Phase-space approach to the bosonic mean field dynamics: a review

Francis Nier,
IRMAR, Univ. Rennes 1
Joint works with Z. Ammari
cont'd with S. Breteaux, M. Falconi, Q. Liard, B. Pawilowski, M. Zerzeri

Zürich, june 14th 2014

Outline

- Semiclassical and mean field asymptotics
- Phase-space geometry/projections
- Wigner (semiclassical) measures
- Wick quantization, (PI)-condition, BBGKY hierarchy
- Propagation results

Finite dimensional semiclassical asymptotics

Semiclassical annihilation-creation operators:

$$(PDE) \quad a_j = h\partial_{\nu_j} + \nu_j \quad , \quad a_j^* = -h\partial_{\nu_j} + \nu_j \quad , \quad \nu \in \mathbb{R}^d$$

For $w \in \mathcal{Z} = \mathbb{C}^d$ set $a(w) = \sum_j \overline{w}_j a_j \quad , \quad a^*(w) = \sum_j w_j a_j^*$,

$$\left[a(w), a^*(w')\right] = 2h\langle w\, ,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\, ,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon = 2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) \quad = \sum_{\substack{|\beta| = p \\ |\alpha| = q}} b_{\alpha,\beta}\overline{z}^{\alpha} z^{\beta} = \langle z^{\otimes q}\, ,\, \tilde{b} z^{\otimes p}\rangle \quad , \quad \tilde{b} = \frac{1}{q!p!}\partial_{\overline{z}}^{q}\partial_{z}^{p} b$$

$$\text{the operator } b^{Wick} \quad = \sum_{\alpha,\beta} b_{\alpha,\beta} a^{*\alpha} a^{\beta} \,, \qquad \text{(Wick)}$$

Weyl operator $W(f)$:

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\langle f\, ,\, z\rangle^{Wick} \quad , \quad W(f) = e^{i\Phi(f)}\,.$$

Semiclassical annihilation-creation operators:

$$Unit.eq. \quad a_j = \sqrt{h}(\partial_{\nu_j} + \nu_j) \quad , \quad a_j^* = \sqrt{h}(-\partial_{\nu_j} + \nu_j) \quad , \quad \nu \in \mathbb{R}^d$$

For $w \in \mathcal{Z} = \mathbb{C}^d$ set $a(w) = \sum_j \overline{w}_j a_j \quad , \quad a^*(w) = \sum_j w_j a_j^*$,

$$\left[a(w), a^*(w')\right] = 2h\langle w\,,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\,,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon = 2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) \quad = \sum_{\substack{|\beta| = p \\ |\alpha| = q}} b_{\alpha,\beta}\overline{z}^{\alpha}z^{\beta} = \langle z^{\otimes q}\,,\, \tilde{b}z^{\otimes p}\rangle \quad , \quad \tilde{b} = \frac{1}{q!p!}\partial_{\overline{z}}^{q}\partial_{z}^{p}b$$

the operator $b^{Wick} \quad = \sum_{\alpha,\beta} b_{\alpha,\beta}{a^*}^{\alpha}a^{\beta}$, (Wick)

Weyl operator $W(f)$:

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\langle f\,,\, z\rangle^{Wick} \quad , \quad W(f) = e^{i\Phi(f)}\,.$$

Finite dimensional semiclassical asymptotics

Semiclassical annihilation-creation operators:

$$\textit{Unit.eq.}\quad a_j = \sqrt{h}(\partial_{\nu_j} + \nu_j) \quad , \quad a_j^* = \sqrt{h}(-\partial_{\nu_j} + \nu_j) \quad , \quad \nu \in \mathbb{R}^d$$

For $w \in \mathcal{Z} = \mathbb{C}^d$ set $a(w) = \sum_j \overline{w}_j a_j \quad , \quad a^*(w) = \sum_j w_j a_j^*$,

$$\left[a(w), a^*(w')\right] = 2h\langle w\,,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\,,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon = 2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) = \sum_{\substack{|\beta| = p \\ |\alpha| = q}} b_{\alpha,\beta}\overline{z}^{\alpha} z^{\beta} = \langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p}\rangle \quad , \quad \tilde{b} = \frac{1}{q!p!}\partial_{\overline{z}}^{q}\partial_{z}^{p} b$$

the operator $b^{Wick} = \sum_{\alpha,\beta} b_{\alpha,\beta} {a^*}^{\alpha} a^{\beta}$, (Wick)

Weyl operator $W(f)$:

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\langle f\,,\, z\rangle^{Wick} \quad , \quad W(f) = e^{i\Phi(f)} .$$

Finite dimensional semiclassical asymptotics

Semiclassical annihilation-creation operators:

$$(QFT) \quad a_j = \sqrt{\varepsilon}\frac{\partial_{\nu_j} + \nu_j}{\sqrt{2}} \quad , \quad a_j^* = \sqrt{\varepsilon}\frac{-\partial_{\nu_j} + \nu_j}{\sqrt{2}} \quad , \quad \nu \in \mathbb{R}^d$$

For $w \in \mathcal{Z} = \mathbb{C}^d$ set $a(w) = \sum_j \overline{w}_j a_j \quad , \quad a^*(w) = \sum_j w_j a_j^*$,

$$\left[a(w), a^*(w')\right] = 2h\langle w\,,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\,,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon = 2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) = \sum_{\substack{|\beta| = p \\ |\alpha| = q}} b_{\alpha,\beta}\overline{z}^{\alpha} z^{\beta} = \langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p}\rangle \quad , \quad \tilde{b} = \frac{1}{q!p!}\partial_{\overline{z}}^{q}\partial_{z}^{p} b$$

the operator $b^{Wick} = \sum_{\alpha,\beta} b_{\alpha,\beta} {a^*}^{\alpha} a^{\beta}$, (Wick)

Weyl operator $W(f)$:

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\langle f\,,\, z\rangle^{Wick} \quad , \quad W(f) = e^{i\Phi(f)}\,.$$

Semiclassical annihilation-creation operators:

$$(QFT) \quad a_j = \sqrt{\varepsilon}\frac{\partial_{\nu_j} + \nu_j}{\sqrt{2}} \quad , \quad a_j^* = \sqrt{\varepsilon}\frac{-\partial_{\nu_j} + \nu_j}{\sqrt{2}} \quad , \quad \nu \in \mathbb{R}^d$$

For $w \in \mathcal{Z} = \mathbb{C}^d$ set $a(w) = \sum_j \overline{w}_j a_j \quad , \quad a^*(w) = \sum_j w_j a_j^*$,

$$\left[a(w), a^*(w')\right] = 2h\langle w\,,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\,,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon = 2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) \quad = \sum_{\substack{|\beta| = p \\ |\alpha| = q}} b_{\alpha,\beta}\overline{z}^\alpha z^\beta = \langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p}\rangle \quad , \quad \tilde{b} = \frac{1}{q!p!}\partial_{\overline{z}}^q \partial_z^p b$$

the operator $b^{Wick} \quad = \sum_{\alpha,\beta} b_{\alpha,\beta} {a^*}^\alpha a^\beta , \qquad \text{(Wick)}$

Weyl operator $W(f)$:

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\langle f\,,\, z\rangle^{Wick} \quad , \quad W(f) = e^{i\Phi(f)} .$$

Semiclassical annihilation-creation operators:

$$(QFT)\quad a_j=\sqrt{\varepsilon}\frac{\partial_{\nu_j}+\nu_j}{\sqrt{2}}\quad,\quad a_j^*=\sqrt{\varepsilon}\frac{-\partial_{\nu_j}+\nu_j}{\sqrt{2}}\quad,\quad \nu\in\mathbb{R}^d$$

For $w\in\mathcal{Z}=\mathbb{C}^d$ set $a(w)=\sum_j\overline{w}_j a_j \quad,\quad a^*(w)=\sum_j w_j a_j^*$,

$$[a(w),a^*(w')]=2h\langle w\,,\,w'\rangle_{\mathcal{Z}}=\varepsilon\langle w\,,\,w'\rangle_{\mathcal{Z}}\quad,\quad \varepsilon=2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z)\quad=\quad\sum_{\substack{|\beta|=p\\|\alpha|=q}} b_{\alpha,\beta}\overline{z}^{\alpha}z^{\beta}=\langle z^{\otimes q}\,,\,\tilde{b}z^{\otimes p}\rangle\quad,\quad \tilde{b}=\frac{1}{q!p!}\partial_{\overline{z}}^{q}\partial_{z}^{p}b$$

the operator $b^{Wick} \;=\; \sum_{\alpha,\beta} b_{\alpha,\beta}a^{*\alpha}a^{\beta}\,,\qquad$ (Wick)

$$b^{A-Wick}\;=\;\sum_{\alpha,\beta} b_{\alpha,\beta}a^{\beta}a^{*\alpha}\,,\qquad \text{(anti-Wick)}$$

Weyl operator $W(f)$:

$$\Phi(f)=\frac{a(f)+a^*(f)}{\sqrt{2}}=\sqrt{2}\,\mathrm{Re}\,\langle f\,,\,z\rangle^{Wick}\quad,\quad W(f)=e^{i\Phi(f)}\,.$$

Semiclassical annihilation-creation operators:

$$(QFT)\quad a_j=\sqrt{\varepsilon}\frac{\partial_{\nu_j}+\nu_j}{\sqrt{2}}\quad,\quad a_j^*=\sqrt{\varepsilon}\frac{-\partial_{\nu_j}+\nu_j}{\sqrt{2}}\quad,\quad \nu\in\mathbb{R}^d$$

For $w\in\mathcal{Z}=\mathbb{C}^d$ set $a(w)=\sum_j\overline{w}_ja_j\quad,\quad a^*(w)=\sum_j w_ja_j^*$,

$$[a(w),a^*(w')]=2h\langle w\,,\,w'\rangle_{\mathcal{Z}}=\varepsilon\langle w\,,\,w'\rangle_{\mathcal{Z}}\quad,\quad \varepsilon=2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z)=\sum_{\substack{|\beta|=p\\|\alpha|=q}}b_{\alpha,\beta}\overline{z}^{\alpha}z^{\beta}=\langle z^{\otimes q}\,,\,\tilde{b}z^{\otimes p}\rangle\quad,\quad \tilde{b}=\frac{1}{q!p!}\partial_{\overline{z}}^{q}\partial_{z}^{p}b$$

the operator $b^{Wick}=\sum_{\alpha,\beta}b_{\alpha,\beta}{a^*}^{\alpha}a^{\beta}$, (Wick)

Example: $\mathbf{N}=(|z|^2)^{Wick}=\sum_j a_j^*a_j=\varepsilon\mathbf{N}_{\varepsilon=1}$, $\mathbf{N}\varphi_\alpha=\varepsilon|\alpha|\varphi_\alpha$ when φ_α is the α-th Hermite function $\alpha\in\mathbb{N}^d$, $|\alpha|=\sum_j\alpha_j$. $\mathbf{N}=\mathcal{O}(1)\leftrightarrow|\alpha|=\mathcal{O}(\frac{1}{\varepsilon})$.

