# Exponentially small eigenvalues of Witten Laplacians 4: the case of $p$-forms

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# Outline



- Result
- Extending the strategy used for $p = 0$
- Restriction to $f_a^b$
- Quasimodes
- Final computation

# Result



REF: Le Peutrec-N.-Viterbo(13)
$(M,g)$ compact (oriented) manifold without boundary.
Consider $f^{\lambda}=\{x\in M, f(x)<\lambda\}$ and $f_{\lambda}=\{x\in M, f(x)>\lambda\}$.

$$\Delta_{f,h}=(d_{f,h}+d_{f,h}^{*})^{2}=d_{f,h}^{*}d_{f,h}+d_{f,h}d_{f,h}^{*}=\bigoplus_{p=0}^{\dim M}\Delta_{f,h}^{(p)}\,.$$

There is a one to one correspondance $j_p$ between $\mathcal{U}^{(p)}$ and the set of eigenvalues (counted with multiplicities) of $\Delta_{f,h}^{(p)}$ lying in $[0,h^{3/2})$ such that

$$j_p(U^{(p)})=0 \quad \text{if} \quad U^{(p)}\in\mathcal{U}_H^{(p)}$$

$$j_p(U^{(p)})=\kappa^2(U^{(p+1)})\frac{h}{\pi}\frac{|\lambda_1^{(p+1)}\dots\lambda_{p+1}^{(p+1)}|}{|\lambda_1^{(p)}\dots\lambda_p^{(p)}|}\frac{|\mathrm{Hess} f(U^{(p)})|^{1/2}}{|\mathrm{Hess} f(U^{(p+1)})|^{1/2}}(1+\mathcal{O}(h))e^{-2\frac{f(U^{(p+1)})-f(U^{(p)})}{h}}$$
$$\text{if } \partial_B U^{(p+1)}=U^{(p)}$$

$$j_p(U^{(p)})=\kappa^2(U^{(p)})\frac{h}{\pi}\frac{|\lambda_1^{(p)}\dots\lambda_p^{(p)}|}{|\lambda_1^{(p-1)}\dots\lambda_{p-1}^{(p-1)}|}\frac{|\mathrm{Hess} f(U^{(p-1)})|^{1/2}}{|\mathrm{Hess} f(U^{(p)})|^{1/2}}(1+\mathcal{O}(h))e^{-2\frac{f(U^{(p)})-f(U^{(p-1)})}{h}}$$
$$\text{if } \partial_B U^{(p)}=U^{(p-1)}$$

Here the $\lambda$'s denote the negative eigenvalues of the $\mathrm{Hess} f$ at the corresponding points.

# Extending the strategy used for $p = 0$



Witten Laplacians: We know that the number of $\mathcal{O}(h^{3/2})$-eigenvalues of $\Delta^{(p)}_{f,h}$ is $m_p = \sharp\mathcal{U}^{(p)} = \sharp\mathcal{C}^{(p)}$. Set $F^{(p)} = \mathrm{Ran} 1_{[0,h^{3/2})}(\Delta^{(p)}_{f,h})$, $F = 1_{[0,h^{3/2})}(\Delta_{f,h})$ and $\beta^{(p)}_{f,h} = d_{f,h}\big|_{F^{(p)}} : F^{(p)} \to F^{(p+1)}$,. Then
$\Delta_{f,h}\big|_F = (\beta_{f,h} + \beta^*_{f,h})^2 = \beta^*_{f,h}\beta_{f,h} + \beta_{f,h}\beta^*_{f,h}\,.$

Singular values: When $\Delta^{(p)}_{f,h}u = \lambda u$, $u \in F^{(p)}$ there are three possibilities:

- $\lambda = 0$ and $\beta_{f,h}u = 0$, $\beta^*_{f,h}u = 0$
- $\lambda \neq 0$ and $\beta^*_{f,h}u \neq 0$. Then $\beta^*_{f,h}u \in F^{(p-1)}$ and $\Delta^{(p-1)}_{f,h}(\beta^*_{f,h}u) = \lambda(\beta^*_{f,h}u) = (\beta^*_{f,h}\beta_{f,h})(\beta^*_{f,h}u)\,.$
- $\lambda \neq 0$ and $\beta^*_{f,h}u = 0$. Then $\lambda u = \Delta_{f,h}u = \beta^*_{f,h}\beta_{f,h}u\,.$

In all cases $\lambda$ is the square of a singular value of $\beta_{f,h}$.

The pairing of critical points is given by Barannikov complex: $\partial_B U^{(p)} = U^{(p-1)}$, $U^{(p)} \in \mathcal{U}^{(p)}_U$, $U^{(p-1)} \in \mathcal{U}^{(p-1)}_L$. Homological critical points $U \in \mathcal{U}^{(p)}_H$ will be associated with eigenvalues 0 of $\Delta^{(p)}_{f,h}$ and harmonic forms ($\dim = \beta_p = \sharp\mathcal{U}^{(p)}_H$).

