Multiscale analysis and mean field asymptotics

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

Rennes, 15/03/2017

Outline

- Wick observables
- Reduced density matrices
- Multiscale measures
- Multiscale analysis of reduced density matrices
- Examples

Wick observables

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

$\mathcal{Z}$ complex separable Hilbert space.

$$\Gamma_{\pm}(\mathcal{Z}) = \oplus_{n\in\mathbb{N}} \mathcal{S}_{\pm}^{n} \mathcal{Z}^{\otimes n}$$

$$\mathcal{S}_{\pm}^{n}(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} s_{\pm}(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \,.$$

$s_{+}(\sigma) = +1$ (bosons) $\quad s_{-}(\sigma) =$signature of σ (fermions).

Definition

For $\tilde{b} \in \mathcal{L}(\mathcal{S}_{\pm}^{p}\mathcal{Z}^{\otimes p}; \mathcal{S}_{\pm}^{q}\mathcal{Z}^{\otimes q})$,

$$\tilde{b}^{Wick}\big|_{\mathcal{S}_{\pm}^{n+p}\mathcal{Z}^{\otimes n+p}} = \varepsilon^{\frac{p+q}{2}} \frac{\sqrt{(n+p)!(n+q)!}}{n!} \mathcal{S}_{\pm}^{n+q}(\tilde{b} \otimes \mathrm{Id}_{\mathcal{Z}^{\otimes n}})\mathcal{S}_{\pm}^{n+p} \,.$$

Wick observables

Examples:

- $a_\pm(f) = (\langle f| : \mathcal{Z} \to \mathbb{C})^{Wick}$; $a_\pm^*(f) = (|f\rangle : \mathbb{C} \to \mathcal{Z})^{Wick}$;

$$[a_\pm(f_1), a_\pm(f_2)]_\pm = [a_\pm^*(f_1), a_\pm^*(f_2)]_\pm = 0$$
$$[a_\pm(f_1), a_\pm^*(f_2)]_\pm = \varepsilon\langle f_1, f_2\rangle \,.$$

- $A \in \mathcal{L}(\mathcal{Z})$, $d\Gamma_\pm(A) = A^{Wick}$.

$$d\Gamma_\pm(A) = i\frac{d}{dt}\Gamma_\pm(^{-i\varepsilon tA}) \quad \text{when } A = A^*$$
$$\mathbf{N}_\pm = (\mathrm{Id}_\mathcal{Z})^{Wick} = d\Gamma(\mathrm{Id}_\mathcal{Z}) \quad \mathbf{N}_\pm\big|_{\mathcal{S}_\pm^n \mathcal{Z}^{\otimes n}} = \varepsilon n \,.$$

Wick observables

Examples:

- $a_\pm(f) = (\langle f| : \mathcal{Z} \to \mathbb{C})^{Wick}$; $a_\pm^*(f) = (|f\rangle : \mathbb{C} \to \mathcal{Z})^{Wick}$;

$$[a_\pm(f_1), a_\pm(f_2)]_\pm = [a_\pm^*(f_1), a_\pm^*(f_2)]_\pm = 0$$
$$[a_\pm(f_1), a_\pm^*(f_2)]_\pm = \varepsilon\langle f_1, f_2\rangle\,.$$

- $A \in \mathcal{L}(\mathcal{Z})$, $d\Gamma_\pm(A) = A^{Wick}$.

$$d\Gamma_\pm(A) = i\frac{d}{dt}\Gamma_\pm(^{-i\varepsilon tA}) \quad \text{when } A = A^*$$
$$\mathbf{N}_\pm = (\mathrm{Id}_\mathcal{Z})^{Wick} = d\Gamma(\mathrm{Id}_\mathcal{Z}) \quad \mathbf{N}_\pm\big|_{\mathcal{S}_\pm^n \mathcal{Z}^{\otimes n}} = \varepsilon n\,.$$

Consequences:

- Mean field asymptotics $n \to \infty$ same as considering $\mathbf{N}_\pm = \mathcal{O}(1)$ and $\varepsilon \to 0$.
- In the bosonic setting, mean field asymptotics = infinite semiclassical analysis with small parameter " h" $= \frac{\varepsilon}{2}$.

