An introduction to disordered systems: random polymers in random environment

Quentin Berger*

(Lecture notes under construction)

January 28, 2026

Table of contents

General introduction

The goal of these lectures is to give an introduction to disordered systems. But first, let us give in this informal introduction a vague idea of what is a disordered systems and what questions one may try to answer.

Disordered systems: the recipe

The general recipe for constructing disordered systems is very simple (we give examples below):

(i) Take a physical system with "homogeneous" interactions, *i.e.* with physical parameters (such as temperature, external field, etc.) that are fixed throughout the system. If the system exhibits a phase transition with respect to one of the parameter, this is even better.

(ii) Sprinkle some disorder, that is add a (tiny) bit of randomness into the system. There are several ways to do it, but essentially, this means that the physical parameters are not constant throughout the system but have some (random) fluctuations.

The question of disorder relevance

You now have a disordered system and you are left with many questions. The main one is whether the addition of disorder has changed the properties of your initial system. More precisely, the question of disorder relevance is to determine whether an arbitrarily small quantity of disorder modifies drastically the properties of the system. In other words, one wonders whether the homogeneous system remains stable if one adds a grain of sand...

For instance, if the homogeneous system had a phase transition, the question my be asked in two steps: First, does the disordered system still exhibit a phase transition? Second, are the characteristics of the disordered phase transition similar to that of the homogeneous one? The question of the influence of disorder can then be framed (informally) as follows: if one denotes by ε the "intensity" of disorder, one says that

- disorder is *irrelevant* if there is some $\varepsilon_0 > 0$ such that, for $\varepsilon \leq \varepsilon_0$, the disordered and homogeneous systems behave similarly;
- disorder is *relevant* if for any $\varepsilon > 0$ the disordered and homogeneous systems have different behaviors.

This question is disorder relevance/irrelevance is central in the literature on disordered systems, and the physicist A.B. Harris [Har74] has devised a criterion to predict whether disorder should be relevant or irrelevant. We will come back to this later on.

Some examples

Let us now give a few examples of disordered systems that have been studies in the literature. We will review briefly the first two examples in Chapter 1, which will also serve as an introduction to the general framework of the question of disorder relevance. The two following examples will be at the center of the lecture notes, in Part I and Part II respectively.

The Ising model

The Ising model is a simplified model to describe a magnetic material: spins take value in $\{+1, -1\}$ and the interactions are nearest-neighbor and ferromagnetic (*i.e.* attractive).

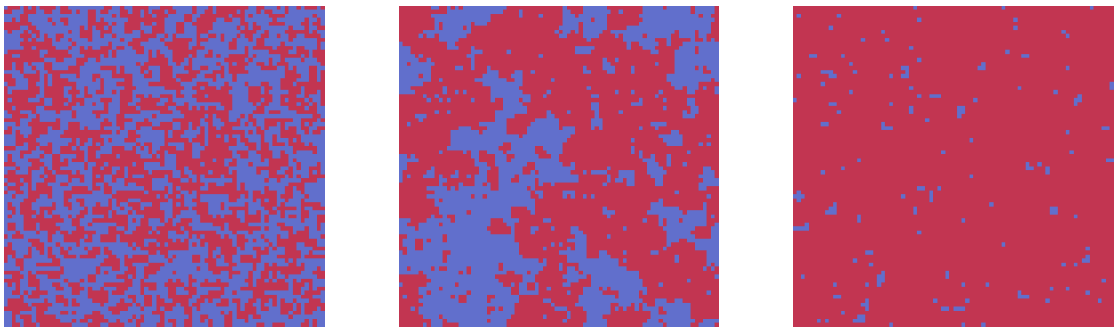

Figure 1: Realizations of the Ising model on a grid, with '+' spins in red and '−' spins in blue. Three different temperatures are represented, with from left to right: below the critical temperature, at the critical temperature, above the critical temperature.

It is a famous result by R. Peierls [Pei36] that the (homogeneous) Ising model on $\mathbb{Z}^d$ undergoes a phase transition in dimension $d \geq 2$. One can then ask what is the effect of adding a *random* external field, with which the spins have a tendency to align. We discuss this in Chapter 1 (Section 1.2).

Models of random interfaces

Random surfaces models have been considered in the literature, for instance as effective interfaces models. An important class that has been extensively studied are the so-called $\nabla\varphi$-interfaces, where a surface is penalized by a function of its gradient, see [Vel06] for an overview; two important examples are the lattice Gaussian Free Field and the Solid-On-Solid model.

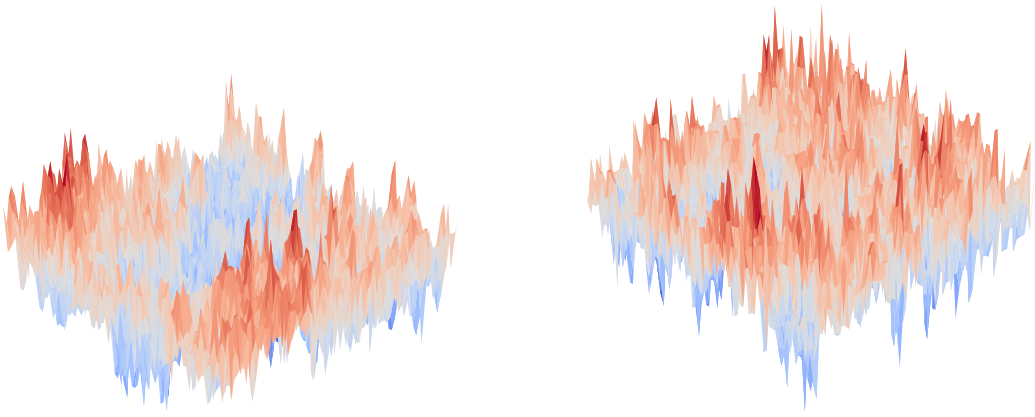

Figure 2: Two examples of random interfaces: on the left, a realization of the Gaussian Free Field; on the right, a realization of the Solid-On-Solid model.

Again, one can then wonder about the effect of adding a *random* external field, which "pulls" the interface in a given direction. We discuss this in Chapter 1 (Section 1.3).

The (polymer) pinning model

Polymer pinning models describe a polymer interacting with a defect line. The model has been studied in a slightly different form in the '60s to model the DNA denaturation phenomenon, see [PS66], or in the physics literature in [Fis84]. The model is very well understood when the interactions are homogeneous (see e.g. [Gia07, Ch. 2]), and one may wonder about the effect of making the interactions inhomogeneous, for instance if the polymer is made of different types of monomers.

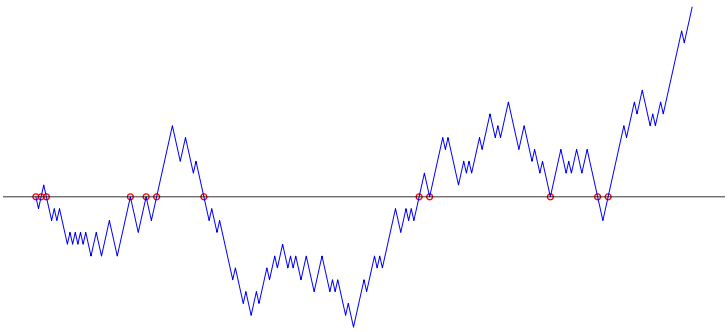

Figure 3: Illustration of the pinning model: a directed polymer (a random walk path) interacts with a defect line when it comes into contact with it, *i.e.* when the random walk returns to 0.

Let us mention that the question of disorder relevance has seen a very intense activity for the pinning model, first in the physics literature and then in the mathematics literature over the past decades, which finally managed to put the Harris predictions on rigorous ground. This is the main model of interest of Part I: the disordered pinning model will be introduced in Chapter **??**; the question of disorder relevance for this model will be addressed in Chapter **??**.

The directed polymer model

The directed polymer model can be used to describe a polymer placed in some heterogeneous medium. In that case, the homogeneous model is simply a random walk on $\mathbb{Z}^d$, which is indeed very well understood. Then, the question is to understand the effect of placing this random walk in some heterogeneous solvent.

