A note on convolution over double quotients of group schemes

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Abstract

Let H be a group scheme over a field k, $K < H$ a smooth subgroup scheme, and let Λ be a torsion ring. Then the category of K-K-equivariant constructible sheaves $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}({}_K\backslash^H/_K;\Lambda)$ over H with coefficients in Λ has a natural monoidal structure, different from the derived tensor product, called convolution. We explicitly prove associativity of this monoidal structure, both at the level of triangulated and stable ∞-categories.

We obtain a similar results in the case of a subgroup acting on an ind-group scheme as a pro-group. A well-known example is the case when $H = G_{\mathcal{K}}$ and $K = G_{\mathcal{O}}$, respectively the loop group and the arc group associated to a reductive group G over k.

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1 Quotients of group schemes by smooth subgroups

Notation 1.1. Let k be a field, H a group scheme over k, and $K < H$ a smooth subgroup scheme.

We also define

$$\mathcal{X} = \left[{}_K\backslash^H/_K \right]$$

as the double quotient stack by the two multiplication actions (left and right). In the rest of the paper, we will often drop the square brackets.

Finally, we define

$$H \times^K H$$

to be the scheme arising as the quotient of $H \times H$ by the K-action

$$(k, h_1, h_2) \mapsto (h_1 k^{-1}, k h_2).$$

Remark 1.2. Recall that the multiplication map $\mu : H \times H \to H$ is *not* a morphism of group schemes, unless H is abelian. Note that it factors as

$$H \times H \to H \times^K H \to H.$$

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Definition 1.3. Let Λ be a torsion commutative ring with unit. Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(-,\Lambda) : \mathrm{Stk}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ be the functor

$$X \mapsto \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$$

associating to a stack X its derived constructible category sheaves with coefficients in Λ-modules, in the sense of [LO08], and with pullback functoriality.

Our aim is to establish a monoidal structure on the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X};\Lambda)$, given on objects by the formula

$$\mathcal{A} \star \mathcal{B} = m_! p^*(\mathcal{A} \boxtimes \mathcal{B})$$

where p and m are the maps in the diagram

$$\begin{array}{ccc} & {}_K\backslash H \times^K H/_K & \\ \swarrow^{p} & & \searrow^{m} \\ \mathcal{X} \times \mathcal{X} & & \mathcal{X}. \end{array}$$

More precisely, p is the map that associates to a class $[h_1, h_2]$ the pair of classes $([h_1],[h_2])$, and m is the map induced by the multiplication $H \times H \to H$.

This monoidal structure is illustrated in [Wed22]. The purpose of this note is to prove in detail that it is associative.

We remark that, by using the formalism of [GR17] or [Man22], our proof could be generalized to any three-functor-formalism

$$\mathrm{Corr}(\mathrm{Sch}_k)_{\mathsf{all,smooth}} \to \widehat{\mathcal{C}\mathrm{at}_\infty}$$

(then Kan extended to stacks by means of [GR17, Chapter 8, Theorem 1.1.9]) in place of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(-;\Lambda)$. Here $\mathrm{Corr}(\mathrm{Sch}_k)_{\mathsf{all,smooth}}$ is the category of correspondences where every left leg is a smooth morphism of schemes over k. We will not include this generalized approach in the present note.

Definition 1.4. We define the functor

$$\mathcal{X}_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Stk}_k$$

as

$$\mathcal{X}_0 = \operatorname{Spec} k$$
$$\mathcal{X}_n = {}_K\backslash \underbrace{H \times^K \cdots \times^K H}_{n} /_K$$

on objects, and as follows on morphisms.

The face map δ_i associated to the inclusion $d_i : [n-1] \to [n]$ that misses i is:

- If $i = 0$, then the map is defined as the composition

$$_K\backslash \underbrace{H \times^K \cdots \times^K H}_{n} /_K \to \mathcal{X} \times ({}_K\backslash \underbrace{H \times^K \cdots \times^K H}_{n-1} /_K) \to {}_K\backslash \underbrace{H \times^K \cdots \times^K H}_{n-1} /_K$$

  where the first map projects onto the quotient and the second one is the projection onto the second factor.

