A model for the $\mathbb{E}_3$ fusion-convolution product of constructible sheaves on the affine Grassmannian

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September 24, 2024

Abstract

Let G be a complex reductive group. The spherical Hecke category of G can be presented as the category of $G_{\mathcal{O}}$-equivariant constructible sheaves on the affine Grassmannian Gr_G. This category admits a convolution product, extending the convolution product of equivariant perverse sheaves. In this paper, we upgrade the mentioned convolution product to a left t-exact $\mathbb{E}_3$-monoidal structure in ∞-categories. The construction is intrinsic to the automorphic side. Our main tools are the Beilinson–Drinfeld Grassmannian, Lurie's characterization of $\mathbb{E}_k$-algebras via the topological Ran space, the homotopy theory of stratified spaces and the formalism of correspondences.

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[...] Non c'eravamo accorti
di un buco tra i rampicanti.

E. MONTALE, A pianterreno, *Satura II*

1 Introduction

1.1 Main results

Let G be a complex reductive group, and R a commutative ring of coefficients. The aim of this paper is to provide an extension of the convolution product in the Satake category of equivariant perverse sheaves on the affine Grassmannian ([MV07, §4]) to the spherical Hecke category, and endow this extension with an $\mathbb{E}_3$-algebra structure in ∞-categories. This upgrade is the derived avatar of Mirkovic and Vilonen's commutativity constraint [MV07, §5]. We briefly illustrate the main results of the paper below.

Definition 1.1 (Affine Grassmannian)**.** Let G be a reductive group. The *arc group* $G_{\mathcal{O}}$ (also denoted by $G[\![t]\!]$ or L^+G) is defined as the functor

$$\begin{aligned} \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} &\to \mathrm{Grp} \\ R &\mapsto G(R[\![t]\!]) = \mathrm{Hom}(R[\![t]\!], G). \end{aligned}$$

The *loop group* $G_{\mathcal{K}}$ (also denoted by $G(\!(t)\!)$ or $\mathrm{L}G$) is defined as the ind-representable functor

$$\begin{aligned} \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} &\to \mathrm{Grp} \\ R &\mapsto G((\!(t)\!)) = \mathrm{Hom}(R(\!(t)\!), G). \end{aligned}$$

The *affine Grassmannian* Gr_G is the fpqc quotient stack

$$\mathrm{Gr}_G = \left[G_{\mathcal{K}}/G_{\mathcal{O}}\right].$$

When there is no ambiguity, we usually denote Gr_G by just Gr.

Remark 1.2. The stack Gr is actually ind-representable. As such, it admits an underlying complex-analytic space $\mathrm{Gr}^{\mathrm{an}}$.

Notation 1.3. Let $\mathcal{E}$ be a symmetric monoidal presentable stable ∞-category, and $\mathcal{E}^{\omega}$ its stable subcategory of compact objects. Let $\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L},\otimes}$ be the symmetric monoidal ∞-category of $\mathcal{E}$-linear presentable ∞-categories and left adjoint functors between them; let also $\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{R},\otimes}$ be the symmetric monoidal ∞-category with the same objects but with right adjoint functors as morphisms (see Notation B.50, Remark B.51). Let also $\mathcal{C}\mathrm{at}_{\infty,\mathcal{E}^{\omega}}^{\times}$ be the ∞-category of $\mathcal{E}^{\omega}$-linear small ∞-categories (see Notation B.52) with its cartesian symmetric monoidal structure.

When R is a commutative ring and $\mathcal{E} = \mathrm{Mod}_R$ is the ∞-category of R-modules, we use the notations $\mathcal{P}\mathrm{r}_{R}^{\mathrm{L},\otimes}, \mathcal{P}\mathrm{r}_{R}^{\mathrm{R},\otimes}, \mathcal{C}\mathrm{at}_{\infty,R}$.

Theorem 1.4 (Main result, Theorem 4.13)**.** *Let G be a complex reductive group and $\mathcal{E}$ a symmetric monoidal presentable stable ∞-category. Then there exists an object*

$$\mathrm{Sph}(G;\mathcal{E})^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_3}(\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{R},\otimes})$$

whose underlying ∞-category is

$$\mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}};\mathcal{E}),$$

i.e. the unbounded derived ∞-category of topological $G_{\mathcal{O}}^{\mathrm{an}}$-equivariant constructible sheaves over $\mathrm{Gr}^{\mathrm{an}}$*, with coefficients in $\mathcal{E}$.*

Corollary 1.5 (Corollary 4.14, Remark 4.17)**.** *In the same setting as Theorem 1.4, there exists an object*

$$\mathrm{Sph}(G;\mathcal{E})^{\mathrm{loc.c},\otimes} \in \mathrm{Alg}_{\mathbb{E}_3}(\mathcal{C}\mathrm{at}_{\infty,\mathcal{E}^{\omega}}^{\times})$$

whose underlying ∞-category is the small spherical Hecke category of G (see Definition A.22).

Let R be a discrete ring. For $\mathcal{E} = \mathrm{Mod}_R$, this $\mathbb{E}_3$-structure is left t-exact for the perverse t-structure (and exact if R is a field). It canonically induces a symmetric monoidal structure on the abelian subcategory of equivariant perverse sheaves, coinciding with the classical convolution product of [MV07, §4].

Remark 1.6. Let R be a ring. As explained in Section A.3, the category $\mathrm{Sph}(G;R)^{\mathrm{loc.c}} = \mathrm{Sph}(G;\mathrm{Mod}_R^{\omega})$ has several presentations.

One is the one that we use as definition, namely the ∞-category of $G_{\mathcal{O}}^{\mathrm{an}}$-equivariant constructible sheaves over $\mathrm{Gr}_G^{\mathrm{an}}$ with values in R, with bounded finitely presented stalks.

When $R = \mathbb{C}$, then the Riemann-Hilbert correspondence implies that $\mathrm{Sph}(G;R)^{\mathrm{loc.c}}$ can be presented as the subcategory of $\mathrm{DMod}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ spanned by objects whose underlying D-module is compact (i.e. which become compact after forgetting the equivariant structure), which agrees with [AG15, 12.2.3], see Remark A.25.

If R is finite, profinite or ℓ-adic (i.e. an algebraic extension of $\mathbb{Q}_{\ell}$), then $\mathrm{Sph}(G;R)^{\mathrm{loc.c}}$ can be presented as a category of *étale* sheaves over the algebro-geometric object Gr_G, see Remark A.24.

Another direct corollary of Theorem 1.4 regards the renormalized spherical Hecke category:

Corollary 1.7 (Corollary 4.15)**.** *In the same setting as Corollary 1.5, there is also an object*

$$\mathrm{Sph}(G;\mathcal{E})^{\mathrm{ren},\otimes} \in \mathrm{Alg}_{\mathbb{E}_3}(\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L},\otimes})$$

whose underlying ∞-category is the renormalized spherical ∞-category

$$\mathrm{Sph}(G;\mathcal{E})^{\mathrm{ren}} = \mathrm{Ind}(\mathrm{Sph}(G;\mathcal{E})^{\mathrm{loc.c}})$$

appearing e.g. in [AG15, §12].

We are now going to provide some context and motivation for these results (Section 1.2) and an outline of the paper (Section 1.3).

1.2 Motivation: the Derived Satake Theorem

A classical problem in representation theory is the study of a reductive group G over a local field (e.g. GL_n, SL_n, $\mathbb{P}\mathrm{GL}_n$) and its Langlands dual $\check{G}$ (e.g. $\check{\mathrm{GL}}_n = \mathrm{GL}_n, \check{\mathrm{SL}}_n = \mathbb{P}\mathrm{GL}_n$).

A celebrated result in the study of the Langlands duality is the Satake theorem [Sat63] which, given a reductive group G over $\mathbb{F}_p$, establishes an isomorphism between the $\mathbb{C}$-algebra of complex compactly supported $G(\mathbb{Z}_p)$-biinvariant functions on $G(\mathbb{Q}_p)$, called the *(spherical) Hecke algebra* of G, and the (complexified) Grothendieck ring of finite-dimensional representations of $\check{G}$. Ginzburg [Gin95] and later Mirkovic and Vilonen [MV07, (13.1)] provided a "sheaf theoretic" analogue (actually a categorification) of this theorem, called the *Geometric Satake Equivalence*: here G is a *complex* reductive group, and the statement has the form of an equivalence of symmetric monoidal abelian categories between the category of *equivariant perverse sheaves* $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ and the category of finite dimensional representations of $\check{G}$. The key new object here is the affine Grassmannian Gr_G, whose definition we recalled above (Definition 1.1). This is an infinite dimensional algebro-geometric object with the property that $\mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$. The Grothendieck group of $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ is the analogue of the Hecke algebra appearing in the original Satake theorem, but now for a reductive group over $\mathbb{C}$.[1]

Theorem 1.8 (Geometric Satake Equivalence, [MV07, (1.1)])**.** *Fix a reductive algebraic group G over $\mathbb{C}$, and a discrete commutative ring R, noetherian and of finite global dimension. There exists a symmetric monoidal structure $\star$ on* $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G; R)$*, called* convolution*, and an equivalence of symmetric monoidal abelian categories*

$$(\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G; R), \star) \simeq (\mathrm{Rep}^{\mathrm{fd}}(\check{G}_R, R), \otimes) \tag{1.1}$$

where $\check{G}_R$ is the Langlands dual of G over R [MV07, Beginning of Sec. 12] and $\otimes$ denotes the standard tensor product of finite-dimensional $\check{G}_R$-representations with coefficients in R.

We recall the meaning of this statement, together with various theoretical recollections necessary for this paper, in Appendix A. We refer the reader seeking for a complete survey to [Zhu16] and [BR18].

Theorem 1.9 (Derived Satake Theorem, [BF07, Theorem 5])**.** *Let G be a complex reductive group and k a field of coefficients of characteristic zero. There is a* monoidal *equivalence of triangulated categories*[2]

$$\mathrm{hCons}^{\mathrm{fd}}_{G_\mathcal{O}}(\mathrm{Gr}_G; k) \simeq \mathrm{hPerf}_{\check{G}_k}(\mathrm{Sym}(\check{\mathfrak{g}}_k[-2])) \tag{1.2}$$

where $\check{G}_k$ is the Langlands dual of G over k and $\check{\mathfrak{g}}_k$ is its Lie algebra.

Remark 1.10. Here $\mathrm{Cons}^{\mathrm{fd}}_{G_\mathcal{O}}(\mathrm{Gr}; k)$ is the bounded derived ∞-category of $G_\mathcal{O}$-equivariant constructible sheaves over Gr with coefficients in k and finitely presentable stalks. The category $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k)$ is the heart of a t-structure on $\mathrm{Cons}^{\mathrm{fd}}_{G_\mathcal{O}}(\mathrm{Gr}_G, k)$ (and hence on the homotopy category). Indeed, this t-stucture is inherited from the presentation of the equivariant constructible category à la Bernstein-Lunts, see Remark 4.17. As explained in Remark 4.17, the Geometric Satake Theorem can be formally recovered from the Derived Satake Theorem by passing to the heart, up to a detail: a priori, the induced statement

[1] A closer analogue to the original Satake isomorphism is given by the geometric Satake theorem in mixed characteristic, see [Zhu17].

[2] For the sake of coherence with the rest of the work, we adopt the notation h— in order to refer to "the homotopy category of a stable ∞-category". Of course, in the original paper both sides are defined directly as triangulated categories.

will only be a monoidal equivalence of monoidal abelian categories, and not a symmetric monoidal equivalence.

Both the left and right-hand side, as triangulated categories, carry a symmetric monoidal structure (for the left-hand-side, see [AR23, Section 3.3]; on the right-hand-side, it is the tensor product described in [BF07, 2.7]). However, the equivalence is *not* symmetric (or even braided) monoidal (cf. [AG15, Remark 12.4.3]).

In the following, the notion of $\mathbb{E}_k$-center of an $\mathbb{E}_k$-∞-category we are referring to is [Lur17, Definition 5.3.1.6, Example 5.3.1.13], and generalizes the notion of Drinfeld center.

Theorem 1.11. *There is a monoidal equivalence of ∞-categories*

$$\mathrm{Mod}_{\check{G}_k}(\mathrm{Sym}(\check{\mathfrak{g}}_k[-2])) \simeq \mathrm{Z}_{\mathbb{E}_2}(\mathrm{DRep}(\check{G};k))$$

where $\mathrm{Z}_{\mathbb{E}_2}$ stays for "$\mathbb{E}_2$-center" and $\mathrm{DRep}(\check{G};k)$ is the derived ∞-category of representations seen as an $\mathbb{E}_2$-∞-category by forgetting its $\mathbb{E}_\infty$-monoidal structure $\otimes$ along the map of operads $\mathbb{E}_2 \to \mathbb{E}_\infty$.

One can make this result follow from [AG15, Proposition 12.4.2] combined with work of Ben-Zvi, Francis, Nadler and Preygel on centers [BZFN10], [BZNP17]: see [BZSV23, (17.1.2)] for a discussion on this matter. We stress that also this latter equivalence is only monoidal and not symmetric monoidal.

Note that the left-hand-side of Theorem 1.11 is the Ind-completion of the $\mathrm{Perf}_{\check{G}_k}(\mathrm{Sym}(\check{\mathfrak{g}}_k[-2]))$, i.e. of the ∞-category whose homotopy category appears in the right-hand-side of Theorem 1.9.

Remark 1.12. In recent work appeared after during the revision of the present paper, Campbell and Raskin proved the following result [CR23, Theorem 6.6.1]. Assume $k = \mathbb{C}$ (or more generally, that G is a reductive group over a field k of characteristic zero). Then there is an equivalence of *factorizable monoidal* ∞-categories

$$\mathrm{Sph}(G;k)^{\mathrm{ren}} \simeq \mathrm{Sph}(\check{G};k)^{\mathrm{spec}}, \tag{1.3}$$

where the right-hand-side is a suitable renormalization of the right-hand-side of Theorem 1.11.

This is the correct statement of a result announced by Gaitsgory and Lurie several years ago, and originally conjectured by Drinfeld (see e.g. [AG15, footnote 19]).

Remark 1.13 (Role of the present paper)**.** The right-hand side of (1.3) has a natural $\mathbb{E}_3$-monoidal structure coming from the fact that it is a renormalization of the $\mathbb{E}_2$-center of $\mathrm{DRep}(\check{G}_k;k)$.

What we do in this paper is rather to endow the left-hand-side of (1.3) (which is also the left-hand-side of Theorem 1.9) with an $\mathbb{E}_3$-monoidal structure: see Corollary 1.7. Our construction is intrinsic to the automorphic side, i.e. it does not use (1.3). In contrast to [CR23], we need to assume that G is defined over $\mathbb{C}$ (and not over an arbitrary field of characteristic zero): the reason for this will be evident from Remark 1.14. However, this also gives us freedom in the choice of coefficients, see Remark 1.6.

Remark 1.14. Note that an $\mathbb{E}_3$-monoidal structure is a slightly stronger notion than being factorizable monoidal. More precisely, by the Dunn–Lurie Additivity Theorem [Lur17, Theorem 5.1.2.2] an $\mathbb{E}_3$-monoidal structure decomposes into an $\mathbb{E}_1$-monoidal and an $\mathbb{E}_2$-monoidal structure on the same ∞-category, which distribute with one another (a higher avatar of the Eckmann-Hilton principle). An $\mathbb{E}_1$-monoidal structure is just a monoidal structure. An $\mathbb{E}_2$-structure is the same as a braided monoidal structure. In the expression "factorizable monoidal", the "monoidal" part corresponds to the $\mathbb{E}_1$-monoidal structure mentioned above. The "factorizable" part is related to the mentioned $\mathbb{E}_2$-monoidal structure as follows: the existence of an $\mathbb{E}_2$-monoidal structure implies the existence of a structure of factorizable category, but not vice-versa: the gap lies precisely in a notion of "local constancy": a "locally constant" factorizable structure induces an $\mathbb{E}_2$-monoidal structure. Formally, this is exactly the constructibility property appearing in Recall 1.17 below.

Remark 1.15. Our result is somehow in the same spirit of the Tannakian reconstruction principle used in the proof of the Geometric Satake Theorem (Theorem 1.8), where the existence of a symmetric monoidal structure on $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr};R)$ is a part of the structure needed to apply the reconstruction machinery, and only a posteriori it is interpreted as corresponding to the tensor product in $\mathrm{Rep}^{\mathrm{fd}}(\check{G}_R;R)$.

Remark 1.16. In the light of Remark 1.13, it is natural to expect an $\mathbb{E}_3$-monoidal equivalence between the two sides of (1.3), refining the factorizable monoidal equivalence proved by Campbell and Raskin.

1.3 Outline of the work

Let G be complex reductive group. Recall Definition 1.1.

In *Section 2* we recall that, for any choice of a smooth complex curve X, there exists a presheaf $\mathrm{Ran}(X)$, defined as the colimit in $\mathrm{PSh}(\mathrm{Sch}_{\mathbb{C}})$ of the diagram

$$\begin{aligned} \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}} &\to \mathrm{IndSch}_{\mathbb{C}} \\ I &\mapsto X^I \end{aligned}$$

where $\mathrm{Fin}_{\geq 1,\mathrm{surj}}$ is the category of nonempty finite sets with surjections between them, and the diagram sends a surjection $I \twoheadrightarrow J$ to the corresponding diagonal $X^J \to X^I$.

We also recall that there exists a presheaf $\mathrm{Gr}_{\mathrm{Ran}}$ called the Ran Grassmannian, living over $\mathrm{Ran}(X)$ and such that for any choice of $x_0 \in X(\mathbb{C})$ the singleton map $\{x_0\}: \mathrm{Spec}\,\mathbb{C} \to \mathrm{Ran}(X)$ induces a pullback square

$$\begin{tikzcd} \mathrm{Gr} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathrm{Ran}} \arrow[d] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r, "\{x_0\}"] & \mathrm{Ran}(X) \end{tikzcd} . \tag{1.4}$$

Such a presheaf arises as the colimit in $I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}$ of the *Beilinson-Drinfeld Grassmannians*

$$\mathrm{Gr}_I = \{x_I \in X^I, \mathcal{F} \in \mathrm{Bun}_G(X), \alpha : \mathcal{F}|_{X\setminus x_I} \xrightarrow{\sim} \mathcal{T}|_{X\setminus x_I}\}. \tag{1.5}$$

Here $\mathcal{T}$ is the trivial G-bundle, see Definition 2.3. The so-called "moduli interpretation" of the affine Grassmannian (Proposition A.10) implies the existence of the pullback diagram (1.4).

The point of view involving $\mathrm{Ran}(X)$ is already used for instance in [Zhu16], [GL] and [Tao20]. We consider an equivariant version of this phenomenon: first of all, recall that Gr admits an action of $G_{\mathcal{O}}$ by left multiplication (see Recall A.1). We define

$$\mathrm{Hck}$$

as an ind-pro-stack whose realization is the fpqc quotient

$$[G_{\mathcal{O}}\backslash \mathrm{Gr}],$$

see Definition A.11, Definition A.17. Just like in the case of $\mathrm{Gr}_{\mathrm{Ran}}$ in (1.4), there is an object $\mathrm{Hck}_{\mathrm{Ran}}$ fitting in a pullback square

$$\begin{tikzcd} \mathrm{Hck} \arrow[r] \arrow[d] & \mathrm{Hck}_{\mathrm{Ran}} \arrow[d] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r, "\{x_0\}"] & \mathrm{Ran}(X) \end{tikzcd} . \tag{1.6}$$

Here Hck and $\mathrm{Hck}_{\mathrm{Ran}}$ are objects of the category $\widehat{\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}}$, a suitable pro-completion and free cocompletion of the category of stratified stacks (see Definition B.5): this is the right environment for our constructions since we want to keep track of the fact that our objects can be approximated by finite-dimensional objects at various levels. In particular, this will allow later to define categories of constructible sheaves in the right way (e.g. as a colimit along the coweight filtration on the affine Grassmannian). Also, all objects in sight have natural stratifications, ultimately coming from the fact that the classical stratification in Schubert cells of the affine Grassmannian (Recall A.1) can be extended to the Beilinson-Drinfeld Grassmannian. Stratifications are crucial in this work because of the stratified-homotopy-invariance features enjoyed by the procedure of taking constructible sheaves with respect to a given stratification (as opposed to *some* stratification, cf. Recall B.7, Recall B.37 for the distinction). For this reason, we always work with *stratified* stacks and variations thereof.

There exists a span

$$\begin{tikzcd} & \mathrm{Hck}_2 \arrow[dl, "\overline{p}"'] \arrow[dr, "\overline{m}"] & \\ \mathrm{Hck} \times \mathrm{Hck} & & \mathrm{Hck} \end{tikzcd} \tag{1.7}$$

which we call "convolution diagram". This span admits a "Ran version" of the form

$$\begin{tikzcd} & \mathrm{Hck}_{\mathrm{Ran},2} \arrow[dl, "\overline{p}_{\mathrm{Ran}}"'] \arrow[dr, "\overline{m}_{\mathrm{Ran}}"] & \\ \mathrm{Hck}_{\mathrm{Ran}} \times \mathrm{Hck}_{\mathrm{Ran}} & & \mathrm{Hck}_{\mathrm{Ran}} \end{tikzcd} . \tag{1.8}$$

The main reason for the existence of this Ran version of the convolution diagram is the fact that the so-called *convolution Grassmannian* admits a Ran version (see Section 2.3), allowing to define the upper vertex of this diagram. From this, we prove that $\mathrm{Hck}_{\mathrm{Ran}}$ carries a nonunital $\mathbb{E}_1$-algebra structure in correspondences, i.e. there exists an object

$$\mathrm{Hck}^{\otimes}_{\mathrm{Ran}} \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_1}(\mathrm{Corr}(\widehat{\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}})^{\times}) \tag{1.9}$$

whose underlying object is $\mathrm{Hck}_{\mathrm{Ran}}$. The target it the 1-category of correspondences on $\widehat{\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}}$ in the sense of [GR17], [Man22], together with the monoidal structure induced by the Cartesian monoidal structure on $\widehat{\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}}$. Associativity ($\mathbb{E}_1$) here is a consequence of the existence of n-fold convolution Grassmannians, see Section 2.3.

The reason we are interested in extending (1.7) to (1.8) is the following. Informally, push-and-pull of perverse (or, in our case, constructible) sheaves along (1.7) induces the convolution product of [MV07, §4], which corresponds to an $\mathbb{E}_1$-algebra structure on the chosen category of sheaves over Hck. As we will see, the existence of the Ran version (1.8) allows to add an additional "$\mathbb{E}_2$-direction", corresponding to the commutativity constraint appearing in [MV07, §5] (which itself uses the existence of the Beilinson-Drinfeld Grassmannian). However, in order to carry out the latter step, we choose to pass to the complex-analytic world: this allows to use Lurie's characterization of factorization algebras [Lur17, Theorem 5.5.4.10]. Note that this latter ingredient is absent in [MV07], where the properties of perverse sheaves allow to establish the commutativity constraint "on the nose".

More precisely, in §3.1 we apply the stratified analytification functor $(-)^{\mathrm{an}}$ of Corollary B.46 to the objects constructed in the previous section, with the goal of deducing the existence of an $\mathbb{E}_3$-algebra structure on the category of *topological* constructible sheaves over the resulting complex-analytic objects.[3] The functor appearing in Corollary B.46 is an upgraded version of Raynaud's original analytification functor [Ray71], which takes into account stratifications and the formation of pro-objects and free colimits. In particular, the analytification of the object

$$\mathrm{Hck}_{\mathrm{Ran}} \in \widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}}$$

belongs to the category $\widehat{\mathrm{StrTStk}}$, which arises in a totally similar way to $\widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}}$ as a pro-completion and free cocompletion of the category of *topological* stratified stacks, see Definition B.25.

The algebra structure in correspondences (1.9) is transferred via this procedure to an object

$$\mathrm{Hck}_{\mathrm{Ran}}^{\mathrm{an},\otimes} \in \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{nu}}(\mathrm{Corr}(\widehat{\mathrm{StrTStk}})). \tag{1.10}$$

Additionally, we are able to build a factorization algebra structure on $\mathrm{Hck}_{\mathrm{Ran}}^{\mathrm{an}}$, in the sense that there is a map of operads (Theorem 3.13)

$$\mathrm{Hck}^{\mathrm{fact}} : \mathrm{Fact}(\mathbb{R}^2) \to \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{nu}}(\mathrm{Corr}(\widehat{\mathrm{StrTStk}}))^{\times}. \tag{1.11}$$

Recall 1.17. Here $\mathrm{Fact}(\mathbb{R}^2)$ is a certain operad whose algebras correspond include nonunital $\mathbb{E}_2$-algebras under Lurie's criterion [Lur17, Theorem 5.5.4.10]; the needed conditions in order to obtain a nonunital $\mathbb{E}_2$-algebra are essentially three (see [Lur17, Theorem 5.5.4.10] for the meaning of the words in italic):

- *factorizability*, corresponding to the factorization property of the Beilinson-Drinfeld Grassmannian Proposition 2.11;
- *constructibility* up to stratified homotopy, corresponding to the fact that the analytification of $\mathrm{Gr}_{\mathrm{Ran}}$ is homotopy invariant under dilation of coordinates of $\mathbb{A}_{\mathbb{C}}^1$ ([NP24b]);
- codescent with respect to the euclidean topology of $\mathbb{R}^2$ (i.e. wrt the complex-analytic topology on $\mathbb{A}_{\mathbb{C}}^1$). This condition is "almost" satisfied: the defect is due to the presence of pro-objects in the story, an issue which is completely solved after taking constructible sheaves: see Remark 3.11.

Let now $\mathcal{E}$ be a presentable stable stable ∞-category, which will be our category of coefficients. We want to give a meaning to the expression

$$\mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran}}^{\mathrm{an}}; \mathcal{E}).$$

[3]"Topological" here is to be understood as opposed to "algebraic". It is actually an interesting question whether an $\mathbb{E}_3$-structure can be established on a category of constructible étale sheaves over Hck without using the theory of topological factorization algebras. One should however keep in mind that this would only make sense for finite or ℓ-adic coefficients, since we would be looking at étale sheaves. In this case, the category of algebraic constructible sheaves on Hck and the category of topological constructible sheaves on $\mathrm{Hck}^{\mathrm{an}}$ coincide (see (A.8)), so it is really a matter of techniques used, not of the result. For other coefficients, the topological model is less replaceable: in the case of complex coefficients, for instance, one looks at $\mathrm{Cons}(\mathrm{Hck}^{\mathrm{an}}; \mathbb{C})$, which corresponds to DMod(Hck), see Remark 1.6.

This point of view is also underlined in [MV07, end of page 2].

The idea is to define this by colimits and limits from the finite-dimensional terms involved in the construction of $\mathrm{Hck}_{\mathrm{Ran}}$. To this end, we ideally would like to build a symmetric monoidal functor

$$\mathrm{Cons}(-;\mathcal{E}) : \mathrm{Corr}(\widehat{\mathrm{StrTStk}}) \to \widehat{\mathcal{C}\mathrm{at}_\infty},$$

the target being the category of large categories. Such a functor would transfer all the desired properties in one go. However, the functorialities needed to build such a functor are somehow only understood for a strict subcategory of $\widehat{\mathrm{StrTStk}}$, built out of what are known as *conically* stratified spaces (Definition B.14): these are spaces with some mild topological conditions and the crucial requirement that the stratification satisfies a certain equisingularity condition (called the conical stratification condition). Whitney stratifications are a standard example of such, and indeed we use the fact that the analytification of the Beilinson-Drinfeld Grassmannian (1.5) is Whitney (due to David Nadler in his PhD thesis) to prove that it is conical.

This construction is performed in Appendix B. More precisely, there is a functor

$$\begin{aligned} \mathrm{StrTop}_{\mathrm{con}} &\to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}} \\ (X,s) &\mapsto \mathrm{Cons}(X,s;\mathcal{E}) \end{aligned} \tag{1.12}$$

where $\mathrm{StrTop}_{\mathrm{con}}$ is the category of conically stratified spaces, $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$ is the category of $\mathcal{E}$-linear presentable stable categories, and $\mathrm{Cons}(X,s;\mathcal{E})$ is the category of $\mathcal{E}$-valued constructible sheaves on (X,s). This functor also carries a symmetric monoidal structure, see Corollary B.56. The existence of (1.12) relies on the formalism of exit paths as developed in [Lur17, Appendix A] and later in [PT22].

This functor can then be extended to $\widehat{\mathrm{StrTStk}}_{\mathrm{con}}$, which is the subcategory of $\widehat{\mathrm{StrTStk}}$ built out of $\mathrm{StrTop}_{\mathrm{con}}$ instead of StrTop (Definition B.25). The resulting functor further extends to a category of correspondences via the formalism of [GR17, Part III] and [Man22]. Again, the category of correspondences appearing in the result is a strict subcategory of $\mathrm{Corr}(\widehat{\mathrm{StrTStk}}_{\mathrm{con}})$, in that it has less morphisms: we are looking at

$$\mathrm{Corr}(\widehat{\mathrm{StrTStk}}_{\mathrm{con}})_{\mathsf{all},\mathsf{subm}},$$

whose morphisms are spans

$$\begin{tikzcd} & \mathcal{Y} \arrow[dl, "h"'] \arrow[dr, "v"] & \\ \mathcal{X} & & \mathcal{Z} \end{tikzcd}$$

of morphisms in $\widehat{\mathrm{StrTStk}}_{\mathrm{con}}$ where the arrow h belongs to a certain class of "smooth submersions" (Definition B.28). This restriction is necessary in order to have the necessary base change properties (also known as Beck-Chevalley conditions) for the extension to correspondences. The final output is a symmetric monoidal functor

$$\mathrm{Corr}(\widehat{\mathrm{StrTStk}}_{\mathrm{con}})^{\times}_{\mathsf{all},\mathsf{subm}} \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}},$$

see Theorem B.66. Here $\mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}}$ is the symmetric monoidal ∞-category of $\mathcal{E}$-linear presentable ∞-categories with right adjoint functors (Remark B.51).

In Section 3.2 we prove that the analytification of the $\mathrm{Hck}_{\mathrm{Ran}}$ and its variations do belong to $\widehat{\mathrm{StrTStk}}_{\mathrm{con}}$, and that the analytification of the left leg in (1.8) belongs to subm. This implies that the functor $\mathrm{Hck}^{\mathrm{fact}}$ from (1.11) factors via the subcategory $\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_1}(\mathrm{Corr}(\widehat{\mathrm{StrTStk}}_{\mathrm{con}})^{\times}_{\mathsf{all},\mathsf{subm}})$, as desired.

In *Section 4* we study the categories of sheaves $\mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran}};\mathcal{E})$ and $\mathrm{Cons}(\mathrm{Hck};\mathcal{E})$. By composing $\mathrm{Hck}^{\mathrm{fact}}$ with $\mathrm{Cons}(-;\mathcal{E})$, we can apply Lurie's criterion [Lur17, Theorem 5.5.4.10] mentioned above (whose conditions are now completely satisfied) and obtain a map of operads $\mathbb{E}_2^{\mathrm{nu}} \to (\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_1}(\mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}}))$ whose underlying category is $\mathrm{Cons}(\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran}};\mathcal{E})$: see Construction 4.7. In other words, there are two nonunital monoidal structures on $\mathrm{Cons}(\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran}};\mathcal{E})$, one of which is braided, which distribute one with the other.

Section 4.2 is devoted to transfer these two structures from

$$\mathrm{Cons}(\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran}};\mathcal{E})$$

to

$$\mathrm{Cons}(\mathrm{Hck}^{\mathrm{an}};\mathcal{E}),$$

which is done by specializing to any chosen point $x_0 \in \mathbb{A}^1_{\mathbb{C}}$ (cf. (1.6)). This procedure amounts to some base change verifications, enabled by the fact that the right leg of the convolution diagram (1.8) is ind-proper and the left leg is pro-smooth.

We prove that, after this specialization, both the $\mathbb{E}_1^{\mathrm{nu}}$- and $\mathbb{E}_2^{\mathrm{nu}}$-structures gain units, and therefore the Dunn-Lurie Additivity Theorem [Lur17, Theorem 5.1.2.2] can be applied, thus combining the two algebra structures into an $\mathbb{E}_3$-structure on $\mathrm{Cons}(\mathrm{Hck}^{\mathrm{an}};\mathcal{E})$.

The latter category is precisely $\mathrm{Sph}(G;\mathcal{E})$, i.e. the ∞-category of $G_{\mathcal{O}}^{\mathrm{an}}$-equivariant constructible sheaves on $\mathrm{Gr}^{\mathrm{an}}$ with coefficients in $\mathcal{E}$. We obtain obtain Theorem 1.4. When $\mathcal{E} = \mathrm{Mod}_R$ for a discrete ring R, this structure is left t-exact (t-exact if R is a field) and therefore restricts canonically to a symmetric monoidal structure on perverse sheaves, which is the classical one used in [MV07]: see Remark 4.17.

Remark 1.18. Note that $\mathrm{Gr}^{\mathrm{an}}$ is homotopy equivalent to the loop space $\Omega G^{\mathrm{an}} \simeq \Omega^2\mathrm{B}(G^{\mathrm{an}})$, which carries a standard $\mathbb{E}_2$-algebra structure in spaces. This is not sufficient to derive the $\mathbb{E}_2$-algebra structure on the spherical category though (at least not with our techniques), because the functor taking constructible sheaves is stratified homotopy invariant but not homotopy invariant (for example, constructible sheaves on $\mathbb{R}$ and on the point are not the same). For recent developments on the loop space perspective, see [CN18, CN24], which also take stratifications into account.

The application of Lurie's [Lur17, Theorem 5.5.4.10] to the affine Grassmannian also appears in [HY19], though in that paper the authors are interested in a purely topological problem and do not take constructible sheaves. Up to our knowledge, the formalism of constructible sheaves via exit paths and exodromy has never been applied to the study of the affine Grassmannian and the spherical Hecke category.

Acknowledgments

This paper constitutes the core chapter of my thesis as a graduate student at Scuola Normale Superiore di Pisa and Université de Strasbourg (2018-2022) under the supervision of Mauro Porta and Gabriele Vezzosi.

During the last phase of revision I was supported by the ERC Starting Grant *Foundations of motivic real K-theory* (2020-2025) held by Yonatan Harpaz.

I am indebted to Pramod Achar, Pierre Baumann, Dario Beraldo, Justin Campbell, Robert Cass, Ivan Di Liberti, Andrea Gagna, Dennis Gaitsgory, Jeremy Hahn, Yonatan Harpaz, Vasily Krylov, Jacob Lurie, Andrea Maffei, Lucas Mann, David Nadler, Cédric Pépin, Michele Pernice, Sam Raskin, Simon Riche, James Tao, Angelo Vistoli and Allen Yuan for their suggestions and explanations.

I especially thank Peter Haine, Mark Macerato, Emanuele Pavia, Morena Porzio and Marco Volpe for the extensive and fruitful discussions carried out with them during various phases of the work, and Roman Bezrukavnikov for hosting me at MIT during April and May 2022.

2 Convolution over the Ran space

Throughout this whole work, G will be a complex reductive group and X a complex smooth curve.

Notation 2.1. When defining a presheaf over the category of complex affine schemes, we will usually drop the dependance on $\operatorname{Spec} R$ when it does not cause confusion. A point $x \in X(R)$ will just be denoted by $x \in X$, and its graph in $X \times \operatorname{Spec} R$ by Γ_x.

Two R-points of X will be declared "equal" if they coincide as maps $\operatorname{Spec} R \to X$, "distinct" if they do not coincide (but their graphs may intersect nontrivially inside X_R), and "disjoint" if their graphs do not intersect.

Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$ and $x_I \in X^I(R)$. Let $\mathrm{pr}_i : X^I \to X$ be the projection on the i-th coordinate and denote by x_i the composite $\mathrm{pr}_i \circ x_I$. We denote by $\Gamma_{x_i} \subset X_R$ the closed subscheme given by the graph of x_i. We denote by Γ_{x_I} the closed (possibly nonreduced) subscheme of X_R corresponding to the composition

$$\operatorname{Spec} R \to X^I \to \mathrm{Sym}_X^{|I|} \simeq \mathrm{Hilb}_X^{|I|}$$

where the last isomorphisms comes from the fact that X is a curve. This subscheme is supported at the union of the graphs Γ_{x_i}. For instance, if $R = \mathbb{C}$, $I = \{1,2\}$ and $x_1 = x_2$ is a closed point of X, then Γ_{x_I} is the only closed subscheme supported at the point and of length 2.

The definition of the affine and punctured formal neighbours of a closed subscheme Γ of a scheme S, denoted by $\widetilde{S}_\Gamma$ and $\mathring{S}_\Gamma$ respectively, is recalled in Recall A.4. When there is no risk of confusion about the ambient scheme, we will also denote them by $\widetilde{\Gamma}$ and $\mathring{\Gamma}$ respectively.

A G-torsor $\mathcal{F} \in \mathrm{Bun}_G(X_R)$ will just be denoted by $\mathcal{F} \in \mathrm{Bun}_G(X)$, and the trivial G-torsor over a scheme S will be denoted by $\mathcal{T}_S$.

Finally, the symbol $\mathrm{PSh}(-)$ denotes groupoid-valued presheaves, whereas $\mathcal{P}(-)$ denotes space-valued presheaves.

2.1 The Beilinson–Drinfeld setting

The following definitions also appear, in various forms, in [Rei12], [Ric14] and [CvdHS22], and are natural generalizations of the characterizations recalled in Proposition A.10.

Definition 2.2. Let I, I_1, I_2 be nonempty finite sets. We recall the following definitions.

- The Beilinson–Drinfeld arc group

$$G_{\mathcal{O},I} = \{x_I \in X^I, g \in G(\widetilde{X}_{x_I})\}.$$

Note that $G(\widetilde{X}_{x_I}) \simeq \mathrm{Aut}(\mathcal{T}_{\widetilde{X}_{x_I}})$.

- The Beilinson–Drinfeld loop group

$$G_{\mathcal{K},I} = \{x_I \in X^I, \mathcal{F} \in \mathrm{Bun}_G(X), \alpha \text{ trivialization of } \mathcal{F} \text{ on } X \setminus x_I, \mu \text{ trivialization of } \mathcal{F} \text{ on } \widetilde{X}_{x_I}\}$$

with its "decoupled version"

$$G_{\mathcal{K},I_1,I_2} = \{x_{I_1} \in X^{I_1}, x_{I_2} \in X^{I_2}, \mathcal{F} \in \mathrm{Bun}_G(X),$$
$$\alpha \text{ trivialization of } \mathcal{F} \text{ on } X \setminus x_{I_1}, \mu \text{ trivialization of } \mathcal{F} \text{ on } \widetilde{X}_{x_{I_2}}\},$$

Definition 2.3. The Beilinson–Drinfeld Grassmannian is defined as

$$\mathrm{Gr}_I = \{x_I \in X^I, \mathcal{F} \in \mathrm{Bun}_G(X), \alpha \text{ trivialization of } \mathcal{F} \text{ on } X \setminus x_I\}.$$

Remark 2.4. The objects $G_{\mathcal{K},I}, G_{\mathcal{K},I_1,I_2}, \mathrm{Gr}_I$ are ind-schemes over X^I by [Zhu16, Theorem 3.1.3, Proposition 3.1.9 and variations thereof]. The object $G_{\mathcal{O},I}$ is representable [Zhu16, Proposition 3.1.6], and has the structure of an infinite-dimensional group scheme relative to X^I. The object Gr_I is often denoted by $\mathrm{Gr}_{G,X^I}, \mathrm{Gr}_{G,I}, \mathrm{Gr}_{X^I}$. The notations $G_{\mathcal{O},X}, G_{\mathcal{K},X}, \mathrm{Gr}_X$, respectively for $G_{\mathcal{O},\{1\}}, G_{\mathcal{K},\{1\}}, \mathrm{Gr}_{\{1\}}$, are also common and we will use them often.