Weyl operator $W(f)$:

$$\Phi(f)=\frac{a(f)+a^*(f)}{\sqrt{2}}=\sqrt{2}\,\mathrm{Re}\,\langle f\,,\,z\rangle^{Wick}\quad,\quad W(f)=e^{i\Phi(f)}\,.$$

Finite dimensional semiclassical asymptotics

Semiclassical annihilation-creation operators:

$$(QFT) \quad a_j = \sqrt{\varepsilon}\frac{\partial_{\nu_j}+\nu_j}{\sqrt{2}} \quad , \quad a_j^* = \sqrt{\varepsilon}\frac{-\partial_{\nu_j}+\nu_j}{\sqrt{2}} \quad , \quad \nu\in\mathbb{R}^d$$

For $w\in\mathcal{Z}=\mathbb{C}^d$ set $a(w)=\sum_j \overline{w}_j a_j \quad , \quad a^*(w)=\sum_j w_j a_j^*$,

$$\left[a(w), a^*(w')\right] = 2h\langle w\,,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\,,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon=2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) \quad = \sum_{\substack{|\beta|=p\\|\alpha|=q}} b_{\alpha,\beta}\overline{z}^{\alpha}z^{\beta} = \langle z^{\otimes q}\,,\, \tilde{b}z^{\otimes p}\rangle \quad , \quad \tilde{b}=\frac{1}{q!p!}\partial_{\overline{z}}^q\partial_z^p b$$

$$\text{the operator } b^{Wick} \quad = \sum_{\alpha,\beta} b_{\alpha,\beta}a^{*\,\alpha}a^{\beta}\,, \qquad \text{(Wick)}$$

Weyl operator $W(f)$:

$$\Phi(f)=\frac{a(f)+a^*(f)}{\sqrt{2}}=\sqrt{2}\,\mathrm{Re}\,\langle f\,,\,z\rangle^{Wick} \quad , \quad W(f)=e^{i\Phi(f)}\,.$$

Finite dimensional semiclassical asymptotics

Semiclassical annihilation-creation operators:

$$(QFT) \quad a_j = \sqrt{\varepsilon}\frac{\partial_{\nu_j} + \nu_j}{\sqrt{2}} \quad , \quad a_j^* = \sqrt{\varepsilon}\frac{-\partial_{\nu_j} + \nu_j}{\sqrt{2}} \quad , \quad \nu \in \mathbb{R}^d$$

For $w \in \mathcal{Z} = \mathbb{C}^d$ set $a(w) = \sum_j \overline{w}_j a_j \quad , \quad a^*(w) = \sum_j w_j a_j^*$,

$$\left[a(w), a^*(w')\right] = 2h\langle w\,,\, w'\rangle_{\mathcal{Z}} = \varepsilon\langle w\,,\, w'\rangle_{\mathcal{Z}} \quad , \quad \varepsilon = 2h$$

The Wick (resp. anti-Wick) quantization associates with the polynomial

$$b(z) = \sum_{\substack{|\beta| = p \\ |\alpha| = q}} b_{\alpha,\beta}\overline{z}^\alpha z^\beta = \langle z^{\otimes q}\,,\, \tilde{b}z^{\otimes p}\rangle \quad , \quad \tilde{b} = \frac{1}{q!p!}\partial_{\overline{z}}^q\partial_z^p b$$

the operator $b^{Wick} = \sum_{\alpha,\beta} b_{\alpha,\beta}{a^*}^\alpha a^\beta$, (Wick)

Weyl operator $W(f)$:

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} = \sqrt{2}\,\mathrm{Re}\,\langle f\,,\, z\rangle^{Wick} \quad , \quad W(f) = e^{i\Phi(f)}\,.$$

If $\hat{b}(\zeta) = \int_{\mathcal{Z}} b(z)e^{-2i\pi\,\mathrm{Re}\,\langle\zeta,z\rangle}\,dL_{\mathcal{Z}}(z)$ then $b(z) = \int_{\mathcal{Z}} \hat{b}(\zeta)e^{2i\pi\,\mathrm{Re}\,\langle\zeta,z\rangle}\,dL_{\mathcal{Z}}(\zeta)$

$$\text{and} \quad b^{Weyl} = b^{Weyl}(\sqrt{h}\nu, \sqrt{h}D_\nu) = \int_{\mathcal{Z}} \hat{b}(\zeta)W(\sqrt{2}\pi\zeta)\,d\zeta\,.$$

Finite dimensional semiclassical asymptotics

Weyl-quantization: $b^{Weyl} = b^{Weyl}(\sqrt{h}\nu, \sqrt{h}D_\nu)$ unit. eq. $b^{Weyl}(\nu, hD_\nu)$.
Weyl-Hörmander classes of $\mathcal{C}^\infty$-symbols: $S(1, |dz|^2) = \mathcal{C}_b^\infty(\mathbb{R}^{2d})$ or $\cup_{s\in\mathbb{R}} S(\langle z\rangle^s, \frac{|dz|^2}{\langle z\rangle^2})$ (harmonic oscillator: $-h^2\Delta + x^2 = (|z|^2)^{Weyl}(\tau, hD_\tau)$)
Algebra of $\mathcal{C}^\infty$-symbol classes, asymptotic expansion in h (or $\varepsilon = 2h$).

Anti-Wick quantization: non-negative quantization, well defined for (polynomially weighted) L^∞-symbols. No obvious algebra of $\mathcal{C}^\infty$-functions

Wick quantization: well defined for some classes of real analytic symbols (polynomials OK!). Algebra of polynomial symbols.

In good cases $b^{Weyl} \equiv b^{A-Wick} \equiv b^{Wick}$ mod $\mathcal{O}(h) = \mathcal{O}(\varepsilon)$.
For $\varrho_\varepsilon \geq 0$ with $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$, e.g. $\varrho_\varepsilon = |\psi_\varepsilon\rangle\langle\psi_\varepsilon|$ with $\|\psi_\varepsilon\|_{L^2(\mathbb{R}^d)} = 1$,
the asymptotic value of $\mathrm{Tr}\,[b^Q\varrho_\varepsilon]$ indep of $Q=$ Weyl, Wick, A-Wick.

Egorov theorem: When U_h is a Fourier integral operator associated with the canonical transform χ on $(\mathbb{C}^d; \mathrm{Im}\,\langle\ ,\ \rangle_{\mathbb{C}^d})$ with amplitude 1 , then

$$U_h^{-1} a^Q(\nu\,, hD_\nu) U_h \equiv (a\circ\chi)^Q(\nu, hD_\nu) \quad \text{mod } \mathcal{O}(h) = \quad \text{mod } \mathcal{O}(\varepsilon)\,.$$

By duality this provides the semiclassical propagation of ϱ_ε (semi-classical or Wigner measures).

Finite dimensional semiclassical asymptotics

Weyl-quantization: $b^{Weyl} = b^{Weyl}(\sqrt{h}\nu, \sqrt{h}D_\nu)$ unit. eq. $b^{Weyl}(\nu, hD_\nu)$.
Weyl-Hörmander classes of $\mathcal{C}^\infty$-symbols: $S(1, |dz|^2) = \mathcal{C}_b^\infty(\mathbb{R}^{2d})$ or
$\cup_{s\in\mathbb{R}} S(\langle z\rangle^s, \frac{|dz|^2}{\langle z\rangle^2})$ (harmonic oscillator: $-h^2\Delta + x^2 = (|z|^2)^{Weyl}(\tau, hD_\tau)$)
Algebra of $\mathcal{C}^\infty$-symbol classes, asymptotic expansion in h (or $\varepsilon = 2h$).

Anti-Wick quantization: non-negative quantization, well defined for (polynomially weighted) L^∞-symbols. No obvious algebra of $\mathcal{C}^\infty$-functions

Wick quantization: well defined for some classes of real analytic symbols (polynomials OK!). Algebra of polynomial symbols.

In good cases $b^{Weyl} \equiv b^{A-Wick} \equiv b^{Wick}$ mod $\mathcal{O}(h) = \mathcal{O}(\varepsilon)$.
For $\varrho_\varepsilon \geq 0$ with $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$, e.g. $\varrho_\varepsilon = |\psi_\varepsilon\rangle\langle\psi_\varepsilon|$ with $\|\psi_\varepsilon\|_{L^2(\mathbb{R}^d)} = 1$,
the asymptotic value of $\mathrm{Tr}\,[b^Q\varrho_\varepsilon]$ indep of $Q =$ Weyl, Wick, A-Wick.

Egorov theorem: When U_h is a Fourier integral operator associated with the canonical transform χ on $(\mathbb{C}^d; \mathrm{Im}\,\langle\ ,\ \rangle_{\mathbb{C}^d})$ with amplitude 1, then

$$U_h^{-1}a^Q(\nu\,, hD_\nu)U_h \equiv (a\circ\chi)^Q(\nu, hD_\nu) \mod \mathcal{O}(h) = \mod \mathcal{O}(\varepsilon)\,.$$

By duality this provides the semiclassical propagation of ϱ_ε (semi-classical or Wigner measures).

Finite dimensional semiclassical asymptotics

Weyl-quantization: $b^{Weyl} = b^{Weyl}(\sqrt{h}\nu, \sqrt{h}D_\nu)$ unit. eq. $b^{Weyl}(\nu, hD_\nu)$.
Weyl-Hörmander classes of $\mathcal{C}^\infty$-symbols: $S(1, |dz|^2) = \mathcal{C}_b^\infty(\mathbb{R}^{2d})$ or $\cup_{s\in\mathbb{R}} S(\langle z\rangle^s, \frac{|dz|^2}{\langle z\rangle^2})$ (harmonic oscillator: $-h^2\Delta + x^2 = (|z|^2)^{Weyl}(\tau, hD_\tau)$)
Algebra of $\mathcal{C}^\infty$-symbol classes, asymptotic expansion in h (or $\varepsilon = 2h$).

Anti-Wick quantization: non-negative quantization, well defined for (polynomially weighted) L^∞-symbols. No obvious algebra of $\mathcal{C}^\infty$-functions

Wick quantization: well defined for some classes of real analytic symbols (polynomials OK!). Algebra of polynomial symbols.

In good cases $b^{Weyl} \equiv b^{A-Wick} \equiv b^{Wick} \mod \mathcal{O}(h) = \mathcal{O}(\varepsilon)$.
For $\varrho_\varepsilon \geq 0$ with $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$, e.g. $\varrho_\varepsilon = |\psi_\varepsilon\rangle\langle\psi_\varepsilon|$ with $\|\psi_\varepsilon\|_{L^2(\mathbb{R}^d)} = 1$,
the asymptotic value of $\mathrm{Tr}\,[b^Q\varrho_\varepsilon]$ indep of $Q =$ Weyl, Wick, A-Wick.

Egorov theorem: When U_h is a Fourier integral operator associated with the canonical transform χ on $(\mathbb{C}^d; \mathrm{Im}\,\langle\ ,\ \rangle_{\mathbb{C}^d})$ with amplitude 1, then

$$U_h^{-1} a^Q(\nu, hD_\nu) U_h \equiv (a\circ\chi)^Q(\nu, hD_\nu) \mod \mathcal{O}(h) = \mod \mathcal{O}(\varepsilon).$$

By duality this provides the semiclassical propagation of ϱ_ε (semi-classical or Wigner measures).

Weyl-quantization: $b^{Weyl}=b^{Weyl}(\sqrt{h}\nu,\sqrt{h}D_\nu)$ unit. eq. $b^{Weyl}(\nu,hD_\nu)$.
Weyl-Hörmander classes of $\mathcal{C}^\infty$-symbols: $S(1,|dz|^2)=\mathcal{C}^\infty_b(\mathbb{R}^{2d})$ or
$\cup_{s\in\mathbb{R}}S(\langle z\rangle^s,\frac{|dz|^2}{\langle z\rangle^2})$ (harmonic oscillator: $-h^2\Delta+x^2=(|z|^2)^{Weyl}(\tau,hD_\tau)$)
Algebra of $\mathcal{C}^\infty$-symbol classes, asymptotic expansion in h (or $\varepsilon=2h$).

Anti-Wick quantization: non-negative quantization, well defined for (polynomially weighted) L^∞-symbols. No obvious algebra of $\mathcal{C}^\infty$-functions

Wick quantization: well defined for some classes of real analytic symbols (polynomials OK!). Algebra of polynomial symbols.