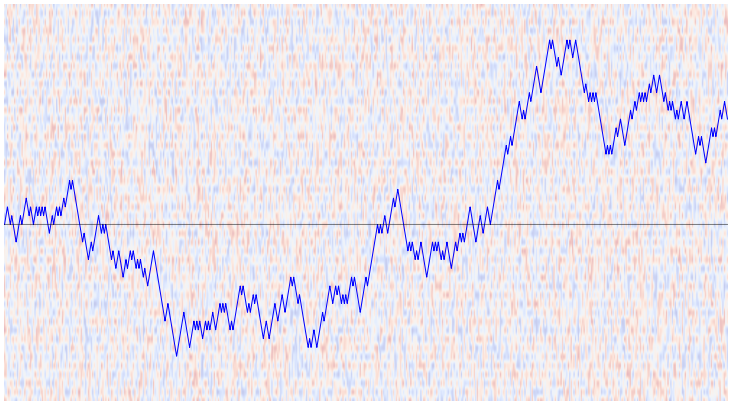

Figure 4: An illustration of the directed polymer model: a directed polymer (a random walk path) which is placed in some heterogeneous medium.

This model has been introduced in the 80's in the physics literature [HH85], and has since then been studied in depth, see [Com17] for a recent mathematical reference. This will be at the center of Part II: we introduce the directed polymer model and we give its first properties in Chapter **??**; the question of disorder relevance and some features of its phase transition will be discussed in chapter **??**; we will also study its scaling limit and its relation to the Stochastic Heat Equation (SHE) in Chapter **??**.

Chapter 1

First examples of disordered systems

1.1 The general statistical mechanics setting

In this section, we start by giving a somehow general framework in which we will study the question of the role of disorder in physical systems: we will mostly study disordered systems that arise as small perturbations of Gibbs measures.

Gibbs measures with homogeneous parameters

First of all, we introduce a class of models that are defined through Gibbs measures. For this, we need to introduce a few notations. We let $N \in \mathbb{N}$ be a parameter that determines the "size" of the system.

- Ω_N = set of all possible configurations of the system (this comes with a σ-algebra $\mathcal{G}_N$);
- μ_N^{ref} = a reference measure on Ω_N (the most "natural" one, if possible);
- θ = some fixed parameter(s), which may include the temperature, the external field, etc.;
- $H_{N,\theta}$ = a Hamiltonian, *i.e.* a function $\Omega_N \to \mathbb{R}$, which determines the energy of any configuration.

Then, we define a Gibbs probability measure $\mathbf{P}_{N,\theta}$ on Ω_N with Hamiltonian $H_{N,\theta}$ as a modification of the reference measure μ_N^{ref}, as follows:

$$\mathrm{d}\mathbf{P}_{N,\theta}(\sigma) = \frac{1}{Z_{N,\theta}} \, \mathrm{e}^{-H_{N,\theta}(\sigma)} \, \mathrm{d}\mu_N^{\mathrm{ref}}(\sigma) \, . \tag{1.1}$$

Here, the quantity $Z_{N,\theta}$ is used to normalize $\mathbf{P}_{N,\theta}$ to a probability measure: it is called the *partition function*, and can be written explicitly as

$$Z_{N,\theta} = \int_{\Omega_N} \mathrm{e}^{-H_{N,\theta}(\sigma)} \, \mathrm{d}\mu_N^{\mathrm{ref}}(\sigma) \, . \tag{1.2}$$

The intuition behind (1.1) is that $\mathbf{P}_{N,\theta}$ gives more weights to configurations $\sigma \in \Omega_N$ with lower energy*. The specific form (1.1) is in fact very natural, since Gibbs measures are the probability measure that *maximize the entropy at a given energy.*

Example 1 (Maxwell–Boltzmann distribution: Gaussian velocities)**.** Consider a system of N non-interacting particles, that are described by their velocities $v_1, \dots, v_N \in \mathbb{R}^3$. In this case, the configuration space is $\Omega_N = (\mathbb{R}^3)^N$, the reference measure μ_N^{ref} is the Lebesgue measure on Ω_N. The parameters are the mass m of the particles and the temperature T (we write $\theta = (m, T)$), and the Hamiltonian is defined by the sum of the kinetic energy of the particles, $H_{N,\theta} = \frac{1}{k_B T} \sum_{i=1}^N \frac{1}{2} m \|v_i\|^2$. (Note that we have included the temperature in the Hamiltonian,

*Usually, one finds in the literature that the Gibbs measure is of the form $\propto \mathrm{e}^{-\frac{1}{k_B T} H(\sigma)} \mathrm{d}\mu_N^{\mathrm{ref}}(\sigma)$, with T the inverse temperature and k_B the Boltzmann constant, but we have chosen in this presentation to include the term $1/k_B T$ in the definition of the Hamiltonian, more precisely in the parameter θ.

see footnote ∗.) In that case, the Gibbs measure (1.1) is known as the Maxwell–Boltzmann distribution: for $(v_1, \dots, v_N) \in \Omega_N = (\mathbb{R}^3)^N$, we have

$$\mathbf{P}_{N,m,T}(\mathrm{d}(v_1, \dots, v_N)) = \Big(\frac{m}{2\pi k_B T}\Big)^{3N/2} \exp\Big(-\frac{m}{2k_B T}\sum_{i=1}^{N} \|v_i\|^2\Big)\, \mathrm{d}v_1 \cdots \mathrm{d}v_N\,.$$

In particular, the velocities of the particles are independent Gaussian[†] random variables $\mathcal{N}(0, \frac{k_B T}{m} I_3)$.

Example 2 (Some polymer models)**.** Let us give other examples, more in relation with the polymer models that we will consider afterwards. One may consider as a configuration space the set of all possible (nearest-neighbor) simple random walk trajectories of length N on $\mathbb{Z}^d$, *i.e.* $\Omega_N := \{(s_0, s_1, \dots, s_N) \in (\mathbb{Z}^d)^{N+1}, s_0 = 0, \|s_i - s_{i-1}\|_1 = 1\}$. Such a trajectory can be interpreted as a *polymer configuration*, where $s_i \in \mathbb{Z}^d$ can be thought as the position of the i-th monomer. It is then natural to take μ_N^{ref} as the uniform measure on Ω_N, *i.e.* $\mu_N^{\mathrm{ref}}(s) = \frac{1}{(2d)^N}\mathbf{1}_{\Omega_N}(s)$ since $|\Omega_N| = (2d)^N$.

One can then consider several polymer models, depending on how the polymer interact with its environment or with itself. Let us simply give two "simple" models that can be considered:

- If the polymer is placed in a confining potential: this corresponds to considering the Hamiltonian $H^V_{N,\beta}(s) = \beta \sum_{i=0}^{N} V(s_i)$ for $s = (s_0, s_1, \dots, s_N) \in \Omega_N$, where $V : \mathbb{R}^d \to \mathbb{R}_+$ is symmetric and diverging as $\|x\| \to \infty$ (a classical example is the quadratic potential $V(x) = \|x\|^2$) and $\beta \geq 0$ is some parameter tuning the strength of the potential $V(\cdot)$.
- If the polymer is self-repellent (or weakly self-avoiding): this corresponds to considering the Hamiltonian $H_{N,\gamma}(s) = \gamma \sum_{0 \leq i < j \leq N} \mathbf{1}_{\{s_i = s_j\}}$ for $s = (s_1, \dots, s_N) \in \Omega_N$, where $\gamma \geq 0$ tunes the strength of the self-interactions.

We can in fact consider both models simultaneously, by considering the Gibbs measure: for $\beta, \gamma \geq 0$.

$$\mathbf{P}^V_{N,\beta,\gamma}(s_0, \dots, s_N) = \frac{1}{Z^V_{N,\beta,\gamma}} \exp\Big(-\gamma \sum_{0 \leq i < j \leq N} \mathbf{1}_{\{s_i = s_j\}} - \beta \sum_{i=0}^{N} V(s_i)\Big) \frac{1}{(2d)^N}\mathbf{1}_{\Omega_N}(s)\,. \tag{1.3}$$

In particular, the potential $V(\cdot)$ has the effect to penalize trajectories that wander to far from the origin, whereas the term $\gamma \mathbf{1}_{\{s_i = s_j\}}$ penalizes (or rewards if $\gamma \leq 0$) monomers that sit on the same site. We refer to [dH07] for an overview of different related polymer models.

