Tits Algebras

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Those are preparation notes for a serie of two talks given in Lausanne in July 2005. They are quite informal and not intended for publication.

Part I

1. Introduction and examples of Tits algebras

The main purpose of these talks is to make some advertisement for Tits paper Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque, *J. Reine Angew. Math.* **247** (1971), 196-220.
We won't give any proof. We will only try and describe the situation and study some examples. Proofs and details may be found in Tits original paper as well as in [**MPW98**, §2] and [**KMRT98**, §27].
As the title of the paper shows, Tits algebras appear in the study of representations of algebraic groups.
Let G be a semi-simple algebraic group over an arbitrary base field F.

Definition 1.1. — A representation of G is a morphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$ for some vector space V over F.
It is said to be irreducible if V does not contain any non trivial G-submodule.

If G is split (that is G contains a split maximal torus defined over F), irreducible representations of G are classified and this classification does not depend on the base field (see section 6.3).
Assume now that G is non split. Denote by F_{sep} a separable closure of F. The group $G_{sep} := G_{F_{sep}}$ is split and irreducible representations of G_{sep} are classified.
Consider any such representation $\rho_s : G_{sep} \to \mathrm{GL}(V)$ for some vector space V over F_{sep}, and assume it is invariant under the action of the Galois group $\Gamma_F = \mathrm{Gal}(F_{sep}/F)$ up to isomorphism. Then, Tits proves it admits a descent to the base field, but this descent need not in general be a representation in the usual sense. It is what we call here an *algebra representation*, ie a morphism of algebraic groups $\rho : G \to \mathrm{GL}_1(A)$ for some central simple algebra A over F, which is called a Tits algebra for the group G.
The fact ρ is a descent of ρ_s means that there exists an isomorphism $A \otimes_F F_{sep} \simeq \mathrm{End}_{F_{sep}}(V)$ such that after scalar extension to F_{sep}, the representation ρ gives rise to a morphism $\rho_{F_{sep}} : G_{sep} \to \mathrm{GL}_1(A \otimes_F F_{sep}) \simeq \mathrm{GL}_1(\mathrm{End}_{F_{sep}}(V)) = \mathrm{GL}(V)$ which is ρ_s.

Example 1.2. — Consider a quadratic space (V, q) over a field F of characteristic different from 2.
We denote by $\mathcal{C}(V, q)$ the Clifford algebra of (V, q). It can be defined as the quotient of the tensor algebra $T(V) = \oplus_{i \geq 0} V^{\otimes i}$ by the ideal generated by $v \otimes v - q(v)$ for any vector $v \in V$. The $\mathbb{Z}$-grading of $T(V)$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{C}(V, q)$ and we let $\mathcal{C}_0(V, q)$ be the even part.
The structure of the Clifford algebra is well known and depends on the parity of the dimension of V and the value of the signed discriminant $d(q) \in F^\times/F^{\times 2}$ of the quadratic form (see [**Sch85**, chap.9, thm 2.10]). In particular, if $\dim(V)$ is odd, then the even Clifford algebra $\mathcal{C}_0(V, q)$ is a central simple algebra over F. If now $\dim(V)$ is even, the center of $\mathcal{C}_0(V, q)$ is $F[X]/(X^2 - d(q))$. When $d(q)$ is trivial, $\mathcal{C}_0(V, q)$ splits into a direct product of two central simple algebras over F, $\mathcal{C}_0(V, q) = \mathcal{C}_+ \times \mathcal{C}_-$.
Clearly, there is a natural embedding $V \subset \mathcal{C}(V, q)$. The quadratic form on V extends to a norm on $\mathcal{C}(V, q)$, which we denote by N. For any $v_1, \dots, v_r \in V$, the norm of the image in $\mathcal{C}(V, q)$ of $v_1 \otimes \cdots \otimes v_r$ is $q(v_1) \dots q(v_r)$.
Consider now any invertible element s in $\mathcal{C}_0(V, q)^\times$. If $sVs^{-1} \subset V$, it can be shown that the map $V \to V$ given by $v \mapsto svs^{-1}$ actually is in the group $SO(V, q)$ of special isometries of (V, q). We then define the Spin group as follows:

$$\mathrm{Spin}(V, q) := \{s \in \mathcal{C}_0(V, q)^\times,\ sVs^{-1} \subset V \text{ and } N(s) = 1\}$$

There is an exact sequence of algebraic groups $1 \to \mu_2 \to \mathrm{Spin}(V, q) \to SO(V, q) \to 1$, in which the map $\mathrm{Spin}(V, q) \to SO(V, q) \subset \mathrm{GL}(V)$ is given by $s \mapsto (v \mapsto svs^{-1})$. This map is a representation of $\mathrm{Spin}(V, q)$, called the vector representation.
We also have, from the definition of the Spin group, a canonical embedding $\mathrm{Spin}(V, q) \mapsto \mathrm{GL}_1(\mathcal{C}_0(V, q))$. If $\dim(V)$ is odd, this map is an algebra representation of the Spin group, and $\mathcal{C}_0(V, q)$ is a Tits algebra for $\mathrm{Spin}(V, q)$. In the split case, that is when V contains a totally isotropic subspace of dimension $[\frac{\dim(V)}{2}]$, this map is the so-called spinor representation.
Assume now that $\dim(V)$ is even and $d(q)$ is trivial. We get two natural maps $\mathrm{Spin}(V, q) \to \mathrm{GL}_1(\mathcal{C}_0(V, q)) \to \mathrm{GL}_1(\mathcal{C}_\pm(V, q))$ which are algebra representations of the Spin group. The algebras $C_+(V, q)$ and $\mathcal{C}_-(V, q)$ are Tits algebras for $\mathrm{Spin}(V, q)$. In the split case (that is when (V, q) is hyperbolic), these maps are known as half-spin representations.

