# Artificial gauge and adiabatic Ansatz for Bose-Einstein condensates

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# Outline

- The model
- Remarks about Born-Oppenheimer Hamiltonians
- Rescaling
- Result

# The model

- We want to minimize the energy

$$\mathcal{E}_\kappa(\phi) = \int_{\mathbb{R}^2} |\nabla\phi|^2 + V_\kappa(x)|\phi|^2 + \frac{G}{2}|\phi|^4 + \kappa\langle\phi\,, M(\frac{x_1}{\ell}, kx_2)\phi\rangle_{\mathbb{C}^2} \quad dx_1 dx_2$$

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- $\phi = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \in \mathbb{C}^2$ and $|\phi|^2(x) = |\phi_1|^2(x) + |\phi_2|^2(x)$

$$M(x_1, x_2) = \Omega(x_1)\begin{pmatrix}\cos(\theta(x_1)) & e^{i\varphi(x_2)}\sin(\theta(x_1))\\ e^{-i\varphi(x_2)}\sin(\theta(x_1)) & -\cos(\theta(x_1))\end{pmatrix}.$$

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-

$$\Omega(x_1) = \sqrt{1+x_1^2}\,,$$

$$\cos(\theta(x_1)) = \frac{x_1}{\sqrt{x_1^2+1}} \quad , \quad \sin(\theta(x_1)) = \frac{1}{\sqrt{x_1^2+1}}$$

$$\varphi(x_2) = x_2\,.$$

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- Example of numerical values

$$\kappa \sim 10^6 \quad , \quad G \sim 600 \quad , \quad \ell \sim 25 \quad , \quad k \sim 50\,,$$

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- The eigenvalues of $M(x)$ are $\pm\Omega(x_1)$ with the eigenvectors

$$\psi_+(x) = \begin{pmatrix} C \\ Se^{-i\varphi} \end{pmatrix}\;,\; \psi_-(x) = \begin{pmatrix} Se^{i\varphi} \\ -C \end{pmatrix} \quad C = \cos(\frac{\theta}{2})\;,\; S = \sin(\frac{\theta}{2})\,.$$

$V_\kappa$ is a scalar external potential to be adjusted.
One may think of taking $\phi(x) \sim u(x)\psi_-(x)\,.$

# Remarks about Born-Oppenheimer Hamiltonians

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- Consider first the quadratic part of the energy

$$\int_{\mathbb{R}^2} \frac{1}{\kappa}|\nabla\phi|^2 + \frac{1}{\kappa}V_\kappa|\phi|^2 + \langle\phi\,,M(x)\phi\rangle \; dx_1 dx_2$$

# Remarks about Born-Oppenheimer Hamiltonians

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$$\int_{\mathbb{R}^2} \epsilon^2 |\nabla\phi|^2 + V|\phi|^2 + \langle \phi\,, M\phi\rangle \;dx_1 dx_2\,.$$

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- Set $\phi(x) \sim u(x) U_0(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with

$$U_0(x) = \begin{pmatrix} C & Se^{i\varphi} \\ Se^{-i\varphi} & C \end{pmatrix} \,.$$

$$U_0^*(\partial_{x_k} U_0) = -i \begin{pmatrix} A_k(x) & X_k(x) \\ \overline{X}_k(x) & -A_k(x) \end{pmatrix}$$

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- Born Oppenheimer Hamiltonian: There exists a unitary operator $U(x, \epsilon D_x) = U_0(x) + \epsilon U_1(x, \epsilon D_x) + \epsilon^2 U_2(x, \epsilon D_x, \epsilon)$ such that

$$U^* H_\epsilon U = \begin{pmatrix} H_{+,\epsilon} & 0 \\ 0 & H_{-,\epsilon} \end{pmatrix} + \epsilon^2 R(\epsilon)\,.$$

$$H_{\pm,\epsilon} = \epsilon^2 \left[ -(\nabla \mp i \underbrace{A(x)}_{ad.connection})^2 + \underbrace{|X(x)|^2}_{Born-Huang} \right] + V(x) \pm \Omega(x_1)\,.$$

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- If one wants to produce the effective potential of an harmonic trap $a|x|^2$, in the lowest energy band, one must take

$$V(x) = \epsilon^2 a|x|^2 + \Omega(x_1) - \epsilon^2 |X(x)|^2\,.$$

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- Without the correcting terms in $U$ there are $\mathcal{O}(\epsilon,\epsilon D_x)$ off-diagonal terms.

