theory and persistent homology

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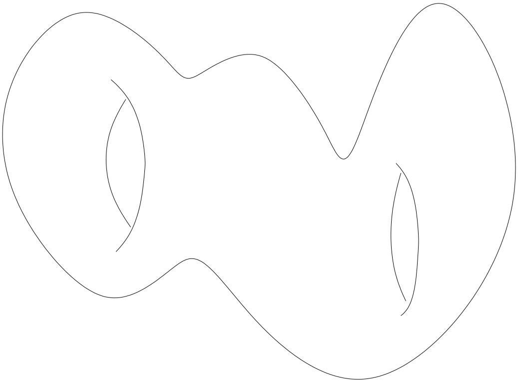
Beijing 01/06/2017

Outline

- Our problem
- Persistent homology
- Barannikov presentation of Morse theory
- Bar codes, persistence diagrams

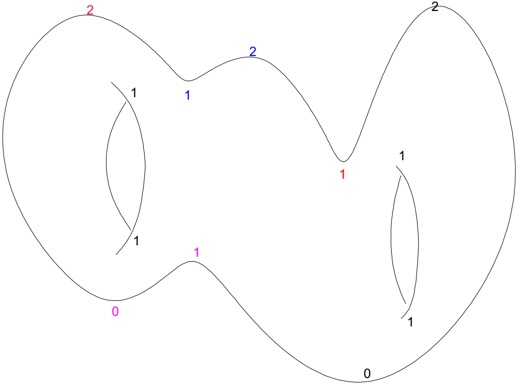
Our problem

Pairing of critical points ? e.g. $f(x)$ =height of x on a surface.



Our problem

Solution(to be explained): pairs encoded by colors, black $\rightarrow$ no pairing.



2 exp small eigenvalues for $\Delta^{(0)}_{f,h}$, among which 0 with multiplicity 1.

7 exp small eigenvalues for $\Delta^{(1)}_{f,h}$, among which 0 with multiplicity 4.

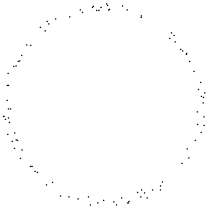
3 exp small eigenvalue for $\Delta^{(2)}_{f,h}$, among which 0 with multiplicity 1.

Persistent homology

REF: Carlsson(05), Carlsson-Zomorodian(05)

> The most robust and concise global information that we can get about the shape of an object in large dimension, is about its topology.
>
> In statistical data analysis, such object are usually given as a cloud of points.

Question: What suggest that the picture below is a circle ?



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Answer: Replace points by balls with varying radius.

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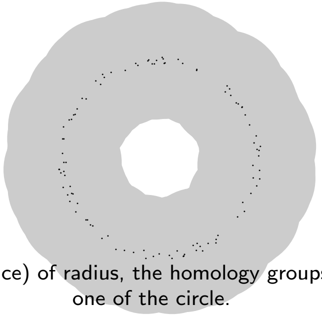

For a wide range (persistence) of radius, the homology groups of the grey area are the one of the circle.

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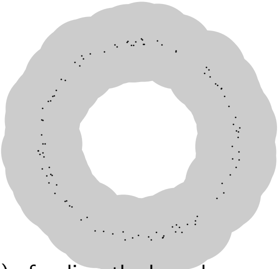

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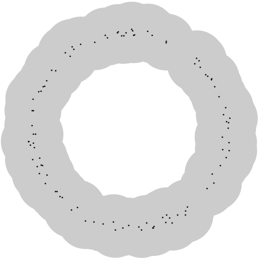

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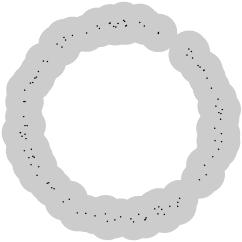

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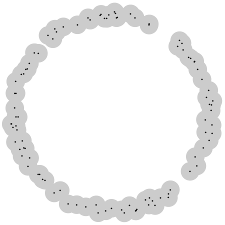

The structure eventually disappear.

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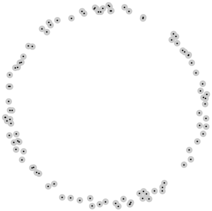

Note that intermittently small irrelevant structure may appear (for a small range of parameter).

Persistent homology

Idea: For a set of point $S \in \mathbb{R}^D$, find the p-cycles which persist in $\cup_{y\in S} B(y,r)$ for a wide range of r.