The group scheme $G_{\mathcal{O},I}$ acts on $G_{\mathcal{K},I}$ relatively to X^I by modification of μ, and there is an equivalence

$$\mathrm{Gr}_I \simeq G_{\mathcal{K},I}/G_{\mathcal{O},I}$$

where the right-hand-side is the fpqc quotient relative to X^I. Analogously, there is an action of $G_{\mathcal{O},I_2}$ on $G_{\mathcal{K},I_1,I_2}$ relative to X^{I_2}, and the quotient is $\mathrm{Gr}_{I_1} \times X^{I_2}$.

Notation 2.5. We denote Let $x : \mathrm{Spec}\,\mathbb{C} \to X$ be a closed point. We denote by

$$\mathrm{Gr}_x = \mathrm{Gr}_X \times_{X,x} \mathrm{Spec}\,\mathbb{C}$$

and similarly $G_{\mathcal{K},x}, G_{\mathcal{O},x}$.

Proposition 2.6 (Translational invariance)**.** *Let $X = \mathbb{A}^1_{\mathbb{C}}$. Then any choice of a closed point $x \in \mathbb{A}^1_{\mathbb{C}}$ induces splittings*

$$\mathrm{Gr}_{\mathbb{A}^1_{\mathbb{C}}} \simeq \mathrm{Gr}_x \times \mathbb{A}^1_{\mathbb{C}}$$

$$G_{\mathcal{O},\mathbb{A}^1_{\mathbb{C}}} \simeq G_{\mathcal{O},x} \times \mathbb{A}^1_{\mathbb{C}}.$$

Proof. The case of Gr is proven as follows. The definition of Gr_X is functorial in X, and hence the translation action on $\mathbb{A}^1_{\mathbb{C}}$ lifts to $\mathrm{Gr}_{\mathbb{A}^1}$ as a map

$$\mathrm{Gr}_{\mathbb{A}^1_{\mathbb{C}}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \mathrm{Gr}_{\mathbb{A}^1_{\mathbb{C}}}.$$

The choice of any point x induces a map

$$\mathrm{Gr}_x \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \mathrm{Gr}_{\mathbb{A}^1_{\mathbb{C}}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to \mathrm{Gr}_{\mathbb{A}^1_{\mathbb{C}}}$$

which provides the splitting.

The case of $G_{\mathcal{O}}$ is straightforward from the definition. □

Warning 2.7. Such splittings only hold for $I = \{1\}$.

Definition 2.8. We denote by $\mathrm{Fin}_{\geq 1,\mathrm{surj}}$ the category of nonempty finite sets and surjections between them.

Remark 2.9. Fix $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. The group $G_{\mathcal{O},I}$ acts on Gr_I over X^I as follows:

$$(x_I, g).(x_I, \mathcal{F}, \alpha) := (x_I, \mathcal{F}, g|_{\mathring{X}_{x_I}} \circ \alpha|_{\mathring{X}_{x_I}}).$$

This is definition is well-posed thanks to the Beuville-Laszlo theorem (cf. Construction A.7), which implies that the datum of a trivialization on the punctured affine formal neighbourhood is equivalent to one on the complement of the point. We will implicitly use this argument in the rest of the paper while writing similar expressions.

Suppose $I = I_1 \sqcup I_2$. The relative group scheme $G_{\mathcal{O},I}$ acts on $G_{\mathcal{K},I_1,I_2}$ relatively over X^I again by modification of α. Note that α is a trivialization away from x_{I_1}, and we are modifying it at all points of x_I (not just those of x_{I_1}).

Let now I_1, I_2, I_3 be nonempty finite sets. The relative group scheme $G_{\mathcal{O},I_2}$ acts on $G_{\mathcal{K},I_1,I_2} \times_{X^{I_2}} G_{\mathcal{K},I_2,I_3}$ by simultaneous modification of μ in the first component and α in the second one, respectively over $\widetilde{X}_{\Gamma_{x_{I_2}}}$ and $\mathring{X}_{\Gamma_{x_{I_2}}}$. The same relative group scheme acts on $G_{\mathcal{K},I_1,I_2} \times_{X^{I_2}} \mathrm{Gr}_{I_2}$ in a similar way.

Construction 2.10. Let $I, J \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$ and $[\phi : I \twoheadrightarrow J]$ a J-partition of I, i.e. the equivalence class of a surjection $\phi : I \twoheadrightarrow J$ modulo autobijections of J. Following [Nad05, §4.2] and [CvdHS22, (4.2)], let

$$X^\phi = \{x_I = (x_1, \dots, x_{|I|}) \in X^I \mid \phi(i) = \phi(i') \Rightarrow x_i = x_{i'},$$

$$\phi(i) \neq \phi(i') \Rightarrow \Gamma_{x_i} \cap \Gamma_{x_{i'}} = \varnothing, i, i' \in I\} \subset X^I.$$

This partition of X^I forms a stratification which is called the *incidence stratification* of X^I ([Nad05, §4.2]).

Proposition 2.11 (Factorization property)**.** *With the above notation, there is an isomorphism*

$$\mathrm{Gr}_I \times_{X^I} X^\phi \simeq \big(\prod_J \mathrm{Gr}_X\big) \times_{X^I} X^\phi$$

where the map $\prod_J \mathrm{Gr}_X \to X^I$ is induced by the diagonal $X^J \to X^I$ associated to ϕ. Note that the right-hand-side is also isomorphic to $\big(\prod_J \mathrm{Gr}_X\big) \times_{X^J} X^{\mathrm{id}}$, id *being the partition induced by the identity of J and $X^{\mathrm{id}} \subset X^J$ being the associated stratum, i.e. the open stratum of pairwise distinct coordinates in X^J.*

Proof. See [Nad05, Proposition 4.2.1] or [CvdHS22, Proposition 4.6] (which refers directly to [Zhu16, Proposition 3.1.13]). To be precise, the proof in [Zhu16] is performed for $X = \mathbb{A}^1_{\mathbb{C}}$ (see Corollary 2.45), but it is literally the same in the general case. □

Proposition 2.12. *With the above notations, there is an isomorphism*

$$G_{\mathcal{O},I} \times_{X^I} X^\phi \simeq \prod_J G_{\mathcal{O},X} \times_{X^I} X^\phi$$

and the right-hand side is in turn isomorphic to $\prod_J G_{\mathcal{O},X} \times_{X^J} X^{\mathrm{id}}$ as above.

Proof. Straightforward from the definition. □

Remark 2.13. Under the identifications of Proposition 2.11, we note the following. Let $x \in X(\mathbb{C})$. Then we can perform pullbacks along $\operatorname{Spec}\mathbb{C} \xhookrightarrow{(x,\dots,x)} X \hookrightarrow X^I$ and obtain isomorphisms

$$\begin{gathered}\operatorname{Spec}\mathbb{C} \times_{X^I} \mathrm{Gr}_I \simeq \mathrm{Gr}\\ \operatorname{Spec}\mathbb{C} \times_{X^I} G_{\mathcal{O},I} \simeq G_{\mathcal{O}}\\ \operatorname{Spec}\mathbb{C} \times_{X^I} G_{\mathcal{K},I} \simeq G_{\mathcal{K}}\end{gathered}$$

and the actions appearing in Remark 2.9 become the ones from Recall A.1 and Construction A.9.

Remark 2.14. The stratification in Schubert cells of Gr (Recall A.1) naturally induces a stratification on Gr_X with the same stratifying poset $\mathbb{X}_\bullet(T)^+$, as showed in [Zhu16, (3.1.11)]. If $(x, \mathcal{F}, \alpha) \in \mathrm{Gr}_X(\mathbb{C})$, we denote by

$$\operatorname{Inv}_x(\mathcal{F}, \alpha) \in \mathbb{X}_\bullet(T)^+$$

the associated coweight (we will often abbreviate this by $\operatorname{Inv}_x(\alpha)$). We also denote the stratum with coweight μ by

$$\mathrm{Gr}_{X,\mu},$$

and set

$$\mathrm{Gr}_{X,\leq\mu} = \bigcup_{\nu\leq\mu} \mathrm{Gr}_{X,\nu}.$$

Remark 2.15. The notion of $\operatorname{Inv}_x(\alpha)$ admits the following generalization. Let $x \in X(\mathbb{C}), \mathcal{F}_0, \mathcal{F}_1 \in \mathrm{Bun}_G(X)(\mathbb{C})$, and let $\eta : \mathcal{F}_1|_{X\setminus x} \simeq \mathcal{F}_0|_{X\setminus x}$. Fix a trivialization λ of $\mathcal{F}_0$ on the formal neighbourhood of x. Such a trivialization always exists because all G-torsors are trivial on $\operatorname{Spec}\mathbb{C}[\![t]\!]$. Then one can compute

$$\operatorname{Inv}_x(\lambda|_{\mathring{X}_x} \circ \eta|_{\mathring{X}_x}) \in \mathbb{X}_\bullet(T)^+$$

and check, by uniqueness of the Cartan decomposition, that this coweight is independent of the choice of λ. We denote it by

$$\operatorname{Inv}_x(\eta).$$

Recall 2.16. There is a well-defined stratification on Gr_I whose explicit description is provided in [Nad05, §4.2] or also [CvdHS22, Definition 4.18]. We recall it here, just to fix notations for the generalization to the convolution Grassmannian. The indexing poset of the stratification is

$$\{[\phi : I \twoheadrightarrow J] \text{ partition of } I, \mu_J = (\mu_1, \dots, \mu_{|J|}) \in (\mathbb{X}_\bullet(T)^+)^J\}$$

where the order relation is given by:

$$(\phi, \mu_J) \leq (\phi', \mu'_{J'})$$

if and only if there exists a refinement $[\psi : J' \twoheadrightarrow J]$ such that for every $h \in J$

$$\mu_h \leq \sum_{h'\in J', \psi(h')=h} \mu'_{h'}.$$

The stratification is then defined by setting

$$\mathrm{Gr}_{I,\phi,\mu_J} = \prod_{h\in J} \mathrm{Gr}_{X,\mu_h} \times_{X^J} X^\phi \hookrightarrow \mathrm{Gr}_I$$

where the embedding is induced by Proposition 2.11.

For $[\phi : I \twoheadrightarrow J]$ partition of I, $\mu_J \in (\mathbb{X}_\bullet(T)^+)^J$, we consider the Zariski closure

$$\overline{\mathrm{Gr}_{I,\phi,\mu_J}}.$$

By [CvdHS22, Lemma 4.20], this is a union of strata.

Let $(\phi, \mu_J) \leq (\phi', \mu'_{J'})$ as above. Then we have a natural closed embedding

$$\overline{\mathrm{Gr}_{I,\phi,\mu_J}} \hookrightarrow \overline{\mathrm{Gr}_{I,\phi',\mu'_{J'}}}.$$

Recall 2.17. We recall the definition of the standard filtration of the Beilinson-Drinfeld Grassmannian ([Zhu16, Theorem 3.1.2], [Ric14, Lemma 3.4]). Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, $n, N \in \mathbb{N}$. Define

$$\mathrm{Gr}^{(N)}_{\mathrm{GL}_n,I} = \{(x_I, \mathcal{F}, \alpha) \in \mathrm{Gr}_{\mathrm{GL}_n,I} \mid \mathcal{O}^n_X(-N\Gamma_{x_I}) \subset \mathcal{F} \subset \mathcal{O}^n_X(N\Gamma_{x_I})\}$$

(here we are implicitly identifying GL_n-torsors with locally free sheaves of rank n). Then this is a projective scheme relative to X^I, and $\mathrm{Gr}_{\mathrm{GL}_n,I}$ is filtered by the $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n,I}$'s. For the case of a general G, one chooses a faithful representation $\rho : G \to \mathrm{GL}_n$ and defines $\mathrm{Gr}^{(N)}_{G,I}$ via the closed embedding between Beilinson-Drinfeld Grassmannians induced by ρ (cf. [Zhu16, Propositions 1.2.5, 1.2.6]). Finally, one checks that the ind-scheme structure of $\mathrm{Gr}_{G,I}$ does not does not depend on ρ.

Remark 2.18. Note that $\mathrm{Gr}^{(N)}_{G,I}$ is a union of strata of $\mathrm{Gr}_{G,I}$ by a principle similar to Remark A.8.

For instance, let $G = \mathrm{GL}_n, I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$ and $N \in \mathbb{N}$. Choose $T = (\mathbb{G}_\mathrm{m})^n$ given by the diagonal matrices in $\mathbb{G}_\mathrm{m}$. This induces an embedding of posets $\mathbb{X}_\bullet(T)^+ \hookrightarrow \mathbb{N}^n$ whose image is spanned by those n-uples $(\mu_1, \ldots, \mu_n)$ where $\mu_1 \geq \cdots \geq \mu_n$ and $\nu_I \in (\mathbb{X}_\bullet(T)^+)^I$ given by ν in all components. Via this identification, it makes sense to define $\nu = (N, \ldots, N) \in \mathbb{X}_\bullet(T)^+$. Then

$$\mathrm{Gr}^{(N)}_I = \overline{\mathrm{Gr}_{I,\mathrm{id}:I\to I,\nu_I}}.$$

Remark 2.19. Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. Then the pullback $X \times_{X^I} \mathrm{Gr}^{(N)}_I$ along the principal diagonal $X \to X^I$ is isomorphic to $\mathrm{Gr}^{N\cdot|I|}_X$.

Recall 2.20. Recall now from Recall A.1 that the action of $G_\mathcal{O}$ on Gr restricts to $\mathrm{Gr}_{\leq\mu}$ for each $\mu \in \mathbb{X}_\bullet(T)^+$ and that this restriction factors through a quotient

$$G_\mathcal{O} \twoheadrightarrow G^{(j_\mu)}_\mathcal{O}$$

for a sufficiently large natural number j_μ, where

$$G^{(j_\mu)}_\mathcal{O} := G(\mathbb{C}[\![t]\!]/t^{j_\mu}) \simeq G(\mathbb{C}[t]/t^{j_\mu})$$

is now a group scheme of finite type. In a totally similar way, the action of $G_\mathcal{O}$ on Gr restricts to $\mathrm{Gr}^{(N)}$ for each $N \in \mathbb{N}$ and that this restriction factors through the quotient

$$G_\mathcal{O} \twoheadrightarrow G^{(j_N)}_\mathcal{O}$$

for a sufficiently large j_N. In what follows, we will privilege the filtration by $\mathrm{Gr}^{(N)}$'s in that it extends in a slightly simpler way to the Beilinson-Drinfeld Grassmannian.

For j a natural number, and $Z \subset X$ a closed subscheme, let $Z_{(j)}$ denote the j-thickening of Z, i.e. the scheme $(Z, \mathcal{O}_X/\mathcal{I}_Z^j)$.

Definition 2.21. Let j be a natural number, and $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. We define

$$G_{\mathcal{O},I}^{(j)}$$

as the group scheme, relative to X^I, classifying

$$\{x_I \in X^I, g \in G((\Gamma_{x_I})_{(j)})\}.$$

Remark 2.22. Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, $X \to X^I$ be the diagonal morphism, and $j \in \mathbb{N}$. Then we have isomorphisms

$$G_{\mathcal{O},I}^{(j)} \times_{X^I} X \simeq G_{\mathcal{O},X}^{(|I|\cdot j)}$$
$$G_{\mathcal{O},I}^{(j)} \times_{X^I} X^{\mathrm{id}} \simeq \left(G_{\mathcal{O},X}^{(j)}\right)^I \times_{X^I} X^{\mathrm{id}}.$$

Proof. The first part follows from the fact that, if $x_I = (x, \dots, x) \in X^I$ for some $x \in X$, the subscheme Γ_{x_I} is defined as the sum of $|I|$ copies of the divisor $\{x\} \subset X$. Hence, its ideal of definition is $\mathcal{I}_x^{|I|}$, and thus $\Gamma_{x_I}^{(j)}$ is the closed subscheme supported at x and with structure sheaf

$$(x, \mathcal{O}_X/\mathcal{I}_x^{|I|\cdot j}).$$

The second part is straightforward from Proposition 2.12. $\square$

Remark 2.23. It is easy to see that

$$G_{\mathcal{O},I} \simeq \lim_j G_{\mathcal{O},I}^{(j)}.$$

The functor $G_{\mathcal{O},I}^{(j)}$ is a smooth group scheme of finite type over X^I (by [Ras18, Lemma 2.5.1]). Smoothness may seem a bit counter-intuitive, since the fiber of $G_{\mathcal{O},I}$ (say $I = \{1,2\}$) over a point in the diagonal $X \subset X^2$ is given by a copy of $G_\mathcal{O}$, while for instance the fiber over a point in the disjoint locus of X^2 is given by $G_\mathcal{O} \times G_\mathcal{O}$. However, one cannot argue that this contradicts flatness, because we are dealing with infinite-dimensional objects. And in fact, when one truncates to $G_{\mathcal{O},I}^{(j)}$, the following happens. The fiber of $G_{\mathcal{O},I}^{(j)}$ at a point (x,x) on the diagonal is

$$G(\mathbb{C}[t]/(t^{2j}))$$

by Remark 2.22. On the other hand, the fiber at a point (x_1, x_2) outside the diagonal is

$$G_\mathcal{O}^{(j)} \times G_\mathcal{O}^{(j)}.$$

Therefore, the dimensions of these fibers are

$$\dim G(\mathbb{C}[t]/(t^{2j})) = (\dim G)^{2j}$$

and

$$\dim(G(\mathbb{C}[t]/t^j))^2 = (\dim G)^{2j}.$$

Remark 2.24. Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, N \in \mathbb{N}$. The action of $G_{\mathcal{O},I}$ on Gr_I over X^I described in Remark 2.9 restricts to each $\mathrm{Gr}_I^{(N)}$. By the same proof of [Ric14, Corollary 3.7], the restriction factors through a quotient $G_{\mathcal{O},I}^{(j_N)}$ for a sufficiently large natural number j_N. Note also, again by the same proof, that j_N is independent of I.

The following definitions are inspired by [AR23, 2.4.3]:

Definition 2.25. Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, N \in \mathbb{N}, j \geq j_N$. We define

$$\mathrm{Hck}_I^{(N,j)} = G_{\mathcal{O},I}^{(j)} \backslash \mathrm{Gr}_I^{(N)}.$$

as the fpqc quotient stack in the category $\mathrm{Stk}_{/X^I}$.

Since the action of $G_{\mathcal{O},I}^{(j)}$ respects the stratification of $\mathrm{Gr}_I^{(N)}$, each $\mathrm{Hck}_I^{(N,j)}$ is a stratified étale stack over X^I, locally of finite type, in the sense of (B.1). Also, its structure map to X^I is stratified when we endow X^I with the incidence stratification. Therefore, for $j \geq j_N$, we obtain a well-defined object

$$\mathrm{Hck}_I^{(N,j)} \in \mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}\,/X^I}.$$

Note 2.26. From now on, we will fix a function $\mathbb{N} \to \mathbb{N}, N \mapsto j_N$, witnessing a choice of index such that the action of $G_{\mathcal{O},I}$ on $\mathrm{Gr}_I^{(N)}$ factors through $G_{\mathcal{O},I}^{(j_N)}$. As remarked above, we can fix a uniform choice which works for every I.

Definition 2.27. We define, for $j \geq j_1$,

$$\mathrm{Hck}_I^{(j)} = \mathrm{Hck}_I^{(1,j)}$$

and

$$\mathrm{Hck}_I = \text{“}\lim_{j \geq j_1}\text{”}\, \mathrm{Hck}_I^{(1,j)} \in \mathrm{Pro}(\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}).$$

Proposition 2.28. *For $j' \leq j$, the transition maps $\mathrm{Hck}_I^{(1,j)} \to \mathrm{Hck}_I^{(1,j')}$ belong to the class* uni *defined in Definition B.3.*

Proof. First of all, the maps are smooth by [SPA, `Tag 02K5`] (see also [Wed22, 2.3], [Ras18, Lemma 2.5]). More precisely, they are smooth quotients relative to X^I, in particular they are representable.

Note then that that the kernel $K_{j,j'}$ of the map of group schemes $G_{\mathcal{O}}^{(j)} \to G_{\mathcal{O}}^{(j')}$ is unipotent. Moreover, the Beilinson-Drinfeld version of K_{ij}, i.e. the relative kernel of $G_{\mathcal{O},I}^{(j)} \to G_{\mathcal{O},I}^{(j')}$, splits as $K_{j,j'}^{|J|} \times_{X^I} X^\phi$ over each stratum X^ϕ of X^I. Hence, by factorization Proposition 2.12, over each stratum X^ϕ the map $G_{\mathcal{O},I}^{(j)} \to G_{\mathcal{O},I}^{(j')}$ is a quotient map with fiber $K_{j,j'}^{|J|}$, which achieves the proof. Note that the pullbacks of $G_{\mathcal{O},I}^{(j)}$ and $G_{\mathcal{O},I}^{(j')}$ to strata of X^I are themselves strata inside $G_{\mathcal{O},I}^{(j)}$ and $G_{\mathcal{O},I}^{(j')}$ respectively (i.e. their stratification is trivial). $\square$

Hence, the object Hck_I belongs to the full subcategory

$$\mathrm{Pro}_{\mathsf{uni}}(\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}) \subset \mathrm{Pro}(\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}})$$

from Definition B.3.

2.2 The Hecke stack over the Ran space

Definition 2.29. The *Ran presheaf* of X is the colimit

$$\mathrm{Ran}(X) = \underset{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}}{\mathrm{colim}} X^I$$

in the category $\mathrm{Fun}(\mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}}, \mathrm{Set})$, where the diagram is the one that associates to a map $I \to J$ the induced diagonal map $X^J \to X^I$.

The formation of this colimit loses any kind of descent, see e.g. [GL, Warning 2.4.4].

Remark 2.30. The functor of points of $\mathrm{Ran}(X)$ can be described as

$$\mathrm{Ran}(X)(\mathrm{Spec}\, R) = \{S \subset X(R) \text{ nonempty unordered finite subset}\}.$$

Notation 2.31. For $S \in \mathrm{Ran}(X)(R)$, we denote by Γ_S the divisor $\sum_{x_i \in S} \Gamma_{x_i}$.

We now want to promote the association $I \mapsto \mathrm{Hck}_I$ to a functor $\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}} \to \mathrm{Pro}(\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}})$.

Lemma 2.32. *A surjection $\tau : I \to J$ in $\mathrm{Fin}_{\geq 1,\mathrm{surj}}$ induces a closed immersion $\mathrm{Gr}^{(1)}_J \hookrightarrow \mathrm{Gr}^{(1)}_I$, and this determines a functor*

$$\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}} \to \mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$$
$$I \mapsto \mathrm{Gr}^{(1)}_I$$

whose colimit

$$\mathrm{Gr}_{\mathrm{Ran}} = \underset{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}}{\mathrm{colim}} \mathrm{Gr}^{(1)}_I \in \mathrm{PSh}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$$

lives over the algebraic Ran space of X and classifies the datum of

$$(S \subset X(R), \mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{T}|_{X_R \setminus \Gamma_S}).$$

Proof. Let $\tilde{\tau}$ be the diagonal $X^J \to X^I$ induced by τ. We define the sought-after closed embedding as

$$(x_J, \mathcal{F}, \alpha) \mapsto (\tilde{\tau} \circ x_J, \mathcal{F}, \alpha). \tag{2.1}$$

One can easily see that this is stratified. To prove the functor-of-points description, let us define the following category $\mathcal{J}$:

$$\mathcal{J} = \{I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, \underline{N} \in \mathbb{N}^I\}$$
$$\mathrm{Hom}((I,\underline{N}),(J,\underline{M})) = \{\phi : J \twoheadrightarrow I \mid N_i \leq \sum_{j \in J \mid \phi(j)=i} M_j \ \forall i \in I\}$$

For $(I,\underline{N}) \in \mathcal{J}$, let $\mathrm{Gr}^{(\underline{N})}_I$ be the closed subscheme of the ind-scheme Gr_I defined by setting

$$\mathrm{Gr}^{(\underline{N})}_I = \{x_I \in X^I, \mathcal{F}, \alpha \mid \mathcal{O}^n_X(-\sum_{i \in I} N_i \Gamma_{x_i}) \subset \mathcal{F} \subset \mathcal{O}^n_X(\sum_{i \in I} N_i \Gamma_{x_i})\}$$

for GL_n and then proceeding as in Recall 2.17. There is a functor

$$\begin{aligned}\mathcal{J}^{\mathrm{op}} &\to \mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}}\\ I &\mapsto \mathrm{Gr}_I^{(\underline{N})}\end{aligned}$$

sending a map in $\mathcal{J}$ to the restriction of (2.1) to $\mathrm{Gr}_I^{(\underline{N})}$ (the condition that $N_i \leq \sum_{j\in J|\phi(j)=i} M_j$ ensures that this restrictions takes values in $\mathrm{Gr}_I^{(\underline{M})}$). One can see that the functor of points appearing in the statement is equivalent to the colimit

$$\operatorname*{colim}_{(I,\underline{N})\in\mathcal{J}^{\mathrm{op}}} \mathrm{Gr}_I^{(\underline{N})}.$$

Now, the functor

$$\begin{aligned}F : \mathrm{Fin}_{\geq 1,\mathrm{surj}} &\to \mathcal{J}\\ I &\mapsto (I, \mathrm{const}_1)\end{aligned}$$

is initial. Indeed, for any $(I,\underline{N}) \in \mathcal{J}$ we can consider the object $J = \sqcup_{i\in I} N_i$ and the canonical surjection $J \to I$ induced by the definition of J. This induces a morphism in $\mathcal{J}$ between (J, const_1) and $(I,\underline{N})$, hence the overcategory $F/(I,\underline{N})$ is nonempty. For any other morphism $\tilde{\tau} : (J', \mathrm{const}_1) \to (I,\underline{N})$ in $\mathcal{J}$ with underlying surjection $\tau : J' \to I$, we have that for every $i \in I$ then $N_i \leq \sum_{j\in J'|\tau(j)=i} 1$. Hence there exist surjections $\nu_i : \tau^{-1}(i) \to N_i$ for each $i \in I$, which assemble to a surjection $\nu : J' \to \sqcup_{i\in I} N_i$. By construction, $\tilde{\tau}$ factors through the image of ν under F.

Therefore, we have an induced isomorphism at the level of colimits, which concludes the proof. □

Construction 2.33. Observe now that, given a surjection $I \to J$, for any $j \in \mathbb{N}$ there exists j' and a map of relative group schemes

$$G_{\mathcal{O},J}^{(j')} \to G_{\mathcal{O},I}^{(j)}$$

over the diagonal $X^J \to X^I$. The index j' need not be the same as j: for instance, take $I = \{1,2\}, J = \{1\}$. Then we are looking at the map

$$G_{\mathcal{O},X}^{(2j)} \simeq G_{\mathcal{O},X^2}^{(j)} \times_{X^2,\Delta} X \hookrightarrow G_{\mathcal{O},X^2}^{(j)}.$$

We thus obtain a map of pro-relative group schemes

$$\text{“}\lim_{j\in\mathbb{N}}\text{”} G_{\mathcal{O},J}^{(j)} \to \text{“}\lim_{j\in\mathbb{N}}\text{”} G_{\mathcal{O},I}^{(j)} \tag{2.2}$$

over the diagonal $X^J \to X^I$. This, together with Lemma 2.32 and the fact that the map $\mathrm{Gr}_J^{(1)} \to \mathrm{Gr}_I^{(1)}$ from Lemma 2.32 is equivariant relatively to the map (2.2), induces a map

$$\text{“}\lim_{j\geq j_1}\text{”} \mathrm{Hck}_J^{(j)} \to \text{“}\lim_{j\geq j_1}\text{”} \mathrm{Hck}_I^{(j)}$$

in $\mathrm{Pro}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}})$. We therefore have a well-defined functor

$$\begin{aligned}\mathrm{Fin}_{\geq 1,\mathrm{surj}}^{\mathrm{op}} &\to \mathrm{Pro}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}})\\ I &\mapsto \text{“}\lim_{j\geq j_1}\text{”} \mathrm{Hck}_I^{(j)}.\end{aligned}$$

We can now consider the colimit

$$\mathrm{Hck}_{\mathrm{Ran}} = \operatorname*{colim}_{I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}} \text{“}\lim_{j \geq j_1}\text{”} \mathrm{Hck}_I^{(j)}$$

in the category $\mathrm{PSh}(\mathrm{Pro}_{\mathrm{uni}}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}))$, which we denote by $\widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}}$ (Definition B.5). We have a natural map $\mathrm{Hck}_{\mathrm{Ran}} \to \mathrm{Ran}(X)$ in $\widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}}$.

2.3 The convolution Beilinson-Drinfeld Grassmannian

Our goal now is to transfer the convolution diagram from Remark A.13 to the “Beilinson–Drinfeld” setting.

Definition 2.34. Given $I_1, I_2 \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, [\phi : I_1 \twoheadrightarrow J]$ a partition, $\mu_J \in (\mathbb{X}_\bullet(T)^+)^J, N, j \in \mathbb{N}$. We define

$$G_{\mathcal{K},I_1,I_2}^{(\infty,j)} = \{x_{I_1} \in X^{I_1}, x_{I_2} \in X^{I_2}, \mathcal{F} \in \mathrm{Bun}_G(X), \alpha : \mathcal{F}|_{X\setminus\Gamma_{x_J}} \xrightarrow{\sim} \mathcal{T}|_{X\setminus\Gamma_{x_J}}, \mu : \mathcal{F}|_{(\Gamma_{x_{I_2}})_{(j)}} \xrightarrow{\sim} \mathcal{T}|_{(\Gamma_{x_{I_2}})_{(j)}}\}$$

$$G_{\mathcal{K},I_1,I_2,\phi,\mu_J}^{(j)} = G_{\mathcal{K},I_1,I_2}^{(\infty,j)} \times_{\mathrm{Gr}_{I_1}} \mathrm{Gr}_{I_1,\phi,\mu_J}$$

$$G_{\mathcal{K},I_1,I_2}^{(N,j)} = G_{\mathcal{K},I_1,I_2}^{(\infty,j)} \times_{\mathrm{Gr}_{I_1}} \mathrm{Gr}_{I_1}^{(N)},$$

where the map $G_{\mathcal{K},I_1,I_2}^{(j)} \to \mathrm{Gr}_{I_1}$ is the one that only remembers $(x_{I_1}, \mathcal{F}, \alpha)$. We also define

$$G_{\mathcal{K},I_1,I_2,\phi,\mu_J} = G_{\mathcal{K},I_1,I_2} \times_{\mathrm{Gr}_{I_1}} \mathrm{Gr}_{I_1,\phi,\mu_J}$$

$$G_{\mathcal{K},I_1,I_2}^{(N)} = G_{\mathcal{K},I_1,I_2} \times_{\mathrm{Gr}_{I_1}} \mathrm{Gr}_{I_1}^{(N)}.$$

Definition 2.35. Let $k \geq 1, I_1, \ldots, I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, N \geq 0$. We define the scheme

$$\mathrm{Conv}_{I_1,\ldots,I_k}^{(N)} = G_{\mathcal{K},I_1,I_2}^{(N)} \times^{G_{\mathcal{O},I_2}} G_{\mathcal{K},I_2,I_3}^{(N)} \times^{G_{\mathcal{O},I_3}} \cdots \times^{G_{\mathcal{O},I_k}} \mathrm{Gr}_{I_k}^{(N)}$$

as the quotient of

$$G_{\mathcal{K},I_1,I_2}^{(N)} \times_{X^{I_2}} \cdots \times_{X^{I_k}} \mathrm{Gr}_{I_k}^{(N)}$$

with respect to the action of $\prod_{i=2,\ldots,k} G_{\mathcal{O},I_i}$ described as follows. For $i = 2, \ldots, k-1$, each $G_{\mathcal{O},I_i}$ acts on $G_{\mathcal{K},I_{i-1},I_i}^{(N)} \times_{X^{I_i}} G_{\mathcal{K},I_i,I_{i+1}}^{(N)}$, relatively to X^{I_i}, as in Remark 2.9. For $i = k$, $G_{\mathcal{O},I_k}$ acts on $G_{\mathcal{K},I_{k-1},I_k}^{(N)} \times_{X^{I_k}} \mathrm{Gr}_{I_k}^{(N)}$ again like in Remark 2.9.

In particular, this is an fpqc quotient, but it is also a schematic quotient.

Remark 2.36. For any $j \geq j_1$, the expression above can be rewritten as

$$G_{\mathcal{K},I_1,I_2}^{(N,j)} \times^{G_{\mathcal{O},I_2}^{(j)}} G_{\mathcal{K},I_2,I_3}^{(N,j)} \times^{G_{\mathcal{O},I_3}^{(j)}} \cdots \times^{G_{\mathcal{O},I_k}^{(j)}} \mathrm{Gr}_{I_k}^{(N)}$$

(cf. [Zhu16, Discussion after Lemma 5.2.3]).

Definition 2.37. We define the *convolution Grassmannian* as

$$\mathrm{Conv}_{I_1,\ldots,I_k} = G_{\mathcal{K},I_1,I_2} \times^{G_{\mathcal{O},I_2}} G_{\mathcal{K},I_2,I_3} \times^{G_{\mathcal{O},I_3}} \cdots \times^{G_{\mathcal{O},I_k}} \mathrm{Gr}_{I_k}$$

where the notation with the superscripts has the same meaning as in Definition 2.35.

Remark 2.38. This object is filtered by the $\mathrm{Conv}^{(N)}_{I_1,\ldots,I_k}$'s, hence the convolution Grassmannian is an ind-scheme.

Remark 2.39. The convolution Grassmannian classifies the datum

$$\begin{gathered}\{(x_{I_1},\ldots,x_{I_k}), x_{I_j} \in X^{I_j} \text{ for each } j=1,\ldots,k, \mathcal{F}_1,\ldots,\mathcal{F}_k,\\ \alpha : \mathcal{F}_1|_{X\setminus x_{I_1}} \simeq \mathcal{T}|_{X\setminus x_{I_1}}, \eta_j : \mathcal{F}_j|_{X\setminus x_{I_j}} \simeq \mathcal{F}_{j-1}|_{X\setminus x_{I_j}}, j=2,\ldots,k\}.\end{gathered}$$

Notation 2.40. Let us fix the following notation. A general element of $\mathrm{Conv}_{I_1,\ldots,I_k}$ will be denoted by

$$\Big(\mathcal{T} \xleftarrow[X\setminus x_{I_1}]{\alpha} \mathcal{F}_1 \xleftarrow[X\setminus x_{I_2}]{\eta_2} \mathcal{F}_2 \xleftarrow[X\setminus x_{I_2}]{\eta_3} \ldots \xleftarrow[X\setminus x_{I_k}]{\eta_k} \mathcal{F}_k \Big).$$

Of course this is just a symbolic notation, in that each of the arrows drawn here is defined over a (potentially) different open set.

Let now $k \geq 1$, $I_1,\ldots,I_k, I = I_1 \sqcup \cdots \sqcup I_k, J \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, and $[\phi : I \twoheadrightarrow J]$ a partition. Define X^{ϕ} as in Construction 2.10.

Proposition 2.41. *There is an isomorphism*

$$\mathrm{Conv}_{I_1,\ldots,I_k} \times_{X^I} X^{\phi} \simeq \prod_{j\in J} \mathrm{Conv}_{\Delta,m_j} \times_{X^I} X^{\phi}$$

where:

- $m_j = \#\{h \mid 1 \leq h \leq k, \phi^{-1}(j) \cap I_h \neq \varnothing\}$
- $\mathrm{Conv}_{\Delta,m_j} := \mathrm{Conv}_{\{*\},\ldots\ m_j \text{ times } \ldots,\{*\}} \times_{X^{m_j}} X$ *(the map from X to X^{m_j} being the diagonal)*
- *the map $\prod_{j\in J} \mathrm{Conv}_{\Delta,m_j} \to X^{I_1\sqcup\cdots\sqcup I_k}$ is induced by the diagonal map $X^J \to X^{I_1\sqcup\cdots\sqcup I_k}$ associated to ϕ.*

Proof. (Sketch). This proof has been suggested to us by Robert Cass. We treat the case $k=3$, $I_1 = I_2 = I_3 = 1$. We have three essentially distinct cases:

- $J = \{1,2,3\}$, $\phi = \mathrm{id}$. The isomorphism (adopting Notation 2.40) is given by

$$\mathrm{Conv}_{I_1,I_2,I_3} \times_{X^3} X^{\phi} \simeq (\mathrm{Gr}_X \times \mathrm{Gr}_X \times \mathrm{Gr}_X) \times_{X^3} X^{\phi}$$

$$(\mathcal{T} \xleftarrow[X\setminus x_1]{\alpha} \mathcal{F}_1 \xleftarrow[X\setminus x_2]{\eta_2} \mathcal{F}_2 \xleftarrow[X\setminus x_3]{\eta_3} \mathcal{F}_3) \mapsto$$

$$(\mathcal{T} \xleftarrow[X\setminus x_1]{\alpha} \mathcal{F}_1 \ , \ \mathcal{T} \xleftarrow[X\setminus x_2]{\alpha|_{\mathring{X}_{x_2}} \circ \eta_2|_{\mathring{X}_{x_2}}} \mathcal{F}_2 \ , \ \mathcal{T} \xleftarrow[X\setminus x_3]{\alpha|_{\mathring{X}_{x_3}} \circ \eta_2|_{\mathring{X}_{x_3}} \circ \eta_3|_{\mathring{X}_{x_3}}} \mathcal{F}_3)$$

whose inverse is given by gluing sheaves (which can be done since the points are distinct).

- $J = \{1,2\}, \phi(1) = \phi(3) = 1, \phi(2) = 2$ (we treat this case and not the case $\phi(1) = \phi(2) = 1, \phi(3) = 3$ since we want to show that our argument works even when the two equal coordinates are not adjacent to one another). The isomorphism is given by

$$\mathrm{Conv}_{I_1,I_2,I_3} \times_{X^3} X^{\phi} \simeq (\mathrm{Conv}_{\Delta,2} \times \mathrm{Gr}_X) \times_{X^3} X^{\phi}$$

$$(\ \mathcal{T} \xleftarrow[X\backslash x_1]{\alpha} \mathcal{F}_1 \xleftarrow[X\backslash x_2]{\eta_2} \mathcal{F}_2 \xleftarrow[X\backslash x_1]{\eta_3} \mathcal{F}_3 \ ) \mapsto$$

$$(\ \mathcal{T} \xleftarrow[X\backslash x_1]{\alpha} \mathcal{F}_1 \xleftarrow[X\backslash x_1]{\eta_2|_{\mathring{X}|_{x_1}} \circ \eta_3|_{\mathring{X}_{x_1}}} \mathcal{F}_3 \ , \ \mathcal{T} \xleftarrow[X\backslash x_2]{\alpha|_{\mathring{X}_{x_2}} \circ \eta_2|_{\mathring{X}_{x_2}}} \mathcal{F}_2 \ )$$

with inverse

$$(\ \mathcal{T} \xleftarrow[X\backslash x_1]{\alpha} \mathcal{F}_1 \xleftarrow[X\backslash x_1]{\eta} \mathcal{F}_3 \ , \ \mathcal{T} \xleftarrow[X\backslash x_2]{\beta} \mathcal{F}_2 \ ) \mapsto$$

$$(\ \mathcal{T} \xleftarrow[X\backslash x_1]{\alpha} \mathcal{F}_1 \xleftarrow[X\backslash x_2]{\alpha|^{-1}_{\mathring{X}_{x_2}} \circ \beta|_{\mathring{X}_{x_2}}} \mathcal{F}_2 \xleftarrow[X\backslash x_1]{\eta|_{\mathring{X}_{x_1}} \circ \alpha|^{-1}_{\mathring{X}_{x_1}} \circ \beta|_{\mathring{X}_{x_1}}} \mathcal{F}_3 \ )$$

- $J = \{1\}$. The isomorphism is the identity.

□

Definition 2.42. Let $x \in X$ be a closed point. We define

$$\mathrm{Conv}_{x,k} = \mathrm{Conv}_{\Delta,k} \times_X \{x\}.$$

Remark 2.43. The object $\mathrm{Conv}_{x,k}$ is isomorphic to

$$\mathrm{Conv}_k = \overbrace{G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \cdots \times^{G_{\mathcal{O}}} G_{\mathcal{K}}}^{k-1} \times^{G_{\mathcal{O}}} \mathrm{Gr}$$

(notation as in Construction A.9).