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- The diagonal remainders have actually the form

$$\sum_{i,j \in \{1,2\}} \epsilon^2 C_{ij}(x)(\epsilon D_{x_i})(\epsilon D_{x_j}) + \epsilon^3 R_3(x, \epsilon D_x, \epsilon)\,.$$

The approximation is valid for low frequencies (or wave-vectors).
Necessity of pseudodifferential calculus.
Consider wave vectors of order $\ll 1/\epsilon$.

# Remarks about Born-Oppenheimer Hamiltonians

- If one starts with

$$H_\epsilon(x,p) = f(p) + M(x) \quad \text{with} \quad p \leftrightarrow \epsilon D_x = \frac{\epsilon}{i}\partial_x\,.$$

One finds

$$H_{\pm,\epsilon}(x,p) = \pm\Omega(x_1) + f(p) \mp \partial_p f(p).A(x) + \frac{\epsilon^2\partial^2_{p_k p_\ell} f(p)}{2} A_k A_\ell + \frac{\epsilon^2\partial^2_{p_k p_\ell} f(p)}{2} X_k \overline{X_\ell} + \frac{\epsilon^2(\partial_{p_k} f)(\partial_{p_\ell} f)}{2\Omega(x)_1} X_k \overline{X_\ell} + \epsilon^3 \mathcal{O}(\partial_p^{(\geq 3)} f(p)) + \cdots\,.$$

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- Take $f(p) = \epsilon^{2\delta} f_1(p)$. In the region where $f_1(p) = |p|^2$

$$H_{\pm,\epsilon}(x,p) = \pm\Omega(x_1) + \epsilon^{2\delta}(p \mp \epsilon A(x))^2 + \epsilon^{2+2\delta}|X|^2 + \mathcal{O}(\epsilon^{2+4\delta}) + \mathcal{O}(\epsilon^\infty)$$

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- Rather take

$$\kappa^{-1} = \varepsilon^{2+2\delta} \quad \delta > 0 \quad (\ell = 1, k = 1, G = 0)\,.$$

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- $$\begin{aligned}\mathcal{E}_\kappa(\phi) &= \int_{\mathbb{R}^2} |\nabla\phi|^2 + V_\kappa(x)|\phi|^2 + \frac{G}{2}|\phi|^4 \\ &\quad +\kappa\langle\phi\,, M(\frac{x_1}{\ell}, kx_2)\phi\rangle_{\mathbb{C}^2} \quad dx_1 dx_2\end{aligned}$$

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- Change of scale $\phi(x,y) = \sqrt{\frac{k}{\ell}}\psi(\sqrt{\frac{k}{\ell}}x) \Rightarrow \mathcal{E}_\kappa(\phi) = \kappa\mathcal{E}_\varepsilon(\psi)$ with

$$\mathcal{E}_\varepsilon(\psi) = \int \varepsilon^{2+2\delta}\tau|\nabla\psi|^2 + V_{\varepsilon,\tau}(x)|\psi|^2 + \frac{G_{\varepsilon,\tau}}{2}|\psi|^4 + \langle\psi\,,\,M(\sqrt{\tau}x_1,\frac{x_2}{\sqrt{\tau}})\psi\rangle\; dx_1\,dx_2$$

$$\varepsilon^{2+2\delta} = \frac{k^2}{\kappa} \ll 1 \quad , \quad \delta > 0 \quad , \quad \tau = \frac{1}{\ell k} \ll 1$$

$$G_{\varepsilon,\tau} = G\tau\varepsilon^{2+2\delta} \quad , \quad V_{\varepsilon,\tau}(x) = \kappa^{-1}V_\kappa(\sqrt{\frac{\ell}{k}}x)\,.$$