Wick observables

Some properties when $p = q$, $\tilde{b} \in \mathcal{L}(\mathcal{S}_{\pm}^{p}\mathcal{Z}^{\otimes p})$, $\tilde{b}_1 \in \mathcal{L}(\mathcal{Z})$.

- $(\tilde{b} = \tilde{b}^*) \Rightarrow (\tilde{b}^{Wick}$ symmetric)
- $(\tilde{b} \geq 0) \Rightarrow (\tilde{b}^{Wick} \geq 0)$
- Number estimates, $m + m' \geq p$

$$\|(1 + \mathbf{N}_{\pm})^{-m}\tilde{b}^{Wick}(1 + \mathbf{N}_{\pm})^{-m'}\| \leq C_{m,m'}\|\tilde{b}\|$$

$$\|(1 + \mathbf{N}_{\pm})^{-m}[d\Gamma_{\pm}(\tilde{b}_1)^p - (\tilde{b}_1^{\otimes p})]^{Wick}(1 + \mathbf{N}_{\pm})^{-m'}\| \leq \varepsilon B_p\|\tilde{b}_1\|^p$$

- finite ε-expansion for composition formulas.

Reduced density matrices

Non normalized reduced density matrices

Definition

Assume $\varrho_\varepsilon \in \mathcal{L}^1(\Gamma_\pm(\mathcal{Z}))$, $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$ and $\mathrm{Tr}\,\left[\varrho_\varepsilon e^{c\mathbf{N}_\pm})\right] < +\infty$.
For $p \in \mathbb{N}$, $\gamma_\varepsilon^{(p)}$ is defined by

$$\forall \tilde{b} \in \mathcal{L}(\mathcal{S}_\pm \mathcal{Z}^{\otimes p}) , \quad \mathrm{Tr}\,\left[\gamma_\varepsilon^{(p)} \tilde{b}\right] = \mathrm{Tr}\,\left[\varrho_\varepsilon \tilde{b}^{Wick}\right]$$

Reduced density matrices

Non normalized reduced density matrices

Definition

Assume $\varrho_\varepsilon \in \mathcal{L}^1(\Gamma_\pm(\mathcal{Z}))$, $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$ and $\mathrm{Tr}\,\left[\varrho_\varepsilon e^{c\mathbf{N}_\pm)}\right] < +\infty$.
For $p \in \mathbb{N}$, $\gamma_\varepsilon^{(p)}$ is defined by

$$\forall \tilde{b} \in \mathcal{L}(\mathcal{S}_\pm \mathcal{Z}^{\otimes p}), \quad \mathrm{Tr}\,\left[\gamma_\varepsilon^{(p)} \tilde{b}\right] = \mathrm{Tr}\,\left[\varrho_\varepsilon \tilde{b}^{Wick}\right]$$

Properties:

- In the bosonic case, if $\varrho_\varepsilon = |\varphi^{\otimes n}\rangle\langle\varphi^{\otimes n}|$, $\|\varphi\| = 1$, $\gamma_\varepsilon^{(p)} = 1_{[0:n]}(p)\varepsilon^p \frac{n!}{(n-p)!} |\varphi^{\otimes p}\rangle\langle\varphi^{\otimes p}|$.
- Symmetrization: $\gamma_\varepsilon^{(p)}$ is completely determined by the quantities $\mathrm{Tr}\,\left[\varrho_\varepsilon (\tilde{b}_1^{\otimes p})^{Wick}\right]$, $\tilde{b}_1 \in \mathcal{L}(\mathcal{Z})$ (or $\tilde{b}_1 \in \mathcal{L}^\infty(\mathcal{Z})$).
- The sequence $(\gamma_\varepsilon^{(p)})_{p\in\mathbb{N}}$ is determined by the family of generating functions $z \mapsto \mathrm{Tr}\,\left[\varrho_\varepsilon \Gamma_\pm(e^{\varepsilon z \tilde{b}_1})\right]$, $|z| < \frac{c}{\|\tilde{b}_1\|}$.