In a similar fashion as Proposition 2.6, one can prove:

Proposition 2.44. *Let $X = \mathbb{A}^1_{\mathbb{C}}, k \geq 1$. With the notations of Proposition 2.41, the choice of a point $x \in X$ induces a splitting*

$$\mathrm{Conv}_{\Delta,k} \simeq \mathrm{Conv}_{x,k} \times \mathbb{A}^1_{\mathbb{C}}.$$

Corollary 2.45. *In the case when $X = \mathbb{A}^1_{\mathbb{C}}$, Proposition 2.41 specializes to*

$$\mathrm{Gr}_I|_{X^\phi} \simeq \Big(\prod_J \mathrm{Gr}\Big) \times X^{\phi}$$

$$\mathrm{Conv}_{I_1,\ldots,I_k}|_{X^\phi} \simeq \Big(\prod_{j \in J} \mathrm{Conv}_{m_j}\Big) \times X^{\phi}$$

for any chosen point $x \in X$.

Proof. It suffices to apply Proposition 2.6 and Proposition 2.44 respectively. □

Construction 2.46. One can define a stratification of $\mathrm{Conv}_{\Delta,k}$ (the case $k=1$ being Gr_X), as follows. The stratifying poset is $(\mathbb{X}_\bullet(T)^+)^k$. First of all, $G_{\mathcal{K},\Delta} := G_{\mathcal{K},\{*\},\{*\}} \times_{\mathrm{Gr}_{\{*\},\{*\}}} \mathrm{Gr}_{\{*\}}$ (the map $\mathrm{Gr}_{\{*\}} \to \mathrm{Gr}_{\{*\},\{*\}}$ being induced by the diagonal of X^2) inherits a stratification over $\mathbb{X}_\bullet(T)^+$ from $\mathrm{Gr}_{\{*\}} = \mathrm{Gr}_X$. This induces a stratification on the product

$$G_{\mathcal{K},\Delta} \times_X \cdots \times G_{\mathcal{K},\Delta} \times_X \mathrm{Gr}_X$$

and one can check that this stratification passes to the multiple quotient

$$G_{\mathcal{K},\Delta} \times^{G_{\mathcal{O},X}} G_{\mathcal{K},\Delta} \times^{G_{\mathcal{O},X}} \cdots \times^{G_{\mathcal{O},X}} \mathrm{Gr}_X.$$

In other words, let $\mu_1,\dots,\mu_k \in \mathbb{X}_\bullet(T)^+$. The set

$$\mathrm{Conv}_{\Delta,k,\mu_1,\dots,\mu_k}(\mathbb{C})$$

is the subset of $\mathrm{Conv}_{\Delta,k}(\mathbb{C})$ where one imposes the condition that (with the notations of Remark 2.39) $\mathrm{Inv}_x(\alpha)=\mu_1, \mathrm{Inv}_x(\eta_i)=\mu_i$ for every $i \geq 2$.

Let $x \in X$ be a closed point. Then each stratum of $\mathrm{Conv}_{x,k}$ can be identified with $G_{\mathcal{K},\mu_1} \times^{G_\mathcal{O}} \cdots \times^{G_\mathcal{O}} \mathrm{Gr}_{\mu_k}$, where $G_{\mathcal{K},\mu_i}$ is the preimage of Gr_{μ_i} along the quotient map. This definition is the direct generalization from $k=2$ to arbitrary k of [MV07, after Lemma 4.3].

These strata are not orbits for a group action, but they are smooth.

Proposition 2.47. *Each stratum*

$$G_{\mathcal{K},\mu_1} \times^{G_\mathcal{O}} \cdots \times^{G_\mathcal{O}} G_{\mathcal{K},\mu_{k-1}} \times^{G_\mathcal{O}} \mathrm{Gr}_{\mu_k}$$

is a smooth locally closed subscheme of Conv_k.

Proof. The following proof has been suggested to us by Mark Macerato. First of all, we note that for j sufficiently large, then

$$G_{\mathcal{K},\mu_1} \times^{G_\mathcal{O}} \cdots \times^{G_\mathcal{O}} G_{\mathcal{K},\mu_{k-1}} \times^{G_\mathcal{O}} \mathrm{Gr}_{\mu_k} \simeq G^{(j)}_{\mathcal{K},\mu_1} \times^{G^{(j)}_\mathcal{O}} \cdots \times^{G^{(j)}_\mathcal{O}} G^{(j)}_{\mathcal{K},\mu_{k-1}} \times^{G^{(j)}_\mathcal{O}} \mathrm{Gr}_{\mu_k}$$

(with the same argument as Remark 2.36). Now,

$$G^{(j)}_{\mathcal{K},\mu_1} \times G^{(j)}_{\mathcal{K},\mu_2} \times \cdots \times G^{(j)}_{\mathcal{K},\mu_{k-1}} \times \mathrm{Gr}_{\mu_k} \to \mathrm{Gr}_{\mu_1} \times \mathrm{Gr}_{\mu_2} \times \cdots \times \mathrm{Gr}_{\mu_{k-1}} \times \mathrm{Gr}_{\mu_k}$$

is a torsor with fiber $(G^{(j)}_\mathcal{O})^{\times k-1}$, which is a smooth group scheme. Since the base is smooth ([Zhu16, Proposition 2.1.5 (1)]), the total space is smooth as well. Now, the map $G^{(j)}_{\mathcal{K},\mu_1} \times G^{(j)}_{\mathcal{K},\mu_2} \times \cdots \times \mathrm{Gr}_{\mu_k} \to G^{(j)}_{\mathcal{K},\mu_1} \times^{G^{(j)}_\mathcal{O}} G^{(j)}_{\mathcal{K},\mu_2} \times^{G^{(j)}_\mathcal{O}} \cdots \times^{G^{(j)}_\mathcal{O}} \mathrm{Gr}_{\mu_k}$ is the fpqc schematic quotient of a smooth scheme with respect to the group $(G^{(j)}_\mathcal{O})^{\times k-1}$. In particular, it is an fpqc covering, and therefore by [SPA, `Tag 02VL`] we conclude. □

Construction 2.48. We can now define a stratification for $\mathrm{Conv}_{I_1,\dots,I_k}$. Recall the notation in Proposition 2.41. The stratifying poset will be

$$\mathrm{Cw}_{I_1,\dots,I_k} = \{[\phi : I_1 \sqcup \cdots \sqcup I_k \twoheadrightarrow J], \mu_J = (\mu_j^{h_1^j},\dots,\mu_j^{h_{m_j}^j}) \in \prod_{j\in J} (\mathbb{X}_\bullet(T)^+)^{m_j}\}$$

where "Cw" stays for "coweight" and, for each $j \in J$, the h_i^j's are those indexes for which $\phi^{-1}(j) \cap I_{h_i} \neq \varnothing$. The stratification for $\mathrm{Conv}_{I_1,\dots,I_k}$ is then defined as

$$\mathrm{Conv}_{I_1,\dots,I_k,\phi,\mu_J} = \prod_{j \in J} \mathrm{Conv}_{\Delta, m_j, \mu_j^{h_1^j}, \dots, \mu_j^{h_{m_j}^j}} \times_{X^I} X^\phi \hookrightarrow \mathrm{Conv}_{I_1,\dots,I_k}$$

where the embedding is induced by Proposition 2.41.

Remark 2.49. Let $I_1, \dots, I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, I = I_1 \sqcup \dots \sqcup I_k, N \in \mathbb{N}, j \geq j_N$. There is an action of $G_{\mathcal{O},I}$ over $\mathrm{Conv}_{I_1,\dots,I_k}$, relative to X^I, which modifies the first trivialization at all points $x_{I_1}, \dots, x_{I_k}$.

This factors as an action of $G_{\mathcal{O},I}^{(j)}$ on $\mathrm{Conv}_{I_1,\dots,I_k}^{(N)}$ relative over X^I.

Remark 2.50. Let us inspect the behaviour on different strata of the action of $G_{\mathcal{O},I}^{(j)}$ on $\mathrm{Conv}_{I_1,\dots,I_k}^{(N)}$ defined above. For simplicity, we look at $k = 2, I_1 = I_2 = \{*\}$, and we distinguish the two cases of equal points $x_1 = x_2 = x$ and of two distinct points x, y. In the first case, the action is just the action of $G_{\mathcal{O}}^{(2j)}$ on the first component of $G_{\mathcal{K}}^{(2N)} \times^{G_{\mathcal{O}}} \mathrm{Gr}^{(2N)}$. In the second case, with the notations of Remark 2.39, the modification of α at y propagates to $\eta = \eta_2$ through factorization: more precisely, up to the isomorphism $\mathrm{Conv}_{x,y} \simeq \mathrm{Gr}_x \times \mathrm{Gr}_y$ induced by Proposition 2.41 the action splits as the canonical componentwise left action

$$G_{\mathcal{O},x}^{(j)} \times G_{\mathcal{O},y}^{(j)} \circlearrowright \mathrm{Gr}_x^{(N)} \times \mathrm{Gr}_y^{(N)}.$$

This follows directly from inspecting the proof of Proposition 2.41.

In particular, the action at y is not trivial: this may seem in contradiction with the principle applied for instance in the proof of Lemma 2.57, where it is said that modifying a trivialization away from its "critical points" (x in this case) does not change the datum up to some isomorphism Φ. The point here is that this isomorphism Φ need not be compatible with the rest of the datum, specifically with the isomorphism η_2 in the notations of Remark 2.39, which is defined on $X \setminus \{y\}$.

Definition 2.51. Let $k \geq 1, I_1, \dots, I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, I = I_1 \sqcup \dots \sqcup I_k, j \geq j_1$. We define

$$\mathrm{Hck}_{I_1,\dots,I_k}^{(j)} = G_{\mathcal{O},I}^{(j)} \backslash \mathrm{Conv}_{I_1,\dots,I_k}^{(1)}$$

Remark 2.52. Definition 2.51 is functorial in $(I_1, \dots, I_k) \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}^{\mathrm{op}}$, in the sense that given surjections $I_1 \to J_1, \dots I_k \to J_k$ we have a map of pro-objects "$\lim_{j \geq j_1}$"$\mathrm{Hck}_{J_1,\dots,J_k}^{(j)} \to$ "$\lim_{j \geq j_1}$"$\mathrm{Hck}_{I_1,\dots,I_k}^{(j)}$, exactly as in Construction 2.33.

This yields a functor

$$\begin{aligned}(\mathrm{Fin}_{\geq 1,\mathrm{surj}}^{\mathrm{op}})^{\times k} &\to \mathrm{Pro}_{\mathsf{uni}}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}) \\ (I_1, \dots, I_k) &\mapsto \text{“}\lim_{j \geq j_1}\text{”}\mathrm{Hck}_{I_1,\dots,I_k}^{(j)}.\end{aligned}$$

Definition 2.53. For $k \geq 1$, we define

$$\mathrm{Hck}_{\mathrm{Ran},k} = \operatorname*{colim}_{I_1,\dots,I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}^{\mathrm{op}}} \text{“}\lim_{j \geq j_1}\text{”}\mathrm{Hck}_{I_1,\dots,I_k}^{(j)}$$

as the colimit in the category $\mathrm{PSh}(\mathrm{Pro}_{\mathsf{uni}}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}})) = \widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}}$.

We also define $\mathrm{Hck}_{\mathrm{Ran},0} = \mathrm{Spec}\,\mathbb{C}$.

Let $k \in \mathbb{N}, x \in X$. We define $\mathrm{Hck}_{\Delta,k}$ and $\mathrm{Hck}_{x,k}$ in a similar way to Definition 2.42.

Notation 2.54. Let $k \geq 1$. We denote by $(\mathrm{Ran}(X)^{\times k})_{\mathrm{disj}}$ the subfunctor of $(\mathrm{Ran}(X)^{\times k})$ spanned by k-uples of systems of points $S_1, \ldots, S_k \subset X$ such that $\Gamma_{S_i} \cap \Gamma_{S_j} = \varnothing$ for all $1 \leq i \neq j \leq k$.

The results regarding the convolution Grassmannian imply the following:

Proposition 2.55. *In the notation of Proposition 2.41, we have stratified equivalences*

$$\mathrm{Hck}_{I_1,\ldots,I_k} \times_{X^I} X^{\phi} \simeq \prod_{j \in J} \mathrm{Hck}_{\Delta,m_j} \times_{X^I} X^{\phi}.$$

If $X = \mathbb{A}^1_{\mathbb{C}}$ *we also have*

$$\mathrm{Hck}_{\Delta,k} \simeq \mathrm{Hck}_{x,k} \times \mathbb{A}^1_{\mathbb{C}}$$
$$\mathrm{Hck}_{I_1,\ldots,I_k} \times_{X^I} X^{\phi} \simeq \Big(\prod_{j \in J} \mathrm{Hck}_{x,m_j}\Big) \times (\mathbb{A}^1_{\mathbb{C}})^{\phi}.$$

Finally,

$$\mathrm{Hck}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} (\mathrm{Ran}(X)^{\times k})_{\mathrm{disj}} \simeq \mathrm{Hck}_{\mathrm{Ran},k} \times_{\mathrm{Ran}(X)^{\times k}} (\mathrm{Ran}(X)^{\times k})_{\mathrm{disj}} \simeq \mathrm{Hck}^{\times k}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)^{\times k}} (\mathrm{Ran}(X)^{\times k})_{\mathrm{disj}}$$

where in the first fiber product the map $(\mathrm{Ran}(X)^{\times k})_{\mathrm{disj}} \to \mathrm{Ran}(X)$ *is the union map.*

Proof. The first three points are straightforward from Proposition 2.41. The third point follows by passing to the colimit in the other ones and in Proposition 2.11, Proposition 2.12, but not immediately, since $\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}$ is not filtered. The following argument, suggested by Emanuele Pavia, circumvents this problem (we explain it with $k = 2$ for simplicity). Let $I_1, I_2 \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, I = I_1 \sqcup I_2$, and denote by $(X^{I_1} \times X^{I_2})_{\mathrm{disj}}$ the open subscheme $\{(x_{I_1}, x_{I_2}) \in X^{I_1} \times X^{I_2} \mid \Gamma_{x_a} \cap \Gamma_{x_b} = \varnothing \ \forall a \in I_1, b \in I_2\}$. Note that

$$\operatorname*{colim}_{I_1,I_2 \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} (X^{I_1} \times X^{I_2})_{\mathrm{disj}} = (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}.$$

Now, in the diagram

$$\begin{tikzcd}
\mathrm{Hck}_{I_1,I_2} \times_{X^{I_1 \sqcup I_2}} (X^{I_1} \times X^{I_2})_{\mathrm{disj}} \arrow[r] \arrow[d] & \mathrm{Hck}_{I_1} \times \mathrm{Hck}_{I_2} \arrow[d] \\
(X^{I_1} \times X^{I_2})_{\mathrm{disj}} \arrow[r] \arrow[d] & X^{I_1} \times X^{I_2} \arrow[d] \\
(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \arrow[r] & \mathrm{Ran}(X) \times \mathrm{Ran}(X)
\end{tikzcd}$$

the top square is Cartesian by the first part of the proposition (one has to assemble the statements for various partitions of ϕ of I), whereas the bottom square is Cartesian by straightforward verification. Therefore, the outer square is Cartesian for every $I_1, I_2 \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, and by universality of colimits we get

$$(\mathrm{Hck}_{\mathrm{Ran}} \times \mathrm{Hck}_{\mathrm{Ran}}) \times_{\mathrm{Ran}(X) \times \mathrm{Ran}(X)} (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \simeq$$
$$\mathrm{Hck}_{\mathrm{Ran},2} \times_{\mathrm{Ran}(X) \times \mathrm{Ran}(X)} (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}.$$

The other equivalence is proven in the same way. □

2.4 The BD-convolution diagram as a 2-Segal object

Remark 2.56. Let $k,N\in\mathbb{N}, I_1,\dots,I_k\in\mathrm{Fin}_{\geq 1,\mathrm{surj}}, I=I_1\sqcup\cdots\sqcup I_k, j\geq j_N$. We have a diagram

$$\begin{tikzcd}
G^{(N,j)}_{\mathcal{K},I_1,I_2}\times_{X^{I_2}}\cdots\times_{X^{I_{k-1}}} G^{(N,j)}_{\mathcal{K},I_{k-1},I_k}\times_{X^{I_k}}\mathrm{Gr}^{(N)}_{I_k} \arrow[r, "q^{(N,j)}_{I_1,\dots,I_k}"] \arrow[d, "p^{(N,j)}_{I_1,\dots,I_k}"'] & G^{(N,j_N)}_{\mathcal{K},I_1,I_2}\times^{G^{(j)}_{\mathcal{O},I_2}}\cdots\times^{G^{(j_N)}_{\mathcal{O},I_{k-1}}} G^{(N,j_N)}_{\mathcal{K},I_{k-1},I_k}\times^{G^{(j_N)}_{\mathcal{O},I_k}}\mathrm{Gr}^{(N)}_{I_k} \arrow[d, "m^{(N)}_{I_1,\dots,I_2}"] \\
\prod_{I=1}^k \mathrm{Gr}^{(N)}_{I_i} & \mathrm{Gr}^{(N)}_I
\end{tikzcd} \tag{2.3}$$

where:

- the left vertical map is the projection to the quotient of the action of $\prod_{i=2}^k G_{\mathcal{O},I_i}$ (relative to $X^{I_2\sqcup\cdots\sqcup I_k}$) induced by Remark 2.4;
- the horizontal map is the quotient map by the actions defined in Remark 2.9 (see also the definition of Definition 2.35);
- the right vertical map arises as follows. Let $I_1,I_2,I_3\in\mathrm{Fin}_{\geq 1,\mathrm{surj}}, I=I_1\sqcup I_2\sqcup I_3, j\geq j_1$. Then there is a map
$$G_{\mathcal{K},I_1,I_2}\times_{X^{I_2}} G_{\mathcal{K},I_2,I_3}\to G_{\mathcal{K},I_1\sqcup I_2,I_3} \tag{2.4}$$
which sends the datum
$$(x_{I_1},x_{I_2},x_{I_3},\mathcal{F},\mathcal{G}\in\mathrm{Bun}_G(X),\alpha:\mathcal{F}|_{X\setminus\Gamma_{x_{I_1}}}\xrightarrow{\sim}\mathcal{T}|_{X\setminus\Gamma_{x_{I_1}}},\beta:\mathcal{F}|_{X\setminus\Gamma_{x_{I_2}}}\xrightarrow{\sim}\mathcal{T}|_{X\setminus\Gamma_{x_{I_2}}},$$
$$\mu:\mathcal{F}|_{\widetilde{X}_{\Gamma_{x_{I_2}}}}\xrightarrow{\sim}\mathcal{T}|_{\widetilde{X}_{\Gamma_{x_{I_2}}}},\nu:\mathcal{F}|_{\widetilde{X}_{\Gamma_{x_{I_3}}}}\xrightarrow{\sim}\mathcal{T}|_{\widetilde{X}_{\Gamma_{x_{I_3}}}})$$
to the datum
$$(x_{I_1}+x_{I_2},x_{I_3},\mathcal{H},\gamma:\mathcal{H}|_{X\setminus\Gamma_{x_{I_1}}+\Gamma_{x_{I_2}}}\xrightarrow{\sim}\mathcal{T}|_{X\setminus\Gamma_{x_{I_1}}+\Gamma_{x_{I_2}}},\nu)$$
where $\mathcal{H}$ is the G-bundle obtained by gluing $\mathcal{F}$ and $\mathcal{G}$ along $\alpha|_{\mathring{X}_{\Gamma_{x_{I_2}}}}\circ\mu^{-1}|_{\mathring{X}_{\Gamma_{x_{I_2}}}}$ (this makes use of the Beauville-Laszlo theorem, cf. Proposition A.6), and γ is the trivialization inherited from α via the gluing procedure.
It is easy to see that this map passes to the quotient
$$G_{\mathcal{K},I_1,I_2}\times^{G_{\mathcal{O},I_2}} G_{\mathcal{K},I_2,I_3}.$$
Analogously, there is a map $G_{\mathcal{K},I_1,I_2}\times\mathrm{Gr}_{I_2}$ which also passes to the quotient $G_{\mathcal{K},I_1,I_2}\times\mathrm{Gr}_{I_2}$. These maps induce a map
$$m_{I_1,\dots,I_k}:G_{\mathcal{K},I_1,I_2}\times^{G_{\mathcal{O},I_2}}\cdots\times^{G_{\mathcal{O},I_k}}\mathrm{Gr}_{I_k}\to\mathrm{Gr}_{I_1\sqcup\cdots\sqcup I_k}.$$
It is easy to check that this restricts to a map $m^{(N)}_{I_1,\dots,I_k}$ like in (2.3).

Note that $q^{(N)}_{I_1,\dots,I_k}$ and $m^{(N)}_{I_1,\dots I_k}$ do not depend on j. Note that, for $x \in X$, the pullback of this whole diagram along the diagonal $\operatorname{Spec}\mathbb{C} \overset{x}{\hookrightarrow} X \xrightarrow{\Delta} X^I$ is isomorphic to a diagram

$$\begin{tikzcd}
G_{\mathcal{K}}^{(|I_1|\cdot N,j)} \times \cdots \times G_{\mathcal{K}}^{(|I_{k-1}|\cdot N,j)} \times \mathrm{Gr}^{(N)} \arrow[r, "q^{(N)}_{I_1,\dots,I_k}"] \arrow[d, "p^{(N)}_{I_1,\dots,I_k}"'] & G_{\mathcal{K}}^{(|I_1|\cdot N,|I_2|j)} \times^{G_{\mathcal{O}}^{(|I_2|j)}} \cdots \times^{G_{\mathcal{O}}^{(|I_{k-1}|j)}} G_{\mathcal{K}}^{(|I_{k-1}|N,|I_k|j)} \times^{G_{\mathcal{O}}^{(|I_k|j)}} \mathrm{Gr}^{(|I_k|N)} \arrow[d, "m^{(N)}_{I_1,\dots,I_2}"] \\
\prod_{i=1}^k \mathrm{Gr}^{(|I_i|N)} & \mathrm{Gr}^{(|I|N)}
\end{tikzcd} \tag{2.5}$$

naturally generalizing (A.2).

We also have a diagram of groups, relative to X^I,

$$\begin{tikzcd}
G^{(j)}_{\mathcal{O},I} \times_{\prod_{i=2}^k X^{I_i}} \prod_{i=2}^k G^{(j)}_{\mathcal{O},I_i} \arrow[r] \arrow[d] & G^{(j)}_{\mathcal{O},I} \arrow[d, equal] \\
\prod_{i=1}^k G^{(j)}_{\mathcal{O},I_i} & G^{(j)}_{\mathcal{O},I}
\end{tikzcd} \tag{2.6}$$

where the left map is induced by the map $G^{(j)}_{\mathcal{O},I} \to G^{(j)}_{\mathcal{O},I_1} \times_{X^{I_1}} X^I$ associated to the embedding $\Gamma_{x_{I_1}} \hookrightarrow \Gamma_{x_I}$, and the horizontal map is the projection.

Lemma 2.57. *The vertices of* (2.6) *act respectively on the vertices of* (2.3)*, and the maps in* (2.3) *are equivariant with respect to the maps in* (2.6)*.*

Proof. The only part that requires some work is equivariance of the leftmost vertical arrow. The proof is an application of the Bauville-Laszlo theorem: in a few words, modifying a trivialization α defined on $X \setminus \Gamma_{x_{I_1}}$ around some points not included in x_{I_1} produces an equivalent datum in Gr_I. We provide all the details of the proof here below.

First of all, by definition of the various actions, it suffices to prove the claim for $N = \infty$ and without the index j. Equivariance in the last $k-1$ components is straightforward, and witnesses the phenomenon that allows to "shift" the right multiplication action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}}$ to the "antidiagonal" action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}} \times G_{\mathcal{K}}$.

We are left to check equivariance of the quotient map $G_{\mathcal{K},I_1,I_2} \to \mathrm{Gr}_{I_1} \times_{X^{I_1}} X^I$ with respect to the restriction map $G_{\mathcal{O},I} \to G_{\mathcal{O},I_1} \times_{X^{I_1}} X^I$. Let thus $x_I \in X(R)^I$, $g \in G(\widetilde{\Gamma}_{x_I}), \mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus \Gamma_{x_I}} \xrightarrow{\sim} \mathcal{T}|_{X_R \setminus \Gamma_{x_I}}$. We want to compare the modification of the datum $(\mathcal{F}, \alpha)$ by g and the modification of the same datum by $g|_{\widetilde{\Gamma}_{x_I}}$. Equivalently, let us assume that g restricts to the identity element on $\widetilde{\Gamma}_{x_{I_1}}$, so that the second modification is trivial.

Let thus $\mathcal{G}$ be the first modification, arising as the gluing of the data

$$\mathcal{F}|_{\widetilde{\Gamma}_{x_I}}, \quad \mathcal{T}_{X \setminus \Gamma_{x_I}}$$

along the isomorphism $g|_{\widetilde{\Gamma}_{x_I}} \circ \alpha|_{\widetilde{\Gamma}_{x_I}}$. Let β be the inherited trivialization on $X_R \setminus \Gamma_{x_{I_1}}$. Let also $\widecheck{\mathrm{can}}$ be the canonical isomorphism between $\mathcal{G}|_{X_R \setminus \Gamma_{x_I}}$ and $\mathcal{T}_{X_R \setminus \Gamma_{x_I}}$, and $\widehat{\mathrm{can}}$ be the canonical isomorphism between

$\mathcal{G}|_{\widetilde{\Gamma}_{x_I}}$ and $\mathcal{F}|_{\widetilde{\Gamma}_{x_I}}$. Note that by construction

$$\widetilde{\mathrm{can}}|_{\mathring{\Gamma}_{x_I}} \simeq g|_{\mathring{\Gamma}_{x_I}} \alpha|_{\mathring{\Gamma}_{x_I}} \circ \widehat{\mathrm{can}}|_{\mathring{\Gamma}_{x_I}}. \tag{2.7}$$

We want to exhibit an isomorphism Φ between $\mathcal{F}$ and $\mathcal{G}$, commuting with α and β on $X_R \setminus \Gamma_{x_{I_1}}$. We define it bia the Beauville-Laszlo theorem, by assigning isomorphisms ϕ, ψ, ξ between the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to the schemes

$$\widetilde{\Gamma}_{x_{I_1}}, \quad \widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_{I_1}}, \quad X_R \setminus \Gamma_{x_I}$$

respectively, and by checking that they are pairwise compatible.

We define

$$\phi : \mathcal{F}|_{\widetilde{\Gamma}_{x_{I_1}}} \xrightarrow{\widehat{\mathrm{can}}|^{-1}_{\widetilde{\Gamma}_{x_{I_1}}}} \mathcal{G}|_{\widetilde{\Gamma}_{x_{I_1}}}$$

$$\psi : \mathcal{F}|_{\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_{I_1}}} \xrightarrow{\widehat{\mathrm{can}}|^{-1}_{\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_{I_1}}} \circ \alpha|^{-1}_{\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_{I_1}}} \circ g^{-1}|_{\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_{I_1}}} \circ \alpha|_{\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_{I_1}}}} \mathcal{G}|_{\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_{I_1}}}$$

$$\xi : \mathcal{F}|_{X_R \setminus \Gamma_{x_I}} \xrightarrow{\alpha|_{X_R \setminus \Gamma_{x_I}}} \mathcal{T}|_{X_R \setminus \Gamma_{x_I}} \xrightarrow{\widetilde{\mathrm{can}}|^{-1}_{X_R \setminus \Gamma_{x_I}}} \mathcal{G}|_{X_R \setminus \Gamma_{x_I}}$$

Note that, in the definition of ψ, we have used that α is defined outside of $\Gamma_{x_{I_1}}$ and not just outside of Γ_{x_I}.

By using that $g|_{\widetilde{\Gamma}_{x_{I_1}}}$ is the identity by assumption, and the identity (2.7), one checks that these three isomorphisms are pairwise compatible.

The verification that the resulting Φ commutes with α and β can be done along the same lines. $\square$

Lemma 2.58. *In the notations of* (2.3)*, the map $q^{(N,j)}_{I_1,\dots,I_k}$ induces an equivalence after passing to the quotient by the actions of* (2.6)*.*

Proof. It suffices to check this after pulling back to the strata of $X^{I_1} \times \cdots \times X^{I_k}$, and therefore by factorization it is sufficient to prove it over the diagonal, hence over a single point $x \in X$. But in that setting, $q^{(N,j)}_{I_1,\dots,I_k}$ restricts to the map

$$G_{\mathcal{K}}^{(|I_1|N,|I_2|j)} \times \cdots \times G_{\mathcal{K}}^{(|I_{k-1}|N,|I_k|j)} \times \mathrm{Gr}^{(|I_k|N)} \to G_{\mathcal{K}}^{(|I_1|N,|I_2|j)} \times^{G_{\mathcal{O}}^{(|I_2|j)}} \cdots \times^{G_{\mathcal{O}}^{(|I_{k-1}|j)}} G_{\mathcal{K}}^{(|I_{k-1}|N,|I_k|j)} \times^{G_{\mathcal{O}}^{(|I_k|j)}} \mathrm{Gr}^{(|I_k|N)} \tag{2.8}$$

which exhibits the target as the quotient of the source with respect to the action of $\prod_{i=2}^k G_{\mathcal{O}}^{(|I_i|j)}$ inducing the twisted product on the right-hand-side. The map (2.8) is therefore equivariant with respect to the morphism of groups $G_{\mathcal{O}}^{(|I|j)} \times \prod_{i=2}^k G_{\mathcal{O}}^{(|I_i|j)} \to G_{\mathcal{O}}^{(|I|j)}$ given by projection on the first component, which concludes the proof. $\square$

Remark 2.59. Let $N \geq 0, j \geq j_N, I_1, \dots, I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. As a consequence of the observations made in Remark 2.56, by passing to the quotient in (2.3) with respect to the actions listed in Remark 2.56, we obtain a diagram of quotient stacks

$$
\begin{tikzcd}
 & \mathrm{Hck}^{(N,j)}_{I_1,\dots,I_k} \arrow[dl, "\overline{p}^{(N,j)}_{I_1,\dots,I_k}"'] \arrow[dr, "\overline{m}^{(N)}_{I_1,\dots,I_k}"] & \\
\mathrm{Hck}^{(N,j)}_{I_1} \times \cdots \times \mathrm{Hck}^{(N,j)}_{I_k} & & \mathrm{Hck}^{(N,j)}_{I_1 \sqcup \cdots \sqcup I_2}.
\end{tikzcd}
$$

Definition 2.60. We define $(\mathrm{Ran}(X)\times\mathrm{Ran}(X))_{\mathrm{disj}}$ as the open subfunctor of $\mathrm{Ran}(X)\times\mathrm{Ran}(X)$ parametrizing

$$\{(S,T)\in\mathrm{Ran}(X)\times\mathrm{Ran}(X)\mid\Gamma_S\cap\Gamma_T=\varnothing\}.$$

More generally, we define $\mathrm{Ran}(X)^{2k}_{\mathrm{disj}}=\{S_1,\dots,S_k,T_1,\dots,T_k\in\mathrm{Ran}(X)\mid\Gamma_{S_i}\cap\Gamma_{T_i}=\varnothing\ \forall i=1,\dots,k\}\subset\mathrm{Ran}(X)^{2k}$.

Remark 2.61. Recall that we have a natural map in $\widehat{\mathrm{StrStk}}^{\mathrm{lft}}_{\mathbb{C}}$ from $\mathrm{Hck}_{\mathrm{Ran},k}$ to $\mathrm{Ran}(X)^k$ (the one that only "remember" the systems of points).

We will now establish a semisimplicial structure on the collection of the $\mathrm{Hck}_{\mathrm{Ran},k}$'s.

Construction 2.62. Let $k\geq 1, 0\leq i\leq k$, and let d_i be the injective ordered map $[k-1]\to[k]$ missing the index i. For each $I_1,\dots,I_k\in\mathrm{Fin}_{\geq 1,\mathrm{surj}}$, $j\geq j_1$, we have maps

$i=0)$ $\delta^{(j)}_{0,I_1,\dots,I_k}:\mathrm{Hck}^{(j)}_{I_1,\dots,I_k}\to\mathrm{Hck}^{(j)}_{I_2,\dots,I_k}$ induced by the projection

$$G^{(j)}_{\mathcal{K},I_1,I_2}\times_{X^{I_2}}G^{(j)}_{\mathcal{K},I_2,I_3}\times_{X^{I_3}}\cdots\times_{X^{I_k}}\mathrm{Gr}^{(1)}_{I_k}\to G^{(j)}_{\mathcal{K},I_2,I_3}\times_{X^{I_3}}\cdots\times_{X^{I_k}}\mathrm{Gr}^{(1)}_{I_k}$$

that forgets the first component.

$i=k)$ $\delta^{(j)}_{k,I_1,\dots,I_k}:\mathrm{Hck}^{(j)}_{I_1,\dots,I_k}\to\mathrm{Hck}^{(j)}_{I_1,\dots,I_{k-1}}$ induced by the composition of the projection

$$G^{(j)}_{\mathcal{K},I_1,I_2}\times_{X^{I_2}}G^{(j)}_{\mathcal{K},I_2,I_3}\times_{X^{I_3}}\cdots\times_{X^{I_k}}\mathrm{Gr}^{(1)}_{I_k}\to G^{(j)}_{\mathcal{K},I_1,I_2}\times_{X^{I_3}}\cdots\times_{X^{I_{k-1}}}G^{(j)}_{\mathcal{K},I_{k-1},I_k}$$

with the map induced by the quotient $G^{(j)}_{\mathcal{K},I_{k-1},I_k}\to\mathrm{Gr}^{(1)}_{I_k}$ (recall that, as usual, $G^{(j)}_{\mathcal{K},I_{k-1},I_k}$ stays for $G^{(1,j)}_{\mathcal{K},I_{k-1},I_k}$).

$i\neq 0,k)$ $\delta^{(j)}_{i,I_1,\dots,I_k}:\mathrm{Hck}^{(j)}_{I_1,\dots,I_k}\to\mathrm{Hck}^{(j)}_{I_1,\dots,I_i\sqcup I_{i+1},\dots,I_k}$ induced by the maps

$$G^{(j)}_{\mathcal{K},I_i,I_{i+1}}\times G_{\mathcal{K},I_{i+1},I_{i+2}}\to G^{(j)}_{\mathcal{K},I_i\sqcup I_{i+1}}$$

(for $i<k-1$) or

$$G^{(j)}_{\mathcal{K},I_{k-1},I_k}\times\mathrm{Gr}_{\mathcal{K},I_k}\to\mathrm{Gr}^{(1)}_{\mathcal{K},I_{k-1}\sqcup I_k}$$

(for $i=k-1$) defined in (2.4) and following.

We can thus define a map

$$\delta_i=\underset{I_1,\dots,I_k\in\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}}{\mathrm{colim}}\ \text{“}\lim_{j\geq j_1}\text{”}\,\delta^{(j)}_{i,I_1,\dots,I_k}:\mathrm{Hck}_{\mathrm{Ran},k}\to\mathrm{Hck}_{\mathrm{Ran},k-1}.$$

Remark 2.63. The maps defined in Construction 2.62 are stratified. Ultimately, this can be reduced to the following statement: for any point $x \in X$, the map

$$\delta_{i,x,m_j} : \mathrm{Hck}_{x,m_j} \to \mathrm{Hck}_{x,m_j-1}$$

which glues the data in the places $i, i+1$ is stratified. This is implied by [MV07, Lemma 4.4].

Proposition 2.64. *Construction 2.62 defines a semisimplicial object, i.e. the given maps satisfy the simplicial identities.*

Proof. We need to prove that, for every $k \geq 1, 0 \leq i < h \leq k$, there are isomorphisms

$$\delta_i \delta_h \simeq \delta_{h-1} \delta_i.$$

Equivalently, we prove that the statement is true for the $\delta^{(j)}_{i,I_1,\ldots,I_k}$'s, and moreover, it suffices to prove it after pulling back to strata of X^I, where $I = I_1 \sqcup \cdots \sqcup I_k$. This last reduction step is not completely obvious, since for instance, the equality of two morphism of schemes which are not reduced cannot, in general, be checked on strata. However, by [Tao20, Theorem 1.2.1], $\mathrm{Gr}_{\mathrm{Ran}}$ can be written ad

$$\operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{Gr}^{\mathrm{red}}_I$$

and therefore passing to reductions does not change the object at the Ran level. The same argument holds for the $\mathrm{Hck}_{\mathrm{Ran},k}$, because we have a map $\mathrm{Hck}_{\mathrm{Ran},k} \to \mathrm{Gr}^{\times k}_{\mathrm{Ran}}$ which is a relative $G_{\mathcal{O},\mathrm{Ran}}$ torsor, in particular a smooth map. In other words,

- For $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, $G_{\mathcal{O},I}$ is smooth by Remark 2.23, hence reduced.
- For $I_1, I_2 \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, $G_{\mathcal{K},I_1,I_2}$ is a $G_{\mathcal{O},I_2}$-torsor over $\mathrm{Gr}_{I_1} \times X^{I_2}$, relative to X^{I_2}. If we pull this torsor back along $\mathrm{Gr}^{\mathrm{red}}_{I_1} \to \mathrm{Gr}_{I_1}$, we obtain another $G_{\mathcal{O},I_2}$-torsor, whose source is now reduced by étale-localness [SPA, `Tag 06QM`], smoothness of the fiber and reducedness of the base. By the universal property of the reduction, we get that

$$G_{\mathcal{K},I_1,I_2} \times_{\mathrm{Gr}_{I_1}} \mathrm{Gr}^{\mathrm{red}}_{I_1} \simeq G^{\mathrm{red}}_{\mathcal{K},I_1,I_2}.$$

- By a similar argument,

$$\mathrm{Conv}_{I_1,\ldots,I_k} \simeq \mathrm{Conv}_{I_1,\ldots,I_k} \times_{\mathrm{Gr}_{I_1} \times \cdots \times \mathrm{Gr}_{I_k}} \mathrm{Gr}^{\mathrm{red}}_{I_1} \times \cdots \times \mathrm{Gr}^{\mathrm{red}}_{I_k}.$$

  More precisely, we use the previous point to deduce the statement for

$$G_{\mathcal{K},I_1,I_2} \times_{X^{I_2}} \cdots \times_{X^{I_k}} \mathrm{Gr}_{I_k},$$

  and then again étale descent for reducedness to obtain the statement for the convolution Grassmannian.

- We have

$$\begin{aligned} \mathrm{Hck}^{(j)}_{I_1,\ldots,I_k})^{\mathrm{red}} \simeq G^{(j)}_{\mathcal{O},I_1 \sqcup \cdots \sqcup I_k} \backslash \mathrm{Conv}^{\mathrm{red}}_{I_1,\ldots,I_k} &\simeq G^{(j)}_{\mathcal{O},I_1,\ldots,I_k} \backslash (\mathrm{Conv}_{I_1,\ldots,I_k} \times_{\mathrm{Gr}_{I_1} \times \cdots \times \mathrm{Gr}_{I_k}} \mathrm{Gr}^{\mathrm{red}}_{I_1} \times \cdots \times \mathrm{Gr}^{\mathrm{red}}_{I_k} \\ &\simeq \mathrm{Hck}^{(j)}_{I_1,\ldots,I_k} \times_{\mathrm{Gr}_{I_1} \times \cdots \times \mathrm{Gr}_{I_k}} \mathrm{Gr}^{\mathrm{red}}_{I_1} \times \cdots \times \mathrm{Gr}^{\mathrm{red}}_{I_k} \end{aligned}$$

  where we used étale descent for reducedness for the first equivalence, the previous point for the second equivalence, and universality of quotients for the last equivalence.

- Finally, we apply the previous point, the fact that the functor $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ preserves finite limits, that limits commute with limits, and again universality of colimits to obtain

$$\mathrm{Hck}_{\mathrm{Ran},k} = \operatorname*{colim}_{I_1,\dots,I_k \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \text{“}\lim_{j \geq j_1}\text{”}(\mathrm{Hck}^{(j)}_{I_1,\dots,I_k})^{\mathrm{red}}.$$

We can thus prove that the statement is true for the reduction of the $\delta^{(j)}_{i,I_1,\dots,I_k}$'s, and this statement can be checked on strata. Moreover, by [Tao20, Lemma 4.2.2], the factorization property holds for $\mathrm{Gr}^{\mathrm{red}}_{\mathrm{Ran}}$ as well (and with a similar proof for its variations), which allows us to use factorization also in the reduced setting.

