About the method of characteristics

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After joint work with Z. Ammari
and improvements by Q. Liard and C. Rouffort

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Outline

- The problem
- An example
- Infinite dimensional measure transportation
- Why a probabilistic trajectory picture ?
- Improvements by Q. Liard and C. Rouffort

The problem

Let E be a separable real Hilbert space.
We want to prove that a Borel probability measure μ_t on E, which depends continuously on the time $t \in \mathbb{R}$ and fulfills

$$\partial_t \mu + \operatorname{div}\left(v(t,x)\mu\right) = 0 \quad , \quad \mu_{t=0} = \mu_0$$

equals

$$\mu_t = \Phi(t,0)_* \mu_0 \, ,$$

when $\Phi(t,s)$ is a well-defined flow on E, associated with the ODE

$$\dot{x} = v(t,x) \, .$$

Finite dimensional case:

Assume that $\Phi(t,s)$ is a diffeomorphism on E for all $t,s\in\mathbb{R}$.
The transport equation is

$$\int_{\mathbb{R}}\int_E(\partial_t\varphi+\langle v(t,.)\,,\,\nabla_x\varphi\rangle_E)\;d\mu_t(x)dt=0\,,\quad\forall\varphi\in\mathcal{C}_0^\infty(\mathbb{R}\times E)\,,$$

or

$$\int_0^T\int_E(\partial_t\varphi+\langle v(t,x)\,,\,\nabla_x\varphi\rangle_E)\;d\mu_t(x)dt+\int_E\varphi(0,x)\;d\mu_0(x)$$
$$-\int_E\varphi(T,x)\;d\mu_T(x)=0\,,\quad\forall\varphi\in\mathcal{C}_0^\infty([0,T]\times E)\,.$$

Finite dimensional case:

The method of characteristics consists in taking

$$\varphi(x,t) = a(\Phi(T,t)x) \quad a \in \mathcal{C}_0^\infty(E)\,.$$

We get

$$(\partial_t\varphi + \langle v, \nabla\varphi\rangle)(\Phi(t,0)x,t) = \frac{d}{dt}\left[a(\Phi(T,0)x)\right] = 0$$

and

$$\int_E a(\Phi(T,0)x)\ d\mu_0(x) = \int_E a(x)\ d\mu_T(x)\,, \quad \forall a \in \mathcal{C}_0^\infty(E)\,,$$

which is

$$\mu_T = \Phi(T,0)_*\mu_0\,.$$

The problem

Infinite dimensional case:

The weak formulation

$$\int_{\mathbb{R}} \int_{E} (\partial_t \varphi + \langle v(t,.) \,,\, \nabla_x \varphi \rangle_E) \; d\mu_t(x) dt = 0 \,,$$

is given for a class of test functions: the cylindrical functions $\varphi \in \mathcal{C}^{\infty}_{0,cyl}(\mathbb{R} \times E)$ or some polynomial functions on E .

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Problem: A nonlinear flow does not preserve these classes. We cannot take $\varphi(x,t) = a(\Phi(T,t)x)$!!!

An example, the Hartree flow

The Hartree equation is given by

$$i\partial_t z = -\Delta z + (V * |z|^2)z \quad , \quad z_{t=0} = z_0 \,.$$

with $V(-x) = V(x)$.
Example : $V(x) = \pm\frac{1}{|x|}$ in dimension 3.

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The equation can be written $i\partial_t z = \partial_{\bar{z}}\mathcal{E}(z)$ with

$$\begin{aligned} \mathcal{E}(z) &= \int_{\mathbb{R}^d} |\nabla z(x)|^2 \; dx + \frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y)|z(x)|^2|z(y)|^2 \; dxdy \\ &= \langle z \,, -\Delta z\rangle + \frac{1}{2}\langle z^{\otimes 2} \,, V(x-y) z^{\otimes 2}\rangle \,. \end{aligned}$$

An example, the Hartree flow

Set $\tilde{z}_t = e^{-it\Delta} z_t$ and the equation becomes

$$
\begin{aligned}
& i\partial_t \tilde{z}_t = \partial_{\overline{z}} h(\tilde{z}, t) \\
& h(\tilde{z}, t) = \langle \tilde{z}^{\otimes 2}\,,\, V_t \tilde{z}^{\otimes 2} \rangle \\
& V_t = e^{-it(\Delta_x + \Delta_y)} (V(x-y)\times) e^{it(\Delta_x + \Delta_y)}
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When $V = \pm \frac{1}{|x|}$, $d = 3$, Hardy's inequality leads to

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\begin{aligned}
\left| \langle z_1 \,,\, V_t z_2 \otimes z_3 \rangle \right|_{L^2} &\leq C \min_{\sigma \in \mathcal{S}_3} \left(|z_{\sigma(1)}|_{H^1} |z_{\sigma(2)}|_{L^2} |z_{\sigma(3)}|_{L^2} \right) \\
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\end{aligned}
$$

More generally it works if V satisfies $V(-x) = V(x)$ and $V(1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$.