Reduced density matrices

Non normalized reduced density matrices

Definition

Assume $\varrho_\varepsilon \in \mathcal{L}^1(\Gamma_\pm(\mathcal{Z}))$, $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$ and $\mathrm{Tr}\,\left[\varrho_\varepsilon e^{c\mathbf{N}_\pm}\right] < +\infty$.
For $p \in \mathbb{N}$, $\gamma_\varepsilon^{(p)}$ is defined by

$$\forall \tilde{b} \in \mathcal{L}(\mathcal{S}_\pm \mathcal{Z}^{\otimes p}), \quad \mathrm{Tr}\,\left[\gamma_\varepsilon^{(p)} \tilde{b}\right] = \mathrm{Tr}\,\left[\varrho_\varepsilon \tilde{b}^{Wick}\right]$$

Properties:

- In the bosonic case, if $\varrho_\varepsilon = |\varphi^{\otimes n}\rangle\langle\varphi^{\otimes n}|$, $\|\varphi\| = 1$, $\gamma_\varepsilon^{(p)} = 1_{[0:n]}(p)\varepsilon^p \frac{n!}{(n-p)!}|\varphi^{\otimes p}\rangle\langle\varphi^{\otimes p}|$.
- Symmetrization: $\gamma_\varepsilon^{(p)}$ is completely determined by the quantities $\mathrm{Tr}\,\left[\varrho_\varepsilon(\tilde{b}_1^{\otimes p})^{Wick}\right]$, $\tilde{b}_1 \in \mathcal{L}(\mathcal{Z})$ (or $\tilde{b}_1 \in \mathcal{L}^\infty(\mathcal{Z})$).
- The sequence $(\gamma_\varepsilon^{(p)})_{p\in\mathbb{N}}$ is determined by the family of generating functions $z \mapsto \mathrm{Tr}\,\left[\varrho_\varepsilon \Gamma_\pm(e^{\varepsilon z \tilde{b}_1})\right]$, $|z| < \frac{c}{\|\tilde{b}_1\|}$.

Reduced density matrices

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

Non normalized reduced density matrices

Definition

Assume $\varrho_\varepsilon \in \mathcal{L}^1(\Gamma_\pm(\mathcal{Z}))$, $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$ and $\mathrm{Tr}\,\left[\varrho_\varepsilon e^{c\mathbf{N}_\pm})\right] < +\infty$.
For $p \in \mathbb{N}$, $\gamma_\varepsilon^{(p)}$ is defined by

$$\forall \tilde{b} \in \mathcal{L}(\mathcal{S}_\pm \mathcal{Z}^{\otimes p}), \quad \mathrm{Tr}\,\left[\gamma_\varepsilon^{(p)} \tilde{b}\right] = \mathrm{Tr}\,\left[\varrho_\varepsilon \tilde{b}^{Wick}\right]$$

Properties:

- In the bosonic case, if $\varrho_\varepsilon = |\varphi^{\otimes n}\rangle\langle\varphi^{\otimes n}|$, $\|\varphi\| = 1$, $\gamma_\varepsilon^{(p)} = 1_{[0:n]}(p)\varepsilon^p \frac{n!}{(n-p)!} |\varphi^{\otimes p}\rangle\langle\varphi^{\otimes p}|$.
- Symmetrization: $\gamma_\varepsilon^{(p)}$ is completely determined by the quantities $\mathrm{Tr}\,\left[\varrho_\varepsilon (\tilde{b}_1^{\otimes p})^{Wick}\right]$, $\tilde{b}_1 \in \mathcal{L}(\mathcal{Z})$ (or $\tilde{b}_1 \in \mathcal{L}^\infty(\mathcal{Z})$).
- The sequence $(\gamma_\varepsilon^{(p)})_{p\in\mathbb{N}}$ is determined by the family of generating functions $z \mapsto \mathrm{Tr}\,\left[\varrho_\varepsilon \Gamma_\pm(e^{\varepsilon z \tilde{b}_1})\right]$, $|z| < \frac{c}{\|\tilde{b}_1\|}$.

