Exotic aspherical manifolds

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Abstract

In this note we explain the construction of an n-dimensional topological manifold, for $n \geq 4$, with following two properties: $\pi_i(M) = 0$ for $i \geq 2$ but whose universal covering is not homeomorphic to $\mathbb{R}^n$. The sketch of this proof has been provided to us by Roberto Frigerio during our masters.

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1 Introduction

Definition 1.1. A topological manifold is **closed** if it is compact and without boundary.

Definition 1.2. A connected topological manifolds X is **aspherical** if $\pi_n(X) = 0$ for every $n \geq 2$.

Definition 1.3. A topological manifold is **piecewise linear** (PL) if there exists a choice of charts such that the changes of charts are piacewise linear.

We recall that every topological manifold is semilocally simply connected, hence it admits a universal covering.

Definition 1.4. An aspherical closed topological manifold of dimension n is **exotic** if its universal covering is not homeomorphic to $\mathbb{R}^n$.

Remark 1.5. If X is a topological manifold whose universal covering is homotopy equivalent to $\mathbb{R}^n$, then X is aspherical, since coverings induce isomorphisms on π_i's with $i \geq 2$.

The existence of exotic manifolds is not guaranteed in every dimension: in particular, there are none in dimensions 1, 2 and 3.

In contrast:

Theorem 1.6 (Main result)**.** *For every $n \geq 4$ there exists an aspherical closed topological n-manifold whose universal covering is not homeomorphic to the euclidean space $\mathbb{R}^n$.*

One proof of this theorem is due to Michael Davis [Dav01, Theorem 9.5], and therefore such manifolds are sometimes referred to as *Davis manifolds*. We illustrate a somehow different proof, which involves the theory of CAT(0) and CAT(1) spaces and cubical complexes Both proofs, however, start from the existence of non-simply connected homology spheres.

Definition 1.7. A **homology sphere** of dimension m is a topological m-manifold X such that, abstractly, $\mathrm{H}_\bullet(X) \simeq \mathrm{H}_\bullet(\mathbb{S}^m)$.

Non-simply connected homology spheres only exist in dimension ≥ 4: their existence is a nontrivial result, proven separately in dimension ≥ 5 and 4 (see [Maz61], [Ker69]).

2 CAT(0) and CAT(1) spaces

Definition 2.1. Let (X, d_X) be a metric space. A **triangle** in X is a set of three vertices which are pairwise connected by a geodesic path.

We denote triangles by their vertices, with the expression $[abc]$.

Definition 2.2. Let (X, d_X) be a metric space, and $[pxy]$ a triangle. Its **euclidean model triangle** $\mathcal{E}([pxy]) = [p'x'y']$ is the triangle in the euclidean plane having edges of length respectively $d_X(p,x), d_X(x,y)$ and $d_X(p,y)$.

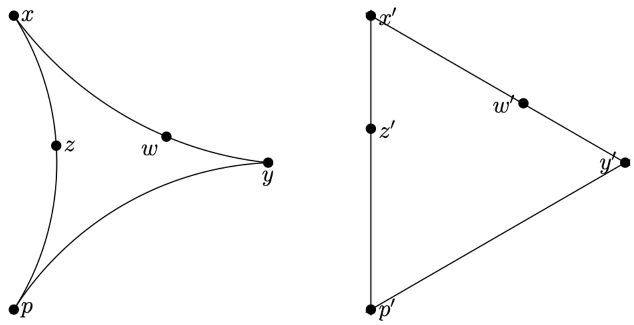
The existence of the euclidean model triangle is guaranteed by the fact that the edges of the original triangle, being geodesic, satisfy the triangle inequality. It is well-defined up to an isometry of the euclidean plane.

Definition 2.3. We let $\mathcal{E}_{[pxy]} : [pxy] \to [p'x'y']$ denote the map sending p to p', x to x', y to y', and a point x e.g. in $[px]$ in the unique point in the edge $[p'x']$ at distance $d_X(p,z)$ from p (and analgously for points lying in other edges).

Definition 2.4. A triangle $[pxy]$ is **thin** if the map $\mathcal{E}$ does not decrease distances, i.e. for every $z, w \in [pxy]$

$$d_X(z,w) \leq d_{\mathbb{R}^2}(\mathcal{E}(z), \mathcal{E}(w)).$$

(see picture).



Definition 2.5. A metric space (X, d_X) is **CAT(0)** if every triangle is thin.

