Pfister forms in the algebraic and geometric theory of quadratic forms

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Abstract. Nearly sixty years ago, Pfister defined what are now called Pfister forms in quadratic form theory. In addition to their remarkable intrinsic properties, Pfister forms are related to symbols in Galois cohomology and K-theory modulo 2, and are at the heart of the Milnor conjecture. In this paper, we intend to show their importance by means of three explicit examples. They also illustrate the evolution of quadratic form theory from its algebraic aspects to geometry of quadrics - more generally of varieties of isotropic subspaces of a given dimension - and their Grothendieck-Chow motives.

Mathematics Subject Classification (2010). 11E81, 14C15, 20G15.

1. Introduction

The study of quadratic forms in the last century can be divided into three different stages. First, in the context of number theory, mathematicians were interested in representation questions, such as which integral number can be written as a sum of a certain number of squares. In the 1930's, Witt started the so-called algebraic theory of quadratic forms, notably by defining the Witt ring, whose stucture is related to the behavior of quadratic forms over a given field. During the last thirty-five years, the topic has been completely renewed with the use of new methods, of a geometric nature. The aim is to study projective quadrics, as well as varieties of isotropic subspaces of a given dimension, and more precisely the algebraic correspondences on these projective varieties. These new tools have allowed significant progress on classical questions and have led to the use of Chow motives, not only in quadratic form theory, but also in the broader context of algebraic groups and related algebraic structures, such as hermitian forms and algebras with involution.

Among quadratic forms, the norm form of a quadratic étale algebra, a quaternion algebra, or an octonion algebra, are examples of dimension 2, 4 and 8, respectively, admitting what are called 'composition formulas'. In particular, the set of non-zero values represented by such a form is a subgroup of the multiplicative group of the base field, so that the form is called multiplicative. In [23], published in Archiv der Mathematik nearly sixty years ago, Pfister gave a concrete description of multiplicative quadratic forms, which exist in all 2-power dimensions, proving that they admit a specific diagonalization. He also proved that these forms, which were named Pfister forms by Elman and Lam in [6], have a remarkable splitting property: they are either anisotropic or hyperbolic. In this survey paper, we wish to illustrate the importance of Pfister forms in the theory of quadratic forms through three significant examples, which also illustrate the evolution of the algebraic theory to geometric and motivic aspects.

In the first part of the paper, we introduce some basic facts and recall the content of Pfister's paper [23]. Section 3 recalls the subform theorem, a classical and famous result in the algebraic theory of quadratic forms, as well as its geometric interpretation. Section 4 is on Milnor's conjecture, with particular attention to the relation between symbols in Milnor's K-theory and Pfister forms. Finally, section 5 describes the so-called Rost-motive, which played a crucial role in the proof of the Milnor conjecture, and pioneered the study of motives of projective homogeneous varieties. All the results are well documented, so most of the proofs are omitted; nevertheless, we have tried to emphasize the importance of Pfister's results, and notably the splitting property (Corollary 2.2 below), by explaining where and how they appear in the arguments.

2. Preliminaries and notation

Throughout this paper, we work over a base field F of characteristic different from 2. All quadratic forms are finite dimensional and non degenerate. Two quadratic forms q and q' over F are called similar if there exists $\lambda \in F^\times$ such that q and $\lambda q'$ are isomorphic. We use the notations $q \simeq q'$ for isomorphic forms; if $q \simeq \lambda q$, the scalar λ is called a similarity factor for q. When q has dimension at least 3, we denote by X_q the associated projective quadric and by $F(q) = F(X_q)$ the corresponding function field.

The quadratic form q is called isotropic if it represents 0 non-trivially, that is if there exists a non-trivial vector v in the underlying vector space V such that $q(v) = 0$, and anisotropic otherwise. It is called split if q is identically 0 over some subspace $W \subset V$ of dimension the integer part of $\dim(V)/2$. An even dimensional split form is also called hyperbolic. Witt's theorem asserts that a quadratic form q admits a decomposition

$$q = q_{\text{an}} + r\mathbb{H},$$

where q_{an} is an anistropic quadratic form called the anisotropic part of q, r is a nonnegative integer called the Witt index of q, $\mathbb{H}$ is a 2-dimensional hyperbolic form $\mathbb{H} \simeq \langle 1, -1\rangle$ and $+$ stands for direct orthogonal sum. The anisotropic part q_{an} of q is uniquely defined up to isomorphism, and q is split if and only if $q_{\rm an}$ is trivial or has dimension 1. Two quadratic forms q and q' are called Witt-equivalent if they have isomorphic anisotropic parts. The Witt ring of F is the set $W(F)$ of Witt-equivalence classes of quadratic forms over F, endowed with the laws induced by direct orthogonal sum and tensor product on quadratic forms.

