Minimization of $\|\varphi\|_4/\|\varphi\|_2$ for polarizations

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Outline

- Motivations
- Some ample line bundles
- Some critical points for a fixed τ
- Minimization w.r.t τ for principal polarizations
- 2-adic version

Motivations

Although experiments or numerical simulations, in Ginzburg-Landau or Gross-Pitaevski models (supraconductivity, superfluidity, rotating BEC), provide some evidence of the optimality of the Abrikosov (hexagonal) lattice, very little is known theoretically in this quantum sphere packing problem.
Solving a model problem, is important for the development of the analysis.

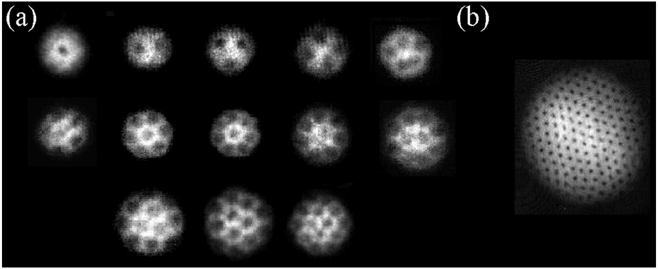

Figure: Experiments on Bose-Einstein condensates obtained a) in ENS (Dalibard et al.) b) in MIT (Ketterle et al.). Chevy-Dalibard, Europhysics News 2008

In this specific case the energy functional is

$$\int_{\mathbb{C}} |z|^2|f(z)|^2 e^{-\frac{|z|^2}{h}} + G|f(z)|^4 e^{-\frac{2|z|^2}{h}}\ L(dz)\,,$$

under the constraint

$$f \quad \text{holomorphic} \qquad \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{h}}\ L(dz) = 1\,.$$

Some remarks

- The physical plane is actually a phase-space if one considers the Bargmann transform

$$(B_h\psi)(z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{z^2}{h}} \int_{\mathbb{R}} e^{-\frac{(\sqrt{2}z-y)^2}{h}} \psi(y)\ dy\ ,$$

which is a unitary transform from $L^2(\mathbb{R}, dy)$ to

$$\mathcal{F}_h = \left\{ f \text{ entire on } \mathbb{C}\,, \int_{\mathbb{C}} e^{-\frac{|z|^2}{h}} |f(z)|^2\ L(dz) < +\infty \right\}\ .$$

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- $h > 0$ is a semiclassical parameter. The repartition of zeroes is constrained by the uncertainty principle, $\Delta q \Delta p \geq h$ when $z = \frac{1}{\sqrt{2}}(q + ip)$: 1 zero per box of volume h.

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- Semiclassical (microlocal) analysis = cotangent geometry of amplitude and frequency modulations. Typically $\psi(q) = a(q)e^{iS(q)/h}$, $q \in \mathbb{R}^d$, encoded by $\mu = |a(q)|^2\delta(p - dS(q))$. Model $e^{i\langle p,q\rangle/h}$, $q \in V$, $p \in V^*$ (or possibly $q \in V/\mathcal{L} \sim \mathbb{T}^d$ $p \in (2\pi\mathbb{Z})^d$).

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$$\mathcal{F}_h = \left\{ f \text{ entire on } \mathbb{C}\,, \int_{\mathbb{C}} e^{-\frac{|z|^2}{h}} |f(z)|^2 \; L(dz) < +\infty \right\}.$$

- Here the confining potential $|z|^2$ provides the compactness in the minimization of $\int_{\mathbb{C}} |z|^2 |u(z)|^2 + G|u(z)|^4 \; L(dz)$ when $u(z) = e^{-\frac{|z|^2}{2h}} f(z)\,,\ f$ entire, $\int_{\mathbb{C}} |u(z)|^2 \; L(dz) = 1\,.$

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- **Semiclassical Toeplitz operators:**

$$\Pi_h(a(z,\bar{z})\Pi_h \circ \Pi_h(b(z,\bar{z}))\Pi_h = \Pi_h ab(z,\bar{z})\Pi_h + \mathcal{O}(h) \qquad (0.1)$$

when $a, b \in \mathcal{C}_b^\infty(\mathbb{C})$ acts by multiplication and

$$\Pi_h = \frac{1}{\pi h} \int_{\mathbb{C}} e^{-\frac{z\bar{z'} - |z'|^2}{h}} f(z', \bar{z}')\ L(dz')\,.$$

Nonlinear pb studied in this way in (Aftalion-Blanc-N. JFA 2006), with other confining potentials (Correggi-Tanner-Yngvason JMP 2007, Aftalion-Blanc-Lerner JFA 2009, related works by Serfaty, Sandier, and Sigal-Tzaneteas 2011).