Carlsson(05) and Carlson-Zomorodian(05) follow this presentation based on Cech cohomology (cohomology of ball coverings, dual to some singular homology) and introduce the general algebraic setting.

Another point of view consist in studying the homology groups of the sublevel set of $x \mapsto d(x,S)$ with e.g. $d(.,.)$ given by the euclidean distance.
When $d(.S)$ is replaced f a Morse function, studying the homology of sublevel sets $f^\lambda = \{x \in M\,, f(x) < \lambda\}$ amounts to Morse theory.
This presentation is detailed by Cohen Steiner-Edelsberg-Harer (07) who also prove the stability of bar codes or persistence diagrams (see also Chazal-Cohen Steiner-et al. (09-12))

Remember that when $X \subset Y$ there is a natural mapping of $H_*(X) \to H_*(Y)$ (homology vector spaces, the ring of coefficients is a field $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ for Witten Lapl.). Applying this with $X = f^s$ and $Y = f^t$ with $s < t$, the persistent homology groups in degree p are defined as the ranges of $F_s = H_p(f^s)$ in $F_t = H_p(f^t)$ via the natural mapping $\varphi_s^t : H_*(f^s) \to H_*(f^t)$,

$$F_s^t = \varphi_s^t H_*(f^s) \quad , t > s\,,$$

and one studies all the family $(F_s^t)_{s<t}$.

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Barannikov presentation of Morse theory

REF: Barannikov(94), Le Peutrec-N.-Viterbo(13)

Generic Assumption: $\#\mathcal{U} = \#\left\{f(U), U \in \mathcal{U}\right\}(=\mathcal{C})$.
$U \in \mathcal{U}$ and $c = f(U) \in \mathcal{C}$ are identified.
$f^a = \{f < a\}$, $f^{-\infty} = \emptyset$ and $f^{+\infty} = M$.

For $c \in \mathcal{C}$ the basis of Morse theory says (under the Generic Assumption)
$H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) = \mathbb{K}$ when c is critical value (point) with index p and
$H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) = 0$ otherwise (ε is a small positive number)

It means the alternative (long exact sequence)

$$\left\{\begin{array}{l} H_p(f^{c-\varepsilon}) \stackrel{\sim}{\longrightarrow} H_p(f^{c+\varepsilon}) \quad \text{and} \\ 0 \longrightarrow H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \longrightarrow H_{p-1}(f^{c-\varepsilon}) \longrightarrow H_{p-1}(f^{c+\varepsilon}) \to 0 \end{array}\right. ,$$

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Duality: U is a critical point of index $\dim M - p$ of $-f$ and use
$f_\lambda = \{x, f(x) > \lambda\} = (-f)^{-\lambda}$ with now $f_{c+\varepsilon} \subset f_{c-\varepsilon}$
Note also that $H_{d-*}(f_b, f_a)$ is the dual of $H_*(f^a, f^b)$ (Alexander duality).

There are a priori 2×2 cases.

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Actually there are only 3 cases:

Definition

- A critical value (resp. point) c of f is called an upper critical value (resp. point), if the natural mapping

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- In all other cases the critical value (resp. point) c, is called an homological critical value (resp. point).

This makes a partition of $\mathcal{U}$ (see next slide)

$$\mathcal{U} = \mathcal{U}_U \sqcup \mathcal{U}_L \sqcup \mathcal{U}_H$$
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Barannikov presentation of Morse theory

Remember that if $B \subset A \subset X$ and $B' \subset A' \subset X'$ and $\varphi : X(\text{resp.}A, B) \to X'(\text{resp.}A', B')$ continuous, then

$$\begin{array}{ccccccccc}
\longrightarrow & H_*(A,B) & \longrightarrow & H_*(X,B) & \longrightarrow & H_*(X,A) & \stackrel{\partial}{\longrightarrow} & H_{*-1}(A,B) & \longrightarrow \\
 & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \overline{\varphi_*} & & \downarrow \varphi_* & \\
\longrightarrow & H_*(A',B') & \longrightarrow & H_*(X',B') & \longrightarrow & H_*(X',A') & \stackrel{\partial}{\longrightarrow} & H_{*-1}(A',B') & \longrightarrow
\end{array}$$