In good cases $b^{Weyl}\equiv b^{A-Wick}\equiv b^{Wick}\mod \mathcal{O}(h)=\mathcal{O}(\varepsilon)$.
For $\varrho_\varepsilon\geq 0$ with $\mathrm{Tr}\,[\varrho_\varepsilon]=1$, e.g. $\varrho_\varepsilon=|\psi_\varepsilon\rangle\langle\psi_\varepsilon|$ with $\|\psi_\varepsilon\|_{L^2(\mathbb{R}^d)}=1$,
the asymptotic value of $\mathrm{Tr}\,[b^Q\varrho_\varepsilon]$ indep of $Q=$ Weyl, Wick, A-Wick.

Egorov theorem: When U_h is a Fourier integral operator associated with the canonical transform χ on $(\mathbb{C}^d;\mathrm{Im}\,\langle\ ,\ \rangle_{\mathbb{C}^d})$ with amplitude 1, then

$$U_h^{-1}a^Q(\nu\,,hD_\nu)U_h\equiv(a\circ\chi)^Q(\nu,hD_\nu)\mod\mathcal{O}(h)=\quad\mod\mathcal{O}(\varepsilon)\,.$$

By duality this provides the semiclassical propagation of ϱ_ε (semi-classical or Wigner measures).

Finite dimensional semiclassical asymptotics

Weyl-quantization: $b^{Weyl} = b^{Weyl}(\sqrt{h}\nu, \sqrt{h}D_\nu)$ unit. eq. $b^{Weyl}(\nu, hD_\nu)$.
Weyl-Hörmander classes of $\mathcal{C}^\infty$-symbols: $S(1, |dz|^2) = \mathcal{C}_b^\infty(\mathbb{R}^{2d})$ or
$\cup_{s\in\mathbb{R}} S(\langle z\rangle^s, \frac{|dz|^2}{\langle z\rangle^2})$ (harmonic oscillator: $-h^2\Delta + x^2 = (|z|^2)^{Weyl}(\tau, hD_\tau)$)
Algebra of $\mathcal{C}^\infty$-symbol classes, asymptotic expansion in h (or $\varepsilon = 2h$).

Anti-Wick quantization: non-negative quantization, well defined for (polynomially weighted) L^∞-symbols. No obvious algebra of $\mathcal{C}^\infty$-functions

Wick quantization: well defined for some classes of real analytic symbols (polynomials OK!). Algebra of polynomial symbols.

In good cases $b^{Weyl} \equiv b^{A-Wick} \equiv b^{Wick}$ mod $\mathcal{O}(h) = \mathcal{O}(\varepsilon)$.
For $\varrho_\varepsilon \geq 0$ with $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$, e.g. $\varrho_\varepsilon = |\psi_\varepsilon\rangle\langle\psi_\varepsilon|$ with $\|\psi_\varepsilon\|_{L^2(\mathbb{R}^d)} = 1$,
the asymptotic value of $\mathrm{Tr}\,[b^Q\varrho_\varepsilon]$ indep of $Q =$ Weyl, Wick, A-Wick.

Egorov theorem: When U_h is a Fourier integral operator associated with the canonical transform χ on $(\mathbb{C}^d; \mathrm{Im}\,\langle\ ,\ \rangle_{\mathbb{C}^d})$ with amplitude 1, then

$$U_h^{-1}a^Q(\nu\,, hD_\nu)U_h \equiv (a\circ\chi)^Q(\nu, hD_\nu) \quad \text{mod } \mathcal{O}(h) = \quad \text{mod } \mathcal{O}(\varepsilon)\,.$$

By duality this provides the semiclassical propagation of ϱ_ε (semi-classical or Wigner measures).

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z,\overline{z}) = \langle z\,,\, -\Delta z\rangle + \frac{1}{2}\iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2 \, dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}}\mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}\,, V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

$$\mathcal{S}_n(f_1 \otimes \cdots\cdots f_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \,.$$

Energy: $\mathcal{E}(z, \overline{z}) = \langle z\,, -\Delta z\rangle + \frac{1}{2} \iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2 |z(y)|^2 \; dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}} \mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2} \langle z^{\otimes 2}\,, V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$
$\Psi(t) = e^{-i\frac{t}{\varepsilon} H_\varepsilon} \Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$
"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z,\overline{z}) = \int_{\mathbb{R}^D} |\nabla_x z(x)|^2 \; dx + \frac{1}{2} \int\int_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2 \; dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}} \mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}, V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z, \overline{z}) = \langle z \, , \, -\Delta z \rangle + \frac{1}{2} \iint_{\mathbb{R}^{2D}} V(x-y) |z(x)|^2 |z(y)|^2 \; dxdy$

$$V(-x) = V(x)$$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}} \mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z \rangle^{Wick} + \frac{1}{2} \langle z^{\otimes 2} \, , V(x-y) z^{\otimes 2} \rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon} H_\varepsilon} \Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$
"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z, \bar{z}) = \langle z\,, -\Delta z\rangle + \frac{1}{2} \iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2 \, dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\bar{z}} \mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}, V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z,\overline{z}) = \langle z\,,\,-\Delta z\rangle + \frac{1}{2}\iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2\,dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}}\mathcal{E}$

Hartree (NLS) equation: $i\partial_t z = -\Delta z + (V * |z|^2)z$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon}a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}\,, V(x-y)z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1,\ldots,x_n,t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z, \overline{z}) = \langle z \, , \, -\Delta z\rangle + \frac{1}{2} \iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2 \, dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}} \mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2} \, , V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon} H_\varepsilon} \Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty}\mathcal{S}_n\mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty}\bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n;\mathbb{C})$$

Energy: $\mathcal{E}(z,\overline{z}) = \langle z\,,\,-\Delta z\rangle + \frac{1}{2}\iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2\,dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}}\mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon}a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}\,, V(x-y)z^{\otimes 2}\rangle^{Wick}$$

$$H_\varepsilon = \underbrace{\int_{\mathbb{R}^D}\nabla a^*(x)\nabla a(x)\,dx}_{d\Gamma(-\Delta)} + \frac{1}{2}\int_{\mathbb{R}^{2D}} V(x-y)a^*(x)a^*(y)a(x)a(y)\,dxdy\,.$$

$$d\Gamma(A) = \varepsilon d\Gamma_{\varepsilon=1}(A) \quad , \quad d\Gamma(\mathrm{Id}) = \mathbf{N} = \varepsilon\mathbf{N}_{\varepsilon=1}\,.$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n;\mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1,\ldots,x_n,t)$ solves

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$

"Second quantization" (name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z, \overline{z}) = \langle z \, , -\Delta z \rangle + \frac{1}{2} \iint_{\mathbb{R}^{2D}} V(x-y) |z(x)|^2 |z(y)|^2 \; dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}} \mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon} a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z \rangle^{Wick} + \frac{1}{2} \langle z^{\otimes 2} \, , V(x-y) z^{\otimes 2} \rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$
$\Psi(t) = e^{-i \frac{t}{\varepsilon} H_\varepsilon} \Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

$$i\varepsilon \partial_t \Psi = -\varepsilon \sum_{j=1}^{n} \Delta_{x_j} \Psi + \frac{\varepsilon^2}{2} \sum_{1 \leq i,j \leq n} V(x_i - x_j) \Psi$$

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$
"Second quantization" (name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z,\overline{z}) = \langle z\,, -\Delta z\rangle + \frac{1}{2}\iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2\ dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}}\mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon}a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}\,, V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$
$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

$$i\partial_t \Psi = -\sum_{j=1}^{n} \Delta_{x_j}\Psi + \frac{\varepsilon}{2}\sum_{1\leq i,j\leq n} V(x_i - x_j)\Psi$$

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$
"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Bosonic mean field asymptotics

Bosonic Fock space: Consider now the one particle (separable) complex Hilbert space $\mathcal{Z} = L^2(\mathbb{R}^D, dx; \mathbb{C})$.

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{Z}^{\otimes n} = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \bigvee^n \mathcal{Z} = L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C})$$

Energy: $\mathcal{E}(z, \overline{z}) = \langle z\,,\, -\Delta z\rangle + \frac{1}{2}\iint_{\mathbb{R}^{2D}} V(x-y)|z(x)|^2|z(y)|^2\, dxdy$

Nonlinear Hamiltonian dynamics: $i\partial_t z = \partial_{\overline{z}}\mathcal{E}$

Wick quantized Hamiltonian : Take $a = \sqrt{\varepsilon}a_{\varepsilon=1}$ with $\varepsilon > 0$ and set

$$H_\varepsilon = \mathcal{E}(z)^{Wick} = \langle z, -\Delta z\rangle^{Wick} + \frac{1}{2}\langle z^{\otimes 2}, V(x-y) z^{\otimes 2}\rangle^{Wick}$$

n-body evolution: For $\Psi_0 \in L^2_{sym}((\mathbb{R}^D)^n; \mathbb{C}) = \bigvee^n \mathcal{Z}$

$\Psi(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_0 = \Psi(x_1, \ldots, x_n, t)$ solves

$$i\partial_t \Psi = -\sum_{j=1}^{n} \Delta_{x_j}\Psi + \frac{1}{2n}\sum_{1\leq i,j\leq n} V(x_i - x_j)\Psi \quad \text{if } \varepsilon = \frac{1}{n}$$

Formally "mean field limit"="semiclassical limit" with $\varepsilon = \frac{1}{n}$
"Second quantization"(name given by Dirac in the 50's) understood as a quantization of a possibly infinite dimensional phase-space since Wigner (32) Bogoliubov (47) Berezin (60's)

Frameworks

Points of view on the bosonic Fock space:

1) Fock representation. Number operator, creation and annihilation operators, combinatorics ;
2) Phase-space without specifying position-momentum (Segal,Berezin). Bargmann representation: complex variables z and $\overline{z}$;
3) Schrödinger representation (position variable), Functional integral (Glimm-Jaffe), (gaussian) random fields on a Hilbert space (Skorohod) or a loc. conv. vector space (Schwartz, Minlos).

Relationship with the bosonic mean field:

Phase-space geometry

Infinite dimensional ΨDO calculus
Séminaire Krée Paris (74-78)
Krée-Raczka (78) B. Lascar (77)
Hilbert-Schmidt condition on $\tilde{b}$

Large dimensional ΨDO calculus
Helffer-Sjöstrand (92), Nourrigat-Amour-Cancelier-Kerdelhué-Lévy Bruhl (00's)
Thermodynamic limit, inductive exploration of the phase-space

Projections

Stochastic processes, marginal of probability measures
Functional Integral. Glimm-Jaffe (70-80's)
self-adjointness for physical models
Euclidean case

Reduced density matrices:
Spohn(80)Adami-Bardos-Golse-Gottlieb-Mauser, Erdös-Yau-Schlein, Elgart-Schlein, Klainerman-Machedon,

Frameworks

Relationship with the bosonic mean field:

Phase-space geometry

Infinite dimensional ΨDO calculus
Séminaire Krée Paris (74-78)
Krée-Raczka (78) B. Lascar (77)
Hilbert-Schmidt condition on $\tilde{b}$

Large dimensional ΨDO calculus
Helffer-Sjöstrand (92), Nourrigat-Amour-Cancelier-Kerdelhué-Lévy Bruhl (00's)
Thermodynamic limit, inductive exploration of the phase-space

Hepp method- Coherent states:
Hepp(74) Ginibre-Velo(79)
Rodnianski-Schlein (09)
Coherent states
mean field propagation

Projections

Stochastic processes, marginal of probability measures
Functional Integral. Glimm-Jaffe (70-80's)
self-adjointness for physical models
Euclidean case

Reduced density matrices:
Spohn(80)Adami-Bardos-Golse-Gottlieb-Mauser, Erdös-Yau-Schlein, Elgart-Schlein, Klainerman-Machedon, Fröhlich-Graffi-Knowles-Schwarz, Chen-Pavlovic, Pickl, Anapolitanos... (00's)
Hermite states $\Psi_0 = \varphi^{\otimes n}$,
BBGKY hierarchy or other methods.