Reduced density matrices

Asymptotics of $\gamma_\varepsilon^{(p)}$ as $\varepsilon \to 0$: Assume $\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_\pm^{(p)}\right] = c_p$.

1) Bosonic case: Wigner measure= probability measure μ on $\mathcal{Z}$ such that after extraction $\varepsilon \in \mathcal{E}$, $0 \in \overline{\mathcal{E}}$,

$$\forall f \in \mathcal{Z}\,, \quad \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathcal{E}}} \mathrm{Tr}\left[W(\sqrt{2}\pi f)\varrho_\varepsilon\right] = \int e^{2i\pi \mathrm{Re}\langle f\,,z\rangle}\, d\mu(z)\,.$$

Then the weak*-limit of $\gamma_\varepsilon^{(p)}$ is

$$\gamma_0^{(p)} = \int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\, d\mu(z)\,.$$

Reduced density matrices

Asymptotics of $\gamma_\varepsilon^{(p)}$ as $\varepsilon \to 0$: Assume $\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_\pm^{(p)}\right] = c_p$.

1) Bosonic case: Wigner measure= probability measure μ on $\mathcal{Z}$ such that after extraction $\varepsilon \in \mathcal{E}$, $0 \in \overline{\mathcal{E}}$,

$$\forall f \in \mathcal{Z}\,, \quad \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathcal{E}}} \mathrm{Tr}\left[W(\sqrt{2}\pi f)\varrho_\varepsilon\right] = \int e^{2i\pi \mathrm{Re}\langle f\,,z\rangle}~d\mu(z)\,.$$

Then the weak*-limit of $\gamma_\varepsilon^{(p)}$ is

$$\gamma_0^{(p)} = \int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|~d\mu(z)\,.$$

But

Reduced density matrices

Asymptotics of $\gamma_\varepsilon^{(p)}$ as $\varepsilon \to 0$: Assume $\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_{\pm}^{(p)}\right] = c_p$.
1) Bosonic case: Wigner measure= probability measure μ on $\mathcal{Z}$ such that after extraction $\varepsilon \in \mathcal{E}$, $0 \in \overline{\mathcal{E}}$,

$$\forall f \in \mathcal{Z}\,, \quad \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathcal{E}}} \mathrm{Tr}\left[W(\sqrt{2}\pi f)\varrho_\varepsilon\right] = \int e^{2i\pi \mathrm{Re}\langle f\,, z\rangle}\, d\mu(z)\,.$$

Then the weak*-limit of $\gamma_\varepsilon^{(p)}$ is

$$\gamma_0^{(p)} = \int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\; d\mu(z)\,.$$

But $\mathrm{Tr}\left[\gamma_0^{(p)}\right] = \int_{\mathcal{Z}} d\mu < c_p = \lim_{\varepsilon\to} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_+^p\right]$ may happen.