Example 1.3. — Let us go a little bit further is the non split direction, and consider now a central simple algebra A over F endowed with an orthogonal involution σ.
If the degree of A is odd, then the algebra A is split, $A = M_n(F)$, and the involution is given by $X \mapsto B^{-1}X^tB$, where B is a symmetric matrix, ie the matrix of a quadratic form. The Spin group in that case is the Spin group of the underlying quadratic space, and there is nothing more than in the previous example.

Asume now that the degree of the algebra is even. We can associate to (A,σ) an orthogonal group, a Special orthogonal group, an even Clifford algebra and a Spin group as we did before for quadratic spaces (see for instance [**KMRT98**]). In particular, we define $O(A,\sigma)=\{a\in A,\ \sigma(a)a=1\}$, and $SO(A,\sigma)=\{a\in O(A,\sigma),\ \mathrm{Nrd}_A(a)=1\}$. The group $\mathrm{Spin}(A,\sigma)$ is the corresponding simply connected cover. It satisfies

$$1\to\mu_2\to\mathrm{Spin}(A,\sigma)\to SO(A,\sigma)\to 1.$$

Hence the 'vector representation' for the group $\mathrm{Spin}(A,\sigma)$ now is a map

$$\mathrm{Spin}(A,\sigma)\to SO(A,\sigma)\subset \mathrm{GL}_1(A).$$

The algebra A itself is a Tits algebra for the group $\mathrm{Spin}(A,\sigma)$.

In the second talk, we will see that using the Brauer classes of Tits algebras, one can define a morphism $\alpha:(\Lambda/\Lambda_r)^{\Gamma_F}\to\mathrm{Br}(F)$, where Λ and Λ_r are the weight and root lattices of G (see sections 5 and 6.2 for a definition of Λ/Λ_r and section 7.2 for a definition of α). This quotient $C=\Lambda/\Lambda_r$ is a finite group, which Tits calls the cocenter of the group G. In particular, this implies that there are only finitely many possibilities for the Brauer class of a Tits algebra of a given algebraic group. For Spin group, we have described all of them. We will come back to this example at the end of the second lecture, and see how some well known relations on Brauer classes of Clifford algebras follow at once from some obvious relations in the cocenter of the Spin group, using this morphism α.
Before that, we give some motivation for studying Tits algebras by presenting two important and recent papers in which they are used.

2. Motivation 1 : Index Reduction Formulas

2.1. Basic facts on central simple algebras. — All the algebras we consider are supposed to be finite dimensional. Recall an F-algebra A is called central simple if the center of A is F and A admits no non-trivial two-sided ideal.

Example 2.1. — Split algebra : $A=\mathrm{End}_F(V)=M_n(F)$
Quaternion algebra : $(a,b)_F=F\oplus Fi\oplus Fj\oplus Fij$, with $i^2=a$, $j^2=b$ and $ij=-ji$.

The structure of central simple algebras is described in the following theorem, essentially due to Wedderburn :

Theorem 2.2. — *The following are equivalent :*
(i) A is central simple over F;
(ii) $A_{F_{sep}}:=A\otimes_F F_{sep}$ is isomorphic to $\mathrm{End}_{F_{sep}}(V)$ for some vector space V over F_{sep};
(iii) $A_{F_{sep}}$ is isomorphic to $M_n(F_{sep})$ for some integer n;
(iv) A is isomorphic to $M_r(D)$ for some integer r and some division algebra D, central over F.

(Note that D is uniquely determined by A ; $D \simeq \mathrm{End}_A(M)$ for any simple left A-module M; D is division by Schur's lemma).
From this, we get the following definition :

Definition 2.3. — The degree of A is $\deg(A) := \sqrt{\dim_F(A)}(= n)$.
The index of A is $\mathrm{ind}(A) := \deg(D)(= n/r)$.

Example 2.4. — The degree of $(a, b)_F$ is 2 and its index is either 1 in which case $(a, b)_F \simeq M_2(F)$ or 2 in which case $(a, b)_F$ is division.

Example 2.5. — It is known that the tensor product of two central simple algebras is again central simple. A famous example are the so called biquaternion algebras which are tensor products of two quaternions, $A = (a, b)_F \otimes_F (c, d)_F$. It has degree 4. Its index, depending on the base field F and the values of a, b, c and d can be 1, 2 or 4. In the first case, A is isomorphic to $M_4(F)$. In the last one it is division. And if $\mathrm{ind}(A) = 2$, $A \simeq M_2(Q)$ for some division quaternion algebra Q.

The Brauer group. — On the set of isomorphism classes of central simple algebras over F, we define an equivalence relation called 'Brauer equivalence' by $A \sim A'$ if and only if $D \simeq D'$, where D and D' are division algebras respectively associated to A and A' by Wedderburn's theorem.
The tensor product induces a product on the set of Brauer classes of algebras, and endow this set with a group structure : the Brauer group.
$\mathrm{Br}(F) = \{[A],\ A \text{ central simple over } F\}$
$= \{\text{isom. classes of division algebras central over } F\}$.
To have an idea of the group structure, note that if A is central simple over F, then the map $A \otimes_F A^{op} \to \mathrm{End}_F(A)$, $a \otimes b \mapsto (x \mapsto axb)$ is an isomorphism. Hence $[A]^{-1} = [A^{op}]$.
One may also prove that $A^{\otimes n}$ is split. Hence $[A^{\otimes n}] = [A]^n = 1 \in \mathrm{Br}(F)$.