Given $a \in F^\times$, the binary quadratic form $q = \langle 1, -a\rangle$ may be identified with the norm form of the quadratic étale algebra $K = F[t]/(t^2 - a)$. In particular, it admits a composition formula: for all v_1 and v_2 in the F-vector space $V = K$, there exists a vector v_3, namely $v_3 = v_1v_2$, such that $q(v_3) = q(v_1)q(v_2)$. It follows that for all field extensions L/F, the set of nonzero values

$$D(q_L) = \{\lambda \in L^\times,\ \exists v \in V_L,\ q_L(v) = \lambda\}$$

is a subgroup of $L^\times$. Moreover, since the product in K is F-bilinear, given two pairs of indeterminates $x = (x_1, x_2)$ and $y = (y_1, y_2)$, there exists some bilinear functions $z_1(x, y)$ and $z_2(x, y)$ such that

$$q_L(z_1, z_2) = q_L(x_1, x_2)q_L(y_1, y_2),$$

where $L = F(x_1, x_2, y_1, y_2)$.

Similar identities are available for the norm forms of quaternion and octonion algebras, providing examples of quadratic forms with composition formulae in dimension 4 and 8. To obtain examples of larger dimension, one must relax the assumption on the functions z_i, and forget the algebra structure on the underlying vector space.

Let q be an n-dimensional quadratic form over F, $x = \{x_1, \dots, x_n\}$ and $y = \{y_1, \dots, y_n\}$ two set of indeterminates, and denote by $F(x)$ and $F(x, y)$ the corresponding purely transcendental field extensions of F. The quadratic form q is called multiplicative if there exists rational functions $z_i \in F(x, y)$ such that

$$q_{F(x,y)}(z_1, \dots, z_n) = q_{F(x,y)}(x_1, \dots, x_n)q_{F(x,y)}(y_1, \dots, y_n).$$

If in addition the functions z_i can be chosen to be linear in the variables y_j, then q is called strongly multiplicative. One may easily check that q is strongly multiplicative if and only if $q_{F(x)}(x_1, \dots, x_n) \in F(x)^\times$ is a similarity factor for $q_{F(x)}$. Theorem 1 in Pfister's paper [23] states:

Theorem 2.1 (Pfister). *For all $a_1, \dots, a_n \in F^\times$, the quadratic form*

$$q \simeq \otimes_{i=1}^n \langle 1, -a_i\rangle$$

is strongly multiplicative.

Such a form is called an n-fold Pfister form. It has dimension 2^n and always represents 1. We use the following notation, departing here from standard textbooks such as [25] and [15], where the minus signs are omitted:

$$\langle\langle a_1, \dots, a_n\rangle\rangle = \otimes_{i=1}^n \langle 1, -a_i\rangle.$$

The reason for this convention will be clear in § 4, see also [9, 2.1.3].

Another important result in Pfister's paper is that an isotropic quadratic form is strongly multiplicative if and only if it is hyperbolic (see [23, Thm. 2(b)]). The next result follows immediately and plays an important role in the sequel:

Corollary 2.2 (Pfister). *Let q be an n-fold Pfister form. If q is isotropic, then it is hyperbolic.*

In particular, for any Pfister form q, the form $q_{F(q)}$ is hyperbolic, where $F(q)$ is $F(\sqrt{a})$ if q is the 1-fold Pfister form $\langle\langle a\rangle\rangle = \langle 1, -a\rangle$ and the function field of the projective quadric X_q otherwise. In the anisotropic case, combining Satz 3 and 5 and Theorems 1 and 2 in [23], we get the following, which provides a converse to Theorem 2.1:

Proposition 2.3 (Pfister). *Let q be a quadratic form over F. Assume q is anisotropic. Then the following are equivalent:*

1. *q is multiplicative.*
2. *q is strongly multiplicative.*
3. *For all field extensions L/F, the set $D(q_L)$ is a subgroup of $L^\times$.*
4. *q is an n-fold Pfister form.*

We recover the norm forms of quadratic étale algebras, quaternion algebras and octonion algebras as n-fold Pfister forms for $n = 1, 2$ and 3 respectively.

3. Representation and subform theorems

This section covers briefly very classical material, already explained in details in [25, Chap. 4] and [15, Chap. IX]. The starting point (Satz 1 in [23]), which is due to Cassels, deals with representation questions, in the setting of polynomials and rational functions. Let q be a quadratic form over F and $f \in F[t]$ a polynomial in one variable over F. If $q_{F(t)}$ represents f, then $q_{F[t]}$ also does. So, for instance, any polynomial in one variable that is a sum of squares of rational functions is also a sum of squares of polynomials. This is no longer true for polynomials in several variables, see [15, p. 302 and § XIII.5], except for homogeneous polynomials of degree 2, which can be considered as quadratic forms. Indeed, Pfister's Satz 3 in [23] is stated as follows:

Proposition 3.1 (Pfister). *Let q' be an n-dimensional quadratic form, $\{x_1, \dots, x_n\}$ a set of indeterminates, and $F(x)/F$ the corresponding purely transcendental field extension. Any anisotropic quadratic form q over F which represents $q'_{F(x)}(x_1, \dots, x_n)$ over $F(x)$ contains q' as a subform (over F).*

It follows that the polynomial $q'_{F(x)}(x_1, \dots, x_n)$ admits a representation in which the entries are not only polynomial, but linear functions in the x_i. As a consequence, Arason proved the following result, known as the subform theorem (see [1, Satz 1.3]):

Theorem 3.2 (Arason). *Let q and q' be two quadratic forms over F, with q anisotropic and q' of dimension at least 3. Assume that $q_{F(q')}$ is hyperbolic. Then the following holds:*

(a) *There exists* $\lambda \in F^\times$ *such that* $\lambda q'$ *is a subform of* q.
(b) *If in addition* q' *is a Pfister form, then there exists a quadratic form* q'' *such that* $q \simeq q' \otimes q''$.

In particular, the quadratic form q has dimension larger than the dimension of q'. The proof of (a) uses the previous proposition. To deduce (b), one may proceed as follows. Assume q' is a Pfister form and $q_{F(q')}$ is hyperbolic. Assertion (a) provides the conclusion if q and q' have the same dimension. Otherwise, it provides an additive decomposition $q = \lambda q' + \varphi$ for some quadratic form φ over F. Moreover, as we already explained, see Corollary 2.2, since q' is a Pfister form, $q'_{F(q')}$ is hyperbolic. Hence $\varphi_{F(q')}$ is hyperbolic and we conclude by induction.

This result, although considered as fundamental in the algebraic theory of quadratic forms, is already geometrical in nature: to a quadratic form q over F, besides the projective quadric X_q, one may associate the variety $\mathrm{Gr}(q)$ of totally isotropic subspaces of dimension the integer part of $\dim(V)/2$. By definition, the quadratic form q over F is split if and only if $\mathrm{Gr}(q)$ has a rational point. Therefore, when q is even-dimensional and q' is a Pfister form, the subform theorem gives a necessary and sufficient condition on q and q' for the variety $\mathrm{Gr}(q)$ to have a rational point over the function field of the quadric $X_{q'}$. Note that these varieties have an action of the special orthogonal groups of q and q', respectively; they are examples of projective homogeneous varieties. Thus, the subform theorem answers a particular case of a very general question : when does a projective homogeneous variety have a rational point over the function field of another projective homogeneous variety? This question is still largely open in general, with the notable exception of generalized Severi-Brauer varieties, for which the existence of a rational point over the function field of any projective homogeneous variety is given by the so-called index-reduction formulas [16, 17].

4. Filtration of the Witt ring and Milnor's conjecture

The dimension modulo 2 is an augmentation map for the Witt ring $W(F)$:

$$\overline{\dim} : W(F) \to \mathbb{Z}/2.$$

The augmentation ideal $IF = \ker(\overline{\dim})$ is called the fundamental ideal. It consists of Witt classes of even-dimensional quadratic forms, and its powers give a filtration of $W(F)$:

$$W(F) = I^0F \supset IF \supset I^2F \supset \cdots \supset I^nF \supset \ldots.$$

Since any binary form $\langle a, b\rangle$ is Witt equivalent to

$$\langle 1, a\rangle - \langle 1, -b\rangle \simeq \langle\langle -a\rangle\rangle - \langle\langle b\rangle\rangle,$$

the fundamental ideal IF is additively generated by 1-fold Pfister-forms and its nth power I^nF is additively generated by n-fold-Pfister forms. The next result, known as the Arason-Pfister Hauptsatz [2], shows that the filtration of $W(F)$ has the following nice property:

$$\bigcap_{n\in\mathbb{N}} I^nF = \{0\}.$$

Theorem 4.1 (Arason-Pfister). *Let q be an anisotropic quadratic form over F. If the Witt class of q belongs to the n-th power I^nF of the fundamental ideal of $W(F)$ for some integer $n \geq 1$, then q has dimension greater than or equal to 2^n.*

Proof. The result follows from the subform theorem (3.2) as follows. Since I^nF is additively generated by n-fold Pfister forms, we may write

$$q = \pi_1 \pm \pi_2 \pm \cdots \pm \pi_r,$$

where the equality holds in $W(F)$, for some integer $r \geq 1$ and some n-fold Pfister forms $\pi_1, \dots, \pi_r$. We proceed by induction on r. If $r = 1$, $q \simeq \pi_1$ is an anisotropic n-fold Pfister form, so it has dimension 2^n. If $r \geq 2$, consider the function field $K = F(\pi_r)$. If q_K is hyperbolic, then $q \simeq \pi_r \otimes q'$ for some quadratic form q' over F and the result follows. Otherwise, we have $q_K = (\pi_1)_K \pm \cdots \pm (\pi_{r-1})_K$, and we conclude by induction. □