Construction 2.6. Let us consider the sphere $\mathbb{S}^2$ and the distance $d(p,q) = \arccos(\langle p \rangle)$. Given a geodetic triangle $[pxy]$ with respect to this distance, if the sum of the length of the edges is less than 2π, we define the spherical model triangle of $[pxy]$ as the only triangle in $\mathbb{S}^2$, up to isometries, having geodesic sides of the corresponding lenghts. Without the given condition, such a triangle does not exists or, in the limit case, it is not unique.

Definition 2.7. A metric space (X, d_X) is said to be **CAT(1)** if every geodesic triangle in it is such that the map $\mathcal{S} : [pxy] \to \mathbb{S}^2$, defined analogously to the euclidean model case, does not decrease distances (or, as we can say, every triangle is "spherical-thin").

Note 2.8. The terminology CAT, introduced by Mikhail Gromov, is an acronym for "Cartan, Alexandrov, Toponogov".

A very useful connection between CAT(0) and CAT(1) is given by the metric cone.

Definition 2.9 (Metric cone)**.** Given a metric space X, its cone Cone(X) is the space $[0,\infty] \times X$ quotiented by the relation

$$(0,p) \sim (0,q)$$

for all p, q, and endowed with the distance

$$d_{\text{Cone}(X)}((p,s),(q,t)) = \sqrt{s^2 + t^2 - 2st\cos\alpha},$$

where $\alpha = \min\{\pi, d_X(p,q)\}$.

Proposition 2.10. Cone(X) *is CAT(0) if and only if X is CAT(1).*

Lemma 2.11 (Inheritance lemma)**.** *Let $[pxy]$ be a geodesic triangle in a metric space, and x a point on the edge $[xy]$. If the triangles $[pxz]$ and $[pyz]$ are both thin, then also $[pxy]$ is. Moreover, in case the perimeter of $[pxy]$ is less than 2π, this is true also if we replace "thin" with "spherical-thin".*

3 Locally CAT(0) spaces

Definition 3.1. A metric space is **locally CAT(0)** if every point admits a closed ball centered at it which is CAT(0) with the subspace distance.

In this context, one studies the properties of *local* geodesics, as we will see in many of the proof that we shall examine.

Remark 3.2. The notion of being locally CAT(0) has an interpretation in terms of curvature: indeed, it measures how geodesics exiting a point become further from one another, and in particular ensures that they become further from one another at least as slow as in the euclidean space: for example, this is what happens in the hyperbolic space. Locally CAT(0) space may also be called "metric spaces of nonpositive curvature" ([Dav01]).

For the record, the analogy with Riemannian geometry also involves the following:

Theorem 3.3 (Globalization theorem, B. Bowditch, 1995)**.** *Every proper length space, locally CAT(0) and simply connected is CAT(0).*

Proposition 3.4. *Every proper length space which is CAT(0) is contractible.*

Indeed, the CAT(0) inequality implies that, under the given hypotheses, such a space is uniquely geodesic with continuous dependence of the geodesics from their extremal points. Therefore, the space can be retracted onto a point by connecting such a chosen point to every other via geodesics and getting advantage of continuous dependance to define a retraction along those geodesics.

In light of this last result, one can relate the Globalization Theorem to the Hadamard Theorem.

Theorem 3.5 (Hadamard Theorem)**.** *A Riemannian m-manifold (M,g) whose sectional curvature is never positive, and which is simply connected, is diffeomorphic to $\mathbb{R}^n$ (hence contractible).*

4 Spherical distance on triangulations

Let S be a finite simplicial complex (in our case, it will be a triangulation of a compact manifold).

Of course, every n-simplex of it is in bijection with the standard simplex of $\mathbb{R}^n$. However, we will adopt the following variant of this representation. The euclidean simplex is homeomorphic to a triangle in $\mathbb{S}^n$ whose angles are right angles (for $n=2$, we are talking about a "quarter of dome" in $\mathbb{S}^2$). One can straightforwardly check that this coherently defines new lengths of edges in the original complex. With this new lengths, edges of all simplices have lentgh $\frac{\pi}{2}$.

One can therefore define a *length metric* on S:

$$d(p,q) = \inf\{\text{length}(\gamma) \mid \gamma \text{ connecting } p \text{ and } q\}.$$

Definition 4.1. A **flag complex** is a complex where given $\{v_0,\ldots,v_k\}$ vertices connected pairwise by edges, these belong to the same k-simplex.

Proposition 4.2. *A complex S endowed with the distance defined above is CAT(1) if and only if it is a flag complex.*

This is an interesting criterion, in that it connects a metric property to a combinatorial one.

Example 4.3. The first barycentric subdivision of any triangulation is a flag complex.

Therefore, in the following we will always suppose our complexes to be CAT(1) with the spherical distance defined above, up to taking a barycentric subdivision.