Definition 2.6. — The exponent of A is the order of $[A]$ in $\mathrm{Br}(F)$.

As we just noticed, $\exp(A) | \deg(A)$.

2.2. Index reduction formulas.

— (See Merkurjev, Panin and Wadsworth [**MPW98**]).
The index of a central simple algebra gives a measure of the size of the division part of the algebra. Hence it is an important invariant, hard to compute in general. We are interested here in the following natural question :
Question : How does $\mathrm{ind}(A)$ behave under scalar extension?

Example 2.7. — $\mathrm{ind}(A_{F_{sep}}) = 1$;
It is known that if $F(t)$ is a purely transcendental extension of F, then $\mathrm{ind}(A_{F(t)}) = \mathrm{ind}(A)$.

Remark 2.8. — Note that the index necessarily decreases under scalar extension. We even have $\operatorname{ind}(A_L)|\operatorname{ind}(A)$, and this is where the name 'index reduction formulas' comes from.

We do not have a general answer to this question, but we do know the answer for some particular fields. The first result on this question deals with the field F_B:=function field of the Severi-Brauer variety of a central simple algebra B over F (this is the variety of minimal right ideals in the algebra B, that is right ideals of dimension $\deg(B)$, corresponding to lines when B is split). It is known that F_B is a generic splitting field for B. Hence $\operatorname{ind}(B_{F_B}) = 1$, whatever $\operatorname{ind}(B)$ is. Also $\operatorname{ind}(B_{F_B}^{\otimes i}) = \operatorname{ind}((B_{F_B})^{\otimes i}) = 1$.
Amitsur proved in the 50's that the kernel of the natural map $\operatorname{Br}(F) \to \operatorname{Br}(F_B)$ is the subgroup of $\operatorname{Br}(F)$ generated by the class of B. In other words, $\operatorname{ind}(A_{F_B}) = 1$ if and only if $A \sim B^{\otimes i}$ for some i, $i \leq \exp(B)$.
This result was generalized Schofield and van den Bergh in 1992. They proved that $\operatorname{ind}(A_{F_B}) = gcd_{1\leq i\leq \exp(B)} \operatorname{ind}(A \otimes B^{\otimes i})$. This answers completely the question for such a field F_B.
After this very nice paper, some other computations were made by various people, until Merkurjev Panin and Wadsworth proved a general formula wich includes the previous ones, in a serie of two papers called 'Index Reduction Formulas for Twisted Flag Varieties' I and II. This is where Tits algebras come into the picture.
A twisted flag variety is a projective variety X endowed with an action of a semi-simple adjoint algebraic group, satisfying certain properties. Essentially, we want the group $G(F_{sep})$ to act transitively on $X(F_{sep})$.

Example 2.9. — Take $G = \operatorname{PGL}_1(B)$, defined by the exact sequence $1 \to G_m \to \operatorname{GL}_1(B) \to \operatorname{PGL}_1(B) \to 1$. It is the group of automorphisms of B, and it acts naturally on the Severi-Brauer variety $SB(B)$, which is a twisted flag variety.

Let us now assume for simplicity that G is of inner type, so that the Galois group Γ_F acts trivially on the group $C = \Lambda/\Lambda_r$ (see sections 5 and 6.2 for a definition of Λ/Λ_r). The index reduction formula can be written, in that case, as follows:

$$\operatorname{ind}(A \otimes_F F(X)) = gcd_{\psi\in C}(n_{\psi,P,F} \operatorname{ind}(A \otimes_F A_G(\psi)),$$

where, for any $\psi \in C$, $A_G(\psi)$ is a Tits algebra for G whose Brauer class is $\alpha(\psi)$.

Example 2.10. — Take again $G = \operatorname{PGL}_1(B)$ and $X = SB(B)$. In that case, the group C is $\mathbb{Z}/n\mathbb{Z}$ and the morphism $C \to \operatorname{Br}(F)$ is given by $\bar{i} \mapsto [A^{\otimes i}]$. Hence, one recovers Schofield and van den Bergh formula in that case.

3. Motivation 2 : Central part of the Rost invariant

For any algebraic group G over F, the Galois group $\Gamma_F = \mathrm{Gal}(F_{sep}/F)$ acts on the group of F_{sep} points $G(F_{sep})$. We will denote by $H^1(F,G)$ the corresponding Galois cohomology set $H^1(F,G) = H^1(\Gamma_F, G(F_{sep}))$. In general, it is only a pointed set. But if G is abelian, then it is a group.

In this section, we assume for simplicity $\mathrm{char}(F) = 0$. Let G be an absolutely simple simply connected algebraic group. We call a morphism of pointed sets $H^1(F,G) \to H^3(F,\mathbb{Q}/\mathbb{Z}(2))$ a degree 3 invariant of G. It has been proven by Rost that the set of such invariants is a cyclic group of finite order. This group has a canonical generator called the Rost invariant (See [**KMRT98**, chap. VII] or [**GMS03**]).

For instance, the group $H^1(F,\mathrm{Spin}(q_0))$ classifies isomorphism classes of quadratic forms over F having same dimension, discriminant and Hasse invariant as q_0, ie such that the difference $q - q_0$ belongs to the third power of the fundamental ideal $I(F)$. The Rost invariant for this group maps the class of q to the Arason invariant $e^3(q-q_0)$. But in general, we do not have such a nice description of the Rost invariant.