This result motivates the study of the graded Witt ring defined by

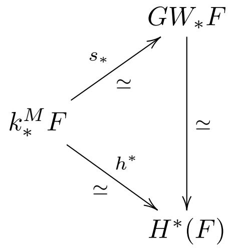
$$GW_*(F) = \bigoplus_{n\in\mathbb{N}} I^nF/I^{n+1}F.$$

In his famous paper [18], published in 1970, Milnor studies the relation between $GW_*(F)$ and two other graded rings, k_*^MF and $H^*(F)$, respectively provided by Milnor's K-theory modulo 2 and Galois cohomology with $\mathbb{Z}/2$ coefficients. There exists non-trivial homomorphisms

$$\begin{array}{ccc} & & GW_*F \\ & \overset{s_*}{\nearrow} & \\ k_*^MF & & \\ & \underset{h^*}{\searrow} & \\ & & H^*(F). \end{array}$$

The definition of both maps is explained in Milnor's paper; the second is due to Bass-Tate. The statement that they are isomorphisms is known as Milnor's conjecture. Milnor does not formally state this as a conjecture in his paper. But he asks in question 4.3 whether the first map is an isomorphism in every degree; and he writes in the introduction that "Section 6 describes the conjecture that k_*^MF is canonically isomorphic to $H^*(F)$". He also gives several examples of fields for which both maps are isomorphisms.

In 2003, Voevodsky [28] published a proof of the second statement, that is the map $h^* : k_*^MF \to H^*(F)$ is an isomorphism. Using Voevodsky's result, several authors provided proofs that the other map also is an isomorphism, or equivalently that we actually have a commutative triangle of isomorphisms



(see Orlov-Vishik-Voevodsky [21], Morel [19, 20] and Kahn-Sujatha [10]).

In the sequel, we focus on the relation between K-theory and the graded Witt ring for which, as we proceed to explain, Pfister forms and their properties played an important role. Recall K_1F is the multiplicative group $F^\times$, written additively. Its elements are denoted by $\{a\}$, for $a \in F^\times$; they satisfy $\{ab\} = \{a\} + \{b\}$. In particular, $\{1\} = 0$. Milnor's K ring $K_*^M F$ is the quotient of the tensor algebra

$$\oplus_{n\geq 0}(K_1F)^{\otimes n}$$

by the ideal generated by the elements $\{a\} \otimes \{1-a\}$, for all $a \in F^\times$, $a \neq 1$. The graded ring we are interested in is Milnor's K-theory modulo 2, defined by $k_*^M F = K_*^M F/2K_*^M F$. In other words, $k_*^M F$ is the associative ring with unit generated by the elements $\{a\}$, for all $a \in F^\times$, subject to the relations

1. $\{ab\} = \{a\} + \{b\}$ for all $a, b \in F^\times$
2. $\{a\}\{1-a\} = 0$ for all $a \in F^\times,\ a \neq 1$
3. $2\{a\} = 0$.

It satisfies $k_0^M F = \mathbb{Z}/2\mathbb{Z}$ and $k_1^M F \simeq F^\times/F^{\times 2}$.

We use the notation $\{a_1, \ldots, a_n\}$ for the class of the product

$$\{a_1, \ldots, a_n\} = \{a_1\}\{a_2\}\ldots\{a_n\} \in k_n^M F.$$

Such an element is called a symbol. For all $a \in F^\times$, $a \neq 1$, we have

$$\{a, -a\} = \{a, \frac{1-a}{1-a^{-1}}\} = \{a, 1-a\} - \{a, 1-a^{-1}\} = \{a(a^{-1})^2, 1-a^{-1}\} = 0.$$

Hence, $\{a, -a\} = 0 \in k_2^M F$ for all $a \in F^\times$.

We now prove the following:

Theorem 4.2 ([18])**.** *There is a homomorphism*

$$s_* : k_*F \to GW_*F,$$

mapping the symbol $\{a\}$ to the 1-fold Pfister form $\langle\langle a\rangle\rangle$. Moreover, the map s_* is surjective.

Proof. If such a homomorphism exists, it maps the n-symbol $\{a_1, \ldots, a_n\}$ to the n-fold Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$. Since I^nF is generated by n-fold Pfister forms, surjectivity is clear and we only have to prove that the map s_* is well defined. In view of the presentation of $k_*^M F$ by generators and relations, it is enough to check that the relations (1), (2) and (3) above also hold in the graded Witt ring. Consider $a, b \in F^\times$. If $a \neq 1$, the form

$$\langle\langle a, 1-a\rangle\rangle = \langle 1, -a, -(1-a), a(1-a)\rangle$$

is isotropic, hence hyperbolic by Corollary 2.2. So we have $\langle\langle a, 1-a\rangle\rangle = 0$ in the Witt ring $W(F)$. The other relations do not hold in the Witt ring, but they are valid in GW_*F. Indeed, one may easily check that

$$\langle\langle a\rangle\rangle + \langle\langle b\rangle\rangle = \langle\langle ab\rangle\rangle + \langle\langle a, b\rangle\rangle \in W(F).$$