Frameworks

Relationship with the bosonic mean field:

Phase-space geometry

Infinite dimensional ΨDO calculus
Séminaire Krée Paris (74-78)
Krée-Raczka (78) B. Lascar (77)
Hilbert-Schmidt condition on $\tilde{b}$

Large dimensional ΨDO calculus
Helffer-Sjöstrand (92), Nourrigat-Amour-Cancelier-Kerdelhué-Lévy Bruhl (00's)
Thermodynamic limit, inductive exploration of the phase-space

Hepp method- Coherent states:
Hepp(74) Ginibre-Velo(79)
Rodnianski-Schlein (09)
Coherent states
mean field propagation

Projections

Stochastic processes, marginal of probability measures
Functional Integral. Glimm-Jaffe (70-80's)
self-adjointness for physical models
Euclidean case

Reduced density matrices:
Spohn(80)Adami-Bardos-Golse-Gottlieb-Mauser, Erdös-Yau-Schlein, Elgart-Schlein, Klainerman-Machedon, Fröhlich-Graffi-Knowles-Schwarz, Chen-Pavlovic, Pickl, Anapolitanos... (00's)
Hermite states $\Psi_0 = \varphi^{\otimes n}$,
BBGKY hierarchy or other methods.

Frameworks

Relationship with the bosonic mean field:

Phase-space geometry

Infinite dimensional ΨDO calculus
Séminaire Krée Paris (74-78)
Krée-Raczka (78) B. Lascar (77)
Hilbert-Schmidt condition on $\tilde{b}$

Large dimensional ΨDO calculus
Helffer-Sjöstrand (92), Nourrigat-Amour-Cancelier-Kerdelhué-Lévy Bruhl (00's)
Thermodynamic limit, inductive exploration of the phase-space

Hepp method- Coherent states:
Hepp(74) Ginibre-Velo(79)
Rodnianski-Schlein (09)
Coherent states
mean field propagation

Projections

Stochastic processes, marginal of probability measures
Functional Integral. Glimm-Jaffe (70-80's)
self-adjointness for physical models
Euclidean case

Reduced density matrices:
Spohn(80)Adami-Bardos-Golse-Gottlieb-Mauser, Erdös-Yau-Schlein, Elgart-Schlein, Klainerman-Machedon, Fröhlich-Graffi-Knowles-Schwarz, Chen-Pavlovic, Pickl, Anapolitanos... (00's)
Hermite states $\Psi_0 = \varphi^{\otimes n}$,
BBGKY hierarchy or other methods.

Frameworks

Relationship with the bosonic mean field:

Phase-space geometry

Infinite dimensional ΨDO calculus
Séminaire Krée Paris (74-78)
Krée-Raczka (78) B. Lascar (77)
Hilbert-Schmidt condition on $\tilde{b}$

Large dimensional ΨDO calculus
Helffer-Sjöstrand (92), Nourrigat-Amour-Cancelier-Kerdelhué-Lévy Bruhl (00's)
Thermodynamic limit, inductive exploration of the phase-space

Hepp method- Coherent states:
Hepp(74) Ginibre-Velo(79)
Rodnianski-Schlein (09)
Coherent states
mean field propagation

Projections

Stochastic processes, marginal of probability measures
Functional Integral. Glimm-Jaffe (70-80's)
self-adjointness for physical models
Euclidean case

Reduced density matrices:
Spohn(80)Adami-Bardos-Golse-Gottlieb-Mauser, Erdös-Yau-Schlein, Elgart-Schlein, Klainerman-Machedon, Fröhlich-Graffi-Knowles-Schwarz, Chen-Pavlovic, Pickl, Anapolitanos... (00's)
Hermite states $\Psi_0 = \varphi^{\otimes n}$,
BBGKY hierarchy or other methods.

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures↔reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Difficulty 1: Various asymptotics have to be considered:

- the behaviour as $|z| \to \infty$ handled with weights $\langle z \rangle^s$, $s \in \mathbb{R}$ or $s \in \mathbb{N}$ (polynomial functions);
- the mean field limit, i.e. $\varepsilon \to 0$ or $n \to \infty$;
- the behaviour w.r.t dimension (see e.g. Hilbert-Schmidt conditions).

Difficulty 2: Weyl (or anti-Wick) quantization defined only for cylindrical observables (Fourier transform, Lebesgue measure), can be extended with gaussian integration (Hilbert-Schmidt condition).
What are the reasonable classes of Wick ($(\overline{z}, z)$-homogeneous) polynomials ?

Difficulty 3: All the natural classes of symbols are not preserved by physically relevant nonlinear symplectic transforms. Cylindrical (obvious), polynomial (obvious). Hilbert-Schmidt condition: answered by L. Gross in 1960.

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures↔reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Difficulty 1: Various asymptotics have to be considered:

- the behaviour as $|z| \to \infty$ handled with weights $\langle z\rangle^s$, $s \in \mathbb{R}$ or $s \in \mathbb{N}$ (polynomial functions);
- the mean field limit, i.e. $\varepsilon \to 0$ or $n \to \infty$;
- the behaviour w.r.t dimension (see e.g. Hilbert-Schmidt conditions).

Difficulty 2: Weyl (or anti-Wick) quantization defined only for cylindrical observables (Fourier transform, Lebesgue measure), can be extended with gaussian integration (Hilbert-Schmidt condition).
What are the reasonable classes of Wick ($(\overline{z}, z)$-homogeneous) polynomials ?

Difficulty 3: All the natural classes of symbols are not preserved by physically relevant nonlinear symplectic transforms. Cylindrical (obvious), polynomial (obvious). Hilbert-Schmidt condition: answered by L. Gross in 1960.

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures↔reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Difficulty 1: Various asymptotics have to be considered:

- the behaviour as $|z| \to \infty$ handled with weights $\langle z \rangle^s$, $s \in \mathbb{R}$ or $s \in \mathbb{N}$ (polynomial functions);
- the mean field limit, i.e. $\varepsilon \to 0$ or $n \to \infty$;
- the behaviour w.r.t dimension (see e.g. Hilbert-Schmidt conditions).

Difficulty 2: Weyl (or anti-Wick) quantization defined only for cylindrical observables (Fourier transform, Lebesgue measure), can be extended with gaussian integration (Hilbert-Schmidt condition).
What are the reasonable classes of Wick ($(\overline{z}, z)$-homogeneous) polynomials ?

Difficulty 3: All the natural classes of symbols are not preserved by physically relevant nonlinear symplectic transforms. Cylindrical (obvious), polynomial (obvious). Hilbert-Schmidt condition: answered by L. Gross in 1960.

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures↔reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Difficulty 1: Various asymptotics have to be considered:

- the behaviour as $|z| \to \infty$ handled with weights $\langle z \rangle^s$, $s \in \mathbb{R}$ or $s \in \mathbb{N}$ (polynomial functions);
- the mean field limit, i.e. $\varepsilon \to 0$ or $n \to \infty$;
- the behaviour w.r.t dimension (see e.g. Hilbert-Schmidt conditions).

Difficulty 2: Weyl (or anti-Wick) quantization defined only for cylindrical observables (Fourier transform, Lebesgue measure), can be extended with gaussian integration (Hilbert-Schmidt condition).
What are the reasonable classes of Wick ($(\overline{z}, z)$-homogeneous) polynomials ?

Difficulty 3: All the natural classes of symbols are not preserved by physically relevant nonlinear symplectic transforms. Cylindrical (obvious), polynomial (obvious). Hilbert-Schmidt condition: answered by L. Gross in 1960.

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures$\leftrightarrow$reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Difficulty 1: Various asymptotics have to be considered:

- the behaviour as $|z| \to \infty$ handled with weights $\langle z \rangle^s$, $s \in \mathbb{R}$ or $s \in \mathbb{N}$ (polynomial functions);
- the mean field limit, i.e. $\varepsilon \to 0$ or $n \to \infty$;
- the behaviour w.r.t dimension (see e.g. Hilbert-Schmidt conditions).

Difficulty 2: Weyl (or anti-Wick) quantization defined only for cylindrical observables (Fourier transform, Lebesgue measure), can be extended with gaussian integration (Hilbert-Schmidt condition).
What are the reasonable classes of Wick $((\bar{z}, z)$-homogeneous) polynomials ?

Difficulty 3: All the natural classes of symbols are not preserved by physically relevant nonlinear symplectic transforms. Cylindrical (obvious), polynomial (obvious). Hilbert-Schmidt condition: answered by L. Gross in 1960.

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures↔reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Question 3: If it is not possible to propagate observables, is it still possible to get propagation results for states ?

Semiclassical or Wigner measures studied in the 90's: Schnirel'man (74), Colin de Verdière (85), Helffer-Martinez-Robert (87), Tartar (90), P. Gérard (91), Lions-Paul (93) Gérard-Markowich-Mauser-Poupaud (97), N. (96)

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures↔reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Question 3: If it is not possible to propagate observables, is it still possible to get propagation results for states ?

Semiclassical or Wigner measures studied in the 90's: Schnirel'man (74), Colin de Verdière (85), Helffer-Martinez-Robert (87), Tartar (90), P. Gérard (91), Lions-Paul (93) Gérard-Markowich-Mauser-Poupaud (97), N. (96)

Questions and known problems

Question 1: Is it possible to make a synthesis between the phase-space geometry (Hamiltonian flow) and a projective point of view (marginals or moments of probability measures↔reduced density matrices) ?

Question 2: Is it possible to identify general classes of n-body states for which the mean field ($n \to \infty$) propagation holds (convergence at time t coherent with assumptions at time $t = 0$) ?

Question 3: If it is not possible to propagate observables, is it still possible to get propagation results for states ?

Semiclassical or Wigner measures studied in the 90's: Schnirel'man (74), Colin de Verdière (85), Helffer-Martinez-Robert (87), Tartar (90), P. Gérard (91), Lions-Paul (93) Gérard-Markowich-Mauser-Poupaud (97), N. (96)

The a priori estimates $\mu \geq 0$, $\int_{\mathcal{Z}} d\mu = 1$ may be used to compensate the limitations of a restricted ΨDO calculus.

Link with the probabilistic (projective) point of view.

Definition of infinite dimensional Wigner measures

Remember: $\mathcal{Z}$ is a separable complex Hilbert space (1 part. space)

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \mathbf{N} z^{\otimes n} = \varepsilon n z^{\otimes n} \,,$$

$$a(f) z^{\otimes n} = \sqrt{\varepsilon n} \langle f\,,\, z\rangle z^{\otimes n-1} \quad , \quad a^*(f) z^{\otimes n} = \sqrt{\varepsilon(n+1)} \mathcal{S}_{n+1}[f \otimes z^{\otimes n}] \,,$$

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} \quad , \quad W(f) = e^{i\Phi(f)} \,.$$

Consider a normal state in $\mathcal{H}$, $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$, $\varrho_\varepsilon \geq 0$, $\mathrm{Tr}\, [\varrho_\varepsilon] = 1$.

Definition

For $\mathcal{E} \in (0, +\infty)$, $0 \in \overline{\mathcal{E}}$, and a family $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$ of normal states in $\mathcal{H}$, $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$ is the set of Borel probability measures μ on $\mathcal{Z}$ for which there exists $\mathcal{E}' \subset \mathcal{E}$ such that

$$0 \in \overline{\mathcal{E}'} \,,$$

$$\forall f \in \mathcal{Z} \,, \quad \lim_{\varepsilon \to 0\,, \varepsilon \in \mathcal{E}'} \mathrm{Tr}\, \left[\varrho_\varepsilon W(\sqrt{2}\pi f)\right] = \int_{\mathcal{Z}} e^{2i\pi\, \mathrm{Re}\, \langle f\,,\, z\rangle}\, d\mu(z)$$

When $\mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$, μ is called a Wigner measure of $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$.