Let Z be the center of G. Clearly, the Rost invariant induces a degree 3 invariant of Z, denoted R_Z, namely :

$$H^1(F,Z) \to H^1(F,G) \to H^3(F,\mathbb{Q}/\mathbb{Z}(2))$$

This map R_Z, which factors through the pointed set $H^1(F,G)$, actually is a group homomorphism. This follows from Gille's result that the Rost invariant is compatible with twisting (see [**Gil00**]), and was noticed by Garibaldi in [**Gar01**]. In [**MPT02**], Merkurjev Parimala and Tignol compute this invariant for classical groups. For each type of group, they give a precise formula, which involves corestriction morphisms and some cup products with Brauer classes of some Tits algebras of G.

Part II

We begin with well known facts on algebraic groups, representations and root systems. More details and proofs will be found for instance in Waterhouse [**Wat79**] for the group scheme point of view, Humphreys [**Hum75**] for classical algebraic groups theory, as well as chapter VI of [**KMRT98**].

4. Generalities on affine group schemes and their representations

4.1. Affine group schemes. — In this talk, we call algebraic group a smooth algebraic affine group scheme. This means, in particular, that we think of an algebraic group as a functor from the category Alg_F of unital commutative F-algebras to the category of groups.

Example 4.1. — (i) $G_m : R \mapsto R^\times$.
(ii) For any central simple F-algebra A, we define $\mathrm{GL}_1(A) : R \mapsto (A \otimes_F R)^\times$;
If $A = F$, we get G_m.
If $A = \mathrm{End}_F(V)$, we get $\mathrm{GL}(V) : R \mapsto \mathrm{GL}_R(V_R)$.
If $A = M_n(F)$, we get $\mathrm{GL}_n : R \mapsto \mathrm{GL}_n(R)$.
(iii) $T : R \mapsto \{\text{diagonal matrices in } \mathrm{GL}_n(R)\} \simeq R^\times \times \cdots \times R^\times$;
Thus T is a product of n copies of G_m. It is a split torus.
(iv) The reduced norm of A induces a group scheme morphism $\mathrm{GL}_1(A) \to G_m$. We define $\mathrm{SL}_1(A)$ as the kernel of this morphism. Hence for any R,
$\mathrm{SL}_1(A)(R) = \{a \in A_R^\times,\ \mathrm{Nrd}_{A_R}(a) = 1\}$.
(v) We have already seen $O(A, \sigma) := \{a \in A,\ \sigma(a)a = 1\}$. The corresponding algebraic group is given by $O(A, \sigma)(R) = \{a \in A_R,\ \sigma_R(a)a = 1\}$.
(vi) The same way, we deduce from the definitions of the groups $\mathrm{Spin}(V, q)$ (resp. $\mathrm{Spin}(A, \sigma)$) as a subgroup of $\mathcal{C}_0(V, q)^\times$ (resp. $\mathcal{C}(A, \sigma)^\times$) the definition of the corresponding group scheme, which is contained in $\mathrm{GL}_1(\mathcal{C}_0(V, q))$ (resp. $\mathrm{GL}_1(\mathcal{C}(A, \sigma))$).

Definition 4.2. — Let G be an algebraic group over F and L/F a field extension. For any $R \in \mathrm{Alg_L}$, R can also be viewed as an F-algebra and we define the group scheme G_L by $G_L(R) := G(R)$.

Definition 4.3. — Let L/F be a finite separable field extension, and G over L an algebraic group. The corestriction $R_{L/F}(G)$ is the group scheme over F defined by $R_{L/F}(G)(R) = G(R \otimes_F L)$ for any $R \in \mathrm{Alg}_F$.

Example 4.4. — By definition, a torus is an algebraic group T which is isomorphic to a product of G_m after scalar extension to a separable closure of the base field.
The group T/F defined by $T = R_{L/F}(G_m)$ is an example of a non split torus. Indeed, $T(F) = L^\times$ which is not isomorphic to a product of n copies of $F^\times$. While after scalar

extension to F_{sep}, we have for any F_{sep} algebra R,

$$T_{F_{sep}}(R) = T(R) = G_m(R \otimes_F L) = (R \otimes_F L)^{\times}.$$

But since R contains F_{sep}, this is isomorphic to $R^{\times} \times \cdots \times R^{\times}$. This proves that $T_{F_{sep}}$ is a split torus.

4.2. Representations. — Let ρ be a representation of the algebraic group G over F, that is a morphism of algebraic groups $G \to \mathrm{GL}(V)$ for some vector space V over F. Hence, we have for any F-algebra R a group morphism $G(R) \mapsto \mathrm{GL}(V \otimes_F R)$.
If $\dim(V) = 1$, then $\mathrm{GL}(V) = G_m$. Hence one-dimensional representations of G are morphisms $G \to G_m$. They are called characters of the group G. The multiplication of G_m, viewed as a morphism $G_m \times G_m \to G_m$ induces a law on the set of characters of G, and endow this set with an abelian group structure.
Notation: We will denote by $G^{\star}$ the abelian group of characters of G.

Example 4.5. — The character group of G_m is $\mathbb{Z}$ ($x \mapsto x^r$).
The character group of the split torus $G_m \times \cdots \times G_m$ is $\mathbb{Z}^r$. (The character corresponding to $(n_1, \dots n_r) \in \mathbb{Z}^r$ is given by $\chi_{(n_1,\dots,n_r)}(t_1, \dots, t_r) = t_1^{n_1} \dots t_r^{n_r}$.)

We will also use the adjoint representation of G, $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$.

Example 4.6. — If $G = \mathrm{GL}(V)$, then $\mathrm{Lie}(G) = \mathrm{End}_F(V)$ and the adjoint representation is given by $\mathrm{Ad}(\alpha)(\beta) = \alpha\beta\alpha^{-1}$.

4.3. Representations of a split torus. — Let T be a split torus $T = (G_m)^r$, and let $\rho : T \to \mathrm{GL}(V)$ be a representation of T over F.