5 Cubical analogue of a finite simplicial complex

Construction 5.1. Let S be a finite simplicial complex with N vertices and of dimension n. Let $C^N \subset \mathbb{R}^N$ the unit cube. We now select some of its faces with the following procedure: for every$k$-simplex $[v_{i_0}, \ldots, v_{i_k}]$ of S, we consider the $(k+1)$-vector subspace of $\mathbb{R}^N$ given by $\mathrm{Span}(e_{i_0}, \ldots, e_{i_k})$ and we select all $(k+1)$-faces of C^N parallel to this subspace. Letting k vary from 0 to n and this considering all simplices of S, we obtain a subset Q of C^N which we define **cubical analogue** of S. This is a "cubical complex", i.e. a union of faces of C^N.

Remark 5.2. If S has dimension n, Q has dimension $n+1$ as a cubical complex. Moreover, since each face can be decomposed in simplices, Q also admits a structure of a simpplicial complex.

Remark 5.3. One can put a metric structure on Q, which is the length metric induced by $\mathbb{R}^N$: $d(p,q)$ will be the infimum of the length of curves (typically piecewise linear) which conect p and q in Q.

One checks directly from the definitions that:

Lemma 5.4. *Given S a finite flag complex, the link* $\mathrm{Lk}(v)$ *of each vertex of Q is isompetric to S with the above defined distances.*

Corollary 5.5. *Let S be a finite flag complex. Then its cubical analogue Q is locally CAT(0).*

Proof. One can cover Q by the "open stars" of its vertices, i.e. by the open subsets

$$\mathrm{st}(v) \simeq \mathrm{Cone}(\mathrm{Lk}(v)) \simeq \mathrm{Cone}(S)$$

(all $\simeq$ are isometries). This latter space is CAT(0) by the combination of Proposition 4.2 and Proposition 2.10. □

6 Universal coverings of cubical analogues

Proposition 6.1. *Let S be a finite simplicial complex and Q its cubical analogue. Then Q is locally CAT(0) and aspherical.*

Proof. By Corollary 5.5, Q is CAT(0). Its universal cover $\tilde{Q}$ inherits a distance from Q by pullback, as follows: one defines the lentgh of a curve in $\tilde{Q}$ to be the length of its image in Q, and defines a distance from this as a length distance (i.e. $d(x,y) = \inf\{\mathrm{length}(\gamma) \mid \gamma$ curve connecting x and $y\}$). Since a covering is a local homeomorphism, this metric structure on $\tilde{Q}$ is also locally CAT(0), it is a lentgh metric by construction, and the space is proper (which one can make follow from compactness of Q). Therefore, by Theorem 3.3 and Proposition 3.4, $\tilde{Q}$ is CAT(0) and contractible, and thus Q is aspherical. □

Definition 6.2. A topological space X is **simply connected at infinity** if for every K compact in X there exists $K' \supset K$ compact s.t. every class in $\pi_1(X \setminus K')$ becomes trivial in $\pi_1(X \setminus K)$.

Remark 6.3. $\mathbb{R}^n$ is simply connnected at infinity for every $n \geq 3$. Indeed, for every K one can select as K' any closed ball containing K.

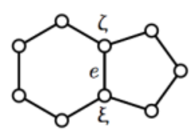
Proposition 6.4. *Let S be a finite simplicial complex and Q its cubical analogue. Let $\tilde{Q}$ be the universal covering of Q. If $\pi_1(S) \neq 0$, then $\tilde{Q}$ is not simply connected at infinity.*

Proof. Let γ be a nontrivial loop in S. Up to deformation, we can suppose that it lies in the 1-skeleton, i.e. it is made of edges. Let us choose γ of minimal lentgh along all nontrivial loops formed by edges of the complex. This is a one-dimensional subcomplex of S. Let G be its cubical analogue, which is in turn a cubical subcomplex of Q, of dimension 2. Let v be a vertex of G and G_v the connected component of G containing v. Let $\tilde{G}_v$ be a connected component in the inverse image of G_v inside $\tilde{Q}$, and $\tilde{v}$ a point in the fiber of v lying in $\tilde{G}_v$.

Claim: $\tilde{G}_v$ is convex in $\tilde{Q}$, and hencefore CAT(0) (since $\tilde{Q}$ is).

Proof. A standard argument tells us that every closed connected and locally convex subset of a CAT(0) space is convex. Let us thus show that every $\tilde{G}_v$ is locally convex in $\tilde{Q}$, or alternatively that G is locally convex in Q.