There are already many questions that one may ask about the "homogeneous" Gibbs measure (1.1). In general, one wishes to understand the properties of typical configurations drawn under $\mathbf{P}_{N,\theta}$, and how they differ from the reference measure μ_N^{ref} depending on the parameter(s) θ. For instance, one may ask whether there is some *infinite volume* limit of the Gibbs measure: in the case where $\Omega_N \uparrow \Omega$ as $N \to \infty$, can one make sense of a limit $\lim_{N\to\infty} \mathbf{P}_{N,\theta}$? Another important question is that of the phase transition: informally, can one show that some of the properties of the system drastically change when the parameter(s) θ crosses some critical threshold?

Remark 1.1. Let us mention that, as we will see in these lecture notes, a lot of information is contained in the partition function (1.2). For instance, if $A \subset \Omega_N$ is measurable, we have that

$$\mathbf{P}_{N,\theta}(A) := \frac{1}{Z_{N,\theta}} \int_A \mathrm{e}^{-H_{N,\theta}(\sigma)}\, \mathrm{d}\mu_N^{\mathrm{ref}}(\sigma) =: \frac{Z_{N,\theta}(A)}{Z_{N,\theta}}\,,$$

where $Z_{N,\theta}(A)$ is the partition function restricted to A. In other words, the probabilities $\mathbf{P}_{N,\theta}(A)$ are ratios are partition functions (or of integrals), and understanding the behavior of the partition functions turns out to be a central issue in statistical mechanics.

[†]Let us also refer to [BCDP21, Ex. 6.34] for an exercise that shows that the normal distribution is the one that maximizes the entropy with a given variance.

Adding a random external field

Let us now turn to the definition of (one possible version of) a disordered system, by introducing some additional source of randomness in the Gibbs measure $\mathbf{P}_{N,\theta}$ in (1.1). We do this by adding a random external field, which is a very common way in the literature of introducing some disorder (but there are various other ways).

To simplify, let us consider the following framework:

- Let $\Lambda_N \Subset \mathbb{Z}^d$ for some $d \geq 3$, where $B \Subset \mathbb{Z}^d$ means that B is a finite subset of $\mathbb{Z}^d$;
- Assume that $\Omega_N \subseteq \mathbb{R}^{\Lambda_N}$.

For instance, in the examples of Sections 1.2 and 1.3 below, we will consider $\Lambda_N = \{-N, \dots, N\}^d$ and either $\Omega_N = \{-1, +1\}^{\Lambda_N}$ (in which case a configuration $\sigma \in \Omega_N$ is interpreted as a spin configuration, where σ_x gives the value of the spin at position $x \in \Lambda_N$) or $\Omega_N = \mathbb{R}^{\Lambda_N}$ (in which case a configuration $\varphi \in \Omega_N$ is interpreted as an interface, where φ_x represents the height of the interface above the point $x \in \Lambda_N$). Example 2 also fall in this framework, since the Hamiltonian only depends on the local time[‡] $\ell_x^N(s) = \sum_{i=0}^N \mathbf{1}_{\{s_i = x\}}$ of the trajectory s at point x after time N (note that $\ell_x^N(s) = 0$ if $x \notin \Lambda_N$). Thus the Gibbs measure (1.3) can be interpreted as a Gibbs measure on the local time field $(\ell_x^N)_{x \in \Lambda_N} \in \hat{\Omega}_N := \mathbb{N}^{\Lambda_N}$, but the reference measure μ_N^{ref} is then of course much more involved.

We are now ready to add a random external field. Let us introduce some field of i.i.d. random variables $\eta = (\eta_x)_{x \in \mathbb{Z}^d}$, whose law we denote by $\mathbb{P}$. Assume that $\mathbb{E}[\eta_0] = 1$ and $\mathbb{V}(\eta_0) = 1$; in fact, one can think of $\eta_0 \sim \mathcal{N}(0,1)$ in the following. Let us also introduce some parameter $\varepsilon \geq 0$ that will play the role of *disorder intensity*. We can then consider a Gibbs measure on Ω_N with some ε-disordered Hamiltonian $H_{N,\theta}^{\varepsilon,\eta}$, where in th case of a random external field is most commonly written as

$$H_{N,\theta}^{\varepsilon,\eta}(\sigma) := H_{N,\theta}(\sigma) - \varepsilon \sum_{x \in \Lambda_N} \eta_x \sigma_x \,, \qquad \text{for } \sigma \in \Omega_N \subseteq \mathbb{R}^{\Lambda_N} \,.$$

Then, the disordered system is given by this new Gibbs measure: for a given realization of $\eta = (\eta_x)_{x \in \mathbb{Z}^d}$

$$\mathrm{d}\mathbf{P}_{N,\theta}^{\varepsilon,\eta}(\sigma) = \frac{1}{Z_{N,\theta}^{\varepsilon,\eta}} \, \mathrm{e}^{-H_{N,\theta}^{\varepsilon,\eta}(\sigma)} \, \mathrm{d}\mu_N^{\mathrm{ref}}(\sigma) = \frac{1}{\tilde{Z}_{N,\theta}^{\varepsilon,\eta}} \exp\Big(\varepsilon \sum_{x \in \Lambda_N} \eta_x \sigma_x\Big) \mathrm{d}\mathbf{P}_{N,\theta}(\sigma) \,, \tag{1.4}$$

where in the second identity we simply have used the form of the Hamiltonian and of the Gibbs measure $\mathbf{P}_{N,\theta}$, and used a different partition function.

Quenched vs. annealed. Notice that there are two sources of randomness:

- the external field $\eta = (\eta_x)_{x \in \mathbb{Z}^d}$, also called *environment*, is random, with law denoted by $\mathbb{P}$;
- given an environment η, a configuration $\sigma \in \Omega_N$ is drawn randomly under $\mathbf{P}_{N,\theta}^{\varepsilon,\eta}$ (which is a probability measure on Ω_N which depends on η!).

The *quenched* setting consists in studying the measure $\mathbf{P}_{N,\theta}^{\varepsilon,\eta}$ from (1.4) with a fixed (quenched), typical, environment. In other words, one studies the behavior of configurations under $\mathbf{P}_{N,\theta}^{\varepsilon,\eta}$ *conditionally on the environment* η. For instance, the goal is to show statements with the following flavor: $\mathbb{P}$-a.s. in the environment η, $\mathbf{P}_{N,\theta}^{\varepsilon,\eta}$ exhibits some given behavior in the large N limit.

The *annealed* setting consists in considering the joint law of (σ, η), which, when only considering the marginal law of the configuration σ, amounts to averaging over η. The annealed law[§] is given by

$$\mathrm{d}\mathbf{P}_{N,\theta}^{\varepsilon,\mathrm{ann}}(\sigma,\eta) = \frac{1}{Z_{N,\theta}^{\varepsilon,\mathrm{ann}}} \, \mathrm{e}^{-H_{N,\theta}^{\varepsilon,\eta}(\sigma)} \, \mathrm{d}\mu_N^{\mathrm{ref}}(\sigma) \mathrm{d}\mathbb{P}(\eta) \,. \tag{1.5}$$

In particular, the annealed partition function is $Z_{N,\theta}^{\varepsilon,\mathrm{ann}} = \mathbb{E}[Z_{N,\theta}^{\varepsilon,\eta}]$.

[‡] More precisely, we can rewrite $\sum_{i=0}^N V(s_i) = \sum_{x \in \Lambda_N} \ell_x^{(N)}(s) V(x)$ and also $\sum_{0 \leq i < j \leq N} \mathbf{1}_{\{s_i = s_j\}} = \frac{1}{2} \sum_{x \in \Lambda_N} \ell_x^N(s)(\ell_x^N(s) - 1)$.

[§] As mentioned above, in practice, one often considers only the first marginal in (1.5).

Disorder relevance and Harris criterion

The quenched and annealed models may have very different behaviors, since the quenched measure $\mathbf{P}_{N,\theta}^{\varepsilon,\eta}$ is studied only for "typical" environments η, whereas it is possible that atypical environment contribute to typical events under the annealed measure $\mathbf{P}_{N,\theta}^{\varepsilon,\mathrm{ann}}$, which may help make the Hamiltonian $H_{N,\theta}^{\varepsilon,\eta}$ much smaller. One of the main question is therefore to compare the quenched and annealed model, to understand whether they have similar properties, at least for ε small enough.

This is a very broad subject, and A.B. Harris [Har74] proposed some criterion to predict whether disorder should be relevant or irrelevant. The criterion is in fact based only on the understanding of the homogeneous model (*i.e.* from (1.1)). If the homogeneous model undergoes a phase transition around some critical point, then if one "knows" the behavior near criticality one can predict whether disorder is relevant or not, that is whether adding some disorder with an arbitrarily small intensity modifies the characteristics of the phase transition.