Reduced density matrices

Asymptotics of $\gamma_\varepsilon^{(p)}$ as $\varepsilon \to 0$: Assume $\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_\pm^{(p)}\right] = c_p$.
1) Bosonic case: Wigner measure= probability measure μ on $\mathcal{Z}$ such that after extraction $\varepsilon \in \mathcal{E}$, $0 \in \overline{\mathcal{E}}$,

$$\forall f \in \mathcal{Z}, \quad \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathcal{E}}} \mathrm{Tr}\left[W(\sqrt{2}\pi f)\varrho_\varepsilon\right] = \int e^{2i\pi \mathrm{Re}\langle f, z\rangle}\, d\mu(z).$$

Then the weak*-limit of $\gamma_\varepsilon^{(p)}$ is

$$\gamma_0^{(p)} = \int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\, d\mu(z).$$

But $\mathrm{Tr}\left[\gamma_0^{(p)}\right] = \int_{\mathcal{Z}} d\mu < c_p = \lim_{\varepsilon\to} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_+^p\right]$ may happen.
Example: $\varrho_\varepsilon = |\varphi_\varepsilon^{\otimes n_\varepsilon}\rangle\langle\varphi_\varepsilon^{\otimes n_\varepsilon}|$ with $\|\varphi_\varepsilon\| = 1$, $\lim_{\varepsilon\to 0} \varepsilon n_\varepsilon = 1$, w-$\lim_{\varepsilon\to 0} \varphi_\varepsilon = 0$. Then $\mu = \delta_0$ while $\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_+^p\right] = 1$.

Reduced density matrices

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

Asymptotics of $\gamma_\varepsilon^{(p)}$ as $\varepsilon \to 0$: Assume $\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}_{\pm}^{(p)}\right] = c_p$.

2) Fermionic case:

For any fixed $p \in \mathbb{N}$, the weak* limit of $\gamma_\varepsilon^{(p)}$ is always 0 while $\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\gamma_\varepsilon^{(p)}\right] = c_p$.

Multiscale measures

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

$\gamma_h \in \mathcal{L}^1(L^2(\mathbb{R}^d))$,
$x \in \mathbb{R}^D$, $X = (x, \xi) \in \mathbb{R}^{2D}$, used with $D = pd$ when $\gamma_h = \gamma^{(p)}_{\varepsilon(h)}$.

Multiscale measures

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

$\gamma_h \in \mathcal{L}^1(L^2(\mathbb{R}^d))$,
$x \in \mathbb{R}^D$, $X = (x, \xi) \in \mathbb{R}^{2D}$, used with $D = pd$ when $\gamma_h = \gamma^{(p)}_{\varepsilon(h)}$.
Double scale class of symbols $a \in \mathcal{S}^{(2)}$: $a \in \mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{R}^{2D})$,

- $\exists C_a > 0, \forall Y \in \mathbb{R}^{2D}, a(., Y) \in \mathcal{C}_0^\infty(B(0, C_a))$.
- There exists $a_\infty \in \mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$ such that $a(X, R\omega) \overset{R\to\infty}{\to} a_\infty(X, \omega)$ in $\mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$.

Multiscale measures

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

$\gamma_h \in \mathcal{L}^1(L^2(\mathbb{R}^d))$,
$x \in \mathbb{R}^D$, $X = (x, \xi) \in \mathbb{R}^{2D}$, used with $D = pd$ when $\gamma_h = \gamma^{(p)}_{\varepsilon(h)}$.

Double scale class of symbols $a \in \mathcal{S}^{(2)}$: $a \in \mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{R}^{2D})$,

- $\exists C_a > 0, \forall Y \in \mathbb{R}^{2D}, a(., Y) \in \mathcal{C}_0^\infty(B(0, C_a))$.
- There exists $a_\infty \in \mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$ such that $a(X, R\omega) \overset{R\to\infty}{\to} a_\infty(X, \omega)$ in $\mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$.

Quantization:

$$a^{(2),h} = a^{Weyl}(\sqrt{h}x, \sqrt{h}D_x, x, D_x) = [a(., h^{-1/2}.)]^{Weyl}(\sqrt{h}x, \sqrt{h}D_x).$$

Multiscale measures

$\gamma_h \in \mathcal{L}^1(L^2(\mathbb{R}^d))$,
$x \in \mathbb{R}^D$, $X = (x,\xi) \in \mathbb{R}^{2D}$, used with $D = pd$ when $\gamma_h = \gamma^{(p)}_{\varepsilon(h)}$.