By contradiction, one assumes G not locally convex, and proves that we can reduce to a setting of the following kind:



where γ is the external loop and e is an edge not contained in γ which connects two of the vertices of γ, ζ and ξ. If any of the internal loops is nontrivial, this contradicts minimality of γ. But if both are trivial, then so is γ, contradiction. □

We can thus say that $\tilde{G}_v$ is a manifold of dimension 2 without boundary and convex in $\tilde{Q}$. Hence, it is contractible. By classification of surfaces, $\tilde{G}_v$ is homeomorphic to th real euclidean plane.

Let now R be a real number, and C_R the circle of radius R in $\tilde{G}_v$ (up to a fixed identification with the real euclidean plane) cnetered at $\tilde{v}$. If R varies, all C_R are homotopy equivalent in $\tilde{G}_v \setminus \{\tilde{v}\}$ and hence in $\tilde{Q} \setminus \{\tilde{v}\}$. But now, there is a well-defined map $\tilde{Q} \setminus \tilde{v} \to \mathrm{Lk}_Q(v)$ which associates to $x \neq \tilde{v}$ the corresponding point in the link in Q. When restricted to $\tilde{G}_v \setminus \{\tilde{v}\}$, this maps takes values in S, and its further restriction to any C_R is surjective onto $\mathrm{Lk}_G(v)$. Now, $\mathrm{Lk}_G(v)$ is homeomorphic to γ inside $\mathrm{Lk}_Q(v)$, which is to say that the image of C_R is homeomorphic to γ inside S.

As a consequence, C_R is nontrivial not only in $\tilde{G}_v \setminus \{\tilde{v}\}$, but also in $\tilde{\setminus}\{\tilde{v}\}$ (otherwise its image along the continuous map described above would be trivial as well, contradicting the fact that γ is not trivial).

Let us now suppos, by contradiction, that $\tilde{Q}$ is simply connected at infinity. If we choose $K = \{\tilde{v}\}$, there must exist a compact K' containing $\tilde{v}$ such that each loop in $\tilde{Q} \setminus K'$ becomes trivial in $\tilde{Q} \setminus \{\tilde{v}\}$. However, for R big enough, C_R is contained in $\tilde{Q} \setminus K'$, since $K' \cap \tilde{G}_v$ is compact, and is nontrivial in $\tilde{Q} \setminus \{\tilde{v}\}$. □

7 Construction of an aspherical manifold

Theorem 7.1. *For every $n \geq 4$ there exists a piecewise linear aspherical closed n-manifold whose universal covering is not homeomorphic to the euclidean space $\mathbb{R}^n$.*

Proof. Let Z, W two smooth compact manifolds with boundary, of dimension $n-1$ and n respectively, such that

- W is contractible
- Z has no boundary, and is the boundary of W
- $H_\bullet(Z) \simeq H_\bullet(\mathbb{S}^{n-1})$
- $\pi_1(Z) \neq 0$.

The existence of such a pair is proven in [Maz61], [Ker69].

Being a smooth manifold, W admits a triangulation. Let S be the restriction of such a triangulation to Z. Up to passing to the first barycentric subdivision $S^{(1)}$, we may assume that S is a flag complex. Therefore, with its spherical metric structure, it is CAT(1). Let Q be the cubical analogue of S endowed with the length metric induced by $\mathbb{R}^n$ (see Section 5). The following hold:

- By Proposition 6.4, $\tilde{Q}$ is not simply connected at infinity and therefore it is not homeomorphic to $\mathbb{R}^n$.
- Q is a homology n-manifold, i.e. for any $x \in Q$ $H_\bullet(Q, Q \setminus x) \simeq H_\bullet(\mathbb{S}^{n-1})$. In particular it is a PL-manifold away from its vertices.

  Proof. The first part follows from the fact that the link of each vertex is homeomorphic to Z, and we know by hypothesis that Z is a homology $n-1$-sphere. The second part follows from standard theory of polyhedral manifolds. □

- There exists a compact PL n-manifold M homotopy equiavalent to Q.

  Proof. Since the link of each vertex of Q is homeomorphic to Z, we can glue copies of W along their boundary Z on boundaries of neighbourhoods ov vertices of Q. Such a gluing yields a compact PL n-manifold. It is also homotopy equivalent to Q, since W is contractible and therefore the pair (W, Z) satisfies the Homotopy Extension Property. One checks that the maps inducing the global homotopy equivalence can be lifted to proper maps between the universal coverings of Q and M. Hence M is aspherical and exotic (since proper homotopy equivalences with proper homotopy inverses preserve (non-)simply connectedness at infinity). □

□

References

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