By factorization, we reduce to two cases:

- proving the statement over the stratum

$$\mathcal{S}_{I_1,\dots,I_k,\mathrm{disj}} = \{x_I \mid \forall i', h' \leq k, x_a \neq x_b \forall a \in I_{i'}, b \in I_{h'}\}.$$

  This case follows again by factorization.

- proving the statement after pullback to the diagonal $X \to X^I$. This case follows from [NP24a].

□

Definition 2.65. We denote the semisimplicial object established in Proposition 2.64 by

$$\mathrm{Hck}_{\mathrm{Ran},\bullet} : \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \widehat{\mathrm{StrStk}}^{\mathrm{lft}}_{\mathbb{C}}.$$

The crucial property of this structure, in order to encode the associativity of the convolution product, is the following:

Proposition 2.66. *The semisimplicial object* $\mathrm{Hck}_{\mathrm{Ran},\bullet}$ *enjoys the 2-Segal property, that is the equivalent conditions of [DK19, Proposition 2.3.2].*

Proof. We will prove the case $k = 3$, for simplicity. First of all, we prove that we can reduce to proving that we can reduce to proving that the map

$$\mathrm{Hck}^{(j)}_{I_1,I_2,I_3} \to \mathrm{Hck}^{(j)}_{I_2,I_3} \times_{\mathrm{Hck}^{(j)}_{I_2 \sqcup I_3}} \mathrm{Hck}^{(j)}_{I_1,I_2 \sqcup I_3} \tag{2.9}$$

is an equivalence for every $I_1, I_2, I_3, j \geq j_1$, where the maps in the pullback are respectively $(\delta^{(j)}_{1,I_2,I_3} \circ \delta^{(j)}_{0,I_1,I_2,I_3})$ and $\delta^{(j)}_{2,I_1,I_2,I_3}$. Indeed, since limits commute with limits, we only need to prove that we can check the statement at the level of pro-objects (i.e. before passing to colimits in $I_1, I_2, I_2 \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}$). Consider the square

$$\begin{tikzcd}
\mathrm{Hck}_{\mathrm{Ran},I_2,I_3} \arrow[r] \arrow[d] & \mathrm{Hck}_{\mathrm{Ran},I_2 \sqcup I_3} \arrow[r] \arrow[d] & \mathrm{Hck}_{\mathrm{Ran},2} \arrow[d] \\
\mathrm{Hck}_{I_2,I_3} \arrow[r] & \mathrm{Hck}_{I_2 \sqcup I_3} \arrow[r] & \mathrm{Hck}_{\mathrm{Ran}}
\end{tikzcd} \tag{2.10}$$

where $\mathrm{Hck}_{\mathrm{Ran},I_2,I_3} = \mathrm{colim}_{I_1 \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{Hck}_{I_1,I_2,I_3}$ and similarly for $\mathrm{Hck}_{\mathrm{Ran},I_2 \sqcup I_3}$. If we assume the statement about (2.9), then the left-hand square is Cartesian by universality of colimits. The right-hand square is also Cartesian, because front, bottom and back faces of the commutative cube

$$\begin{tikzcd}
& \mathrm{Hck}_{\mathrm{Ran},I_2\sqcup I_3} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{Hck}_{\mathrm{Ran},2} \arrow[dl] \arrow[dd] \\
\mathrm{Hck}_{I_2\sqcup I_3} \arrow[rr, crossing over] \arrow[dd] & & \mathrm{Hck}_{\mathrm{Ran}} & \\
& \mathrm{Ran}(X)\times X^{I_2\sqcup I_3} \arrow[rr] \arrow[dl, "\mathrm{pr}_2"] & & \mathrm{Ran}(X)\times\mathrm{Ran}(X) \arrow[dl, "\mathrm{pr}_2"] \\
X^{I_2\sqcup I_3} \arrow[rr] & & \mathrm{Ran}(X) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

are Cartesian. Hence the outer square in (2.10) is Cartesian, and by universality of colimits in a category of presheaves we conclude.

Let us thus prove the statement for the $\mathcal{S}^{(j)}_{I_1,\dots,I_k}$'s. By factorization (and the workaround including passing to reductions explained in the proof of Proposition 2.64), it suffices to:

- prove the statement after restricting to the stratum

  $$\mathcal{S}_{I_1,I_2,I_3,\mathrm{disj}} = \{x_{I_1}, x_{I_2}, x_{I_3} \mid x_a \neq x_b \forall a \in I_2, b \in I_2 \text{ or } a \in I_2, b \in I_3 \text{or } a \in I_1, b \in I_3\}.$$

  There, by Proposition 2.11, Proposition 2.12 and Proposition 2.55, the statement is equivalent to proving that the square

  $$\begin{tikzcd}
(\mathrm{Hck}^{(j)}_{I_1} \times \mathrm{Hck}^{(j)}_{I_2} \times \mathrm{Hck}^{(j)}_{I_3})|_{\mathcal{S}_{I_1,I_2,I_3,\mathrm{disj}}} \arrow[r, equal] \arrow[d] & (\mathrm{Hck}^{(j)}_{I_1} \times \mathrm{Hck}^{(j)}_{I_2} \times \mathrm{Hck}^{(j)}_{I_3})|_{\mathcal{S}_{I_1,I_2,I_3,\mathrm{disj}}} \arrow[d] \\
(\mathrm{Hck}^{(j)}_{I_2} \times \mathrm{Hck}^{(j)}_{I_3})|_{\mathcal{S}_{I_2,I_3,\mathrm{disj}}} \arrow[r, equal] & (\mathrm{Hck}^{(j)}_{I_2} \times \mathrm{Hck}^{(j)}_{I_3})|_{\mathcal{S}_{I_2,I_3,\mathrm{disj}}}
\end{tikzcd}$$

  is Cartesian, which is of course true.

- prove the statement after restriction to the diagonal. We can as well assume $|I_1| = |I_2| = |I_3|$, up to considering suitable surjections $I_i' \to I_i, i = 1,2,3$. Then we need to prove that, for any $N \in \mathbb{N}, j \geq j_N$, the diagram

  $$\begin{tikzcd}
G^{(j)}_{\mathcal{O}}\backslash G^{(N,j)}_{\mathcal{K}} \times^{G^{(j)}_{\mathcal{O}}} G^{(N,j)}_{\mathcal{K}} \times^{G^{(j)}_{\mathcal{O}}} \mathrm{Gr}^{(N)} \arrow[r, "m_{23}"'] \arrow[d, "p_{23}"] & G^{(j)}_{\mathcal{O}}\backslash G^{(N,j)}_{\mathcal{K}} \times^{G^{(j)}_{\mathcal{O}}} \mathrm{Gr}^{(2N)} \arrow[d, "p_2"'] \\
G^{(j)}_{\mathcal{O}}\backslash G^{(N,j)}_{\mathcal{K}} \times^{G^{(j)}_{\mathcal{O}}} \mathrm{Gr}^{(N)} \arrow[r, "m"] & G_{\mathcal{O},j}\backslash \mathrm{Gr}^{(2N)}
\end{tikzcd}$$

  is Cartesian (note that here we do need N to be arbitrary, since pulling back to diagonals takes into account the cardinalities of the I_i's). This latter claim is true by [NP24a, Proposition 2.3] with $H = G_{\mathcal{K}}, K = G_{\mathcal{O}}$.

□

Theorem 2.67. *Let G be a complex reductive group and X a complex smooth curve. The object*

$$\mathrm{Hck}_{\mathrm{Ran}} \in \widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}}$$

from Construction 2.33 carries a nonunital associative algebra structure in $\mathrm{Corr}(\widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}})^{\times}$*, extending the convolution diagram* (2.59).

The fiber of this algebra structure at any singleton $\{x\} \in \mathrm{Ran}(X)$ *encodes the convolution diagram for the Hecke stack (Remark A.13) and its associativity.*

Proof. It suffices to apply [DK19, Proposition 8.1.5] to the 2-Segal semisimplicial object $\mathrm{Hck}_{\mathrm{Ran},\bullet}$ (Construction 2.62, Proposition 2.66). □

3 Topological factorization of the Hecke stack

3.1 Consequences of analytification

Definition 3.1. Let M be a topological manifold of dimension m. The *Ran space* of M is the set of nonempty finite subsets of M, endowed with its so-called *metric topology*, i.e. the topology induced by the following base:

$$\{\prod_{i=1}^{k} \mathrm{Ran}(D_i) \mid \{D_i\} \text{ finite family of pairwise disjoint disks in } M\}.$$

Note that the given family is actually a family of subsets of $\mathrm{Ran}(M)$, in the sense that the union map $\prod_{i=1}^{k} \mathrm{Ran}(D_i) \to \mathrm{Ran}(M)$ is injective whenever the disks are pairwise disjoint. We denote by $\star_i \mathrm{Ran}(D_i)$ the image of this map.

Remark 3.2. The set $\mathrm{Ran}(M)$ can be equivalently presented as the $\mathrm{colim}_{I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}} M^I$ in Set. This carries a natural colimit topology, which is finer than the one presented above. For a thorough comparison between those, and many further considerations, see [CL21] and [DL23]. When unspecified, by $\mathrm{Ran}(M)$ we will always mean the Ran space with its metric topology.

One of the main features of the Ran space is to encode the notion of *factorization algebra* in a particularly efficient way. Actually, a theorem by Jacob Lurie says even more, showing that a family over the Ran space with some conditions induces a "sheaf of $\mathbb{E}_m$-algebras" over M (m being the dimension of M).

Theorem 3.3 (part of [Lur17, Theorem 5.5.4.10])**.** *Let M be a topological manifold of dimension m, and $\mathcal{C}^{\otimes}$ a symmetric monoidal ∞-category where the tensor product preserves colimits separately in both variables. Any* constructible factorizable *cosheaf over* $\mathrm{Ran}(M)$ *with values in $\mathcal{C}$ induces a nonunital $\mathbb{E}_m$-algebra structure on its stalk at any singleton* $\{x\} \in \mathrm{Ran}(M)$.

Recall 3.4. We refer to [Lur17, Section 5.5.4] for the definitions. Here we just recall that:

- two open subsets U, V of $\mathrm{Ran}(M)$ are declared to be *independent* if for every $S \in U, T \in V$, then $S \cap T = \varnothing$. For example, for any two open subsets D, D' of M, $\mathrm{Ran}(D)$ and $\mathrm{Ran}(D')$ are independent if and only if $D \cap D' = \varnothing$. For U, V independent open subsets of $\mathrm{Ran}(M)$, one denotes $U \star V = \{S \sqcup T \mid S \in U, T \in V\}$. Again, this is homeomorphic to $U \times V$.

- there is an operadic structure $\mathrm{Fact}(M)^{\otimes}$ over the category of open subsets of $\mathrm{Ran}(M)$, encoding the "partial operation" $U \star V$ for independent U, V.
- a *factorizable cosheaf* over $\mathrm{Ran}(M)$ is a map of operads

$$A^{\otimes} : \mathrm{Fact}(M)^{\otimes} \to \mathcal{C}^{\otimes}$$

  satisfying the following conditions:

  - the underlying functor $A : \mathrm{Open}(\mathrm{Ran}(M)) \to \mathcal{C}$ is a cosheaf;
  - for any two independent open subsets U, V of $\mathrm{Ran}(M)$, the map $A(U) \otimes^{\mathcal{C}} A(V) \to A(U \star V)$ induced by the fact that A is a map of operads is an equivalence in $\mathcal{C}$.

- such a functor is *constructible* if, as a cosheaf over $\mathrm{Ran}(M)$ with values in $\mathcal{C}$, it is hypercomplete and satisfies the following (cf. [Lur17, Proposition 5.5.1.14]). Let $\{D_i\}_{i\in I}$, $\{D_i'\}_{i\in I}$ be finite sequences of disks in M such that $D_i \cap D_j = D_i' \cap D_j' = \varnothing$ for $i \neq j$ and $D_i \subset D_i'$. Then A sends the induced map

$$\star_i \mathrm{Ran}(D_i) \hookrightarrow \star_i \mathrm{Ran}(D_i') \tag{3.1}$$

  into an equivalence in $\mathcal{C}$.

Recall the definition of the functor

$$(-)^{\mathrm{an}} : \widehat{\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}} \to \widehat{\mathrm{StrTStk}},$$

from Corollary B.46. By construction, this functor preserves finite limits. By composing the semisimplicial object $\mathrm{Hck}_{\mathrm{Ran},\bullet}$ from Definition 2.65 with $(-)^{\mathrm{an}}$ we obtain a semisimplicial object $\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},\bullet}$, which is again 2-Segal because $(-)^{\mathrm{an}}$ preserves finite limits. This object comes with a natural map towards $\mathrm{Ran}(X)^{\mathrm{an}}$ with the colimit topology (by functoriality) and therefore towards $\mathrm{Ran}(X^{\mathrm{an}})$ with the metric topology described in Definition 3.1. For instance, if we consider $\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},1}$, this is given by

$$\operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \text{“}\lim\text{”}_{j \geq j_1} (G^{(j)}_{\mathcal{O},I})^{\mathrm{an}} \backslash (\mathrm{Gr}^{(1)}_{G,I})^{\mathrm{an}}.$$

Note that the single terms appearing in this formula are quotients in the category of stratified topological stacks (Notation B.16).

From now on, when proving a property for $\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},k}$ we will always argue as "we prove the property for each term and the property is stable under the relevant colimits and formal limits".

Definition 3.5. For any $U \subset \mathrm{Ran}(X^{\mathrm{an}})$ open subset, and $k \geq 1$, let

$$\mathrm{Conv}_{U,k} = \mathrm{Conv}^{\mathrm{an}}_{\mathrm{Ran},k} \times_{\mathrm{Ran}(X^{\mathrm{an}})^k} U^k \in \widehat{\mathrm{StrTStk}}$$
$$\mathrm{Hck}_{U,k} = \mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},k} \times_{\mathrm{Ran}(X^{\mathrm{an}})^k} U^k \in \widehat{\mathrm{StrTStk}}.$$

We define functors

$$\mathrm{Open}(\mathrm{Ran}(M)) \times \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \widehat{\mathrm{StrTStk}}$$
$$(U, [k]) \mapsto \mathrm{Conv}_{U,k}$$

and

$$\begin{aligned}\mathrm{Open}(\mathrm{Ran}(M)) \times \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} &\to \widehat{\mathrm{StrTStk}}\\ (U,[k]) &\mapsto \mathrm{Hck}_{U,k}\end{aligned}$$

on objects, and in the natural way on morphisms.

Indeed, an inclusion $U \subset V$ naturally induces embeddings

$$\begin{aligned}\mathrm{Conv}_{U,k} &\hookrightarrow \mathrm{Conv}_{V,k}\\ \mathrm{Hck}_{U,k} &\hookrightarrow \mathrm{Hck}_{V,k}\end{aligned}$$

which are clearly natural in k.

Remark 3.6. Note that by universality of colimits

$$\mathrm{Conv}_{U,k} \simeq \operatorname*{colim}_{I_1,\dots,I_k} \mathrm{Conv}^{\mathrm{an}}_{I_1,\dots,I_k} \times_{\mathrm{Ran}(X^{\mathrm{an}})^k} U^k.$$

resp.

$$\mathrm{Hck}_{U,k} \simeq \operatorname*{colim}_{I_1,\dots,I_k} \mathrm{Hck}^{\mathrm{an}}_{I_1,\dots,I_k} \times_{\mathrm{Ran}(X^{\mathrm{an}})^k} U^k.$$

We denote the terms appearing in the colimit by $\mathrm{Conv}_{U,I_I,\dots,I_k}$ resp. $\mathrm{Hck}_{U,I_1,\dots,I_k}$ or, when $U = \mathrm{Ran}(D)$ for a disk D, by $\mathrm{Conv}_{D^{I_1},\dots,D^{I_k}}$ resp. $\mathrm{Hck}_{D^{I_1},\dots,D^{I_k}}$, to stress the fact that each of them is equivalent to

$$\mathrm{Conv}^{\mathrm{an}}_{I_1,\dots,I_k} \times_{X^{I_1\sqcup\cdots\sqcup I_k}} D^{I_1\sqcup\cdots\sqcup I_k}$$

resp.

$$\mathrm{Hck}^{\mathrm{an}}_{I_1,\dots,I_k} \times_{X^{I_1\sqcup\cdots\sqcup I_k}} D^{I_1\sqcup\cdots\sqcup I_k}.$$

Definition 3.5 is the first step in the direction of building a factorization algebra out of $\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},\bullet}$. The following step is to upgrade $\mathrm{Hck}^{\mathrm{fact}}_{\bullet}$ to a functor

$$\mathrm{Fact}(M)^{\otimes} \times \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \widehat{\mathrm{StrTStk}}^{\times}$$

suitably encoding the "factorization property" of the Beilinson–Drinfeld Grassmannian.

Notation 3.7. Let M be a topological manifold. We denote by

$$(\mathrm{Ran}(M)^k \times \mathrm{Ran}(M)^k)_{\mathrm{disj}}$$

the topological space

$$\{S_1,\dots,S_k,T_1\dots,T_k \mid S_i \cap T_i = \varnothing, i = 1,\dots,k\} \subset \mathrm{Ran}(M)^k \times \mathrm{Ran}(M)^k.$$

Remark 3.8. Consider two independent open subsets U and V of $\mathrm{Ran}(X^{\mathrm{an}})$. We have the following diagram

$$\begin{tikzcd}
\mathrm{Hck}_{U,k} \times \mathrm{Hck}_{V,k} \arrow[r] \arrow[d, "\pi"] & (\mathrm{Hck}_{\mathrm{Ran},k} \times \mathrm{Hck}_{\mathrm{Ran},k})^{\mathrm{an}}_{\mathrm{disj}} \arrow[r] \arrow[d] & \mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},k} \arrow[d] \\
U^k \times V^k \arrow[r, "\subset"] & (\mathrm{Ran}(X^{\mathrm{an}})^k \times \mathrm{Ran}(X^{\mathrm{an}})^k)_{\mathrm{disj}} \arrow[r, "\mathrm{union}"] & \mathrm{Ran}(X^{\mathrm{an}})^k,
\end{tikzcd} \tag{3.2}$$

where the left hand square is by definition a pullback in $\widehat{\mathrm{StrTStk}}$, and the right top horizontal map is induced by Construction 2.62 and factorization with $k=2$, and then by applying $(-)^{\mathrm{an}}$. In concrete, this map is induced by the maps

$$\mathrm{Gr}_{I_1} \times \mathrm{Gr}_{I_2}|_{(X^{I_1}\times X^{I_2})_{\mathrm{disj}}} \to \mathrm{Gr}_{I_1 \sqcup I_2}$$
$$(x_{I_1}, \mathcal{F}, \alpha, x_{I_2}, \mathcal{G}, \beta) \mapsto (x_{I_1} + x_{I_2}, \mathcal{H}, \gamma)$$

where $\mathcal{H}$ is the gluing of $\mathcal{F}$ and $\mathcal{G}$ along the isomorphism $\beta^{-1}|_{X\setminus\Gamma_{x_{I_1}+x_{I_2}}} \circ \alpha|_{X\setminus\Gamma_{x_{I_1}+x_{I_2}}}$ and γ is the trivialization inherited from α and β via the gluing procedure.

Note also that the bottom composition in (3.2) coincides with $U \times V \to U \star V \hookrightarrow \mathrm{Ran}(X^{\mathrm{an}})$, the first map being the one taking unions of systems of points; hence, by the universal property of the fibered product, $\mathrm{Hck}_{U,k} \times \mathrm{Hck}_{V,k}$ admits a map towards $\mathrm{Hck}_{U\star V,k}$, which we call $\chi_{U,V,k}$.

Proposition 3.9. *Let $k \in \mathbb{N}$. There is a well-defined map of operads*

$$\mathrm{Hck}_k^{\mathrm{fact}} : \mathrm{Fact}(M)^{\otimes} \to \widehat{\mathrm{StrTStk}}^{\times}$$

assigning

$$(U_1, \dots, U_m) \mapsto \mathrm{Hck}_{U_1,k} \times \dots \times \mathrm{Hck}_{U_m,k}.$$
$$(U \subset V) \mapsto [\mathrm{Hck}_{U,k} \hookrightarrow \mathrm{Hck}_{V,k}]$$
$$((U,V) \to (U \star V)) \mapsto \chi_{U,V,k} : \mathrm{Hck}_{U,k} \times \mathrm{Hck}_{V,k} \to \mathrm{Hck}_{U\star V,k}.$$

Proof. To prove that this is this association is functorial, it suffices to prove that the diagram

$$\begin{tikzcd}
 & \mathrm{Conv}_{U\star V,k} \times \mathrm{Conv}_{W,k} \arrow[dr, "\chi_{U\star V,W,k}"] & \\
\mathrm{Conv}_{U,k} \times \mathrm{Conv}_{V,k} \times \mathrm{Conv}_{W,k} \arrow[ur, "\chi_{U,V,k}\times \mathrm{id}"] \arrow[dr, "\mathrm{id}\times\chi_{V,W,k}"'] & & \mathrm{Conv}_{U\star V\star W,k} \\
 & \mathrm{Conv}_{U,k} \times \mathrm{Conv}_{V\star W,k} \arrow[ur, "\chi_{U,V\star W,k}"'] &
\end{tikzcd}$$

(notations as in Definition 3.5) commutes in $\widehat{\mathrm{StrTStk}}$. Now this is true because the operation of gluing is associative, as it is easily checked by means of the defining property of the gluing of sheaves.
Finally, to prove that the functor $\mathrm{Hck}_k^{\mathrm{fact}}$ is a map of operads, we use the characterization of inert morphisms in a Cartesian structure provided by [Lur17, Proposition 2.4.1.5]. Note that:

- An inert morphism in $\mathrm{Fact}(M)^{\otimes}$ is a morphism of the form

$$(U_1, \dots, U_m) \to (U_{\phi^{-1}(1)}, \dots U_{\phi^{-1}(n)})$$

  covering some inert arrow $\phi : \langle m \rangle \to \langle n \rangle$ where every $i \in \langle n \rangle^{\circ}$ has exactly one preimage $\phi^{-1}(i)$.

- An inert morphism in $\widehat{\mathrm{StrTStk}}^{\times}$ is a morphism of functors $\overline{\alpha}$ between $f : \mathscr{P}(\langle m \rangle^{\circ})^{\mathrm{op}} \to \widehat{\mathrm{StrTStk}}$ and $g : \mathscr{P}(\langle n \rangle^{\circ})^{\mathrm{op}} \to \widehat{\mathrm{StrTStk}}$, covering some $\alpha : \langle m \rangle \to \langle n \rangle$, and such that, for any $S \subset \langle n \rangle$, the map induced by $\overline{\alpha}$ from $f(\alpha^{-1}S) \to g(S)$ is an equivalence in $\widehat{\mathrm{StrTStk}}$.

By definition, $\mathrm{Hck}_k^{\mathrm{fact}}(U_1, \dots, U_m)$ is the functor f assigning

$$T \subset \langle m \rangle^\circ \mapsto \prod_{j \in T} \mathrm{Hck}_{U_j, k},$$

and analogously $\mathrm{Hck}_k^{\mathrm{fact}}(U_{\phi^{-1}(1)}, \dots, U_{\phi^{-1}(m)})$ is the functor g assigning

$$S \subset \langle n \rangle^\circ \mapsto \prod_{i \in S} \mathrm{Hck}_{U_{\phi^{-1}(i)}, k}.$$

But now, if $\alpha = \phi$ and $T = \phi^{-1}(S)$, we have the desired equivalence.

□

For the following result, we specialize to the case $X = \mathbb{A}^1_{\mathbb{C}}$. Consider the class $\mathrm{Ball}(\mathbb{C})$ of metric balls in $\mathbb{A}^1_{\mathbb{C}}$, i.e. open subspaces of the form

$$\{z \in (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \mid |z - z_0| < r\}$$

for $z_0 \in (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$, $r \in \mathbb{R}_{>0}$. Let $D_i' \subset D_i$, $i \in I$ nonempty finite set, be elements in $\mathrm{Ball}(\mathbb{C})$. There is an inclusion

$$\star_{i \in I} \mathrm{Ran}(D_i') \subset \star_{i \in I} \mathrm{Ran}(D_i) \tag{3.3}$$

which in turn induces a map

$$\mathrm{Hck}_{\mathrm{Ran}(D')} \to \mathrm{Hck}_{\mathrm{Ran}(D)}$$

in $\widehat{\mathrm{StrTStk}}$.

Recall the notion of stratified homotopy equivalence (she and $\widehat{\mathsf{she}}$) from Definition B.28.

Proposition 3.10. *Let $X = \mathbb{A}^1_{\mathbb{C}}$, $k \in \mathbb{N}$. The functor $\mathrm{Hck}_k^{\mathrm{fact}}$ from Proposition 3.9 satisfies the factorization property (see Recall 3.4) and sends maps of the form* (3.3) *to* $\widehat{\mathsf{she}}$ *in* $\mathrm{Mor}(\widehat{\mathrm{StrTStk}})$.

Proof. The fact that the analytification functor preserves finite limits, together with Proposition 2.55 for $k = 2$ and the consideration that the diagram

$$\begin{array}{ccc} (\mathrm{Ran}(X)^{\mathrm{an}} \times \mathrm{Ran}(X)^{\mathrm{an}})_{\mathrm{disj}} & \longrightarrow & \mathrm{Ran}(X)^{\mathrm{an}} \\ \downarrow & & \downarrow \\ (\mathrm{Ran}(X^{\mathrm{an}}) \times \mathrm{Ran}(X^{\mathrm{an}}))_{\mathrm{disj}} & \longrightarrow & \mathrm{Ran}(X^{\mathrm{an}}) \end{array}$$

(where in the first row we are considering the colimit topology and in the second row we are considering the metric topology) is Cartesian, achieves the factorization property by pulling back everything along $U^k \times V^k \to (\mathrm{Ran}(X^{\mathrm{an}})^k \times \mathrm{Ran}(X^{\mathrm{an}})^k)_{\mathrm{disj}}$.

As for the second property, by using factorization we reduce to the case of a single inclusion of disks $i : D' \subset D$. That is, we need to prove that the induced map

$$\mathrm{Hck}_{\mathrm{Ran}(D'), k} \to \mathrm{Hck}_{\mathrm{Ran}(D), k}$$

is a stratified homotopy equivalence in $\widehat{\mathrm{StrTStk}}$. This amounts to proving that for each $I_1, \dots, I_k \in \mathrm{Fin}_{\geq 1, \mathrm{surj}}$, $N \in \mathbb{N}$, the maps

$$\mathrm{Conv}^{(N)}_{(D')^{I_1}, \dots, (D')^{I_k}} \to \mathrm{Conv}^{(N)}_{D^{I_1}, \dots, D^{I_k}}$$
$$G_{\mathcal{O}, (D')^{I_1}} \to G_{\mathcal{O}, D^{I_1}}$$

are resp. stratified homotopy equivalences and homotopy equivalences, and in a compatible way (i.e. the homotopy inverse of the first map is equivariant with respect to the second one, and one can choose homotopies which are compatible with respect to the actions on source and target). A sketch of the proof for the first one, in the case $k=1$, is given in [HY19, Proposition 3.17]. The full statement is proven in [NP24b]. $\square$

Remark 3.11. From the properties of Recall 3.4, we did not prove that the association $U \mapsto \mathrm{Hck}_{U,k}$ is a hypercomplete cosheaf. This property becomes true after taking categories of constructible sheaves, that is after applying the functor Cons from Theorem B.66: see Proposition 4.5. Note that for $I_1,\dots,I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, $j \geq j_1$, the functor $U \mapsto \mathrm{Hck}^{(j)}_{I_1,\dots,I_k}$ is indeed a hypercomplete cosheaf by Lemma 4.4. However, this does not trivially extend to the functor $U \mapsto \mathrm{Hck}_{I_1,\dots,I_k}$, because cofiltered limits do not commute with sifted colimits and hence one cannot formally transfer descent to the level of pro-objects.

Note as well that we only proved that maps of the form (3.3) are sent to stratified homotopy equivalences, which are not equivalences in $\widehat{\mathrm{StrTStk}}$. However, stratified homotopy equivalences are sent to equivalences of ∞-categories under $\mathrm{Cons}(-;\mathcal{E})$, the functor which takes constructible sheaves with coefficients in a symmetric monoidal presentable stable ∞-category $\mathcal{E}$ (Proposition B.57). Also, proving the property for maps of the form (3.3) instead of (3.1) is sufficient for our purposes, as we will see in the proof of Theorem 4.6.

Remark 3.12. The constructions performed in the proof of Proposition 3.9 are compatible with the face maps of $\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},\bullet}$ defined in Construction 2.62, since for any $k \in \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$, $i=1,\dots k$, the square

$$\begin{tikzcd}
(\mathrm{Hck}_{\mathrm{Ran},k} \times \mathrm{Hck}_{\mathrm{Ran},k})_{\mathrm{disj}} \arrow[r] \arrow[d, "\delta_i \times \delta_i"] & \mathrm{Hck}_{\mathrm{Ran},k} \arrow[d, "\delta_i"] \\
(\mathrm{Hck}_{\mathrm{Ran},k-1} \times \mathrm{Hck}_{\mathrm{Ran},k-1})_{\mathrm{disj}} \arrow[r] & \mathrm{Hck}_{\mathrm{Ran},k-1}.
\end{tikzcd}$$

commutes.

Therefore:

Theorem 3.13. *Let G be a reductive complex group and $X=\mathbb{A}^1_{\mathbb{C}}$. Proposition 3.9 induces a well-defined map of operads*

$$\mathrm{Hck}^{\mathrm{fact}} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\widehat{\mathrm{StrTStk}})^{\times}$$

such that:

- *in the first variable, it satisfies the factorization property in the sense of Recall 3.4, and sends maps of the form* (3.3) *to stratified homotopy equivalences;*
- *for every open $U \subset \mathrm{Ran}(X^{\mathrm{an}})$, the restriction to $\{U\} \times \mathbb{E}_1^{\mathrm{nu}}$ yields the map of operads defined in Theorem 2.67, analytified and pulled back from* $\mathrm{Ran}(X^{\mathrm{an}})$ *to U.*

Proof. Remark 3.12 yields a functor

$$\mathrm{Fact}(M)^{\otimes} \times \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \widehat{\mathrm{StrTStk}}^{\times}$$

which is a map of operads in the first variable. This in turn induces a functor[4]

$$\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathrm{Map}_{\mathrm{Op}_\infty}(\mathrm{Fact}(M)^{\otimes}, \widehat{\mathrm{StrTStk}}^{\times}).$$

[4]The notation $\mathrm{Map}_{\mathrm{Op}_\infty}$ follows [Lur17] and stays for “maps of ∞-operads”.

By invoking [DK19, Proposition 8.1.5] we obtain a map of operads

$$\mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\mathrm{Map}_{\mathrm{Op}_\infty}(\mathrm{Fact}(M)^{\otimes}, \widehat{\mathrm{StrTStk}}^{\times}))^{\times}.$$

Note that the target admits a map of operads to

$$\mathrm{Map}_{\mathrm{Op}_\infty}(\mathrm{Fact}(M)^{\otimes}, \mathrm{Corr}(\widehat{\mathrm{StrTStk}})^{\times})^{\times}$$

(it is easy to provide a map towards the category of functors, and then one can check that it actually takes values in the full subcategory spanned by maps of operads). Finally, we get a functor

$$\mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\widehat{\mathrm{StrTStk}})^{\times}$$

which is a map of operads separately in both variables. The verification that the claimed properties hold is straightforward by restricting to the two separate variables. □

3.2 Setup for taking constructible sheaves

In the next section we will take constructible sheaves over the geometric objects introduced up to now and prove that this induces the sought-after $\mathbb{E}_3$-algebra structure on $\mathrm{Sph}(G)$. The first step, carried out in the present subsection, will be to check that the mentioned geometric objects (and maps between them) indeed belong to the source category of Theorem B.66.

Remark 3.14. For $G = \mathrm{GL}_n, k \geq 1, I_1, \ldots, I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, N \in \mathbb{N}$ fixed, the stratifying poset of $\mathrm{Conv}^{(N)}_{I_1,\ldots,I_k}$ is given, with the notations of Construction 2.48, by

$$\left\{ J, [\phi : I_1 \sqcup \cdots \sqcup I_k \twoheadrightarrow J], (\nu_j^{h_i^j})_{j\in J, i=1,\ldots,m_j} \in \prod_{j\in J} (\mathbb{X}_\bullet(T)^+)^{m_j} \mid \sum_{j, \phi^{-1}(j)\cap I_h \neq \varnothing} \nu_j^h \leq (N, \ldots, N)\ \forall h = 1, \ldots, k \right\}$$

which is finite, the bounds being induced by N and the cardinality of $I_1 \sqcup \cdots \sqcup I_k$. For the case of a general G, choose a faithful representation $G \to \mathrm{GL}_n$: the bounds on coweights will be induced by the bounds inherited from the case of GL_n.

Proposition 3.15. *Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, N \in \mathbb{N}$. The stratified space $(\mathrm{Gr}_I^{(N)})^{\mathrm{an}}$ belongs to $\mathrm{StrTop}_{\mathrm{con}}$.*

Proof. It suffices to prove that the stratification is Whitney. Indeed, this implies that the stratification is conical, since strata are smooth manifolds and possess tubular neighbourhoods. From this, we also obtain that the conical neighbourhoods of each point can be chosen to be contractible: hence, the space is locally of singular shape. The existence of tubular neighbourhoods has been proven by Mather [Mat70]. Together with Marco Volpe, we provided an explicit reformulation in the language of conically stratified spaces [NV23, Construction 3.4].

The fact that $(\mathrm{Gr}_I^{(N)})^{\mathrm{an}}$ is Whitney is proven in [Nad05, Proposition 4.5.1]. □

Lemma 3.16. *Let $(X, s) \to (Y, t)$ be a map of stratified spaces embedded in $\mathbb{R}^N$ for some N. Suppose that the map is surjective, and that it is a smooth stratified submersion in the sense of Definition B.21. Then (X, s) satisfies the Whitney conditions if an only if (Y, t) does.*

Proof. Since the Whitney condition is local, and the map is surjective (hence any point in y admits a neighbourhood which is the image of a trivializable neighbourhood in X) we can reduce the problem to the case of a product of a space $Y \subset \mathbb{R}^N$ with a stratification s and a euclidean space $\mathbb{R}^n$ (considered with its trivial stratification), with projection $\pi : Y \times \mathbb{R}^n \to Y$. If (Y, s) is Whitney, then the product $Y \times \mathbb{R}^n$ with the trivial stratification on $\mathbb{R}^n$ is Whitney. Conversely, suppose that $Y \times \mathbb{R}^n$ is Whitney with the trivial stratification on $\mathbb{R}^n$. Pick two strata $W, Z \subset Y$, and two sequences $(w_i) \subset W \subset Y, (y_i) \subset Z \subset Y$ both converging to some $y \in Z$, such that also $w_i y_i \to v, T_{w_i} W \to \tau$ for some line v and some vector space τ. We can define liftings of the w_i, y_i's by $(w_i, 0), (y_i, 0)$, where 0 is the origin of $\mathbb{R}^n$. The sequences of the secants and of the tangent spaces of these new points converge respectively to $v \times 0$ and $\tau \times 0$: therefore, we obtain that $v \subset \tau$ by applying the hypothesis that $Y \times \mathbb{R}^n$ is Whitney. □

Lemma 3.17. *Let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, j \in \mathbb{N}$. The stratified space $(G^{(j)}_{\mathcal{O},I})^{\mathrm{an}}$ belongs to* $\mathrm{StrTop}_{\mathrm{con}}$.

Proof. Since the structure map $G^{(j)}_{\mathcal{O},I} \to X^I$ is smooth (Remark 2.23) and surjective, by Example B.43 the map $(G^{(j)}_{\mathcal{O},I})^{\mathrm{an}} \to (X^{\mathrm{an}})^I$ is a stratified smooth submersion. Since the incidence stratification on X^I is Whitney, we can apply Lemma 3.16. □

Proposition 3.18. *The stratified space* $(\mathrm{Conv}^{(N)}_{I_1,\dots,I_k})^{\mathrm{an}}$ *belongs to* $\mathrm{StrTop}_{\mathrm{con}}$.

Proof. The following proof has been suggested to us by Robert Cass.

First of all, strata are smooth by factorization (Proposition 2.41) and Proposition 2.47. Note then that $(\mathrm{Gr}^{(N)}_{I_1})^{\mathrm{an}} \times \cdots \times (\mathrm{Gr}^{(N)}_{I_k})^{\mathrm{an}}$ and $(\mathrm{Conv}^{(N)}_{I_1,\dots,I_k})^{\mathrm{an}}$ admit a common smooth cover

$$G^{(N,j_N)}_{\mathcal{K},I_1,I_2} \times_{X^{I_2}} \cdots \times_{X^{I_{k-1}}} G^{(N,j_N)}_{\mathcal{K},I_{k-1},I_k} \times_{X^{I_k}} \mathrm{Gr}^{(N)}_{I_k},$$

which projects onto each of them as a smooth bundle having as fiber the smooth unstratified relative group scheme $G^{(j_N)}_{\mathcal{O},I_2} \times \cdots \times G^{(j_N)}_{\mathcal{O},I_k}$. If we pass to analytifications, $(\mathrm{Gr}^{(N)}_{I_1})^{\mathrm{an}} \times \cdots \times (\mathrm{Gr}^{(N)}_{I_k})^{\mathrm{an}}$ is Whitney by Proposition 3.15, and by Example B.44 the two covers become surjective topological submersions. It now suffices to apply Lemma 3.16 twice, one in each direction. □

Corollary 3.19. *For any k, the object* $\mathrm{Hck}^{\mathrm{an}}_{\mathrm{Ran},k}$ *lies in* $\widehat{\mathrm{StrTStk}}_{\mathrm{con}}$.

Proof. We need to prove that each term in the formula

$$\operatorname*{colim}_{I_1,\dots,I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}} \text{“}\lim\text{”}_{j \geq j_1} \operatorname*{colim}_{[n] \in \Delta^{\mathrm{op}}} (G^{\mathrm{an}}_{\mathcal{O},I,(j)})^{\times_{X^I} n} \times_{X^I} (\mathrm{Conv}^{(1)}_{I_1,\dots,I_k})^{\mathrm{an}}$$

(where the inner colimit is the colimit along the usual simplicial diagram encoding the action, and is taken in StrTStk) belongs to $\mathrm{StrTop}_{\mathrm{con}}$. This statement is a formal consequence of the previous ones: the map $G^{(j)}_{\mathcal{O},I} \times_{X^I} \mathrm{Conv}^{(1)}_{I_1,\dots,I_k} \to \mathrm{Conv}^{(1)}_{I_1,\dots,I_k}$ is a smooth stratified submersion, hence the source is Whitney. □

Recall now Definition B.21. We need to prove that:

Proposition 3.20. *The functor* $\mathrm{Hck}^{\mathrm{fact}}$ *from Theorem 3.13 takes values in the subcategory*

$$\mathrm{Corr}(\widehat{\mathrm{StrTStk}}_{\mathrm{con}})^{\times}_{\mathit{all,subm}},$$

that is, that we have the right class of horizontal morphisms in order to apply Theorem B.66.

Proof. We need to describe the image of an arbitrary morphism $(\alpha,\phi):(U_1,\dots,U_m,\langle k\rangle)\to(V_1,\dots,V_n,\langle h\rangle)$ in $\mathrm{Fact}(M)^{\otimes}\times\mathbb{E}_1^{\mathrm{nu}}$ under the functor $\mathrm{Hck}^{\mathrm{fact}}$. Suppose first that ϕ is of the form $\phi:\langle 2\rangle\to\langle 1\rangle$ (either one of the two "projections" or the "multiplication" map). Since the image of (α,ϕ) can be factored, by definition, as the composition of the image of (α,id) under $\mathrm{Hck}^{\mathrm{fact}}(-,\langle 2\rangle)$ and the image of (id,ϕ) under $\mathrm{Hck}^{\mathrm{fact}}((V_1,\dots,V_n),-)$, let us inspect what happens on each component.