In order to extend the strategy used for $p = 0$ with singular values, we need to construct local quasimodes around upper critical points (WKB following Helffer-Sjöstrand) and global quasimodes for lower critical points. The explicit form $\psi^{(0)}_k(h) = \chi_k \exp[-\frac{f(x)-f(U^{(0)}_k)}{h}]$ which is no more possible for $U \in \mathcal{U}^{(p>0)}_L$, but $d_{f,h}(\chi\omega) = (hd\chi) \wedge \omega$ holds for any $\omega$ which satisifies $\Delta_{f,h}\omega = 0$.

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$$F^{(p)}=\mathrm{Ran}(\beta_{f,h}^{(p-1)})\overset{\perp}{\oplus}\ker(\Delta_{f,h}^{(p)})\overset{\perp}{\oplus}\mathrm{Ran}(\beta_{f,h}^{(p),*})\quad \text{Hodge decomposition}.$$

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# Restriction to $f_a^b$



The persistent homology (classification and pairing of critical points via $\partial_B$) for the Morse function $f$ on $M$ is a way to understand the homology groups $H_*(M) = H_*(f^{+\infty}, f^{-\infty})$ . Actually it is a particular case of the $H_*(f^b, f^a)$ , $a < b$ , $a, b \notin \mathcal{U}$ , and those constructions have natural restriction properties.

When $a \leq a' < b' \leq b$, the definitions of $c \in \mathcal{C}_{U,L,H}$ and $\partial_B c = c'$ yield:

- if $c \in \mathcal{C}_H(f^b, f^a)$ then $c \in \mathcal{C}_H(f^{b'}, f^{a'})$ .
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Translation in terms of Witten Laplacians on $f_a^b = \{x \in M, a < f(x) < b\}$: The Neumann boundary condition corresponds to the absolute homology and the Dirichlet boundary condition to the relative homology. So the BC realization of $\Delta_{f,h}$ to $f_a^b$ which encodes $H_*(f^b, f^a)$ is the one with Dirichlet boundary condtions on $\{f = a\}$ ($f < a$ is replaced by $f = -\infty$) and Neumann boundary conditions on $\{f = b\}$ ($f > b$ is replaced by $f = +\infty$) , denoted by $\Delta_{f,h}^{DN}$ .

Since $\partial_n f < 0$ on $\{f = a\}$ and $\partial_n f > 0$ on $\{f = b\}$ there will be no generalized critical points on the boundary $\partial f_a^b$ and the critical points involved in the asymptotic analysis of $\Delta_{f,h}^{DN}$ are the critical points of $f$ belonging to $(a, b)$ .

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This can be formulated by saying that the sheaf $I \to H_*(f^{\sup I}, f^{\inf I})$ of vector spaces is a sum of one dimensional sheaves (bar codes).

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# Quasimodes



If $U\in\mathcal{U}_H^{(p)}$, there exists $\psi_U=\tilde{v}_U\in\ker\Delta_{f,h}^{(p)}$ localized near $U$.

If $U\in\mathcal{U}_U^{(p)}$, take $\psi_U=\chi_U\tilde{v}_U$ where $\chi_U$ localizes in the neighborhood of $U$ and $\tilde{v}_U$ is an eigenmode on $f(U)-\varepsilon<f<f(U)+\varepsilon$.

If $U\in\mathcal{U}_L^{(p)}$ take $\psi_U=\chi_U\tilde{v}_U$ where $\chi_U$ and $\tilde{v}_U$ correspond to a local truncation just below $U'$ such that $\partial_B U'=U$.

By Helffer-Sjöstrand WKB techniques, we have a local approximation of $\tilde{v}_U$ in $B(U,\varepsilon_1)$ for all $U\in\mathcal{U}$ and therefore can compute the normalisation constants for $\psi_U$ as $h$-power asymptotic expansion by Laplace methods in term of $\mathrm{Hess}\, f(U)$ like in the case $p=0$ or $p=1$.

Nevertheless we have no explicit form of $\psi_U$ near $U'\in\mathcal{U}_U^{(p+1)}$ when $U\in\mathcal{U}_L^{(p)}$.

In all cases $d_{f,h}\tilde{v}_U=d_{f,h}^{*}\tilde{v}_U=0$. In particular when $U\in\mathcal{U}_L$, this property valid near $U'$, $\partial_B U'=U$, combined with Stokes formula allows to bypass the explicit approximation of $\psi_U$ near $U'$.

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# Quasimodes

If $U \in \mathcal{U}_H^{(p)}$, there exists $\psi_U = \tilde{v}_U \in \ker \Delta_{f,h}^{(p)}$ localized near $U$.

If $U \in \mathcal{U}_U^{(p)}$, take $\psi_U = \chi_U \tilde{v}_U$ where $\chi_U$ localizes in the neighborhood of $U$ and $\tilde{v}_U$ is an eigenmode on $f(U) - \varepsilon < f < f(U) + \varepsilon$.

If $U \in \mathcal{U}_L^{(p)}$ take $\psi_U = \chi_U \tilde{v}_U$ where $\chi_U$ and $\tilde{v}_U$ correspond to a local truncation just below $U'$ such that $\partial_B U' = U$.