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(0.1) allows amplitude modulations and partitions of unity for the non linear problem. Geometrical deformations Sjöstrand, Sjöstrand-Boutet de Monvel, Boutet de Monvel-Guillemin

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- **Model problem:** minimize $\frac{⨍ |u(z)|^4}{(⨍ |u(z)|^2)^2}$ for f or φ entire on $\mathbb{C}$ with

$$u(z) = e^{-\frac{|z|^2}{2h}} f(z) = e^{-\frac{|z|^2 - z^2}{2h}} \varphi(z)\,,$$

$$|u| \quad \mathcal{L}-\text{periodic} \quad , \quad ⨍ F = \lim_{R\to\infty} \frac{\int_{|z|<R} F \; L(dz)}{\int_{|z|<R} 1 \; L(dz)}.$$

Some ample line bundles

- Nondegenerate ample line bundles over complex tori are parametrized by $\tau = {}^t\tau \in \mathcal{M}_g(\mathbb{C})$, $\mathrm{Im}\,\tau > 0$ and $d \in \mathcal{M}_g(\mathbb{Z})$, diagonal $d_1|d_2\cdots|d_g$ $\rightarrow$ dimension of the set of sections $\prod_{i=1}^{g} d_i$ (ref e.g: Debarre, Lion-Vergne, Igusa, Beauville, Mumford, Bierkenhake-Lange)

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- Fix the period matrix $\tau = x + iy$ and with φ entire on $\mathbb{C}^g$ associate $u_\varphi(z) = e^{-\frac{\pi}{2}(y^{-1}\{z\} - y^{-1}[z])}\varphi(z)$, $|u_\varphi(z)| = e^{-\pi y^{-1}[\mathrm{Im}\, z]}|\varphi(z)|$ (Siegel's notations $A[B] = {}^tBAB$ and $A\{B\} = {}^t\bar{B}AB$).

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- **Phase translations:** For $z_0 \in \mathbb{C}^g$ and $\lambda \in S^1$, $(z_0, \lambda).\varphi(z) = \lambda e^{-i\pi {}^t(\mathrm{Im}\, z_0) y^{-1}(2z - z_0)}\varphi(z - z_0)$.

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- **Weyl relations:** $(z_1,\lambda_1)\circ(z_2,\lambda_2) = (z_1+z_2, e^{i\pi\,\mathrm{Im}\,({}^t\bar{z}_1 y^{-1} z_2)}\lambda_1\lambda_2)$.

$$|u_{(z_0,\lambda).\varphi}|(z) = |u_\varphi|(z-z_0).$$

Particular case (Mumford's notations): $a, b \in \mathbb{R}^g$,

$$S_b\varphi = (-b,0).\varphi = \varphi(z+b) \quad T_a\varphi = (-\tau a,0).\varphi = e^{i\pi\tau[a]+2i\pi {}^taz}\varphi(z+\tau a)$$

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- **Model problem:** Fix the period matrix $\tau = x + iy$ and consider the set of entire functions φ such that $|u_\varphi(z)| = e^{-\pi y^{-1}[\mathrm{Im}\, z]}|\varphi(z)|$ is $\ell(\mathbb{Z}^g + \tau\mathbb{Z}^g)$-periodic, $\ell \in \mathbb{N}$. The limit $h \to 0$ corresponds to $\ell \to \infty$.