Definition 4.7. — A character $\lambda \in T^{\star}$ is called a weight for ρ if there exist a non-trivial vector $v \in V$ such that for any $t \in T(F)$, $\rho(t)(v) = \lambda(t).v$. The multiplicity m_λ of a weight λ is the dimension of the corresponding weight subspace
$V_\lambda = \{v \in V,\ \forall t \in T(F),\ \rho(t)(v) = \lambda(t).v\}$.

Example 4.8. — Look at the adjoint representation for $G = \mathrm{GL}(V)$, and its restriction to the torus of diagonal matrices. We have $\rho(t_1, \dots, t_n)(m_{ij}) = (t_i t_j^{-1} m_{ij})$. Hence the action is trivial on diagonal matrices, which means that 0 is a weight of mutiplicity n. Any n-tuple of $\mathbb{Z}^r$ of the form $(\dots, 1, \dots, -1, \dots)$ or $(\dots, -1, \dots, 1, \dots)$ is a weight of multiplicity 1.

Theorem 4.9. — *For any representation $\rho : T \to \mathrm{GL}(V)$, the vector space V decomposes as a direct sum of the weight subspaces $V = \oplus_\lambda V_\lambda$.*

Hence a representation of a split torus is entirely determined by its weights and their multiplicities.

5. Root systems

A root system is a geometric data. It is defined as follows:
Let V be a finite dimensional $\mathbb{R}$ vector space. For any $\alpha \in V$, $\alpha \neq 0$, a reflection of V with respect to α is any endomorphism s of V satisfying $s(\alpha) = -\alpha$ and $s_{|W} = \mathrm{Id}_W$ for some hyperplane $W \subset V$. Hence, for any $v \in V$, we have $s(v) - v = x\alpha$ for some $x \in \mathbb{R}$, which we denote by $s^\star(v)$.

Definition 5.1. — A root system in V is a finite subset $\Phi \subset V$ satisfying :
(i) $0 \notin \Phi$ and Φ spans V.
(ii) For any $\alpha \in \Phi$, $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$.
(iii) For any $\alpha \in \Phi$, there exists a reflection s_α with respect to α such that $s_\alpha(\Phi) \subset \Phi$.
(iv) For each $\alpha, \beta \in \Phi$, $s_\alpha^\star(\beta) \in \mathbb{Z}$.

Example 5.2. — Consider the set of vertices of a regular hexagon in $\mathbb{R}^2$. It is a root system, reflections s_α being orthogonal reflections with respect to some of the symetry axis of the hexagon.

An automorphism of the root system Φ is any automorphism f of V which preserves Φ. The automorphism group $\mathrm{Aut}(V, \Phi)$ contains a subgroup generated by the reflections s_α for any $\alpha \in \Phi$, which is called the Weyl group of Φ.

Example 5.3. — In our situation, the automorphism group is the group of isometries preserving the hexagon. It is isomorphic to the dihedral group D_6. The Weyl subgroup is a subgroup of index 2.

Given such a root system, we define two lattices in $\mathbb{R}^n$:
The root lattice is the lattice generated by the roots, $\Lambda_r = \{\sum_{\alpha\in\Phi} m_\alpha \alpha,\ m_\alpha \in \mathbb{Z}\}$.
The weight lattice is the dual lattice of Λ_r, $\Lambda = \{v \in V,\ s_\alpha^\star(v) \in \mathbb{Z}\ \ \forall \alpha \in \Phi\}$.
Clearly, $\Lambda_r \subset \Lambda$. Moreover, it is known that the quotient Λ/Λ_r is a finite group.

Example 5.4. — In our situation, the weight lattice contains the centers of the triangles of the root lattice. The quotient is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

Definition 5.5. — A subset $\Pi \subset \Phi$ is called a basis of Φ (or a set of simple roots) if it is a basis of V and any $\alpha \in \Phi$ is written $\alpha = \sum_{\gamma\in\Pi} m_\gamma \gamma$ with either $m_\gamma \geq 0$ for all $\gamma \in \Pi$, or $m_\gamma \leq 0$ for all $\gamma \in \Pi$

Example 5.6. — Pick two vertices α and γ in the hexagon which are neither consecutive nore opposite. The other roots are $\alpha + \gamma$, $-\alpha$, $-\gamma$ and $-\alpha - \gamma$. Hence $\{\alpha, \gamma\}$ is a basis.

As one may easily check on this example, the Weyl group acts simply transitively on the set of basis of Φ.

Definition 5.7. — Given a basis $\Pi \subset \Phi$, the cone of dominant weights is the subset $\Lambda^+ \subset \Lambda$ defined by $\Lambda^+ = \{v \in \Lambda,\ s^\star_\alpha(v) \geq 0,\ \forall \alpha \in \Pi\}$.

Example 5.8. — In our situation, the axis of the three reflections s_α, s_β and s_γ divides $\mathbb{R}^2$ in six sectors. The cone of dominant weights with respect to the basis $\{\alpha, \gamma\}$ consists of the points of Λ which belongs to the sector containing $\alpha + \gamma$.

Given a root system Φ and a base $\Pi \subset \Phi$, we define the Dynkin diagram of Φ as follows :
Π is the set of vertices ;
Two vertices α and β are connected by $s^\star_\alpha(\beta)s^\star_\beta(\alpha)$ edges, with an orientation from α to β if $s^\star_\alpha(\beta) > s^\star_\beta(\alpha)$.

Example 5.9. — In our example, the diagram has two vertices connected with one edge. This root system is denoted by A_2 in the litterature.

