Sheaves-functions dictionary on the stack of coherent sheaves

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Abstract

In this note we review some properties of the sheaves-functions dictionary for algebraic varieties, in particular proving that it is an injective map of vector spaces and that commutes with tensor product, pullback and proper pushforward of sheave/functions. We also illustrate a generalization to algebraic stacks of finite type, and to general algebraic stacks via pro-objects. We apply this theory to show that for an algebraic variety X, the Hall product of constructible sheaves on $\mathbf{Coh}(X)$ agrees with the convolution product of functions via the generalized sheaves-functions dictionary.

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1 Injectivity

The sheaves-functions correspondence is the function (not actually a correspondence, i.e. it is not bijective) described by

$$\chi : \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)) \otimes \overline{\mathbb{Q}_\ell} \to \mathrm{Hom}_{\mathrm{Set}}(\coprod X(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell}),$$

$$[F] \mapsto \chi_{\mathcal{F}}$$

where $\chi_{\mathcal{F}}$ sends $x \in X(\mathbb{F}_{q^n})$ to

$$\sum_{i=0}^{2\dim X} (-1)^i \mathrm{Tr}(\mathrm{Fr}_x, \mathcal{H}^i(\mathcal{F})).$$

We assume that $\mathbb{C}$ and $\overline{\mathbb{Q}_\ell}$ are identified by means of a chosen isomorphism. Let us explain the above expression.

The complex being constructible, the cohomology sheaves $\mathcal{H}^i(\mathcal{F})$ are constructible on X by definition. Therefore there exists a stratification $\{j_\alpha : X_\alpha \to X\}$ of X by locally closed subschemes such that $j_\alpha^*(\mathcal{H}^i(\mathcal{F}))$ is locally constant.
The monodromy action (see [Lov12]) establishes a correspondence between local systems on X_α and ℓ-adic representations of the fundamental group $\pi_1(X_\alpha, \overline{x}_\alpha)$ for any basepoint x and any choice of a geometric point $\overline{x}$ over x. We denote the representation associated to $j_\alpha^*\mathcal{H}^i(\mathcal{F})$ by $\rho_\alpha : \pi_1(X_\alpha, \overline{x}_\alpha) \to GL_n(\mathbb{Q}_\ell)$.
For every $x \in X_\alpha(\mathbb{F}_{q^n})$, n arbitrary, we have an induced map $\pi_1(x, \overline{x}) = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^n}) \to GL_n(\mathbb{Q}_\ell)$. Note that $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^n})$ is topologically generated by the Frobenius element Fr^n. Call σ_x the conjugacy class induced by Fr^n in $\pi_1(X_\alpha, \overline{x}_\alpha)$ and $\rho_\alpha(\sigma_x)$ the conjugacy class in $GL_n(\mathbb{Q}_\ell)$.
We define

$$\mathrm{Tr}(\mathrm{Fr}_x, \mathcal{H}^i(\mathcal{F})) := \mathrm{Tr}(\rho_\alpha(\sigma_x))$$

(this is well defined because the trace is conjugacy-invariant).

Remark 1.1. We will prove in a moment that the function χ is well-defined on the Grothendieck group. Before, it is important to note that the Grothendieck group above can be identified with the Grothendieck group of $Perv(X)$: this is because $Perv(X)$ is, by definition, the heart of the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}_\ell})$, hence by Quillen's heart theorem the two groups coincide.
Moreover, since $Perv(X)$ is an abelian category, the Grothendieck group is (abelian) freely generated by the isomorphism classes of simple perverse sheaves. More precisely, since we are in the Grothendieck group we can assume that our perverse objects are semisimple (every object is equivalent to its semisimplification), and write $[{}^p\mathcal{H}^i(\mathcal{F})] = \sum_j [S_j]$, where the S_j are simple perverse sheaves. By [BBDG83, Theorem 4.3.1 (ii)] these are of the form $i_* j_{!*} L^i_j[\dim Z^i]$ for some *irreducible* local systems L^i_j on an open set $j : U^i \to Z^i$ in a closed irreducible subscheme $i : Z^i \to X$.

Theorem 1.2. *The function χ is well-defined, and it is injective.*

Proof. The function is well-defined on the Grothendieck group. Indeed, for an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of perverse sheaves, we consider the long exact sequence induced on the cohomology sheaves, and then we take stalks at the point x; now we can apply the additivity of the trace of a morphism on exact sequences of vector spaces.
Let us prove now that the function is injective. Suppose by contradiction that $\chi_{\mathcal{F}} = \chi_{\mathcal{G}}$ for two different objects in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. By Remark 1.1 we can suppose that $\mathcal{F}$ and $\mathcal{G}$ are perverse and semisimple.