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- If $(e_1, \ldots, e_g)$ be the canonical basis of $\mathbb{R}^g$ then $\frac{S_{\ell e_i}\varphi}{\varphi}$ and $\frac{T_{\ell e_i}\varphi}{\varphi}$ are holomorphic functions of modulus 1:

$$S_{\ell e_i}\varphi = e^{2i\pi\phi_i} g \quad , \quad T_{\ell e_i}\varphi = e^{2i\pi\psi_i}\varphi\,.$$

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- $$V_\ell = \{\varphi \text{ entire}\,, S_{\ell e_i}\varphi = \varphi\,, T_{\ell e_i}\varphi = \varphi\}\ .$$

  Irreducible representation of

$$\left\{(-b - \tau a, \lambda)\,, (a, b) \in (\ell^{-1}\mathbb{Z})^{2g}\,, \lambda \in \exp(2i\pi\ell^{-2}\mathbb{Z})\right\}$$

Some ample line bundles

- For $\varphi \in V_\ell$ the condition $S_{\ell e_i}\varphi(z) = \varphi(z + \ell e_i) = \varphi(z)$ implies $\varphi(z) = \sum_{n\in(\ell^{-1}\mathbb{Z})^g} c_n e^{i\pi\tau[n]} e^{2i\pi {}^t nz}$ while the condition $T_{\ell e_i}\varphi = \varphi$ implies $c_m = c_n$ when $m - n \in (\ell\mathbb{Z})^g$. Hence

$$V_\ell = \bigoplus_{\omega\in(\ell^{-1}\mathbb{Z}/\ell\mathbb{Z})^g} \mathbb{C}e_\omega \,, \quad e_\omega = \ell^{g/2} \sum_{n\in(\ell\mathbb{Z})^g} e^{i\pi\tau[n+\omega]} e^{2i\pi {}^t(n+\omega)z} \,.$$

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- Combining $S_b e_\omega = e^{2i\pi^t b\omega} e_\omega$ and $T_a e_\omega = e_{\omega+a}$ with the Poisson formula leads to

$$\det(2y)^{1/2}|u_\varphi|^2(q + \tau p) = \sum_{k_1,k_2\in(\ell^{-1}\mathbb{Z})^g} \hat{U}_{k_1,k_2} e^{2i\pi({}^t k_1 q + {}^t k_2 p)}$$

with $$\hat{U}_{k_1,k_2} = \langle\varphi\,,\, S_{k_2} T_{-k_1}\varphi\rangle e^{i\pi^t k_1 k_2} e^{-\frac{\pi}{2} y^{-1}\{k_2 - \tau k_1\}} \,.$$

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  with $\quad \hat{U}_{k_1,k_2} = \langle \varphi \,, \, S_{k_2} T_{-k_1} \varphi \rangle e^{i\pi {}^t k_1 k_2} e^{-\frac{\pi}{2} y^{-1}\{k_2 - \tau k_1\}} \,.$
- Consequence: Set $\|\varphi\|_r = \det(2y)^{1/4} (\fint |u_\varphi|^r)^{1/r}$,

$$\frac{\fint |u_\varphi|^4}{\left(\fint |u_\varphi|^2\right)^2} = \frac{\|\varphi\|_4^4}{\|\varphi\|_2^4} = \sum_{k_1, k_2 \in (\ell^{-1}\mathbb{Z})^g} e^{-\pi y^{-1}\{k_2 - \tau k_1\}} \frac{|\langle \varphi \,, \, S_{k_2} T_{-k_1} \varphi \rangle|^2}{\|\varphi\|_2^4} \,.$$

Remarks:

- For $\ell = 1$ one gets $\frac{\|\varphi\|_4^4}{\|\varphi\|_2^4} = \sum_{k_1,k_2\in\mathbb{Z}^g} e^{-\pi y^{-1}\{k_2-\tau k_1\}} \overset{def}{=} \gamma(\tau)$.

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- The minimization problem $\min_\tau \gamma(\tau)$ is well-posed. When $g = 1$ the minimum is achieved for $\tau = e^{2i\pi/3}$ with $\gamma(e^{2i\pi/3}) \sim 1.1596$ while $\gamma(i) \sim 1.18$.

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- The matrix of the quadratic form $(q,p) \to y^{-1}\{q - \tau p\}$ is nothing but $s_\tau = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix}$ and $\tau \to s_\tau$ is the homeomorphism between Siegel's upper half plane and the set of symplectic positive definite matrices. ref: Siegel, Freitag, Klingen

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- Splitting of the general minimization problem:
  1. minimize w.r.t τ for $\ell = 1$;
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- Extending to higher $(g = 2)$ dimension has been motivated by the corresponding works of Beckner, Brascamp-Lieb and Lieb about the maximality of gaussian functions in $L^p \to L^q$ estimates for gaussian kernels.