Since $\langle\langle a, b\rangle\rangle \in I^2F$, we get $\langle\langle a\rangle\rangle + \langle\langle b\rangle\rangle = \langle\langle ab\rangle\rangle \in IF/I^2F$ as required. Similarly, one has $2\langle\langle a\rangle\rangle = \langle\langle a, -1\rangle\rangle \in I^2F$, so that $2\langle\langle a\rangle\rangle = 0 \in IF/I^2F$. □

In his paper, Milnor proves that s_1 and s_2 are bijective. In fact, using an ad-hoc version of Stiefel-Whitney invariant, he constructs a 'stable' section for s_n, which is a section if $n = 1, 2$. Using properties of Pfister forms, one can prove the following partial injectivity result, which says that s_n induces a bijection between n-symbols modulo 2 and n-fold Pfister forms modulo $I^{n+1}F$ (see [9, § 9.4]):

Proposition 4.3. *Restricted to symbols, the map s_n is injective.*

This result is of course much weaker than the injectivity of s_n. For further use, we prove here an even weaker version:

Lemma 4.4. *If the Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ belongs to $I^{n+1}F$, then the corresponding symbol $\{a_1, \dots, a_n\} \in k_n^M F$ is 0.*

Proof. By the Arason-Pfister Haupsatz mentioned above, every anisotropic form with Witt class in $I^{n+1}F$ has dimension at least 2^{n+1}. Hence, if the Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ belongs to $I^{n+1}F$, then it is hyperbolic. So we want to prove that $\{a_1, \dots, a_n\} = 0 \in k_n^M F$ if the form $\langle\langle a_1, \dots, a_n\rangle\rangle$ is hyperbolic. We proceed by induction on n, and we let $\pi_n = \langle\langle a_1, \dots, a_n\rangle\rangle$ for $n \geq 1$ and $\pi_0 = \langle 1\rangle$. If $n = 1$, and if the Pfister form $\pi_1 = \langle\langle a_1\rangle\rangle$ is hyperbolic then $a_1 \in F^{\times 2}$, so we get $\{a_1\} = 0 \in k_1^M F$. For $n \geq 2$, let us assume that π_n is hyperbolic. Since $\pi_n = \pi_{n-1} - \langle a_n\rangle \otimes \pi_{n-1}$, the hypothesis says that a_n is a similarity factor for π_{n-1}. The form π_{n-1} being a Pfister form, it represents 1, hence also all its similarity factors. Since $\pi_{n-1} = \pi_{n-2} - \langle a_{n-1}\rangle\pi_{n-2}$, we get two scalars x and y, represented by π_{n-2}, such that $a_n = x - a_{n-1}y$.

If $y = 0$, then $a_n = x$ is represented by π_{n-2}, so the Pfister form

$$\pi_{n-2} \otimes \langle\langle a_n\rangle\rangle = \pi_{n-2} \oplus \langle -a_n\rangle\pi_{n-2}$$

is isotropic, hence hyperbolic by Corollary 2.2. By the induction hypothesis we get $\{a_1, \dots, a_{n-2}, a_n\} = 0 \in k_{n-1}^M F$ which implies the conclusion. If x is 0, the same argument shows that $\{a_1, \dots, a_{n-2}, -a_{n-1}a_n\} = 0 \in k_{n-1}^M F$. Multiplying on the right by $\{a_n\}$, and using the fact that $\{-a_n, a_n\} = 0$ in $k_2^M F$, we get

$$\{a_1, \dots, a_{n-2}, -a_{n-1}a_n, a_n\} = \{a_1, \dots, a_n\} = 0 \in k_n^M F.$$

In general, one may write $a_n = x - a_{n-1}y = x(1 - a_{n-1}z)$, where $z = y/x$ also is represented by π_{n-2} by Proposition 2.3. So we have

$$\{a_1, \ldots, a_n\} = \{a_1, \ldots, a_{n-1}, x\} + \{a_1, \ldots, a_{n-1}, 1 - a_{n-1}z\}.$$

The same argument as before shows that the symbols

$$\{a_1, \ldots, a_{n-2}, x\} \text{ and } \{a_1, \ldots, a_{n-2}, z\}$$

are both trivial in $k_n^M F$. In particular, we have

$$\{a_1, \ldots, a_{n-1}\} = \{a_1, \ldots, a_{n-1}z\},$$

so that $\{a_1, \ldots, a_{n-1}, 1 - a_{n-1}z\} = \{a_1, \ldots, a_{n-1}z, 1 - a_{n-1}z\} = 0 \in k_n^M F$. We finally get $\{a_1, \ldots, a_n\} = 0 \in k_n^M F$, as expected. □