An example, the Hartree flow

Theorem

Assume $V(x) = V(-x)$ *and* $V(1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$ *. The equations*

$$i\partial_t z = \partial_{\bar{z}}\mathcal{E}(z) \quad et \quad i\partial_t \tilde{z} = \partial_{\bar{z}} h(\tilde{z}, t)$$

define flows $\Phi(t)$ *and* $\tilde{\Phi}(t,s)$ *on* $H^1(\mathbb{R}^d; \mathbb{C})$ *.*
The norm $|\ |_{L^2}$ *and the energy* $\mathcal{E}$ *are invariant under* Φ *. The norm* $L^2(\mathbb{R}^d)$ *is invariant under* $\tilde{\Phi}$ *and the velocity field* $v(z,t) = \frac{1}{i}\partial_{\bar{z}} h(z,t)$ *, satisfies*

$$|v(z,t)|_{H^1} \leq C|z|_{H^1}^2 |z|_{L^2} \,.$$

The mean field analysis for bosons interacting via a pair potential $V(x-y)$, leads to Borel probability measures on $H^1 = H^1(\mathbb{R}^d;\mathbb{C})$ which verifies

$$\int_{\mathbb{R}}\int_{H^1}(\partial_t\varphi + i(\partial_z h.\partial_{\overline{z}}\varphi - \partial_z\varphi.\partial_{\overline{z}}h)\ d\mu_t(z)dt = 0$$

for all $\varphi \in \mathcal{C}^{\infty}_{0,cyl}(\mathbb{R}\times H^1;\mathbb{R})$.

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for all $\varphi \in \mathcal{C}^{\infty}_{0,cyl}(\mathbb{R}\times H^1;\mathbb{R})$.

For a cylindrical function $f \in \mathcal{C}^{\infty}_{0,cyl}(H^1;\mathbb{R})$ on H^1, we define $\nabla_{\overline{z}} f$ by

$$\forall u \in H^1(\mathbb{R}^d)\,, \quad \langle u\,,\,\nabla_{\overline{z}} f\rangle_{H^1} = \langle u\,,\,\partial_{\overline{z}} f\rangle\,.$$

Its gradient for the real structure on $H^1 = H^1(\mathbb{R}^d;\mathbb{C})$ with the scalar product $\langle u_1, u_2\rangle_{H^1_{\mathbb{R}}} = \mathrm{Re}\,\langle u_1, u_2\rangle_{H^1}$ is given by $\nabla = 2\nabla_{\overline{z}}$

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for all $\varphi \in \mathcal{C}^{\infty}_{0,cyl}(\mathbb{R}\times H^1;\mathbb{R})$.

The above equation is the weak version of

$$\partial_t\mu + \mathrm{div}\,(v_t\mu) = 0 \quad , \quad v_t = \frac{1}{i}\partial_{\bar z}h(z,t)$$

with cylindrical test functions on $\mathbb{R}\times H^1_{\mathbb{R}}$, $H^1_{\mathbb{R}} = H^1(\mathbb{R}^d;\mathbb{C})$ being the real Hilbert space with the scalar product $\langle u,v\rangle_{H^1_{\mathbb{R}}}$ and $|z|_{H^1_{\mathbb{R}}} = |z|_{H^1}$.

An example, the Hartree flow

Remark : With good assumptions (on the initial mean field data), one verifies that the measure μ_t is continuous w.r.t the Wasserstein distance [*]

$$W_2(\mu_1, \mu_2) = \left(\inf \left\{ \int_{H^1_{\mathbb{R}} \times H^1_{\mathbb{R}}} |z_2 - z_1|^2_{H^1} d\mu(z_1, z_2), \quad \Pi_{j,*}\mu = \mu_j \right\} \right)^{1/2} .$$

Also, for all $t \in \mathbb{R}$

$$\int_{H^1_{\mathbb{R}}} |z|^4_{H^1} |z|^2_{L^2} \; d\mu_t(z) \leq C$$

and

$$\int_0^T \int_{H^1_{\mathbb{R}}} |v(t,z)|^2_{H^1} \; d\mu_t(z) \leq C_T \,.$$

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[*] This strong continuity property requires intermediate steps

