About the method of characteristics

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Joint work with Z. Ammari about bosonic mean-field dynamics

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Outline

- The problem
- An example
- Infinite dimensional measure transportation
- Why a probabilistic trajectory picture ?

The problem

Let E be a separable real Hilbert space.
We want to prove that a Borel probability measure μ_t on E, which depends continuously on the time $t \in \mathbb{R}$ and fulfills

$$\partial_t \mu + \mathrm{div}\,(v(t,x)\mu) = 0 \quad , \quad \mu_{t=0} = \mu_0$$

equals

$$\mu_t = \Phi(t,0)_* \mu_0 \,,$$

when $\Phi(t,s)$ is a well-defined flow on E, associated with the ODE

$$\dot{x} = v(t,x)\,.$$

The problem

Finite dimensional case:

Assume that $\Phi(t,s)$ is a diffeomorphism on E for all $t, s \in \mathbb{R}$.
The transport equation is

$$\int_{\mathbb{R}} \int_E (\partial_t \varphi + \langle v(t,.)\,,\, \nabla_x \varphi \rangle_E)\; d\mu_t(x) dt = 0\,, \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}\times E)\,,$$

or

$$\int_0^T \int_E (\partial_t \varphi + \langle v(t,x)\,,\, \nabla_x \varphi \rangle_E)\; d\mu_t(x) dt + \int_E \varphi(0,x)\; d\mu_0(x) - \int_E \varphi(T,x)\; d\mu_T(x) = 0\,, \quad \forall \varphi \in \mathcal{C}_0^\infty([0,T] \times E)\,.$$

Finite dimensional case:

The method of characteristics consists in taking

$$\varphi(x,t) = a(\Phi(T,t)x) \quad a \in \mathcal{C}_0^\infty(E)\,.$$

We get

$$(\partial_t \varphi + \langle v, \nabla\varphi\rangle)(\Phi(t,0)x, t) = \frac{d}{dt}\left[a(\Phi(T,0)x)\right] = 0$$

and

$$\int_E a(\Phi(T,0)x)\; d\mu_0(x) = \int_E a(x)\; d\mu_T(x)\,, \quad \forall a \in \mathcal{C}_0^\infty(E)\,,$$

which is

$$\mu_T = \Phi(T,0)_* \mu_0\,.$$

The problem

Infinite dimensional case:

The weak formulation

$$\int_{\mathbb{R}} \int_{E} (\partial_t \varphi + \langle v(t,.)\,,\, \nabla_x \varphi \rangle_E)\; d\mu_t(x) dt = 0\,,$$

is given for a class of test functions: the cylindrical functions $\varphi \in \mathcal{C}^{\infty}_{0,cyl}(\mathbb{R} \times E)$ or some polynomial functions on E .

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Problem: A nonlinear flow does not preserve these classes. We cannot take $\varphi(x,t) = a(\Phi(T,t)x)$!!!

An example, the Hartree flow

The Hartree equation is given by

$$i\partial_t z = -\Delta z + (V * |z|^2)z \quad , \quad z_{t=0} = z_0 \,.$$

with $V(-x) = V(x)$.
Example : $V(x) = \pm\frac{1}{|x|}$ in dimension 3.

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The equation can be written $i\partial_t z = \partial_{\overline{z}}\mathcal{E}(z)$ with

$$\begin{aligned}\mathcal{E}(z) &= \int_{\mathbb{R}^d} |\nabla z(x)|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y)|z(x)|^2|z(y)|^2 \, dxdy \\ &= \langle z \,, -\Delta z\rangle + \frac{1}{2}\langle z^{\otimes 2} \,, V(x-y) z^{\otimes 2}\rangle \,.\end{aligned}$$

An example, the Hartree flow

Set $\tilde{z}_t = e^{it\Delta} z_t$ and the equation becomes

$$
\begin{aligned}
& i\partial_t \tilde{z}_t = \partial_{\overline{z}} h(\tilde{z}, t) \\
& h(z,t) = \langle z^{\otimes 2}\,,\, V_t z^{\otimes 2}\rangle \\
& V_t = e^{it(\Delta_x+\Delta_y)}(V(x-y)\times)e^{-it(\Delta_x+\Delta_y)}
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When $V = \pm\frac{1}{|x|}$, $d = 3$, Hardy's inequality leads to

$$\begin{aligned} \left|\langle z_1\,,\, V_t z_2 \otimes z_3\rangle\right|_{L^2} &\leq C \min_{\sigma\in\mathcal{S}_3}\left(|z_{\sigma(1)}|_{H^1}|z_{\sigma(2)}|_{L^2}|z_{\sigma(3)}|_{L^2}\right) \\ \left|\langle z_1\,,\, V_t z_2 \otimes z_3\rangle\right|_{H^1} &\leq C \min_{\sigma\in\mathcal{S}_3}\left(|z_{\sigma(1)}|_{H^1}|z_{\sigma(2)}|_{H^1}|z_{\sigma(3)}|_{L^2}\right) \end{aligned}$$

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\end{aligned}$$

More generally it works if V satisfies $V(-x) = V(x)$ and $V(1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$.

