Ind-representability of affine Grassmannians for curves and surfaces

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Abstract

Let G be a complex reductive group. In this note we recall the proof of ind-properness of its affine Grassmannian Gr_G, and apply this result to the study of the Hecke endofunctors of the category of coherent sheaves over Gr_G. We also study two naïve versions of the affine Grassmannian for a surface S: one parametrizes G-torsors over S with a trivialization away from a point x, and is isomorphic to the group G itself (hence "trivial"). The second one parametrizes G-torsors over S with a trivialization away from a divisor, and is ind-quasiprojective.

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1 Ind-properness of the affine Grassmannian of GL_k

Throughout the whole note, G will be equal to some $\mathrm{GL}_{k,\mathbb{C}}$ for simplicity. In general, this should be replaced with an arbitrary complex reductive group.

Definition 1.1. Let C be a smooth complex curve, and x a closed point. Define the Hecke stack $\mathbf{Hecke}_x^{\mathrm{Bun}_G}(C)$ as the functor

$$\mathrm{Sch} \to \mathcal{G}\mathrm{rpd}$$

sending

$$T \mapsto \{(\mathcal{F}, \mathcal{G}, \phi) \mid \mathcal{F}, \mathcal{G} \in \mathrm{Bun}_G(C \times T), \phi : \mathcal{F}|_{(C \setminus \{x\}) \times T} \to \mathcal{G}|_{(C \setminus \{x\}) \times T} \text{ an isomorphism}\}$$

where morphisms on the right are isomorphisms of pairs of G-torsors compatible with the isomorphisms on the open set. Alternatively, this can be written as $\mathbf{Bun}_G(C) \times_{\mathbf{Bun}_G(C \setminus \{x\})} \{\mathcal{O}^k\}$.

Theorem 1.2. *Let C be a smooth projective curve. The fibers of any of the two projection maps $\mathbf{Hecke}_x^{\mathrm{Bun}_G}(C) \to \mathbf{Bun}_G(C)$ are representable by ind-proper schemes.*

The fiber at the trivial bundle is isomorphic to the so-called the **affine Grassmannian** of G (hence the title of this section), see [Zhu16, 1.2].

Proof. This proof is essentially the same as the one of [Zhu16, Theorem 1.2]. We recall it to fix the arguments and notations for the proof of Theorem 3.5.

Let T be a scheme, and consider a fixed $V^0 \in \mathrm{Bun}_G((C \setminus x) \times T)$. We want to parametrize all pairs $(\mathcal{F}, \tau)$ where $\mathcal{F} \in \mathrm{Bun}_G(C \times T)$ and $\tau : \mathcal{F}|_{(C \setminus \{x\}) \times T} \to V^0$ is an isomorphism. Let $X = C \times T, Y = \{x\} \times T, U = X \setminus Y$.
In this setting, the following lemma holds:

Lemma 1.3. *If $\mathcal{I}$ is the sheaf of ideals defining $\{x\} \times T$ inside $C \times T$, then we have the following correspondence: a triple $\mathcal{F}, \mathcal{G} \in \mathrm{Bun}_G(X)$, $\tau : \mathcal{F}|_U \xrightarrow{\sim} \mathcal{G}|_U$ corresponds to a quadruple $\mathcal{F}, \mathcal{G} \in \mathrm{Bun}_G(X), i_1 : \mathcal{I}^n \mathcal{F} \hookrightarrow \mathcal{G}, i_2 : \mathcal{I}^n \mathcal{G} \hookrightarrow \mathcal{F}$ for some n.*