One can prove that this diagram does not depend on the choice of a basis $\Pi \subset \Phi$. Moreover, it contains all information on the root system, as the following theorem shows :

Theorem 5.10. — *Two root systems are isomorphic if and only if they have the same Dynkin diagrams.*

Root systems are classified and to describe this classification, it is enough to describe the corresponding diagrams. A root system is called irreducible if its diagram is connected. Any root system is a sum of irreducible ones (by this, we mean that we take the sum of the underlying vector spaces and the union of the corresponding root systems). The corresponding diagram is the union of the diagrams of the summands. Hence we only have to describe irreducible root systems, or equivalently connected Dynkin diagrams.

Theorem 5.11. — *Any connected Dynkin diagram is one of the following:* A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 *and* G_2.

This classification is described in many books, and you may find there all information about the corresponding root systems. For instance, a description of Λ, Λ_r, the value of the finite group Λ/Λ_r... (See for instance [**KMRT98**]).

6. Split semi-simple algebraic groups

6.1. Definition. — Given an algebraic group G, the group $G_{F_{alg}}$ is an algebraic group in the classical sense (that is an affine variety over an algebraically closed field endowed with a group structure).
We say that G is semi-simple if this group $G_{F_{alg}}$ is semi-simple in the classical sense. More precisely, this means that G is non trivial and connected, and $G_{F_{alg}}$ does not

contain any non trivial normal connected solvable subgroup. (An algebraic group H is said to be solvable if $H(F_{alg})$ is solvable as an abstract group).

Example 6.1. — $\mathrm{SL}_1(A)$, $\mathrm{PGL}_1(A)$, $SO(A,\sigma)$, $\mathrm{Spin}(A,\sigma)$ are examples of semi-simple groups.
$GL_1(A)$ and $O(A,\sigma)$ are not semi-simple.

A torus $T \subset G$ is called maximal if it is not contained in a larger torus $T' \subset G$. It is known that maximal tori remain maximal under any scalar extension. Moreover, they are conjugate over F_{alg} by elements of $G(F_{alg})$.

Definition 6.2. — The semi-simple group G is said to be split if it contains a split maximal torus T.

Example 6.3. — $SL(V)$, $PGL(V)$, $SO(V,q)$ and $\mathrm{Spin}(V,q)$ for some hyperbolic quadratic space (V,q) are examples of split semi-simple groups.

From now on, we assume G is a split semi-simple algebraic group and we fix a split maximal torus $T \subset G$.

6.2. Classification by root systems. — To the data of a split semi-simple group G and a split maximal torus $T \subset G$, we can associate a root system as follows.
Consider the adjoint representation $\mathrm{Ad} : G \mapsto \mathrm{GL}(\mathrm{Lie}(G))$, and its restriction $\mathrm{Ad}_{|T}$ to the torus T. Among the weights of $\mathrm{Ad}_{|T}$, there is
0 with a certain multiplicity (namely the dimension of the corresponding weight space $V_0 = \{v \in V,\ \forall t \in T(F),\ \rho(t)(v) = v\}$);
some characters of T which appear to have multiplicity 1.

Definition 6.4. — A root of G is a non trivial weight of the adjoint representation. The roots of G form a finite set $\Phi(G)$ in $T^\star = \mathbb{Z}^r$.

Theorem 6.5. — *(i) $\Phi(G)$ is a root system in $T^\star \otimes_{\mathbb{Z}} \mathbb{R}$; we have $\Lambda_r \subset T^\star \subset \Lambda$.*
(ii) The group G is uniquely determined, up to isomorphism by its root system $\Phi(G)$ and the quotient $T^\star/\Lambda_r$ which is a finite subgroup of $C = \Lambda/\Lambda_r$.

Definition 6.6. — The group G is adjoint if $T^\star = \Lambda_r$ and simply connected if $T^\star = \Lambda$.

It follows from the theorem that simply-connected and adjoint groups are uniquely determined by their root system.

Example 6.7. — (i) *Groups of type A_n.* For A_n, the quotient $C = \Lambda/\Lambda_r$ is equal to $\mathbb{Z}/(n+1)\mathbb{Z}$. The subgroups of $\mathbb{Z}/(n+1)\mathbb{Z}$ are $\mathbb{Z}/k\mathbb{Z}$ for any k dividing $n+1$. The corresponding split semi-simple group is SL_{n+1}/μ_k. In particular, the split simply connected group of type A_n is SL_{n+1} and the split adjoint group of type A_n is $\mathrm{SL}_{n+1}/\mu_{n+1} \simeq PGL_{n+1}$.

(ii) *Groups of type D_n.* Among the split semi-simple groups of type D_n, one finds $\mathrm{Spin}(V,q)$(simply connected), $\mathrm{SO}(V,q)$ and $\mathrm{SO}(V,q)/\mu_2$ (adjoint), where (V,q) is a hyperbolic quadratic space of dimension $2n$.
If n is odd, then $C=\mathbb{Z}/4\mathbb{Z}$, which admits only one non trivial subgroup and the classification is done in that case. If n is even, then $C=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ has three different non trivial subgroups. The missing groups in this case are the images of the half spin representations $\mathrm{Spin}(V,q)\to\mathrm{GL}_1(\mathcal{C}_0(V,q))\to\mathrm{GL}_1(\mathcal{C}_\pm(V,q))$.

