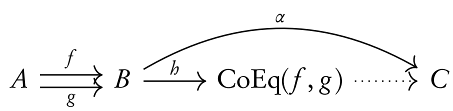
Homotopie II 2023-2024 - TD sheet A

Guglielmo Nocera

Exercise 1. Prove that the construction of $L_S\mathcal{C}$ below Definition 1.27 in prof. Horel's notes actually satisfies the universal property of a localization. Hint: prove and use that epimorphisms of categories are stable under cobase change. Note that epimorphisms "of categories" do not mean "epimorphisms in the category of categories".

Exercise 2 (by Victor Saunier)**.** For simplicity, all rings are commutative.

1. Let R be a commutative ring. Show that $A \otimes_C B \simeq A \coprod_C B$ in the category of R-algebras.
2. Let $f, g : A \to B$ be two morphisms of a category $\mathcal{C}$. The coequalizer of f, g is the map $h : B \to X$ given by the following universal property: for every $\alpha : B \to C$ such that $\alpha \circ f \simeq \alpha \circ g$, α factors through h:



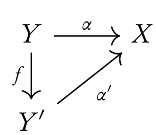
   (a) Compute the coequalizer of two morphisms in Set. Is it different from the pushout $B \coprod_A B$ of f and g?
   (b) Suppose $\mathcal{C}$ admits a zero object. Show the coequalizer of f and 0 is the cokernel of f.
   (c) Let $\mathcal{A}$ be an abelian category. Show that $\mathrm{CoEq}(f, g) \simeq \mathrm{coker}(f - g)$.

3. Let M be a commutative monoid, $t \in M$, and X a set with an action of M. We denote $\phi_t : X \to X$ the morphism $x \mapsto xt$.

   (a) Compute the colimit of the following diagram in the category of M-sets:
   $$X \xrightarrow{\phi_t} X \xrightarrow{\phi_t} X \xrightarrow{\phi_t} X \xrightarrow{\phi_t} \dots$$
   and show that it is canonically endowed with an action of $M[t^{-1}]$.
   (b) Show that the functor $M[t^{-1}]\text{-Set} \to M\text{-Set}$ which forgets the underlying action admits a left adjoint L, and then show that $L(X)$ is the above colimit endowed with its $M[t^{-1}]$-action.
   (c) Show that L is a localisation functor.

4. Show that a functor L admitting a right adjoint preserves colimits.

Exercise 3 (by Victor Saunier)**.** Let $X \in \mathcal{A}$ and (C, F, W) be a model structure on $\mathcal{A}$. We denote by $\mathcal{A}_{/X}$ the category whose objects are maps $\alpha : Y \to X$ of $\mathcal{A}$ and whose morphisms are commutative triangles



Similarly, we denote $C_{/X}$ (resp. $F_{/X}, W_{/X}$) the morphisms of $\mathcal{A}_{/X}$ as above where $f \in C$ (resp. F, W).

Show that $(C_{/X}, F_{/X}, W_{/X})$ determines a model structure on $\mathcal{A}_{/X}$. We call it the *slice model structure*.

What are the fibrant objects on the above described model structure? The cofibrant objects ?

Exercise 4 (Rezk's model structure on Cat)**.** Denote Cat the category of small categories. We assume that it is complete and cocomplete. We let W denote equivalences of categories and C denote functors that are injective on objects. We let F denote functors $H : \mathcal{A} \to \mathcal{B}$ such that for every isomorphism $g : F(a) \to b$ of $\mathcal{B}$, there is a map $f : a \to a'$ with $g = F(f)$; such functors are called *isofibrations*.

Denote $*$ the category with one object and no non-trivial arrows, and I the category with two objects $0, 1$ and exactly one isomorphism in each direction. Let $i : * \to I$ be the inclusion at 0. Show that $F = (\{i\})^{\perp}$.

Show that W verifies 2-out-of-3, and that W, C and F are stable under retracts.

1. (a) Show that every functor H of $C \cap W$ has a left inverse G which is also a quasi-inverse and such that the natural transformation $FG \simeq \mathrm{id}$ is equal to the identity on the image of F.

   (b) Deduce that $F \subset RLP(C \cap W)$.

2. Show that $C \subset^{\perp} (F \cap W)$.

3. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Denote $\mathrm{Path}(F) := \mathcal{A} \times_{\mathcal{B}} \mathcal{B}^I$ where the map $\mathcal{B}^I \to \mathcal{B}$ is the source map, and $\mathrm{Cyl}(F) := (\mathcal{A} \times I) \coprod_{\mathcal{A}} \mathcal{B}$, the functor $\mathcal{A} \to I$ being the functor sending everything to $\{1\}$. Show that F factors through $\mathrm{Path}(F)$ and $\mathrm{Cyl}(F)$; deduce that (C, F, W) is a model structure on Cat.

4. What are the fibrant objects, the cofibrant objects?