- any pair (α,id) will be sent to a vertical morphism.
- suppose that ϕ is inert. Then the pair $(\mathrm{id}_{(U_1,\dots,U_m)},\phi)$ is sent to a vertical morphism, namely one of the two projections
$$(\mathrm{Hck}_{U_1}\times\cdots\times\mathrm{Hck}_{U_m})\times(\mathrm{Hck}_{U_1}\times\cdots\times\mathrm{Hck}_{U_m})\to\mathrm{Hck}_{U_1}\times\cdots\times\mathrm{Hck}_{U_m}.$$
- suppose finally that ϕ is the active morphism. There is the following list of progressive simplifications:
  - We can assume that $m=1$. Indeed, since the object $\mathrm{Hck}_{\mathrm{Ran},\bullet}$ is 2-Segal, any other case can be recovered via pullback from the cases treated above (in the same spirit, see the definition before [DK19, Proposition 8.1.7 of the arXiv version]), and since our classes of vertical and horizontal morphisms are stable under pullbacks we conclude.
  We are thus dealing with the correspondence
$$\begin{tikzcd} & \mathrm{Hck}_{U,2} \arrow[dl] \arrow[dr] & \\ \mathrm{Hck}_U\times\mathrm{Hck}_U & & \mathrm{Hck}_U \end{tikzcd}$$
  and we want to prove that the left leg belongs to subm.
  - It suffices to prove the claim for $U=\mathrm{Ran}(X^{\mathrm{an}})$.
  - It suffices to prove that the map
$$\mathrm{Hck}_{\mathrm{Ran},2}\to\mathrm{Hck}_{\mathrm{Ran}}\times\mathrm{Hck}_{\mathrm{Ran}}$$
  is representable and smooth, i.e. its pullback to any $\mathcal{X}\in\mathrm{Pro}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}})$ is a pro-smooth map of pro-stacks, in the sense of algebraic geometry. By definition (Definition B.28), the combination of this and Corollary 3.19 will ensure that its analytification is in subm.
  - It is sufficient to prove that for every $I_1,I_2\in\mathrm{Fin}_{\geq 1,\mathrm{surj}}, I=I_1\sqcup I_2, j\geq j_1$
$$\overline{p}^{(j)}_{I_1,I_2}:\mathrm{Hck}^{(j)}_{I_1,I_2}\to\mathrm{Hck}^{(j)}_{I_1}\times\mathrm{Hck}^{(j)}_{I_2} \tag{3.4}$$
  (notations as in Section 2.3) is smooth. Indeed, let $\mathcal{X}$ be any object in $\mathrm{Pro}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}})$ together with a map $\mathcal{X}\to\mathrm{Hck}_{\mathrm{Ran}}\times\mathrm{Hck}_{\mathrm{Ran}}$ in $\widehat{\mathrm{StrStk}}_{\mathbb{C}}^{\mathrm{lft}}$. Then there is a map
$$\mathcal{X}\to\mathrm{Ran}(X)\times\mathrm{Ran}(X),$$
  which will factor via some $X^{I_1}\times X^{I_2}, I_1,I_2\in\mathrm{Fin}_{\geq 1,\mathrm{surj}}$, because any representable object in a category of presheaves is atomic. Hence, the map $\mathcal{X}\to\mathrm{Hck}_{\mathrm{Ran}}\times\mathrm{Hck}_{\mathrm{Ran}}$ factors as
$$\mathcal{X}\to\mathrm{Hck}_{I_1}\times\mathrm{Hck}_{I_2}$$

and therefore we have an equivalence

$$\mathcal{X} \times_{\mathrm{Hck}_{\mathrm{Ran}} \times \mathrm{Hck}_{\mathrm{Ran}}} \mathrm{Hck}_{\mathrm{Ran},2} \simeq \mathcal{X} \times_{\mathrm{Hck}_{I_1} \times \mathrm{Hck}_{I_2}} \mathrm{Hck}_{I_1,I_2}.$$

Note that this fiber product belongs to $\mathrm{Pro}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}})$ as wanted, and the projection from it to $\mathcal{X}$ is pro-smooth whenever the map (3.4) is a smooth map of stacks for every j.

– Recall that there is a morphism of schemes

$$p_{I_1,I_2}^{(j)} : G_{\mathcal{K},I_1,I_2}^{(1,j)} \times_{X^{I_2}} \mathrm{Gr}_{I_2}^{(1)} \to \mathrm{Gr}_{I_1}^{(1)} \times \mathrm{Gr}_{I_2}^{(1)}$$

which takes the right $G_{\mathcal{O},I_2}^{(j)}$-quotient in the first component and is the identity in the second one. This fits into a commutative square of stacks

$$\begin{tikzcd} G_{\mathcal{K},I_1,I_2}^{(1,j)} \times_{X^{I_2}} \mathrm{Gr}_{I_2}^{(1)} \arrow[r, "p_{I_1,I_2}^{(j)}"] \arrow[d] & \mathrm{Gr}_{I_1}^{(1)} \times \mathrm{Gr}_{I_2}^{(1)} \arrow[d] \\ \mathrm{Hck}_{I_1,I_2}^{(j)} \arrow[r, "\bar{p}_{I_1,I_2}^{(j)}"] & \mathrm{Hck}_{I_1}^{(j)} \times \mathrm{Hck}_{I_2}^{(j)} \end{tikzcd}$$

where the leftmost vertical arrow exhibits the target as the quotient of the source, relative to X^I, with respect to the action of $G_{\mathcal{O},I}^{(j)}$ from Remark 2.56), and the rightmost vertical arrow exhibits the target as the quotient of the source, relative to X^I, with respect to the action of $G_{\mathcal{O},I_1}^{(j)} \times G_{\mathcal{O},I_2}^{(j)}$ Now, the map $p_{I_1,I_2}^{(j)}$ is a $G_{\mathcal{O},I_2,(j)}$-torsor, which implies that $\bar{p}_{I_1,I_2}^{(j)}$ is smooth in the sense of [SPA, `Tag 075U`]. We need the topological version of this: namely, by applying Remark 2.23 and Example B.43 to $p_{I_1,I_2}^{(j)}$, we obtain that $\left(\bar{p}_{I_1,I_2}^{(j)}\right)^{\mathrm{an}}$ belongs to subm' in the sense of Definition B.28.

□

4 The $\mathbb{E}_3$-structure

4.1 Spherical Hecke category over the Ran space

Construction 4.1. Let $\mathcal{E}$ a presentable stable symmetric monoidal ∞-category. We consider the composition

$$\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \xrightarrow{\mathrm{Hck}^{\mathrm{fact}}} \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})^{\times}_{\mathsf{all},\mathsf{subm}} \xrightarrow{\mathrm{Cons}_{\mathcal{E}}^{\otimes,\mathrm{corr}}} \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{R},\otimes}$$

where the first functor is the one from Proposition 3.20 and the second functor is the one constructed in Theorem B.66. Note that $\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}}$ sends a morphism of the form $(\alpha, \mathrm{id}_{\langle k \rangle})$, where α is of the form Eq. (3.3), to an equivalence of ∞-categories by Theorem 3.13 and Theorem B.66. Moreover, it satisfies the factorization property from Recall 3.4 since $\mathrm{Hck}^{\mathrm{fact}}$ does (Proposition 3.10) and $\mathrm{Cons}_{\mathcal{E}}^{\otimes}$ is symmetric monoidal: in other words, by Theorem B.66 the functor

$$\mathrm{Cons}(\mathrm{Hck}_U;\mathcal{E}) \otimes_{\mathcal{E}} \mathrm{Cons}(\mathrm{Hck}_V;\mathcal{E}) \to \mathrm{Cons}(\mathrm{Hck}_U \times \mathrm{Hck}_V;\mathcal{E})$$

is an equivalence for each independent $U, V \subset \mathrm{Ran}(X^{\mathrm{an}})$.

We now want to prove that the functor $\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}}$ is a hypercomplete cosheaf when restricted to $\mathrm{Open}(\mathrm{Ran}(M))$.

Proposition 4.2. *Fix $I_1,\dots,I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, and $j \geq j' \geq j_1$. Let $f : \mathrm{Hck}^{(j)}_{I_1,\dots,I_k} \to \mathrm{Hck}^{(j')}_{I_1,\dots,I_k}$ be the transition map. Then the functor*

$$f^* : \mathrm{Cons}((\mathrm{Hck}^{(j')}_{I_1,\dots,I_k})^{\mathrm{an}};\mathcal{E}) \to \mathrm{Cons}((\mathrm{Hck}^{(j)}_{I_1,\dots,I_k})^{\mathrm{an}};\mathcal{E})$$

is an equivalence.

Proof. This follows from the fact that the transition maps are in uni (Proposition 2.28), hence their analytification is in tri, and Proposition B.58. □

Remark 4.3. The ∞-category (4.3) is, by the construction in Theorem B.66, equivalent to

$$\operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}}{}^{\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathcal{E}}} \mathrm{Cons}(\mathrm{Hck}^{(j)}_{D^I};\mathcal{E}) \tag{4.1}$$

where j is any number greater than j_1. The transition functors, up to a change of index j (which induces an equivalence on the categories of constructible sheaves by Proposition 4.2) in I are given by $*$-pushforward along the closed embeddings[5] $\mathrm{Hck}_{D^I} \to \mathrm{Hck}_{D^J}$ induced by $J \twoheadrightarrow I$. This colimit is not filtered. As a consequence of [Lur09, Theorem 5.5.3.13], it corresponds to the limit in $\widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{E}}$[6] taken with *-pullback functoriality.

This means that (4.1) can be rewritten as

$$\lim_{I\in \mathrm{Fin}_{\geq 1,\mathrm{surj}}}{}^{(-)^*} \mathrm{Cons}(\mathrm{Hck}^{(j)}_{D^I};\mathcal{E}). \tag{4.2}$$

Here $\mathrm{Cons}(\mathrm{Hck}_{D^I};\mathcal{E})$ coincides with $\mathrm{Cons}(\mathrm{Hck}^{(j)}_{D^I};\mathcal{E})$ for any $j \geq j_1$. The reason we chose to adopt the pro-object perspective is that, without the possibility of increasing j, some maps are impossible to define (notably, the ones associated to surjections $I \to J$ in Construction 2.33).

Finally, each term $\mathrm{Cons}(\mathrm{Hck}^{(j)}_{D^I};\mathcal{E})$ is computed, again by construction, à la Bernstein-Lunts, i.e. as

$$\operatorname*{colim}_{[n]\in\Delta}^{\mathcal{P}\mathrm{r}^{\mathrm{R}}_{R},\dashv} \mathrm{Cons}((G^{(j)}_{\mathcal{O},D^I})^{\times_{D^I} n} \times_{D^I} \mathrm{Gr}^{(1)}_{D^I};\mathcal{E}) \simeq \lim_{[n]\in\Delta}^{\widehat{\mathcal{C}\mathrm{at}}_{\infty,\mathcal{E}},-^*} \mathrm{Cons}((G^{(j)}_{\mathcal{O},D^I})^{\times_{D^I} n} \times_{D^I} \mathrm{Gr}^{(1)}_{D^I};\mathcal{E})$$

where the limit is taken along the simplicial diagram encoding the action of $G^{(j)}_{\mathcal{O},D^I}$ on $\mathrm{Gr}^{(1)}_{D^I}$, with pullback functoriality.

Lemma 4.4. *Let $p : Y \to Z$ be a map of topological spaces, and let $\mathcal{F}$ be the functor*

$$\begin{aligned} \mathrm{Open}(Z) &\to \mathrm{Top} \\ U &\mapsto p^{-1}(U). \end{aligned}$$

Then $\mathcal{F}$ is a hypercomplete cosheaf on Z.

[5] Recall that these are closed embeddings of unions of strata, hence $*$-pushforward preserves constructible sheaves.
[6] This is the ∞-category of large $\mathcal{E}$-linear ∞-categories, defined similarly to Notation B.52.

Proof. The claim amounts to the fact that, for every $U \in \mathrm{Open}(Z)$ and any hypercovering $\mathcal{U} : \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathrm{Open}(Z)$ of U, the map

$$\operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}} p^{-1}(\mathcal{U}_n) \to p^{-1}(U)$$

between open subsets of Y is a homeomorphism, which is true. □

Proposition 4.5. *The restriction of the functor* $\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}}$ *from Construction 4.1 to* $\mathrm{Open}(\mathrm{Ran}(M)) \times \{<k>\}$ *is a hypercomplete cosheaf for every* $<k> \in \mathbb{E}_1^{\mathrm{nu}}$.

Proof. Note that, for each $I_1, \ldots, I_k \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}, N \in \mathbb{N}, j \geq j_1$, the functors $U \mapsto \mathrm{Conv}^{(N)}_{U,I_1,\ldots,I_k}$ and $U \mapsto G^{(j)}_{\mathcal{O},U,I}$ are of the form appearing in Lemma 4.4, and hence hypercomplete cosheaves. Therefore, the functor $U \mapsto G^{(j)}_{\mathcal{O},U,I} \backslash \mathrm{Conv}^{(1)}_{U,I_1,\ldots,I_k}$ (for $j \geq j_N$) is again hypercomplete, since it arises as a colimit of hypercomplete cosheaves.

Therefore, by Proposition B.59, the functor $U \mapsto \mathrm{Cons}(\mathrm{Hck}^{(j)}_{U,I_1,\ldots,I_k};\mathcal{E})$ is a hypercomplete cosheaf as well.

By the discussion in Remark 4.3, this functor coincides with $U \mapsto \mathrm{Cons}(\mathrm{Hck}_{U,I_1,\ldots,I_k};\mathcal{E})$. Finally, the functor $U \mapsto \mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran},k};\mathcal{E})$ is a hypercomplete cosheaf because it arises as a colimit of hypercomplete cosheaves. □

Summing up:

Theorem 4.6. *Let G be a complex reductive group, $X = \mathbb{A}^1_{\mathbb{C}}, M = X^{\mathrm{an}} = \mathbb{C}$, and $\mathcal{E}$ a presentable stable symmetric monoidal ∞-category.*

The functor

$$\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathcal{E}}$$

from Construction 4.1 has the following properties:

- *It is a hypercomplete cosheaf in the first variable, satisfying constructibility and the factorization property in the sense of [Lur17, Definition 5.5.4.1], [Lur17, Theorem 5.5.4.10].*
- *It is a map of operads in the second variable.*

Proof. The only part that requires justification is the constructibility property. We know that the functor sends maps of the form (3.3) to equivalences of ∞-categories, by Proposition 3.10 and Proposition B.57, and that it is a hypercomplete cosheaf by Proposition 4.5. Note that this is sufficient to apply [Lur17, Proposition 5.5.1.14] and conclude that $\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}}$ is constructible in the first variable, in the sense of [Lur17, Definition 5.5.4.1]. Indeed, the proof of the "if" direction of [Lur17, Proposition 5.5.1.14] works in the same way if one restricts to a subclass of $\mathrm{Disk}(M)$ forming a base of M, which is the case for $\mathrm{Ball}(\mathbb{C}) \subset \mathrm{Disk}(\mathbb{C})$. □

Construction 4.7. We can now apply [Lur17, Theorem 5.5.4.10] to $\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}}$ and obtain an $\mathbb{E}_M^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}}$-algebra with values in $\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathcal{E}}$, where $M = \mathbb{C}$, therefore a 2-dimensional real manifold. Its restriction to any disk $D \subset \mathbb{C}$ has the same value for different D's (up to equivalence in $\mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}}$), since an $\mathbb{E}_M^{\mathrm{nu}}$-algebra is in particular locally constant, and by [Lur17, Example 5.4.5.3], this restriction is naturally an $\mathbb{E}_2^{\mathrm{nu}}$-algebra. This means that the stalk of $\mathrm{Sph}(G;\mathcal{E})^{\mathrm{fact}}$ in the first variable at any point $\{x\} \in \mathrm{Ran}(X^{\mathrm{an}})$, which is just its value at any disk D containing x, induces a map of operads

$$\mathbb{E}_2^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}}.$$

Its underlying ∞-category is

$$\mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran}(D)};\mathcal{E}) \tag{4.3}$$

for any (irrelevant) choice of a disk D around x. Also the choice of x is irrelevant, in the sense that different choices give noncanonically equivalent ∞-categories with equivalent algebra structures (this is easily seen by choosing a path between any two points in X^{an} a finite number of disks covering the image of this path).

We will refer to the operation implicit in the $\mathbb{E}_2^{\mathrm{nu}}$-component as *fusion* and to the one implicit in the $\mathbb{E}_1^{\mathrm{nu}}$-component as *convolution*.

4.2 Specialization to a point

Recall that we denote by Hck_x or just Hck the fiber of $\mathrm{Hck}_{\{1\}}$ at any point $x\in X(\mathbb{C})$. Our goal in this subsection is to transfer the algebra structure established in Construction 4.7 to the ∞-category

$$\mathrm{Cons}(\mathrm{Hck}^{\mathrm{an}};\mathcal{E}),$$

i.e. $\mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}};\mathcal{E})$ with the notation of Definition B.38.

Remark 4.8. Let $j\geq j_1, N\in\mathbb{N}$. Let $D\hookrightarrow \mathrm{Ran}(D)$ denote the closed embedding corresponding to the cardinality 1 stratum. Recall that we have defined $\mathrm{Gr}_D^{(N)}=\mathrm{Gr}_{\{1\}}^{\mathrm{an}}\times_{X^{\mathrm{an}}}D\simeq \mathrm{Gr}^{\mathrm{an}}\times D$, $G_{\mathcal{O},D}^{(j)}=G_{\mathcal{O},\{1\}}^{\mathrm{an}}\times_X D\simeq G_{\mathcal{O}}^{\mathrm{an}}\times D$ and $\mathrm{Hck}_D=\mathrm{Hck}_{\{1\}}^{\mathrm{an}}\times_{X^{\mathrm{an}}}D\in \widehat{\mathrm{StrTStk}}_{\mathrm{con}}$ (the equivalences follow from Proposition 2.6).

First of all, notice that for any choice of a disk D in X^{an} there is an equivalence of ∞-categories

$$\mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}};\mathcal{E})\simeq \mathrm{Cons}_{G_{\mathcal{O},D}^{\mathrm{an}}}(\mathrm{Gr}_D^{\mathrm{an}};\mathcal{E}) \tag{4.4}$$

induced by pulling back along the projection $\mathrm{Gr}_D^{\mathrm{an}}\to\mathrm{Gr}^{\mathrm{an}}$ (the inverse functor is given by pulling back along the embedding $\mathrm{Gr}^{\mathrm{an}}\times\{x_0\}\to\mathrm{Gr}^{\mathrm{an}}\times D$ (for any choice of a point x_0 in D). This is true by Proposition B.57. Therefore, to give an $\mathbb{E}_3$-algebra structure on $\mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}};\mathcal{E})$ is the same as giving an $\mathbb{E}_3$-algebra structure on $\mathrm{Cons}_{G_{\mathcal{O},D}}(\mathrm{Gr}_D;\mathcal{E})$. By the very nature of this identification, the equivalence does not actually depend on the choice of x and D.

Now, there is a pullback diagram in $\widehat{\mathrm{StrTStk}}_{\mathrm{con}}$

$$\begin{tikzcd} \mathrm{Hck}_D \arrow[r, hook, "i"] \arrow[d] & \mathrm{Hck}_{\mathrm{Ran}(D)} \arrow[d] \\ D \arrow[r, hook] & \mathrm{Ran}(D) \end{tikzcd}$$

and an adjunction

$$\begin{tikzcd} \mathrm{Cons}(\mathrm{Hck}_D;\mathcal{E}) \arrow[r, shift right, "i_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran}(D)};\mathcal{E}) \arrow[l, shift right, "i^*"'] \end{tikzcd} \tag{4.5}$$

where i_* is fully faithful because i is an equivariant closed embedding of a union of strata.

Theorem 4.9. *There is a canonical* $\mathbb{E}_2^{\mathrm{nu}}\times\mathbb{E}_1^{\mathrm{nu}}$*-algebra structure on* $\mathrm{Cons}(\mathcal{H}\mathrm{ck}_D;\mathcal{E})$ *such that the functor* i^* *becomes lax-monoidal.*

Proof. The idea is to transfer the $\mathbb{E}_2^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}}$-algebra structure to $\mathrm{Cons}_{G_{\mathcal{O},D}}(\mathrm{Gr}_D; \mathcal{E})$ by applying [Lur17, Proposition 2.2.1.9] with $\mathcal{C}^{\otimes} = \mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran}(D)}; \mathcal{E})$ and each L_X induced by i^*.

Recall from Construction 4.7 that there are two compatible product structures on $\mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran}(D)}; \mathcal{E})$, which we call $\star$ (convolution, the one parametrized by the $\mathbb{E}_1^{\mathrm{nu}}$-variable) and $\odot$ (fusion, the one parametrized by the $\mathbb{E}_2^{\mathrm{nu}}$-variable). In order to apply Lurie's result, we need to verify that for any

$$\mathcal{A}, \mathcal{A}', \mathcal{B} \in \mathrm{Cons}(\mathrm{Hck}_{\mathrm{Ran}(D)}; \mathcal{E}),$$

and a morphism

$$f : \mathcal{A} \to \mathcal{A}'$$

such that $i^* f$ is an equivalence in $\mathrm{Cons}(\mathrm{Hck}_D; \mathcal{E})$, the natural maps

$$i^*(\mathcal{A} \star \mathcal{B}) \to i^*(\mathcal{A}' \star \mathcal{B})$$

and

$$i^*(\mathcal{A} \odot \mathcal{B}) \to i^*(\mathcal{A} \odot \mathcal{B})$$

are equivalences.

First of all, by the description in Construction 4.7, we can fix I_1, I_2 and assume that $\mathcal{A}, \mathcal{A}' \in \mathrm{Cons}(\mathrm{Hck}_{D^{I_1}}; \mathcal{E}), \mathcal{B} \in \mathrm{Cons}(\mathrm{Hck}_{D^{I_2}}; \mathcal{E})$ (actually, we could even assume $I_1 = I_2$, but it is instructive to see what happens in the general case).

We need to prove two things:

- $\star$ For the convolution case, we need to prove that

$$i^*(m_{I_1,I_2*} p^*_{I_1,I_2}(\mathcal{A} \boxtimes \mathcal{B})) \to i^*(m_{I_1,I_2*} p^*_{I_1,I_2}(\mathcal{A}' \boxtimes \mathcal{B}))$$

is an equivalence, where the notations are as in the following diagram:

$$\begin{tikzcd}
\mathrm{Hck}_D \times \mathrm{Hck}_D \arrow[d, "j'"'] & \mathrm{Hck}_{D,D} \arrow[l, "p"'] \arrow[r, "m"] \arrow[d, "\tilde{j}"'] & \mathrm{Hck}_{D^2} \arrow[d, "j"'] & \mathrm{Hck}_D \arrow[l, "d"'] \arrow[dl, "i"] \\
\mathrm{Hck}_{D^{I_1}} \times \mathrm{Hck}_{D^{I_2}} & \mathrm{Hck}_{D^{I_1},D^{I_2}} \arrow[l, "p_{I_1,I_2}"'] \arrow[r, "m_{I_1,I_2}"] & \mathrm{Hck}_{D^{I_2 \sqcup I_2}} &
\end{tikzcd}$$

Here i stays for the map i read at the $I_1 \sqcup I_2$-level, and d for the map pulled back from the diagonal of D^2. Note that both squares and the triangle commute, the second square is a pullback and the map m_{I_1,I_2} is proper[7]. Moreover, we have equivalences $m_* \simeq m_{\vdash}, m_{I_1,I_2*} \simeq m_{I_1,I_2\vdash}$. To see this, it suffices to see that m_* and m_{I_1,I_2*} preserve constructible sheaves. By properness of m, m_{I_1,I_2} and proper base change for sheaves (not necessarily constructible), it suffices to check this after pullback to each stratum of $X^{I_1 \sqcup I_2}$. There, the map can be realized as a product of copies of the multiplication map $m : \mathrm{Hck}_{x,2} \to \mathrm{Hck}_x$ for some point x. This latter map descends from a $G_{\mathcal{O}}$-equivariant map $\mathrm{Conv}_{x,2} \to \mathrm{Gr}_x$; therefore, pushforward along it preserves equivariant sheaves which are constructible with respect to *some* stratification. But equivariant constructible sheaves over Gr_x with respect to *some* stratification are automatically constructible with respect with to *the* stratification by Schubert cells by Proposition A.20.

[7] In the sense that each $m^{(j)}_{I_1,I_2}$ is proper.

Therefore, we can apply proper base change and conclude that

$$\begin{aligned} i^*(m_{I_1,I_2*}p^*_{I_1,I_2}(\mathcal{A}\boxtimes\mathcal{B})) &\simeq d^*j^*m_{I_1,I_2*}p^*_{I_1,I_2}(\mathcal{A}\boxtimes\mathcal{B}) \\ &\simeq d^*m_*\widetilde{j}^*p^*_{I_1,I_2}(\mathcal{A}\boxtimes\mathcal{B})\simeq d^*m_*p^*j'^*(\mathcal{A}\boxtimes\mathcal{B}). \end{aligned}$$

Note now that j' corresponds to the map $i\times i$ read at the (I_1,I_2)-level, and therefore the last expression equals

$$d^*m_*p^*(i^*\mathcal{A}\boxtimes i^*\mathcal{B}).$$

If we read this construction functorially in $\mathcal{A}$, we conclude that the map $i^*(\mathcal{A}\odot\mathcal{B})\to i^*(\mathcal{A}\odot\mathcal{B})$ induced by f is an equivalence.

$\odot$ For the fusion product, the product law is depicted by the diagram

$$\begin{tikzcd} \mathrm{Hck}_D\times\mathrm{Hck}_D \arrow[d, hook] & \mathrm{Hck}_{D_1}\times\mathrm{Hck}_{D_2} \arrow[l, "\sim"'] \arrow[r, hook, "u_{1,1}"] \arrow[d, hook, "j'"'] & \mathrm{Hck}_{D^2} \arrow[d, hook, "j"] & \mathrm{Hck}_D \arrow[l, hook', "d"'] \arrow[dl, hook, "d_{I_1\sqcup I_2}"] \\ \mathrm{Hck}_{D^{I_1}}\times\mathrm{Hck}_{D^{I_2}} & \mathrm{Hck}_{D_1{}^{I_1}}\times\mathrm{Hck}_{D_2{}^{I_2}} \arrow[l, "\sim"'] \arrow[r, hook, "u_{I_1,I_2}"] & \mathrm{Hck}_{D^{I_1\sqcup I_2}} & \end{tikzcd} \tag{4.6}$$

Note first of all that $d_{I_1\sqcup I_2}=dj$ corresponds to the map i read at the $I_1\sqcup I_2$-level. Up to identifying the first column with the second one, $\mathcal{A}\odot\mathcal{B}$ corresponds to $(u_{I_1,I_2})_\vdash(\mathcal{A}\boxtimes\mathcal{B})$ and

$$i^*(\mathcal{A}\odot\mathcal{B})\simeq d^*j^*(u_{I_1,I_2})_\vdash(\mathcal{A}\boxtimes\mathcal{B}).$$

Therefore, it suffices to show that the base-change map

$$j^*(u_{I_1,I_2})_\vdash\to u_\vdash j'^* \tag{4.7}$$

induces an equivalence

$$d^*_{I_1\sqcup I_2}(u_{I_1,I_2})_\vdash(\mathcal{A}\boxtimes\mathcal{B})\simeq d^*j^*(u_{I_1,I_2})_\vdash(\mathcal{A}\boxtimes\mathcal{B})\simeq d^*u_\vdash j'^*(\mathcal{A}\boxtimes\mathcal{B}).$$

Let us consider the diagram

$$
\begin{tikzcd}
\mathrm{Hck}_{D_1} \times \mathrm{Hck}_{D_2} \arrow[r, hook, "u_{1,1}"] \arrow[d, equal] & \mathrm{Hck}_{D^2} & \mathrm{Hck}_D \arrow[l, hook', "d"'] \arrow[ddddd, equal, bend left=60] \\
\mathrm{Hck}_{D_1} \times \mathrm{Hck}_{D_2} \arrow[r, hook, "\widetilde{u}_{1,1}"] \arrow[d, equal] & \mathrm{Hck}_{D,D} \arrow[u, "\overline{m}"'] \arrow[d, "\overline{p}"] & \mathrm{Hck}_{D,D} \times_{D^2} D \arrow[l, hook', "\widetilde{d}"'] \arrow[u, "\overline{m}_\Delta"'] \arrow[d, "\overline{p}_\Delta"] \arrow[ddd, equal, bend left=60] \\
\mathrm{Hck}_{D_1} \times \mathrm{Hck}_{D_2} \arrow[r, hook, "u'_{1,1}", "\text{s.h.e.}"'] \arrow[d, hook, "j'"] & \mathrm{Hck}_D \times \mathrm{Hck}_D \arrow[d, "\delta"] & \mathrm{Hck}_D \times_D \mathrm{Hck}_D \arrow[l, hook', "d'"'] \arrow[d, equal] \\
\mathrm{Hck}_{D_1^{I_1}} \times \mathrm{Hck}_{D_2^{I_2}} \arrow[r, hook, "u'_{I_1,I_2}", "\text{s.h.e.}"'] \arrow[d, equal] & \mathrm{Hck}_{D^{I_1}} \times \mathrm{Hck}_{D^{I_2}} & \mathrm{Hck}_D \times_D \mathrm{Hck}_D \arrow[l, hook', "d'_{I_1,I_2}"'] \\
\mathrm{Hck}_{D_1^{I_1}} \times \mathrm{Hck}_{D_2^{I_2}} \arrow[r, hook, "\widetilde{u}_{I_1,I_2}"] \arrow[d, equal] & \mathrm{Hck}_{D^{I_1},D^{I_2}} \arrow[u, "\overline{p}_{I_1,I_2}"'] \arrow[d, "\overline{m}_{I_1,I_2}"] & \mathrm{Hck}_{D^{I_1},D^{I_2}} \times_{D^{I_1 \sqcup I_2}} D \arrow[l, hook', "\widetilde{d}_{I_1,I_2}"'] \arrow[u, "\overline{p}_\Delta"'] \arrow[d, "\overline{m}_\Delta"] \\
\mathrm{Hck}_{D_1^{I_1}} \times \mathrm{Hck}_{D_2^{I_2}} \arrow[r, hook, "u_{I_1,I_2}"] & \mathrm{Hck}_{D^{I_1 \sqcup I_2}} & \mathrm{Hck}_D \arrow[l, hook', "d_{I_1,I_2}"']
\end{tikzcd}
\tag{4.8}
$$

where the maps are defined as follows:

- $u_{1,1}, u_{I_1,I_2}$ are defined in (4.6) and come from the factorization property for $\mathrm{Hck}_{I_1 \sqcup I_2}$.
- $\widetilde{u}_{1,1}, \widetilde{u}_{I_1,I_2}$ come from Proposition 2.41 and Remark 2.50.
- $u'_{1,1}, u'_{I_1,I_2}$ are the open embeddings induced by the inclusions $D_1 \subset D, D_2 \subset D$.
- d, d_{I_1,I_2} are defined in (4.6).
- d', d'_{I_1,I_2} are the maps induced by the diagonal inclusion $D \subset D^2, D \subset D^{I_1} \times D^{I_2}$.
- $\widetilde{d}, \widetilde{d}_{I_1,I_2}$ are the maps induced by the diagonal inclusion $D \subset D^2, D \subset D^{I_1} \times D^{I_2}$ and by Proposition 2.41.
- j' comes from (4.6).
- δ is the map induced by the diagonal inclusions $D \subset D^{I_1}, D^{I_2}$.
- $\overline{p}, \overline{m}, \overline{p}_\Delta, \overline{m}_\Delta, \overline{p}_{I_1,I_2}, \overline{m}_{I_1,I_2}$ are (or are induced by) the maps in (A.7).

The goal is to prove that if we start with a sheaf $\mathcal{A} \boxtimes \mathcal{B}$ over $\mathrm{Hck}_{D_1^{I_1}} \times \mathrm{Hck}_{D_2^{I_2}}$, then pulling back along the first column and performing push-pull along the first row is the same as performing push-pull along the last row. Note that all squares are pullback squares. By properness of the maps $\overline{m}_\Delta, \overline{m}_{I_1,I_2}$, we have equivalences

$$
d^*_{I_1,I_2}(u_{I_1,I_2})_\vdash \simeq d^*_{I_1,I_2}(\overline{m}_{I_1,I_2})_\vdash(\widetilde{u}_{I_1,I_2})_\vdash \simeq (\overline{m}_\Delta)_\vdash \widetilde{d}^*_{I_1,I_2}(\widetilde{u}_{I_1,I_2})_\vdash
$$

(the second one is implied by proper base change. Note that $\overline{m}_\vdash = \overline{m}_*$ by the discussion in the convolution case). Then, by smoothness[8] of $\overline{p}_{I_1,I_2}$, we have

$$(\overline{m}_\Delta)_* \widetilde{d}^*_{I_1,I_2}(\widetilde{u}_{I_1,I_2})_\vdash \simeq (\overline{m}_\Delta)_* \overline{p}^*_\Delta d'^*_{I_1,I_2}(u'_{I_1,I_2})_\vdash \simeq (\overline{m}_\Delta)_* \overline{p}^*_\Delta d'^* \delta^*(u'_{I_1,I_2})_\vdash.$$

But now, since $u'_{1,1}$ and u'_{I_1,I_2} are stratified homotopy equivalences, the last expression is equivalent to

$$(\overline{m}_\Delta)_* \overline{p}^*_\Delta (d')^*(u'_{1,1})_\vdash j'^*.$$

Then by applying again smooth and then proper base change on the squares on the top half of the diagram we conclude that the original expression is equivalent to

$$d^*(u_{1,1})_\vdash j'^*(\mathcal{A} \boxtimes \mathcal{B}).$$

Like in the convolution case, we can deduce from this that the map f induces an equivalence as desired.

□

Notation 4.10. We denote the algebra structure inherited by $\mathrm{Cons}(\mathrm{Hck}_x^{\mathrm{an}};\mathcal{E})$ by means of (4.4) as

$$\mathrm{Sph}(G)_x^\otimes \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_2}(\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_1}(\mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}})).$$

4.3 Main result and t-exactness

By convenience, given three ∞-operads $\mathcal{O}, \mathcal{O}', \mathcal{O}''$, we will call "bilinear maps" those maps $\mathcal{O} \times \mathcal{O}' \to \mathcal{O}''$ which are maps of operads separately in each variable.

Proposition 4.11. *Let x be any point in X^{an}. The bilinear map $\mathrm{Sph}(G;\mathcal{E})_x^\otimes : \mathbb{E}_2^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}}$ extends to a bilinear map $\mathbb{E}_2 \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}}$, which we again denote by $\mathrm{Sph}(G;\mathcal{E})_x^\otimes$.*

Proof. We can apply [Lur17, Theorem 5.4.4.5]: that is, it suffices to exhibit a quasi-unit for any

$$\mathrm{Sph}(G;\mathcal{E})_x^\otimes(-,\langle k\rangle)$$

functorial in $\langle k\rangle \in \mathbb{E}_1^{\mathrm{nu}}$. Consider the map (natural in k) $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Hck}_{x,k}$ represented by the sequence $(\mathcal{T},\ldots,\mathcal{T},\mathrm{id}|_{X\setminus\{x\}},\ldots,\mathrm{id}_{X\setminus\{x\}}) \in \mathrm{Hck}_{x,k}$. Note now that this induces a map

$$e_k : * \to \mathrm{Hck}^{\mathrm{an}}_{x,k}.$$

We want to check that the map e_k is a quasi-unit in the sense of [Lur17, Definition 5.4.3.1.] for any k, functorially in k. But both maps $\mathrm{Hck}^{\mathrm{an}}_{x,k} \to \mathrm{Hck}^{\mathrm{an}}_{x,k} \times \mathrm{Hck}^{\mathrm{an}}_{x,k} \to \mathrm{Hck}^{\mathrm{an}}_{x,k}$ induced by e_k (by targeting respectively the first or the second factor of the product) are the identity, since gluing with the trivial G-torsor along with its trivial trivialization does not change the original torsor. At the level of constructible sheaves, the unit is given by the pushforward of the constant sheaf $\mathbf{1}_{\mathcal{E}}$ along e_k. □

Proposition 4.12. *The bilinear map $\mathrm{Sph}(G;\mathcal{E})_x^\otimes : \mathbb{E}_2 \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_R$ extends to a bilinear map $\mathrm{Sph}(G;\mathcal{E})_x^\otimes$: $\mathbb{E}_2 \times \mathbb{E}_1 \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_R$.*

[8] Here, as usual, we take advantage of the definition of our categories of constructible sheaves as colimits of categories of sheaves over "truncated" objects of the form $\mathrm{Hck}_I^{(j)}$, which allows us to work with smooth maps instead of pro-smooth, cf. Proposition 3.20.

Proof. Again, it suffices to exhibit a quasi-unit. Let us denote by $\mathbf{1}$ the pushforward along the trivial section $t : * \to \mathrm{Hck}^{\mathrm{an}}_{x,1}$, $t(*) = (\mathcal{T}, \mathcal{T}, \mathrm{id}|_{X \setminus x})$, of the constant sheaf with value $\mathbf{1}_{\mathcal{E}}$.
The proof is given in [Rei12, Proposition IV.3.5]. We just rewrite it in our notation. We drop the superscript $(-)^{\mathrm{an}}$ everywhere for simplicity. Let us assume that the entry in the variable $\mathbb{E}_2$ is $\langle 1 \rangle$ by simplicity (the general case is just "a direct power" of this one). We denote by $\star$ the $\mathbb{E}_1$-product of equivariant constructible sheaves on Gr_x described by $\mathrm{Sph}(G; \mathcal{E})^{\otimes}_x(\langle 1 \rangle, -)$. For any $F \in \mathrm{Cons}_{G_{\mathcal{O}}}(\mathrm{Gr}, R)$ we can compute the product via the convolution diagram

$$\begin{tikzcd}
 & G_{\mathcal{K}} \times \mathrm{Gr} \arrow[r, "q"] \arrow[dl, "p"'] & G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr} \arrow[dr, "m"] & \\
\mathrm{Gr} \times \mathrm{Gr} & & & \mathrm{Gr}.
\end{tikzcd}$$

This diagram extends to

$$\begin{tikzcd}
 & & G_{\mathcal{K}} \times \mathrm{Gr} \arrow[r, "q"] \arrow[dl, "p"'] & G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr} \arrow[dr, "m"] & \\
* \times \mathrm{Gr} \arrow[r, "t \times \mathrm{id}"] \arrow[urrr, bend left, "j"] & \mathrm{Gr} \times \mathrm{Gr} & & & \mathrm{Gr},
\end{tikzcd}$$

where j is the closed embedding $(\mathcal{F}, \alpha) \mapsto (\mathcal{T}, \mathrm{id}|_{X \setminus x}, \mathcal{F}, \alpha)$ whose image is canonically identified with Gr. Let $F \in \mathrm{Cons}_{G_{\mathcal{O}}}(\mathrm{Gr}; \mathcal{E})$. We want to prove that $\mathbf{1} \widetilde{\boxtimes} F \simeq j_*(\mathbf{1}_{\mathcal{E}} \boxtimes F)$, i.e. that

$$q^* j_*(\mathbf{1}_{\mathcal{E}} \boxtimes F) \simeq p^*(t \times \mathrm{id})_*(\mathbf{1}_{\mathcal{E}} \boxtimes F).$$

Note that because of the consideration about the image of j the support of both sides lies in $G_{\mathcal{O}} \times \mathrm{Gr} \subset G_{\mathcal{K}} \times \mathrm{Gr}$, and this yields a restricted diagram

$$\begin{tikzcd}
 & G_{\mathcal{O}} \times \mathrm{Gr} \arrow[r, "\tilde{q}"] \arrow[dl, "\tilde{p}"', "\sim"] & \mathrm{Gr} \arrow[dr, "\tilde{m}"] & \\
\mathrm{Gr} \arrow[urr, "j"'] & & & \mathrm{Gr}.
\end{tikzcd}$$

This proves the claim. By applying m_* we obtain

$$\mathbf{1} \star F \simeq m_*(j_*(\mathbf{1}_{\mathcal{E}} \boxtimes F)) = \mathbf{1}_{\mathcal{E}} \boxtimes F = F$$

since $mj = \mathrm{id}$. $\square$

Thanks to these results, our functor $\mathrm{Sph}(G; \mathcal{E})^{\otimes}_x$ from Notation 4.10 is finally promoted to a bilinear map $\mathbb{E}_2 \times \mathbb{E}_1 \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_R$. By the Additivity Theorem ([Lur17, Theorem 5.1.2.2]), this is the same as an $\mathbb{E}_3$-algebra object in $\mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_R$.