Now, in the Grothendieck group we have that for any perverse sheaf $\mathcal{F}_0$,

$$[\mathcal{F}_0] = \sum_i (-1)^i [\mathcal{H}^i(\mathcal{F}_0)]$$

and also

$$[\mathcal{F}_0] = \sum_i (-1)^i [\,{}^p\mathcal{H}^i(\mathcal{F}_0)].$$

The non-injectivity hypothesis tells us that the local traces of the first expression (the one with the ordinary cohomology sheaves) coincide for $\mathcal{F}$ and $\mathcal{G}$; therefore, the same holds for

the second one, if we write it in a way such that the trace of the Frobenius makes sense, i.e. if we decompose the perverse cohomologies into summands related to local systems, which we will do immediatly.
Let us call $L^i = \oplus_j L^i_j$ the semisimple local systems corresponding to $\mathcal{F}$ and $N^i = \oplus_j N^i_j$ the local systems corresponding to $\mathcal{G}$ as in Remark 1.1.
One can now follow the proof in [MSV18] and proceed by noetherian induction, separating the supports. We are thus dealing with two irreducible local systems L^i and N^i, defined over a common open set U^i inside a closed irreducible set Z^i. By restricting to U^i, we can deduce from our noninjectivity hypothesis that for every $x \in U^i$

$$\sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_x, L^i) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_x, N^i).$$

By linearity of the trace, we can rewrite the equalities as

$$\mathrm{Tr}(\mathrm{Fr}^n_x, \sum_i (-1)^i L^i) = \mathrm{Tr}(\mathrm{Fr}^n_x, \sum_i (-1)^i N^i),$$

where the algebraic sum is meant to be seen as an algebraic sum of representations.
Let us just call the two local systems of which we are taking the trace L and N. They correspond to two continuous representations $\rho_1, \rho_2 : G := \pi_1(U, \overline{u}) \to GL_n(\overline{\mathbb{Q}_\ell})$. We want to prove that the morphisms

$$\mathrm{tr}_1, \mathrm{tr}_2 : G \to \overline{\mathbb{Q}_\ell}$$

$$g \mapsto \mathrm{Tr}(\rho_1(g)), \mathrm{Tr}(\rho_2(g))$$

are the same. By the Brauer-Nesbitt theorem ([CR62]) this implies that the two representations have the same semisimplification. But since we are dealing with simple (i.e. irreducible) local systems, we can conclude that they are isomorphic.
Now, we know that if we consider a closed point $x : \mathrm{Spec}\, \mathbb{F}_{q^n} \to U$ and its fundamental group $\pi_1(x, \overline{x}) = \mathrm{Gal}(k(\overline{x})/k)$, we have a conjugacy class in $\pi_1(U, \overline{u})$ induced by the Frobenius element $\sigma_x \in \mathrm{Gal}(k(\overline{x})/k(x))$. The Chebotarev theorem (as stated in [Pin97, Theorem B.9]) asserts precisely that these conjugacy classes are dense in $\pi_1(U, \overline{u})$ when x varies over the closed points of U. On the other hand, for every closed point x, $\mathrm{Tr}(\mathrm{Fr}^n_{\overline{x}}, \mathcal{L}_x)$ is the same as $\mathrm{Tr}(\rho_{1x}(\sigma_x))$ as trace of a matrix, where ρ_{1x} is the continuous representation $\mathrm{Gal}(\overline{k(x)}/k(x)) \to GL_n(\overline{\mathbb{Q}_\ell})$ induced by L on x (and the same for N). This follows from the correspondence between local systems and continuous representations.
Since the representations are continuous and the non-injectivity assumption tells us that the two traces $\mathrm{tr}_1, \mathrm{tr}_2$ coincide on these conjugacy classes, we conclude. □

Remark 1.3. It can be proved that

$$\mathrm{Tr}(\mathrm{Fr}_x, \mathcal{H}^i(\mathcal{F})) = \mathrm{Tr}(\mathrm{Fr}_{\overline{X}, \overline{x}}, \mathcal{H}^i_{\overline{x}}(\overline{\mathcal{F}}))$$

where $\mathrm{Fr}_{\overline{X}} = \mathrm{Fr}_X \otimes id_{\overline{\mathbb{F}_q}}$ is the automorphism of $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ induced by the Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, $\overline{\mathcal{F}}$ is the pullback of $\mathcal{F}$ to $\overline{X}$, and the action on the cohomology vector space is induced by functoriality. This slightly different version will be useful later.

Definition 1.4 (External product of constructible complexes on schemes)**.** Define the external product of two constructible complexes sheaves $\mathcal{F}$ and $\mathcal{G}$ on two (possibly different) schemes X and Y as $\mathcal{F} \boxtimes \mathcal{G} := p_1^* \mathcal{F} \otimes p_2^* \mathcal{G}$, referring to the diagram

$$\begin{array}{ccccc} & & X \times Y & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X & & & & Y. \end{array}$$

Note that the tensor product is the derived tensor product in the derived category of constructible sheaves.

Observe now the following two facts:

Proposition 1.5. *The correspondence sends the tensor product of simple perverse sheaves to the product of functions.*

Proof. This follows from the fact that the tensor product of simple perverse sheaves corresponds to the product of representations. □

Theorem 1.6. *The correspondence commutes with the pullback.*

Proof. Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ a constructible complex on Y. We write f^* for the pullback on both sides. We have that $(\chi_{f^*(\mathcal{F})})(x) = \sum (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_{\overline{X}}, \mathcal{H}^i_{\overline{x}}(f^* \mathcal{F}))$, while

$$f^* \chi_{\mathcal{F}} = f^*(y \mapsto \sum (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_{\overline{Y}}, \mathcal{H}^i_{\overline{y}}(\mathcal{F})))(x) = \sum (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_{\overline{Y}}, \mathcal{H}^i_{f(\overline{x})}(\mathcal{F}))$$

but $\mathcal{H}^i(\mathcal{F}) = \frac{Ker^i \mathcal{F}}{Im^i \mathcal{F}}$ and $\mathcal{H}^i_{f(\overline{x})}(\mathcal{F}) = \frac{Ker^i(\mathcal{F}_{f(\overline{x})})}{Im^i(\mathcal{F}_{f(\overline{x})})} = \frac{(Ker^i(f^*\mathcal{F}))_{\overline{x}}}{(Im^i(f^*\mathcal{F}))_{\overline{x}}} = \mathcal{H}^i_{\overline{x}}(f^* \mathcal{F})$.