Some critical points for a fixed τ

- Remember

$$\gamma(\tau) = \sum_{k_1,k_2\in\mathbb{Z}^g} e^{-\pi y^{-1}\{k_2-\tau k_1\}}$$

and
$$e_\omega = \ell^{g/2} \sum_{n\in(\ell\mathbb{Z})^g} e^{i\pi\tau[n+\omega]} e^{2i\pi^t(n+\omega)z}\,, \quad \omega \in (\ell^{-1}\mathbb{Z}/\ell\mathbb{Z})^g\,.$$

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- **Theta function with characteristics:** $a, b \in \left\{0, \ldots, \frac{\ell-1}{\ell}\right\}^g$.

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \ell^{-g/2} \sum_{\omega\in(\mathbb{Z}/\ell\mathbb{Z})^g} e^{2i\pi^t b(a+\omega)} e_{a+\omega} = \sum_{n\in\mathbb{Z}^g} e^{i\pi\tau[n+a]} e^{2i\pi^t(n+a)(z+b)}\,.$$

form an orthormal basis of V_ℓ. For $k \in (\ell^{-1}\mathbb{Z})^g$:

$$T_k\theta\begin{bmatrix} a \\ b \end{bmatrix} = e^{-2i\pi^t b\lfloor a+k\rfloor}\theta\begin{bmatrix} \{a+k\} \\ b \end{bmatrix} \quad S_k\theta\begin{bmatrix} a \\ b \end{bmatrix} = e^{2i\pi^t\lfloor b+k\rfloor a}\theta\begin{bmatrix} a \\ \{b+k\} \end{bmatrix}$$

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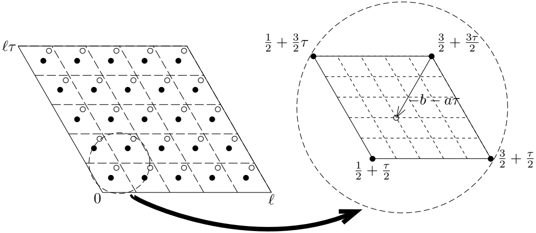

Fig 1. Zeroes of θ_{00} ($\bullet$) and of $\theta_{a,b}$ ($\circ$), for $a, b \in \left\{0, \frac{1}{\ell}, \ldots, \frac{\ell-1}{\ell}\right\}$.

$$\|\theta \begin{bmatrix} a \\ b \end{bmatrix}\|_4^4 \Big/ \|\theta \begin{bmatrix} a \\ b \end{bmatrix}\|_2^4 = \gamma(\tau) = \sum_{k_1, k_2 \in \mathbb{Z}^g} e^{-\pi y^{-1}\{k_2 - \tau k_1\}}$$

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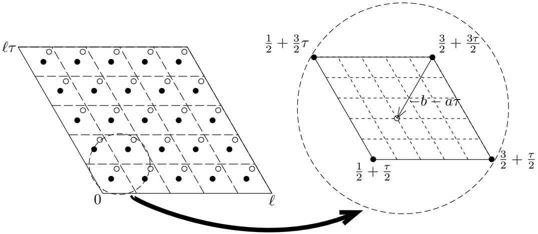

Fig 1. Zeroes of θ_{00} ($\bullet$) and of $\theta_{a,b}$ ($\circ$), for $a, b \in \left\{0, \frac{1}{\ell}, \ldots, \frac{\ell-1}{\ell}\right\}$.

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The theta functions with characteristics are critical points of the functional $\Phi_\tau(\varphi) = \|\varphi\|_4^4 - \gamma(\tau)\|\varphi\|_2^2$ on $\{\varphi \in V_\ell, \|\varphi\|_2 = 1\}$.
Consider $\Phi_\tau\left(\theta\begin{bmatrix}0\\0\end{bmatrix} + \varphi'\right)$ for $\varphi' \in \left(\theta\begin{bmatrix}0\\0\end{bmatrix}\right)^\perp$

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- When $g = 1$ and $\tau = e^{2i\pi/3}$, the theta functions with characterisitics are local minima for $\Phi_\tau\big|_{\|g\|_2=1}$.
  Actually the eigenvalues of Hess $\Phi_\tau \left(\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} + g'\right)\Big|_{g'=0}$ are:

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with $\Omega = \frac{i}{\sqrt{3}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $(0,0) \neq (a,b) \in \left\{0, \ldots, \frac{\ell-1}{\ell}\right\}^2$.