Note also that, by the convolution presentation of the monoidal law, the functor $\mathrm{Sph}(G, \mathcal{E})^{\otimes}_x$ takes values in $\mathcal{P}\mathrm{r}^{\mathrm{LR},\otimes}_{\mathcal{E}}$: indeed, by the same argument as in the first part of the proof of Theorem 4.9 and by properness, the functor $\overline{m}_{\vdash}$ coincides both with m_* and with $m_!$, and hence it is a right and left adjoint. For convenience, we will say that our algebra takes values in $\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathcal{E}}$ from now on.

Summing up:

Theorem 4.13 (Main theorem)**.** *Let G be a complex reductive group and $\mathcal{E}$ be a presentable stable symmetric monoidal ∞-category. There is an object* $\mathrm{Sph}(G;\mathcal{E})^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_3}(\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L},\otimes})$ *having as underlying object the topological spherical Hecke category*

$$\mathrm{Sph}(G;\mathcal{E})^{\mathrm{top}} = \mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}_G^{\mathrm{an}};\mathcal{E})$$

(see Definition A.22).

Note that, when R is a commutative ring, either discrete, prodiscrete or ℓ-adic (i.e. an algebraic extension of $\mathbb{Q}_\ell$), this specializes to Theorem 1.4 for $\mathcal{E} = \mathrm{Mod}_R^{\mathrm{cont}}$ in the sense of Definition B.35.

Corollary 4.14 (Small spherical Hecke category)**.** *In the same setting as Theorem 4.13, there is an induced $\mathbb{E}_3$-monoidal structure in $\mathcal{C}\mathrm{at}^{\times}_{\infty,\mathcal{E}^{\omega}}$ on*

$$\mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}_G^{\mathrm{an}};\mathcal{E}^{\omega}).$$

Let R be a commutative ring, noetherian and of finite global dimension, and $\mathcal{E} = \mathrm{Mod}_R$. Then, on objects, the restriction of this product to equivariant perverse sheaves coincides with the classical (commutative) convolution product of perverse sheaves [MV07], up to the perverse truncation of the derived tensor product appearing in the definition of the latter (cf. also Remark 4.17).

Proof. The (not full) inclusion $\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{R},\otimes} \to \widehat{\mathcal{C}\mathrm{at}}^{\times}_{\infty,\mathcal{E}}$ (the ∞-category of large $\mathcal{E}$-linear ∞-categories) is lax monoidal, i.e. it is a map of operads. Therefore, $\mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}};\mathcal{E})$ has an induced $\mathbb{E}_3$-algebra structure in $\widehat{\mathcal{C}\mathrm{at}}^{\times}_{\infty,\mathcal{E}}$. By using the convolution formula, one sees that the convolution product restricts to $\mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}},\mathcal{E}^{\omega})$: again, this follows from the fact that $\overline{p}^*$ preserves $\mathcal{E}^{\omega}$-valued sheaves (since it is induced by restriction along functors between categories of exit paths), and that the same is true for $\overline{m}_*$[9]. Since the inclusion[10] $\mathcal{C}\mathrm{at}^{\times}_{\infty,\mathcal{E}^{\omega}} \subset \widehat{\mathcal{C}\mathrm{at}}^{\times}_{\infty,\mathcal{E}^{\omega}}$ is strong symmetric monoidal, we obtain the first part of the statement. The claim regarding perverse sheaves follows from what observed in Remark A.13. □

Corollary 4.15 (Renormalization)**.** *Let us assume that we are in the same setting of Corollary 4.14, and that $\mathcal{E}$ is compactly generated. Then there is an induced $\mathbb{E}_3$-monoidal structure on*

$$\mathrm{Sph}(G;\mathcal{E})^{\mathrm{ren}} = \mathrm{Ind}(\mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr};\mathcal{E}))$$

(see Definition A.23) as an object of $\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L},\otimes}$.

Proof. Since $\mathcal{E}$ is compactly generated, we have that $\mathrm{Ind}(\mathcal{E}^{\omega}) \simeq \mathcal{E}$. Recall first of all that the functor

$$\mathrm{Ind} : \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{ex}} \to \mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}}$$

is symmetric monoidal: this follows from [Lur17, Proposition 4.8.1.8]) with $\mathcal{K} = \varnothing$ and

$$\mathcal{K}' = \{\varkappa\text{-filtered simplicial sets, for some regular cardinal } \varkappa\}$$

[9]Indeed, let $f : X \to Y$ be a map between topological spaces, $\mathcal{E}$ an ∞-category. If $\mathcal{F} \in \mathrm{Shv}(X;\mathcal{E})$ and U is open in Y, then

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

which belongs to $\mathcal{E}^{\omega}$ if $\mathcal{F}$ takes values in $\mathcal{E}^{\omega}$.

[10]Here we mean the functor sending a small category to itself.

and the fact that colimits are generated by filtered colimits and finite colimits. Therefore, there is an induced symmetric monoidal functor

$$\mathrm{Mod}_{\mathcal{E}^\omega}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}) \to \mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{\mathrm{L}}).$$

This functor factors through $\mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L}}$, because Ind of an exact functor F is strongly continuous, i.e. its right adjoint G preserves colimits, and hence by [Lur11, Remark 6.6] G is automatically $\mathcal{E}$-linear. □

Again, the latter results specialize to Corollary 1.5 and Corollary 1.7 in the ring case.

Remark 4.16. Let $\mathcal{C}^\otimes$ be an $\mathbb{E}_k$-algebra in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$, the ∞-category of stable ∞-categories and exact functors between them. Suppose given a t-structure on $\mathcal{C}$ which is compatible with the algebra structure, in the sense of [Lur17, Example 2.2.1.3] (intuitively, the subcategory $\mathcal{C}_{\geq 0}$ should be closed under tensor). Then by [Lur17, Proposition 2.2.1.8, Proposition 2.2.1.9] the heart $\mathcal{C}^\heartsuit$ of the t-structure canonically inherits an $\mathbb{E}_k$-algebra structure (the proof goes along the same lines of [Lur17, Example 2.2.1.10], which deals with the case $\mathbb{E}_\infty$).

Note that this "induced $\mathbb{E}_k$-structure" procedure is functorial: given a stable-exact and t-left exact (in the sense of [Lur17, Definition 1.3.3.1]) $\mathbb{E}_k$-monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$, one obtains an additive functor $\mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ between abelian categories, which can be viewed as the composition of the $\mathbb{E}_k$-monoidal functors

$$\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}_{\leq 0} \xrightarrow{F|_{\mathcal{C}_{\leq 0}}} \mathcal{D}_{\leq 0} \xrightarrow{\tau_{\leq 0}} \mathcal{D}^\heartsuit$$

(notice that F restricts to the coconnective parts by left-t-exactness).

Remark 4.17. Let R be a commutative discrete ring. The small spherical Hecke category carries a canonical t-structure inherited from the perverse t-structure on bounded categories of (finite-dimensional) constructible sheaves. Indeed, the Bernstein-Lunts presentation

$$\mathrm{Sph}(G;R)^{\mathrm{loc.c}} \simeq \lim_n \mathcal{D}_{\mathrm{c}}^{\mathrm{fd}}(G_{\mathcal{O}}^{\times n} \times \mathrm{Gr}; R)$$

of Recall B.9 establishes a canonical t-structure on the limit by [BL94, Section 2.5].

The convolution product on $\mathrm{Sph}(G;R)^{\mathrm{loc.c}}$ is perverse left t-exact, and t-exact when R is a field. This follows from the considerations recalled in Construction A.9 ,

Thus, one can apply Remark 4.16 to $\mathrm{Sph}(G;R)^{\mathrm{loc.c}}$ with $k=3$ and recover the convolution product of perverse sheaves and its commutativity constraint, as mentioned in Remark 1.10. Indeed, although in the original formula the perverse truncation appears right after performing the derived tensor product (and not at the end of the process), there is no actual difference with our formula in this respect, since the rest of the procedure defining the convolution product is perverse t-exact.

A Recollections and complements in Geometric Langlands

A.1 The Satake category

Let us resume from the definition of affine Grassmannian, recalled in Definition 1.1.

Recall A.1 (see [Zhu16, Theorem 1.1.3]). There is a natural action of $G_{\mathcal{O}}$ on Gr_G by left multiplication, whose orbits define an algebraic stratification of Gr_G over the poset $\mathbb{X}_\bullet(T)^+$ of dominant coweights of the Cartan group T of G. When viewed from the point of view of the complex-analytic topology on Gr_G, this stratification satisfies the so-called *Whitney conditions* (for a proof, see [MO14]). One can characterize the stratification as follows. The Cartan decomposition[11]

$$G_{\mathcal{K}} = \bigsqcup_{\mu \in \mathbb{X}_\bullet(T)^+} G_{\mathcal{O}} t^{\mu} G_{\mathcal{O}}$$

induces a partition

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in \mathbb{X}_\bullet(T)^+} G_{\mathcal{O}} t^{\mu}.$$

For an element g of $G_{\mathcal{K}}$, the associated μ is denoted by

$$\mathrm{Inv}(g)$$

and is the same for every g' in the same right $G_{\mathcal{O}}$-class, i.e. $\mathrm{Inv}(-)$ factors through $G_{\mathcal{K}} \to \mathrm{Gr}_G$.

If $G = \mathrm{GL}_n$, $\mathbb{X}_\bullet(T)^+$ can be realized (noncanonically) as the set

$$\{(\mu_1, \dots, \mu_n) \mid \mu_1 \geq \dots \geq \mu_n\}$$

and via this identification t^{μ} is exactly the diagonal matrix $\mathrm{diag}(t^{-\mu_i})$.

For $\mu \in \mathbb{X}_\bullet(T)^+$, one defines

$$\mathrm{Gr}_{\mu} = \{\Lambda \in \mathrm{Gr} \mid \mathrm{Inv}(\Lambda) = \mu\}.$$

There is a natural filtration of Gr by finite-dimensional projective schemes $\mathrm{Gr}_{\leq \mu} = \bigcup_{\nu \leq \mu} \mathrm{Gr}_{\nu}$. Moreover, the action $G_{\mathcal{O}} \circlearrowright \mathrm{Gr}_G$ preserves $\mathrm{Gr}_{\leq\mu}$, and actually each $G_{\mathcal{O}} \circlearrowright \mathrm{Gr}_{\leq\mu}$ factors through the quotient $G_{\mathcal{O}} \twoheadrightarrow G_{\mathcal{O}}^{(j)} = G(\mathbb{C}[\![t]\!]/t^j\mathbb{C}[\![t]\!])$ for any j larger than some j_{μ} ([Rei12, Lemma IV.1.4]). This is actually the reason why the orbits form a stratification in the first place ([MO14]).

Definition A.2. Let $j \geq 1$. We define

$$G_{\mathcal{O}}^{(j)} = G(\mathbb{C}[\![t]\!]/t^j\mathbb{C}[\![t]\!]).$$

Definition A.3. Let R be a ring. The category of $G_{\mathcal{O}}$-equivariant perverse sheaves on Gr_G (or *Satake category*) with values in R-modules is

$$\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}; R) := \operatorname*{colim}_{\mu \in \mathbb{X}_\bullet(T)^+} \mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}^{(j_\mu)}}(\mathrm{Gr}_{\leq\mu}; R)$$

(see [Zhu16, 5.1 and A.1]).

The definition of each term is independent of j_{μ} because of [Zhu16, Lemma A.1.4].

Recall A.4 (cf. [Zhu16, Lemma 3.1.7])**.** Let X be a smooth complex curve, R a complex ring, and $x \in X(R)$ an R-point. There is a well-defined formal completion of Γ_x, the graph of x in X_R, which is a formal scheme whose (discrete) ring of functions is $\widehat{\mathcal{O}}_{\Gamma_x}$. We consider the *affine formal neighbourhood* of x, defined as the map

$$\widetilde{(X_R)}_{\Gamma_x} = \mathrm{Spec}\, \widehat{\mathcal{O}}_{\Gamma_x} \to X_R$$

[11] The symbol $\bigsqcup$ should only be understood as a set-theoretical decomposition, not a topological or scheme-theoretic one.

coming from [Zhu16, Lemma 3.1.7], [BD05, Proposition 2.12.6]. Note that the source of this map is always isomorphic, étale-locally on R (and noncanonically, since the isomorphism depends on the map x), to $\operatorname{Spec} R[\![t]\!]$. In particular, each closed point x of X admits an affine formal neighbourhood

$$\widetilde{X}_x \simeq \operatorname{Spec} \mathbb{C}[\![t]\!] \to X.$$

In general, for a R a commutative ring and $x \in X(R)$, consider the square

$$\begin{tikzcd} (\mathring{X}_R)_{\Gamma_x} \arrow[r] \arrow[d] & \widetilde{(X_R)}_{\Gamma_x} \arrow[d] \\ X_R \setminus \Gamma_x \arrow[r] & X_R \end{tikzcd}$$

where the upper left vertex is by definition the pullback of the span. This pullback is again an affine scheme, called the *punctured affine formal neighbourhood* of x. If $R = \mathbb{C}$ and x is a closed point of X, we obtain

$$\mathring{X}_x \simeq \operatorname{Spec} \mathbb{C}(\!(t)\!) \to X.$$

Definition A.5. Let $\mathbf{Bun}_G$ be the moduli stack of G-torsors over $\mathbb{C}$. If a scheme Z over $\mathbb{C}$ is given, we define the relative version

$$\mathbf{Bun}_G^Z : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpd}$$

$$R \mapsto \{G\text{-torsors over } Z \times \operatorname{Spec} R\} = \mathrm{Bun}_G(Z_R).$$

In the language of mapping stacks, we can write

$$\mathbf{Bun}_G^Z \simeq \mathbf{Map}_{\mathrm{Stk}_{\mathbb{C}}}(Z, \mathbf{Bun}_G).$$

Proposition A.6. *For any closed point x of a smooth projective complex curve X, the functor Gr_G is equivalent to the following:*

$$\mathrm{Gr}_G^{\mathrm{loc}} : R \to \{\mathcal{F} \in \mathrm{Bun}_G(\widetilde{(X_R)}_{x \times \operatorname{Spec} R}), \alpha : \mathcal{F}|_{(\mathring{X}_R)_{x \times \operatorname{Spec} R}} \xrightarrow{\sim} \mathcal{T}_{(\mathring{X}_R)_{\{x\} \times \operatorname{Spec} R}}\}. \tag{A.1}$$

Proof. The proof is explained for instance in [Zhu16, Proposition 1.3.6]. □

We will need the following version of the affine Grassmannian as well.

Construction A.7. Let $G = \mathrm{GL}_n$. Define $\mathrm{Gr}_G^{\mathrm{glob}}$ as the fiber of the restriction map $\mathbf{Bun}_G^X \to \mathbf{Bun}_G^{X \setminus \{x\}}$ at the trivial G-torsor, i.e. as the functor

$$R \to \{\mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus (\{x\} \times \operatorname{Spec} R)} \xrightarrow{\sim} \mathcal{T}_{X_R \setminus (\{x\} \times \operatorname{Spec} R)}\}.$$

Indeed, in the diagram of groupoids

$$\begin{tikzcd} \mathrm{Gr}_G^{\mathrm{glob}}(R) \arrow[r] \arrow[d] & \mathrm{Bun}_G(X_R) \arrow[r] \arrow[d] & \mathrm{Bun}_G(\widetilde{(X_R)}_{\{x\} \times \operatorname{Spec} R}) \arrow[d] \\ \{\mathcal{T}|_{X \setminus \{x\}}\} \arrow[r] & \mathrm{Bun}_G(X_R \setminus (\{x\} \times \operatorname{Spec} R)) \arrow[r] & \mathrm{Bun}_G((\mathring{X}_R)_{\{x\} \times \operatorname{Spec} R}) \end{tikzcd}$$

the right-hand square is Cartesian by the Beauville-Laszlo Theorem [BL95], more precisely in the form of [BD05, Remark 2.3.7]. Since the left-hand square is Cartesian by definition, the outer square is Cartesian. Therefore, $\mathrm{Gr}_G^{\mathrm{glob}}(R)$ is isomorphic to the fiber of the restriction map $\mathrm{Bun}_G((\widetilde{X_R})_{\{x\}\times \operatorname{Spec} R}) \to \mathrm{Bun}_G((\mathring{X}_R)_{\{x\}\times \operatorname{Spec} R})$ at the trivial bundle, and this is exactly $\mathrm{Gr}_G^{\mathrm{loc}}(R)$. For more details, and for the case of an arbitrary reductive G, see [Zhu16, Theorem 1.4.2].

Remark A.8. Let G be a complex reductive group, X a smooth complex curve, $x \in X(\mathbb{C})$. In the case $G = \mathrm{GL}_n$, one can filter Gr by

$$\mathrm{Gr}^{(N)} = \{\mathcal{F} \in \mathrm{Bun}_G(X), \alpha : \mathcal{F}|_{X\setminus\{x\}} \xrightarrow{\sim} \mathcal{T}|_{X\setminus\{x\}} \mid \mathcal{O}_X^n(-N) \subset \mathcal{F} \subset \mathcal{O}_X^n(N)\},$$

$N \in \mathbb{N}$. This filtration is compatible with the stratification and the filtration appearing in Recall A.1: see e.g. [KMW18, §2.3]. In the case of a general G, a similar filtration is achieved by means of the choice of a faithful representation $G \to \mathrm{GL}_n$ for some n (see [Zhu16, Propositions 1.2.5, 1.2.6]), and a similar compatibility result holds (see again [KMW18, §2.3]).

Construction A.9. Let R be a commutative discrete ring. We recall now the tensor structure given by **convolution product** on $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G; R)$. A more detailed account is given in [Zhu16, Section 1, Section 5.1, 5.4]. Consider the diagram

$$\begin{tikzcd}[column sep=small] & G_\mathcal{K} \times \mathrm{Gr}_G \arrow[r, "q"] \arrow[dl, "p"'] & G_\mathcal{K} \times^{G_\mathcal{O}} \mathrm{Gr}_G \arrow[dr, "m"] & \\ \mathrm{Gr}_G \times \mathrm{Gr}_G & & & \mathrm{Gr}_G \end{tikzcd} \tag{A.2}$$

where $G_\mathcal{K} \times^{G_\mathcal{O}} \mathrm{Gr}_G$ (or Conv_G) is the stack quotient of the product $G_\mathcal{K} \times \mathrm{Gr}_G$ with respect to the "anti-diagonal" left action of $G_\mathcal{O}$ defined by $\gamma \cdot (g, [h]) = (g\gamma, [\gamma^{-1}h])$. The map p is the projection to the quotient on the first factor and the identity on the second one, the map q is the projection to the quotient by the mentioned action of $G_\mathcal{O}$.

Note that the left multiplication action of $G_\mathcal{O}$ on $G_\mathcal{K}$ and on Gr_G induces a left action of $G_\mathcal{O} \times G_\mathcal{O}$ on $\mathrm{Gr}_G \times \mathrm{Gr}_G$. It also induces an action of $G_\mathcal{O}$ on $G_\mathcal{K} \times \mathrm{Gr}_G$ given by (left multiplication, id) which canonically projects to an action of $G_\mathcal{O}$ on $G_\mathcal{K} \times^{G_\mathcal{O}} \mathrm{Gr}_G$. Note that p, q and m are equivariant with respect to these actions (more precisely, p is $G_\mathcal{O} \times G_\mathcal{O}$-eqivariant, whereas q and m are $G_\mathcal{O}$-equivariant).

Now if $\mathcal{A}_1, \mathcal{A}_2$ are two $G_\mathcal{O}$-equivariant perverse sheaves on Gr_G, one can define a convolution product

$$\mathcal{A}_1 \star \mathcal{A}_2 = m_* \widetilde{\mathcal{A}} \tag{A.3}$$

where m_* is the derived direct image functor, and $\widetilde{\mathcal{A}}$ is a perverse sheaf on $G_\mathcal{K} \times^{G_\mathcal{O}} \mathrm{Gr}_G$ which is equivariant with respect to the left action of $G_\mathcal{O}$ and such that $q^*\widetilde{\mathcal{A}} = p^*({}^{\mathrm{p}}\mathcal{H}^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2))$. [12] Note that such an $\widetilde{\mathcal{A}}$ exists because q is the projection to the quotient and $\mathcal{A}_2$ is $G_\mathcal{O}$-equivariant, and one can prove that $\widetilde{\mathcal{A}}$ is again perverse.

Note that m_* carries perverse sheaves to perverse sheaves: indeed, it can be proven that m is ind-proper, i.e. it can be represented by a filtered colimit of proper maps of schemes, and also semi-small. By [KW01, Lemma III.7.5], and the definition of $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G; R)$ as a filtered colimit, this ensures that m_* carries perverse sheaves to perverse sheaves.

[12] The tensor product denotes the derived tensor product in the derived category. If the stalks of the two sheaves are flat, e.g. when the ring of coefficients is a field, the external tensor product is already perverse, see [MV07, Lemma 4.1]. In general, one needs to consider the perverse truncation, as in the formula.

It is important to stress that, at every step, we are implicitly assuming our sheaves to be supported on some $\mathrm{Gr}_{\leq\mu}$, and we are considering equivariant structures with respect to the truncations $G_{\mathcal{O}}^{(j_\mu)}$.

Observations similar to Construction A.7 prove the following:

Proposition A.10. *We have the following equivalences of groupoids:*

$$G_{\mathcal{O}}(R) \simeq \mathrm{Aut}((\widetilde{X_R})_{x\times \mathrm{Spec}\, R}, \mathcal{T})$$

$$G_{\mathcal{K}}(R) \simeq \{\mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{(X\setminus\{x\})\times \mathrm{Spec}\, R} \simeq \mathcal{T}|_{(X\setminus\{x\})\times \mathrm{Spec}\, R}, \\ \mu : \mathcal{F}|_{(\widetilde{X_R})_{x\times \mathrm{Spec}\, R}} \simeq \mathcal{T}|_{(\widetilde{X_R})_{x\times \mathrm{Spec}\, R}}\}$$

$$(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr})(R) \simeq \{\mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{(X\setminus\{x\})\times \mathrm{Spec}\, R} \simeq \mathcal{T}|_{(X\setminus\{x\})\times \mathrm{Spec}\, R}, \\ \mathcal{G} \in \mathrm{Bun}_G(X_R), \eta : \mathcal{F}|_{(X\setminus\{x\})\times \mathrm{Spec}\, R} \simeq \mathcal{G}_{(X\setminus\{x\})\times \mathrm{Spec}\, R}\}$$

A.2 The convolution product via quotient stacks

Definition A.11. We define the complex stack

$$G_{\mathcal{O}}\backslash \mathrm{Gr}$$

as the fpqc quotient stack of Gr by the left action of $G_{\mathcal{O}}$. We also define

$$G_{\mathcal{O}}\backslash (G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr})$$

as the fpqc quotient stack of $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}$ by the left action of $G_{\mathcal{O}}$ on the first factor $G_{\mathcal{K}}$.

Proposition A.12. *There is an equivalence of stacks between $G_{\mathcal{O}}\backslash \mathrm{Gr}$ and the functor*

$$\mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Grpd}$$

$$\{\mathcal{F}_0, \mathcal{F}_1 \in \mathrm{Bun}_G(\mathrm{Spec}\, R[\![t]\!]), \eta : \mathcal{F}_0|_{\mathrm{Spec}\, R(\!(t)\!)} \xrightarrow{\sim} \mathcal{F}_1|_{\mathrm{Spec}\, R(\!(t)\!)}\}.$$

Proof. There is a map π from Gr to the moduli space appearing in the statement, described as follows. Let $\mathcal{T}$ be the trivial G-bundle on $\mathrm{Spec}\, R[\![t]\!]$; then $\pi(\mathcal{F}, \alpha) := (\mathcal{T}, \mathcal{F}, \alpha^{-1})$. Let us show that this map is essentially surjective. Since any G-torsor on $\mathrm{Spec}\, R[\![t]\!]$ is trivializable locally in $\mathrm{Spec}\, R$, for any triple $(\mathcal{F}_0, \mathcal{F}_1, \eta)$ as in the statement one can find, locally in $\mathrm{Spec}\, R$, $\mu : \mathcal{F}_0 \xrightarrow{\sim} \mathcal{T}$ and consequently $\alpha : \mathcal{F}_1|_{\mathrm{Spec}\, R(\!(t)\!)} \xrightarrow{\sim} \mathcal{T}|_{\mathrm{Spec}\, R(\!(t)\!)}$ such that $\eta = \alpha^{-1} \circ \mu|_{\mathring{D}}$. The fact that this is local in $\mathrm{Spec}\, R$ is not a problem, since we are considering the quotient stack on the left-hand-side. Thus, the triple $(\mathcal{T}, \mathcal{F}_1, \alpha^{-1})$ is isomorphic to $(\mathcal{F}_0, \mathcal{F}_1, \eta)$ by means of the isomorphism $(\mu, \mathrm{id}) : (\mathcal{F}_0, \mathcal{F}_1, \eta) \to (\mathcal{T}, \mathcal{F}_1, \alpha^{-1})$.

To conclude the proof, it suffices to prove that the fiber of π at each R-point of the right-hand side is $G_{\mathcal{O}} \times_{\mathbb{C}} \mathrm{Spec}\, R$. But the fiber at $(\mathcal{F}_0, \mathcal{F}_1, \eta)$ is the set of those (α, μ), $\alpha : \mathcal{F}_1|_{\mathrm{Spec}\, R(\!(t)\!)} \xrightarrow{\sim} \mathcal{T}|_{\mathrm{Spec}\, R(\!(t)\!)}$, $\mu : \mathcal{F}_0 \xrightarrow{\sim} \mathcal{T}$ such that $\alpha^{-1} \circ \mu|_{\mathrm{Spec}\, R(\!(t)\!)} = \eta$, which in turn amounts to the set (in particular a 0-truncated groupoid) of μ's since α is completely determined by η and μ. But this is $G_{\mathcal{O}}$, since any two trivializations on $\mathrm{Spec}\, R[\![t]\!]$ are connected by a unique automorphism of $\mathcal{T}$ on $\mathrm{Spec}\, R[\![t]\!]$. $\square$

In a similar way, one can prove that

$$G_{\mathcal{O}}\backslash (G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}) \simeq \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Bun}_G(\hat{D}), \eta_1 : \mathcal{F}_0|_{\mathring{D}} \simeq \mathcal{F}_1|_{\mathring{D}}, \eta_2 : \mathcal{F}_0|_{\mathring{D}} \simeq \mathcal{F}_2|_{\mathring{D}}\}.$$

Remark A.13. Consider the diagram of stacks

$$\begin{tikzcd} & (G_{\mathcal{O}} \times G_{\mathcal{O}})\backslash(G_{\mathcal{K}} \times \mathrm{Gr}) \arrow[r, "\sim"] \arrow[dl, "\overline{p}"'] & G_{\mathcal{O}}\backslash(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}) \arrow[dr, "\overline{m}"] \arrow[dll, dotted, "r"] & \\ G_{\mathcal{O}}\backslash\mathrm{Gr} \times G_{\mathcal{O}}\backslash\mathrm{Gr} & & & G_{\mathcal{O}}\backslash\mathrm{Gr} \end{tikzcd} \tag{A.4}$$

where the action of the second copy of $G_{\mathcal{O}}$ on $G_{\mathcal{K}} \times G_{\mathcal{O}}$ is the antidiagonal one described in (A.2), all other actions are induced by the left multiplication action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}}$. Then:

- the horizontal map is an equivalence (and therefore a map r as above is defined such that the diagram commutes);
- a $G_{\mathcal{O}}$-equivariant perverse sheaf on Gr is the same thing as a perverse sheaf on $G_{\mathcal{O}}\backslash\mathrm{Gr}$, in the following sense. Note that the latter is an ind-stack of finite type, hence we can define

$$\mathcal{P}\mathrm{erv}(G_{\mathcal{O}}\backslash\mathrm{Gr}) := \operatorname*{colim}_{\mu} \mathcal{P}\mathrm{erv}(G_{\mathcal{O}}^{(j_\mu)}\backslash\mathrm{Gr}_{\leq\mu}),$$

  where the expression in the colimit is meant in the sense of [LO06, Section 4]. Finally, the claimed equivalence follows from [LO06, Remark 5.5]. Similar considerations can be done for the other vertices of the diagram (A.4);
- under the identification at the previous point, the convolution product is equivalently described (up to the perverse truncations of the derived tensor product) by

$$\mathcal{A}_1 \star \mathcal{A}_2 = \overline{m}_*(r^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2)). \tag{A.5}$$

  Note that pullbacks and pushforwards are defined, at the level of the terms in the colimit at the previous point, as pullback and pushforwards of elements of the derived category.

  In particular, we can say that the diagram of stacks

$$\begin{tikzcd} & G_{\mathcal{O}}\backslash(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}) \arrow[dl, "r"'] \arrow[dr, "\overline{m}"] & \\ G_{\mathcal{O}}\backslash\mathrm{Gr} \times G_{\mathcal{O}}\backslash\mathrm{Gr} & & \mathrm{Hck} \end{tikzcd} \tag{A.6}$$

  "correctly models" the convolution product of $G_{\mathcal{O}}$-equivariant perverse sheaves over the affine Grassmannian (up to the perverse truncation of the derived tensor product appearing in the original definition).

Lemma A.14. *The map r can be described as*

$$r(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \eta_1, \eta_2) = ((\mathcal{F}_0, \mathcal{F}_1, \eta_1), (\mathcal{F}_1, \mathcal{F}_2, \eta_2)).$$

Proof. A priori, r works as follows: choose

$$\mu_0 : \mathcal{F}_0 \xrightarrow{\sim} \mathcal{T}$$

$$\mu_1 : \mathcal{F}_1 \xrightarrow{\sim} \mathcal{T}$$

$$\alpha_1 : \mathcal{F}_1|_{\operatorname{Spec}\mathbb{C}((t))} \xrightarrow{\sim} \mathcal{T}|_{\operatorname{Spec}\mathbb{C}((t))}$$
$$\alpha_2 : \mathcal{F}_2|_{\operatorname{Spec}\mathbb{C}((t))} \xrightarrow{\sim} \mathcal{T}|_{\operatorname{Spec}\mathbb{C}((t))}$$

such that $\alpha_1^{-1}\mu_0|_{\operatorname{Spec}\mathbb{C}((t))} = \eta_1, \alpha_2^{-1} \circ \mu_1|_{\operatorname{Spec}\mathbb{C}((t))} \simeq \eta_2$.

Then by definition

$$r(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \eta_1, \eta_2) = ((\mathcal{T}, \mathcal{F}_1, \alpha_1^{-1}), (\mathcal{T}, \mathcal{F}_2, \alpha_2^{-1})).$$

But now, there are squares of isomorphisms on $\operatorname{Spec}\mathbb{C}((t))$

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\alpha_1^{-1}"] & \mathcal{F}_1 \\ \mathcal{F}_0 \arrow[u, "\mu_0|_{\operatorname{Spec}\mathbb{C}((t))}"] \arrow[r, "\eta_1"] & \mathcal{F}_1 \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\alpha_2^{-1}"] & \mathcal{F}_2 \\ \mathcal{F}_1 \arrow[u, "\mu_1|_{\operatorname{Spec}\mathbb{C}((t))}"] \arrow[r, "\eta_2"] & \mathcal{F}_2 \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

where the vertical maps are induced by the isomorphisms on $\operatorname{Spec}\mathbb{C}[[t]]$, and we conclude. □

Remark A.15. Note that this map does not coincide with the map induced by the isomorphism (id, m) : $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr} \to \mathrm{Gr} \times \mathrm{Gr}$ (cf. [Zhu16, (1.2.14)]), which is instead described by

$$(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \eta_1, \eta_2) \mapsto ((\mathcal{F}_0, \mathcal{F}_1, \eta_1), (\mathcal{F}_0, \mathcal{F}_2, \eta_1 \circ \eta_2)).$$

Indeed, to prove that this is the same map we would need to build a commuting square

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\alpha_2^{-1}"] & \mathcal{F}_2 \\ \mathcal{F}_0 \arrow[u] \arrow[r, "\eta_2\eta_1"] & \mathcal{F}_2 \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

but here there is no reason why $\alpha_2\eta_2\eta_1$ would extend to the complete disk: we know that there exist $\widetilde{\mu}, \widetilde{\alpha}$ such that $\eta_2\eta_1 = \widetilde{\alpha}^{-1}\widetilde{\mu}|_{\mathring{D}}$, which in general lies in $G_{\mathcal{K}}$ and not in $G_{\mathcal{O}}$.

Remark A.16. Let $N \in \mathbb{N}, j \geq j_N$. Let $G_{\mathcal{K}}^{(N,j)} = G_{\mathcal{K}} \times^{G_{\mathcal{O}}} G_{\mathcal{O}}^{(j)}$. The diagram (A.6) admits a truncated version

$$\begin{tikzcd} & \mathrm{Hck}_2^{(N,j)} \arrow[dl, "r"'] \arrow[dr, "\overline{m}"] & \\ \mathrm{Hck}^{(N,j)} \times \mathrm{Hck}^{(N,j)} & & \mathrm{Hck}^{(2N,2j)} \end{tikzcd} \tag{A.7}$$

where

$$\mathrm{Hck}^{(N,j)} = G_{\mathcal{O}}^{(j)} \backslash \mathrm{Gr}^{(N)}$$
$$\mathrm{Hck}_2^{(N,j)} = G_{\mathcal{O}}^{(j)} \backslash (G_{\mathcal{K}}^{(N,j)} \times^{G_{\mathcal{O}}^{(j)}} \mathrm{Gr}^{(N)}).$$

and similar.

We can assemble these objects into the following:

Definition A.17. The ind-pro-stack

$$\text{“}\operatorname*{colim}_{N\in\mathbb{N}}\text{”“}\lim_{j\geq j_N}\text{”}\mathrm{Hck}^{(N,j)}$$

is denoted by

$$\mathrm{Hck},$$

and similarly

$$\mathrm{Hck}_2 = \text{“}\operatorname*{colim}_{N\in\mathbb{N}}\text{”“}\lim_{j\geq j_N}\text{”}\mathrm{Hck}_2^{(N,j)}.$$

One can check that the convolution formula arising from push-pull along this diagram and (A.5) agree. In particular, the choice of j_N, j does not change the formula.

To see this, it suffices to note the following. First of all, by definition of the categories of perverse sheaves, the original formula for the convolution product is actually the one arising from the diagram

$$\begin{tikzcd} & \mathrm{Hck}_2^{(N)} \arrow[dl, "r"'] \arrow[dr, "\overline{m}"] & \\ \mathrm{Hck}^{(N)}\times\mathrm{Hck}^{(N)} & & \mathrm{Hck}^{(2N)} \end{tikzcd}$$

where

$$\mathrm{Hck}^{(N)} = G_{\mathcal{O}}\backslash\mathrm{Gr}^{(N)}$$
$$\mathrm{Hck}_2^{(N)} = G_{\mathcal{O}}\backslash(G_{\mathcal{K}}^{(N)}\times^{G_{\mathcal{O}}}\mathrm{Gr}^{(N)}).$$

and similar.

If $j \geq j_{2N}$, we can further truncate the diagram to (A.7), and the convolution formula is again the same because

- $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}^{(N)};R) \simeq \mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}^{(j)}}(\mathrm{Gr}^{(N)};R)$.
- $G_{\mathcal{K}}^{(N)}\times^{G_{\mathcal{O}}}\mathrm{Gr}^{(N)} \simeq G_{\mathcal{K}}^{(N,j)}\times^{G_{\mathcal{O}}^{(j)}}\mathrm{Gr}^{(N)}$ by definition.

Notice that the same arguments work at the level of equivariant constructible sheaves (see e.g. [AR23, 2.4.3]).

A.3 Models for the spherical Hecke category

The main result of this paper Corollary 4.14 is about $\mathrm{Sph}(G;\mathcal{E})^{\mathrm{loc.c}}$, a category of equivariant constructible sheaves on $\mathrm{Gr}^{\mathrm{an}}$, the analytification of the affine Grassmannian, with very general coefficients: namely, any presentable stable ∞-category $\mathcal{E}$ works. In the present subsection, we will remark how that category specializes to familiar ones in two special cases: with coefficients in finite/profinite/ℓ-adic rings, or with complex coefficients. In the first case, there is an interpretation of the mentioned category in terms of algebraic constructible sheaves over the affine Grassmannian. Hence, let us gather a couple observations about those.

Let $\mathscr{S}$ be the stratification by Schubert cells of the affine Grassmannian (Recall A.1). For $N\in\mathbb{N}$, let $\mathscr{S}^{(N)}$ be its restriction to $\mathrm{Gr}^{(N)}$.

Definition A.18. Let G be a reductive group over $\mathbb{C}$, and R a finite ring. We define

$$\mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}^{(N)}, \mathscr{S}^{(N)}; R) = \mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}^{(j_N)}}(\mathrm{Gr}^{(N)}, \mathscr{S}^{(N)}; R)$$

with the notation of Definition B.8.

By Corollary B.49, this category is equivalent to its counterparts defined in the complex-analytic world

$$\mathrm{Cons}^{\mathrm{fd}}_{(G_{\mathcal{O}}^{(j_N)})^{\mathrm{an}}}((\mathrm{Gr}^{(N)})^{\mathrm{an}}, (\mathscr{S}^{(N)})^{\mathrm{an}}; R)$$

(Remark B.36).

As a consequence, by the same proof of Proposition 4.2, the definition is independent of j_N. One could also reach the same conclusion directly on the algebro-geometric side ([AR23, Proposition 10.2.8 with $K = (0)$]).

We can thus define

$$\begin{aligned}
\mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathscr{S}; R) &= \operatorname*{colim}_{N \geq 0} \mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}^{(N)}, \mathscr{S}^{(N)}; R) \\
\mathcal{D}^{\mathrm{fd}}_{\mathrm{c}, G_{\mathcal{O}}}(\mathrm{Gr}^{(N)}; R) &= \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}, G_{\mathcal{O}}^{(j_N)}}(\mathrm{Gr}^{(N)}; R) \\
\mathcal{D}^{\mathrm{fd}}_{\mathrm{c}, G_{\mathcal{O}}}(\mathrm{Gr}; R) &= \operatorname*{colim}_{N \geq 0} \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}, G_{\mathcal{O}}}(\mathrm{Gr}^{(N)}; R).
\end{aligned}$$

which again will agree with their complex-analytic counterparts (by definition):

$$\begin{aligned}
\mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, \mathscr{S}; R) &\simeq \mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}}, \mathscr{S}^{\mathrm{an}}; R) \\
\mathcal{D}^{\mathrm{fd}}_{\mathrm{c}, G_{\mathcal{O}}}(\mathrm{Gr}; R) &\simeq \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}, G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}}; R).
\end{aligned} \tag{A.8}$$

Remark A.19. Note that the forgetful functor

$$\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}, \mathscr{S}; R) \to \mathcal{P}\mathrm{erv}(\mathrm{Gr}, \mathscr{S}; R)$$

is an equivalence (see for example [BR18, Section 4.4]), but

$$\mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}, \mathscr{S}; R) \to \mathrm{Cons}^{\mathrm{fd}}(\mathrm{Gr}, \mathscr{S}; R)$$

is not.

On the other hand:

Proposition A.20. *The map*

$$\mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}, \mathscr{S}; R) \to \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}, G_{\mathcal{O}}}(\mathrm{Gr}; R)$$

is an equivalence.