Double scale class of symbols $a \in \mathcal{S}^{(2)}$: $a \in \mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{R}^{2D})$,

- $\exists C_a > 0, \forall Y \in \mathbb{R}^{2D}, a(., Y) \in \mathcal{C}_0^\infty(B(0, C_a))$.
- There exists $a_\infty \in \mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$ such that $a(X, R\omega) \overset{R\to\infty}{\to} a_\infty(X,\omega)$ in $\mathcal{C}^\infty(\mathbb{R}^{2D} \times \mathbb{S}^{2D-1})$.

Quantization:
$$a^{(2),h} = a^{Weyl}(\sqrt{h}x, \sqrt{h}D_x, x, D_x) = [a(., h^{-1/2}.)]^{Weyl}(\sqrt{h}x, \sqrt{h}D_x).$$

Multiscale measures: Assume $\mathrm{Tr}\,[\gamma_h] = 1$ there exists a subset $\mathcal{E}$, $0 \in \overline{\mathcal{E}}$, two non negative measures ν on $\mathbb{R}^{2D}$, ν_I on $\mathbb{S}^{2D-1}$ and a trace class operator γ_0 such that

$$\lim_{\substack{h\to 0\\ h\in\mathcal{E}}} \mathrm{Tr}\left[\gamma_h a^{(2),h}\right] = \int_{\mathbb{R}^{2D}\setminus\{0\}} a(X, \frac{X}{|X|})\, d\nu(X) + \int_{\mathbb{S}^{2D-1}} a_\infty(0,\omega)\, d\nu_I(\omega) + \mathrm{Tr}\left[\gamma_0 a^{Weyl}(0, x, D_x)\right].$$

Multiscale measures

Notations: Wigner measures $\nu \in \mathcal{M}(\gamma_h, h \in \mathcal{E})$.
Multiscale measures $(\nu, \nu_I, \gamma_0) \in \mathcal{M}^{(2)}(\gamma_h, h \in \mathcal{E})$.

Definition

The scaling $a^{Weyl}(\sqrt{h}x, \sqrt{h}D_x)$, $h \to 0$, is said adapted to the family $(\gamma_h)_{h\in\mathcal{E}}$,, $\gamma_h \in \mathcal{L}^1(L^2(\mathbb{R}^D))$, $\gamma_h \geq 0$, if for some $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2D})$, $0 \leq \chi \leq 1$, $\chi(0) = 1$,

$$\lim_{R\to\infty} \limsup_{\substack{h\to 0 \\ h\in\mathcal{E}}} \mathrm{Tr}\,\left[(1-\chi(R^{-1}.))^{Weyl,h}\gamma_h\right] = 0\,.$$

After extraction one can assume $c = \lim_{h\to 0} \mathrm{Tr}\,[\gamma_h]$ and if the scale is adapted all Wigner measures have the total mass c.

Definition

The scale $h \to 0$ is said separating if for all $(\nu, \nu_I, \gamma_0) \in \mathcal{M}^{(2)}(\gamma_h, h \in \mathcal{E})$, $\nu_I = 0$.

Multiscale measures

Francis Nier, LAGA, Univ. Paris 13
Joint work with Z. Ammari and S. Breteaux

When the scale is adapted, the separating property can be checked by a simple mass argument: After extraction assume that γ_0 is the weak* limit of γ_h and $\mathcal{M}(\gamma_h, h \in \mathcal{E}) = \{\nu\}$. Then the scale h is separating iff $\nu(\{0\}) = \mathrm{Tr}\,[\gamma_0]$ and then $\mathcal{M}^{(2)}(\gamma_h, h \in \mathcal{E}) = \{(\nu, 0, \gamma_0)\}$.