Definition of infinite dimensional Wigner measures

Remember: $\mathcal{Z}$ is a separable complex Hilbert space (1 part. space)

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \bigvee^{n} \mathcal{Z} \quad , \quad \mathbf{N} z^{\otimes n} = \varepsilon n z^{\otimes n} ,$$

$$a(f) z^{\otimes n} = \sqrt{\varepsilon n} \langle f \, , \, z \rangle z^{\otimes n-1} \quad , \quad a^*(f) z^{\otimes n} = \sqrt{\varepsilon(n+1)} \mathcal{S}_{n+1}[f \otimes z^{\otimes n}] ,$$

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} \quad , \quad W(f) = e^{i\Phi(f)} .$$

Consider a normal state in $\mathcal{H}$, $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$, $\varrho_\varepsilon \geq 0$, $\mathrm{Tr}\, [\varrho_\varepsilon] = 1$.
Example: $\varrho_\varepsilon = |\Psi_\varepsilon\rangle\langle\Psi_\varepsilon|$, $\Psi_\varepsilon \in \mathcal{H}$,
Mean field coherent state $\Psi_\varepsilon = E(f) = W(\frac{\sqrt{2}}{i\varepsilon} f)|\Omega\rangle$
Mean field Hermite (atomic coherent) state: $\Psi_\varepsilon = \varphi^{\otimes n}$ with $\varepsilon = \frac{1}{n}$.

Definition

For $\mathcal{E} \in (0, +\infty)$, $0 \in \overline{\mathcal{E}}$, and a family $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$ of normal states in $\mathcal{H}$, $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$ is the set of Borel probability measures μ on $\mathcal{Z}$ for which there exists $\mathcal{E}' \subset \mathcal{E}$ such that

$$0 \in \overline{\mathcal{E}'} ,$$

$$\forall f \in \mathcal{Z}, \quad \lim_{\varepsilon \to 0, \varepsilon \in \mathcal{E}'} \mathrm{Tr}\, \left[\varrho_\varepsilon W(\sqrt{2}\pi f) \right] = \int_{\mathcal{Z}} e^{2i\pi \,\mathrm{Re}\, \langle f \, , \, z \rangle} \, d\mu(z)$$

When $\mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$, μ is called a Wigner measure of $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$.

Definition of infinite dimensional Wigner measures

Remember: $\mathcal{Z}$ is a separable complex Hilbert space (1 part. space)

$$\mathcal{H} = \Gamma_b(\mathcal{Z}) = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} \quad , \quad \mathbf{N} z^{\otimes n} = \varepsilon n z^{\otimes n} \,,$$

$$a(f) z^{\otimes n} = \sqrt{\varepsilon n} \langle f \,,\, z\rangle z^{\otimes n-1} \quad , \quad a^*(f) z^{\otimes n} = \sqrt{\varepsilon(n+1)} \mathcal{S}_{n+1}[f \otimes z^{\otimes n}] \,,$$

$$\Phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}} \quad , \quad W(f) = e^{i\Phi(f)} \,.$$

Consider a normal state in $\mathcal{H}$, $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$, $\varrho_\varepsilon \geq 0$, $\mathrm{Tr}\,[\varrho_\varepsilon] = 1$.

Definition

For $\mathcal{E} \in (0, +\infty)$, $0 \in \overline{\mathcal{E}}$, and a family $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$ of normal states in $\mathcal{H}$, $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$ is the set of Borel probability measures μ on $\mathcal{Z}$ for which there exists $\mathcal{E}' \subset \mathcal{E}$ such that

$$0 \in \overline{\mathcal{E}'} \,,$$

$$\forall f \in \mathcal{Z} \,, \quad \lim_{\varepsilon \to 0 \,, \varepsilon \in \mathcal{E}'} \mathrm{Tr}\,\left[\varrho_\varepsilon W(\sqrt{2}\pi f)\right] = \int_{\mathcal{Z}} e^{2i\pi \, \mathrm{Re}\, \langle f \,,\, z\rangle} \, d\mu(z)$$

When $\mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in \mathcal{E})$, μ is called a Wigner measure of $(\varrho_\varepsilon)_{\varepsilon \in \mathcal{E}}$.

Wigner measures:Existence

Th. (Ammari-N. AHP 08)

If there exists $\delta > 0$ *and* $C_\delta > 0$ *s.t.*

$$\forall \varepsilon \in \mathcal{E}\,, \quad \mathrm{Tr}\left[\varrho_\varepsilon \langle \mathbf{N} \rangle^\delta\right] \leq C_\delta \tag{3.1}$$

then $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E}) \neq \emptyset$ *and all* $\mu \in \mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \mathcal{E})$ *satisfies*

$$\int_{\mathcal{Z}} (1 + |z|^2)^\delta \; d\mu(z) \leq C_\delta\,.$$

Definition

$b \in \mathcal{S}_{cyl}(\mathcal{Z})$ if there exist a finite rank orth. proj. p and $a \in \mathcal{S}(p\mathcal{Z})$ s.t. $b = a \circ p$.

Corollary

Under the condition (3.1) *with* $\mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \mathcal{E}) = \{\mu\}$,

$$\forall b \in \mathcal{S}_{cyl}(\mathcal{Z})\,, \quad \lim_{\varepsilon \to 0\,, \varepsilon \in \mathcal{E}} \mathrm{Tr}\left[\varrho_\varepsilon\, b^{Weyl}\right] = \int_{\mathcal{Z}} b(z)\; d\mu(z)\,.$$

Wigner measures:Existence

Th. (Ammari-N. AHP 08)

If there exists $\delta > 0$ *and* $C_\delta > 0$ *s.t.*

$$\forall \varepsilon \in \mathcal{E}\,, \quad \mathrm{Tr}\left[\varrho_\varepsilon \langle \mathbf{N}\rangle^\delta\right] \leq C_\delta \tag{3.1}$$

then $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E}) \neq \emptyset$ *and all* $\mu \in \mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E})$ *satisfies*

$$\int_{\mathcal{Z}} (1+|z|^2)^\delta \, d\mu(z) \leq C_\delta\,.$$

Main ideas of the proof:

1. Separation of variables:

$$\begin{array}{ccccccc} \mathcal{Z} & = & \mathcal{Z}_1 & \overset{\perp}{\oplus} & \mathcal{Z}_2 & & \\ \mathcal{H} & = & \mathcal{H}_1 & \otimes & \mathcal{H}_2\,, & \mathcal{H}_* = \Gamma_b(\mathcal{Z}_*) & \\ W(f_1 \oplus f_2) & = & W(f_1) & \otimes & W(f_2) & = W(f_1)\otimes \mathrm{Id}_{\mathcal{H}_2} & \text{if } f_2 = 0\,. \end{array}$$

2. $\mathcal{Z}$ is separable $- >$ Borel σ-set and diagonal extraction.
3. Condition (3.1) is a tightness condition (see Prokhorov criterion)

Definition

$b \in \mathcal{S}_{cyl}(\mathcal{Z})$ if there exist a finite rank orth. proj. p and $a \in \mathcal{S}(p\mathcal{Z})$ s.t. $b = a \circ p$.

Corollary

Under the condition (3.1) *with* $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E}) = \{\mu\}$,

Wigner measures:Existence

Th. (Ammari-N. AHP 08)

If there exists $\delta > 0$ *and* $C_\delta > 0$ *s.t.*

$$\forall \varepsilon \in \mathcal{E}\,, \quad \mathrm{Tr}\left[\varrho_\varepsilon \langle \mathbf{N}\rangle^\delta\right] \leq C_\delta \tag{3.1}$$

then $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E}) \neq \emptyset$ *and all* $\mu \in \mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E})$ *satisfies*

$$\int_{\mathcal{Z}} (1 + |z|^2)^\delta \; d\mu(z) \leq C_\delta\,.$$

Remark: After a subsequence extraction we can assume $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E}) = \{\mu\}$.

Definition

$b \in \mathcal{S}_{cyl}(\mathcal{Z})$ if there exist a finite rank orth. proj. p and $a \in \mathcal{S}(p\mathcal{Z})$ s.t. $b = a \circ p$.

Corollary

Under the condition (3.1) *with* $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E}) = \{\mu\}$,

$$\forall b \in \mathcal{S}_{cyl}(\mathcal{Z})\,, \quad \lim_{\varepsilon \to 0\,, \varepsilon \in \mathcal{E}} \mathrm{Tr}\left[\varrho_\varepsilon b^{Weyl}\right] = \int_{\mathcal{Z}} b(z)\; d\mu(z)\,.$$

Wigner measures:Existence

Th. (Ammari-N. AHP 08)

If there exists $\delta > 0$ *and* $C_\delta > 0$ *s.t.*

$$\forall \varepsilon \in \mathcal{E}\,, \quad \mathrm{Tr}\left[\varrho_\varepsilon \langle \mathbf{N}\rangle^\delta\right] \leq C_\delta \tag{3.1}$$

then $\mathcal{M}(\varrho_\varepsilon\,, \varepsilon \in \mathcal{E}) \neq \emptyset$ *and all* $\mu \in \mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \mathcal{E})$ *satisfies*

$$\int_{\mathcal{Z}} (1 + |z|^2)^\delta \; d\mu(z) \leq C_\delta\,.$$

Definition

$b \in \mathcal{S}_{cyl}(\mathcal{Z})$ if there exist a finite rank orth. proj. p and $a \in \mathcal{S}(p\mathcal{Z})$ s.t. $b = a \circ p$.

Corollary

Under the condition (3.1) *with* $\mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \mathcal{E}) = \{\mu\}$,

$$\forall b \in \mathcal{S}_{cyl}(\mathcal{Z})\,, \quad \lim_{\varepsilon \to 0\,, \varepsilon \in \mathcal{E}} \mathrm{Tr}\left[\varrho_\varepsilon\, b^{Weyl}\right] = \int_{\mathcal{Z}} b(z)\; d\mu(z)\,.$$

Wigner measures: Examples

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\ \exists C_k>0\,,\ ,\forall\varepsilon\in\mathcal{E}\,,\ \mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then for any cylindrical polynomial and with $Q=$*Weyl,Wick or anti-Wick*

$$\lim_{\varepsilon\to 0\,,\,\varepsilon\in\mathcal{E}}\mathrm{Tr}\,\left[\varrho_\varepsilon b^Q\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)\,.$$

Examples

Coherent states: $f\in\mathcal{Z}\,,|f|_{\mathcal{Z}}=1\,,\ E(f)=W(\frac{\sqrt{2}}{i\varepsilon}f)|\Omega\rangle=e^{\frac{a^*(f)-a(f)}{\varepsilon}}|\Omega\rangle\,,$
$\varrho_\varepsilon^C(f)=|E(f)\rangle\langle E(f)|\,,\ \mathrm{Tr}\,\left[\varrho_\varepsilon^C(f)b^{Wick}\right]=b(f)\,,\quad\mathcal{M}(\varrho_\varepsilon^C(f),\varepsilon\in\mathcal{E})=\{\delta_f\}\,.$

Hermite (atomic coherent) states: $f\in\mathcal{Z}\,,\ |f|_{\mathcal{Z}}=1\,,$
$\varrho_\varepsilon^H(f)=|f^{\otimes n}\rangle\langle f^{\otimes n}|\,,\varepsilon=\frac{1}{n}\,,\ \mathcal{E}=\left\{\frac{1}{n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}(\varrho_\varepsilon^H(f)\,,\varepsilon\in\mathcal{E})=\left\{\delta_f^{\mathbb{S}^1}=\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}f}\,d\theta\right\}\,.$$

mixed Hermite (twin Fock) states: $f_1,f_2\in\mathcal{Z}\,,\ \langle f_i\,,\,f_j\rangle=\delta_{ij}\,,$
$\varrho_\varepsilon^H(f_1,f_2)=|f_1^{\otimes n}\rangle\langle f_1^{\otimes n}|\otimes|f_2^{\otimes n}\rangle\langle f_2^{\otimes n}|\,,\varepsilon=\frac{1}{2n}\,,\ \mathcal{E}=\left\{\frac{1}{2n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}\left\{\varrho_\varepsilon^H(f_1,f_2)\,,\,\varepsilon\in\mathcal{E}\right\}=\left\{\delta_{2^{-1/2}f_1}^{\mathbb{S}^1}\otimes\delta_{2^{-1/2}f_2}^{\mathbb{S}^1}\right\}\,.$$