□

In the next section, we will prove that χ "preserves" the derived direct image with proper support $Rf_!$ in a suitable sense.

2 Commutation with proper pushforward

Definition 2.1. Let X be a scheme of finite type over $\mathbb{F}_q$. Let $\mathcal{F}$ be a constructible complex over X, i.e. an object of the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \overline{\mathbb{Q}_\ell})$. Define the function

$$\chi_X(\mathcal{F}) : X(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}_\ell}$$

$$\chi_X(\mathcal{F})(x) := \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_{\overline{X}}, \mathcal{H}^i_{\overline{x}}(\mathcal{F}))$$

where an algebraic closure of $\mathbb{F}_q$ is chosen, so that $\overline{X}$ is the fiber product $X \times_{\mathbb{F}_q} \mathrm{Spec}\ \overline{\mathbb{F}_q}$ and $\overline{x}$ is a geometric point corresponding to x. Recall that Fr is the geometric Frobenius acting on $\overline{X}$ and therefore on $\mathcal{H}^i_{\overline{x}}(\mathcal{F})$, the cohomology stalks of $\mathcal{F}$, where for simplicity we write $\mathcal{F}$ for the pullback to the algebraic closure. The whole construction is independent of the choice of $\overline{x}$.

Proposition 2.2 ([Lau87, 1.1.1.3], without proof)**.** *Let $f: X \to Y$ be a morphism of schemes of finite type over $\mathbb{F}_q$. Then the diagram*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X) & \xrightarrow{\chi_X} & \mathrm{Hom}_{\mathrm{Set}}(X(\mathbb{F}_{q^n}), \mathcal{C}) \\ \downarrow{\scriptstyle Rf_!} & & \downarrow{\scriptstyle f_!} \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y) & \xrightarrow{\chi_Y} & \mathrm{Hom}_{\mathrm{Set}}(Y(\mathbb{F}_{q^n}), \mathcal{C}) \end{array}$$

commutes. Here $f_!$ is the following operator:

$$f_!: \mathrm{Hom}_{\mathrm{Set}}(X(\mathbb{F}_{q^n}), \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}}(Y(\mathbb{F}_{q^n}), \mathcal{C})$$

$$f_!(\phi)(y) := \sum_{x \in X(\mathbb{F}_{q^n}), f(x)=y} \phi(x).$$

Recall that the sets $X(\mathbb{F}_{q^n})$ and $Y(\mathbb{F}_{q^n})$ are finite, hence the above sum makes sense.

Proof. We want to prove that, for any $y \in Y$,

$$\sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_{\overline{Y}} \mid \mathcal{H}^i_{\overline{y}}(Rf_!\mathcal{F})) = \sum_{x \in X(\mathbb{F}_{q^n}) \cap f^{-1}(y)} \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_{\overline{X}} \mid \mathcal{H}^i_{\overline{x}}(\mathcal{F})).$$

We begin by exchanging the two sums in the right-hand side (for brevity, from now on, RHS) and applying the Grothendieck-Lefschetz trace formula to $Z := \mathrm{Spec}\,\overline{\mathbb{F}_q} \times f^{-1}(y)$:

Proposition 2.3 (Grothendieck-Lefschetz trace formula)**.** *Let $\mathcal{G}$ be a constructible complex over a scheme $Z/\mathbb{F}_q$. Then*

$$\sum_{z \in Z(\mathbb{F}_{q^n})} \mathrm{Tr}(\mathrm{Fr}^n_Z \mid \mathcal{G}_z) = \sum_k (-1)^k \mathrm{Tr}(\mathrm{Fr}^n_Z \mid H^k_c(Z, \mathcal{G})).$$

Note that on the right we have the compact support (étale) cohomology *group* of the sheaf (not the stalk of the cohomology sheaf of the complex): this is defined using a Nagata compactification $j: Z \to \tilde{Z}$ (see e.g. [Tam, Definition 2.2]); in that setting, if $\mathcal{F}$ is a torsion sheaf[1] on $f: Z \to S$ is a morphism and S the spectrum of a separably closed field (in our case, $S = \overline{y}$ and f is the restriction to the geometric fiber Z of the pullback of our original f to the algebraic closure) one can define

$$H^n_c(Z, \mathcal{F}) = H^n(Z, j_!\mathcal{F})$$

$$R\Gamma_c(X, \mathcal{F}) = \Gamma(S, Rf_!\mathcal{F}),$$

check that this is independent from f, S and also see that

$$H^n_c(Z, \mathcal{F}) = H^n(R\Gamma_c(Z, \mathcal{F})) = \Gamma(S, R^n f_!\mathcal{F}). \tag{1}$$

Note that the Frobenius automorphism is well behaved with respect to morphisms of schemes over $\mathbb{F}_q$ (see [G$^+$66], XV, no. 1-2), hence we can interchange $\mathrm{Fr}^n_{\overline{X}}$ and Fr^n_Z while looking at

[1] The case of ℓ-adic sheaves is dealt with by taking the relevant limits.

the action on cohomology stalks at points of Z: i.e., $\mathrm{Tr}(\mathrm{Fr}^n_{\overline{X}} \mid \mathcal{H}^i_{\overline{x}}(\mathcal{F})) = \mathrm{Tr}(\mathrm{Fr}^n_Z \mid \mathcal{H}^i_{\overline{x}}(\mathcal{F}|_Z))$. We then get

$$RHS = \sum_i \sum_k (-1)^{i+k} \mathrm{Tr}(\mathrm{Fr}^n_Z \mid H^k_c(Z, \mathcal{H}^i \mathcal{F}|_Z)).$$