- If $i = n$, the definition is analogous but by forgetting the last coordinate.
- If $i \neq 0, n$, then δ_i is the map induced by the multiplication of the coordinates i and $i+1$.

The degeneration map σ_i associated to the surjection $s_i : [n] \to [n-1], i < n$ that sends both i and $i+1$ to i is the composition

$$_K\backslash \underbrace{H \times^K \cdots \times^K H}_{n-1} /_K \simeq {}_K\backslash \underbrace{H \times^K \cdots \times^K \underset{(i+1)}{K} \times^K \cdots \times^K H}_{n} /_K \to {}_K\backslash \underbrace{H \times^K \cdots \times^K H}_{n} /_K .$$

The simplicial identities are easily verified.

Proposition 1.5. *The square*

$$\begin{array}{ccc} _K\backslash^{H\times^K H \times^K H}/_K & \xrightarrow{m_{23}} & _K\backslash^{H\times^K H}/_K \\ \downarrow{\scriptstyle (p_1,p_{23})} & & \downarrow{\scriptstyle (p_1,p_2)} \\ _K\backslash^{H\times^K H}/_K & \xrightarrow[(\mathrm{id},m)]{} & _K\backslash^{H}/_K \end{array}$$

is Cartesian.

Proof. We start with the observation that the square

$$\begin{array}{ccc} H \times H \times H & \xrightarrow[m_{23}]{} & H \times H \\ \downarrow{\scriptstyle (p_2,p_3)} & & \downarrow{\scriptstyle p_2} \\ H \times H & \xrightarrow{m} & H \end{array} \tag{1.1}$$

is cartesian. Thus, we start by taking quotients with respect to various actions with respect to which all maps in the diagram are equivariant. This procedure yields again a cartesian diagram.

Lemma 1.6. *Suppose given a Cartesian diagram of smooth sheaves, and suppose there are actions of a smooth group scheme G on all vertices such that all maps are equivariant. Then the induced square of quotient sheaves is Cartesian.*

Proof. The transformation from the original square of sheaves to the square of quotients is a smooth covering, hence Cartesianness of the latter, which amounts to the comparison map from the initial vertex to the actual pullback being an isomorphism, follows from smooth descent. □

1. Notice that all arrows in (1.1) are equivariant with respect to the actions of $K \times K$ inducing the following quotients:

$$\begin{array}{ccc} H \times H \times^K {}^H/_K & \xrightarrow[m_{23}]{} & H \times ({}^H/_K) \\ \downarrow{\scriptstyle (p_2,p_3)} & & \downarrow{\scriptstyle p_2} \\ H \times^K {}^H/_K & \xrightarrow{m} & {}^H/_K \end{array} \tag{1.2}$$

More specifically, the action of $K \times K$ on the first column is evident by the notation, and the one on the second column factors through the projection $K \times K \to K$ onto the second factor.

2. All arrows in (1.2) are equivariant with respect to the actions of K inducing the following quotients:

$$\begin{array}{ccc} H \times^K H \times^K {}^{H}/_{K} & \xrightarrow{m_{23}} & H \times^K {}^{H}/_{K} \\ \big\downarrow{\scriptstyle (p_2,p_3)} & & {\scriptstyle p_2}\big\downarrow \\ {}_{K}\backslash^{H} \times^K {}^{H}/_{K} & \xrightarrow{m} & {}_{K}\backslash^{H}/_{K} \end{array} \tag{1.3}$$

3. Finally, all arrows in (1.3) are equivariant with respect to the actions of K inducing the following quotients:

$$\begin{array}{ccc} {}_{K}\backslash^{H} \times^K H \times^K {}^{H}/_{K} & \xrightarrow{m_{23}} & {}_{K}\backslash^{H} \times^K {}^{H}/_{K} \\ \big\downarrow{\scriptstyle (p_2,p_3)} & & {\scriptstyle p_2}\big\downarrow \\ {}_{K}\backslash^{H} \times^K {}^{H}/_{K} & \xrightarrow{m} & {}_{K}\backslash^{H}/_{K}. \end{array} \tag{1.4}$$

   This latter square is the desired one.