Though Harris' criterion dates back to the 1970's, there does not seem to be a (rigorous) general theory that indeed proves disorder relevance or irrelevance. The literature consists more of case by case studies. We give two examples in the rest of this chapter, and then focus on two other (class of) examples in the bulk of these lecture notes.

1.2 The Ising model (with random external field)

We now go a bit more into the details of the Ising model, in order to illustrate better how the question of the influence of disorder haa been addressed in this setting. For an introduction to the Ising model, we refer to the excellent book of Friedli and Velenik [FV17, Chapter 3].

1.2.1 The standard Ising model on $\mathbb{Z}^d$

Let $\Lambda_N := \{-N, \dots, N\}^d$ be the box of radius N in $\mathbb{Z}^d$, and let us denote $x \sim y$ if x, y are neighbor in $\mathbb{Z}^d$. The set of all spin configurations is $\Omega_N := \{-1, +1\}^{\Lambda_N}$.

Definition 1.2 (Ising model)**.** The Ising model in Λ_N with coupling constant $J > 0$ and *boundary conditions* ζ is given by the Gibbs measure

$$\mathbf{P}_{N,J}^{\zeta}(\sigma) := \frac{1}{Z_{N,J}^{\zeta}} \mathrm{e}^{-H_{N,J}^{\zeta}(\sigma)} \qquad \forall\, \sigma \in \Omega_N = \{-1, +1\}^{\Lambda_N}\,, \tag{1.6}$$

where the Hamiltonian is given by

$$H_{N,J}^{\zeta}(\sigma) = -J\Bigg(\sum_{\substack{x,y \in \Lambda_N \\ x \sim y}} \sigma_x \sigma_y + \sum_{\substack{x \in \Lambda_N, y \notin \Lambda_N \\ x \sim y}} \sigma_x \zeta_y \Bigg)\,. \tag{1.7}$$

In other words, this is the definition (1.1) with reference measure μ_N^{ref} the counting measure on Ω_N. The Ising measure (1.6) then favors configurations where spins are aligned (*i.e.* such that $\sigma_x = \sigma_y$ so that $\sigma_x \sigma_y = +1$), and the intensity of interactions is tuned by the coupling constant J (this is usually written $\frac{J'}{k_B T}$, but here we have included the term $1/k_B T$ in the definition of J).

The boundary condition ζ, in view of the Hamiltonian (1.7), has some influence on the spins inside Λ. There are three central examples:

- *Free boundary conditions*: $\zeta \equiv 0$. This is equivalent to removing the second term in (1.7), and corresponds to removing the effect of the boundary[¶]. We simply denote by $\mathbf{P}_{N,J}$ (resp. $Z_{N,J}$) the corresponding measure (resp. partition function).

[¶]We could also consider periodic boundary conditions by considering the torus $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^d$ instead of a box, but we do not develop on this.

Note that under this free boundary condition, there are two configurations with minimal energy, called *ground states*: the configurations $\sigma_+ \equiv +1$ and $\sigma_- \equiv -1$ in Λ_N.

- *Plus boundary condition*: $\zeta \equiv +1$. This corresponds to setting all spins outside of the box Λ_N to be equal to $+1$. We denote by $\mathbf{P}^+_{N,J}$ (resp. $Z^+_{N,J}$) the corresponding measure (resp. partition function). Here, the only ground state is $\sigma_+ \equiv +1$.
- *Minus boundary condition*: $\zeta \equiv +1$. This corresponds to setting all spins outside of the box Λ_N to be equal to -1. We denote by $\mathbf{P}^-_{N,J}$ (resp. $Z^-_{N,J}$) the corresponding measure (resp. partition function). Here, the only ground state is $\sigma_- \equiv -1$.

There are now a series of questions that one may ask. For instance: can one let $N \to \infty$ and obtain some infinite volume limit? what does a typical spin configuration look like under $\mathbf{P}_{N,J}$, $\mathbf{P}^{+/-}_{N,J}$ or $\mathbf{P}_{N,J,h}$? How does this depend on the coupling constant J (which we recall includes the inverse temperature)?

Remark 1.3 (With an external field)**.** We can also consider Ising model with coupling constant J and external field of strength $h \in \mathbb{R}$, whose Hamiltonian is given by the following:

$$H_{N,J,h}(\sigma) := -J \sum_{x,y \in \Lambda_N, x \sim y} \sigma_x \sigma_y - h \sum_{x \in \Lambda_N} \sigma_x \,.$$

We also let $\mathbf{P}_{N,J,h}$ denote the corresponding Gibbs measure on $\Omega_N = \{-1,+1\}^{\Lambda_N}$ (we could also add some boundary condition, but that will not be necessary for our purpose). We have the same heuristic picture as above: spins are favored when then align with each other *and* with the external field, in such a way that $h\sigma_x > 0$.

Some important physicial quantities

In order to answer some of the previous questions, let us introduce some physical quantities that are of interest.

The magnetization. For a boundary condition ζ, we introduce the magnetization of the central spin of Λ_N:

$$m^\zeta_N(J) := \mathbf{E}^\zeta_{N,J}[\sigma_0] \,.$$

Note that, with free boundary conditions we have $m_N(J) = 0$ by symmetry, but for any $N \geq 1$ and $J > 0$ we have $m^+_N(J) > 0$, and $m^-_N(J) = -m^+_N(J) < 0$ by symmetry. We can then define (using some monotonicity properties) the so-called *spontaneous magnetization* as

$$m^*(J) = \lim_{N \to \infty} m^+_N(J) \geq 0 \,,$$

which also turns out to the the asymptotic magnetization of the whole system, that is

$$m^*(J) = \lim_{N \to \infty} \mathbf{E}^+_{N,J}\Big[\frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \sigma_x\Big] \,.$$

The question is then to determine whether one has $m^*(J) = 0$, which means that there is no spontaneous magnetization and that the influence of the boundary vanishes in the large N limit, or whether one has $m^*(J) > 0$,

Remark 1.4. Alternatively, the spontaneous magnetization can also be defined by adding some external field with strength $h > 0$, consider the limiting magnetization and then let $h \downarrow 0$. In other words, recalling Remark 1.3 which defines the Ising model with coupling constant $J > 0$ and external field $h > 0$, we have

$$m^*(J) = \lim_{h \downarrow 0} m(J,h) \qquad \text{with} \qquad m(J,h) = \lim_{N \to \infty} \mathbf{E}_{N,J,h}[\sigma_0] = \lim_{N \to \infty} \mathbf{E}_{N,J,h}\Big[\frac{1}{|\Lambda_N|} \sum_{x \in \Lambda_N} \sigma_x\Big] \,.$$

The pressure or free energy. The pressure, or free energy, of the Ising model with coupling constant J and external field h is given by the following limit:

$$\mathbf{f}(J,h) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z_{N,J,h}\,,$$

where $Z_{N,J,h}$ is the partition function of the Ising model with free boundary condition. Note that the limit above does not depend on the boundary condition, so that one could replace $Z_{N,J,h}$ by $Z^{+/-}_{N,J,h}$.

In practice, the pressure contains important physical information on the system. Indeed, let us denote

$$\mathbf{f}_N(J,h) := \frac{1}{|\Lambda_N|} \log Z_{N,J,h}\,.$$

Then, recalling that

$$Z_{N,J,h} = \sum_{\sigma\in\{-1,+1\}^{\Lambda_N}} \mathrm{e}^{-H_{N,J,h}} \qquad \text{with} \qquad H_{N,J,h} = -J \sum_{x,y\in\Lambda_N, x\sim y} \sigma_x\sigma_y - h\sum_{x\in\Lambda_N}\sigma_x\,,$$

we get that

$$\frac{\partial}{\partial J}\mathbf{f}_N(J,h) = \sum_{\sigma\in\{-1,+1\}^{\Lambda_N}} \frac{1}{|\Lambda_N|} \sum_{x,y\in\Lambda_N,x\sim y} \sigma_x\sigma_y \frac{1}{Z_{N,J,h}}\mathrm{e}^{-H_{N,J,h}} = \mathbf{E}_{N,J,h}\left[\frac{1}{|\Lambda_N|}\sum_{x,y\in\Lambda_N,x\sim y}\sigma_x\sigma_y\right],$$

so that it corresponds to (minus) the *internal energy* of the system under $\mathbf{P}_{N,J,h}$. A similar calculation gives that

$$\frac{\partial}{\partial h}\mathbf{f}_N(J,h) = \mathbf{E}_{N,J,h}\left[\frac{1}{|\Lambda_N|}\sum_{x\in\Lambda_N}\sigma_x\right] = m_N(J,h)\,,$$

which corresponds to the *magnetization* under $\mathbf{P}_{N,J,h}$. After some justification, one can then pass to the limit $N\to\infty$, and the derivatives of $\mathtt{F}$ (if they exist) all have some physical interpretation:

$-J\frac{\partial}{\partial J}\mathbf{f}$	is the *internal energy*;
$J^2\frac{\partial^2}{\partial J^2}\mathbf{f}$	is called the *specific heat*;
$\frac{\partial}{\partial h}\mathbf{f}$	is the *magnetization*;
$\frac{\partial^2}{\partial h^2}\mathbf{f}$	is called the *susceptibility*.

