# $J$-invariant and triality

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# Notation

$p$ prime number
$k$ field, $\mathrm{char}(k) \neq p$
$G$ absolutely almost simple linear algebraic group over $k$.
$G_0$ corresponding split group.

So $G = {}_{\xi}G_0$ for some cocycle $\xi \in Z^1(k, \mathrm{Aut}(G_0))$;
We assume $\xi \in Z^1(k, G_0)$.

We fix $T_0 \subset B_0 \subset G_0$
$T_0$ split maximal torus ; $B_0$ Borel subgroup
$\mathfrak{X}_0 = G_0/B_0, \quad \mathfrak{X} = {}_{\xi}\mathfrak{X}_0$

## Example

$G = \mathrm{O}^+(\varphi)$ for some $\varphi : V \to k$, $\dim V = 2n$.
$\mathfrak{X} \leftrightarrow V_1 \subset V_2 \subset \cdots \subset V_{n-1}$,
totally isotropic of dimension $1, 2, \ldots, n-1$

# J-invariant of the group G

$$\mathrm{Ch}^\star(\mathfrak{X}) \xrightarrow{\text{res}} \mathrm{Ch}^\star(\mathfrak{X}_0) \xrightarrow{\pi_\star} \mathrm{Ch}^\star(G_0),$$
$$\text{where } \pi : G_0 \longrightarrow \mathfrak{X}_0 = G_0/B_0.$$

V. Kac (1985) $\quad \mathrm{Ch}^\star(G_0) \simeq \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}),$
for some integers $r$, $k_1, \ldots, k_r$, known for each $G_0$.

Petrov-Semenov-Zainoulline (2008)

$$\mathrm{coker}(\pi_\star \circ \mathrm{res}) \simeq \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{j_1}}, \ldots, x_r^{p^{j_r}}),$$

for some $j_1, \ldots, j_r, \quad 0 \leqslant j_i \leqslant k_i$

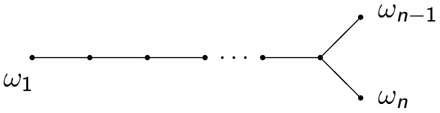
Definition: $\quad J_p(G) = (j_1, \ldots, j_r)$

# More notation

$$\left.\begin{array}{l} A \text{ central simple } k \text{ algebra} \\ \deg(A) = 2n \\ \sigma : A \to A \text{ involution} \\ \quad \text{of orthogonal type} \end{array}\right\} \Leftrightarrow (A,\sigma) \otimes_k k_s \simeq (M_{2n}(k_s), t)$$

$\mathrm{Aut}^+(A,\sigma) = \mathrm{PGO}^+(A,\sigma) = {}_\xi\mathrm{PGO}^+_{2n}$ Adjoint of inner type $D_n$

We assume $\mathrm{disc}(\sigma) = 1 \in k^\star/k^{\star 2}$.

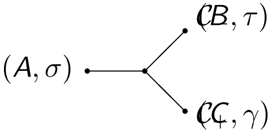



Tits algebras

$(A,\sigma)$ $\qquad\qquad$ $\mathcal{C}(A,\sigma) = \mathcal{C}_+ \times \mathcal{C}_-$

# Triality

$\deg(A) = 8$ $\qquad$ $\mathrm{PGO}^+(A, \sigma)$ of type $D_4$



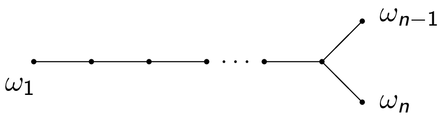

Fact: $\begin{cases} \mathcal{C}(A, \sigma) = (B, \tau) \times (C, \gamma) \\ \mathcal{C}(B, \tau) = (A, \sigma) \times (C, \gamma) \\ \mathcal{C}(C, \gamma) = (A, \sigma) \times (B, \tau) \end{cases}$

Trialitarian triple $\mathcal{T} = \big((A, \sigma), (B, \tau), (C, \gamma)\big)$

# How to order the generators of $\mathrm{Ch}^\star(G_0)$?

- $G$ is not adjoint of type $D_n$, with $n$ even.
  Order by degree $d_1 < d_2 < \cdots < d_r$.

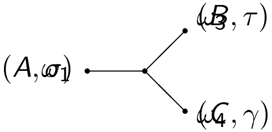
- $G$ adjoint of type $D_{2m}$, that is $G_0 = PGO^+_{4m}$, $m \geqslant 3$
  $\mathrm{CH}^1(G_0) \simeq \Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq \{0, \bar\omega_1, \bar\omega_{n-1}, \bar\omega_n\}$




$x_1 = \bar\omega_1$, $x_2 = \bar\omega_n$ or $\bar\omega_{n-1}$, then order by degree.

Notation : $J(A,\sigma) = J_2\big(\mathrm{PGO}^+(A,\sigma)\big)$.

$\deg(A) = 8$




$$\mathrm{PGO}^+(A,\sigma) \simeq \mathrm{PGO}^+(B,\tau) \simeq \mathrm{PGO}^+(C,\gamma)$$

**Definition**

$J(A,\sigma) = J_2\big(\mathrm{PGO}^+(A,\sigma)\big)$, computed with $x_1 = \bar{\omega}_1$, $A_{\omega_1} = A$.