Multiscale analysis of reduced density matrices

We shall consider $\varepsilon = \varepsilon(h)$ with $\lim_{h\to 0}\varepsilon(h)) = 0$ and $\varepsilon(h)$-Wick quantization of h-dependent semiclassical observables,
$\tilde{b} = a^{Weyl}(\sqrt{h}x, \sqrt{h}D_x)$, $a \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ or $\tilde{b} = a^{(2)}$, when $a \in \mathcal{S}^{(2)}$.

Multiscale analysis of reduced density matrices

We shall consider $\varepsilon = \varepsilon(h)$ with $\lim_{h\to 0}\varepsilon(h)) = 0$ and $\varepsilon(h)$-Wick quantization of h-dependent semiclassical observables, $\tilde{b} = a^{Weyl}(\sqrt{h}x, \sqrt{h}D_x)$, $a \in \mathcal{C}_0^\infty(\mathbb{R}^{2d})$ or $\tilde{b} = a^{(2)}$, when $a \in \mathcal{S}^{(2)}$.

Assumptions: $\varrho_{\varepsilon(h)} \in \mathcal{L}^1(\Gamma_\pm(\mathcal{Z}))$, $\varrho_{\varepsilon(h)} \geq 0$.

- $\mathrm{Tr}\,\left[\varrho_{\varepsilon(h)}\right] = 1$.
- $\mathrm{Tr}\,\left[\varrho_{\varepsilon(h)}e^{c\mathbf{N}_\pm}\right] \leq C$.
- There exist $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^{2D})$ and $0 < c' < c$, $0 \leq \chi \leq 1$, $\chi(0) = 1$, such that, with $\chi_\delta = \chi(\delta.)$,

$$\lim_{\delta\to 0}\limsup_{h\to 0}\,\mathrm{Tr}\,\left[\varrho_{\varepsilon(h)}(e^{c'\mathbf{N}_\pm} - e^{c' d\Gamma_\pm(\chi_\delta)^{W,h}})\right] = 0\,.$$

Multiscale analysis of reduced density matrices

Proposition

The set $\mathcal{E}$ *can be chosen such that for all* $p \in \mathbb{N}$,
$\mathcal{M}(\gamma^{(p)}_{\varepsilon(h)}, h \in \mathcal{E}') = \left\{\nu^{(p)}\right\}$ *with* $\int_{\mathbb{R}^{2dp}} d\nu^{(p)} = \lim_{h\to 0} \mathrm{Tr}\left[\gamma^{(p)}_h\right]$.
Moreover for any $a \in \mathcal{C}^\infty_0(\mathbb{R}^{2d})$, *there exists* $r_a > 0$ *such that*
$\Phi_{a,h}(s) = \mathrm{Tr}\left[\varrho_{\varepsilon(h)} e^{sd\Gamma_\pm(a^{W,h})}\right]$ *is uniformly bounded in*
$H^\infty(\{|s| < r_a\})$ *and*

$$\lim_{\substack{h\to 0\\ h\in\mathcal{E}'}} \Phi_{a,h}(s) = \Phi_{a,0}(s) = \sum_{p=0}^{\infty} \frac{s^p}{p!} \int_{\mathbb{R}^{2dp}} a^{\otimes p}(X)\, d\nu^{(p)}(X).$$

Multiscale analysis of reduced density matrices

Proposition

For all $K \in \mathcal{L}^\infty(L^2(\mathbb{R}^d))$ *, there exists* $r_K > 0$ *such that* $\Psi_{K,h}(s) = \mathrm{Tr}\,\left[\varrho_{\varepsilon(h)} e^{s d\Gamma_\pm(K)}\right]$ *is uniformly bounded in* $H^\infty(\{|s| < r_K\})$ *. The pointwise convergence (or any weak convergence) of* $\Psi_{K,h}$ *to* $\Psi_{K,0}$ *for all* $K \in \mathcal{L}^\infty(L^2(\mathbb{R}^d))$ *is equivalent to* $w^* - \lim_{h\to 0} \gamma^{(p)}_{\varepsilon(h)} = \gamma^{(p)}_0$ *with*

$$\Psi_{K,0}(s) = \sum_{p=0}^{\infty} \frac{s^p}{p!}\,\mathrm{Tr}\,\left[\gamma_0^{(p)} K^{\otimes p}\right] .$$