Solving a transport equation in $\mathrm{Prob}_2(E)$

E real separable Hilbert space.
$\mathrm{Prob}_2(E)$ is the space of Borel probability measures on E, μ, such that

$$\int_E |x|^2 \, d\mu(x) < +\infty$$

The Wasserstein distance W_2 on $\mathrm{Prob}_2(E)$ is given by

$$W_2^2(\mu_1, \mu_2) = \inf \left\{ \int_{E^2} |x_2 - x_1|^2 d\mu(x_1, x_2), \quad (\Pi_j)_* \mu = \mu_j \right\} .$$

Solving a transport equation in $\mathrm{Prob}_2(E)$

For $T > 0$, set $\Gamma_T = \mathcal{C}^0([-T, T]; E)$ endowed with the norm $|\gamma|_\infty = \max_{t\in[-T,T]} |\gamma(t)|$
or the distance
$d(\gamma, \gamma') = \max_{t\in[-T,T]} \left(\sum_{n\in\mathbb{N}^*} |\langle\gamma(t) - \gamma'(t), e_n\rangle|^2 2^{-n}\right)^{1/2}$ with $(e_n)_{n\in\mathbb{N}^*}$ ONB of E
For a Borel probability measure η on $E \times \Gamma_T$ define the evaluation map at time $t \in [-T, T]$ by

$$\int_E \varphi \; d\mu_t^\eta = \int_{\Gamma_T} \varphi(\gamma(t)) \; d\eta(x, \gamma)\,, \quad \forall \varphi \in \mathcal{C}^\infty_{0,cyl}(E)\,.$$

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Actually, $\mu_t^\eta = (e_t)_* \eta$ with

$$e_t : (x, \gamma) \in E \times \Gamma_T \to \gamma(t) \in E \,.$$

Solving a transport equation in $\mathrm{Prob}_2(E)$

The following result is the infinite dimensional version of a result by Ambrosio-Gigli-Savaré.

Proposition

If $\mu_t : [-T, T] \to \mathrm{Prob}_2(E)$ is a W_2-continuous solution to the equation

$$\partial_t \mu + \mathrm{div}\,(v(t,x))\mu) = 0$$

on $(-T, T)$, for a Borel velocity field $v(t,x) = v_t(x)$ on E such that $|v_t|_{L^2(E,\mu_t)} \in L^1([-T, T])$; then there exists a Borel probability measure η on $E \times \Gamma_T$ which satisfies

- *η is carried by the set of pairs (x, γ) such that $\gamma \in AC^2([-T, T]; E)$ solves $\dot{\gamma}(t) = v_t(\gamma(t))$ for almost all $t \in (-T, T)$ and $\gamma(0) = x$.*
- *$\mu_t = \mu_t^\eta$ for all $t \in [-T, T]$.*

Corollary

If for all $x \in E$, the Cauchy problem

$$\dot{\gamma}(t) = v_t(\gamma(t))\,, \quad \gamma(0) = x$$

is well posed and defines a flow $\tilde{\Phi}(t,s)$ on E, then

$$\mu_t = \tilde{\Phi}(t,0)_* \mu_0\,.$$

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Proof:

$$\int_E \varphi \, d\mu_t = \int_{E\times\Gamma_T} \varphi(\gamma(t))\, d\eta(x,\gamma) = \int_{E\times\Gamma_T} [\varphi\circ\tilde{\Phi}(t,s)](\gamma(s))\, d\eta(x,\gamma)$$

$$= \int_E \varphi \circ \tilde{\Phi}(t,s)\; d\mu_s\,.$$

Why a probabilistic trajectory picture ?

Notations:

$(e_n)_{n\in\mathbb{N}^*}$ Hilbert basis of E .

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$$E \ni x \mapsto \pi^d(x) = (\langle e_n, x\rangle \, , n \leq d) \in \mathbb{R}^d \, ,$$

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$\mu_t^d = \pi_*^d\mu_t \qquad \hat{\mu}_t^d = \hat{\pi}_*^d\mu_t = \mu_t^d\otimes\delta_0$ when $E = F_d\oplus F_d^\perp$, $F_d\sim\mathbb{R}^d$.
$\left\{\mu_{t,y}\,, y\in\mathbb{R}^d\right\}$ is the disintegration of μ_t w.r.t μ_t^d .