Phase transition

We are now ready to state results on the phase transition for the Ising model. The case of dimension $d=1$ is in fact solvable, and exact calculations are possible; this is the content of Ising's thesis [Isi25].

Theorem 1.5 (Dimension $d=1$)**.** *For any $J\geq 0$ and any $h\in\mathbb{R}$, we have*

$$\mathbf{f}(J,h) = \log\left(\mathrm{e}^J\cosh(h) + \sqrt{\mathrm{e}^{2J}\cosh^2(h) - 2\sinh(2J)}\right).$$

In particular, there is no phase transition and $m^*(J) = \lim_{h\downarrow 0}\frac{\partial}{\partial h}\mathbf{f}(J,h) = 0$ for all $J>0$.

The case of dimension $d\geq 2$ is much richer, and the existence of a phase transition has been shown by Peierls [Pei36]. We refer to Figure 1 on p.2 for an illustration.

Theorem 1.6 (Phase transition in dimension $d\geq 2$)**.** *There exists some $J_c = J_c(d)$ such that:*

- $m^*(J) = 0$ *if* $J < J_c$ *(this is called a* disordered phase*, and correlations decay exponentially);*
- $m^*(J) > 0$ *if* $J > J_c$ *(this is called an* ordered phase*, with long-range correlations).*

Let us mention that the dimension $d = 2$ also turned out to be exactly solvable, notably thanks to transfer-matrix techniques. Let us state for completeness the famous Onsager solution [Ons44] for the free energy when $h = 0$, and who also stated a formula for the spontaneous magnetization without a proof, given a few years later by Yang [Yan52].

Theorem 1.7 (Dimension $d = 2$)**.** *For the Ising model with coupling constant $J > 0$ and no external field ($h = 0$), one has*

$$\mathbf{f}(J, 0) = \log 2 + \frac{1}{8\pi^2} \int_{[0,2\pi)^2} \log \Big(\cosh^2(2J) - \sinh(2J)(\cos\theta_1 + \cos\theta_2) \Big) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \,.$$

Additionally, we have

$$m^*(J) = \big(1 - \sinh(2J)^{-4}\big)_+^{1/8} \,.$$

In particular, we have that $J_c = \frac{1}{2}\log(1+\sqrt{2})$ and $m^*(J) \sim (8\sqrt{2})^{1/8}(J - J_c)^{1/8}$ as $J \downarrow J_c$.

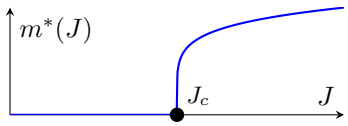

Figure 1.1: Schematic view of the spontaneous magnetization $m^*(J)$ as a function of J in $d = 2$, where the exact formula for $m^*(J)$ is known. The question in higher dimensions are whether one still have a similar picture: do we have continuity at $J = J_c$? can we describe the critical behavior as $J \downarrow J_c$?

There are of course many interesting questions for the Ising model, which remains a very active field of investigation. We refer to [DC23] for an overview of the history of the model and of recent results, in particular regarding its properties at criticality.

1.2.2 The random field Ising model (RFIM)

We now introduce some random external field to the model. Let $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$ be a field of i.i.d. random variables, with law denoted by $\mathbb{P}$. We assume that $\mathbb{E}[\eta_x] = 0$ and $\mathbb{E}[\eta_x^2] = 1$ and that the η_x have all their exponential moments. For simplicity, let us in fact assume that $\eta_x \sim \mathcal{N}(0, 1)$.

Then, we consider the Ising model with coupling constant $J > 0$ with $+$ boundary condition and *random external field* $\varepsilon\eta$, where $\varepsilon > 0$ is the intensity of the external field. The Hamiltonian is

$$H_{N,J}^{+;\varepsilon,\eta}(\sigma) = -J\Big(\sum_{\substack{x,y\in\Lambda_N \\ x\sim y}} \sigma_x\sigma_y + \sum_{\substack{x\in\Lambda_N, y\notin\Lambda_N \\ x\sim y}} \sigma_x \Big) - \varepsilon \sum_{x\in\Lambda_N} \eta_x\sigma_x \,,$$

and so

$$\mathbf{P}_{N,J}^{+;\varepsilon,\eta}(\sigma) = \frac{1}{Z_{N,J}^{+;\varepsilon,\eta}} \exp\left(J\Big(\sum_{\substack{x,y\in\Lambda_N \\ x\sim y}} \sigma_x\sigma_y + \sum_{\substack{x\in\Lambda_N, y\notin\Lambda_N \\ x\sim y}} \sigma_x \Big) + \varepsilon \sum_{x\in\Lambda_N} \eta_x\sigma_x \right)$$

Then, there are two competing effects: (i) the interaction $J\sigma_x\sigma_y$ makes σ_x trying to align with its neighboring spins; (ii) the external field $\varepsilon\eta_x\sigma_x$ makes σ_x trying to align with the external field at site x.

The Imry-Ma argument

The first question is whether this random field Ising model still has a phase transition in dimension $d \geq 2$, at least when ε is fixed small enough. The physicists Imry and Ma [IM75] made the following prediction, which amounts to considering the question of disorder relevance.

Prediction (Imry–Ma)**.** *For the random field Ising model:*

- *In dimension $d = 2$, there is no phase transition and there is* no ordered phase*: in particular, for any $\varepsilon > 0$, the spontaneous magnetization is zero for any $J > 0$. This is an instance of* disorder relevance*, and one can interpret the above as follows: "for any $\varepsilon > 0$, we have $J_c(\varepsilon) = +\infty$".*
- *In dimension $d \geq 3$, there is still a phase transition. One can interpret this as having disorder irrelevance: "there is some ε_0 small enough such that for $\varepsilon \leq \varepsilon_0$ we have $J_c(\varepsilon) < +\infty$".*

Let us briefly present some heuristic for the prediction. The idea is to compare two strategies and understand which one is more likely: either align with the + boundary condition, or align with the external field.

Consider $\sigma_+ \equiv +1$ in Λ_N, which is the ground state (and align with the boundary condition), and σ^{flip} where we have flipped all spins of σ_+ inside a box B_L of width L (in order to align with the external field, which may be negative in this box). Then, we can compare the probabilities $\mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma_+)$ and $\mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma^{\text{flip}})$ by considering their ratios: since the partition functions cancel out, we have

$$\frac{\mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma^{\text{flip}})}{\mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma_+)} = \exp\left(H^{+;\varepsilon,\eta}_{N,J}(\sigma_+) - H^{+;\varepsilon,\eta}_{N,J}(\sigma^{\text{flip}})\right).$$

We easily have that

$$H^{+;\varepsilon,\eta}_{N,J}(\sigma_+) - H^{+;\varepsilon,\eta}_{N,J}(\sigma^{\text{flip}}) = -2J\,|\partial B_L| + 2\varepsilon \sum_{x \in B_L} \eta_x \tag{1.8}$$

where: (i) the first term simply counts how many terms $\sigma_x\sigma_y$ have been changed from $+1$ to -1 (*i.e.* $|\partial B_L|$ with $\partial B_L = \{x \sim y, x \in B_L, y \notin B_L\}$), which then contribute to the difference by $2J$ each; (ii) the second term accounts to the swap that has occurred in the contribution of the external field inside B_L (from $\varepsilon \sum_{x \in B_L} \eta_x$ to $-\varepsilon \sum_{x \in B_L} \eta_x$).