An example, the Hartree flow

Theorem

Assume $V(x) = V(-x)$ *and* $V(1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$ *. The equations*

$$i\partial_t z = \partial_{\overline{z}}\mathcal{E}(z) \quad et \quad i\partial_t \tilde{z} = \partial_{\overline{z}} h(\tilde{z}, t)$$

define flows $\Phi(t)$ *and* $\tilde{\Phi}(t,s)$ *on* $H^1(\mathbb{R}^d;\mathbb{C})$ *.*
The norm $|\ |_{L^2}$ *and the energy* $\mathcal{E}$ *are invariant under* Φ *. The norm* $L^2(\mathbb{R}^d)$ *is invariant under* $\tilde{\Phi}$ *and the velocity field* $v(z,t) = \frac{1}{i}\partial_{\overline{z}} h(z,t)$ *, satisfies*

$$|v(z,t)|_{H^1} \leq C|z|^2_{H^1}|z|_{L^2} \,.$$

An example, the Hartree flow

The mean field analysis for bosons interacting via a pair potential $V(x-y)$, leads to Borel probability measures on $H^1 = H^1(\mathbb{R}^d;\mathbb{C})$ which verifies

$$\int_{\mathbb{R}}\int_{H^1} (\partial_t\varphi + i(\partial_z h.\partial_{\overline{z}}\varphi - \partial_z\varphi.\partial_{\overline{z}}h)\ d\mu_t(z)dt = 0$$

for all $\varphi \in \mathcal{C}^{\infty}_{0,cyl}(\mathbb{R}\times H^1;\mathbb{R})$.

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for all $\varphi \in \mathcal{C}^\infty_{0,cyl}(\mathbb{R}\times H^1;\mathbb{R})$.

For a cylindrical function $f \in \mathcal{C}^\infty_{0,cyl}(H^1;\mathbb{R})$ on H^1, we define $\nabla_{\overline{z}}f$ by

$$\forall u \in H^1(\mathbb{R}^d)\,, \quad \langle u\,,\, \nabla_{\overline{z}}f\rangle_{H^1} = \langle u\,,\, \partial_{\overline{z}}f\rangle\,.$$

Its gradient for the real structure on $H^1 = H^1(\mathbb{R}^d;\mathbb{C})$ with the scalar product $\langle u_1, u_2\rangle_{H^1_{\mathbb{R}}} = \mathrm{Re}\,\langle u_1, u_2\rangle_{H^1}$ is given by $\nabla = 2\nabla_{\overline{z}}$

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The above equation is the weak version of

$$\partial_t\mu + \mathrm{div}\,(v_t\mu) = 0 \quad , \quad v_t = \frac{1}{i}\partial_{\bar z}h(z,t)$$

with cylindrical test functions on $\mathbb{R}\times H^1_{\mathbb{R}}$, $H^1_{\mathbb{R}} = H^1(\mathbb{R}^d;\mathbb{C})$ being the real Hilbert space with the scalar product $\langle u, v\rangle_{H^1_{\mathbb{R}}}$ and $|z|_{H^1_{\mathbb{R}}} = |z|_{H^1}$.

Remark : With good assumptions (on the initial mean field data), one verifies that the measure μ_t is continuous w.r.t the Wasserstein distance *

$$W_2(\mu_1, \mu_2) = \left(\inf \left\{ \int_{H^1_{\mathbb{R}} \times H^1_{\mathbb{R}}} |z_2 - z_1|^2_{H^1} d\mu(z_1, z_2), \quad \Pi_{j,*}\mu = \mu_j \right\} \right)^{1/2} .$$