□

Theorem 1.7. *The convolution product $\star$ is associative.*

Proof. Thanks to Proposition 1.5, one can straightforwardly deduce that the square in the diagram

$$\begin{array}{ccccccc} & & & \mathcal{X}_3 & & & \\ & & \swarrow & & \searrow & & \\ & \mathcal{X}_2 \times_k \mathcal{X} & & & & \mathcal{X}_2 & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathcal{X} \times_k \mathcal{X} \times_k \mathcal{X} & & & \mathcal{X} \times_k \mathcal{X} & & & \mathcal{X} \end{array}$$

is Cartesian. Call p' the composition of the two leftmost arrows, and m' the composition of the two rightmost arrows. By smooth base change, one obtains that, for any three $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X}; \Lambda)$,

$$(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \simeq m'_! p'^*(\mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{H}).$$

To complete the proof, it suffices to prove that the square in the diagram

$$\begin{array}{ccccccc} & & & \mathcal{X}_3 & & & \\ & & \swarrow & & \searrow & & \\ & \mathcal{X} \times_k \mathcal{X}_2 & & & & \mathcal{X}_2 & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ \mathcal{X} \times_k \mathcal{X} \times_k \mathcal{X} & & & \mathcal{X} \times_k \mathcal{X} & & & \mathcal{X} \end{array}$$

is Cartesian as well, since the two outer compositions are again p' and m'. The proof of the fact that this square is Cartesian follows from a statement completely symmetric to Proposition 1.5. □

2 Quotients of ind-group schemes by pro-smooth subgroups

Definition 2.1. Let $H = \operatorname{colim}_{N\in\mathbb{N}} H_N$ be an ind-group scheme, and suppose that the multiplication restricts to

$$m : H_N \times H_M \to H_{N+M}.$$

Let K be a normal subgroup of H which is contained in each H_N and which is isomorphic to

$$\lim_{j\in\mathbb{N}} K_j$$

where each K_j is smooth over k and the transition maps are quotients by unipotent subgroups.

Let us suppose that for every N there exists a j_N such that the left multiplication action of K over H_N/K (the quotient is with respect to the right multiplication action) factors through $K \to K_{j_N}$.

We define

$$\mathcal{X} = \operatorname{colim}_N \text{“}\lim_{j\geq j_N}\text{”}\, {}_{K_{j_N}}\backslash H/_K.$$

Remark 2.2. Let $N \in \mathbb{N}, j \geq j_N$. Notice that, if we define $H_{N,j} = H_N \times^K K_j$, we have an isomorphism

$$H_N \times^K H_N \simeq H_{N,j} \times^{K_j} H_N.$$

Proposition 2.3. *For every $N, j \geq j_{2N}$, the square*

$$\begin{array}{ccc} {}_{K_j}\backslash H_N \times^K H_N \times^K H_N/_K & \xrightarrow{m_{23}} & {}_{K_j}\backslash H_N \times^K H_{2N}/_K \\ \downarrow{\scriptstyle p_{23}} & & \downarrow{\scriptstyle p_2} \\ {}_{K_j}\backslash H_N \times^K H_N/_K & \xrightarrow{m} & {}_{K_j}\backslash H_{2N}/_K \end{array}$$

is Cartesian.

Proof. We proceed in a similar way as in the proof of Proposition 1.5. As in that case, we only treat the case $n = 3$. We start with the square

$$\begin{array}{ccc} H_N \times H_N \times H_N & \xrightarrow{m_{23}} & H_N \times H_{2N} \\ \downarrow{\scriptstyle (p_2,p_3)} & & \downarrow{\scriptstyle p_2} \\ H_N \times H_N & \xrightarrow{m} & H_{2N} \end{array} \tag{2.1}$$

which is easily seen to be Cartesian.