There are therefore two competing effects: note that $|\partial B_L| \approx L^{d-1}$ and that $\sum_{x \in B_L} \eta_x \approx L^{d/2}\mathcal{N}(0,1)$, so that

(i) If $\varepsilon L^{d/2}\mathcal{N}(0,1) > JL^{d-1}$ (*i.e.* $d < 2$), then the external field "wins" and the second term in (1.8) is the dominating one, so that $\mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma_+) \ll \mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma^{\text{flip}})$. The spins are favored if they flip in some large regions to align with the external field, and there is no ordered phase.

(ii) If $\varepsilon L^{d/2}\mathcal{N}(0,1) < JL^{d-1}$ (*i.e.* $d > 2$), then the interactions "win" and the first term in (1.8) is the dominating one, so that $\mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma_+) \gg \mathbf{P}^{+;\varepsilon,\eta}_{N,J}(\sigma^{\text{flip}})$. The spins have no incentive to flip in order to align with the external field, and an ordered phase still exists.

Let us stress that, in dimension $d = 2$, the two terms in (1.8) are of the same order, but the fluctuations of the external field will make the second term dominant on several scales. Thus, one expects the external field to "win" again, and there should be no ordered phase.

Mathematical results

It turns out that the Imry–Ma prediction is correct, and has been proved in several papers. There are in fact very recent results on this subject giving for instance quantitative estimates on the decay of correlations in the disordered phase. Let me give a brief overview of the literature.

In dimension $d = 2$. The absence of phase transition in the random field Ising model has first been proven by Aizenman–Wehr [AW90]: they showed that, for any $J > 0$, we have that the magnetization of the central point $m^{+;\varepsilon}_N(J) := \mathbf{E}^{+;\varepsilon,\eta}_{N,J}[\sigma_0]$ converges to 0 in $\mathbb{P}$-probability as $N \to \infty$.

Let us now state some of the recent progress on this question, which estimates how fast $m^{+;\varepsilon}_N(J)$ goes to 0, and also control the decay of the influence of the boundary conditions. The following (simplified) statements are taken from [AHP20, DX21, DHP24].

Theorem 1.8 (Absence of an ordered phase for the RFIM in $d = 2$)**.** *For any $\varepsilon > 0$ and any $J > 0$, there are some constants $c, C > 0$ such that*

$$0 \le \mathbb{E}\mathbf{E}_{N,J}^{+;\varepsilon,\eta}[\sigma_0] \le \frac{1}{c}\,\mathrm{e}^{-c\,N}\,.$$

Additionally, if one considers $M_{N,J}^{+/-;\varepsilon,\eta} = \mathbf{E}_{N,J}^{+/-;\varepsilon,\eta}\big[\frac{1}{|\Lambda_N|}\sum_{x\in\Lambda_N}\sigma_x\big]$ the system magnetization under $\mathbf{P}_{N,J}^{+/-;\varepsilon,\eta}$, we have that

$$\mathbb{E}\big[\big|M_{N,J}^{+;\varepsilon,\eta} - M_{N,J}^{-;\varepsilon,\eta}\big|\big] \le \frac{C}{(\log\log N)^{1/4}}\,.$$

In dimension $d \ge 3$. The fact that, for ε small enough, there is an ordered phase in dimension $d \ge 3$ has been proven by Imbrie [Imb85] (for $J = +\infty$) and Bricmont–Kuppianen [BK88] (for J large enough). Again, there are recent results on this question, see for instance [DZ24] which provides a simple proof. Let us state one the best result up to date, from [DLX24], which proves that one has long-range order for any $J > J_c$ (from Theorem 1.6), provided that ε is small enough. This shows that "$J_c(\varepsilon) \to J_c(0)$ as $\varepsilon \downarrow 0$".

Theorem 1.9 (Ordered phase for the RFIM in $d = 3$)**.** *Let $d = 3$ and take $J > J_c$. Then, for any $\delta > 0$ there is some $\varepsilon_0 > 0$ and $s > 0$ such that for all $\varepsilon \le \varepsilon_0$*

$$\mathbb{P}\big(\mathbf{E}_{N,J}^{+;\varepsilon,\eta}[\sigma_0] \ge s\big) \ge 1 - \delta \qquad \text{for all } N \ge 1\,.$$

A few words on the dimension $d = 1$

In dimension $d = 1$, there is no phase transition, see Theorem 1.5. In fact, one has $\mathbf{f}(J, 0) = \log(\mathrm{e}^J + \mathrm{e}^{-J}) = J + \log(1 + \mathrm{e}^{-2J})$, and one can think of having a "phase transition" when taking $J \to \infty$. One can also consider the disordered case in this setting, in the limit $J \to \infty$; this is referred to the Random Field Ising Chain and has been studied by a series of recent papers by Collin, Giacomin, Greenblatt and Hu (see e.g. [CGH24, CGGH25, Col25] for the most recent references). We also refer to the lecture notes of Giacomin [Gia24] where a nice overview of the techniques is given; in particular since the one-dimensional Ising model can be written in terms of a product of 2×2 transfer matrices, the random field Ising model is intimately related to the products of random 2×2 matrices and has thus a lot of connections with dynamical systems.

It turns out that, in dimension $d = 1$, the random field Ising chain has some "explicit" ground state (*i.e.* a configuration with minimal energy), called the Fisher configuration, where the spins align with ascending or descending stretches of the cumulated external field (see [CGH24] for some discussion[‖]). This leads to a very precise understanding of the large J limit for the one-dimensional random field Ising model. Let us state some results obtained by Collin, Giacomin, Greenblatt and Hu, which quantify the effect of disorder on the model (we assume that $\eta_x \sim \mathcal{N}(0, 1)$ for the simplicity of exposition, but some moment conditions are important to obtain the following).

Theorem 1.10 (Random field Ising model in $d = 1$)**.** *For any $J > 0$ and $\varepsilon > 0$, the* quenched *free energy*[**]

$$\mathbf{f}(J, \varepsilon) := \lim_{N\to\infty} \frac{1}{2N+1} \log Z_{N,J}^{+;\varepsilon,\eta} \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P})\,,$$

exists and is constant $\mathbb{P}$-a.s. Then, there is a constant $\kappa \in \mathbb{R}$ such that, as $J \to \infty$, we have

$$\mathbf{f}(J, \varepsilon) = \frac{\varepsilon^2}{2J + \kappa} + O(\mathrm{e}^{-2J})\,.$$

(Recall that in the homogeneous case we have $\mathbf{f}(J, 0) = J + O(\mathrm{e}^{-2J})$ as $J \to \infty$.)

[‖] The Fisher configuration turns out to be related to the so-called Γ-extrema of a random walk or Brownian motion, which are well understood, see [NP89].

[**] Also referred to as the Lyapunov exponent when using the interpretation in terms of products of random 2×2 matrices.

Let us add more informally that, if σ^{F} denotes the so-called Fisher configuration, [CGH24] proves that

$$\frac{1}{2N+1}\sum_{x=-N}^{N}\mathbf{1}_{\{\sigma_x\neq\sigma_x^{\mathrm{F}}\}}\xrightarrow[N\to\infty]{} D_J \quad \text{“in } \mathbf{P}_{N,J}^{+;\varepsilon,\eta}\text{ probability”},$$

where D_J is a constant and verifies that $D_J = O(\frac{\log\log J}{J})$ as $J\to\infty$. In other words, under $\mathbf{P}_{N,J}^{+;\varepsilon,\eta}$, the configurations agree with the Fisher configuration (*i.e.* with the ground state), with an error on at most a proportion D_J of sites.

1.3 Random interfaces (with random external field)

Let us now turn to another class of models, which have been used as effective interface models. We refer to [Fun05, Vel06] (or even [Ber, Ch. 1])for an overview of interface models, and [DHP23] for the version with a random field.

1.3.1 The $\nabla\varphi$-interface model

Let again $\Lambda_N := \{-N,\dots,N\}^d$ be the box of radius N in $\mathbb{Z}^d$, and let $\Omega_N = \mathbb{R}^{\Lambda_N}$ or $\Omega_N = \mathbb{Z}^{\Lambda_N}$ be the configuration space. An element $\varphi\in\Omega_N$ is thought as a height function, where φ_x denotes the height of the interface above the point x. When $\Omega_N = \mathbb{R}^{\Lambda_N}$, we refer to φ as being a continuous-height interface, whereas when $\Omega_N = \mathbb{Z}^{\Lambda_N}$, we refer to φ as being a discrete-height interface.