6.3. Representations of a split semi-simple group. — This root system data can also be used to classify irreducible representations of a split semi-simple group G. Indeed, let us fix a base Π in the root system $\Phi(G)$. Remember that this determines a subset of Λ called the cone of dominant weights.
We can define an ordering on Λ by $\lambda\geq\mu$ if $\lambda-\mu$ is a non negative linear combination of elements of Π. (Remember that Π is a basis of V, hence $\lambda-\mu$ is a unique linear combination of elements of Π).
Now, let $\rho:\,G\to\mathrm{GL}(V)$ be an irreducible representation of V. The set of weights of ρ is a finite subset in $T^\star\subset\Lambda$. It can be proven that it contains a maximal element, which we call the highest weight of ρ, and which belongs to Λ_+. This gives a classification of irreducible representations of G:

Theorem 6.8. — *The map which associates to any irreducible representation of G its highest weight induces a 1-1 correspondance between isomorphism classes of irreducible representations and $T^\star\cap\Lambda_+$.*

Remark 6.9. — If G is simply connected, we get $T^\star\cap\Lambda_+=\Lambda_+$.
In fact, given a semi-simple group G, it can always be written as $G=\tilde{G}/Z$ where $\tilde{G}$ is the semi-simple simply connected group of the same type as G and Z is a subgroup of the center of $\tilde{G}$. Any dominant weight then corresponds to an irreducible representation $\tilde{\rho}:\,\tilde{G}\to\mathrm{GL}(V)$. This representation factors through G if and only if its dominant weight actually belongs to $T^\star$.

7. Tits algebras

We go back to the general situation. Hence, the group G now is a not necessarily split semi-simple group over a field F, and we fix a maximal torus $T\subset G$. For simplicity, we also assume that G is simply connected.
Clearly, the group G_{sep} is a split semi-simple algebraic group over F.

Definition 7.1. — We call root system of G the root system $\Phi(G_{sep})$.

Hence irreducible representations of G_{sep} are classified by Λ_+.

7.1. Algebra representations. — We give the following definitions.

Definition 7.2. — An algebra representation of G is a morphism $\rho : G \to \mathrm{GL}_1(A)$ for some central simple algebra A over F.
Two such representations $\rho : G \to \mathrm{GL}_1(A)$ and $\rho' : G \to \mathrm{GL}_1(A')$ are called isomorphic if there exist an isomorphism $\phi : A \tilde{\to} A'$ such that $\rho' = \phi \circ \rho$.
The algebra representation ρ is called irreducible if ρ_{sep} is irreducible (as an usual representation).
If ρ is irreducible, the highest weight of ρ is the highest weight of ρ_{sep}.

Note that the Galois group Γ_F acts naturally on $T^{\star}_{sep} = \mathrm{Hom}(T_{sep}, G_{m,F_{sep}})$. This induces an action on $\Phi(G)$, Λ_r and Λ. But this action does not preserve Π and Λ_+.
We define a new action of γ on $T^{\star}_{sep}$ as follows. It is known that the Weyl group permutes the bases of $\Phi(G)$. Hence, for any $\gamma \in \Gamma_F$, there exists a unique element $w_\gamma \in W(\Phi(G))$ such that $w(\gamma(\Pi)) = \Pi$. We let $\gamma \star v := w_\gamma(\gamma(v))$. This new action preserves Π and hence Λ_+.
Moreover, one can check that for any representation $\rho_{sep} : G_{sep} \to \mathrm{GL}(V)$ of dominant weight $\lambda \in \Lambda_+$, and any $\gamma \in \Gamma_F$, the dominant weight of ${}^{\gamma}\rho_{sep}$ is $\gamma \star \lambda$.
In his paper, Tits proves an analogue of the theorem of classification of irreducible representations which is the following :

Theorem 7.3. — *(*[**Tit71**]*) The map which associates to any irreducible algebra representation of G its highest weight induces a 1-1 correspondance between isomorphism classes of irreducible algebra representations and $\Lambda_+^{\Gamma_F}$.*

Hence, given any $\lambda \in \Lambda_+^{\Gamma_F}$, there exists a central simple algebra $A(\lambda)$ unique up to isomorphism and a morphism $\rho : G \to \mathrm{GL}_1(A(\lambda))$ which is, after scalar extension to F_{sep} the irreducible representation of G_{sep} of highest weight λ.

7.2. Brauer classes of Tits algebras. — Tits also proves that there are only very few possibilities for the Brauer classes of those algebras. More precisely :

Proposition 7.4. — *For any $\lambda \in {\Lambda_+}^{\Gamma_F} \cap \Lambda_r$, the algebra $A(\lambda)$ is split.
For any λ and $\mu \in \Lambda_+^{\Gamma_F}$, we have $[A(\lambda + \mu)] = [A(\lambda)] + [A(\mu)]$.*

Combining those two results, we can extend the map $\Lambda_+^{\Gamma_F} \to \mathrm{Br}(F)$ to a group morphism $\Lambda^{\Gamma_F} \to \mathrm{Br}(F)$, and then

$$\alpha : (\Lambda/\Lambda_r)^{\Gamma_F} \to \mathrm{Br}(F).$$

Remember that this group Λ/Λ_r is a finite group. Hence there are only finitely many possibilities for the Brauer class of a Tits algebra for a given group.

Example 7.5. — (i) Consider the group of type A_n, $G = \mathrm{SL}_1(A)$ for some central simple algebra A over F of degree $n+1$. The morphism α is given by
$\alpha : \mathbb{Z}/(n+1)\mathbb{Z} \to \mathrm{Br}(F)$, $\bar{i} \mapsto [A^{\otimes i}]$.

