The brane action and string topology

Young Topologists Meeting

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Outline

Operads and string topology

Brane action: a first look

Brane action: behind the scene

Applications to brane topology

Operads and string topology

Invariants of topological spaces

Problem. Two topological spaces can have the same homology groups, without being weakly homotopy equivalent.

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$\rightsquigarrow$ Need the language of operads.

Operads

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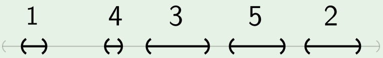
$\mathbb{E}_1$: associativity up to homotopy

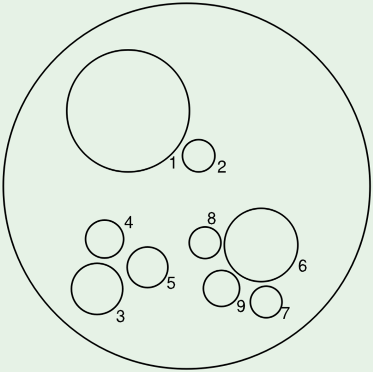
$\mathbb{E}_1(5) = \Big\{$ 

$\Big\}$

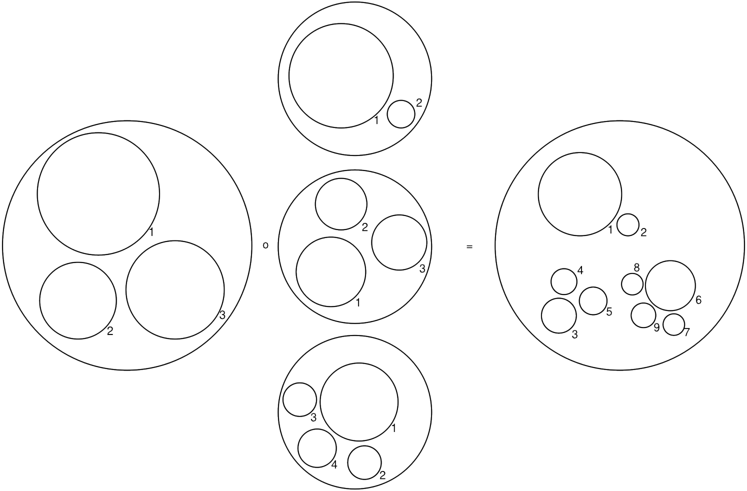
Little disks operad $\mathbb{E}_n$

$\mathbb{E}_n$: more commutativity as $n \to \infty$

$$\mathbb{E}_n(k) = \{\text{configurations of } k \text{ disjoint disks in } \mathbb{D}^n\}$$



Composition of little disks



String topology

For X a closed oriented manifold, the *free loop space of* X is

$$\mathcal{L}X = \mathrm{Map}(S^1, X).$$

Theorem (Chas–Sullivan)

The homology $H_*(\mathcal{L}X)$ is a BV*-algebra.*

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Remark

The operad $\mathbb{E}_2^{\mathrm{fr}}$ is *not reduced*, ie $\mathbb{E}_2^{\mathrm{fr}}(1) \simeq SO(2) \not\simeq *$.

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New approach

The previous structure comes from a general operadic phenomenon: the *brane action*.

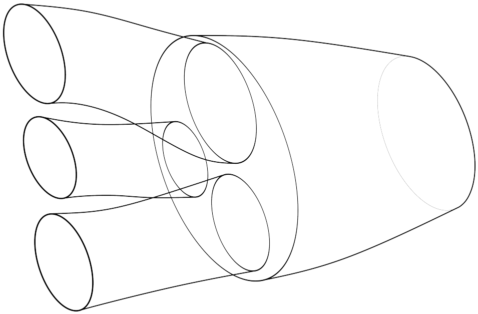
Brane action: a first look

The brane action for $\mathbb{E}_2$

The $\mathbb{E}_2$-structure in string topology comes from cobordisms

$$\amalg^k S^1 \longrightarrow \Sigma \longleftarrow S^1$$

parametrized by configurations of disks $\sigma \in \mathbb{E}_2(k)$.