Then, for $V:\mathbb{R}\to\mathbb{R}_+$ some symmetric convex function (with $V(0)=0$), we consider the following Hamiltonian:

$$H_{N,V}(\varphi) = \frac{1}{4d}\sum_{\substack{x,y\in\Lambda_N\\ x\sim y}} V(\varphi_x-\varphi_y) + \frac{\beta}{2d}\sum_{\substack{x,y\notin\Lambda_N\\ x\sim y}} V(\varphi_x-0) \tag{1.9}$$

which corresponds to giving higher energy to height functions with large gradients $\nabla\varphi$. We then introduce the Gibbs measure, which defines the $\nabla\varphi$-interface with potential $V(\cdot)$:

$$\mathrm{d}\mathbf{P}_{N,V}(\varphi) = \frac{1}{Z_{N,V}}\,\mathrm{e}^{H_{N,V}(\varphi)}\mathrm{d}\mu_N^{\mathrm{ref}}(\varphi)\,,$$

where μ_N^{ref} is the Lebesgue measure on $\mathbb{R}^{\Lambda_N}$ for continuous heights interfaces and μ_N^{ref} is the counting measure on $\mathbb{Z}^{\Lambda_N}$ for discrete heights interfaces.

Let us observe that, in (1.9), we only consider for simplicity:

- The case of 0-boundary condition, *i.e.* we have set $\varphi\equiv 0$ outside of Λ_N; one could also consider any given boundary condition ζ, simply by replacing $V(\varphi_x-0)$ by $V(\varphi_x-\zeta_y)$ in the last sum.
- Nearest neighbor interactions; one could also consider longer-range interactions, simply by replacing in the sums the terms $\frac{1}{2d}\mathbf{1}_{\{x\sim y\}}$ by $q(x-y)$ for some probability $q(\cdot)$.

Example 3 (GFF)**.** The first and most studied example is that of the (lattice) Gaussian Free Field, which is a continuous height interface (φ is $\mathbb{R}$-valued), where one takes as a potential $V(x)=\frac{1}{2}x^2$. One can think of springs that pull closer to each other points with different heights. A large class of interfaces that fall into the same “universality” are when $V(\cdot)$ is uniformly strictly convex, *i.e.* $0<c_1\le V''(x)\le c_2<+\infty$ for all x.

Example 4 (SOS)**.** Another important example that has been studied in the literature is the so-called Solid-On-Solid model, which is used to model some interface in the Ising model. This is a discrete height interface (φ is $\mathbb{Z}$-valued), where one takes as a potential $V(x)=2J|x|$, which amounts to the energy coming from disagreeing spins along the interface. Note that the p-SOS model has also been considered in the literature, where the potential is $V(x)=\beta|x|^p$ for some $p\ge 1$.

Localization vs. delocalization

For simplicity, in the rest of this section, we consider only the case of continuous height interfaces. The first question that one asks about the interfaces concerns the fluctuations of the interface. For instance, one can consider the height φ_0 of the central point, which verifies $\mathbf{E}_{N,V}[\varphi_0] = 0$ by symmetry, and ask about its fluctuations: in particular, one wants to estimate $\mathbf{E}_{N,V}[\varphi_0^2]$.

The case of dimension $d = 1$. In dimension $d = 1$, since the interactions are only nearest neighbor, one realizes that $\mathbf{P}_{N,V}$ is a product measure with the constraint that $\varphi_{-(N+1)} = \varphi_{N+1} = 0$. In particular, letting $(X_i)_{i\geq 1}$ be i.i.d. random variables with density $cst.\, \mathrm{e}^{-V(x)}$ and $S_k := \sum_{i=1}^k X_i$ the associated random walk one can see that the law of $(\varphi_x)_{-(N+1)\leq x\leq N+1}$ is the law of $(S_k)_{0\leq k\leq 2N+3}$ conditioned on having $S_{2N+3} = 0$. In other words, $(\varphi_x)_{-(N+1)\leq x\leq N+1}$ is a random walk bridge, and we get that the fluctuations of the interface are of order $\sqrt{N}$.

In particular, in dimension $d = 1$, we have $\mathbf{E}_{N,V}[\varphi_0^2] \sim c\, N$ as $N \to \infty$, and the diffusively rescaled interface converges to a Brownian bridge.

Dimension $d = 2$ vs. dimension $d \geq 3$. The following result is proven in [BLL75] and [BL76], and somehow describes a generic situation for continuous-height interfaces. We include the case of dimension $d = 1$ to collect all results in a single statement.

Theorem 1.11 (Localization vs. Delocalization)**.** *Assume that $V(\cdot)$ is uniformly strictly convex, that is $V : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $0 < c_1 \leq V''(x) \leq c_2 < +\infty$ for all x.*

(i) If $d = 1$, the interface has fluctuations of order $\sqrt{N}$: there is a constant c such that

$$\mathbf{E}_{N,V}[\varphi_0^2] \sim c\, N \qquad \text{as } N \to \infty\,.$$

(ii) If $d = 2$, the interface has fluctuations of order $\sqrt{\log N}$: there are constants c, C such that for all $N \geq 2$

$$c \log N \leq \mathbf{E}_{N,V}[\varphi_0^2] \leq C \log N\,.$$

(iii) If $d \geq 3$, the interface is rigid*: there is a constant C such that for all $N \geq 2$*

$$\mathbf{E}_{N,V}[\varphi_0^2] \leq C\,.$$

In fact, one can show that: (i) in dimension $d = 1$ the interface, rescaled by $\sqrt{N}$, converges to a Brownian bridge; (ii) in dimension $d = 2$, the interface converges as a distribution towards the continuous GFF (which is a random distribution); (iii) in dimension $d \geq 3$,

Proof. Let us give the proof of Theorem 1.11 in the case of the (lattice) GFF, *i.e.* when $V(x) = \frac{1}{2}x^2$ (let us drop the dependence on V in the notation from now on). In that case, the interface $(\varphi_x)_{x\in\Lambda_N}$ is a Gaussian vector, with covariance given by the Green function of the simple random walk killed when exiting Λ_N:

$$\mathbf{C}\mathrm{ov}_N(\varphi_x, \varphi_y) = G_N(x, y) := \mathrm{E}_x\Big[\sum_{k=0}^{\tau_N} \mathbf{1}_{\{S_k = y\}}\Big]\,,$$

with P_x the law of a simple random walk $(S_k)_{k\geq 0}$ starting from x and $\tau_N := \inf\{k \geq 0, S_k \notin \Lambda_N\}$.

To see this, one only need to make some algebraic manipulations:

$$2H_N(\varphi) = \frac{1}{4d} \sum_{\substack{x,y\in\Lambda_N \\ x\sim y}} (\varphi_x - \varphi_y)^2 + \frac{1}{2d} \sum_{\substack{x\in\Lambda_N, y\notin\Lambda_N \\ x\sim y}} (\varphi_x)^2 = \sum_{x\in\Lambda_N} -\frac{1}{2d} \sum_{\substack{x,y\in\Lambda_N \\ x\sim y}} \varphi_x\varphi_y\,.$$

Therefore, if we define the Laplacian Δ_N on $\Lambda_N \times \Lambda_N$ by

$$\Delta_N(x,y) := \begin{cases} \frac{1}{2d} & \text{if } x \sim y\,, \\ -1 & \text{if } x = y\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

we can rewrite $H_N(\varphi) = \frac{1}{2}\langle \varphi, (-\Delta_N)\varphi\rangle$. Thus, we indeed have that $\varphi = (\varphi_x)_{x\in\Lambda_N}$ is a Gaussian vector, with covariance matrix $(-\Delta_N)^{-1}$. But writing $-\Delta_N = I - Q_N$ with $Q_N(x,y) = \frac{1}{2d}\mathbf{1}_{\{x\sim y\}}$ for $x, y \in \Lambda_N$ (this is the sub-stochastic matrix of the simple random walk killed when exiting Λ_N), it is standard to get that $((-\Delta_N)^{-1}) = \sum_{k=0}^{\infty} Q_N^k$, which is exactly the Green function G_N. (We refer to [FV17, Ch. 8] for details.)