5. Norm quadrics and Rost motives

Throughout this section, we let π_{n-1} and π_n be the Pfister forms

$$\pi_{n-1} = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \text{ and } \pi_n = \langle\langle a_1, \ldots, a_n\rangle\rangle,$$

for some $a_1, \ldots, a_n$ in $F^\times$. The quadric X_φ associated with the quadratic form $\varphi = \pi_{n-1} + \langle -a_n\rangle$ is called a norm quadric. It has the following nice property:

Proposition 5.1 (Norm quadric). *Let $a_1, \ldots, a_n$ be non-zero elements of F. The following assertions are equivalent:*

1. *The quadratic form $\varphi = \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle + \langle -a_n\rangle$ is isotropic.*
2. *The Pfister form $\pi_n = \langle\langle a_1, \ldots, a_n\rangle\rangle$ is isotropic.*
3. *The Pfister form $\pi_n = \langle\langle a_1, \ldots, a_n\rangle\rangle$ is hyperbolic.*
4. *The symbol $\{a_1, \ldots, a_n\} \in k_n^M F$ is 0.*

Proof. Clearly, (1) implies (2) and (2) implies (3) by Corollary 2.2. Assume now that (3) holds. Then, the form φ, which is a subform of π_n of dimension strictly larger than half the dimension of π_n has to be isotropic. The equivalence between (3) and (4) is given by Theorem 4.2 and Lemma 4.4. □

Rephrasing the previous proposition in terms of the norm quadric, we get that the symbol $\{a_1, \ldots, a_n\}$ vanishes in Milnor's K-theory modulo 2 if and only if X_φ has a rational point. We say that X_φ is a splitting variety for the symbol $\{a_1, \ldots, a_n\} \in k_n^M F$, and the function field $F(X_\varphi)$ a generic splitting field for this symbol.

The Rost motive is a summand of the motive of a norm quadric, first introduced by Rost in [24], and which proved to be an important tool in Voevodsky's proof of the Milnor conjecture, see [28, Thm. 4.3], [8, § 8.1]. As we explain in this section, the definition of the Rost motive uses Pfister's forms and their splitting property.

The motives we are talking about here are Grothendieck Chow motives, as described for instance in [5, Chap XII]. Recall that there is a functor denoted by M from the category of smooth projective varieties over F to the category $\mathrm{Mot}_F(\mathbb{Z})$ of Chow motives over F. Given two such varieties X and Y, with X integral of dimension d, the set of homomorphisms between $M(X)$ and $M(Y)$ is equal to the Chow group with $\mathbb{Z}$-coefficients $\mathrm{CH}_d(X\times Y)$. Elements in this group are also called correspondances and composition is defined as in [7, § 16.1]. One of the benefits of moving from varieties to motives is that the target category is pseudo-abelian, providing direct sum decompositions for objects. In the case of quadrics, by a theorem of Vishik [26, Cor. 3.4], we even have a Krull-Schmidt property, so that the motive of a quadric decomposes uniquely into a direct sum of indecomposable motives.

In [24], Rost studies motivic decompositions for quadrics. He proves the following results. Assume first $q=\mathbb{H}\oplus q_1$ is an isotropic quadratic form, and consider the corresponding quadrics $X=X_q$ and $X_1=X_{q_1}$. By [24, Prop. 2], the motive of the quadric X decomposes as follows:

$$M(X)=\mathbb{Z}\{0\}\oplus M(X_1)\{1\}\oplus\mathbb{Z}\{d\},$$

where $\mathbb{Z}\{0\}$ denotes the Tate motive $M(\operatorname{Spec}F)$, $\{i\}$ is the shift operation, for $i\in\mathbb{Z}$, and d is the dimension of X. One remarkable feature in this decomposition is that the middle term is of geometric nature: it is a shift of the motive of a quadric. Surprisingly, a similar decomposition holds for a norm quadric in the anisotropic case, see [24, Thm 17]. More precisely, consider as above the quadratic form $\varphi=\pi_{n-1}+\langle -a_n\rangle$, and denote by π'_{n-1} the pure subform of π_{n-1} defined by $\pi_{n-1}=\langle 1\rangle+\pi'_{n-1}$. There exists a motive R in $\mathrm{Mot}_F(\mathbb{Z})$ and a decomposition

$$M(X_\varphi)=R+M(X_{\pi'_{n-1}})\{1\},$$

with $R_K=\mathbb{Z}\{0\}\oplus\mathbb{Z}\{d\}$ in $\mathrm{Mot}_K(\mathbb{Z})$ for all field extensions K/F such that φ_K is isotropic. This motive R is called the Rost motive of the norm quadric X_φ. Rost also proves that the motive of the Pfister quadric X_{π_n} decomposes as a sum of shifts of the Rost motive:

$$M(X_{\pi_n})=\bigoplus_{0\leq i\leq d} R\{i\},$$

where $d=2^{n-1}-1$ is the dimension of X_φ, and X_{π_n} has dimension $2d$.