We now use the existence of a spectral sequence for hypercohomology with compact support

$$E_2^{ik} = H^k_c(Z, \mathcal{H}^i(\mathcal{F}|_Z)) \to H^{i+k}_c(\mathcal{F}|_Z).$$

Although we do not know if the spectral sequence degenerates at its second leaf (and in general it will not) we can use the fact that the differential of the spectral sequence, whose kernels and images give us the iterated leaves of the sequence, has always degree 1 (hence odd); so taking our alternate sum of traces is independent of the leaf of the spectral sequence: more precisely,

$$\sum_{i,k} (-1)^{i+k} \mathrm{Tr}(\mathrm{Fr}^n_Z \mid H^k_c(\mathcal{H}^i_x(\mathcal{F}|_Z))) = \sum_n (-1)^n \sum_{i+k=n} \mathrm{Tr}(\mathrm{Fr}^n_Z \mid E_2^{ik}) =$$

$$= \sum_n (-1)^n \mathrm{Tr}(\mathrm{Fr}^n_Z \mid \oplus_{i+k=n} E^{ik}_\infty) = \sum_n (-1)^n \mathrm{Tr}(\mathrm{Fr}^n_Z \mid H^n_c(\mathcal{F}|_Z)).$$

Now by (1) the latter is equal to

$$\sum_n (-1)^n \mathrm{Tr}(\mathrm{Fr}^n_Z \mid \Gamma(S, R^n f_!(\mathcal{F}|_Z))) = \sum_n (-1)^n \mathrm{Tr}(\mathrm{Fr}^n_Z \mid \Gamma(S, \mathcal{H}^n(Rf_!(\mathcal{F}|_Z)))).$$

By proper base change on

$$\begin{array}{ccc} Z & \xrightarrow{f|_Z} & \overline{y} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle s} \\ \overline{X} & \xrightarrow{f} & \overline{Y} \end{array}$$

one gets an isomorphism in the derived category (of complexes over $\overline{y}$)

$$Rf_!(\mathcal{F}|_Z) = Rf_! i^* \mathcal{F} \cong s^* Rf_! \mathcal{F}$$

hence $\Gamma(S, \mathcal{H}^n(Rf_!(\mathcal{F}|_Z))) = \Gamma(S, \mathcal{H}^n s^* Rf_! \mathcal{F}) \cong (\mathcal{H}^n Rf_! \mathcal{F})_{\overline{y}}$ (because $\overline{y}$ is the spectrum of a field). But this gives us exactly the LHS $\sum_n (-1)^n \mathrm{Tr}(\mathrm{Fr}^n_{\overline{Y}} \mid \mathcal{H}^n_{\overline{y}}(Rf_! \mathcal{F}))$. □

3 Extension to algebraic stacks

Our goal is now to extend and adapt the results of the previous two sections to the level of algebraic stacks locally of finite type, more precisely to the algebraic stack $\mathbf{Coh}(X), X$ being a smooth proper curve over $\mathbb{F}_q$. Let $\mathcal{X}$ be an algebraic stack of finite type over $\mathbb{F}_q$. Then the Frobenius automorphism is defined and acts by functoriality over $\mathcal{H}^i(\mathcal{F})$ for any complex of sheaves $\mathcal{F}$ over $\mathcal{X}$. Here by "sheaf" we mean "sheaf over the lisse-étale site", but we will soon change our perspective towards the context of constructible sheaves.

Remark 3.1. Let $\mathcal{X}$ be an Artin stack of finite type over $\mathbb{F}_q$, with a smooth atlas $x : X \to \mathcal{X}$. By the finite-type hypothesis, we can assume X quasicompact of finite type as well. Let Λ be a sheaf of rings over the pro-étale site of $\mathcal{X}$ (see [Cho13]). A sheaf of Λ-modules F over the small pro-étale site of $\mathcal{X}$ is *cartesian* if for every morphism $(f, f^b):(T',t')\to(T,t)$ in $(\mathrm{Sch}/\mathcal{X})_{\text{pro-ét}}$, the canonical morphism of pro-étale sheaves of $\Lambda_{(T',t')}$-modules on T'

$$f^*F_{(T,t)} = f^{-1}F_{(T,t)} \otimes_{f^{-1}\Lambda_{(T,t)}} \Lambda_{(T',t')} \to F_{(T',t')}$$

is an isomorphism. We denote by $\mathrm{Mod}_\Lambda^{\mathrm{cart}}(\mathcal{X}_{\text{pro-ét}})$ the full subcategory of $\mathrm{Mod}(\mathcal{X}_{\text{pro-ét}},\Lambda)$ spanned by cartesian sheaves.

Let $X\to\mathcal{X}$ be an atlas, and $X_\bullet$ be its Cech nerve. At the level of ∞-categories, one can alternatively define

$$\mathrm{Mod}_\Lambda^{\mathrm{cart}}(\mathcal{X}) := \varprojlim p,$$

p being the functor $\boldsymbol{\Delta}\to\mathcal{C}\mathrm{at}_\infty$ defined by $p([n])=\mathrm{Mod}_\Lambda(X_n)$, $p(\delta:[n]\to[m])=\delta^*:\mathrm{Mod}_\Lambda(X_n)\to\mathrm{Mod}_\Lambda(X_m)$. By [Lur09, Cor. 3.3.3.2], this definition agrees with the classical definition.