We therefore have that $\mathbf{E}_{N,V}[\varphi_0^2] = G_N(0,0)$ and we therefore need to estimate the Green function $G_N(0,0)$, *i.e.* the mean number of visits to 0 before exiting the box Λ_N. Standard random walk estimates then provide the following estimates: as $N \to \infty$

$$\mathbf{E}_{N,V}[\varphi_0^2] = G_N(0,0) = \begin{cases} N+1 & \text{if } d = 1\,, \\ \frac{2}{\pi}\log N + O(1) & \text{if } d = 2\,, \\ G(0) + O(n^{2-d}) & \text{if } d \geq 3\,. \end{cases}$$

(The case $d = 1$ is proven in [Law96, Thm. 1.6.4], the case $d = 2$ is proven in [Law96, Thm. 1.6.6], the case $d \geq 3$ is proven in [Law96, Prop. 1.5.9].) This concludes the proof of Theorem 1.11 in the case of the Gaussian Free Field. □

Remark 1.12 (The case of discrete height interfaces)**.** The localization/delocalization picture is actually slightly more subtle in the case of discrete height interfaces. Even though the localization of interfaces in dimension $d \geq 3$ seem to be quite general, in dimension $d = 2$ discrete interfaces undergo a localization/delocalization phase transition, depending on the inverse temperature of the model. This has first been observed in [FS81], and has then been proved in several other models, such as the discrete height GFF in [AHPS21] or a large class of models including the SOS model in [LO24] (we refer to the introductions of these two articles for an overview of recent results). For instance, for the SOS model with $V(x) = 2J|x|$ (we denote $\mathbf{P}_{N,J}$ the corresponding Gibbs measure), we can summarize the results as follows: there exist some $0 < J_0 \leq J_1 < +\infty$ such that

$$\begin{aligned} \forall\, t > 0,\ \lim_{N\to\infty} \mathbf{P}_{N,J}\big(|\varphi_0| \leq t\big) = 0 \quad &\text{(delocalization)} \quad \text{if } J < J_0\,, \\ \limsup_{N\to\infty} \mathbf{E}_{N,J}[\varphi_0^2] < +\infty \quad &\text{(localization)} \quad \text{if } J > J_1\,. \end{aligned}$$

1.3.2 Random interfaces with a random external field

Part I

The disordered pinning model

Part II

Directed polymers in random environment

Bibliography

[AHP20] M. Aizenman, M. Harel, and R. Peled. Exponential decay of correlations in the 2 d random field ising model. *Journal of Statistical Physics*, 180(1):304–331, 2020.

[AHPS21] M. Aizenman, M. Harel, R. Peled, and J. Shapiro. Depinning in integer-restricted gaussian fields and bkt phases of two-component spin models. *arXiv:2110.09498*, 2021.

[AW90] M. Aizenman and J. Wehr. Rounding effects of quenched randomness on first-order phase transitions. *Communications in Mathematical Physics*, 130(3):489–528, 1990.

[BCDP21] Q. Berger, F. Caravenna, and P. Dai Pra. *Introduction aux probabilités*. Dunod, 2021.

[Ber] Q. Berger. Random interfaces and pinning models. Lecture notes.

[BK88] J. Bricmont and A. Kupiainen. Phase transition in the 3d random field Ising model. *Comm. Math. Phys.*, 116(4):539–572, 1988.

[BL76] H. J. Brascamp and E. H. Lieb. On extensions of the brunn-minkowski and prékopa-leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *Journal of functional analysis*, 22(4):366–389, 1976.

[BLL75] H. J. Brascamp, E. H. Lieb, and J. L. Lebowitz. The statistical mechanics of anharmonic lattices. In *Statistical Mechanics: Selecta of Elliott H. Lieb*, pages 379–390. Springer, 1975.

[CGGH25] O. Collin, G. Giacomin, R. L. Greenblatt, and Y. Hu. On the lyapunov exponent for the random field ising transfer matrix, in the critical case. *Annales Henri Poincaré*, pages 1–42, 2025.

[CGH24] O. Collin, G. Giacomin, and Y. Hu. The random field ising chain domain-wall structure in the large interaction limit. *arXiv preprint arXiv:2401.03927*, 2024.

[Col25] O. Collin. Further results on the free energy of the random field ising chain in the case of centered disorder. *arXiv:2511.18912*, 2025.

[Com17] F. Comets. *Directed Polymers in Random Environments*, volume 2175 of *Ecole d'Eté de probabilités de Saint-Flour*. Springer International Publishing, 2017.

[DC23] H. Duminil-Copin. 100 years of the (critical) ising model on the hypercubic lattice. In *International Congress of Mathematicians*, pages 164–210. European Mathematical Society-EMS-Publishing House GmbH, 2023.

[dH07] F. den Hollander. *Random polymers*, volume 1974 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2007. Lectures from the 37th Probability Summer School held in Saint-Flour, 2007.

[DHP23] P. Dario, M. Harel, and R. Peled. Random-field random surfaces. *Probability Theory and Related Fields*, 186(1):91–158, 2023.

[DHP24] P. Dario, M. Harel, and R. Peled. Quantitative disorder effects in low-dimensional spin systems. *Communications in Mathematical Physics*, 405(9):212, 2024.

[DLX24] J. Ding, Y. Liu, and A. Xia. Long range order for three-dimensional random field ising model throughout the entire low temperature regime. *Inventiones mathematicae*, 238(1):247–281, 2024.

[DX21] J. Ding and J. Xia. Exponential decay of correlations in the two-dimensional random field ising model. *Inventiones mathematicae*, 224(3):999–1045, 2021.

[DZ24] J. Ding and Z. Zhuang. Long range order for random field ising and potts models. *Communications on Pure and Applied Mathematics*, 77(1):37–51, 2024.

[Fis84] M. E. Fisher. Walks, walls, wetting, and melting. *Journal of Statistical Physics*, 34(5-6):667–729, 1984.

[FS81] J. Fröhlich and T. Spencer. The kosterlitz-thouless transition in two-dimensional abelian spin systems and the coulomb gas. *Communications in Mathematical Physics*, 81(4):527–602, 1981.

[Fun05] T. Funaki. Stochastic interface models. *Lectures on probability theory and statistics*, 1869:103–274, 2005.

[FV17] S. Friedli and Y. Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge University Press, 2017.

[Gia07] G. Giacomin. *Random Polymer Models*. Imperial College Press, World Scientific, 2007.

[Gia24] G. Giacomin. Random matrix products and the statistical mechanics of disordered systems. Lecture notes, available on the author's webpage https://www.math.unipd.it/~giacomin/notesRM-SM_24_10_2024.pdf, 2024.

[Har74] A. B. Harris. Effect of random defects on the critical behaviour of ising models. *Journal of Physics C: Solid State Physics*, 7(9):1671, 1974.

[HH85] D. A. Huse and C. L. Henley. Pinning and roughening of domain walls in Ising systems due to random impurities. *Physics Review Letters*, 54:2708–2711, 1985.

[IM75] Y. Imry and S.-k. Ma. Random-field instability of the ordered state of continuous symmetry. *Physical Review Letters*, 35(21):1399, 1975.

[Imb85] J. Z. Imbrie. The ground state of the three-dimensional random-field ising model. *Communications in mathematical physics*, 98(2):145–176, 1985.

[Isi25] E. Ising. Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik*, 31(1):253–258, 1925.

[Law96] G. F. Lawler. *Intersection of random walks*. Probability and its Applications. Springer, Boston, 1996.

[LO24] P. Lammers and S. Ott. Delocalisation and absolute-value-fkg in the solid-on-solid model. *Probability Theory and related fields*, 188(1):63–87, 2024.

[NP89] J. Neveu and J. Pitman. Renewal property of the extrema and tree property of the excursion of a one-dimensional brownian motion. In *Séminaire de Probabilités XXIII*, pages 239–247. Springer, 1989.

[Ons44] L. Onsager. Crystal statistics. i. a two-dimensional model with an order-disorder transition. *Physical review*, 65(3-4):117, 1944.

[Pei36] R. Peierls. On Ising's model of ferromagnetism. *Mathematical Proceedings of the Cambridge Philosophical Society*, 32(3):477–481, 1936.

[PS66] D. Poland and H. A. Schegara. Occurence of a phase transition in nucleic acid models. *J. Chem. Phys*, 45:1464–1469, 1966.

[Vel06] Y. Velenik. Localization and delocalization of random interfaces. *Probability Surveys*, 3:112 – 169, 2006.

[Yan52] C. N. Yang. The spontaneous magnetization of a two-dimensional ising model. *Physical Review*, 85(5):808, 1952.