Wigner measures: Examples

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,,\forall\varepsilon\in\mathcal{E}\,,\ \mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then for any cylindrical polynomial and with $Q=$*Weyl,Wick or anti-Wick*

$$\lim_{\varepsilon\to0\,,\,\varepsilon\in\mathcal{E}}\mathrm{Tr}\,\left[\varrho_\varepsilon b^Q\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)\,.$$

Remark: Cylindrical polynomial and Schrödinger representation (gaussian processes): related to Malliavin calculus

Examples

Coherent states: $f\in\mathcal{Z}\,,|f|_{\mathcal{Z}}=1\,,\,E(f)=W(\frac{\sqrt{2}}{i\varepsilon}f)|\Omega\rangle=e^{\frac{a^*(f)-a(f)}{\varepsilon}}|\Omega\rangle\,,$
$\varrho_\varepsilon^C(f)=|E(f)\rangle\langle E(f)|\,,\ \mathrm{Tr}\,\left[\varrho_\varepsilon^C(f)b^{Wick}\right]=b(f)\,,\quad\mathcal{M}(\varrho_\varepsilon^C(f),\varepsilon\in\mathcal{E})=\{\delta_f\}\,.$

Hermite (atomic coherent) states: $f\in\mathcal{Z}\,,\,|f|_{\mathcal{Z}}=1\,,$
$\varrho_\varepsilon^H(f)=|f^{\otimes n}\rangle\langle f^{\otimes n}|\,,\varepsilon=\frac{1}{n}\,,\ \mathcal{E}=\left\{\frac{1}{n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}(\varrho_\varepsilon^H(f)\,,\varepsilon\in\mathcal{E})=\left\{\delta_f^{\mathbb{S}^1}=\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}f}\,d\theta\right\}\,.$$

mixed Hermite (twin Fock) states: $f_1,f_2\in\mathcal{Z}\,,\,\langle f_i\,,\,f_j\rangle=\delta_{ij}\,,$
$\varrho_\varepsilon^H(f_1,f_2)=|f_1^{\otimes n}\rangle\langle f_1^{\otimes n}|\otimes|f_2^{\otimes n}\rangle\langle f_2^{\otimes n}|\,,\varepsilon=\frac{1}{2n}\,,\ \mathcal{E}=\left\{\frac{1}{2n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}\left\{\varrho_\varepsilon^H(f_1,f_2)\,,\,\varepsilon\in\mathcal{E}\right\}=\left\{\delta_{2^{-1/2}f_1}^{\mathbb{S}^1}\otimes\delta_{2^{-1/2}f_2}^{\mathbb{S}^1}\right\}\,.$$

Wigner measures: Examples

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,,\forall\varepsilon\in\mathcal{E}\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then for any cylindrical polynomial and with $Q=$*Weyl,Wick or anti-Wick*

$$\lim_{\varepsilon\to 0\,,\,\varepsilon\in\mathcal{E}}\mathrm{Tr}\,\left[\varrho_\varepsilon b^Q\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)\,.$$

Examples

Coherent states: $f\in\mathcal{Z}\,,|f|_{\mathcal{Z}}=1\,,\;E(f)=W(\frac{\sqrt{2}}{i\varepsilon}f)|\Omega\rangle=e^{\frac{a^*(f)-a(f)}{\varepsilon}}|\Omega\rangle\,,$
$\varrho_\varepsilon^C(f)=|E(f)\rangle\langle E(f)|\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon^C(f)b^{Wick}\right]=b(f)\,,\quad\mathcal{M}(\varrho_\varepsilon^C(f),\varepsilon\in\mathcal{E})=\{\delta_f\}\,.$

Hermite (atomic coherent) states: $f\in\mathcal{Z}\,,\;|f|_{\mathcal{Z}}=1\,,$
$\varrho_\varepsilon^H(f)=|f^{\otimes n}\rangle\langle f^{\otimes n}|\,,\varepsilon=\frac{1}{n}\,,\;\mathcal{E}=\left\{\frac{1}{n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}(\varrho_\varepsilon^H(f)\,,\varepsilon\in\mathcal{E})=\left\{\delta_f^{\mathbb{S}^1}=\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}f}\;d\theta\right\}\,.$$

mixed Hermite (twin Fock) states: $f_1,f_2\in\mathcal{Z}\,,\;\langle f_i\,,\,f_j\rangle=\delta_{ij}\,,$
$\varrho_\varepsilon^H(f_1,f_2)=|f_1^{\otimes n}\rangle\langle f_1^{\otimes n}|\otimes|f_2^{\otimes n}\rangle\langle f_2^{\otimes n}|\,,\varepsilon=\frac{1}{2n}\,,\;\mathcal{E}=\left\{\frac{1}{2n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}\left\{\varrho_\varepsilon^H(f_1,f_2)\,,\,\varepsilon\in\mathcal{E}\right\}=\left\{\delta^{\mathbb{S}^1}_{2^{-1/2}f_1}\otimes\delta^{\mathbb{S}^1}_{2^{-1/2}f_2}\right\}\,.$$

Wigner measures: Examples

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,,\forall\varepsilon\in\mathcal{E}\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then for any cylindrical polynomial and with $Q=$*Weyl,Wick or anti-Wick*

$$\lim_{\varepsilon\to 0\,,\,\varepsilon\in\mathcal{E}}\mathrm{Tr}\,\left[\varrho_\varepsilon b^Q\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)\,.$$

Examples

Coherent states: $f\in\mathcal{Z}\,,|f|_{\mathcal{Z}}=1\,,\;E(f)=W(\frac{\sqrt{2}}{i\varepsilon}f)|\Omega\rangle=e^{\frac{a^*(f)-a(f)}{\varepsilon}}|\Omega\rangle\,,$
$\varrho_\varepsilon^C(f)=|E(f)\rangle\langle E(f)|\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon^C(f)b^{Wick}\right]=b(f)\,,\quad\mathcal{M}(\varrho_\varepsilon^C(f),\varepsilon\in\mathcal{E})=\{\delta_f\}\,.$

Hermite (atomic coherent) states: $f\in\mathcal{Z}\,,\;|f|_{\mathcal{Z}}=1\,,$
$\varrho_\varepsilon^H(f)=|f^{\otimes n}\rangle\langle f^{\otimes n}|\,,\varepsilon=\frac{1}{n}\,,\;\mathcal{E}=\left\{\frac{1}{n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}(\varrho_\varepsilon^H(f)\,,\varepsilon\in\mathcal{E})=\left\{\delta_f^{\mathbb{S}^1}=\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}f}\;d\theta\right\}\,.$$

mixed Hermite (twin Fock) states: $f_1,f_2\in\mathcal{Z}\,,\;\langle f_i\,,\,f_j\rangle=\delta_{ij}\,,$
$\varrho_\varepsilon^H(f_1,f_2)=|f_1^{\otimes n}\rangle\langle f_1^{\otimes n}|\otimes|f_2^{\otimes n}\rangle\langle f_2^{\otimes n}|\,,\varepsilon=\frac{1}{2n}\,,\;\mathcal{E}=\left\{\frac{1}{2n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}\left\{\varrho_\varepsilon^H(f_1,f_2)\,,\,\varepsilon\in\mathcal{E}\right\}=\left\{\delta_{2^{-1/2}f_1}^{\mathbb{S}^1}\otimes\delta_{2^{-1/2}f_2}^{\mathbb{S}^1}\right\}\,.$$

Wigner measures: Examples

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,,\forall\varepsilon\in\mathcal{E}\,,\ \mathrm{Tr}\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then for any cylindrical polynomial and with $Q=$*Weyl,Wick or anti-Wick*

$$\lim_{\varepsilon\to0\,,\,\varepsilon\in\mathcal{E}}\mathrm{Tr}\left[\varrho_\varepsilon b^Q\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)\,.$$

Examples

Coherent states: $f\in\mathcal{Z}\,,|f|_{\mathcal{Z}}=1\,,\ E(f)=W(\frac{\sqrt{2}}{i\varepsilon}f)|\Omega\rangle=e^{\frac{a^*(f)-a(f)}{\varepsilon}}|\Omega\rangle\,,$
$\varrho_\varepsilon^C(f)=|E(f)\rangle\langle E(f)|\,,\ \mathrm{Tr}\left[\varrho_\varepsilon^C(f)b^{Wick}\right]=b(f)\,,\quad\mathcal{M}(\varrho_\varepsilon^C(f),\varepsilon\in\mathcal{E})=\{\delta_f\}\,.$

Hermite (atomic coherent) states: $f\in\mathcal{Z}\,,\ |f|_{\mathcal{Z}}=1\,,$
$\varrho_\varepsilon^H(f)=|f^{\otimes n}\rangle\langle f^{\otimes n}|\,,\varepsilon=\frac{1}{n}\,,\ \mathcal{E}=\left\{\frac{1}{n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}(\varrho_\varepsilon^H(f)\,,\varepsilon\in\mathcal{E})=\left\{\delta_f^{\mathbb{S}^1}=\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}f}\,d\theta\right\}.$$

mixed Hermite (twin Fock) states: $f_1,f_2\in\mathcal{Z}\,,\ \langle f_i\,,\,f_j\rangle=\delta_{ij}\,,$
$\varrho_\varepsilon^H(f_1,f_2)=|f_1^{\otimes n}\rangle\langle f_1^{\otimes n}|\otimes|f_2^{\otimes n}\rangle\langle f_2^{\otimes n}|\,,\varepsilon=\frac{1}{2n}\,,\ \mathcal{E}=\left\{\frac{1}{2n}\,,n\in\mathbb{N}^*\right\}\,,$

$$\mathcal{M}\left\{\varrho_\varepsilon^H(f_1,f_2)\,,\,\varepsilon\in\mathcal{E}\right\}=\left\{\delta_{2^{-1/2}f_1}^{\mathbb{S}^1}\otimes\delta_{2^{-1/2}f_2}^{\mathbb{S}^1}\right\}.$$

Definition

Fixed degrees: we say that $b(z) = \langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p}\rangle$ belongs to $\mathcal{P}_{p,q}(\mathcal{Z})$, if

$$\tilde{b} = \frac{1}{q!}\frac{1}{p!}\partial_{\bar{z}}^{q}\partial_{z}^{p} b \in \mathcal{L}(\bigvee^{p}\mathcal{Z}; \bigvee^{q}\mathcal{Z}),$$

Polynomials: $\mathcal{P}(\mathcal{Z}) = \oplus^{alg}_{p,q\in\mathbb{N}} \mathcal{P}_{p,q}(\mathcal{Z})$

For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, and $n \geq 0$,

$$b^{Wick}\big|_{\bigvee^{n+p}\mathcal{Z}} = \frac{\sqrt{(n+p)!(n+p)!}}{n!}\varepsilon^{\frac{p+q}{2}}\mathcal{S}_{n+q}(\tilde{b}\otimes \mathrm{Id}_{\bigvee^{n}\mathcal{Z}}).$$

Properties of $\mathcal{P}(\mathcal{Z})$:

1. $b^{Wick} : \mathcal{H}_{fin} = \oplus^{alg}_{n\in\mathbb{N}} \bigvee^{n}\mathcal{Z} \to \mathcal{H}_{fin}$;
2. number estimates: $\|\langle \mathbf{N}\rangle^{-q/2} b^{Wick} \langle \mathbf{N}\rangle^{-p/2}\| \leq C\|\tilde{b}\| = C|b|_{\mathcal{P}_{p,q}}$ for all $b \in \mathcal{P}_{p,q}(\mathcal{Z})$;
3. Wick ordering: The Wick symbol $b_1 \sharp^{Wick} b_2$ of $b_1^{Wick} \circ b_2^{Wick}$ satisfies

$$b_1 \sharp^{Wick} b_2 = e^{\varepsilon \partial_\omega . \partial_{\bar{z}}} b_1(\omega) b_2(z)\big|_{\omega=z} = \sum_{k=0}^{\min(p_2,q_1)} \frac{(\varepsilon\partial_\omega . \partial_{\bar{z}})^k}{k!} b_1(\omega)b_2(z)\big|_{\omega=z} \quad \text{in } \mathcal{P}(\mathcal{Z})$$

Symbol of the commutator $\frac{1}{\varepsilon}\left[b_1^{Wick}\,,\, b_2^{Wick}\right] = \{b_1, b_2\}^{(1)} + \mathcal{O}(\varepsilon)$.