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- $\theta\begin{bmatrix}a\\b\end{bmatrix}$ global minima when $\ell=2$.
- **Main Question:** Are the $\theta\begin{bmatrix}a\\b\end{bmatrix}$ the only global minima for any ℓ, when $\tau=e^{2i\pi/3}$? (e.g. it is not true for $\tau=i$ and $\ell=2$).

Minimization w.r.t τ for principal polarizations

- Now $\ell = 1$ and the problem is

$$\min_{\tau \in \mathfrak{H}_g} \gamma(\tau) \quad , \quad \gamma(\tau) = \sum_{k_1, k_2 \in \mathbb{Z}^g} e^{-\pi y^{-1}\{k_2 - \tau k_1\}}$$

Although not holomorphic nor meromorphic, this function has the modular $+\ \tau \mapsto -\bar{\tau}$ invariance on $\mathfrak{H}_g$.

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- The mimimization problem is well posed:

$$\gamma(\tau) \geq \sum_{k_2 \in \mathbb{Z}^g} e^{-\pi y^{-1}[k_2]} = \det(y)^{1/2} \sum_{k_2 \in \mathbb{Z}^g} e^{-\pi y[k_2]} \overset{\det(y) \to \infty}{\to} +\infty \, .$$

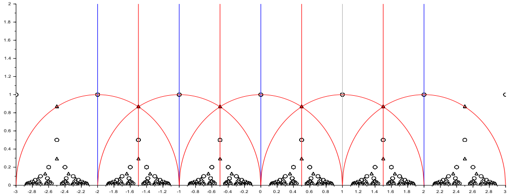
Siegel fundamental domain F_g:

  - $|\det(c\tau + d)| \geq 1$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$ (or finite subset).
  - y is Minkowski reduced
  - $|x_{ij}| \leq \frac{1}{2}$

$g = 1$: well-known ; when $g = 2$: 28 inequations in $\mathcal{H}_2 \subset \mathbb{C}^3 \sim \mathbb{R}^6$ specified by Gottschling (50's).
Subsets of F_g with bounded height ($|\det(y)| \leq C$) are compact.

Minimization w.r.t τ for principal polarizations

- Case $g = 1$: Dutour 99, Nonnemacher-Voros 89, Montgomery 87.
  Improvement: Up to symmetries (or in F_1) $e^{2i\pi/3}$ and i are the only critical points.



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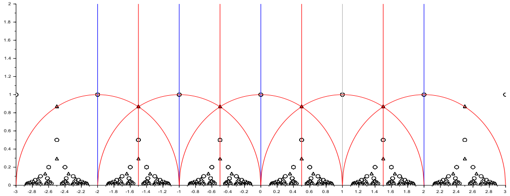

- When $g = 2$ (done with A. Grigis): Combinatorics of Gottschling's (non flat) polytope understood: More than 28 faces (faces associated with some inequations are not connected), 182 vertices,like a 5-dimensional hypercube at infinity (real geometry). Not a face to face tesselation of $\mathcal{H}_2$. F_2 has many (not clear up to now) neighbouring domains. Among the 182 vertices, some of them satisfy more than 6 equations, some edges satisfy more than 5 equations.

Minimization w.r.t τ for principal polarizations

Case $g = 2$, critical points (with A. Grigis):

- Among the vertices, only 20 of them are critical points of $\gamma(\tau)$, some other critical points lie on $\tau = -\overline{\tau}$, like $\tau = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ with index 4 (maximal value ~ 1.3932039 among all the found points).

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- Minimum achieved only (not proved) for $\tau = \tau_{D_4}$ or $\tau = -\overline{\tau_{D_4}}$, index 0 and Gottschling's 28 inequations checked numerically:

$$\tau_{D4} = \frac{1}{3}\begin{pmatrix} -1 + i2\sqrt{2} & 1 + i\sqrt{2} \\ 1 + i\sqrt{2} & -1 + i2\sqrt{2} \end{pmatrix}, \quad \gamma(\tau_{D4}) \sim 1.2858 < 1.34 \sim \gamma(e^{2i\pi/3})^2 .$$

Related works: finite group acting on hyperelliptic curves $y^2 = P(x)$, $\deg P \in \{5, 6\}$,Klein-Kokotov-Korotkin, Silhol-et-al.