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$\left\{\mu_{t,y}\,, y\in\mathbb{R}^d\right\}$ is the disintegration of μ_t w.r.t μ_t^d .

$$\begin{aligned} v_t^d(y) &= \int_{(\pi^d)^{-1}(y)} \pi^d(v_t(x))\ d\mu_{t,y}(x) \quad \text{for } a.e.y\in\mathbb{R}^d\,,\\ \hat{v}_t^d(y) &= \int_{(\hat{\pi}^d)^{-1}(\hat{\pi}^d y)} \hat{\pi}^d(v_t(x))\ d\mu_{t,\pi^d y}(x) \quad \text{for } a.e.y\in E\,. \end{aligned}$$

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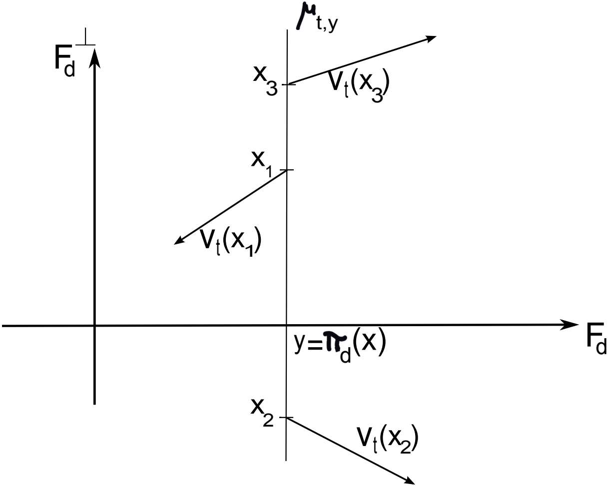

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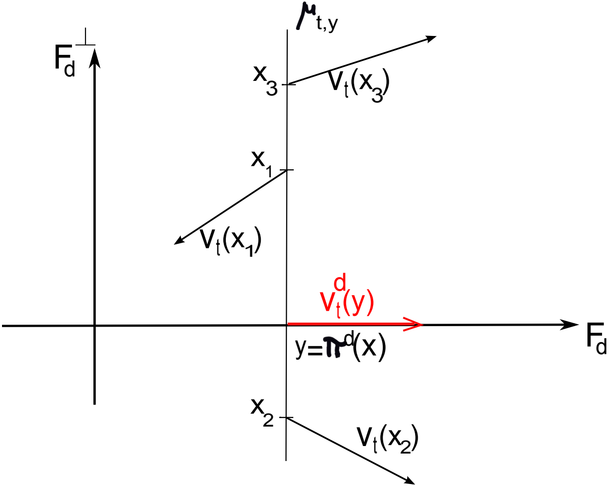

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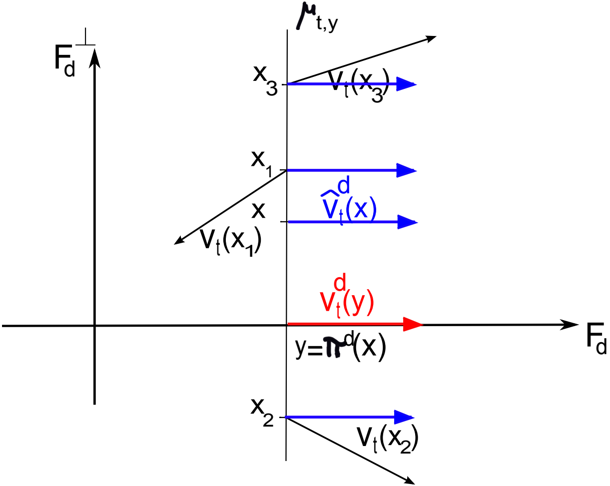

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Why a probabilistic trajectory picture ?

Properties:

$$|\hat{v}_t^d|_{L^2(E,\hat{\mu}_t^d)} = |v_t^d|_{L^2(\mathbb{R}^d,\mu_t^d)} \leq |v_t|_{L^2(E,\mu_t)} \,.$$

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$$W_2(\mu_{t_1}^d, \mu_{t_2}^d) \leq \int_{t_1}^{t_2} |v_t^d|_{L^2(\mathbb{R}^d,\mu_t^d)} dt \leq \int_{t_1}^{t_2} |v_t|_{L^2(E,\mu_t)} dt .$$

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The sequence $(\hat{\mu}_t^d)_{d\in\mathbb{N}^*}$ converges weakly narrowly to μ_t with the estimate

$$W_2(\mu_{t_2},\mu_{t_1}) \leq \liminf_{d\to\infty} W_2(\hat{\mu}_{t_2}^d,\hat{\mu}_{t_1}^d)\,.$$

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Similar properties hold for the families η^d , $\hat{\eta}^d$ after projecting the trajectories.