This is in some way a generalization of the known result [GD71] saying that if a coherent sheaf $\mathcal{F}$ over a scheme X is 0 outside some closed subscheme Y defined by a sheaf of ideals $\mathcal{I}$, then there is a natural number n such that $\mathcal{I}^n \mathcal{F}$ is 0 everywhere. It is a direct consequence of [GD71, Prop. 6.9.17], when one looks at the proof and keeps in mind that our sheaves, being locally free, are in particular torsion free. Note also that the cited result from EGA 1 regards noetherian schemes, but an argument analogous to [Vis07, Prop. 4.37] shows that we can assume to live in that setting.
The groupoid $\{(\mathcal{F}, \tau), \mathcal{F} \in \mathrm{Bun}_G(C \times T), \tau : \mathcal{F}|_U \xrightarrow{\sim} V^0\}$ is equivalent to the groupoid $\mathrm{colim}_n \{(i_1 : \mathcal{I}^n \mathcal{F} \hookrightarrow \mathcal{G}_0, i_2 : \mathcal{I}^n \mathcal{G}_0 \hookrightarrow \mathcal{F})\}$ where $\mathcal{G}_0$ is any fixed locally free sheaf extending V^0 in x (if there is none, the thesis is trivial).
Since the ideal $\mathcal{I}$ is invertible, we can rewrite this as $\mathcal{I}^n \mathcal{G}_0 \subset \mathcal{F} \subset \mathcal{I}^{-n} \mathcal{G}_0$. The following lemma and its proof have been suggested to us by Angelo Vistoli.

Lemma 1.4. *Il $\mathcal{F}$ is locally free, then the quotient $\mathcal{I}^{-n} \mathcal{G}_0 / \mathcal{F}$ is flat over T.*

Proof. Since $\mathcal{I}^{-n} \mathcal{G}_0$ and $\mathcal{I}^n \mathcal{G}_0$ are locally free as well, the local expression for our couple of inclusions takes the form

$$\mathcal{O}_{X,x}^k \overset{j_1}{\hookrightarrow} \mathcal{O}_{X,x}^k \overset{j_2}{\hookrightarrow} \mathcal{O}_{X,x}^k$$

for some k. These two inclusions are represented by two matrices $A(s,t)$ and $B(s,t)$ with values in $\mathcal{O}_{C \times T,(s,t)}$. To say that the quotient of the two copies of $\mathcal{O}^k$ on the right is flat is equivalent to say that the sequence

$$0 \to \mathcal{O}^k \overset{j_2}{\hookrightarrow} \mathcal{O}^k \to \operatorname{coker} B(s,t)$$

remains exact while tensoring by $k(t)$ for every $t \in T$, by symmetry of the Tor functors.
This is equivalent to say that the rank of the matrix $B(s,t)$ remains unaltered while tensoring by $k(t)$. Notice however that the cokernel of the product $A(s,t)B(s,t)$ is supported on Y, since this product of matrices corresponds the inclusion $\mathcal{I}^n \mathcal{G}_0 \hookrightarrow \mathcal{I}^{-n} \mathcal{G}_0$. Therefore for every t_0 the rank of $A(s,t_0)B(s,t_0)$ remains generically the same while tensoring by $k(t_0)$, since for $s \neq x$ we have $A(s,t_0)B(s,t_0) = id$.
Therefore, since the rank can only decrease, also rk B remains unaltered, as desired. □

Thus we have an inclusion of functors between the functor we are studying and the Quot functor

$$\mathrm{Quot}_{(\mathcal{I}^{-n}\mathcal{G}_0/\mathcal{I}^n\mathcal{G}_0)/X/\mathbb{C}}$$

that associates to an affine scheme T the set of flat finitely presented sheaves over $C \times_{\mathbb{C}} T$ which are a quotient of $\mathcal{I}^{-n}\mathcal{G}_0/\mathcal{I}^n\mathcal{G}_0$. Note that this (more precisely, the arguments above) also imply that the fiber appearing in the statement of Theorem 1.2 is set-valued. The mentioned inclusion is actually an equivalence: indeed, if we have a flat quotient $\mathcal{I}^{-n}\mathcal{G}_0/\mathcal{I}^n\mathcal{G}_0 \to \mathcal{H}$, the kernel $\mathcal{F}$ of this quotient is locally a submodule of a free module; we would like to use the fact that C is a curve and that a submodule of a free module over a PID is free. The following lemma allows us to use this argument:

Claim 1.5. *In the above notations, if $\mathcal{H}$ is flat over T, then $\mathcal{F}$ is locally free if and only if it is locally free over each t-fiber, $t \in T$.*

Proof. Consider the exact sequence $0 \to \mathcal{F} \to \mathcal{I}^{-n}\mathcal{G}_0 \to \mathcal{H} \to 0$ (for simplicity, we identify the above sheaves with the ones obtained without quotienting by $\mathcal{I}^n\mathcal{G}_0$). Locally, the claim reduces to the following problem:

Lemma 1.6. *Let $R \to A$ be a map of local rings,*

$$0 \to F \to M \to H \to 0$$

an exact sequence of finitely generated modules, H flat over R and M free over A. Then F is free over A if and only if $M \otimes A/\mathfrak{m}_A)$ is free over $A/\mathfrak{m}_A$.