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The span

$$(\mathcal{L}X)^k \xleftarrow{f} \mathrm{Map}(\Sigma, X) \xrightarrow{g} \mathcal{L}X$$

yields

$$g_* f^! \colon H_*(\mathcal{L}X)^{\otimes k} \longrightarrow H_{*-(k-1)d}(\mathcal{L}X).$$

Extensions

Operads $\rightsquigarrow$ *∞-operads*

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Definition (Extensions)

Let σ be an operation in $\mathcal{O}^{\otimes}$ of arity k. An *extension* of σ is an operation σ^{+} of arity $k+1$ that restricts to σ on the first k inputs:

$$\sigma^{+} \circ (\mathrm{id}, \ldots, \mathrm{id}, \iota) \simeq \sigma.$$

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Functoriality

Every composite $h\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ yields a cospan

$$\mathrm{Ext}(f) \xrightarrow{\text{in}} \mathrm{Ext}(h) \xleftarrow{\text{out}} \mathrm{Ext}(g)$$

Coherent ∞-operads

Definition

We say that $\mathcal{O}^{\otimes}$ is *coherent* if

- every unary operation is invertible, and
- for every composite $h \colon X \xrightarrow{f} Y \xrightarrow{g} Z$, the square

$$\begin{array}{ccc} \mathrm{Ext}(\mathrm{id}_Y) & \longrightarrow & \mathrm{Ext}(g) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(f) & \longrightarrow & \mathrm{Ext}(h). \end{array}$$

  is cocartesian.

The brane action

Theorem (Toën, 2013)

Let $\mathcal{O}^{\otimes}$ be a coherent reduced ∞-operad with unique color c. Then the space

$$\mathcal{O}(2) \simeq \mathsf{Ext}(\mathsf{id}_c)$$

is canonically an $\mathcal{O}$-algebra in $\mathsf{Cospan}(\mathcal{S})$*:*

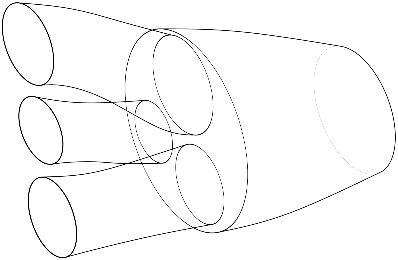
$$\sigma \quad \longmapsto \quad \mathsf{Ext}(\mathsf{id}_c)^{\amalg k} \xrightarrow{\text{in}} \mathsf{Ext}(\sigma) \xleftarrow{\text{out}} \mathsf{Ext}(\mathsf{id}_c)$$

Example 1: little disks

$\mathcal{O} = \mathbb{E}_n$ is coherent, with

$$\mathrm{Ext}(\sigma) \simeq \bigvee^{k} S^{n-1}.$$

$\rightsquigarrow$ cobordisms from before:



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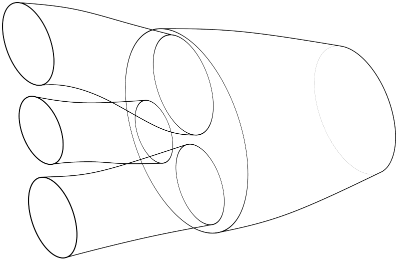

Framed little disks

$\mathcal{O} = \mathbb{E}_n^{\mathrm{fr}}$ is not reduced, so cannot apply Toën's result.

Example 2: Gromov–Witten invariants

X smooth projective variety over $\mathbb{C}$
$\overline{\mathcal{M}}_{g,n}$ moduli of stable curves of genus g with n marked points

GW invariants [Kontsevich–Manin]:

$$H^*(X) \text{ is an } H_*(\overline{\mathcal{M}}_{g,\cdot})\text{-algebra.}$$

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Theorem (Mann–Robalo, 2018)

X is a lax $\overline{\mathcal{M}}_{0,\cdot}$-algebra in spans of derived stacks:

$$\overline{\mathcal{M}}_{0,n+1} \times X^n \xleftarrow{p,\mathrm{ev}_{1,\ldots,n}} \overline{\mathcal{M}}_{0,n+1}(X) \xrightarrow{\mathrm{ev}_{n+1}} X$$

Brane action: behind the scene

Constructions of the brane action

- Toën's approach: relies on strictification arguments.
- Mann–Robalo's approach: more synthetic.

