Homotopy II 2023-2024 - TD sheet B

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Exercise 1 Quillen adjunctions (by Victor Saunier)

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be two left Quillen functors.

1. Show that $G \circ F$ is a left Quillen functor.
2. Show that there is a natural equivalence $\mathbb{L}(G \circ F) \simeq (\mathbb{L}G) \circ (\mathbb{L}F)$.
3. Let (L, R) be an adjoint pair. Show that L is left Quillen if and only if R is right Quillen.
4. Suppose the restriction of a functor F to cofibrant objects preserves trivial cofibrations, show that F is left derivable. (Hint: Ken Brown's lemma).

Exercise 2 Slice model structure II (by Victor Saunier)

Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. Let $f : X \to Y$ be a morphism. Recall that we defined in the last exercise sheet a model structure on every slice category $\mathcal{A}_{/X}$.

1. Show that the functor $f_! : \mathcal{A}_{/X} \to \mathcal{A}_{/Y}$ which postcomposes by f admits a right adjoint f^* and describe it.
2. Show that the pair $(f_!, f^*)$ is a Quillen pair of adjoints.
3. Suppose $\mathcal{A}$ is right proper, i.e. weak equivalences are stable under pullback by fibrations, and that $f \in \mathcal{W}$. Show that the pair $(f_!, f^*)$ is a Quillen equivalence.
4. (Rezk) Suppose that for every weak equivalence f, the pair $(f_!, f^*)$ is a Quillen equivalence. Show that $\mathcal{A}$ is right proper.

Exercise 3 A Lifting Criterion (*HTT A.2.3*)

Let $\mathcal{C}$ be a model category, and denote $\overline{\cdot} : \mathcal{C} \to h\mathcal{C}$ the localisation functor. Suppose we have $i : A \to B$ a cofibration between cofibrant objects, and $f : A \to X$ a map with X fibrant. Suppose there is a commutative diagram in $h\mathcal{C}$

$$\begin{array}{ccc} A & & \\ {\scriptstyle \bar{i}}\downarrow & \searrow^{\bar{f}} & \\ & & X \\ & \nearrow_{h} & \\ B & & \end{array}$$

Show that there exists $g : B \to X$ such that $f = gi$ (in particular, we have $\overline{g} = h$).

Exercise 4 Monoidal model structures (*HTT A.3.1*)

Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ be three model categories and $F : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$ a functor. F is a *left Quillen bifunctor* if it preserves small colimits in each variable and for every cofibration $i : M \to M'$ in $\mathcal{M}$ and $j : N \to N'$ in $\mathcal{N}$, the induced map

$$i \wedge j : F(M, N') \coprod_{F(M,N)} F(M', N) \longrightarrow F(M', N')$$

is a cofibration, which is trivial as soon as either i or j is.

A monoidal model category is a closed[1] monoidal category $(\mathcal{S}, \otimes)$ equipped with a model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ with the following compatibility axioms:

- $\otimes$ is a left Quillen bifunctor
- The unit $\mathbb{1}$ is cofibrant

1. Show that the cartesian product of simplicial sets equipped with the Kan model structure is a left Quillen bifunctor. Deduce that the Kan model structure endows sSet with a monoidal model structure.
2. Let $(\mathcal{A}, \otimes)$ be a monoidal model category. Let $X \in \mathcal{A}$, show that $- \otimes X$ is a left Quillen functor.
3. Let $(\mathcal{A}, \otimes)$ be a monoidal model category.
   a) Show that the left derived tensor product $\otimes^L$ exists. (Hint: use Question 1.4)
   b) Show that $(h\mathcal{A}, \otimes^L)$ is a monoidal category.
   c) Show that the localization functor $\mathcal{A} \to h\mathcal{A}$ acquires a lax-monoidal structure which is strong monoidal on the restriction to cofibrant objects.
4. Show that the category of chain complexes with the usual tensor product:

$$(C_\bullet \otimes D_\bullet)_n := \bigoplus_{i+j=n} C_i \otimes D_j$$

   is a monoidal model category when equipped with the projective model structure.

(Schwede-Shipley) A monoidal model category $\mathcal{A}$ satisfies the *monoid axiom* if, for every acyclic cofibration j, the class of arrows generated by cobase change and transfinite composition by the $j \wedge \mathrm{id}_X$ for every $X \in \mathcal{A}$ is contained in $\mathcal{W}$, the class of weak equivalences.

5. Let $\mathcal{A}$ be a monoidal model category where every object is cofibrant. Show that $\mathcal{A}$ satisfies the monoid axiom.
6. Suppose $\mathcal{A}$ is cofibrantly generated, with $\mathcal{J}$ a set of generating acyclic cofibrations. Show that if for every $j \in \mathcal{J}$, the monoid axiom holds for j, then $\mathcal{A}$ satisfies the monoid axiom for every acyclic cofibration.

Exercise 5 Enriched model categories (by Victor Saunier)

Recall that a $\mathcal{V}$-enriched category $\mathcal{C}$ is tensored and cotensored over a closed monoidal $\mathcal{V}$, if there are two functors $\otimes : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$ and $[-,-] : \mathcal{V}^{op} \times \mathcal{C} \to \mathcal{V}$ such that there are natural equivalences:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X_1 \otimes V, X_2) \simeq \underline{\mathrm{Hom}}_{\mathcal{V}}(V, \underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, X_2))$$
$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, [V, X_2]) \simeq \underline{\mathrm{Hom}}_{\mathcal{V}}(V, \underline{\mathrm{Hom}}_{\mathcal{C}}(X_1, X_2))$$

of objects in $\mathcal{V}$.

1. Let $V \in \mathcal{V}$, show that $[V, -]$ is right adjoint to $V \otimes -$.
   Let $\mathcal{V}$ be a monoidal model category (see above). Let $\mathcal{A}$ be a $\mathcal{V}$-enriched category, cotensored and tensored over $\mathcal{V}$, and equipped with a model structure.
2. (HTT A.3.1.6) Show that the following propositions are equivalent:
   (i) $\otimes : \mathcal{C} \otimes \mathcal{V} \to \mathcal{C}$ is a left Quillen bifunctor
   (ii) For any cofibration $i : D \to D'$ and any fibration $j : X \to Y$ in $\mathcal{A}$, the induced

$$h : \underline{\mathrm{Hom}}_{\mathcal{C}}(C', X) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(C, X) \times_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,Y)} \mathrm{Hom}(C', Y)$$

   is a fibration in $\mathcal{V}$, trivial as soon as i or j is.
   (iii) For any cofibration $i : V \to V'$ in $\mathcal{V}$ and any fibration $j : X \to Y$ in $\mathcal{A}$, the induced

$$k : [V', X] \longrightarrow [V, X] \times_{[V,Y]} [V', Y]$$

   is a fibration in $\mathcal{A}$, trivial as soon as i or j is.

[1] I.e. the one variable tensor $- \otimes X$ has a right adjoint $\underline{\mathrm{Hom}}(X, -)$

A $\mathcal{A}$ satisfying the above is called a $\mathcal{V}$-enriched model category. In particular, every monoidal model category is enriched over itself.

3. Show that $h\mathcal{A}$ inherits a $\mathcal{V}$-enriched structure, such that $\mathcal{A} \to h\mathcal{A}$ is a $h\mathcal{V}$-enriched functor, and where the mapping objects are given by

$$\underline{\mathrm{Hom}}_{h\mathcal{A}}(X, Y) \simeq \overline{\underline{\mathrm{Hom}}_{\mathcal{A}}(X, Y)}$$