This latter lemma is a direct consequence of Nakayama's lemma and the flatness of H over R (one tensors the exact sequence over R by A/m...). □

After this discussion, we can verify the local freeness of $\mathcal{F}$ on the T-fibers of $C \times T$, where the structure sheaf is isomorphic to $\mathcal{O}_C$. But there, locally, we are looking to a submodule of a free module over a PID, as wanted.

Now, the Quot functor is representable by the Quot scheme, which is a proper scheme if there are finitely many Hilbert polynomials. This is true for projective varieties, hence in our setting.

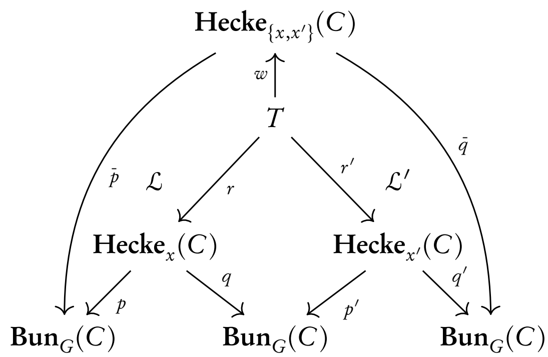
Theorem 1.2 then follows by observing that, because of the above discussion, the fiber at V^0 of the Hecke stack can be expressed as a colimit $\mathrm{colim}_n \mathrm{Quot}_{(\mathcal{I}^{-n}\mathcal{G}_0/\mathcal{I}^n\mathcal{G}_0)/C/\mathbb{C}}$. This colimit is filtered and the transition maps are easily seen to be closed immersions. □

2 Hecke functors

We recall the definition of (quasi)coherent sheaves on algebraic stacks from [Ols16, Section 9.1]. The stack $\mathrm{Bun}_G(C)$ is algebraic because it can be written as the mapping stack between C and BG. This latter is an algebraic stack (see [Ols16, Example 8.1.12]) and therefore the mapping stack is (see [Ols06, Theorem 1.1] or [Lur12, Proposition 3.3.8] for a derived version).

A general three-functor-formalism (pullback, pushforward, tensor product) for sheaves on algebraic stacks can be given. However, in our case we will usually deal with ind-representable maps, and therefore pushforwards can be presented as colimits of pushforwards along representable maps. This implies for example that the projection formula holds *verbatim* as in the schematic case. Moreover, these maps will be ind-proper, and therefore a base-change theorem holds as well.

Definition 2.1. If x, x' are closed points in a smooth projective curve C, and $\mathcal{L}, \mathcal{L}'$ are coherent sheaves over $\mathbf{Hecke}_x(C)$, $\mathbf{Hecke}_{x'}(C')$, respectively, then consider the diagram



and the sheaf

$$\mathcal{L} \otimes^c \mathcal{L}' := w_*(\mathcal{L} \boxtimes \mathcal{L}') = w_*(r^*\mathcal{L} \otimes r'^*\mathcal{L}')$$

where:

- $p, q, p', q', \bar{p}, \bar{q}$ are the different projection maps;
- T is by definition the pullback in the central square (with its canonical maps r, r'), represented by tuples $(\mathcal{F}, \mathcal{G}, \mathcal{G}', \mathcal{H}, \phi\, \psi, \chi)$ where $\mathcal{F}, \mathcal{G}, \mathcal{G}', \mathcal{H} \in \mathrm{Bun}_G(C)$, and $\phi : \mathcal{F}|_{C\setminus\{x\}} \to \mathcal{G}|_{C\setminus\{x\}}, \psi : \mathcal{G}'|_{C\setminus\{x'\}} \to \mathcal{H}|_{C\setminus\{x'\}}, \chi : \mathcal{G} \to \mathcal{G}'$ are isomorphisms;
- w is the map represented by

$$(\mathcal{F}, \mathcal{G}, \mathcal{G}', \mathcal{H}, \phi, \psi, \chi) \mapsto (\mathcal{F}, \mathcal{H}, \psi \circ \chi|_{C\setminus\{x,x'\}} \circ \phi).$$

With these definitions, it is immediate to see that all triangles commute.