4. With coherent states: $\langle E(f)\,,\, b^{Wick} E(f)\rangle = \langle f^{\otimes q}\,,\, \tilde{b} f^{\otimes p}\rangle = b(f)$ for all $b \in \mathcal{P}_{p,q}(\mathcal{Z})$.

Definition

Fixed degrees: we say that $b(z) = \langle z^{\otimes q}\,,\,\tilde{b}z^{\otimes p}\rangle$ belongs to $\mathcal{P}^{r}_{p,q}(\mathcal{Z})$, if

$$\tilde{b} = \frac{1}{q!}\frac{1}{p!}\partial_{\bar{z}}^{q}\partial_{z}^{p}b \in \mathcal{L}^{r}(\bigvee^{p}\mathcal{Z};\bigvee^{q}\mathcal{Z})\,,\ 1 \leq r \leq \infty \text{ Schatten classes}$$

Polynomials: $\mathcal{P}(\mathcal{Z}) = \oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z})$ $\quad\mathcal{P}^{\infty}(\mathcal{Z}) = \oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}^{\infty}_{p,q}(\mathcal{Z})$

For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, and $n \geq 0$,

$$b^{Wick}\big|_{\bigvee^{n+p}\mathcal{Z}} = \frac{\sqrt{(n+p)!(n+p)!}}{n!}\varepsilon^{\frac{p+q}{2}}\mathcal{S}_{n+q}(\tilde{b}\otimes \mathrm{Id}_{\bigvee^{n}\mathcal{Z}})\,.$$

Properties of $\mathcal{P}(\mathcal{Z})$:

1. $b^{Wick} : \mathcal{H}_{fin} = \oplus^{alg}_{n\in\mathbb{N}}\bigvee^{n}\mathcal{Z} \to \mathcal{H}_{fin}$;
2. number estimates: $\|\langle \mathbf{N}\rangle^{-q/2}b^{Wick}\langle \mathbf{N}\rangle^{-p/2}\| \leq C\|\tilde{b}\| = C|b|_{\mathcal{P}_{p,q}}$ for all $b \in \mathcal{P}_{p,q}(\mathcal{Z})$;
3. Wick ordering: The Wick symbol $b_1\sharp^{Wick}b_2$ of $b_1^{Wick}\circ b_2^{Wick}$ satisfies
$$b_1\sharp^{Wick}b_2 = e^{\varepsilon\partial_\omega.\partial_{\bar{z}}}b_1(\omega)b_2(z)\big|_{\omega=z} = \sum_{k=0}^{\min(p_2,q_1)}\frac{(\varepsilon\partial_\omega.\partial_{\bar{z}})^k}{k!}b_1(\omega)b_2(z)\big|_{\omega=z} \quad \text{in } \mathcal{P}(\mathcal{Z})$$
Symbol of the commutator $\frac{1}{\varepsilon}\left[b_1^{Wick}\,,\,b_2^{Wick}\right] = \{b_1, b_2\}^{(1)} + \mathcal{O}(\varepsilon)$.
4. With coherent states: $\langle E(f)\,,\,b^{Wick}E(f)\rangle = \langle f^{\otimes q}\,,\,\tilde{b}f^{\otimes p}\rangle = b(f)$ for all $b \in \mathcal{P}_{p,q}(\mathcal{Z})$.

Definition

Fixed degrees: we say that $b(z)=\langle z^{\otimes q}\,,\,\tilde{b}z^{\otimes p}\rangle$ belongs to $\mathcal{P}_{p,q}(\mathcal{Z})$, if

$$\tilde{b}=\frac{1}{q!}\frac{1}{p!}\partial_{\bar{z}}^{q}\partial_{z}^{p}b\in\mathcal{L}(\bigvee^{p}\mathcal{Z};\bigvee^{q}\mathcal{Z}),$$

Polynomials: $\mathcal{P}(\mathcal{Z})=\oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z})$

For $b\in\mathcal{P}_{p,q}(\mathcal{Z})$, and $n\geq 0$,

$$b^{Wick}\big|_{\bigvee^{n+p}\mathcal{Z}}=\frac{\sqrt{(n+p)!(n+p)!}}{n!}\varepsilon^{\frac{p+q}{2}}\mathcal{S}_{n+q}(\tilde{b}\otimes\mathrm{Id}_{\bigvee^{n}\mathcal{Z}}).$$

Properties of $\mathcal{P}(\mathcal{Z})$:

1. $b^{Wick}:\mathcal{H}_{fin}=\oplus^{alg}_{n\in\mathbb{N}}\bigvee^{n}\mathcal{Z}\to\mathcal{H}_{fin}$;
2. number estimates: $\|\langle\mathbf{N}\rangle^{-q/2}b^{Wick}\langle\mathbf{N}\rangle^{-p/2}\|\leq C\|\tilde{b}\|=C|b|_{\mathcal{P}_{p,q}}$ for all $b\in\mathcal{P}_{p,q}(\mathcal{Z})$;
3. Wick ordering: The Wick symbol $b_1\sharp^{Wick}b_2$ of $b_1^{Wick}\circ b_2^{Wick}$ satisfies

$$b_1\sharp^{Wick}b_2=e^{\varepsilon\partial_\omega.\partial_{\bar{z}}}b_1(\omega)b_2(z)\big|_{\omega=z}=\sum_{k=0}^{\min(p_2,q_1)}\frac{(\varepsilon\partial_\omega.\partial_{\bar{z}})^k}{k!}b_1(\omega)b_2(z)\big|_{\omega=z}\quad\text{in }\mathcal{P}(\mathcal{Z})$$

   Symbol of the commutator $\frac{1}{\varepsilon}\left[b_1^{Wick}\,,\,b_2^{Wick}\right]=\{b_1,b_2\}^{(1)}+\mathcal{O}(\varepsilon)$.
4. With coherent states: $\langle E(f)\,,\,b^{Wick}E(f)\rangle=\langle f^{\otimes q}\,,\,\tilde{b}f^{\otimes p}\rangle=b(f)$ for all $b\in\mathcal{P}_{p,q}(\mathcal{Z})$.

Wick calculus, (PI)-condition, reduced density matrices

Definition

Fixed degrees: we say that $b(z) = \langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p}\rangle$ belongs to $\mathcal{P}_{p,q}(\mathcal{Z})$, if

$$\tilde{b} = \frac{1}{q!}\frac{1}{p!}\partial_{\bar{z}}^{q}\partial_{z}^{p} b \in \mathcal{L}(\bigvee^{p}\mathcal{Z};\bigvee^{q}\mathcal{Z}),$$

Polynomials: $\mathcal{P}(\mathcal{Z}) = \oplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z})$

For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, and $n \geq 0$,
$b^{Wick}\big|_{\bigvee^{n+p}\mathcal{Z}} = \frac{\sqrt{(n+p)!(n+p)!}}{n!}\varepsilon^{\frac{p+q}{2}}\mathcal{S}_{n+q}(\tilde{b}\otimes \mathrm{Id}_{\bigvee^{n}\mathcal{Z}})$.

Properties of $\mathcal{P}(\mathcal{Z})$:

1. $b^{Wick} : \mathcal{H}_{fin} = \oplus^{alg}_{n\in\mathbb{N}}\bigvee^{n}\mathcal{Z} \to \mathcal{H}_{fin}$;
2. number estimates: $\|\langle \mathbf{N}\rangle^{-q/2} b^{Wick}\langle\mathbf{N}\rangle^{-p/2}\| \leq C\|\tilde{b}\| = C|b|_{\mathcal{P}_{p,q}}$ for all $b \in \mathcal{P}_{p,q}(\mathcal{Z})$;
3. Wick ordering: The Wick symbol $b_1\sharp^{Wick}b_2$ of $b_1^{Wick}\circ b_2^{Wick}$ satisfies

$$b_1\sharp^{Wick}b_2 = e^{\varepsilon\partial_\omega.\partial_{\bar{z}}}b_1(\omega)b_2(z)\big|_{\omega=z} = \sum_{k=0}^{\min(p_2,q_1)}\frac{(\varepsilon\partial_\omega.\partial_{\bar{z}})^k}{k!}b_1(\omega)b_2(z)\big|_{\omega=z} \quad \text{in } \mathcal{P}(\mathcal{Z})$$

   Symbol of the commutator $\frac{1}{\varepsilon}\left[b_1^{Wick}\,,\, b_2^{Wick}\right] = \{b_1, b_2\}^{(1)} + \mathcal{O}(\varepsilon)$.
4. With coherent states: $\langle E(f)\,,\, b^{Wick}E(f)\rangle = \langle f^{\otimes q}\,,\, \tilde{b}f^{\otimes p}\rangle = b(f)$ for all $b \in \mathcal{P}_{p,q}(\mathcal{Z})$.

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\ \varepsilon\in\mathcal{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\ \exists C_k>0\,,\ \forall\varepsilon\in\mathcal{E}\,,\ \mathrm{Tr}\ \left[\varrho_\varepsilon \mathbf{N}^k\right]\leq C_k\,,$$

then $\lim_{\varepsilon\to 0\,,\ \varepsilon\in E}\ \mathrm{Tr}\ \left[\varrho_\varepsilon b^{Wick}\right]=\int_{\mathcal{Z}} b(z)\ d\mu(z)$ *for all* $b\in\mathcal{P}^{\infty}(\mathcal{Z})$.

A counter-example with $\tilde{b}$ not compact: Take $\varepsilon=\frac{1}{n}$, $\mathcal{E}=\left\{\frac{1}{n}\,,\,n\in\mathbb{N}^*\right\}$ and consider a normalized sequence $(f_n)_{n\in\mathbb{N}^*}$ converging weakly to 0 . Then

$$\mathcal{M}(\varrho_\varepsilon^C(f_n),\varepsilon\in\mathcal{E})=\{\delta_0\}\ ,$$

$$\mathrm{Tr}\ \left[\varrho_\varepsilon^C(f_n)(|z|^{2p})^{Wick}\right]=|f_n|^{2p}=1\neq 0=\int_{\mathcal{Z}}|z|^{2p}\ \delta_0(z)\,.$$

Polynomial-Identity: The failure of the convergence when $\tilde{b}=\mathrm{Id}_{\bigvee^p\mathcal{Z}}$ is the sole obstruction to the convergence with a general $\tilde{b}\in\mathcal{P}(\mathcal{Z})$.

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E})=\{\mu\}$ *and*

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,\forall\varepsilon\in\mathcal{E}\,,\ \mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then $\lim_{\varepsilon\to0\,,\,\varepsilon\in E}\mathrm{Tr}\,\left[\varrho_\varepsilon b^{Wick}\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)$ *for all* $b\in\mathcal{P}^\infty(\mathcal{Z})$.