Definition 3.2. Define

$$\mathrm{Cons}_\Lambda(\mathcal{X}) = \{\mathcal{F}\in\mathrm{Mod}_\Lambda^{\mathrm{cart}}(\mathcal{X}) \mid \mathcal{F}_{(X,x)} \text{ is constructible}\}.$$

It is very important to note that

Lemma 3.3 ([Cho13, Lemma 6.2.13])**.** $\mathrm{Mod}_\Lambda^{\mathrm{cart}}(\mathcal{X}_{\text{pro-ét}})$ *and* $\mathrm{Cons}_\Lambda(\mathcal{X})$ *are abelian categories.*

Definition 3.4. Define also $\mathrm{D}_c^b(\mathcal{X},\Lambda)=\{\mathcal{F}\in\mathrm{D}^b(\mathrm{Mod}_\Lambda(\mathcal{X}))\mid \mathcal{H}^n(\mathcal{F})\in\mathrm{Cons}_\Lambda(\mathcal{X})\}$.

From now on, Λ will be the constant sheaf $\overline{\mathbb{Q}_\ell}$ (and omit it when clear from the context).

Definition 3.5 (Perverse t-structure on stacks)**.** One can define a perverse t-structure on D_c^b by

$${}^p\mathrm{D}_c^{\leq 0}(\mathcal{X}) = \{\mathcal{F}\in\mathrm{D}_c^b(\mathcal{X}) \mid x^*\mathcal{F}[\dim x]\in {}^p\mathrm{D}^{\leq 0}(X)\}$$

$${}^p\mathrm{D}_c^{\geq 0}(\mathcal{X}) = \{\mathcal{F}\in\mathrm{D}_c^b(\mathcal{X}) \mid x^*\mathcal{F}[\dim x]\in {}^p\mathrm{D}^{\geq 0}(X)\}$$

where ${}^p\mathrm{D}_c^{\leq 0}(X)$ and ${}^p\mathrm{D}_c^{\geq 0}(X)$ refer to the usual perverse structure on schemes: a complex $\mathcal{F}\in\mathrm{D}_c^b(X)$ is in ${}^p\mathrm{D}_c^{\leq 0}(X)$ (resp. ${}^p\mathrm{D}_c^{\geq 0}(X)$) if for every $i\in\mathbb{Z}$

$$\dim\operatorname{supp}(\mathcal{H}^{-i}(\mathcal{F}))\leq i$$

(respectively $\dim\operatorname{supp}(\mathcal{H}^{-i}(\mathbb{D}\mathcal{F}))\leq i$, where $\mathbb{D}$ is the Verdier dual).

It can be checked ([LO09, Sec. 4]) that this defines a t-structure on $\mathrm{D}_c^b(\mathcal{X})$, independent of the atlas X.

Definition 3.6. Define $\mathrm{Perv}(\mathcal{X}) = {}^p\mathrm{D}^{\leq 0}(\mathcal{X})\cap {}^p\mathrm{D}^{\geq 0}(\mathcal{X})$ as full triangulated subcategory of $\mathrm{D}_c^b(\mathcal{X})$.

Remark 3.7. This allows us to define a Grothendieck group $\mathrm{K}_0(\mathrm{Perv}(X))$. By Quillen's Heart theorem, we have $\mathrm{K}_0(\mathrm{Perv}(X))=\mathrm{K}_0(\mathrm{D}_c^b(X))$ but also $\mathrm{K}_0(\mathrm{Perv}(X))=K^0(\mathrm{Cons}_\Lambda(\mathcal{X}))$. This is because every object $\mathcal{F}\in\mathrm{D}_c^b(X)$ has the same class in the Grothendieck group as both $\sum(-1)^i[{}^p\mathcal{H}^i(\mathcal{F})]$ and $\sum(-1)^i[\mathcal{H}^i(\mathcal{F})]$. The summands in the second one are constructible by [Cho13, Notation 6.2.15, Theorem 6.2.16], hence the above statement. Recall that we have already used this kind of argument while dealing with schemes.

Let $\mathcal{X}$ be again an algebraic stack *of finite type* over $\mathbb{F}_q$. Recall that for the constructible sheaf $\mathcal{G} = \mathcal{H}^i(\mathcal{F})$ the χ function reduces to

$$\chi_{\mathcal{G}}(x) = \mathrm{Tr}(\mathrm{Fr}^n_{\mathcal{X}}, \mathcal{G}_{\overline{x}}).$$

More in general, we have a homomorphism

$$\chi : \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X})) \otimes \overline{\mathbb{Q}_\ell} \to \prod_n \mathrm{Hom}_{\mathrm{Set}}(\pi_0(\mathcal{X}(\mathbb{F}_{q^n})), \overline{\mathbb{Q}_\ell}) \tag{2}$$

given by the usual formula

$$\chi_{\mathcal{F}}(x) = \sum (-1)^i \mathrm{Tr}(\mathrm{Fr}^n_{\mathcal{X}}, \mathcal{H}^i_{\overline{x}}(\mathcal{F})).$$

This sum converges because our complexes are bounded. A theory for trace formulas in the unbounded setting is developed in [Sun12].