Apply it with

$$(X, A, B) = (f^{c+\varepsilon}, f^{c-\varepsilon}, \emptyset) \quad \text{and} \quad (X', A', B') = (M, f^{c-\varepsilon}, \emptyset)$$

with the mapping $i^{\infty, c+\varepsilon} : f^{c+\varepsilon} \to M = f^{+\infty}$:

$$\begin{array}{ccccccc}
H_*(f^{c-\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) & \stackrel{\partial}{\longrightarrow} & H_{*-1}(f^{c-\varepsilon}) \\
\downarrow \mathrm{Id} & & \downarrow i_*^{\infty, c+\varepsilon} & & \downarrow \overline{i_*^{\infty, c+\varepsilon}} & & \downarrow \mathrm{Id} \\
H_*(f^{c-\varepsilon}) & \longrightarrow & H_*(M) & \longrightarrow & H_*(M, f^{c-\varepsilon}) & \stackrel{\partial'}{\longrightarrow} & H_{*-1}(f^{c-\varepsilon})
\end{array}.$$

If $\overline{i_*^{\infty, c+\varepsilon}} = 0$ $(U \in \mathcal{U}_L)$, the ∂-map in the first line cannot be one to one $(U \notin \mathcal{U}_U)$ and conversely $U \in \mathcal{U}_U \Rightarrow (U \notin \mathcal{U}_L)$.

Barannikov presentation of Morse theory

Remember that if $B \subset A \subset X$ and $B' \subset A' \subset X'$ and $\varphi : X(\text{resp.}A, B) \to X'(\text{resp.}A', B')$ continuous, then

$$\begin{array}{ccccccccc}
\longrightarrow & H_*(A,B) & \longrightarrow & H_*(X,B) & \longrightarrow & H_*(X,A) & \xrightarrow{\partial} & H_{*-1}(A,B) & \longrightarrow \\
 & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \overline{\varphi_*} & & \downarrow \varphi_* & \\
\longrightarrow & H_*(A',B') & \longrightarrow & H_*(X',B') & \longrightarrow & H_*(X',A') & \xrightarrow{\partial} & H_{*-1}(A',B') & \longrightarrow
\end{array}$$

Apply it with

$$(X, A, B) = (f^{c+\varepsilon}, f^{c-\varepsilon}, \emptyset) \quad \text{and} \quad (X', A', B') = (M, f^{c-\varepsilon}, \emptyset)$$

with the mapping $i^{\infty,c+\varepsilon} : f^{c+\varepsilon} \to M = f^{+\infty}$:

$$\begin{array}{ccccccc}
H_*(f^{c-\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) & \xrightarrow{\partial} & H_{*-1}(f^{c-\varepsilon}) \\
\downarrow \mathrm{Id} & & \downarrow i_*^{\infty,c+\varepsilon} & & \downarrow \overline{i_*^{\infty,c+\varepsilon}} & & \downarrow \mathrm{Id} \\
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holds for $\lambda = -\infty$. It is clearly not true for $\lambda = c - \varepsilon$.

By diagram chasing (a bit more involved than before) one can prove:

- $c' = \sup\{\lambda < c, (0.1) \text{ true}\}$ is a lower critical value.
- $\#\mathcal{C}_H^{(p)} = \beta_p(M) = \dim H_p(M)$.

Definition: On $\bigoplus_{c \in \mathcal{C}} \mathbb{K}c = \bigoplus_{p=0}^{\dim M} \mathrm{Vect}\left(\mathcal{C}^{(p)}\right)$ we define ∂_B by:

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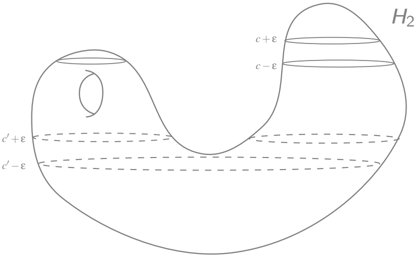
Let us look at the case $p = 2$ on an example



$H_2(f^{c+\varepsilon}, f^{c'-\varepsilon}) = H_2(f^{c+\varepsilon}) = \{0\}$ by retraction to an 8-curve.

$$H_2(f^{c+\varepsilon}, f^{c'+\varepsilon}) = H_2(\mathbb{T}^2) \sim H_2(\mathbb{S}^2) = H^2(f^{c+\varepsilon, c-\varepsilon}).$$

relative homology = *reduced homology of the suspension*

One can also notice that $\dim H_1(f^t)$ is increased by 1 (from 2 to 3) when t goes from $c + \varepsilon$ to $c - \varepsilon$.