Also, for all $t \in \mathbb{R}$

$$\int_{H^1_{\mathbb{R}}} |z|^4_{H^1} |z|^2_{L^2} \; d\mu_t(z) \leq C$$

and

$$\int_0^T \int_{H^1_{\mathbb{R}}} |v(t,z)|^2_{H^1} \; d\mu_t(z) \leq C_T \, .$$

* This strong continuity property is requires intermediate steps

E real separable Hilbert space.
$\mathrm{Prob}_2(E)$ is the space of Borel probability measures on E, μ, such that

$$\int_E |x|^2 \, d\mu(x) < +\infty$$

The Wasserstein distance W_2 on $\mathrm{Prob}_2(E)$ is given by

$$W_2^2(\mu_1, \mu_2) = \inf \left\{ \int_{E^2} |x_2 - x_1|^2 d\mu(x_1, x_2), \quad (\Pi_j)_* \mu = \mu_j \right\} .$$

Solving a transport equation in $\mathrm{Prob}_2(E)$

For $T > 0$, set $\Gamma_T = \mathcal{C}^0([-T, T]; E)$ endowed with the norm $|\gamma|_\infty = \max_{t\in[-T,T]} |\gamma(t)|$
or the distance
$d(\gamma, \gamma') = \max_{t\in[-T,T]} \left(\sum_{n\in\mathbb{N}^*} |\langle\gamma(t) - \gamma'(t), e_n\rangle|^2 2^{-n}\right)^{1/2}$ with $(e_n)_{n\in\mathbb{N}^*}$ ONB of E
For a Borel probability measure η on $E \times \Gamma_T$ define the evaluation map at time $t \in [-T, T]$ by

$$\int_E \varphi \, d\mu_t^\eta = \int_{\Gamma_T} \varphi(\gamma(t)) \, d\eta(x, \gamma) , \quad \forall \varphi \in \mathcal{C}^\infty_{0,cyl}(E) .$$

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Actually, $\mu_t^\eta = (e_t)_*\eta$ with

$$e_t : (x, \gamma) \in E \times \Gamma_T \to \gamma(t) \in E .$$

Solving a transport equation in $\mathrm{Prob}_2(E)$

The following result is the infinite dimensional version of a result by Ambrosio-Gigli-Savaré.

Proposition

If $\mu_t : [-T, T] \to \mathrm{Prob}_2(E)$ is a W_2-continuous solution to the equation

$$\partial_t \mu + \mathrm{div}\,(v(t,x))\mu) = 0$$

on $(-T, T)$, for a Borel velocity field $v(t,x) = v_t(x)$ on E such that $|v_t|_{L^2(E,\mu_t)} \in L^1([-T, T])$; then there exists a Borel probability measure η on $E \times \Gamma_T$ which satisfies

- *η is carried by the set of pairs (x, γ) such that $\gamma \in AC^2([-T, T]; E)$ solves $\dot{\gamma}(t) = v_t(\gamma(t))$ for almost all $t \in (-T, T)$ and $\gamma(0) = x$.*
- *$\mu_t = \mu_t^\eta$ for all $t \in [-T, T]$.*

Solving a transport equation in $\mathrm{Prob}_2(E)$

Corollary

If for all $x \in E$, the Cauchy problem

$$\dot{\gamma}(t) = v_t(\gamma(t))\,, \quad \gamma(0) = x$$

is well posed and defines a flow $\tilde{\Phi}(t,s)$ on E, then

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Proof:

$$\int_E \varphi\, d\mu_t = \int_{E\times\Gamma_T} \varphi(\gamma(t))\, d\eta(x,\gamma) = \int_{E\times\Gamma_T} [\varphi\circ\tilde{\Phi}(t,s)](\gamma(s))\, d\eta(x,\gamma)$$

$$= \int_E \varphi\circ\tilde{\Phi}(t,s)\; d\mu_s\,.$$

Why a probabilistic trajectory picture ?

Notations:

$(e_n)_{n\in\mathbb{N}^*}$ Hilbert basis of E .

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$$E \ni x \mapsto \pi^d(x) = (\langle e_n, x\rangle \,, n \leq d) \in \mathbb{R}^d \,,$$

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$\mu_t^d = \pi_*^d \mu_t$ $\quad \hat{\mu}_t^d = \hat{\pi}_*^d \mu_t = \mu_t^d \otimes \delta_0$ when $E = F_d \oplus F_d^\perp$, $F_d \sim \mathbb{R}^d$.
$\left\{\mu_{t,y} \,, y \in \mathbb{R}^d\right\}$ is the disintegration of μ_t w.r.t μ_t^d .