Theorem 3.8 ([Zhe18, Prop. 4.6])**.** *The function χ is injective.*

Sketch. The proof tries to go along the same lines as in the case of schemes. Given $A \in \mathrm{Ker}\,\chi$, we can suppose as usual that it is (the isomorphism class of) a simple perverse sheaf. We can prove that there exists a stratification of $\mathcal{X}$ by geometrically unibranch substacks where A is lisse, i.e. locally constant. From that, one can reduce straightforwardly to Deligne-Mumford stacks and apply a version of a Chebotarev-like argument proved in [SZ16]. □

4 The sheaves-function dictionary and the Hall product on $\mathbf{Coh}(X)$

Let us now consider a curve X and the algebraic stack $\mathcal{X} = \mathbf{Coh}(X)$. We would like to construct a convolution product in $\mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X}))$, which should be preserved under the sheaves-functions correspondence χ. The construction is as follows: consider the stack $\mathbf{Coh}^{ext}(X)$ as in [PS19, Sec. 3], and the diagram

$$\begin{array}{ccc} & \mathbf{Coh}^{ext}(X) & \\ {\scriptstyle ev_3 \times ev_1} \swarrow & & \searrow {\scriptstyle ev_2} \\ \mathbf{Coh}(X) \times \mathbf{Coh}(X) & & \mathbf{Coh}(X) \end{array}$$

(we borrow the notations from [PS19]). Then take two simple perverse sheaves $\mathcal{F}$ and $\mathcal{G}$ over $\mathbf{Coh}(X)$, and set

$$\mathcal{F} * \mathcal{G} := ev_{2!}(ev_3 \times ev_1)^*(\mathcal{F} \boxtimes \mathcal{G}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbf{Coh}(X)).$$

We extend this product to the whole $\mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbf{Coh}(X))$ by bilinearity.
We now define the counterpart of this convolution product on the "function" side, as in [Toe06]. Starting from the diagram above, we want to build a diagram of the following kind:

$$\begin{array}{ccc} & \mathrm{Func}(\mathbf{Coh}^{ext}(X)(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell}) & \\ {\scriptstyle (ev_3 \times ev_1)^*} \nearrow & & \searrow {\scriptstyle ev_{2!}} \\ \mathrm{Func}((\mathbf{Coh}(X) \times \mathbf{Coh}(X))(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell}) & & \mathrm{Func}(\mathbf{Coh}(X))(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell}) \end{array}$$

and define a convolution product by $f * g = ev_{2!}(ev_3 \times ev_1)^*(fg)$. The pullback is defined easily by precomposition. But the definition of $ev_{2!}$, as we will see in a moment, is not so straightforward. Also, we need to make sense of Func(—,—), because $\mathbf{Coh}^{ext}(X)(\mathbb{F}_{q^n})$ and $(\mathbf{Coh}(X) \times \mathbf{Coh}(X))(\mathbb{F}_{q^n})$ are groupoids (not discrete sets as in the case of schemes). A second problem is the following. In the case of schemes of finite type, we defined the "lower shriek" functor as $f_!(\phi)(y) := \sum_{x \in X(\mathbb{F}_{q^n}), f(x)=y} \phi(x)$. There everything made sense because the schemes were *of finite type* and therefore all sums were finite. But in the case of schemes (or stacks) not necessarily of finite type, like $\mathbf{Coh}(X)$ and $\mathbf{Coh}^{ext}(X)$, the above expression is an infinite sum, that could be divergent in $\overline{\mathbb{Q}_\ell}$.
In fact, there is no reason in general to suppose that the trace function is finitely supported. This problem will be addressed later. For now, let us suppose that our stacks are of finite type. In that setting, we can borrow from [Toe06] the following formula for the proper pushforward. Let $f : \mathcal{X} \to \mathcal{Y}$ be morphism of algebraic stacks of finite type over $\mathbb{F}_q$. Let $\alpha \in \mathrm{Func}(\pi_0 \mathcal{X}(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell})$ where Func is just "function of sets" ($\mathbb{F}_{q^n}$ is an ∞-groupoid). In [Toe06] a finite support condition is required, but an algebraic stack over a finite field has a finite number of $\mathbb{F}_{q^n}$-rational points. Let $y \in \pi_0 \mathcal{Y}(\mathbb{F}_{q^n})$, and let F_y be the homotopy fiber of y, with map $i : F_y \to X$ to X. Define $f_! \alpha \in \mathrm{Func}(\pi_0 \mathcal{Y}(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell})$ as

$$y \mapsto \sum_{x \in \pi_0(F_y)} \frac{\alpha(i(x))}{|\pi_1(F_y, x)|}.$$

Let us now state a crucial result proved by Sun Shenghao, which generalizes the trace formula to algebraic stacks:

Theorem 4.1 ([Sun12, Theorem 4.2, adapted])**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of $\mathbb{F}_q$-algebraic stacks of finite type, and let $\mathrm{K}_0 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{X}, \overline{\mathbb{Q}_\ell})$, K its pullback to the algebraic closure. Then*

$$c_n(\mathcal{X}) = c_n(\mathcal{Y}, Rf_! \mathrm{K}_0)$$

for every integer $n \geq 1$, where

$$c_n(\mathcal{X}, \mathrm{K}_0) = \sum_{x \in \pi_0(\mathcal{X}(\mathbb{F}_{q^n}))} \frac{1}{|\mathrm{Aut}_x \mathbb{F}_{q^n}|} \mathrm{Tr}(\mathrm{Fr}_x, K_{\overline{x}}).$$