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$$V_{\varepsilon,\tau}(x) = \frac{\varepsilon^{2+2\delta}}{\ell_V^2} v(\sqrt{\tau}x_1, \sqrt{\tau}x_2) + \sqrt{1+\tau x_1^2} - \varepsilon^{2+2\delta}\left[\frac{\tau^2}{(1+\tau x_1^2)^2} + \frac{1}{(1+\tau x_1^2)}\right]$$

$$v(x) = |x|^2\chi_v(|x|^2) + (1-\chi_v(|x|^2)) \quad \chi_v(t) = \begin{cases} 1 & \text{for } t \leq 1 \\ 0 & \text{for } t \geq 2 \,. \end{cases}$$

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- Born-Oppenheimer Hamiltonian ($\varepsilon \to 0$): $H_{\varepsilon,-} = \varepsilon^{2+2\delta}\tau \hat{H}_-$ with

$$H_- = -\partial_{x_1}^2 - (\partial_{x_2} - i\frac{x_1}{2\sqrt{1+\tau x_1^2}})^2 + \frac{1}{\ell_V^2\tau} v(\sqrt{\tau}x)$$

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- Harmonic approximation ($\tau \to 0$)

$$H_{\ell_V} = -\partial_{x_1}^2 - (\partial_{x_2} - i\frac{x_1}{2})^2 + \frac{|x|^2}{\ell_V^2}\,.$$

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Three nonlinear energies have to be considered

- $\mathcal{E}_\varepsilon(\psi) = \langle \psi \, , \, H_{\varepsilon,\tau}\psi \rangle + \frac{G\varepsilon^{2+2\delta}\tau}{2} \int |\psi|^4 \, ,$

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- Definition: $\min \mathcal{E}$=minimal value of $\mathcal{E}$
  $\mathrm{Argmin}\ \mathcal{E}$ set of ground states.

# Result

## Theorem

*Fix the constants* $\ell_V$, $G$ *and take* $\delta = \frac{5}{4}$. *Assume* $\varepsilon \leq c_0 \tau^{\frac{2}{3}}$ *and* $\tau \leq \tau_0$ *with* $\tau_0$ *small enough.*
*Let* $\chi = (\chi_1, \chi_2)$ *be a pair of cut-off functions such that* $\chi_1^2 + \chi_2^2 = 1$, $\chi_1 \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ *and* $\chi_1 \equiv 1$ *in* $\{|x| \leq 1\}$. *There exist* $\nu_0 = \nu_{\ell_V, G} \in (0, \frac{1}{2}]$, $C = C(\ell_V, G)$, $C_\chi = C(\ell_V, G, \chi)$ *and a unitary operator* $U$ *such that*

- $|\mathcal{E}_{\varepsilon,min} - \varepsilon^{\frac{9}{2}} \tau \mathcal{E}_{H,min}| \leq C \varepsilon^{9/2} \tau^{5/3} = \mathcal{O}(\tau^{14/3})$
- *If* $\psi \in \mathrm{Argmin}\ \mathcal{E}_\varepsilon$, $\psi = U \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$ *with*

$$\|a_+\|_{L^2} \leq C \varepsilon^{9/2} \tau^{2/3} = \mathcal{O}(\tau^{11/3}) \quad , \quad \|a_+\|_{L^\infty} \leq C \varepsilon^{9/4} \tau^{-1/6} = \mathcal{O}(\tau^{4/3})$$

$$\|\chi_2(\tau^{1/9}.) a_-\|_{L^2} \leq C_{\chi,\delta} \tau^{1/3}$$

$$d_{L^2}(\chi_1(\tau^{1/9}.) a_-, \mathrm{Argmin}\ \mathcal{E}_H) \leq C_\chi \tau^{\frac{2\nu_0}{3}} ,$$

$$d_{L^\infty}(\chi_1(\tau^{1/9}.) a_-, \mathrm{Argmin}\ \mathcal{E}_H) \leq C_\chi \tau^{\frac{\nu_0}{3}} .$$