We have $\mathrm{Ch}^\star(\mathrm{PGO}_8^+) = \mathbb{F}_2[x_1, x_2, x_3]/(x_1^4, x_2^4, x_3^2)$.
$J(A,\sigma) = (j_1, j_2, j_3)$, with $0 \leqslant j_1, j_2 \leqslant 2$ and $0 \leqslant j_3 \leqslant 1$.

$J(B,\tau), J(C,\gamma) \in \{(j_1, j_2, j_3), (j_2, j_1, j_3)\}$

# The value of $J(A,\sigma)$ when $\deg(A) = 8$

$(A,\sigma) = (\mathrm{End}_D(M), \mathrm{ad}_h)$, for some $(M,h)$ hermitian over $(D, ^-)$

Index: $\mathrm{ind}(A) = \deg(D) = 2^{\mathrm{i}_A}$, $0 \leqslant \mathrm{i}_A \leqslant 3$.

The involution $\sigma$ is isotropic / hyperbolic iff the hermitian form $h$ is isotropic / hyperbolic

## Theorem (QSZ)

*Let $\mathcal{T} = \big((A,\sigma),(B,\tau),(C,\gamma)\big)$ be a trialitarian triple, with $\mathrm{ind}(A) \leqslant \mathrm{ind}(B) \leqslant \mathrm{ind}(C)$.*
*Let $j = \min\{\mathrm{i}_A, 2\}$ and $j' = \min\{\mathrm{i}_B, 2\}$*
*Then*

$$J(A,\sigma) = (j, j', j_3),$$
$$J(B,\tau) = J(C,\gamma) = (j', j, j_3).$$

*Moreover,* $j_3 = \begin{cases} 0 \text{ if } \mathcal{T} \text{ is isotropic} \\ 1 \text{ if } \mathcal{T} \text{ is anisotropic} \end{cases}$

# Proof

Based on the main theorem in QSZ, which consists of inequalities:

$$\begin{matrix} j_1 \\ j_1, j_2 \end{matrix} \quad \leftrightarrow \quad \text{indices of Tits algebras of } G.$$

## Example

$J(A,\sigma) = (j_1, j_2, j_3); \quad i_J = \min\{\mathrm{i}_A, \mathrm{i}_B, \mathrm{i}_C\};$

We have:

1. $\begin{cases} j_1 \leqslant \mathrm{i}_A \\ j_2 \leqslant \mathrm{i}_B, \mathrm{i}_C \end{cases}$
2. $\begin{cases} \mathrm{i}_J > 0 \quad \Rightarrow \quad j_1 > 0 \text{ and } j_2 > 0 \\ \mathrm{i}_J > 1 \quad \Rightarrow \quad j_1 > 1 \text{ and } j_2 > 1 \end{cases}$

# Possible values

## Corollary

1. *The values* $(1,2,0)$, $(2,1,0)$ *and* $(2,2,0)$ *are impossible.*
2. *All other values* $(j_1, j_2, j_3)$ *with* $0 \leqslant j_1, j_2 \leqslant 2$ *and* $0 \leqslant j_3 \leqslant 1$ *do occur.*

## Theorem (Tits-Allen 1968)

$(E, \rho)$ *central simple algebra with orthogonal involution,*
*of degree* $\equiv 0 \mod 4$
*If* $\rho$ *is hyperbolic, then* $\mathcal{C}(E, \rho)$ *has a split component.*

# Examples of trialitarian triples

$Q_1, Q_2, Q_3, Q_4$ quaternion algebras, with

$$Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4 \text{ split.}$$

$Q_1 \otimes Q_2 \sim Q_3 \otimes Q_4 \sim D$;   Choose $\rho$ orthogonal involution of $D$.

$$\begin{cases} (Q_1, {}^-) \otimes (Q_2, {}^-) = \mathrm{Ad}_{h_{12}} \\ (Q_3, {}^-) \otimes (Q_4, {}^-) = \mathrm{Ad}_{h_{34}} \end{cases} \qquad h_{12},\ h_{34} \text{ hermitian over } (D, \rho)$$

## Definition (Dejaiffe)

$$(Q_1, {}^-) \otimes (Q_2, {}^-) \boxplus_{\lambda} (Q_3, {}^-) \otimes (Q_4, {}^-) := \mathrm{Ad}_{h_{12} \oplus \lambda h_{34}}$$

## Proposition

$\mathcal{C}(A, \sigma) = (\mathcal{C}_+, \sigma_+) \times (\mathcal{C}_-, \sigma_-)$, *where*

$$\begin{cases} (\mathcal{C}_+, \sigma_+) = (Q_1, {}^-) \otimes (Q_3, {}^-) \boxplus_{\lambda} (Q_2, {}^-) \otimes (Q_4, {}^-), \\ (\mathcal{C}_-, \sigma_-) = (Q_1, {}^-) \otimes (Q_4, {}^-) \boxplus_{\lambda} (Q_2, {}^-) \otimes (Q_3, {}^-). \end{cases}$$