Applying this in the case where $\mathcal{Y}$ is a point yields a generalized trace formula

$$\sum_{x \in \pi_0(\mathcal{X}(\mathbb{F}_{q^n}))} \frac{1}{|\mathrm{Aut}_x \mathbb{F}_{q^n}|} \mathrm{Tr}(\mathrm{Fr}_x, K_{\overline{x}}) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}, H^i_c(\mathrm{K}_0)).$$

This can be applied in the only argument of the proof of Proposition 2.2 that is not easily extendable to algebraic stacks of finite type: in fact, for $y \in \pi_0 \mathcal{Y}(\mathbb{F}_q)$,

$$f_! \chi_{\mathcal{F}}(y) = \sum_{x \in \pi_0(F_y), f(x)=y} \frac{\chi_{\mathcal{F}}(i(x))}{|\pi_1(F_y, x)|}$$

which can be replaced by

$$\sum_{x \in \pi_0(\mathcal{X}(\mathbb{F}_{q^n})), f(x)=y} \frac{1}{|\mathrm{Aut}_x \mathbb{F}_{q^n}|} \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}_x, \mathcal{H}^i_{\overline{x}}(\mathcal{F})).$$

By the trace formula this equals

$$\sum_i (-1)^i \sum_k (-1)^k \mathrm{Tr}(\mathrm{Fr}_{f^{-1}(y)}, H_c^k(\mathcal{H}^i(\mathcal{F})|_{\overline{f^{-1}(y)}})).$$

Then the proof goes on as above. The hypercohomology spectral sequence holds for the derived category of any abelian category, and thus we can apply it to our setting.

Recall from the previous section that, if our stacks are not assumed to be of finite type over $\mathbb{F}_q$, the proper pushforward of functions is not well defined. A possible solution to this problem is to define a "completion" of both sides of the correspondence, which reduces the situation to stacks of finite type. This is strongly related to a notion already considered by O. Schiffmann in his works.

Consider the category $qc(\mathbf{Coh}(X))$ having as objects all $U \subset \mathbf{Coh}(X)$ open quasicompact substacks of $\mathbf{Coh}(X)$, and the inclusions as morphisms. Consider the pro-object

$$A_1 : qc(\mathbf{Coh}(X)) \to \{\overline{\mathbb{Q}_\ell}\text{-vector spaces}\}$$

$$(u : U \to \mathbf{Coh}(X)) \mapsto \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U)) \otimes \overline{\mathbb{Q}_\ell}$$

$$(f : U \hookrightarrow V) \mapsto f^* : \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V)) \otimes \overline{\mathbb{Q}_\ell} \to \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U)) \otimes \overline{\mathbb{Q}_\ell}$$

This is a cofiltered diagram in $\mathbb{Q}_\ell$-vector spaces, because for any two $U \subset \mathbf{Coh}(X), V \subset \mathbf{Coh}(X)$, we have that $U \coprod_{\mathbf{Coh}(X)} V \to \mathbf{Coh}(X)$ is quasicompact, and any two morphisms $U \to V$ must coincide by definition of the category.

The algebra structure on the left is defined as follows: an algebraic stack $U^{ext} \to \mathbf{Coh}(X)$ is defined by the cartesian diagram

$$\begin{array}{ccc} U^{ext} & \longrightarrow & \mathbf{Coh}^{ext}(X) \\ \downarrow & & \downarrow \\ U^{\times 2} & \longrightarrow & \mathbf{Coh}(X)^{\times 2} \end{array}.$$

We can complete this diagram to the following

$$\begin{array}{ccccc} U^{ext} & \xrightarrow{u} & \mathbf{Coh}^{ext}(X) & \xrightarrow{ev_2} & \mathbf{Coh}(X) \\ \downarrow{\scriptstyle p_3 \times p_1} & & \downarrow{\scriptstyle ev_3 \times ev_1} & & \\ U \times U & \longrightarrow & \mathbf{Coh}(X) \times \mathbf{Coh}(X) & & \end{array}$$

and define, for any two $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U)$,

$$\mathcal{F} \star \mathcal{G} = r_! s^*(\mathcal{F} \boxtimes \mathcal{G}). \tag{3}$$

Recall from [PS19] that the right vertical map is of the form $SpecSym$, hence the preimage of the open quasicompact stack given by the image of $U \times U$ inside $\mathbf{Coh}(X) \times \mathbf{Coh}(X)$ is an open quasicompact substack of $\mathbf{Coh}^{ext}$. Moreover, the right horizontal map preserves quasicompact substacks by continuity.

Hence the composition $ev_2 \circ u$ factorizes through a quasicompact substack $V \subseteq \mathbf{Coh}(X)$, which can be embedded into an open quasicompact stack by taking a finite covering of V by

open quasicompact substacks as follows. Take $z : Z \to \mathbf{Coh}(X)$ a smooth atlas; write $z^{-1}(V)$ for the pullback of V to z. By definition of quasicompact substack, $z^{-1}(V)$ is a quasicompact scheme in Z. Hence it can be covered by a finite number of affine subschemes of Z, whose union forms an open scheme $Z' \subset Z$ containing $z^{-1}(V)$. Its projection to $\mathbf{Coh}(X)$ is open and compact (a smooth morphism is open and continuous), and contains V.
Thus, for any $U \in qc(\mathbf{Coh}(X))$ one has an induced "convolution map" $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V)$ for some V given again by (3).
Consider now the pro-object $\lim_{qc(\mathbf{Coh}(X))} A_1$.