Definition 2.2. Define the **Hecke functors** of the category $\mathrm{Coh}^{\mathrm{b}}(\mathbf{Bun}_G(C))$ as

$$\mathcal{H}_{x,\mathcal{L}}(\mathcal{M}) = q_*(p^*\mathcal{M} \otimes \mathcal{L})$$

where x varies among all closed points of C and $\mathcal{L}$ among all coherent sheaves over $\mathbf{Hecke}_x(C)$.

The following theorem and its proof have been suggested by Mauro Porta.

Theorem 2.3. *In the above setting,*

$$\mathcal{H}_{x',\mathcal{L}'} \circ \mathcal{H}_{x,\mathcal{L}} \cong \mathcal{H}_{\{x\}\cup\{x'\},\mathcal{L}\otimes^c\mathcal{L}'}.$$

Proof. This is an application of the proper base change theorem and of the projection formula for quasicoherent sheaves. Note that both make sense in our setting, since the relevant maps are ind-proper.

$$\mathcal{H}_{x',\mathcal{L}'} \circ \mathcal{H}_{x,\mathcal{L}}(\mathcal{M}) = q'_*(p'^*q_*(p^*\mathcal{M} \otimes \mathcal{L}) \otimes \mathcal{L}').$$

By proper base change $p'^* q_* \cong r'_* r^*$, and the pullback commutes with tensor products, so we have

$$\ldots \cong q'_*(r'_* r^*(p^*\mathcal{M} \otimes \mathcal{L}) \otimes \mathcal{L}') \cong q'_*(r'_*(r^* p^*\mathcal{M} \otimes r^*\mathcal{L}) \otimes \mathcal{L}').$$

By the projection formula we have that

$$r'_*(r^* p^*\mathcal{M} \otimes r^*\mathcal{L}) \otimes \mathcal{L}' \cong r'_*(r^* p^*\mathcal{M} \otimes r^*\mathcal{L} \otimes r'^*\mathcal{L}').$$

so the main expression becomes

$$\ldots \cong (q'r')_*((pr)^*\mathcal{M} \otimes (r^*\mathcal{L} \otimes r'^*\mathcal{L}')) \cong (\bar{q}w)_*((\bar{p}w)^*\mathcal{M} \otimes (r^*\mathcal{L} \otimes r'^*\mathcal{L}')) =$$

$$= \bar{q}_* w_*(w^* \bar{p}^*\mathcal{M} \otimes (r^*\mathcal{L} \otimes r'^*\mathcal{L}'))$$

which again by the projection formula becomes

$$\ldots \cong \bar{q}_*(\bar{p}^*\mathcal{M} \otimes w_*(r^*\mathcal{L} \otimes r'^*\mathcal{L}')) = \mathcal{H}_{\{x,x'\},\mathcal{L}\otimes^c\mathcal{L}'}.$$

□

We now recall from [PS19] that for each smooth proper complex scheme X there exists a diagram

$$\begin{array}{ccc} & \mathbf{Coh}^{ext}(X) & \\ {}^{ev_3 \times ev_1}\swarrow & & \searrow^{ev_2} \\ \mathbf{Coh}(X) \times \mathbf{Coh}(X) & & \mathbf{Coh}(X) \end{array}$$

where $\mathbf{Coh}^{ext}(X)$ is defined as the pullback

$$\begin{array}{ccc} \mathbf{Coh}^{ext}(X) & \longrightarrow & \mathbf{Perf}^{ext}(X) \\ \downarrow & & \downarrow{\scriptstyle ev_1 \times ev_2 \times ev_3} \\ \mathbf{Coh}(X) \times \mathbf{Coh}(X) \times \mathbf{Coh}(X) & \longrightarrow & \mathbf{Perf}(X) \times \mathbf{Perf}(X) \times \mathbf{Perf}(X) \end{array}$$

and $\mathbf{Perf}^{ext}(X)$ is the stack

$$T \mapsto \{\mathcal{F} \to \mathcal{G} \to \mathcal{H}, \mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Perf}(X \times T) \mid \text{the sequence is a fiber sequence}\}.$$