Proof. By definition, the claim can be checked on each $\mathrm{Gr}^{(N)}$, where it is true by Lemma B.11. □

Remark A.21. All these results, with the right definitions, are true for profinite and ℓ-adic coefficients (cf. Remark B.10, Remark B.48) as well. We will only mention such coefficients in the following Remark A.24, which serves as a connection with other definitions of the small spherical Hecke category appearing in the literature and only uses the contents of this subsection. Hence, we will not delve into further details about profinite and ℓ-adic coefficients. The interested reader is encouraged to look at the references mentioned in Remark B.10.

Definition A.22. Let G be a complex reductive group and R a discrete, prodiscrete or ℓ-adic ring. The *topological spherical Hecke category* of G with coefficients in R is

$$\mathrm{Sph}(G;R)^{\mathrm{top}} = \mathrm{Cons}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}};R).$$

The *small spherical Hecke category* of G with coefficients in R as

$$\mathrm{Sph}(G;R)^{\mathrm{loc.c.}} = \mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}^{\mathrm{an}}}(\mathrm{Gr}^{\mathrm{an}};R).$$

Usually, in the Geometric Langlands Program, a renormalization of $\mathrm{Sph}(G)$ is used:

Definition A.23. Let G be a complex reductive group and R be a discrete, prodiscrete ring or ℓ-adic ring. The *renormalized spherical Hecke category* of G with coefficients in R is

$$\mathrm{Sph}(G;R)^{\mathrm{ren}} = \mathrm{Ind}(\mathrm{Sph}(G;R)^{\mathrm{loc.c}}).$$

Remark A.24. Let R be finite, profinite or ℓ-adic. By (A.8) and Proposition A.20,

$$\mathrm{Sph}(G;R)^{\mathrm{loc.c}} \simeq \mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr};R) \simeq \mathcal{D}^{\mathrm{fd}}_{\mathrm{c},G_{\mathcal{O}}}(\mathrm{Gr};R)$$

(and same for its Ind-completion). In particular, with these coefficients, the small and the renormalized spherical Hecke category do not distinguish between the algebraic and the analytic setting. Therefore, the same is true for the main result of our paper (Corollary 4.14), although the proof uses features of the analytic setting.

Remark A.25. On the other hand, we can consider discrete infinite rings such as $\mathbb{C}$. In this case, $\mathrm{Sph}(G;\mathbb{C})^{\mathrm{loc.c}}$ and $\mathrm{Sph}(G;\mathbb{C})^{\mathrm{ren}}$ admit an interpretation in terms of D-modules. Namely, by the Riemann-Hilbert correspondence we have that

$$\mathcal{D}^{\mathrm{fd}}_{\mathrm{c}}(\mathrm{Gr}^{\mathrm{an}};\mathbb{C}) \simeq \mathrm{DMod}(\mathrm{Gr})^{\omega}.$$

Let $\mathrm{DMod}_{G_{\mathcal{O}}}(\mathrm{Gr})^{\mathrm{loc.c}} \subset \mathrm{DMod}_{G_{\mathcal{O}}}(\mathrm{Gr})$ be the full subcategory spanned by objects which become compact after forgetting the equivariant structure ([AG15, 12.2.3]). Then we have an equivalence

$$\mathcal{D}^{\mathrm{fd}}_{\mathrm{c},G_{\mathcal{O}}}(\mathrm{Gr}^{\mathrm{an}};\mathbb{C}) \simeq \mathrm{DMod}_{G_{\mathcal{O}}}(\mathrm{Gr})^{\mathrm{loc.c}}$$

Combined with Lemma B.40, this provides an equivalence

$$\mathrm{Sph}(G;\mathbb{C})^{\mathrm{loc.c}} \simeq \mathrm{DMod}_{G_{\mathcal{O}}}(\mathrm{Gr})^{\mathrm{loc.c}}.$$

B Recollections and complements in stratified homotopy theory

B.1 Stratified schemes and stacks

Let us denote by $\mathrm{Sch}_{\mathbb{C}}$ the category of complex schemes, and by $\mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}}$ the full subcategory of complex schemes, locally of finite type.

Definition B.1. The category of *stratified complex schemes* is defined as $\mathrm{StrSch}_{\mathbb{C}} = \mathrm{Sch}_{\mathbb{C}} \times_{\mathrm{Top}} \mathrm{StrTop}$, where the map $\mathrm{Sch}_{\mathbb{C}} \to \mathrm{Top}$ sends a scheme X to its underlying *Zariski* topological space, and the other map is the evaluation at $[0]$.

We also define the category

$$\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}}$$

as the full subcategory of $\mathrm{StrSch}_{\mathbb{C}}$ spanned by those stratified schemes such that the stratification is finite constructible in the sense of [BGH20, Definition 1.2.1], and such that the underlying scheme is locally of finite type.

Remark B.2. By looking at the étale topology, we can consider the site $(\mathrm{Sch}_{\mathbb{C}}, \mathsf{\acute{e}t})$ and the category

$$\mathrm{Stk}_{\mathbb{C}}^{\mathrm{lft}} = \mathrm{Shv}_{\mathsf{\acute{e}t}}(\mathrm{Sch}_{\mathbb{C}}^{\mathrm{lft}}; \mathrm{Grpd})$$

of étale stacks locally of finite type.

We can define $\mathsf{str\acute{e}t}$ as the topology whose coverings are étale coverings whose stratification is induced by the one on the base. This leads to defining the category

$$\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}} = \mathrm{Sh}_{\mathsf{str\acute{e}t}}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}}; \mathrm{Grpd}) \tag{B.1}$$

of stratified étale stacks locally of finite type (stratified stacks for short).

Definition B.3. Let $\mathsf{uni} \subset \mathrm{Mor}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}})$ be the class of those morphisms $f : X \to Y$ which:

- are smooth morphisms on the underlying schemes
- when restricted to each stratum Y_α of Y, can be presented as a quotient by a unipotent smooth group scheme (trivially stratified).

By abuse of notation, we define $\mathsf{uni} \subset \mathrm{Mor}(\mathrm{StrTStk})$ to be the class of morphisms $\mathcal{X} \to \mathcal{Y}$ which are representable and whose pullback to any map $Z \to \mathcal{Y}$, with Z representable, belongs to tri.

We define $\mathrm{Pro}_{\mathsf{uni}}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}})$ to be the full subcategory of $\mathrm{Pro}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}})$ spanned by those pro-objects which can be presented as formal limits of cofiltered diagrams with transition maps belonging to uni.

Remark B.4. The class uni (both at the level of schemes and stacks) is stable under pullbacks.

Definition B.5. We define the $(2,1)$-category

$$\widehat{\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}} = \mathrm{Fun}((\mathrm{Pro}_{\mathsf{uni}}(\mathrm{StrStk}_{\mathbb{C}}^{\mathrm{lft}}))^{\mathrm{op}}, \mathrm{Grpd}).$$

B.2 Constructible sheaves on stratified schemes and stacks

Definition B.6 ([BGH20, Example 13.2.9])**.** Let (Y, s) be a complex stratified scheme of finite dimension, and R a finite ring. The ∞-category of constructible sheaves

$$\mathrm{Cons}^{\mathrm{fd}}(Y, s; R)$$

is defined as the full subcategory of

$$\mathrm{Shv}_{\mathsf{\acute{e}t}}(Y, \mathrm{Mod}_R)$$

spanned by those objects which, after restriction to each stratum, are locally constant and lisse (in the sense of [BGH20, Definition 13.2.6]).

There is also the notion of constructible sheaves with respect to *some* stratification (instead of a fixed one).

Recall B.7 ([BL94, 4.1], [GL18, Proposition 4.2.5]). Let Y be a complex scheme locally of finite type, and R a finite ring. We define

$$\mathcal{D}^{\mathrm{fd}}_{\mathrm{c}}(Y;R) = \operatorname*{colim}_{\mathscr{S} \text{ algebraic stratification of } Y} \mathrm{Cons}(Y,\mathscr{S};R) \subset \mathrm{Shv}_{\text{ét}}(Y;R). \tag{B.2}$$

since the colimit in the right-hand-side is filtered: indeed, one can refine two algebraic stratification by a common one[13]. When Y is quasiprojective, this is the ∞-category of compact objects[14] in $\mathrm{Shv}_{\text{ét}}(Y;R)$.

Constructible sheaves on stratified stacks can be defined by right Kan extension. However, we will only need the special case of quotient stacks (i.e. the case of equivariant constructible shaves on schemes), which we recall below.

Definition B.8. Let (Y,s) be a stratified complex scheme of finite dimension, H a group scheme of finite type acting on Y in such a way that the action sends strata to strata, and let R be a finite ring. There is a simplicial diagram

$$\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; (H\times H\times Y, s_2) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; (H\times Y, s_1) \;\substack{\longleftarrow\\ \longrightarrow}\; (Y,s) \tag{B.3}$$

where s_i is the stratification on $\overbrace{H\times\cdots\times H}^{i}\times Y$ which is trivial on the group factors and s on the last factor. As usual, in the left direction are induced by the identity element of G in various ways, and maps in the right direction are induced by combinations of the action and the projections.

The ∞-category $\mathrm{Cons}^{\mathrm{fd}}_H(Y,s;R)$ of H-equivariant constructible sheaves on Y with respect to the stratification s is defined as the limit of the cosimplicial diagram (induced by pullback of sheaves from (B.3))

$$\cdots \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathrm{Cons}^{\mathrm{fd}}(H\times H\times Y, s_2;R) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathrm{Cons}^{\mathrm{fd}}(H\times Y, s_1;R) \;\substack{\longrightarrow\\ \longleftarrow}\; \mathrm{Cons}^{\mathrm{fd}}(Y,s;R) \tag{B.4}$$

Recall B.9. Let Y be a finite-dimensional scheme, H a group scheme of finite type acting on Y, R a finite ring. The ∞-category $\mathcal{D}^{\mathrm{fd}}_{\mathrm{c},H}(Y;R)$ of H-equivariant constructible sheaves on Y is the limit of the diagram (induced by pullback of sheaves from (B.3))

$$\cdots \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}}(H\times H\times Y;R) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}}(H\times Y;R) \;\substack{\longrightarrow\\ \longleftarrow}\; \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}}(Y;R) \tag{B.5}$$

Remark B.10. One can define constructible sheaves and equivariant constructible sheaves on stratified schemes (both with respect to a fixed stratification or not) with coefficients in profinite or ℓ-adic rings, cf. [Beh04, §6.1], [BGH20, Recollection 13.7.7, 13.8.7]. Unlike the case of constructible sheaves on

[13] This is easy because we did not assume the strata to be smooth. For the smooth case, see [MSE20].

[14] See [GL18, Proposition 4.2.5]. To see the equivalence between constructibility and condition (1) in *loc. cit.*, one can apply noetherianity in order to find suitable finite stratifications of Y.

stratified topological spaces (Definition B.35), this definition is not formal. However, in this paper we are only interested in proving that with the correct definitions a GAGA principle holds (Remark B.48). Since the profinite and ℓ-adic case arise via limits and filtered colimits from the finite case, it will suffice to prove the finite case (Proposition B.47).

In the setting of Recall B.9, if one assumes that there are finitely many orbits of H, these orbits form a stratification themselves ([MO14]), which we denote by s. Let R be a finite ring. We have a pullback square of stable ∞-categories

$$\begin{tikzcd} \mathrm{Cons}^{\mathrm{fd}}_H(Y,s;R) \arrow[r] \arrow[d, hook] & \mathrm{Cons}^{\mathrm{fd}}(Y,s;R) \arrow[d, hook] \\ \mathcal{D}^{\mathrm{fd}}_{\mathrm{c},H}(Y;R) \arrow[r] & \mathcal{D}^{\mathrm{fd}}_{\mathrm{c}}(Y;R). \end{tikzcd} \tag{B.6}$$

Note that the vertical functors are fully faithful because the transition maps in the colimit (B.2) are, and the colimit is filtered. Now, the horizontal arrows in (B.6) are not equivalences. On the contrary:

Lemma B.11. *Let R be a finite ring. Let H be a group scheme acting on a finite-dimensional scheme Y, and suppose that there are finitely many orbits, forming a stratification s of Y. Then the functor* $\mathrm{Cons}^{\mathrm{fd}}_H(Y,s;R) \to \mathcal{D}^{\mathrm{fd}}_{\mathrm{c},H}(Y;R)$ *is an equivalence.*

Proof. We already remarked that the functor is fully faithful. We now argue like in [MO21]. Let us now consider an equivariant constructible sheaf $\mathcal{F}$ with respect to some stratification, and let us prove that it is constructible with respect to the orbit stratification. Let us consider the maximal open subset U of Y where the sheaf is locally constant: this is nonempty since we know that $\mathcal{F}$ is constructible with respect to some stratification, and any stratification of a finite-dimensional scheme has an open stratum. More subtly, there is an open dense stratum in every connected component of Y, and by taking unions of such over all the connected components of Y, we obtain an U such that its complementary is closed of dimension strictly smaller than $\dim Y$. Also, U is unique, since the union of two open subsets where $\mathcal{F}$ is locally constant has again the property that $\mathcal{F}$ is locally constant there. Now, U is H-stable by equivariancy of $\mathcal{F}$ and maximality of U itself, and thus its complementary is H-stable as well and we can apply Noetherian induction. □

B.3 Stratified topological spaces and stacks

The following definition is a particular case of [BGH20, 8.2.1 and ff.].

Definition B.12. Let Top be the 1-category of topological spaces. The category of *stratified topological spaces* is defined as

$$\mathrm{StrTop}_{\mathbb{C}} = \mathrm{Fun}(\Delta^1, \mathrm{Top}) \times_{\mathrm{Top}} \mathrm{Poset},$$

where the map $\mathrm{Fun}(\Delta^1, \mathrm{Top}) \to \mathrm{Top}$ is the evaluation at 1, and $\mathrm{Alex} : \mathrm{Poset} \to \mathrm{Top}$ assigns to each poset P its underlying set with the so-called Alexandrov topology (see [BGH20, Definition 1.1.1]).

Note that StrTop is complete and cocomplete, because Top, $\mathrm{Fun}(\Delta^1, \mathrm{Top})$ and Poset are.

Recall B.13. Let (X,s) be a stratified topological space. The notion of conical stratification as given in [Lur17, Definition A.5.5] amounts to asking that around each point of X there exists a neighbourhood which is stratified homemorphic to $Z \times \mathrm{C}(Y)$ where Z is an unstratified space and $\mathrm{C}(Y)$ is the stratified open cone of a stratified space Y. Being conical is the main condition required to a stratified space in order to make the so-called Exodromy Theorem (Recall B.30) true.

For simplicity, in the present paper a stratified space is called *conical* if satisfies several conditions altogether:

Definition B.14. The category $\mathrm{StrTop}_{\mathrm{con}}$ is the full subcategory of StrTop spanned by those stratified spaces $(X, s : X \to P)$ such that:

- X is locally of singular shape in the sense of [Lur17, Definition A.4.15]
- the strata of X are locally weakly contractible;
- P satisfies the ascending chain condition;
- the stratification is conical in the sense of [Lur17, Definition A.5.5].

This category admits finite products, essentially because the product of two cones is the cone of the join space. Therefore, there is a well-defined symmetric monoidal Cartesian structure $\mathrm{StrTop}^{\times}_{\mathrm{con}}$.

Remark B.15 (Condition of the frontier)**.** One consequence of the conicality assumption is the condition of the frontier, i.e. the fact that given two strata X_p, X_q, such that X_p is connected and such that $\overline{X_q} \cap X_p \neq \varnothing$, then $X_p \subset \overline{X_q}$. This is true locally by inspection of the conical model and is globalized by connectedness of X_p. Notice that, whenever this happens, we automatically get that $p \in s(\overline{X_q}) \subset \overline{\{q\}} = P_{\leq q}$, i.e. $p \leq q$.

Notation B.16. Let us consider the topology of local homeomorphisms on the topological side (which has however the same sheaves as the topology of open embeddings). We have thus the site $(\mathrm{Top}, \mathsf{loc})$ and the category

$$\mathrm{TStk} = \mathrm{Shv}_{\mathsf{loc}}(\mathrm{Top}; \mathrm{Grpd})$$

of topological stacks.

Let strloc be the topology whose coverings are jointly surjective families of local homeomorphisms such that the stratification on the total space is induced by that on the base. This defines $(2,1)$—categories

$$\mathrm{StrTStk} = \mathrm{Sh}_{\mathsf{strloc}}(\mathrm{StrTop}; \mathrm{Grpd})$$
$$\mathrm{StrTStk}_{\mathrm{con}} = \mathrm{Sh}_{\mathsf{strloc}}(\mathrm{StrTop}_{\mathrm{con}}; \mathrm{Grpd})$$

of (conically) stratified topological stacks.

Definition B.17. Let $\alpha, \beta : (X, s) \to (Y, t)$ be two stratified maps between stratified topological spaces. Let $\widetilde{s}$ be the stratification of $[0,1] \times X$ induce by the projection $[0,1] \times X \to X$. A stratified homotopy between α and β is a stratified map

$$H : ([0,1] \times X, \widetilde{s}) \to (Y, t)$$

such that $H(0,—) = f, H(1,—) = g$.

Definition B.18. A stratified homotopy equivalence between stratified topological spaces is a stratified map $f : (X, s) \to (Y, t)$ such that there exist a stratified map $g : (Y, t) \to (X, s)$ and stratified homotopies $fg \simeq \mathrm{id}_Y, gf \simeq \mathrm{id}_X$.

Remark B.19. Note that the class of stratified homotopy equivalences is not closed under pullbacks (just like homotopy equivalences of topological spaces are closed under homotopy pullback but not under pullbacks).

Remark B.20. A stratified homotopy equivalence induces an isomorphism at the level of posets, and homotopy equivalences on each stratum.

Definition B.21. Let $f:(X,s)\to(Y,t)$ be a morphism in $\mathrm{StrTop}_{\mathrm{con}}$. We say that f is a *smooth stratified submersion* if, locally in the topology of stratified local homeomorphisms on X, it is of the form of a projection $(Y,t)\times\mathbb{R}^N\to(Y,t)$ for some N (where $\mathbb{R}^N$ is seen as a trivially stratified space).

We denote this class by subm. When both X and Y are unstratified, we will just speak about *smooth topological submersion*.

Remark B.22. Smooth stratified submersions are closed under pullback (cf. [Vol21, Remark 3.23]). This is of course not true for stratified homotopy equivalence (since for example homotopy equivalences of trivially stratified manifolds are not closed under pullback).

Definition B.23. Let $\mathsf{tri}\subset\mathrm{Mor}(\mathrm{StrTop})$ be the class of those morphisms $f:X\to Y$ which:

- are smooth stratified submersions in the sense of Definition B.21
- when restricted to each stratum Y_α of Y, become trivial Serre fibrations (in particular, the stratification on both source and target becomes trivial).

Let $\mathsf{tri}\subset\mathrm{Mor}(\mathrm{StrTStk})$ be the class of morphisms $\mathcal{X}\to\mathcal{Y}$ which are representable and whose pullback to any map $Z\to\mathcal{Y}$, with Z representable, belongs to tri.

We define $\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk})$ to be the full subcategory of $\mathrm{Pro}(\mathrm{StrTStk})$ spanned by those pro-objects which can be presented as formal limits of cofiltered diagrams with transition maps belonging to tri; we also define $\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}})$ to be the full subcategory of $\mathrm{Pro}(\mathrm{StrTStk}_{\mathrm{con}})$ spanned by those pro-objects which can be presented as formal limits of cofiltered diagrams with transition maps belonging to tri.

Remark B.24. The class tri (both at the level of spaces and stacks) is stable under pullback.

Definition B.25. We define the $(2,1)$-categories

$$\widehat{\mathrm{StrTStk}}=\mathrm{Fun}((\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}))^{\mathrm{op}},\mathrm{Grpd}).$$
$$\widehat{\mathrm{StrTStk}}_{\mathrm{con}}=\mathrm{Fun}((\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}))^{\mathrm{op}},\mathrm{Grpd}).$$

Remark B.26. Note that there is a canonical fully faithful embedding

$$\widehat{\mathrm{StrTStk}}_{\mathrm{con}}\hookrightarrow\widehat{\mathrm{StrTStk}}.$$

This is defined as follows. First of all, there is a functor

$$\mathrm{StrTStk}_{\mathrm{con}}\to\mathrm{StrTStk}$$

given by left Kan extension, which preserves colimits. Then, this can be extended to a functor

$$\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}})\to\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk})$$

by using right Kan extension, and the new functor preserves cofiltered limits. Finally, we can extend the latter functor to

$$\widehat{\mathrm{StrTStk}}_{\mathrm{con}}\to\widehat{\mathrm{StrTStk}}$$

by left Kan extension, and the new functor preserves all colimits.

Definition B.27. A morphism $f : \mathcal{X} \to \mathcal{Y}$ in StrTStk belongs to she' if it is representable and it can be presented as a colimit of maps in she.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk})$ belongs to she'' if it can be presented as

$$\text{“}\lim_{j\in J}\text{”} f_j$$

where J is cofiltered and each f_j is equivalent to a map in she' in StrTStk.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\widehat{\mathrm{StrTStk}}$ belongs to $\widehat{\mathsf{she}}$ if it can be presented as a colimit of maps in she''.

These classes restrict to classes of maps in the correspondent categories formed from $\mathrm{StrTop}_{\mathrm{con}}$ instead of StrTop, and will be denoted in the same way.

Definition B.28. A morphism $f : X \to \mathcal{Y}$ in StrTStk belongs to $\widetilde{\mathsf{subm}}$ if it is representable and its pullback to any representable belongs to subm.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ in StrTStk belongs to subm if there is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

where X, Y are representable, the vertical arrows belong to $\widetilde{\mathsf{subm}}$ and the top horizontal arrow belongs to subm.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk})$ belongs to subm'' if it can be presented as

$$\text{“}\lim_{j\in J}\text{”} f_j$$

where J is cofiltered and each f_j is equivalent to a map in subm' in StrTStk.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\widehat{\mathrm{StrTStk}}$ belongs to $\widehat{\mathsf{subm}}$ if for every $\mathcal{Z} \in \mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}) \hookrightarrow \widehat{\mathrm{StrTStk}}_{\mathrm{con}}$ and a map $\mathcal{Z} \to \mathcal{Y}$, the pullback in $\widehat{\mathrm{StrTStk}}_{\mathrm{con}}$

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$$

belongs to the essential image of the Yoneda embedding $\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}) \hookrightarrow \widehat{\mathrm{StrTStk}}$ and the canonical map

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$$

is equivalent to a map in subm'' in $\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk})$.

These classes restrict to classes of maps in the correspondent categories formed from $\mathrm{StrTop}_{\mathrm{con}}$ instead of StrTop, and will be denoted in the same way.

Remark B.29. The classes of maps $\widehat{\mathsf{subm}}$ in $\widehat{\mathrm{StrTStk}}$ and in $\widehat{\mathrm{StrTStk}}_{\mathrm{con}}$ are closed under pullbacks.

B.4 Constructible sheaves on stratified topological spaces and stacks

Recall B.30. Let $(Y, s) \in \mathrm{StrTop}_{\mathrm{con}}$. The ∞-category

$$\mathrm{Cons}(Y, s; \mathcal{S})$$

of space-valued constructible sheaves is defined in [Lur17, Definition A.5.2] and proven to be equivalent (by [Lur17, Theorem A.9.3] , nowadays known as the *Exodromy Theorem in topology*), to the ∞-category

$$\mathrm{Fun}(\mathrm{Exit}(Y,s),\mathcal{S}).$$

Here $\mathrm{Exit}(Y,s)$ is a small ∞-category called the ∞-category of exit paths on (Y,s) (see [Lur17, Definition A.6.2], where it is denoted by $\mathrm{Sing}^A(Y)$, A being the poset associated to the stratification).

Remark B.31. Every space which is locally of singular shape in the sense of [Lur17, Definition A.4.15] has the property that every locally constant sheaf is automatically hypercomplete ([Lur17, Corollary A.1.17]).

One can consider coefficient categories different than $\mathcal{S}$, namely any presentable stable ∞-category.

Definition B.32. Let $(Y,s)\in \mathrm{StrTop}_{\mathrm{con}}$ and $\mathcal{E}$ a presentable stable ∞-category. We define

$$\mathrm{Cons}(Y,s;\mathcal{E})$$

as the full subcategory of $\mathrm{Shv}(Y;\mathcal{E})$ spanned by constructible sheaves in the sense of [PT22, Definition 2.27] (hypercompleteness in *loc. cit.* can be ignored thanks to Remark B.31).

Theorem B.33 ([PT22])**.** *Let $(Y,s)\in \mathrm{StrTop}_{\mathrm{con}}$, and $\mathcal{E}$ a presentable stable ∞-category. Then*

$$\mathrm{Cons}(Y,s;\mathcal{E})\simeq \mathrm{Fun}(\mathrm{Exit}(Y,s),\mathcal{E}).$$

Proof. This is the content of [PT22, Theorem 5.17] together with [PT22, Remark 5.18] and Remark B.31. □

Remark B.34. Note that, for $\mathcal{C}$ any small ∞-category and $\mathcal{E}$ presentable,

$$\mathrm{Fun}(\mathcal{C},\mathcal{E})\simeq \mathrm{Fun}(\mathcal{C},\mathcal{S})\otimes\mathcal{E}. \tag{B.7}$$

This follows from the fact that if $\mathcal{C}$ is small then $\mathrm{Fun}(\mathcal{C},\mathcal{S})$ is presentable and from the formula

$$\mathcal{A}\otimes\mathcal{B}\simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}},\mathcal{A})$$

[Lur17, Proposition 4.8.1.17] for $\mathcal{A},\mathcal{B}$ presentable. As a consequence, under the hypotheses of Theorem B.33, we have

$$\mathrm{Cons}(Y,s;\mathcal{E})\simeq \mathrm{Cons}(Y,s;\mathcal{S})\otimes\mathcal{E}.$$

Definition B.35. Let R be a discrete ring. We denote

$$\mathrm{Mod}_R^{\mathrm{cont}}=\mathrm{Mod}_R$$
$$\mathrm{Mod}_R^{\mathrm{fd}}=\mathrm{Mod}_R^{\mathrm{cont,fd}}=\mathrm{Perf}_R.$$

Let us also denote by $\mathrm{Mod}_R^{\mathrm{tors}},\mathrm{Mod}_R^{\mathrm{fd,tors}}$ the respective full subcategories of torsion modules.

Let $R=\lim_{i\in\mathcal{I}}R_i$ be a prodiscrete ring, i.e. $\mathcal{I}$ is cofiltered, R_i is discrete for each i and R has the limit topology.

Then we define

$$\mathrm{Mod}_R^{\mathrm{cont}}=\lim_{i\in\mathcal{I}}\mathrm{Mod}_{R_i}$$

as a limit in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (transition maps are given by tensor product) and

$$\mathrm{Mod}_R^{\mathrm{cont,fd}} = \lim_{i\in\mathcal{I}} \mathrm{Perf}_{R_i}.$$

In the same way we define $\mathrm{Mod}^{\mathrm{cont,tors}} = \lim_i \mathrm{Mod}_{R_i}^{\mathrm{tors}}$ and $\mathrm{Mod}^{\mathrm{cont,fd,tors}} = \lim_i \mathrm{Mod}_{R_i}^{\mathrm{fd,tors}}$.

Let R be a finite extension of $\mathbb{Q}_\ell$ (from now on an "ℓ-adic ring"). We define

$$\mathrm{Mod}_R^{\mathrm{cont}} = \mathrm{cofib}[\mathrm{Mod}_{\mathcal{O}_R}^{\mathrm{cont,tors}} \hookrightarrow \mathrm{Mod}_{\mathcal{O}_R}^{\mathrm{cont}}]$$
$$\mathrm{Mod}_R^{\mathrm{cont,fd}} = \mathrm{cofib}[\mathrm{Mod}_{\mathcal{O}_R}^{\mathrm{cont,fd,tors}} \hookrightarrow \mathrm{Mod}_{\mathcal{O}_R}^{\mathrm{cont,fd}}]$$

where the cofibers are taken respectively in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ and in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$. Finally, let R be an algebraic extension of $\mathbb{Q}_\ell$. Then we define

$$\mathrm{Mod}_R^{\mathrm{cont}} = \operatorname*{colim}_{\mathbb{Q}_\ell\subset E\subset R \text{ finite subextension}} \mathrm{Mod}_E^{\mathrm{cont}}$$
$$\mathrm{Mod}_R^{\mathrm{cont,fd}} = \operatorname*{colim}_{\mathbb{Q}_\ell\subset E\subset R \text{ finite subextension}} \mathrm{Mod}_E^{\mathrm{cont,fd}}.$$

Note that since the inclusion functors $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ and $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ are closed under limits and filtered colimits, all of the above are stable ∞-categories.

Remark B.36. If R is a discrete reing, we denote

$$\mathrm{Cons}(Y,s;R) = \mathrm{Cons}(Y,s,\mathrm{Mod}_R)$$
$$\mathrm{Cons}^{\mathrm{fd}}(Y,s;R) = \mathrm{Cons}^{\mathrm{fd}}(Y,s;\mathrm{Mod}_R^{\mathrm{fd}}).$$

Let $R = \lim_i R_i$ be a profinite ring. We can apply Theorem B.33 with $\mathcal{C} = \mathrm{Mod}_R^{\mathrm{cont}}$ and obtain

$$\mathrm{Cons}(Y,s;\mathrm{Mod}_R^{\mathrm{cont}}) \simeq \mathrm{Fun}(\mathrm{Exit}(Y,s),\mathrm{Mod}_R^{\mathrm{cont}}) \simeq \lim_i \mathrm{Cons}(Y,s;R_i) \tag{B.8}$$

which we denote by $\mathrm{Cons}(Y,s;R)$. This equivalence restricts to an equivalence

$$\mathrm{Cons}^{\mathrm{fd}}(Y,s;R) \simeq \mathrm{Fun}(\mathrm{Exit}(Y,s),\mathrm{Mod}_R^{\mathrm{cont,fd}}) \simeq \lim_i \mathrm{Cons}(Y,s;R_i),$$

which we denote by $\mathrm{Cons}^{\mathrm{fd}}(Y,s;R)$. Note that, in general, this is *not* the category of compact objects of $\mathrm{Cons}(Y,s;R)$.

When R is a finite extension of $\mathbb{Q}_\ell$, by Remark B.34 we have

$$\begin{gathered}\mathrm{Cons}(Y,s;\mathrm{Mod}_R^{\mathrm{cont}}) \simeq \mathrm{Fun}(\mathrm{Exit}(Y,s),\mathrm{Mod}_R^{\mathrm{cont}}) \simeq \\ \mathrm{Fun}(\mathrm{Exit}(Y,s),\mathcal{S}) \otimes \mathrm{cofib}(\mathrm{Mod}_{\mathcal{O}_R}^{\mathrm{cont,tors}} \to \mathrm{Mod}_{\mathcal{O}_R}^{\mathrm{cont}}) \simeq \\ \mathrm{cofib}(\mathrm{Cons}^{\mathrm{tors}}(Y,s;\mathcal{O}_R) \to \mathrm{Cons}(Y,s;\mathcal{O}_R)).\end{gathered}$$

When R is an algebraic extension of $\mathbb{Q}_\ell$, we have

$$\mathrm{Cons}(Y,s;\mathrm{Mod}_R^{\mathrm{cont}}) \simeq \operatorname*{colim}_{\mathbb{Q}_\ell\supset E\supset R \text{ finite subextension}} \mathrm{Cons}(Y,s;\mathrm{Mod}_E^{\mathrm{cont}})$$

which we denote by $\mathrm{Cons}(Y,s;R)$. We also define, in all previous cases of R,

$$\mathrm{Cons}^{\mathrm{fd}}(Y,s;R) = \mathrm{Cons}^{\mathrm{fd}}(Y,s;\mathrm{Mod}_R^{\mathrm{cont,fd}}).$$

Recall B.37. Let Y be a topological space, $\mathcal{E}$ a stable presentable ∞-category. We define

$$\mathcal{D}_c(Y;\mathcal{E}) = \operatorname*{colim}_{(Y,s)\text{ conical stratification}} \operatorname{Cons}(Y,s;\mathcal{E}).$$

As we will see in Section B.6, the functor $\operatorname{Cons}(-;\mathcal{E}) : \mathrm{StrTop}_{\mathrm{con}}^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$ can be right Kan extended to $\mathrm{StrTStk}_{\mathrm{con}}$. We consider here the case of quotient stacks, i.e. the case of equivariant constructible sheaves over stratified topological spaces.

Definition B.38. Let (Z,t) be a stratified topological space and K a topological group acting on Z compatibly with t. Let $\mathcal{E}$ be a presentable stable ∞-category. We define the the category of K-equivariant $\mathcal{E}$-valued constructible sheaves on Z

$$\operatorname{Cons}_K(Z,t;\mathcal{E})$$

as the limit of the diagram (induced by pullback of sheaves along (B.3))

$$\cdots \substack{\longleftarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} \operatorname{Cons}(K\times K\times Z,t_2;\mathcal{E}) \substack{\longleftarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow} \operatorname{Cons}(K\times Z,t_1;\mathcal{E}) \substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow} \operatorname{Cons}(Z,t;\mathcal{E}) \tag{B.9}$$

where s_i is the stratification on $\overbrace{H\times\cdots\times H\times Y}^{i}$ which is trivial on the group factors and s on the last factor, and the diagram is the simplicial diagram encoding the action of H on Y.

Definition B.39. Let (Y,s) be a stratified topological space, H a topological group acting on Y compatibly with s, and let a presentable stable ∞-category. We define

$$\operatorname{Cons}^{\mathrm{fd}}_H(Y,s;\mathcal{E}) = \operatorname{Cons}_H(Y,s;\mathcal{E}) \times_{\operatorname{Cons}(Y,s;\mathcal{E})} \operatorname{Cons}^{\mathrm{fd}}(Y,s;\mathcal{E}) \simeq \lim_{n\in\boldsymbol{\Delta}} \operatorname{Cons}^{\mathrm{fd}}(H^n\times Y,s_n;\mathcal{E})$$

Lemma B.40. *Let $\mathcal{E}$ be a symmetric monoidal presentable stable ∞-category. Let K be a topological group acting on a topological space Z locally of singular shape, and suppose that the orbits, form a conical stratification t of Z. Then the functor $\operatorname{Cons}^{\mathrm{fd}}_K(Z,t;\mathcal{E}) \to \mathcal{D}^{\mathrm{fd}}_{c,K}(Z;\mathcal{E})$ is an equivalence.*

Proof. As in the proof of Lemma B.11, we already know that the functor is fully faithful. Let us now consider an equivariant constructible sheaf $\mathcal{F}$ with respect to some conical stratification $(Z, s : Z\to P)$, and let us prove that it is constructible with respect to the orbit stratification. By the ascending chain condition for P, there exists at least an open stratum in s. In fact, all minimal depth strata (i.e. strata associated to a maximal p in P) are open: indeed, let p be a maximal element of P, and consider $\overline{Y\setminus Y_p}$. Suppose for simplicity that Y_p is connected. If this subspace intersects Y_p nontrivially, in particular there exists another stratum $Y_q, q\neq p$, such that $\overline{Y_q}\cap Y_p\neq\varnothing$. By the condition of the frontier Remark B.15, this implies that $Y_p\subset\overline{Y_q}$ and hence $q\geq p$, contradiction.

We define U as the maximal open union of strata such that $\mathcal{F}$ is locally constant over U. It is nonempty because $\mathcal{F}$ is constructible with respect to s and by the above remark about minimal depth strata. Moreover, U is unique because the union of two open subsets where $\mathcal{F}$ is locally constant has again the property that $\mathcal{F}$ is locally constant there. Also, the complementary of U is concentrated in strata of non-minimal depth, again by the above remark. Now, U is K-stable by equivariancy of $\mathcal{F}$ and maximality of U itself[15], and thus its complementary is K-stable as well and we can apply Noetherian induction on the maximum lenght of ascending chains of P. $\square$

[15]To see this, one can argue as follows: suppose U is not K-stable, and let $g\in K$ such that $gU\neq U$. Pick a point $v\in gU$, and an open subset $V\subset U$ such that $gV\ni v$ and $\mathcal{F}|_V$ is constant with value $E\in\mathcal{E}$. Then, by equivariance, we have that $\mathcal{F}(gV)\simeq\mathcal{F}(V)\simeq E$. Hence, $\mathcal{F}$ is locally constant on $U\cup gV$, which contradicts maximality of U.

B.5 Stratified analytification

Theorem B.41 ([Ray71, Théorème et définition 1.1]). *Let X be a scheme locally of finite type over $\mathbb{C}$. Then there exists an associated complex-analytic space X^{an}, whose underlying set is $X(\mathbb{C})$ and which represents the functor*

$$\{\text{complex-analytic spaces}\} \to \mathrm{Set}$$
$$Y \mapsto \mathrm{Hom}_{\text{ringed spaces}}(Y, X).$$

We can forget the structure sheaf of holomorphic functions and recover an underlying Hausdorff topological space (which corresponds to the operation denoted by $|-|$ in [Ray71]). We thus obtain a functor which by abuse of notation we denote by

$$(-)^{\mathrm{an}} : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{lft}} \to \mathrm{Top}$$

(instead of $|(-)^{\mathrm{an}}|$).

This functor preserves finite limits ([Ray71, 1.2]) and sends étale coverings to coverings in the local homeomorphism topology.

Construction B.42. There is a natural stratified version of the functor $(-)^{\mathrm{an}}$, namely the one which accounts for the stratification induced by the map of ringed spaces $u : S^{\mathrm{an}} \to S$ coming with the universal property:

$$\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}} \to \mathrm{StrTop}$$
$$(S, s) \mapsto (S^{\mathrm{an}}, s \circ u).$$

Example B.43 (Smooth algebraic maps). Let X, Y be complex schemes, locally of finite type, and $f : X \to Y$ be a smooth morphism in the sense of algebraic geometry. Then f^{an} is a smooth topological submersion. In particular, the analytification of a smooth scheme locally of finite type is a topological manifold.

Proof. This is [Vak22, Exercise 13.6.A]. □

Example B.44 (Relative stratified torsors with smooth fiber). Let X, Y, S be complex stratified schemes, locally of finite type, and $X \to S, Y \to S$. Suppose that $f : X \to Y$ is a torsor, relative over S, whose fiber is a smooth unstratified scheme over S. Then f^{an} is a smooth stratified submersion.

Proof. Such a map is, in particular, a smooth morphism in the sense of algebraic geometry, hence we can apply Example B.43. □

Proposition B.45. *The stratified analytification functor* $(-)^{\mathrm{an}} : \mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}} \to \mathrm{StrTop}$ *sends stratified étale coverings to stratified coverings in the topology of local homeomorphism. Also, it sends morphisms in* uni *to morphisms in* tri.

Proof. The first part follows from [Ray71, Proposition 3.1].

For the second part, note first of all that the analytification of a smooth morphism of schemes is in subm by Example B.43. After restriction to any stratum of Y, a map in uni is étale-locally trivial (because it is assumed to be a torsor) and hence its analytification is a Serre fibration. Now, a unipotent group scheme is isomorphic (as a scheme) to some affine space $\mathbb{A}_{\mathbb{C}}^{N}$ (see [KMT74, Theorem 8.0]). Therefore, its analytification is contractible. This implies that the analytification of a morphism in uni, when restricted to strata, is a trivial Serre fibration of trivially stratified spaces.