Theorem 4.2. *The above structure makes* $\lim A_1$ *into a pro-algebra.*

Proof. The map $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V)$ induces a map $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \to \lim_W \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(W)$. This is because, for every open quasicompact substack $W \subset \mathbf{Coh}(X)$, one can take the quasicompact open substack $V \coprod_{\mathbf{Coh}(X)} W$ containing both W and V and observe that the triangle in the diagram

$$\begin{array}{ccc}
 & & \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V) \\
 & \nearrow & \uparrow \\
\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) & \longrightarrow & \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V \coprod_{\mathbf{Coh}(X)} V) \\
 & & \downarrow \\
 & & \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(W)
\end{array}$$

easily commutes. (The horizontal map is defined as in (3) but with V replaced by $V \coprod_{\mathbf{Coh}(X)} W$.)
This defines a map

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \to \lim A_1.$$

We want to lift this map to

$$\lim \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) = \lim A_1 \times A_1 \to \lim A_1,$$

and we do this by checking that for every $U, U' \in qc(\mathbf{Coh}(X))$ the maps induced by the projections, namely

$$\lim A_1 \times A_1 \to A_1(U) \times A^1(U) = \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \times \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U) \to \lim A_1$$

and

$$\lim A_1 \times A_1 \to A_1(U') = \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U') \to \lim A_1$$

coincide.

This is true by commutativity of the small triangles and squares in the diagram

$$\begin{array}{ccccc} & & \lim_W \mathrm{D}^b_c(W) \times \mathrm{D}^b_c(W) & & \\ & \swarrow & \downarrow & \searrow & \\ \mathrm{D}^b_c(U)^{\times 2} & \longleftarrow & \mathrm{D}^b_c(U \coprod_{\mathbf{Coh}(X)} U')^{\times 2} & \longrightarrow & \mathrm{D}^b_c(U')^{\times 2} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{D}^b_c(V) & \longleftarrow & \mathrm{D}^b_c(V \coprod_{\mathbf{Coh}(X)} V') & \longrightarrow & \mathrm{D}^b_c(V') \\ & & \vdots & & \\ & & \lim_W \mathrm{D}^b_c(W) & & \end{array}$$

which implies that the two vertical trapezes commute and there is a common induced dotted map towards the limit $\lim A_1$.
This defines a multiplication on $\lim A_1$. □

Now we turn to the "function" side. In analogy to the above construction, we define:

$$A_2 : qc(\mathbf{Coh}(X)) \to \{\overline{\mathbb{Q}_\ell}\text{-vector spaces}\}$$

$$(u : U \to \mathbf{Coh}(X)) \mapsto \prod_n \mathrm{Func}(\pi_0(U(\mathbb{F}_{q^n})), \overline{\mathbb{Q}_\ell})$$

$$j : U \to V \text{ over } \mathbf{Coh}(X) \mapsto j^* : \mathrm{Func}(\pi_0(V(\mathbb{F}_q)), \overline{\mathbb{Q}_\ell}) \to \mathrm{Func}(\pi_0(U(\mathbb{F}_q)), \overline{\mathbb{Q}_\ell}).$$

This is again a cofiltered diagram.

Also, we can define an algebra structure induced by

$$\begin{array}{ccccc} & & U^{ext}(X) & & \\ & \overset{s=ev_3\times ev_1}{\swarrow} & & \overset{r=ev_2}{\searrow} & \\ U \times U & & & & V. \end{array}$$

More precisely, if $f, g \in \mathrm{Fun}(U(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell})$ for some n, define

$$f \star g := r_! s^*(\mathcal{F} \boxtimes \mathcal{G}) \in \mathrm{Func}(\pi_0(V(\mathbb{F}_{q^n})), \overline{\mathbb{Q}_\ell}).$$

This induces a map

$$\prod_n \mathrm{Func}(\pi_0(U(\mathbb{F}_{q^n})), \overline{\mathbb{Q}_\ell}) \times \prod_n \mathrm{Func}(\pi_0(U(\mathbb{F}_{q^n})), \overline{\mathbb{Q}_\ell}) \to \prod_n \mathrm{Func}(\pi_0(V(\mathbb{F}_{q^n})), \overline{\mathbb{Q}_\ell}).$$

Again

Theorem 4.3. *The pro-object* $\lim A_2$ *is a pro-algebra when endowed with the structure described above.*

Proof. We can follow exactly the same proof of Theorem 4.2. □

The sheaves-functions correspondence χ defined in (2) (when $\mathcal{X} = \mathbf{Coh}(X)$) extends naturally to a morphism of pro-algebras $A_1 \to A_2$. To check this, one has to check that for every $f : U \to V$ inclusion of quasicompact substacks of $\mathbf{Coh}(X)$ the induced diagram

$$\begin{array}{ccc} \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(V)) \otimes \overline{\mathbb{Q}_\ell} & \longrightarrow & \mathrm{K}_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(U)) \otimes \overline{\mathbb{Q}_\ell} \\ \downarrow & & \downarrow \\ \prod_n \mathrm{Func}(\pi_0 V(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell}) & \longrightarrow & \prod_n \mathrm{Func}(\pi_0 U(\mathbb{F}_{q^n}), \overline{\mathbb{Q}_\ell}) \end{array}$$

commutes in Set.
But this amounts to say that χ for finite type stacks commutes with the pullback, which is true by previous observations. Hence

Theorem 4.4. *The sheaves-functions correspondence defines a morphism of pro-algebras*

$$\chi : A_1 \to A_2.$$

Question 4.5. The first open question that one may want to tackle is of course injectivity of this map. One would also want to study the two sides of this correspondence and relate them to the constructions made in Lecture 5 of [Sch12].

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