The above convergence can always be achieved after some extraction $h \in \mathcal{E}'$ *,* $0 \in \overline{\mathcal{E}'}$ *,* $\mathcal{E}' \subset \mathcal{E}$ *.*

Examples: Gibbs states

$\alpha(X) = |X|^2 = x^2 + \xi^2$ or $\alpha(X)$ like $|X|^2$ at infinity with a non degenerate minimum at $X = 0$.

$$H = \alpha^{Weyl}(\sqrt{h}x, \sqrt{h}D_x) - \lambda_0(\alpha^{Weyl}(\sqrt{h}x, \sqrt{h}D_x)),$$
$$\varepsilon = \varepsilon(h) = h^d \quad , \quad \mu(\varepsilon) = -\frac{\varepsilon}{\beta\nu_C}, \nu_C, \beta > 0$$
$$\varrho_{\varepsilon(h)} = \frac{\Gamma_\pm(e^{-\beta(H-\mu(\varepsilon))})}{\mathrm{Tr}\left[\Gamma_\pm(e^{-\beta(H-\mu(\varepsilon))})\right]}.$$

Examples: Gibbs states

$\alpha(X) = |X|^2 = x^2 + \xi^2$ or $\alpha(X)$ like $|X|^2$ at infinity with a non degenerate minimum at $X = 0$.

$$H = \alpha^{Weyl}(\sqrt{h}x, \sqrt{h}D_x) - \lambda_0(\alpha^{Weyl}(\sqrt{h}x, \sqrt{h}D_x)),$$
$$\varepsilon = \varepsilon(h) = h^d \quad , \quad \mu(\varepsilon) = -\frac{\varepsilon}{\beta\nu_C}, \nu_C, \beta > 0$$
$$\varrho_{\varepsilon(h)} = \frac{\Gamma_\pm(e^{-\beta(H-\mu(\varepsilon))})}{\mathrm{Tr}\;\left[\Gamma_\pm(e^{-\beta(H-\mu(\varepsilon))})\right]}.$$

1) Fermionic case: For all $p \in \mathbb{N}$,
$\mathcal{M}^{(2)}(\gamma^{(p)}_{\varepsilon(h)}, h \in (0, h_0)) = \left\{\nu^{(p)}, 0, 0\right\}$ with

$$\nu^{(p)} = \left(\frac{e^{-\beta\alpha(X)}}{1 + e^{-\beta\alpha(X)}} \; \frac{dX}{(2\pi)^d}\right)^{\otimes p}.$$

Examples: Gibbs states

2) Bosonic case with Bose-Einstein condensation: Assume $d \leq 2$. Then for all $p \in \mathbb{N}$, $\mathcal{M}^{(2)}(\gamma^{(p)}_{\varepsilon(h)}, h \in (0,h_0)) = \left\{(\nu^{(p)}, 0, \gamma_0^{(p)})\right\}$ with

$$\gamma_0^{(p)} = p! n_C^p |\psi_0^{\otimes p}\rangle\langle\psi_0^{\otimes p}| \quad , \quad \psi_0(x) = U_T\left[\frac{e^{-x^2/2}}{\pi^{d/4}}\right]$$

$$\nu^{(p)} = \sum_{\sigma\in\mathfrak{S}_p} \sigma_*\left[\sum_{k=0}^{p} \frac{p!}{(p-k)!k!}\nu_C^k \delta_0^{\otimes k} \otimes (\nu(\beta,.)^{\otimes p-k})\right]$$

$$d\nu(\beta, X) = \frac{e^{-\beta\alpha(X)}}{1-e^{-\beta\alpha(X)}}\frac{dX}{(2\pi)^d}$$