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Key idea: producing the brane action

$$\mathcal{O}^{\otimes} \longrightarrow \mathsf{Cospan}(\mathcal{S})^{\otimes}$$

is equivalent to constructing a certain right fibration

$$\pi\colon \mathcal{B}\mathcal{O} \longrightarrow \mathsf{Tw}(\mathsf{Env}(\mathcal{O}))^{\otimes}$$

with fibers $\mathcal{B}\mathcal{O}_\sigma \simeq \mathsf{Ext}(\sigma)$.

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Problem: [MR] gave a construction of $\mathcal{B}\mathcal{O}$ but incomplete proof.

Brane action: generalization

Theorem (P.)

Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad. Then the collection of spaces $\{\mathrm{Ext}(\mathrm{id}_X)\}_{X \in \mathcal{O}}$ carries a canonical $\mathcal{O}$-algebra structure in $\mathrm{Cospan}(\mathcal{S})$.

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New examples

- $\mathbb{E}_n^{\mathrm{fr}}$ framed little disks
- $\mathbb{E}_M$ for M a manifold
- More generally, ∞-operad $\mathbb{E}_B$ of B-framed little disks, for $B \to B\mathrm{Top}(n)$
- $\mathrm{SC}_{n,m}$ Swiss–Cheese ∞-operad

Computing spaces of extensions

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Definition (non-colored situation)

For $\sigma \in \mathcal{O}(n)$, define $\mathcal{E}\mathrm{xt}_\sigma$ as the pullback

$$\begin{array}{ccc} \mathcal{E}\mathrm{xt}_\sigma & \longrightarrow & \mathcal{O}(n+1) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{forget}=i^*} \\ * & \xrightarrow{\sigma} & \mathcal{O}(n). \end{array}$$

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Corollary

If $\mathcal{O}(1) \simeq *$, *then* $\mathcal{E}\mathrm{xt}_\sigma \simeq \mathrm{Ext}(\sigma)$ *for every* $\sigma \in \mathcal{O}(n)$.

This corollary is used

- by Lurie to prove coherence of the ∞-operad $\mathbb{E}_n$,
- by Mann–Robalo to compute the homotopy types of $\mathrm{Ext}(\sigma)$.

Applications to brane topology

Operations on spaces of branes

Corollary

Let $\mathcal{X}$ be an ∞-topos and $X \in \mathcal{X}$. Then the space

$$\mathrm{Map}(S^{n-1}, X)$$

of $\mathbb{E}_B$-branes internal to $\mathcal{X}$ has a canonical $\mathbb{E}_B$-algebra structure in $\mathrm{Span}(\mathcal{X})$*:*

$$\mathrm{Map}(S^{n-1}, X)^m \longleftarrow \mathrm{Map}(\mathrm{Ext}(\sigma), X) \longrightarrow \mathrm{Map}(S^{n-1}, X).$$

Inverting spans - algebraic geometry

Using the universal property of spans [Stefanich], we obtain:

Corollary (Toën, Ben-Zvi–Francis–Nadler, P.)

Let X be a perfect stack. Then

$$\mathsf{QCoh}(\mathsf{Map}(S^{n-1}, X))$$

carries a canonical $\mathbb{E}_B$-algebra structure in $\mathrm{dg}\mathcal{C}\mathrm{at}_k^{\mathrm{L}}$.

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(Work in progress)

Problem. Difficult to functoriality construct $\mathrm{in}^!$ for the non-locally compact space $\mathsf{Map}(S^{n-1}, X)$.

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Partial solution (towards Sullivan–Voronov conjecture). Considering the 6 functors formalism of local systems/parametrized spectra:

$$\rightsquigarrow \quad \mathbb{E}_n^{\mathrm{fr}}\text{-monoidal dg-category} \quad \mathrm{Loc}(X^{S^{n-1}}).$$

For X a Poincaré duality space, one can identify the endomorphism of the unit as

$$\mathrm{map}_{\mathrm{Loc}(X^{S^{n-1}})}(\mathbf{1}, \mathbf{1}) \simeq C_*(X^{S^n}, k)[-d].$$

which then inherits an $\mathbb{E}_n^{\mathrm{fr}} \otimes \mathbb{E}_1$-structure.

Some references

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