To conclude the proof, it suffices to apply [HN24, Lemma 4]. More precisely, that theorem is stated with coefficients in spaces, but we recover our version from combining Recall B.30 and Remark B.34. □

Corollary B.46. *We have an extended analytification functor*

$$(-)^{\mathrm{an}} : \widehat{\mathrm{StrStk}^{\mathrm{lft}}_{\mathbb{C}}} \to \widehat{\mathrm{StrTStk}}. \tag{B.10}$$

Proof. The extension from schemes to stacks follows from the first part of Proposition B.45. The extension from stacks to pro-uni-objects follows from the second part of Proposition B.45. The extension from pro-uni-objects is done by left Kan extension (i.e. covariant functoriality of $\mathrm{Fun}(-, \mathrm{Grpd})$). □

Proposition B.47. *Let (Y, s) be a qcqs complex stratified scheme, locally of finite type, and R a finite ring. Then there is an equivalence of ∞-categories*

$$\mathrm{Cons}^{\mathrm{fd}}(Y, s; R) \simeq \mathrm{Cons}^{\mathrm{fd}}(Y^{\mathrm{an}}, s^{\mathrm{an}}; R).$$

Proof. The claim follows from the proof [BGH20, Proposition 12.6.4] (in turn building upon [AGV72, Théorème XVI. 4.1])[16]. □

Remark B.48. This proof only relies on the Exodromy theorem both on the algebraic[17] and topological side. Hence, it extends to the profinite and ℓ-adic case (cf. Remark B.10) by [BGH20, Theorem 13.7.8] (for the finite/profinite case) and [BGH20, Theorem 13.8.8] (for algebraic extensions of $\mathbb{Q}_\ell$).

By forming limits from the statement of Proposition B.47, we obtain, in the same setting, that:

Corollary B.49. *Let (Y, s) be a finite dimensional complex stratified scheme, and H a complex group scheme, locally of finite type, acting on it in a stratified way. There is an equivalence of ∞-categories*

$$\mathrm{Cons}^{\mathrm{fd}}_H(Y, s; R) \simeq \mathrm{Cons}^{\mathrm{fd}}_{H^{\mathrm{an}}}(Y^{\mathrm{an}}, s^{\mathrm{an}}; R).$$

B.6 Monoidality, Kan extensions and correspondences

Notation B.50. The (very large) ∞-category of presentable stable ∞-categories and all functors is denoted by $\mathcal{P}\mathrm{r}_{\mathrm{st}}$. The (very large) ∞-category of presentable stable ∞-categories and left adjoint functors is denoted by $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$. The ∞-category of presentable stable ∞-categories and right adjoint functors is denoted by $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathrm{st}}$.

Let $\mathcal{E}$ be a presentable stable symmetric monoidal ∞-category. We denote:

- by $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$ the ∞-sub-category of $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}^{\otimes}_{\mathrm{st}})$ spanned by the objects belonging to $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{st}})$ and whose morphisms are $\mathcal{E}$-linear functors admitting an $\mathcal{E}$-linear right adjoint. This is, in particular, a non-full subcategory of $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{st}})$;
- by $\mathcal{P}\mathrm{r}^{\mathrm{R}}_{\mathcal{E}}$ the ∞-sub-category of $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}^{\otimes}_{\mathrm{st}})$ spanned by the objects belonging to $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{st}})$ and whose morphisms are $\mathcal{E}$-linear functors admitting an $\mathcal{E}$-linear left adjoint;
- by $\mathcal{P}\mathrm{r}^{\mathrm{LR}}_{\mathcal{E}}$ the ∞-sub-category of $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}^{\otimes}_{\mathrm{st}})$ spanned by the objects belonging to $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{st}})$ and whose morphisms are $\mathcal{E}$-linear functors admitting both an $\mathcal{E}$-linear left adjoint and an $\mathcal{E}$-linear right adjoint.

[16] Although the statement is given for categories of constructible sheaves with respect to some algebraic stratification (and its analytic counterpart) the proof actually proceeds by establishing the equivalence when the stratification is fixed, and then passes to the colimit.

[17] By Exodromy on the algebraic side, we mean [BGH20, Corollary 13.2.12] for finite coefficients, and the references below in this remark for profinite and ℓ-adic coefficients. Note that a "coherent scheme" in the terminology of *loc.cit.* is a qcqs scheme ([BGH20, 0.11.15]).

For $\mathcal{E} = \mathrm{Mod}_R$, R being a discrete commutative ring, we denote this categories by $\mathcal{P}r^{\mathrm{L}}_R, \mathcal{P}r^{\mathrm{R}}_R, \mathcal{P}r^{\mathrm{LR}}_R$. For R prodiscrete or ℓ-adic, we use the same notation while taking $\mathcal{E} = \mathrm{Mod}_R^{\mathrm{cont}}$.

Remark B.51. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. We denote by $\mathcal{C}^{\otimes\text{-op}}$ the "pointwise-opposite" symmetric monoidal structure on $\mathcal{C}^{\mathrm{op}}$ ([MO15], [BGN18]). For example, if $\mathcal{C}$ has finite products, $\mathcal{C}^{\times\text{-op}} \simeq (\mathcal{C}^{\mathrm{op}})^{\amalg}$.

$\mathcal{P}r^{\mathrm{L}}_{\mathcal{E}}$ carries a natural symmetric monoidal structure inherited from $\mathcal{P}r^{\mathrm{L},\otimes}_{\mathrm{st}}$, which we denote by $\mathcal{P}r^{\mathrm{L},\otimes}_{\mathcal{E}}$. The monoidal law is the relative tensor product $-\otimes_{\mathcal{E}} -$.

Let $\mathcal{P}r^{\mathrm{R}}$ be the ∞-category of presentable ∞-categories with right adjoint functors between them. We denote by $\mathcal{P}r^{\mathrm{R},\otimes}$ the symmetric monoidal structure induced by the equivalence

$$(\mathcal{P}r^{\mathrm{L}})^{\mathrm{op}} \simeq \mathcal{P}r^{\mathrm{R}}$$

i.e.

$$\mathcal{P}r^{\mathrm{R},\otimes} = \mathcal{P}r^{\mathrm{L},\otimes\text{-op}}.$$

As usual, for any stable presentable symmetric monoidal ∞-category, we also have the $\mathcal{E}$-linear variant $\mathcal{P}r^{\mathrm{R},\otimes}_{\mathcal{E}}$, whose objects are the same as $\mathrm{Mod}_{\mathcal{E}}(\mathcal{P}r_{\mathrm{st}})$ and whose morphisms are those functors admitting a left adjoint. Note that we also have an equivalence

$$(\mathcal{P}r^{\mathrm{L}}_{\mathcal{E}})^{\mathrm{op}} \simeq \mathcal{P}r^{\mathrm{R}}_{\mathcal{E}}.$$

This does not happen for categories of $\mathcal{E}$-modules, and relies on the fact that we required the right adjoints to be $\mathcal{E}$-linear.

Finally, $\mathcal{P}r^{\mathrm{LR}}_{\mathcal{E}}$ also admits a symmetric monoidal structure, which is auto-dual and makes the inclusion functors $\mathcal{P}r^{\mathrm{LR},\otimes}_{\mathcal{E}} \subset \mathcal{P}r^{\mathrm{L},\otimes}_{\mathcal{E}}$ and $\mathcal{P}r^{\mathrm{LR},\otimes}_{\mathcal{E}} \subset \mathcal{P}r^{\mathrm{R},\otimes}_{\mathcal{E}}$ both symmetric monoidal.

Notation B.52. Let $\mathcal{E}$ be a small stable symmetric monoidal ∞-category. We denote by $\mathcal{C}at_{\infty,\mathcal{E}} = \mathrm{Mod}_{\mathcal{E}}(\mathcal{C}at^{\mathrm{ex}}_{\infty})$ the ∞-category of small stable $\mathcal{E}$-linear ∞-categories and exact $\mathcal{E}$-linear functors. We denote the Cartesian monoidal structure on this ∞-category by $\mathcal{C}at^{\times}_{\infty,\mathcal{E}}$.

For R a commutative ring (discrete, prodiscrete or ℓ-adic) we adopt the notation $\mathcal{C}at_{\infty,R}$ for $\mathcal{C}at_{\infty,\mathrm{Perf}(R)}$.

Recall B.30 implies that

Corollary B.53. *Let $(X,s) \in \mathrm{StrTop}_{\mathrm{con}}$, and $\mathcal{E}$ be presentable stable ∞-category. Then $\mathrm{Cons}(X,s;\mathcal{E})$ is a presentable stable $\mathcal{E}$-linear ∞-category and $\mathrm{Cons}^{\mathrm{fd}}(X,s;\mathcal{E})$ is a small $\mathcal{E}^{\omega}$-linear ∞-category.*

Lemma B.54. *The functor*

$$\mathrm{Exit} : \mathrm{StrTop}_{\mathrm{con}} \to \mathcal{C}at_{\infty}$$
$$(Y, s : X \to P) \mapsto \mathrm{Exit}(Y,s)$$

carries a symmetric monoidal structure when we endow both source and target with the Cartesian symmetric monoidal structure. In other words, Exit *preserves finite products.*

Proof. Given two stratified topological spaces $Y, s : Y \to P$, $W, t : W \to Q$, in the notations of [Lur17, A.6], consider the commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Sing}^{P\times Q}(Y \times W) \arrow[r] \arrow[d] & \mathrm{Sing}(Y \times W) \arrow[r, "\sim"] \arrow[d] & \mathrm{Sing}(Y) \times \mathrm{Sing}(W) \arrow[dd] \\
N(P \times Q) \arrow[r] \arrow[d, "\sim"'] & \mathrm{Sing}(P \times Q) \arrow[dr, "\sim"] & \\
N(P) \times N(Q) \arrow[rr] & & \mathrm{Sing}(P) \times \mathrm{Sing}(Q).
\end{tikzcd}$$

The inner diagram is Cartesian by definition. Therefore the outer diagram is Cartesian, and we conclude that $\mathrm{Sing}^{P\times Q}(Y\times W)$ is canonically equivalent to $\mathrm{Sing}^{P}(Y)\times \mathrm{Sing}^{Q}(W)$. Since $\mathrm{Sing}^{P}(Y)$ models the ∞-category of exit paths of Y with respect to s, and similarly for the other spaces, we conclude. □

Lemma B.55 ([Lur17, Remark 4.8.1.8 and Proposition 4.8.1.15]). *Let $\mathcal{E}$ be a presentable symmetric monoidal ∞-category. There exist symmetric monoidal functors*

$$\mathcal{P}^{(*)}_{\mathcal{E}} : \mathcal{C}\mathrm{at}_{\infty}^{\times\text{-op}} \to \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{LR},\otimes}$$

$$\mathcal{P}_{\mathcal{E},(\dashv)} : \mathcal{C}\mathrm{at}_{\infty}^{\times} \to \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L},\otimes}$$

$$\mathcal{P}_{\mathcal{E},(\vdash)} : \mathcal{C}\mathrm{at}_{\infty}^{\times} \to \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{R},\otimes}$$

sending an ∞-category $\mathcal{C}$ to the ∞-category of $\mathcal{E}$-valued presheaves $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{E})$*, and a functor $F:\mathcal{C}\to\mathcal{D}$ to the functor*

$$F^* : \mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathcal{E}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{E})$$

induced by precomposition by F, resp. to the functor

$$F_{\dashv} : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{E}) \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathcal{E})$$

induced by left Kan extension along F, resp. to the functor

$$F_{\vdash} : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{E}) \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathcal{E})$$

induced by right Kan extension along F.

Proof. The functor $\mathcal{P}_{\mathcal{E},(\dashv)}$ takes values in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$ since for each $F:\mathcal{C}\to\mathcal{D}$ the functor $F_{\dashv}:\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{E})\to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathcal{E})$ admits a right adjoint given by F^*, and both $F_{\dashv}$ and F^* are $\mathcal{E}$-linear.

Let us start with the case $\mathcal{E}=\mathcal{S}$, the ∞-category of spaces. The existence of the (strong) monoidal structure on $\mathcal{P}_{\mathcal{S},(\dashv)}$ follows from [Lur17, Proposition 4.8.1.8]. Indeed, if we take $\mathcal{K}=\varnothing, \mathcal{K}'=\{\text{all simplicial sets}\}$ in *loc.cit.*, the functor $\mathcal{C}\mapsto \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})$ is what we call $\mathcal{P}_{\mathcal{E},(\dashv)}$, since it is the functor sending $F:\mathcal{C}\to\mathcal{D}$ to the left Kan extension of the Yoneda embedding $\mathcal{C}\to\mathcal{P}(\mathcal{C})$ along $\mathcal{C}\xrightarrow{F}\mathcal{D}\to\mathcal{P}(\mathcal{D})$ (see the proof of [Lur09, Proposition 5.3.6.2]).

In the case of a general presentable ∞-category $\mathcal{E}$, this follows from Remark B.34.

As for $\mathcal{P}^{(*)}$ and $\mathcal{P}_{(\vdash)}$, the claim follows from the fact that we have a symmetric monoidal equivalence

$$\mathcal{P}\mathrm{r}^{\mathrm{R},\otimes} \simeq \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes\text{-op}}.$$

□

Corollary B.56. *Let $\mathcal{E}$ be a presentable stable ∞-category. There are well-defined symmetric monoidal functors*

$$\mathrm{Cons}^{(*),\otimes}_{\mathcal{E}} : \mathrm{StrTop}_{\mathrm{con}}^{\times\text{-op}} \to \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{LR},\otimes}$$
$$(Y,s) \mapsto \mathrm{Cons}(Y,s;\mathcal{E})$$
$$f \mapsto f^* = - \circ \mathrm{Exit}(f).$$

$$\mathrm{Cons}^{\otimes}_{(\dashv),\mathcal{E}} : \mathrm{StrTop}_{\mathrm{con}}^{\times} \to \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L},\otimes}$$

$$(Y,s) \mapsto \mathrm{Cons}(Y,s;\mathcal{E})$$
$$f \mapsto f_{\dashv} := \mathrm{Lan}_{\mathrm{Exit}(f)}.$$

$$\mathrm{Cons}^{\otimes}_{(\vdash),\mathcal{E}} : \mathrm{StrTop}^{\times}_{\mathrm{con}} \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}}$$
$$(Y,s) \mapsto \mathrm{Cons}(Y,s;\mathcal{E})$$
$$f \mapsto f_{\vdash} := \mathrm{Ran}_{\mathrm{Exit}(f)}.$$

Proof. The previous constructions provide us with a symmetric monoidal functor

$$\mathrm{StrTop}^{\times\text{-op}}_{\mathrm{con}} \xrightarrow{\mathrm{Exit}(-)} \mathcal{C}\mathrm{at}^{\times\text{-op}}_{\infty} \xrightarrow{(-)^{\mathrm{op}}} \mathcal{C}\mathrm{at}^{\times\text{-op}}_{\infty} \xrightarrow{\mathcal{P}^{(*)}_{\mathcal{E}}} \mathcal{P}\mathrm{r}^{\mathrm{LR},\otimes}$$

sending

$$(X,s) \mapsto \mathrm{Fun}(\mathrm{Exit}(X,s),\mathcal{E}),$$
$$f \mapsto f^{*}$$

and similarly for the other two cases. □

Proposition B.57. *The functors constructed in Corollary B.56 take stratified homotopy equivalences to equivalences of ∞-categories.*

Proof. Indeed, consider a stratified homotopy $H : [0,1] \times Y \to Y$. This map has the property that the compositions $\{0\} \times Y \to [0,1] \times Y \to Y$ is $f \circ g$ and $\{1\} \times Y \to [0,1] \times Y \to Y$ is id_Y. Since the functor $\mathrm{Exit}(-)$ preserves products (Lemma B.54), one gets a map $\mathrm{Exit}([0,1]) \times \mathrm{Exit}(Y) \to \mathrm{Exit}(Y)$ such that

$$\mathrm{Exit}(\{0\}) \times \mathrm{Exit}(Y) \to \mathrm{Exit}([0,1]) \times \mathrm{Exit}(Y) \to \mathrm{Exit}(Y)$$

is $\mathrm{Exit}(f) \circ \mathrm{Exit}(g)$ and

$$\mathrm{Exit}(\{1\}) \times \mathrm{Exit}(Y) \to \mathrm{Exit}([0,1]) \times \mathrm{Exit}(Y) \to \mathrm{Exit}(Y)$$

is $\mathrm{Exit}(\mathrm{id}_Y)$. But since $[0,1]$ is contractible and unstratified, $\mathrm{Exit}([0,1])$ is equivalent to the terminal ∞-category $*$, and the two compositions are equivalent as functors $\mathrm{Exit}(Y) \to \mathrm{Exit}(Y)$. Therefore $\mathrm{Exit}(f)$ is a left inverse to $\mathrm{Exit}(g)$. By repeating the argument on K one obtains that $\mathrm{Exit}(g)$ is a left inverse to $\mathrm{Exit}(f)$. □

Proposition B.58. *The functors constructed in Corollary B.56 take maps in* tri *to equivalences of ∞-categories.*

Proof. [HN24, Lemma 4] proves this statement but with coefficients in spaces. We recover our version by combining Recall B.30 and Remark B.34. □

Proposition B.59. *The functor* $\mathrm{Cons}_{(\dashv),\mathcal{E}}$ *from Corollary B.56 satisfies hyperdescent.*

Proof. This follows from the stratified Seifert-Van Kampen theorem [Lur17, Theorem A.7.1] and the fact that

$$\mathrm{Fun}(-,\mathcal{E}) : \mathcal{C}\mathrm{at}_{\infty} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$$

(with left Kan extension functoriality) preserves colimits. □

We now turn to some essential recalls of the theory of Gaitsgory and Rozenblyum's correspondences.

Let $\mathcal{C}$ be an $(\infty,1)$-category, and $\mathsf{vert}, \mathsf{horiz}, \mathsf{adm}$ classes of morphisms with the properties [GR17, Chapter 7, 1.1.1]. The $(\infty,2)$-category of correspondences $\mathrm{Corr}(\mathcal{C})^{\mathsf{adm}}_{\mathsf{vert},\mathsf{horiz}}$ is defined in [GR17, Chapter 7, 1.2.5].[18]

Remark B.60 ([GR17, Chapter 7, 1.3.3]). If $\mathsf{adm} = \mathsf{isom}$ is the class of equivalences in $\mathcal{C}$, then

$$\mathrm{Corr}(\mathcal{C})^{\mathsf{isom}}_{\mathsf{vert},\mathsf{horiz}}$$

is an $(\infty,1)$-category, which we denote simply by $\mathrm{Corr}(\mathcal{C})_{\mathsf{vert},\mathsf{horiz}}$. We will always be in this situation. However, the results from [GR17] used in the proof of Theorem B.66 rely on $(\infty,2)$-categorical constructions.

Remark B.61. Let $\mathcal{C}_{\mathsf{vert}}$ and $\mathcal{C}_{\mathsf{horiz}}$ the subcategories of $\mathcal{C}$ spanned by all objects and vertical or horizontal morphisms, respectively. There are embeddings

$$\mathcal{C}_{\mathsf{vert}} \to \mathrm{Corr}(\mathcal{C})^{\mathsf{adm}}_{\mathsf{vert},\mathsf{horiz}}$$

$$\mathcal{C}^{\mathrm{op}}_{\mathsf{horiz}} \to \mathrm{Corr}(\mathcal{C})^{\mathsf{adm}}_{\mathsf{vert},\mathsf{horiz}}.$$

Remark B.62 ([GR17, Chapter 9, 2.1.3]). If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category and the classes $\mathsf{vert}, \mathsf{horiz}, \mathsf{adm}$ are closed under tensor product, then there is a symmetric monoidal structure on $\mathrm{Corr}(\mathcal{C})^{\mathsf{adm}}_{\mathsf{vert},\mathsf{horiz}}$. which we denote by

$$\mathrm{Corr}(\mathcal{C})^{\mathsf{adm},\otimes}_{\mathsf{vert},\mathsf{horiz}}.$$

In our specific case, we will always consider Cartesian symmetric monoidal structures. Note that the class $\widehat{\mathsf{subm}}$ from Definition B.28 is closed under cartesian product and pullback (Remark B.29), hence it is a good class for the theory of correspondences.

Notation B.63. If $\mathcal{C}$ is an $(\infty,1)$-category, we denote by all the class of all morphisms in $\mathcal{C}$.

Recall B.64. When $\mathsf{horiz} = \mathsf{all}$ and $\mathsf{adm} = \mathsf{isom}$, the $(\infty,1)$-category $\mathrm{Corr}(\mathcal{C})^{\otimes}_{\mathsf{vert},\mathsf{horiz}}$ is also treated in [Man22] and [Sch23]. In what follows, we will sometimes use some results formulated in this context.

We will need the following lemma:

Lemma B.65. *Let $\mathcal{C}^{\times}$ be a Cartesian symmetric monoidal ∞-category. Let $X, Y \in \mathcal{C}$ be objects. Then the map*

$$\mathcal{C}_{/X} \times \mathcal{C}_{/Y} \to \mathcal{C}_{/X\times Y}$$

is cofinal.

If $F : \mathcal{C}^{\times} \to \mathcal{D}^{\times}$ is a symmetric monoidal functor between symmetric monoidal structures, and $X, Y \in \mathcal{D}$ are objects, then the map $F_{/X} \times F_{/Y} \to F_{/X\times Y}$ is cofinal.

Proof. By Quillen's Theorem A, it suffices to prove that fibers are contractible. This can be straightforwardly checked by considering, for $\mathrm{Z} \to \mathcal{X} \times \mathcal{Y}$, Z representable, the canonical factorization

$$\mathrm{Z} \xrightarrow{\Delta} \mathrm{Z} \times \mathrm{Z} \to \mathcal{X} \times \mathcal{Y}$$

and proving that it is initial amongst all factorizations

$$Z \xrightarrow{\Delta} X \times Y \to \mathcal{X} \times \mathcal{Y}$$

induced by maps $X \to \mathcal{X}, Y \to \mathcal{Y}, Z \to X, Z \to Y$, with X, Y representable. $\square$

[18] Note that we indeed need a definition that works at least for $\mathcal{C}$ a $(2,1)$-category, since for example $\mathrm{Str}\widehat{\mathrm{TStk}}_{\mathrm{con}}$ is such.

Theorem B.66. *Let $\mathcal{E}$ be a presentable stable symmetric monoidal ∞-category. There is a symmetric monoidal functor of ∞-categories*

$$\mathrm{Cons}_{\mathcal{E}}^{\mathrm{corr},\otimes} : \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})^{\times}_{\mathit{all},\widehat{\mathit{subm}}} \to \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{R},\otimes}$$

with the following properties:

1. *its restriction along*

$$(\mathrm{StrTStk}_{\mathrm{con}})_{\mathit{vert}} \hookrightarrow \mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}})_{\mathit{all},\mathit{subm}'} \hookrightarrow \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\mathit{all},\widehat{\mathit{subm}}}$$

   left Kan extends $\mathrm{Cons}_{\mathcal{E},(\vdash)}$ *from Corollary B.56, preserves colimits and sends maps in* tri *and* she' *to equivalences;*

2. *dually, its restriction along*

$$(\mathrm{StrTStk}_{\mathrm{con}})^{\mathrm{op}}_{\mathit{horiz}} \hookrightarrow \mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}})_{\mathit{all},\mathit{subm}'} \hookrightarrow \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\mathit{all},\widehat{\mathit{subm}}}$$

   right Kan extends $\mathrm{Cons}_{\mathcal{E}}^{(*)}$ *from Corollary B.56, preserves limits and sends maps in* tri *and* she' *to equivalences;*

3. *its restriction along*

$$\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}})_{\mathit{vert}} \hookrightarrow \mathrm{Corr}(\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}))_{\mathit{all},\mathit{subm}''} \hookrightarrow \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\mathit{all},\widehat{\mathit{subm}}},$$

   resp.

$$\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}})^{\mathrm{op}}_{\mathit{horiz}} \hookrightarrow \mathrm{Corr}(\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}))_{\mathit{all},\mathit{subm}''} \hookrightarrow \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\mathit{all},\widehat{\mathit{subm}}},$$

   extends the points (1), resp. (2), by tri*-invariance, and sends maps in* she'' *to equivalences;*

4. *its restriction along*

$$(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\mathit{vert}} \hookrightarrow \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\mathit{all},\widehat{\mathit{subm}}},$$

   resp.

$$(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})^{\mathrm{op}}_{\mathit{horiz}} \hookrightarrow \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\mathit{all},\widehat{\mathit{subm}}},$$

   left (resp. right) Kan extends the previous point, preserves colimits (resp. limits) and send morphisms in $\widehat{\mathsf{she}}$ *to equivalences.*

Proof. Our starting datum is the functor $\mathrm{Cons}^{(*)} : \mathrm{StrTop}^{\mathrm{op}}_{\mathrm{con}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$ defined in Corollary B.56. Let us prove that this functor satisfies the right Beck-Chevelley condition [GR17, Chapter 7, Definition 3.1.5] with respect to the classes $\mathsf{vert} = \mathsf{subm}, \mathsf{horiz} = \mathsf{all}, \mathsf{adm} = \mathsf{isom}$. First for all, for each $\beta : (Y', s') \to (Y, s)$ in $\mathsf{subm} \subset \mathrm{Mor}(\mathrm{StrTop}_{\mathrm{con}})$, the functor $\beta^* : \mathrm{Cons}(Y, s) \to \mathrm{Cons}(Y', s')$ admits a left adjoint $\beta_{\dashv}$. Then, consider a cartesian diagram

$$\begin{tikzcd} (X', t') \arrow[r, "\alpha_0"] \arrow[d, "\beta_1"] & (X, t) \arrow[d, "\beta_0"] \\ (Y', s') \arrow[r, "\alpha_1"] & (Y, s) \end{tikzcd}$$

where $\alpha_0, \alpha_1 \in \mathsf{all}, \beta_0, \beta_1' \in \mathsf{subm}$. Then we want to prove that the base-change map

$$\beta_{1\dashv}\alpha_0^* \to \alpha_1^*\beta_{0\dashv} \tag{B.11}$$

is an equivalence of functors $\mathrm{Cons}(X, t; \mathcal{E}) \to \mathrm{Cons}(Y', s'; \mathcal{E})$. Note that, since *all* functors are left adjoints[19], this formula is local in X and X', and therefore we may assume that β_0 is of the form $(X, t) \times \mathbb{R}^N \to (X, t)$ for some N, and analogously for β_1. But in this case β_0^* and β_1^* are stratified homotopy equivalences, which implies that $\beta_{0\dashv}, \beta_{1\dashv}$ are equivalences. Note incidentally that a similar argument (reduction to the local case and the fact that in the local case β_i^* is an equivalence on constructible sheaves) proves that $\beta_{i\dashv}$ coincides with $\beta_{i\#}$ from [Vol21, Lemma 3.24]. So, base change also follows from [Vol21, Lemma 3.25].

We now want to obtain an extension at the level of $(\infty, 2)$-categories

$$\begin{tikzcd} \mathrm{StrTop}_{\mathrm{con}}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L}} \\ \mathrm{Corr}(\mathrm{StrTop}_{\mathrm{con}})_{\mathsf{subm},\mathsf{all}} \arrow[ur, dotted] & \end{tikzcd} . \tag{B.12}$$

which encodes $(-^*, -_\dashv)$-functoriality. As explained to us by Lucas Mann, one can proceed as in the proof of [Man22, Proposition A.5.10] with $I = \mathsf{subm}, P = \mathsf{isom}$, and without the assumption that I satisfies property (d) in *loc. cit.*: one does not need [LZ12b, Theorem 5.4] and instead of $\delta^*_{3,\{\ldots\}}$ one works with $\delta^*_{2,\{\ldots\}}$. Then the main observation is that one can invert edges in dimension 1 using [LZ12a, Lemma 1.4.4].

The extended functor (B.12) carries a canonical symmetric monoidal structure, because $\mathrm{Cons}_{(\dashv)} : \mathrm{StrTop}_{\mathrm{con}}^{\times} \to \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L},\otimes}$ is symmetric monoidal (Corollary B.56) and thus we can apply[20][GR17, Chapter 9, Proposition 3.1.5] with $\mathsf{vert} = \mathsf{subm}, \mathsf{horiz} = \mathsf{adm} = \mathsf{all}, \mathsf{co\text{-}adm} = \mathsf{isom}$. We will now perform a series of extensions:

- Let us consider the Yoneda embedding $F : \mathrm{StrTop}_{\mathrm{con}} \to \mathrm{StrTStk}_{\mathrm{con}}$. We want to obtain a symmetric monoidal extension

  $$\begin{tikzcd} \mathrm{Corr}(\mathrm{StrTop}_{\mathrm{con}})_{\mathsf{subm},\mathsf{all}} \arrow[r] \arrow[d] & \mathcal{P}\mathrm{r}_{\mathcal{E}}^{\mathrm{L}} \\ \mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}})_{\widetilde{\mathsf{subm}},\mathsf{all}} \arrow[ur, dotted] & \end{tikzcd} \tag{B.13}$$

  (see Definition B.28 for the notation) which preserves limits in both vert and horiz.

  The functor F satisfies the conditions of [GR17, Chapter 8, 6.1.1 and Theorem 6.1.5], (with $\mathbf{C} = \mathrm{StrTop}_{\mathrm{con}}, \mathbf{D} = \mathrm{StrTStk}_{\mathrm{con}}, \mathsf{vert} = \mathsf{subm}, \widetilde{\mathsf{subm}}$ respectively, $\mathsf{horiz} = \mathsf{all}, \mathsf{adm} = \mathsf{isom}$ for both $\mathbf{C}$ and $\mathbf{D}$), because:

  - it preserves finite limits;
  - it sends subm to $\widetilde{\mathsf{subm}}$ by definition of $\widetilde{\mathsf{subm}} \subset \mathrm{Mor}(\mathrm{StrTStk}_{\mathrm{con}})$;

[19] Here we use the existence of $\alpha_{0\vdash}, \alpha_{1\vdash}$ implied by Corollary B.56, i.e. that $\mathrm{Cons}_R^{(*)}$ lands in $\mathcal{P}\mathrm{r}_R^{\mathrm{LR}}$ and not just in $\mathcal{P}\mathrm{r}_R^{\mathrm{R}}$.

[20] Alternatively, from [Man22, Proposition A.5.10] we already get a lax-monoidal structure, and one can check that it is indeed strong symmetric monoidal.

– for each $(Y,s)\in \mathrm{StrTop}_{\mathrm{con}}$, the functor $((\mathrm{StrTop}_{\mathrm{con}})_{\mathsf{subm}})_{/(Y,s)} \to ((\mathrm{StrTStk}_{\mathrm{con}})_{\mathsf{subm}'})_{/F(Y,s)}$ is an equivalence. Indeed, since F is fully faithful, it suffices to prove that for any $\mathcal{Y}\in \mathrm{StrTStk}_{\mathrm{con}}, \phi : \mathcal{Y} \to F(Y,s)$ in $\mathrm{StrTStk}_{\mathrm{con}}$, then $\mathcal{Y}$ belongs to the essential image of F. But this is true since any morphism in subm is assumed to be representable, cf. Definition B.28.

Therefore, we may apply [GR17, Theorem 6.1.5] to (B.12) and obtain a horizontal right Kan extension (in the terminology of that theorem) as in (B.13).

In particular, when restricted to $(\mathrm{StrTop}_{\mathrm{con}})^{\mathrm{op}}_{\mathsf{horiz}} \to (\mathrm{StrTStk}_{\mathrm{con}})^{\mathrm{op}}_{\mathsf{horiz}}$, this is the right Kan extension. Also, the extended functor preserves limits in both vert and horiz.

As for the symmetric monoidal structure, we proceed as follows. By [GR17, Chapter 9, Proposition 3.2.4], we obtain a right-lax monoidal functor

$$\mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}})^{\times}_{\mathsf{subm},\mathsf{all}} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathcal{E}}.$$

Incidentally, the same verifications show that [Man22, Proposition A.5.16] can be applied to obtain the same result.

Let us prove that the obtained functor is indeed strongly monoidal. It suffices to prove that, for every $\mathcal{X},\mathcal{Y}\in \mathrm{StrTStk}_{\mathrm{con}}$, the map

$$\mathrm{Cons}(\mathcal{X};\mathcal{E})\otimes_{\mathcal{E}} \mathrm{Cons}(\mathcal{Y};\mathcal{E}) \to \mathrm{Cons}(\mathcal{X}\times\mathcal{Y};\mathcal{E})$$

is an equivalence. But this map can be presented as the chain of equivalences

$$\begin{aligned}\mathrm{Cons}(\mathcal{X};\mathcal{E})\otimes_{\mathcal{E}} \mathrm{Cons}(\mathcal{Y};\mathcal{E}) = \underset{(\mathrm{StrTop}_{\mathrm{con}})_{/\mathcal{X}}}{\overset{\mathcal{P}\mathrm{r}^{\mathrm{L}}}{\mathrm{colim}}} \mathrm{Cons}(X;\mathcal{E}) \otimes_{\mathcal{E}} \underset{(\mathrm{StrTop}_{\mathrm{con}})_{/\mathcal{Y}}}{\overset{\mathcal{P}\mathrm{r}^{\mathrm{L}}}{\mathrm{colim}}} \mathrm{Cons}(Y;\mathcal{E}) \simeq \\ \underset{(\mathrm{StrTop}_{\mathrm{con}})_{/\mathcal{X}}\times(\mathrm{StrTop}_{\mathrm{con}})_{/\mathcal{Y}}}{\overset{\mathcal{P}\mathrm{r}^{\mathrm{L}}}{\mathrm{colim}}} \mathrm{Cons}(X;\mathcal{E})\otimes_{\mathcal{E}}\mathrm{Cons}(Y;\mathcal{E}) \simeq \\ \underset{(\mathrm{StrTop}_{\mathrm{con}})_{/\mathcal{X}}\times(\mathrm{StrTop}_{\mathrm{con}})_{/\mathcal{Y}}}{\overset{\mathcal{P}\mathrm{r}^{\mathrm{L}}}{\mathrm{colim}}} \mathrm{Cons}(X\times Y;\mathcal{E}) \simeq \\ \underset{Z\in(\mathrm{StrTop}_{\mathrm{con}})_{/\mathcal{X}\times\mathcal{Y}}}{\overset{\mathcal{P}\mathrm{r}^{\mathrm{L}}}{\mathrm{colim}}} \mathrm{Cons}(Z;\mathcal{E}) \simeq \mathrm{Cons}(\mathcal{X}\times\mathcal{Y};\mathcal{E})\end{aligned} \tag{B.14}$$

where the second-to-last equivalence holds by Lemma B.65.

- We want to extend the functor (B.13) to $\mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}})_{\mathsf{subm}',\mathsf{all}}$, that is, we want to enlarge the class of vertical morphisms from representable smooth submersions to all smooth submersions. To this end, we want to apply [Man22, Proposition A.5.14] to the functor constructed in (B.13). The target category can be upgraded from $\mathcal{C}\mathrm{at}_\infty$ to $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$: indeed, the upgrade to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is already in the proof of *loc.cit.*, and adding the linear structure is straightforward. Let us thus check the conditions of [Man22, Proposition A.5.14]. In the notations of *loc.cit.*, let $E=\mathsf{subm}, E'=\mathsf{subm}', S\subset E$ be the class of smooth covers, i.e. maps in subm whose source is representable and which are surjective.

  a For every $\mathcal{X}\in\mathrm{StrTStk}_{\mathrm{con}}$, $\mathrm{Cons}(\mathcal{X};\mathcal{E})$ is presentable.

  b $\mathrm{Cons}_{(-),\mathcal{E}} : \mathrm{StrTStk}_{\mathrm{con}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$ satisfies S-descent by construction (it is left-Kan-extended from stratified topological spaces).

c By definition of subm', every map $f : \mathcal{X} \to \mathcal{Y}$ in E' there is a square

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g"] & Y \arrow[d, "g'"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

where the top map is in E (actually in subm) and the vertical arrows are in S. Hence, the composition $f \circ g$ belongs to E.

d We want to prove that pullbacks of edges in S remain smooth covers, and they are computed in the same way in $(\mathrm{StrTStk}_{\mathrm{con}})_{\widetilde{\mathsf{subm}}}$ and in $(\mathrm{StrTStk}_{\mathrm{con}})$. Indeed, the take a square

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"'] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd},$$

Cartesian in $\mathrm{StrTStk}_{\mathrm{con}}$, with $f \in E, g \in S$. Then $g' \in S$ because smooth covers are stable under pullback, and $f' \in E$ because smooth and representable maps are as well. We are left to prove that for any $\mathcal{X}' \in \mathrm{StrTStk}_{\mathrm{con}}$, equipped with maps $\mathcal{X}' \to Y, \mathcal{X}' \to \mathcal{X}$ making the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"'] \\ & \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

commute, the natural map $\mathcal{X}' \to X$ is smooth and representable (i.e. it belongs to E). Representability amounts to representability of $\mathcal{X}'$ (since X is represetnable), which follows from the fact that the map $\mathcal{X}' \to Y$ is representable with representable target. Smoothness follows from the fact that g' is a smooth cover and smoothness passes to smooth covers by definition.

We can thus apply the result and obtain an extension of (B.13) to a lax-monoidal functor

$$\mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}})_{\mathsf{subm}',\mathsf{all}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}. \tag{B.15}$$

Symmetric monoidality is established in a similar way as in the previous point.

- Let us consider the localization functor

$$G : \mathrm{StrTStk}_{\mathrm{con}} \to \mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}].$$

By restriction from (B.15), we get a functor

$$(\mathrm{StrTStk}_{\mathrm{con}})^{\mathrm{op}}_{\mathsf{horiz}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$$

satisfying the right Beck-Chevalley condition for $(\mathsf{vert} = \mathsf{subm}, \mathsf{horiz} = \mathsf{all})$. This functor is right-Kan-extended from $\mathrm{Cons}(-,\mathcal{E})^{(*)} : \mathrm{StrTop}^{\mathrm{op}}_{\mathrm{con}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$, hence sends tri to equivalences by Proposition B.58, hence factors as the composition of G and a functor

$$\mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}]^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$$

which also satisfies the right Beck-Chevalley condition with respect to $(\mathsf{vert} = \mathsf{subm}, \mathsf{horiz} = \mathsf{all})$ (note that $\mathsf{tri} \subset \mathsf{subm}'$). Hence, again by [GR17, Chapter 7, Theorem 3.2.2.(b)], it induces a functor

$$\mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}])_{\mathsf{subm}',\mathsf{all}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}}$$

(where we also call subm' the class generated by subm' in the localization), compatible with all of the above. This functor carries a symmetric monoidal structure by the same arguments as above.

- Note that the functor $\mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}] \to \mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}])$ is an equivalence (since $\mathsf{tri} \subset \mathsf{isom}$ in $\mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}]$), symmetric monoidal with respect to the Cartesian structures. On the other hand, there is a functor $\mathrm{Pro}(G) : \mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}) \to \mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}])$, also symmetric monoidal. This induces a functor

$$\mathrm{Corr}(\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}))_{\mathsf{subm}'',\mathsf{all}} \to \mathrm{Corr}(\mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}])_{\mathsf{subm}',\mathsf{all}}$$

which is symmetric monoidal because pro-objects are compatible with Cartesian monoidal structures. Note that we cannot argue anymore that the restriction to horizontal morphisms

$$\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}})^{\mathrm{op}} \to \mathrm{StrTStk}_{\mathrm{con}}[\mathsf{tri}^{-1}]^{\mathrm{op}}$$

preserves limits, since $\mathrm{Pro}(G)$ need not preserve colimits.

- Consider the Yoneda embedding $H : \mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}) \to \widehat{\mathrm{StrTStk}_{\mathrm{con}}}$. By the exact same arguments used in the case of F above, this induces a horizontal right Kan extension

$$\begin{tikzcd} \mathrm{Corr}(\mathrm{Pro}_{\mathsf{tri}}(\mathrm{StrTStk}_{\mathrm{con}}))_{\mathsf{subm}'',\mathsf{all}} \arrow[r] \arrow[d] & \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathcal{E}} \\ \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})_{\widehat{\mathsf{subm}},\mathsf{all}} \arrow[ur, dotted] & \end{tikzcd} \tag{B.16}$$

which is again symmetric monoidal.

From this, by passing to opposite categories and opposite monoidal structures as in Remark B.51, we obtain the sought-after symmetric monoidal functor of $(\infty, 1)$-categories

$$\mathrm{Cons}^{\mathrm{corr},\otimes}_{\mathcal{E}} : \mathrm{Corr}(\widehat{\mathrm{StrTStk}_{\mathrm{con}}})^{\times}_{\mathsf{all},\mathsf{subm}} \to \mathcal{P}\mathrm{r}^{\mathrm{R},\otimes}_{\mathcal{E}} \tag{B.17}$$

which encodes $(-^*, -_{\vdash})$-functoriality and satisfies the conditions of the statement. □

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