1. This point is identical to the one in Proposition 1.5. Notice that all arrows in (2.1) are equivariant with respect to the actions of $K \times K$ inducing the following quotients:

$$\begin{array}{ccc} H_N \times H_N \times^K H_N/_K & \xrightarrow{m_{23}} & H_N \times H_{2N}/_K \\ \downarrow{\scriptstyle (p_2,p_3)} & & {\scriptstyle p_2}\downarrow \\ H_N \times^K H_N/_K & \xrightarrow{m} & H_{2N}/_K \end{array} \tag{2.2}$$

2. All arrows in (2.2) are equivariant with respect to the actions of K inducing the following quotients (notice that we are using Remark 2.2):

$$\begin{array}{ccc} H_{N,j} \times^{K_j} H_{N,j} \times^{K_j} H_N/_K & \xrightarrow{m_{23}} & H_{N,j} \times^{K_j} H_{2N}/_K \\ \Big\downarrow{\scriptstyle (p_2,p_3)} & & {\scriptstyle p_2}\Big\downarrow \\ {}_{K_j}\backslash H_{N,j} \times^{K_j} H_N/_K & \xrightarrow{\quad m \quad} & {}_{K_j}\backslash H_{2N}/_K \end{array} \tag{2.3}$$

3. Finally, all arrows in (2.3) are equivariant with respect to the actions of K_j inducing the following quotients:

$$\begin{array}{ccc} {}_{K_j}\backslash H_{N,j} \times^{K_j} H_{N,j} \times^{K_j} H_N/_K & \xrightarrow[m_{23}]{} & {}_{K_j}\backslash H_{N,j} \times^{K_j} H_{2N}/_K \\ \Big\downarrow{\scriptstyle (p_2,p_3)} & & {\scriptstyle p_2}\Big\downarrow \\ {}_{K_j}\backslash H_{N,j} \times^{K_j} H_N/_K & \xrightarrow{\quad m \quad} & {}_{K_j}\backslash H_{2N}/_K \end{array} \tag{2.4}$$

which one can show being isomorphic to the desired square by now using Remark 2.2 backwards.

The symmetric case is treated analogously. □

Remark 2.4. Let $N \in \mathbb{N}, j_N \leq i \leq j$. By the hypothesis that the transition maps $K_j \to K_i$ are quotients with unipotent kernel, the functors

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}({}_{K_j}\backslash H_N/_K;\Lambda) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}({}_{K_i}\backslash H/_K;\Lambda)$$

are equivalences ([AR23, Proposition 10.2.8]).

Therefore, we can define

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X};\Lambda) = \mathrm{colim}_N\, \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}({}_{K_j}\backslash H_N/_K;\Lambda)$$

and this definition is independent of the choice of each $j \geq j_N$.

For $N \in \mathbb{N}, j \geq j_{2N}$, we have a diagram:

$$\begin{array}{ccccc} & & {}_{K_j}\backslash H_N \times^K H_N/_K & & \\ & \swarrow^{p_{N,j}} & & \searrow^{m_{N,j}} & \\ {}_{K_j}\backslash H_N/_K \times_k {}_{K_j}\backslash H_N/_K & & & & {}_{K_j}\backslash H_{2N}/_K. \end{array}$$

Theorem 2.5. *The category*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X};\Lambda)$$

carries a monoidal structure given by the following convolution law. Suppose $\mathcal{A}, \mathcal{B}$ are objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X};\Lambda)$ coming from objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}({}_{K_j}\backslash H_N/_K;\Lambda)$ for some $N \in \mathbb{N}, j \geq j_N$. Then

$$\mathcal{A} \star \mathcal{B} = m_{N,j,!}\, p^*_{N,j}(\mathcal{A} \boxtimes \mathcal{B}).$$

Proof. As in the previous Section, associativity amounts to the fact that the square in Proposition 2.3 and its symmetric version are Cartesian. □

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