Duality: Alternatively for the case $(p = 2, p - 1 = 1)$, take the picture upside down and look at $(p = 1, p - 1 = 0)$.

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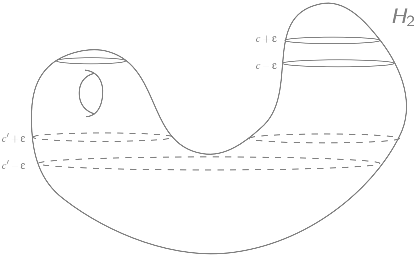

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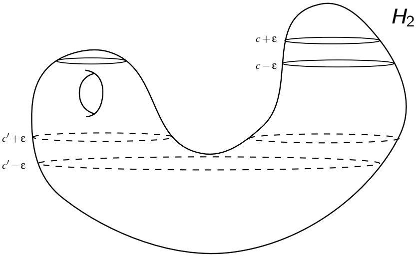

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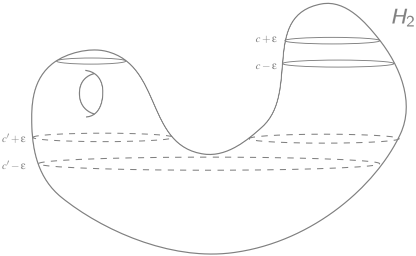

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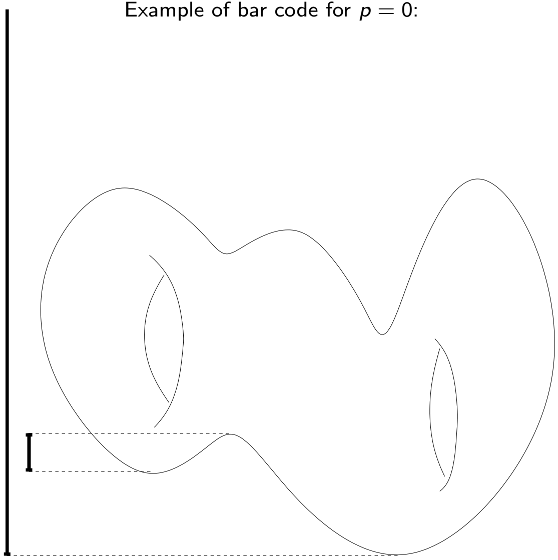
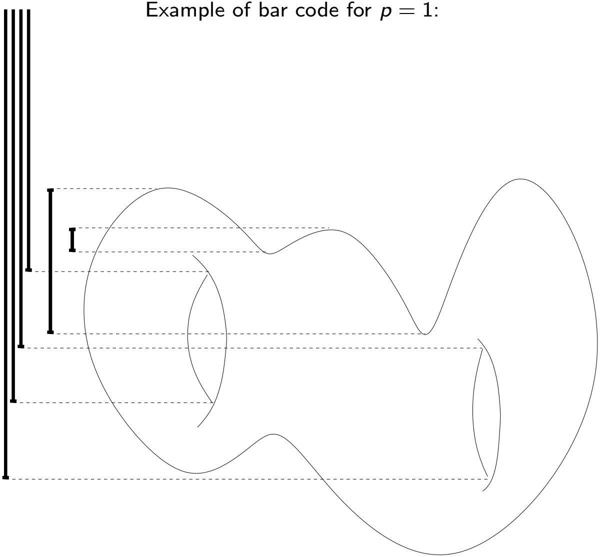
Example of bar code for $p = 0$:



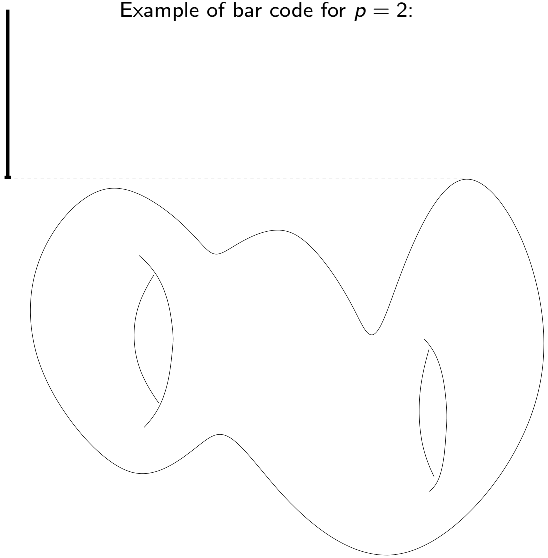
Bar codes, persistence diagrams

Example of bar code for $p = 1$:



Bar codes, persistence diagrams

Example of bar code for $p = 2$:



Bar codes, persistence diagrams

When $c \in \mathcal{C}_U^{(p)}$ and $\partial_B c = c' \in \mathcal{C}_L^{(p-1)} \Rightarrow c' \notin \mathcal{C}_U^{(p-1)}$, we know

$$H_p(f^{c+\varepsilon}) \xrightarrow{=0} H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \xrightarrow{\partial} H_{p-1}(f^{c-\varepsilon}) \longrightarrow H_{p-1}(f^{c+\varepsilon}) \longrightarrow 0$$

and $$0 \longrightarrow H_{p-1}(f^{c'-\varepsilon}) \longrightarrow H_{p-1}(f^{c'+\varepsilon}) \longrightarrow H_{p-1}(f^{c'+\varepsilon, c'-\varepsilon}) \longrightarrow 0\,.$$

By further diagram chasing, one can prove that the range of $H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \xrightarrow{\partial} H_{p-1}(f^{c-\varepsilon})$ defines a non nul element of $H_{p-1}(f^{c'+\varepsilon}, f^{c'-\varepsilon})$: namely let e_p (resp. e_{p-1}) denote the stable manifold of ∇f at $U^{(p)}$ (resp. $U^{(p-1)}$) and let $[e_p]$ (resp $[e_{p-1}]$) denotes its class in $H_p(f^{c+\varepsilon}, f^{c-\varepsilon})$ (resp. $H_{p-1}(f^{c'+\varepsilon}, f^{c'-\varepsilon})$) , then $\partial[e_p] = \kappa[e_{p-1}]$ with $\kappa \neq 0$.
Because the coefficient ring is a field $\mathbb{K}$, H_* are vector spaces and

$$H_{p-1}(f^{c'+\varepsilon}) = H_{p-1}(f^{c'-\varepsilon}) \oplus \mathbb{K}[e_{p-1}]\,,$$
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Playing with the maps $\varphi_s^t : H_{p-1}(f^s) \to H_{p-1}(f^t)$ with range F_s^t, we obtain

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(c' cannot satisfy the condition for $c' \in \mathcal{C}_U^{(p-1)}$ and $\dim H_{p-1}(f^{c'+\varepsilon}, f^{c'-\varepsilon}) = 1$.)

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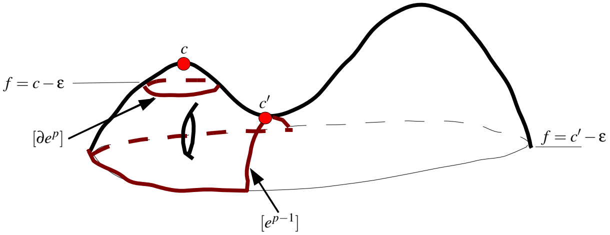

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Because the coefficient ring is a field $\mathbb{K}$, H_* are vector spaces and

$$H_{p-1}(f^{c'+\varepsilon}) = H_{p-1}(f^{c'-\varepsilon}) \oplus \mathbb{K}[e_{p-1}]\,,$$
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Bar codes, persistence diagrams

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DEF: The bar code of (M, f) is the set of intervals $(\partial_B c, c)$ with $c \in \mathcal{C}_U$, or (c', c_f) with $c' \in \mathcal{C}_H$ and c_f any number $> \max f$ (possibly $+\infty$) .
The persistence diagram is the corresponding set in $\mathbb{R}^2$ made of the pairs (a, b) , $a < b$ the extremities of the above intervals to which we add the diagonal $\Delta = \{(x, x)\}$.

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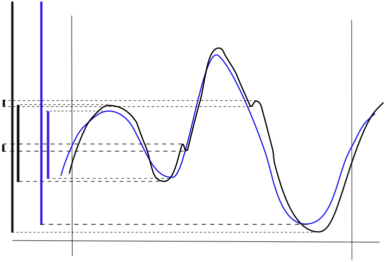
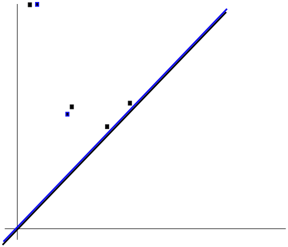


Two periodic functions close to each other and their $p = 0$ bar code



Corresponding persistence diagrams