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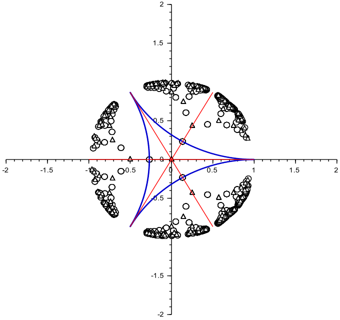
- Among index 1 critical points $\tau_3 = \begin{pmatrix} i & -1/2 + i/2 \\ -1/2 + i/2 & i \end{pmatrix}$ geodesic midpoint between τ_{D_4} and $-\overline{\tau_{D_4}} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.



Questions: When $g = 2$, is it possible to prove that D4 the minimum for γ ? Can we find algebraically all the critical points of γ ? For a general g ? Can we find a Morse stratification, with unstable manifolds of $\nabla\gamma$, of a finite covering of Siegel's fundamental domain ? Is it related with the riemannian structure of the rank g globally symmetric space $\mathcal{H}_g$ (ref: Helgason) ?

2-adic version

After Mumford "On the equations defining abelian varieties II-III":

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(z)=\sum_{n\in\mathbb{Z}^g} e^{i\pi\tau[n]}e^{2i\pi^t nz}\quad,\quad u=e^{\frac{\pi}{2}(y^{-1}[z]-y^{-1}\{z\})}\theta\begin{bmatrix}0\\0\end{bmatrix}$$

$$u_0(z)=u(\ell z)=e^{\frac{\pi}{2}((\ell^{-2}y)^{-1}[z]-(\ell^{-2}y)^{-1}\{z\})}\sum_{n\in(\ell\mathbb{Z})^g} e^{i\pi(\ell^{-2}\tau)[n]}e^{2i\pi^t nz}\,.$$

For $(\alpha_1,\alpha_2)\in\mathbb{Q}_2^g$, $u_0(\alpha_1+(\ell^{-2}\tau)\alpha_2)$ is the algebraic theta function (analytic version) associated with $\tau'=\ell^{-2}\tau$.

$$\begin{aligned}\fint F &= \int_{[0,1]^{2g}} F(\ell(q+\tau p))\,dqdp=\ell^{-2g}\int_{([0,1]\times[0,\ell^2])^g} F(\ell q+\ell^{-1}\tau p)\,dqdp\\ &=\lim_{K\to\infty} 2^{-4gK}\sum_{(q_j,p_j)\in(2^{-K}\mathbb{Z}/2^K\mathbb{Z})^{2g}} F(\ell(q_j+\tau' p_j))\quad,\quad(\ell=2^{N_0})\,.\end{aligned}$$

The minimization of $\|\varphi\|_4^4/\|\varphi\|_2^4=\frac{\fint|u_\varphi|^4}{(\fint|u_\varphi|^2)^2}$ can be transformed into a similar problem for algebraic theta functions on $\mathbb{Q}_2^{2g}$.

2-adic version

- The function $u_0(\alpha_1 + \tau'\alpha_2)$ now denoted $\theta_{an}\begin{bmatrix}0\\0\end{bmatrix}(\alpha)$ is a complex valued locally constant function on $\mathbb{Q}_2^g$, (condition $\ell = 2^{N_0}$). More generally the same holds for $\theta_{an}\begin{bmatrix}a\\b\end{bmatrix}(\alpha_1, \alpha_2)$.

$$\theta_{an}\begin{bmatrix}a\\b\end{bmatrix}(\alpha_1, \alpha_2) = e^{-i\pi^t\alpha_1\alpha_2}\mathcal{F}_{\mathbb{Q}_2^g}\left[1_{a+\alpha_2+\mathbb{Z}_2^g} \times \mu\right](b + \alpha_1),$$

where μ is a finitely additive measure on $\mathbb{Q}_2^g$ and a compactly supported distribution.

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  where μ is a finitely additive measure on $\mathbb{Q}_2^g$ and a compactly supported distribution.
- Even measures μ associated with even theta functions which satisy Riemann's relations are gaussian measures: There exists another measure ν

$$(\mu \times \mu)(U) = (\nu \times \nu)(\xi(U))$$

  with $\xi(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$.
  Analogy with the works of Lieb.