A counter-example with $\tilde{b}$ not compact: Take $\varepsilon=\frac{1}{n}$, $\mathcal{E}=\left\{\frac{1}{n}\,,n\in\mathbb{N}^*\right\}$ and consider a normalized sequence $(f_n)_{n\in\mathbb{N}^*}$ converging weakly to 0. Then

$$\mathcal{M}(\varrho_\varepsilon^C(f_n),\varepsilon\in\mathcal{E})=\{\delta_0\}\,,$$
$$\mathrm{Tr}\,\left[\varrho_\varepsilon^C(f_n)(|z|^{2p})^{Wick}\right]=|f_n|^{2p}=1\neq0=\int_{\mathcal{Z}}|z|^{2p}\,\delta_0(z)\,.$$

Polynomial-Identity: The failure of the convergence when $\tilde{b}=\mathrm{Id}_{\bigvee^p\mathcal{Z}}$ is the sole obstruction to the convergence with a general $\tilde{b}\in\mathcal{P}(\mathcal{Z})$.

Corollary

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E})=\{\mu\}$ and

$$\forall k\in\mathbb{N}\,,\,\exists C_k>0\,,\,\forall\varepsilon\in\mathcal{E}\,,\;\mathrm{Tr}\,\left[\varrho_\varepsilon\mathbf{N}^k\right]\leq C_k\,,$$

then $\lim_{\varepsilon\to 0\,,\,\varepsilon\in E}\mathrm{Tr}\,\left[\varrho_\varepsilon b^{Wick}\right]=\int_{\mathcal{Z}}b(z)\,d\mu(z)$ *for all* $b\in\mathcal{P}^{\infty}(\mathcal{Z})$.

A counter-example with $\tilde{b}$ not compact: Take $\varepsilon=\frac{1}{n}$, $\mathcal{E}=\left\{\frac{1}{n}\,,n\in\mathbb{N}^*\right\}$ and consider a normalized sequence $(f_n)_{n\in\mathbb{N}^*}$ converging weakly to 0. Then

$$\mathcal{M}(\varrho_\varepsilon^C(f_n),\varepsilon\in\mathcal{E})=\{\delta_0\}\,,$$
$$\mathrm{Tr}\,\left[\varrho_\varepsilon^C(f_n)(|z|^{2p})^{Wick}\right]=|f_n|^{2p}=1\neq 0=\int_{\mathcal{Z}}|z|^{2p}\;\delta_0(z)\,.$$

Polynomial-Identity: The failure of the convergence when $\tilde{b}=\mathrm{Id}_{\bigvee^p\mathcal{Z}}$ is the sole obstruction to the convergence with a general $\tilde{b}\in\mathcal{P}(\mathcal{Z})$.

Wick calculus, (PI) condition, reduced density matrices

Remember $(|z|^{2p})^{Wick} = \left(\langle z^{\otimes p}\,,\,\mathrm{Id}\,z^{\otimes p}\rangle\right)^{Wick} = \mathbf{N}(\mathbf{N}-\varepsilon)\cdots(\mathbf{N}-\varepsilon(p-1)) \sim \mathbf{N}^p$

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$, *with*

$$\forall k\in\mathbb{N}\,,\quad \lim_{\varepsilon\to 0,\varepsilon\in\mathcal{E}} \mathrm{Tr}\,\left[\varrho_\varepsilon \mathbf{N}^k\right] = \int_{\mathcal{Z}} |z|^{2k}\,d\mu(z)\,.\quad (PI)$$

Then

1. $\lim_{\varepsilon\to 0,\varepsilon\in\mathcal{E}} \mathrm{Tr}\,\left[\varrho_\varepsilon b^{Wick}\right] = \int_{\mathcal{Z}} b(z)\,d\mu(z)$ *for all* $b\in\mathcal{P}(\mathcal{Z})$ *;*
2. $\lim_{\varepsilon\to 0,\varepsilon\in\mathcal{E}} \|\gamma_\varepsilon^p - \gamma_0^p\|_{\mathcal{L}^1(\bigvee^p \mathcal{Z})} = 0$, *for all* $p\in\mathbb{N}$

with (assuming $\mu\neq\delta_0$*)*

$$\mathrm{Tr}\,\left[\gamma_\varepsilon^p \tilde{b}\right] = \frac{\mathrm{Tr}\,\left[\varrho_\varepsilon b^{Wick}\right]}{\mathrm{Tr}\,\left[\varrho_\varepsilon (|z|^{2p})^{Wick}\right]}\quad,\quad \gamma_0^p = \frac{\int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\,d\mu(z)}{\int_{\mathcal{Z}} |z|^{2p}\,d\mu(z)}\,.$$

Wick calculus, (PI) condition, reduced density matrices

Remember $(|z|^{2p})^{Wick} = \left(\langle z^{\otimes p}\,,\, \mathrm{Id}\, z^{\otimes p}\rangle\right)^{Wick} = \mathbf{N}(\mathbf{N}-\varepsilon)\cdots(\mathbf{N}-\varepsilon(p-1)) \sim \mathbf{N}^p$

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$, *with*

$$\forall k\in\mathbb{N}\,,\quad \lim_{\varepsilon\to 0,\varepsilon\in\mathcal{E}} \mathrm{Tr}\left[\varrho_\varepsilon \mathbf{N}^k\right] = \int_{\mathcal{Z}} |z|^{2k}\, d\mu(z)\,. \qquad (PI)$$

Then

1. $\lim_{\varepsilon\to 0,\varepsilon\in\mathcal{E}} \mathrm{Tr}\left[\varrho_\varepsilon b^{Wick}\right] = \int_{\mathcal{Z}} b(z)\, d\mu(z)$ *for all* $b\in\mathcal{P}(\mathcal{Z})$;
2. $\lim_{\varepsilon\to 0,\varepsilon\in\mathcal{E}} \|\gamma_\varepsilon^p - \gamma_0^p\|_{\mathcal{L}^1(\bigvee^p \mathcal{Z})} = 0$, *for all* $p\in\mathbb{N}$

with (assuming $\mu\neq\delta_0$*)*

$$\mathrm{Tr}\left[\gamma_\varepsilon^p \tilde{b}\right] = \frac{\mathrm{Tr}\left[\varrho_\varepsilon b^{Wick}\right]}{\mathrm{Tr}\left[\varrho_\varepsilon (|z|^{2p})^{Wick}\right]} \quad,\quad \gamma_0^p = \frac{\int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\, d\mu(z)}{\int_{\mathcal{Z}} |z|^{2p}\, d\mu(z)}\,.$$

Remark: When $\varrho_\varepsilon \in \mathcal{L}^1(L^2_{sym}((\mathbb{R}^D)^n))$, $\varepsilon = \frac{1}{n}$,

$$\gamma_\varepsilon^p(x_1,\ldots,x_p;y_1,\ldots,y_p) = \int_{(\mathbb{R}^D)^{N-p}} \varrho_\varepsilon(x_1,\ldots,x_p,X;y_1,\ldots,y_p,X)\, dX$$

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\, Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and the condition (PI) , then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) holds for all times.

Theorem Liard-Pawilowski arXiv 14

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and $Q\in\mathcal{P}^\infty(\mathcal{Z})$, then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

$$\text{and} \qquad ((PI) \text{ at } t=0) \Leftrightarrow ((PI) \text{ at any } t)$$

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\, Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and the condition (PI), then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) holds for all times.

Theorem Liard-Pawilowski arXiv 14

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and $Q\in\mathcal{P}^\infty(\mathcal{Z})$, then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

$$\text{and}\qquad ((PI) \text{ at } t=0) \Leftrightarrow ((PI) \text{ at any } t)$$

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\, Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and the condition (PI) , then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) holds for all times.

Theorem Liard-Pawilowski arXiv 14

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and $Q\in\mathcal{P}^\infty(\mathcal{Z})$, then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

and $\quad ((PI)\text{ at } t=0) \Leftrightarrow ((PI) \text{ at any } t)$

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\, Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon \in \mathcal{E}) = \{\mu\}$ and the condition (PI) , then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon \in \mathcal{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) holds for all times.

Method: Truncated Dyson expansion after (Fröhlich-Graffi-Schwarz 07 and Fröhlich-Knowles-Schwarz 09) combined with a priori information on $\mu(t)$.

Theorem Liard-Pawilowski arXiv 14

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon \in \mathcal{E}) = \{\mu\}$ and $Q \in \mathcal{P}^\infty(\mathcal{Z})$, then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon \in \mathcal{E}) = \{\Phi(t)_*\mu\}$$

and $((PI) \text{ at } t=0) \Leftrightarrow ((PI) \text{ at any } t)$

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\,Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ *and the condition* (PI)*, then*

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) *holds for all times.*

Theorem Liard-Pawilowski arXiv 14

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ *and* $Q \in \mathcal{P}^\infty(\mathcal{Z})$*, then*

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

$$\text{and} \qquad ((PI)\ \text{at}\ t=0) \Leftrightarrow ((PI)\ \text{at any}\ t)$$

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\, Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \mathcal{E}) = \{\mu\}$ and the condition (PI) , then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\, \varepsilon \in \mathcal{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) holds for all times.

Theorem Liard-Pawilowski arXiv 14

Assume $\mathcal{M}(\varrho_\varepsilon\,,\, \varepsilon \in \mathcal{E}) = \{\mu\}$ and $Q \in \mathcal{P}^\infty(\mathcal{Z})$, then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\, \varepsilon \in \mathcal{E}) = \{\Phi(t)_*\mu\}$$

$$\text{and} \qquad ((PI) \text{ at } t=0) \Leftrightarrow ((PI) \text{ at any } t)$$

Method: Like in Ammari-N. to appear in Ann. Sci. Pisa for the pair 3D-Coulombic interaction. Measure transportation adapted from Ambrosio-Gigli-Savaré (book 05).

Mean field propagation of Wigner measures

Problem: After composition with a nonlinear flow, cylindrical (resp. polynomial symbols) do not remain cylindrical (resp. polynomials).

Take $\mathcal{E}(z) = \langle z\,,\, Az\rangle + Q(z)$ with A self-adjoint and $Q \in \mathcal{P}(\mathcal{Z})$ and set $H_\varepsilon = \mathcal{E}^{Wick}$ while Φ is the hamiltonian flow associated with $\mathcal{E}$.

Theorem Ammari-N. (JMPA 11)

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and the condition (PI) , then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

and the condition (PI) holds for all times.

Theorem Liard-Pawilowski arXiv 14

Assume $\mathcal{M}(\varrho_\varepsilon\,,\,\varepsilon\in\mathcal{E}) = \{\mu\}$ and $Q\in\mathcal{P}^\infty(\mathcal{Z})$, then

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}\,,\,\varepsilon\in\mathcal{E}) = \{\Phi(t)_*\mu\}$$

and $\quad ((PI)\text{ at } t=0) \Leftrightarrow ((PI)\text{ at any } t)$

Some compactness is needed either on the interaction or on the initial data. In the 3D-Coulombic case, we used the compactness of $(1-\Delta)^{-1/2}\frac{1}{|x|}(1-\Delta)^{-1/2}$.

Related works

1. S. Breteaux (phD 11, to appear in Ann. Inst. Fourier): 1 particle in a gaussian random potential=1 particle coupled to a bosonic field $\to$ random homogenization. Distinguishing stochastic processes from phase-space geometry is a matter of scaling; see e.g. $W(f)$ versus $W(\frac{f}{\varepsilon})$.
2. Z. Ammari-M. Zerzeri 12: coherent-state propagation with Pauli-Fierz Hamiltonians.
3. Q. Liard (phD in progress): Singular interactions with possibly confining potentials.
4. B. Pawilowski (phD in progress Rennes-Wien): $->$Numerics.
5. Z. Ammari-M. Falconi (arXiv 14): Nelson model.
6. Z. Ammari-M. Falconi-B. Pawilowski (in progress): order of convergence (extends Lewin-Rougerie arXiv 13)

Thank you for your attention !