Let now $X = C$ be a curve, and fix a closed point x in C. Fix the element represented by $k(x)$ in $\mathbf{Coh}(X)$ as the third component of the fiber sequence, i.e. take the fiber of ev_3. We have the following diagram:

$$\begin{array}{ccc} & \mathbf{Hecke}_x^{ext}(C) & \\ \swarrow & & \searrow \\ \{k(x)\} \times \mathbf{Coh}(C) & & \mathbf{Coh}(C) \end{array}$$

where $\mathbf{Hecke}_x^{ext}(X)$ is defined as the fiber

$$T \mapsto \{\mathcal{F} \to \mathcal{G} \to p_C^* k(x) \text{ fiber sequences of coherent sheaves over } C \times T\},$$

$p_C : C \times T \to C$ being the projection.
Let $\mathcal{L}$ be a sheaf in $\mathrm{Coh}^{\mathrm{b}}(\mathbf{Coh}(C))$. We have a diagram

$$\begin{array}{ccc} & \mathbf{Hecke}^{ext}(C) & \\ {}^{p^{ext}}\swarrow & & \searrow^{q^{ext}} \\ \mathbf{Coh}(C) & & \mathbf{Coh}(C) \end{array}$$

that allows us to define the endofunctor $\mathcal{H}^{ext}_{x,\mathcal{L}}$ of $\mathrm{Coh}^{\mathrm{b}}(\mathbf{Coh}(C))$ as

$$\mathcal{H}^{ext}_{x,\mathcal{L}}(\mathcal{M}) = q^{ext}_*(p^{ext*}\mathcal{M} \otimes \mathcal{L}).$$

Let us now define a "coherent" counterpart for the "locally free" Hecke stack defined in Part 1:

$$\mathbf{Hecke}^{coh}_x(C)(T) := \{(\mathcal{F}, \mathcal{G}, \phi) \mid \mathcal{F}, \mathcal{G} \in \mathrm{Coh}(C \times T), \phi : \mathcal{F}|_{(C\setminus\{x\})\times T} \to \mathcal{G}|_{(C\setminus\{x\})\times T} \text{ isomorphism}\}$$

with the analogously defined Hecke endofunctors. There is an evident map

$$h : \mathbf{Hecke}^{ext}_x(C) \to \mathbf{Hecke}^{coh}_x(C),$$

since the fiber condition for $\mathcal{F} \to \mathcal{G}$ is read as an isomorphism condition outside $\{x\} \times T$, because the cokernel is supported on $\{x\} \times T$.
Note that the triangles in the diagram

$$\begin{array}{ccccc} & & \mathbf{Hecke}^{ext}_x(C) & & \\ & {}^{p^{ext}}\swarrow & \downarrow h & \searrow^{q^{ext}} & \\ & & \mathbf{Hecke}^{coh}_x(C) & & \\ & \swarrow_{p^{coh}} & & {}_{q^{coh}}\searrow & \\ \mathbf{Coh}(X) & & & & \mathbf{Coh}(X) \end{array}$$

commute. We note that $\mathcal{H}^{ext}_{x,\mathcal{L}} \cong \mathcal{H}_{x,h_*\mathcal{L}}$. Indeed, by the projection formula we have

$$q^{ext}_*(p^{ext*}\mathcal{M}\otimes\mathcal{L}) \cong (q^{coh}h)_*((p^{coh}h)^*\mathcal{M}\otimes\mathcal{L}) \cong q^{coh}_* h_*(h^* p^{coh*}\mathcal{M}\otimes\mathcal{L}) \cong q^{coh}_*(p^{coh*}\mathcal{M}\otimes h_*\mathcal{L}).$$