Quadrics are examples of projective homogeneous varieties for special orthogonal groups, and direct sum decompositions were studied for the motives of such varieties in a broader context, for semi-simple algebraic groups of arbitrary type. One needs to replace the ring of coefficients $\mathbb{Z}$ by some finite ring $\mathbb{Z}/p\mathbb{Z}$ so that the Krull-Schmidt property holds for these motives, see [3]. Once this is done, the anisotropic case remains significantly more difficult than the isotropic case, and Rost's observations were pioneering in this setting. They were later extended in various ways. In particular, Petrov, Semenov and Zainoulline proved that if the projective homogeneous G-variety X has a function field which splits G, then its motive is a sum of shifts of a given indecomposable motive, which is the Rost motive in the case of a Pfister quadric, see [22, Main Theorem]. Using this, they were able to extend Vishik's J-invariant for quadrics [27, Def. 5.11], [5, § 88] to algebraic groups of arbitrary types and related algebraic structures.

We now briefly sketch a construction of the Rost motive, which is due to Karpenko [11]. Let us denote by X the norm quadric X_φ, of dimension $d = 2^{n-1} - 1$. The Pfister quadric $Y = X_{\pi_{n-1}}$ is a closed subvariety of codimension 1 in X. Let $\bar{F}$ be an algebraic closure of F, and denote by $\bar{Y}$ and $\bar{X}$ the split quadrics $Y_{\bar{F}}$ and $X_{\bar{F}}$, respectively. Let p be an $\bar{F}$-point of $\bar{X}$. The cycles $[p \times \bar{X}]$ and $[\bar{X} \times p]$ are elements of $\mathrm{CH}_d(\bar{X} \times \bar{X})$, hence endomorphisms of $M(\bar{X})$. One may easily check that they are orthogonal projectors, and the corresponding summands of $M(\bar{X})$ are isomorphic to the Tate motives $\mathbb{Z}\{0\}$ and $\mathbb{Z}\{d\}$, respectively. The next statement shows that the motive $M(X)$ of the norm quadric has a summand $\big(M(X), r\big)$, which precisely is the Rost motive:

Theorem 5.2. [24][11] *There exists a projector r in $\mathrm{CH}_d(X \times X)$ such that*

$$r_{\bar{F}} = [p \times \bar{X}] + [\bar{X} \times p].$$

Sketch of Proof. The proof of the theorem has three main steps. The first important ingredient is the Rost nilpotence theorem. Using it, one may prove it suffices to check that the cycle $[p \times \bar{X}] + [\bar{X} \times p]$ is rational, that is belongs to the image of $\mathrm{CH}(X \times X) \to \mathrm{CH}(\bar{X} \times \bar{X})$. Indeed, if a correspondance $r \in \mathrm{CH}(X \times X)$ satisfies the required equality, it is a projector after scalar extension to $\bar{F}$. Hence, replacing r by a certain power, we may assume it is a projector already over F and this proves the theorem (see Lemma 2 in [11]).

The second step uses the embedding of the Pfister quadric Y in X. Using explicit descriptions of the Chow groups of $\bar{Y} \times \bar{Y}$ and $\bar{X} \times \bar{X}$, which are products of split quadrics, together with a clever cycle computation, Karpenko shows it suffices to prove that the cycle

$$[\bar{Y}] \otimes \ell_m + \ell_m \otimes [\bar{Y}] \in \mathrm{CH}_{3m}(\bar{Y} \times \bar{Y}) = \mathrm{CH}^m(\bar{Y} \times \bar{Y})$$

is rational, where $m = 2^{n-2} - 1$ so that Y has dimension $2m$, and ℓ_m in $\mathrm{CH}_m(\bar{Y})$ is given by the class of a maximal totally isotropic subspace for the quadratic form $(\pi_{n-1})_{\bar{F}}$.

Let $\eta : \operatorname{Spec} F(Y) \to Y$ be the generic point of Y. By [5, Cor. 57.11], the pull-back of $\mathrm{id} \times \eta$ induces a surjective map $\mathrm{CH}^m(Y \times Y) \to \mathrm{CH}^m(Y_{F(Y)})$, and the same holds after scalar extension to $\bar{F}$. Combined with restriction maps, we get the following commutative diagram, used for the final step of the proof:

$$\begin{array}{ccc} \mathrm{CH}^m(\bar{Y} \times \bar{Y}) & \xrightarrow{(\mathrm{id} \times \eta)^*} & \mathrm{CH}^m(\bar{Y}_{\bar{F}(\bar{Y})}) \\ {\scriptstyle \mathrm{res}_{\bar{F}/F}} \uparrow & & \uparrow {\scriptstyle \mathrm{res}_{\bar{F}(\bar{Y})/F(Y)}} \\ \mathrm{CH}^m(Y \times Y) & \xrightarrow{(\mathrm{id} \times \eta)^*} & \mathrm{CH}^m(Y_{F(Y)}) \end{array}$$