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Similar properties hold for the families η^d , $\hat{\eta}^d$ after projecting the trajectories.

In this approximation process, the finite dimensional vector field v_t^d may be (is) singular $\to$ no uniqueness of trajectories.

Proposition

If $\mu_t : [-T, T] \to \mathrm{Prob}_2(E)$ *is a* W_2*-continuous solution to the equation*

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on $(-T, T)$*, for a Borel velocity field* $v(t,x) = v_t(x)$ *on* E *such that* $|v_t|_{L^2(E,\mu_t)} \in L^1([-T, T])$ *; then there exists a Borel probability measure* η *on* $E \times \Gamma_T$ *which satisfies*

- η *is carried by the set of pairs* (x, γ) *such that* $\gamma \in AC^2([-T, T]; E)$ *solves* $\dot{\gamma}(t) = v_t(\gamma(t))$ *for almost all* $t \in (-T, T)$ *and* $\gamma(0) = x$ *.*
- $\mu_t = \mu_t^\eta$ *for all* $t \in [-T, T]$ *.*

$$|\langle z_1\,,\ V_t z_2 \otimes z_3\rangle|_{L^2} \quad \leq \quad C \min_{\sigma \in \mathcal{S}_3} \left(|z_{\sigma(1)}|_{H^1} |z_{\sigma(2)}|_{L^2} |z_{\sigma(3)}|_{L^2}\right)$$

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Replacing the exponent 2 by 1:

Working with probability measures μ such that $\int_E |x|_E^2 \, d\mu$ provides compactness, for a family of probability measures μ (tightness) , and relies on estimates of $\int_E |v_t(x)|_E^2 \, d\mu_t(x)$ once this is known at time $t = 0$.

Assuming simply $\int_0^T \int_E |v_t(x)|_E \, d\mu_t(x) \, dt$ a priori provides $\int_E |x|_E \, d\mu_T$ which is not sufficient.

Dunford-Pettis (or de la Vallée Poussin) argument about equiintegrability allows to prove the existence of a strictly convex function Ψ (with superlinear asymptotics as $|x| \to \infty$) a $\int_E \Psi(|v_t(x)|) \, d\mu_t$. This provides the required tightness especially in the limit $d \to \infty$, when we consider the convergence $\eta^d \to \eta$.

Improvements by Q. Liard and C. Rouffort

The functional estimates on V_t: A weak formulation of the nonlinear evolution problem makes sense in the framework of the rigged Hilbert triple $H^1 \subset L^2 \subset H^{-1}$ where the velocity field $v_t(z)$ only belongs to $H^{-1}(\mathbb{R}^d)$.
The corresponding estimate is

$$\begin{aligned} |\langle z_1 \, , \, V_t z_2 \otimes z_3 \rangle|_{H^{-1}} &\leq C \min_{\sigma \in \mathcal{S}_3} \left(|z_{\sigma(1)}|_{H^1} |z_{\sigma(2)}|_{H^1} |z_{\sigma(3)}|_{H^1} \right) \, , \\ \text{or} \quad |\langle u_1 \, , \, V_t u_2 \rangle| &\leq C |u_1|_{H^1(\mathbb{R}^{2d})} |u_2|_{H^1(\mathbb{R}^{2d})} \end{aligned}$$

Handling weak solutions to the boundary value problem

$$\partial_t x = v_t(x) \in H^{-1} \ a.e. \ t \quad , \quad x(t=0) = x_0 \in H^1$$

requires to verify some weak uniqueness property.

Result:

General statement with rigged Hilbert spaces $E_1 \subset E \subset E_{-1}$

Proposition

Consider $v : \mathbb{R} \times E_1 \to E_{-1}$ *a Borel vector field such that* v *is bounded on bounded sets. Let* I *be a bounded interval containing* 0 *. If* $t \to \mu_t \in Prob(E_1)$ *is a probability measure weakly narrowly continuous in* $Prob(E_{-1})$ *solving the Liouville equation with* $\int_I \int_{E_1} |v_t(x)|_{E_{-1}} \, d\mu_t(x)dt$ *, then therere exist a probability measure* $\eta \ldots$
If additionnally $\partial_t x = v_t(x) \, , x(t=0) = x_0$ *has a weak uniqueness property then* $\mu_t = \phi(t)_* \mu_0$ *.*

Application: In $\mathbb{R}^3$ it allows to consider potentials with singularities like $\frac{\pm 1}{|x|^\beta}$,$0 < \beta < 2$.