3 Naïve affine Grassmannians for surfaces

Definition 3.1. Le S be a smooth complex surface, $x \in S(\mathbb{C})$. Let G be a reductive group over $\mathbb{C}$ and, for any complex commutative algebra R, let $S_R = S \times_{\mathbb{C}} \mathrm{Spec} R$. Define

$$\mathrm{Gr}_{S,x} : \mathrm{CAlg} \to \mathcal{G}\mathrm{rpd}$$

as

$$R \mapsto \{\mathcal{F} \in \mathrm{Bun}(S_R), \alpha : \mathcal{F}|_{(S\setminus\{x\})\times \mathrm{Spec} R} \xrightarrow{\sim} \mathcal{T}_{(S\setminus\{x\})\times \mathrm{Spec} R}\}$$

where Γ_x is the graph of x inside S_R and $\mathcal{T}$ is the trivial G-bundle.

Proposition 3.2. $\mathrm{Gr}_{S,x}$ *is strictly equivalent to* G *as a functor* $\mathrm{CAlg} \to \mathcal{G}\mathrm{rpd}$.

More generally

Proposition 3.3. *The functor*

$$\mathrm{Gr}_S : \mathrm{CAlg} \to \mathcal{G}\mathrm{rpd}$$
$$R \mapsto \{x \in S(R), \mathcal{F} \in \mathrm{Bun}(S_R), \alpha : \mathcal{F}|_{S_R \setminus \Gamma_x} \xrightarrow{\sim} \mathcal{T}|_{S_R \setminus \Gamma_S}\}$$

is strictly equivalent to $G \times_{\mathbb{C}} S$.

Note that $\mathrm{Gr}_{S,x}$ is the fiber of Gr_S at the point x. It suffices thus to prove the second result.

Proof. Let X denote S_R. We use the local cohomology exact sequence for $\mathcal{F}$ at $Z = \Gamma_x$. Let $j : X \setminus Z \to X$ be the open embedding. Then the exact sequence has the form

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_* j^* \mathcal{F} \to \mathcal{H}^1_Z(\mathcal{F}).$$

We want to prove that $\mathcal{H}^i_Z(\mathcal{F}) = 0$ as sheaves, for $i = 0, 1$. Since these sheaves are supported at Z, let us choose a point $p \in Z$ and compute the stalks at p.
Recalling the definition of local cohomology, we want to compute

$$\lim_n \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n_Z, \mathcal{F})_p = \lim_n \mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/\mathfrak{m}^n_p, \mathcal{F}_p)$$

for $i = 0, 1$.
We use now a depth argument. $\mathcal{O}_{X,p}$ is local with ideal $\mathfrak{m}_p$. Since $\mathcal{F}$ is locally free, the projective dimension of $\mathcal{F}_p$ over $\mathcal{O}_{X,p}$ is 0, and the Auslander-Buchsbaum formula says that

$$\mathrm{depth}_{\mathfrak{m}_p} \mathcal{F}_p = \mathrm{depth} \mathcal{O}_{X,p}.$$

Now, $\mathcal{O}_{X,p} = A \otimes R'$, where $A = \mathcal{O}_{S,x}$ is a regular local ring of dimension 2 and R is the localization of R at the prime corresponding to p. In particular, A has depth 2. Let (x_1, x_2) be a regular sequence in A, and

$$0 \to A \to A \oplus A \to A$$

be the corrisponding exact sequence. Tensoring by R over $\mathbb{C}$ preserves left exactness, and thus we obtain a sequence

$$0 \to A \otimes R' \to (A \otimes R')^{\oplus 2} \to A \otimes R'$$

where the last map is the one associated to the sequence $(x_1 \otimes 1, x_2 \otimes 1)$, which by exactess is again regular. We conclude that $A \otimes R'$ has depth at least 2.