Since the Pfister form π_{n-1} is hyperbolic over its function field by Corollary 2.2, the restriction map $\mathrm{res}_{\bar{F}(\bar{Y})/F(Y)}$ is an isomorphism. Therefore, by surjectivity of $(\mathrm{id} \times \eta)^*$, there exists a cycle $\alpha \in \mathrm{CH}^m(Y \times Y)$ with image $\mathrm{res}_{\bar{F}(\bar{Y})/\bar{F}}\, \ell_m$ in $\mathrm{CH}^m\big(\bar{Y}_{\bar{F}(\bar{Y})}\big)$. As explained in [11], see also [5, § 68], the Chow group $\mathrm{CH}(\bar{Y})$ is free with basis $\{\ell_i, h^i,\ i \in [0, m]\}$ where h denotes the class of a hyperplane section of $\bar{Y}$ and ℓ_i the class of a linear subspace of dimension i in $\bar{Y}$, with h^i rational and $2\ell_i$ rational for all i. This produces a basis of $\mathrm{CH}(\bar{Y} \times \bar{Y})$ whose elements of codimension m, hence dimension $3m$, are all rational, except for $\ell_m \otimes [\bar{Y}]$ and $[\bar{Y}] \otimes \ell_m$. Since the first one maps to $\mathrm{res}_{\bar{F}(\bar{Y})/\bar{F}}(\ell_m)$ in $\mathrm{CH}^m(\bar{Y}_{\bar{F}(\bar{Y})})$ while the second maps to 0, we get that $\mathrm{res}_{\bar{F}/F}(\alpha)$ is equal either to $\ell_m \otimes [\bar{Y}]$ or to $\ell_m \otimes [\bar{Y}] + [\bar{Y}] \otimes \ell_m$ up to a rational cycle, so that one of them is rational. Finally, if $\ell_m \otimes [\bar{Y}]$ is rational, so is its transpose $[\bar{Y}] \otimes \ell_m$, and we get that $\ell_m \otimes [\bar{Y}] + [\bar{Y}] \otimes \ell_m$ is rational in both cases, as expected. □

As we have explained, if the norm quadric X_φ and the Pfister quadric X_{π_n} are anisotropic, then the Rost motive R is an indecomposable summand of the motives $M(X_\varphi)$ and $M(X_{\pi_n})$. Moreover, for all K/F such that φ_K and $(\pi_n)_K$ are isotropic, we have $R_K \simeq \mathbb{Z}\{0\} + \mathbb{Z}\{d\}$, where $d = 2^{n-1} - 1$. Therefore, R is the upper motive of X_φ and X_{π_n}, in the sense of Karpenko [12, Def. 2.10]. Again, Rost's results maybe viewed as the starting point of the theory of upper motives, which was developed mainly by Karpenko. Replacing the ring of coefficients $\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ so that the Krull-Schmidt property holds, Karpenko proved that any indecomposable summand of the motive of a projective homogeneous variety under an algebraic group of inner type is isomorphic to a shift of the upper motive of a (possibly different) projective homogeneous variety, see [12, Thm. 3.5]. In [13, Thm. 1.1], he proved a more general version, which concerns projective homogeneous varieties under p-inner algebraic groups, that is groups that become of inner type after some scalar extension of p-power degree. Using this, a complete classification of Chow motives of projective homogeneous varieties under groups of p-inner types was recently produced, using the Tate trace, that is the maximal summand of such a motive which is isomorphic to a sum of Tate motives, see [4].

Another strong motivation for studying upper motives is provided by their relation with the canonical dimension. Given a projective variety Y, its canonical dimension is the smallest possible value for the dimension of the image of a rational map $Y \dashrightarrow Y$. Recall that the variety Y is called incompressible if its canonical dimension is equal to its dimension, that is if any rational map $Y \dashrightarrow Y$ is dominant. Thus, in general, the canonical dimension of Y measures its level of compressibility. Consider now a projective homogeneous variety X over F. As explained in [14, Thm. 5.1], its canonical dimension is determined by its upper motive U_X. More precisely, it coincides with the dimension of U_X, which is the largest shift of a Tate summand of U_X after scalar extension to an algebraic closure of the base field. For the norm quadric X_φ and the Pfister quadric X_{π_n}, it reads as follows: they have the same upper motive, namely the Rost motive R, which satisfies

$$R_{\bar{F}} \simeq \mathbb{Z}\{0\} + \mathbb{Z}\{d\},$$

where $\bar{F}$ is an algebraic closure of F, and d is the dimension of X_φ and half the dimension of X_{π_n}. Consequently, the upper motive R has dimension d, the norm quadric is incompressible, and the Pfister quadric X_{π_n} has canonical dimension d. The interested reader will find more details on these topics in Karpenko's paper [14], where the importance of studying upper motives and their relationship with the canonical dimension is explained.

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