(ii) Consider the group of type D_n $(n \neq 4)$, $G = \mathrm{Spin}(A, \sigma)$ for some central simple algebra A of degree $2n$ with orthogonal involution σ.
Let us write $C = \{0, \lambda, \lambda_+, \lambda_-\}$, where λ is the unique element of order 2 if n is odd and $C \simeq \mathbb{Z}/4\mathbb{Z}$ and $\lambda = (1,1)$ if n is even and $C \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
We have seen in the very beginning of the first lecture that $\mathrm{Spin}(A, \sigma) \to \mathrm{GL}_1(A)$, so that A is a Tits algebra for G. One may prove that $\alpha(\lambda) = [A]$.
If $d(\sigma) = 1 \in F^\times/F^{\times 2}$, then we have already seen that $\mathcal{C}(A, \sigma) = \mathcal{C}_+ \times \mathcal{C}_-$, and we have $\alpha(\lambda_\pm) = [\mathcal{C}_\pm]$.
Let us assume now that $d(\sigma) \neq 1$. In that case, one may prove that the elements λ_+ and λ_- are not invariant under Γ_F. They are conjugate under the action of Γ_F.
But if we extend scalars to $E = F(\sqrt{\delta})$, the discriminant becomes trivial. Hence the weights $\lambda_\pm$ belongs to C^{Γ_E} and they are mapped under α_E to some classes in $\mathrm{Br}(E)$. Let us now compute these classes.
The Clifford algebra of (A_E, σ_E) is $\mathcal{C}(A_E, \sigma_E) = \mathcal{C}(A, \sigma) \otimes_F E$. But since E is the center of $\mathcal{C}(A, \sigma)$, we get $\mathcal{C}(A_E, \sigma_E) = \mathcal{C}(A, \sigma) \times \mathcal{C}(A, \sigma)$. Hence $\alpha_E(\lambda_+) = \alpha_E(\lambda_-) = [\mathcal{C}(A, \sigma)] \in \mathrm{Br}(E)$.

This morphism α has a nice behavior with respect to scalar extensions. In particular, Tits proves that the following diagram is commutative for any field extension E/F:

$$\begin{array}{ccc} C^{\Gamma_F} & \to & \mathrm{Br}(F) \\ \downarrow & & \downarrow \\ C^{\Gamma_E} & \to & \mathrm{Br}(E) \end{array}$$

He also proves α commutes with corestriction. Let E/F be a finite Galois extension. Let $\lambda \in C^{\Gamma_E}$, and let $N_{E/F}(\lambda) \in C^{\Gamma_F}$ be the sum of the conjugates of λ under the action of $\mathrm{Gal}(E/F)$.
We then have $\alpha_F(N_{E/F}(\lambda)) = N_{E/F}(\alpha_E(\lambda))$, where the second $N_{E/F}$ denotes the corestriction morphism $\mathrm{Br}(E) \to \mathrm{Br}(F)$.

7.3. Brauer classes of Clifford algebras. — Since α is a group morphism, any relation in the group C induces an analogue relation for the Brauer classes of the corresponding Tits algebras.
For instance, it is proven in [**KMRT98**, (9.12)] that :
If n is even, then
(i) $[\mathcal{C}(A, \sigma)]^2 = 1 \in \mathrm{Br}(E)$ and
(ii) $N_{E/F}([C(A, \sigma)]) = [A] \in \mathrm{Br}(F)$.
If n is odd, then
(iii) $[\mathcal{C}(A, \sigma)]^2 = [A_E] \in \mathrm{Br}(E)$ and
(iv) $N_{E/F}([C(A, \sigma)]) = 1 \in \mathrm{Br}(F)$.

This theorem can be easily deduced from the properties of the morphism α described above. Indeed, we have $[\mathcal{C}(A,\sigma)]^2 = \alpha_E(2\lambda_+)$. Hence (i) and (iii) follows from $2\lambda_+ = \begin{cases} 0 \text{ if } n \text{ is even} \\ \lambda \text{ if } n \text{ is odd} \end{cases}$.

Also, $N_{E/F}([C(A,\sigma)]) = \alpha(N_{E/F}(\lambda_+)) = \alpha_E(\lambda_+ + \lambda_-)$. Hence (ii) and (iv) follows from $\lambda_+ + \lambda_- = \begin{cases} \lambda \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$.

References

[Gar01] R. S. Garibaldi – "The rost invariant has trivial kernel for quasi-split groups of low rank", *Commentarii Mathematici Helvetici* (2001), no. 76, p. 684–711.

[Gil00] P. Gille – "Invariants cohomologiques de rost en caractristique positive", *K-Theory* **21** (2000), no. 1, p. 57–100.

[GMS03] S. Garibaldi, A. Merkurjev & J.-P. Serre – *Cohomological invariants in galois cohomology*, University Lecture Series, vol. 28, Amer. Math. Soc., Providence, RI, 2003.

[Hum75] J. E. Humphreys – *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Springer-Verlag, New York-Heidelberg, 1975.

[KMRT98] M.-A. Knus, S. A. Merkurjev, M. Rost & J.-P. Tignol – *The book of involutions*, Colloquium Publ., vol. 44, Amer. Math. Soc., Providence, RI, 1998.

[MPT02] A. Merkurjev, R. Parimala & J.-P. Tignol – "Invariants of quasi-trivial tori and the rost invariant", *Algebra i Analiz* **14** (2002), no. 5, p. 110–151, Translation in *St Petersburg Math. J.* **14** (2003), no 5, 791-821.

[MPW98] A. S. Merkurjev, I. A. Panin & A. R. Wadsworth – "Index reduction formulas for twisted flag varieties, i", *K-Theory* **14** (1998), no. 2, p. 101–196.

[Sch85] W. Scharlau – *Quadratic and hermitian forms*, Grundlehren der Mathematischen Wissenschaften, vol. 270, Springer, Berlin, 1985.

[Tit71] J. Tits – "Reprsentations linaires irreductibles d'un groupe rductif sur un corps quelconque", *J. Reine Angew. Math.* **247** (1971), p. 196–220.

[Wat79] W. C. Waterhouse – *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York-Berlin, 1979.

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