It suffices now to apply the following couple of arguments: first, by [?, Theorem 16.6] we have that $\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(N, \mathcal{F}_p) = 0$ for $i = 0, 1$ for every finite $\mathcal{O}_{X,p}$-module N with support concentrated in p. This includes the case $N = \mathcal{O}_{X,p}/\mathfrak{m}_p$ and every quotient of the form $\mathfrak{m}^n_p/\mathfrak{m}^{n+1}_p$. As for the quotients of $\mathcal{O}_{X,p}$ by higher powers of the maximal ideal, we can apply induction using the long exact sequence of Ext groups induced by the short exact sequence

$$0 \to \mathfrak{m}^n_p/\mathfrak{m}^{n+1}_p \to \mathcal{O}_{X,p}/\mathfrak{m}^{n+1}_p \to \mathcal{O}_{X,p}/\mathfrak{m}^n_p \to 0$$

and thus conclude that $\mathrm{Ext}^i_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/\mathfrak{m}^n_p, \mathcal{F}_p) = 0, i = 0, 1, \forall n.$ □

We have just proven that the most naïve generalization of the affine Grassmannian to surfaces is "trivial". We therefore introduce a slightly less naïve version, and prove that it is ind-quasiprojective.

Definition 3.4. Let S be a smooth complex surface, and $C \subset S$ a closed subscheme of dimension 1. The stack $\mathbf{Hecke}_C^{\mathrm{Bun}_G}(S)$ is the functor sending

$$T \mapsto \{\mathcal{F}, \mathcal{G} \in \mathrm{Bun}_G(S \times Y), \phi : \mathcal{F}|_{(S\setminus C)\times T} \xrightarrow{\sim} \mathcal{G}|_{(S\setminus C)\times T}\}$$

where the one on the right is a groupoid whose morphisms are isomorphisms of pairs of torsors whose restriction to the open set commute with the given isomorphisms.

Theorem 3.5 (sketched in [Kap00, Proposition 2.2.2]). *The fibers of the map* $\mathbf{Hecke}_C^{\mathrm{Bun}_G}(S) \to \mathbf{Bun}_G(C)$ *are representable by ind-quasiprojective schemes.*

Definition 3.6. We call the fiber at the trivial bundle the **affine Grassmannian** associated to the triple (G, S, C) and denote it by $\mathrm{Gr}_C(S)$ (the group is omitted from the notation).

Proof of Theorem 3.5. The proof goes along the same line of the proof of Theorem 1.2, except for the fact that we cannot extablish a bijective correspondence between $\{\mathcal{F} \mid \mathcal{I}^n\mathcal{G}_0 \subset \mathcal{F} \subset \mathcal{I}^{-n}\mathcal{G}_0\}$ and $\mathrm{Quot}_{\mathcal{I}^{-n}\mathcal{G}_0/\mathcal{I}^n\mathcal{G}_0/S/\mathbb{C}}(T)$, since the local rings of S are not PIDs.
So in general, the condition that the kernel

$$0 \to \mathcal{F} \to \mathcal{I}^{-n}\mathcal{G}_0 \to \mathcal{H} \to 0$$

is locally free is a nontrivial condition, and it is open. To prove this, we can again restrict to the t-fibers of $S \times T$ as above, and note that a sheaf which is locally free at a point is free over an open set around the point. An alternative viewpoint, still after restricting to fibers, is the following: provided that $\mathrm{Coh}(S)$ has enough locally frees, i.e. that each coherent sheaf is a quotient of a locally free sheaf (and this is the case for quasiprojective varieties like S: `https://stacks.math.columbia.edu/tag/0F85`), it is true that $\mathcal{F} \in \mathrm{Coh}(S)$ is locally free if and only if $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{G} \in \mathrm{Coh}(S)$ ([Har77, Exercises 6.4-6.5 on page 238]). One can now invoke the so-called "semicontinuity theorem for cohomology":

Theorem 3.7 ([BPS80, Satz 3]). *Let $X \to W$ be a flat morphism of schemes, $\mathcal{F}, \mathcal{G}$ coherent sheaves on X flat over W. Then the function $W \to \mathbb{N}, w \mapsto \dim \mathrm{Ext}^1_{\mathcal{O}_{X_w}}(\mathcal{F}_w, \mathcal{G}_w)$ is upper semicontinuous.*

It is now possible to conclude the proof in the same way as for the case of the curve. □

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