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# Final computation (simplified version)



The essential element to be computed is $\langle \psi_{U'}\,,\, d_{f,h}\psi_U\rangle$ when $\partial_B U' = U \in \mathcal{U}_L^{(p)}$.
This will provide like for $p=0$ the singular values of $\beta_{f,h}$ up to exponentially small relative errors.
Remember $\psi_{U'} = \chi_{U'}\tilde{v}_{U'}$ and $\psi_U = \chi_U \tilde{v}_U$ with

- $\chi_U$ global cut-off, $\chi_{U'}$ local cut-off.
- $d_{f,h}\tilde{v}_U = 0\,,\; d_{f,h}\tilde{v}_{U'} = 0\,.$

Simplified version: euclidean metric around $U'$ in Morse coordinates $y=(y',y'')$, $y'=(y_1,\ldots,y_{p+1})\,.$

$$\tilde{v}_{U'} \sim C(U',h)e^{-\frac{\Phi_{U'}(y)}{h}} \star (dy_{p+2}\wedge \ldots dy_n) \quad \text{around } U'$$

$$(f(y)-f(U')) = \frac{-\lambda_1 y_1^2 \cdots - \lambda_{p+1}y^{p+1} + \lambda_{p+2}y_{p+2}^2 + \cdots \lambda_n y_n^2}{2}$$

$$\Phi_{U'}(y) = \frac{\sum_{j=0}^n \lambda_j y_j^2}{2}$$

$$d(e^{\frac{f(y)-f(U)}{h}}\,\tilde{v}_U) = 0\,.$$

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$$\begin{aligned} I_{U',U}(h) &\sim \int_{\omega(U')} e^{-\frac{f-f(U')}{h}} e^{-\frac{\Phi_{U'}}{h}}\, dy_{p+2}\wedge\ldots dy_n \wedge (hd\chi_U)\wedge(e^{\frac{f-f(U)}{h}}\tilde{v}_U) \\ &\sim \int_{|y''|\leq r} e^{-\frac{\sum_{j=p+2}^n \lambda_j y_j^2}{h}} |dy''| \underbrace{\int_{B^{p+1}_{y''}} (hd\chi_U)\wedge(e^{\frac{f-f(U)}{h}}\tilde{v}_U)}_{\text{Stokes } d_{f,h}\tilde{v}_U=0} \end{aligned}$$

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Applying again Stokes with $\partial B^{p+1}_{y',r}$ homologous to $\partial B^{p+1}_{0,r} = \partial e^{p+1}$, $e^{p+1}$ the stable cell of $\nabla f$ at $U'$, we obtain

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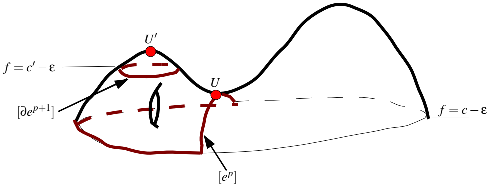
A byproduct of Barannikov says that there exists a constant $\kappa(U') \in \mathbb{R}$ such that $\partial e^{p+1} - \kappa(U')e^p$ is a boundary (relatively to $\{f = f(U) - \varepsilon\}$)




We use again Stokes to get

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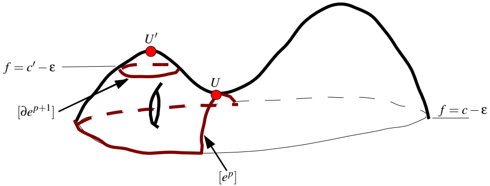


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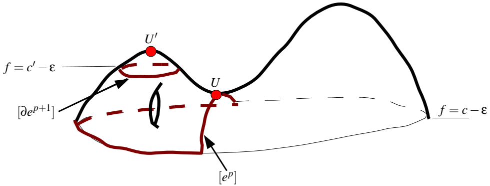


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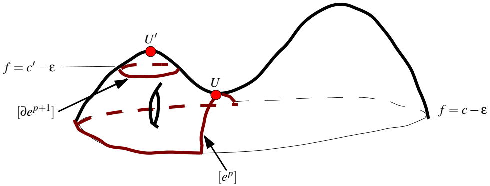


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integration localized around $U$ $\rightarrow$ WKB approx of $\tilde{v}_U$ .

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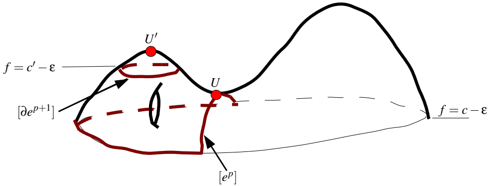


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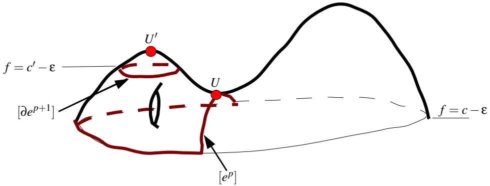


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