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$\left\{\mu_{t,y} \,, y \in \mathbb{R}^d\right\}$ is the disintegration of μ_t w.r.t μ_t^d .

$$v_t^d(y) = \int_{(\pi^d)^{-1}(y)} \pi^d(v_t(x))\, d\mu_{t,y}(x) \quad \text{for } a.e.y \in \mathbb{R}^d \,,$$

$$\hat{v}_t^d(y) = \int_{(\hat{\pi}^d)^{-1}(\hat{\pi}^d y)} \hat{\pi}^d(v_t(x))\, d\mu_{t,\pi^d y}(x) \quad \text{for } a.e.y \in E \,.$$

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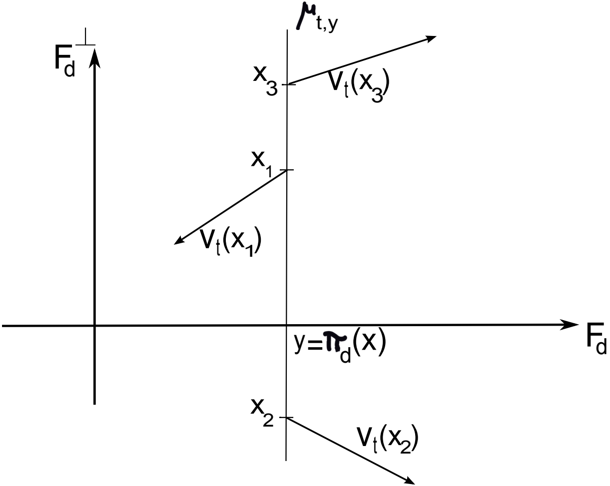

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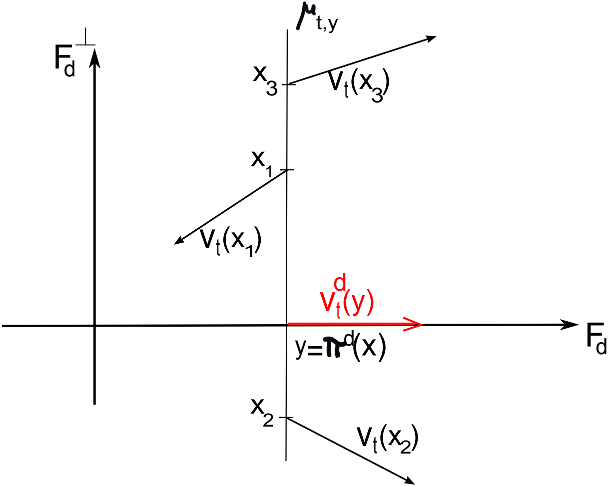

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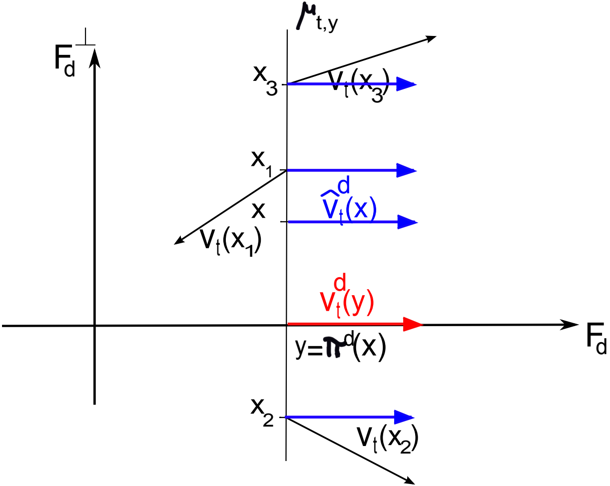

$$\partial_t \hat{\mu}_t^d + \mathrm{div}\left(\hat{v}_t^d \hat{\mu}_t^d\right) = 0 \,.$$

Why a probabilistic trajectory picture ?

Properties:

$$\left|\hat{v}_t^d\right|_{L^2(E,\hat{\mu}_t^d)} = \left|v_t^d\right|_{L^2(\mathbb{R}^d,\mu_t^d)} \leq \left|v_t\right|_{L^2(E,\mu_t)} .$$

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$$W_2(\mu_{t_1}^d,\mu_{t_2}^d) \leq \int_{t_1}^{t_2} |v_t^d|_{L^2(\mathbb{R}^d,\mu_t^d)}\, dt \leq \int_{t_1}^{t_2} |v_t|_{L^2(E,\mu_t)}\, dt \,.$$

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The sequence $(\hat{\mu}_t^d)_{d\in\mathbb{N}^*}$ converges weakly narrowly to μ_t with the estimate

$$W_2(\mu_{t_2}, \mu_{t_1}) \leq \liminf_{d\to\infty} W_2(\hat{\mu}_{t_2}^d, \hat{\mu}_{t_1}^d) .$$

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In this approximation process, the finite dimensional vector field v_t^d may be (is) singular $\to$ no uniqueness of trajectories.