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présentée par

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Polymères aléatoires et modèles reliés: désordre, localisation et phénomènes critiques

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Random polymers and related models: disorder, localization and critical phenomena

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Résumé

Ce manuscrit propose un aperçu de certains modèles de polymères aléatoires, dont les motivations proviennent aussi bien de la physique que de la biologie ou de la chimie, et qui ont attiré l'attention des mathématiciens depuis plusieurs décennies. Pour les modèles considérés, je présente un état de l'art et je passe en revue les résultats récents, afin de mettre en perspective mes contributions, obtenues avec mes collaborateurs. Des idées concernant les outils mathématiques et les méthodes utilisés, ainsi que des commentaires à propos de possibles perspectives de recherche sont aussi présentés. Une question centrale (pour les systèmes désordonnés de manière générale) est celle de l'influence du désordre sur les caractéristiques du système : dans notre contexte, on souhaite comprendre si les propriétés géométriques du polymère sont affectées par la présence d'impuretés dans le milieu. En particulier, dans les modèles présentés ici, un phénomène de *localisation* peut se produire : le polymère adopte une forme spécifique selon la nature de l'interaction, et "s'accroche" aux impuretés environnantes. Notre but est de décrire l'effet du désordre sur ce phénomène de localisation, pour différents modèles de polymères.

- La première partie du manuscrit est dédiée aux modèles de copolymère, d'accrochage, et de Poland-Scheraga généralisé : ces modèles possèdent une transition de phase dite de localisation. De nombreux travaux se sont attachés à la question de la pertinence du désordre, c'est-à-dire de savoir si le désordre modifie les propriétés critiques du système. Le Chapitre 1 présente le modèle de copolymère, et décrit mes contributions [6, 19, 20]. Le Chapitre 2 présente le modèle d'accrochage, et détaille mes travaux [14, 17] (et [19, 20]). Le Chapitre 3 présente le modèle d'ADN de Poland Scheraga (généralisé), et les résultats obtenus dans [3, 10].
- La deuxième partie traite des modèles de polymère dirigé et de percolation de dernier passage, qui peuvent être utilisés pour représenter un polymère dans un solvant possédant des impuretés. Le désordre peut poséder un effet localisant, le polymère "s'étirant" pour atteindre certaines impuretés : on cherche alors à donner une description quantitative de ce phénomène. Le Chapitre 4 présente le modèle de polymère dirigé, et décrit ma contribution [15]. Le Chapitre 5 traite du modèle de polymère dirigé en environnment à queue lourde, et se base sur l'article [4]. Le Chapitre 6 présente le modèle de percolation de dernier passage (controllée par l'entropie), et expose certains des résultats obtenus dans [2, 9].
- La troisième partie se concentre sur les objets probabilistes au centre des différents modèles : les marches aléatoires et les processus de renouvellement. Le Chapitre 7 présente certains de mes résultats sur les processus de renouvellements et leurs intersections [16, 17], ainsi que sur les marches aléatoires dans le domaine d'attraction d'une loi stable [5, 7].

Mots-clés : Modèles de polymères, pertinence du désordre, localisation, transition de phase, phénomènes critiques, modèle de copolymère, modèle d'accrochage, modèle de Poland-Scheraga, polymère dirigé, percolation de dernier passage, marches aléatoires, processus de renouvellement.

Remarque : Ce manuscrit contient deux bibliographies : la première, page 1, fait la liste de mes publications, rangées de la plus récente à la plus ancienne (en commençant par les prépublications), et sont référencées par des nombres, par exemple [1–26] ; la deuxième, page 81, fait la liste des références extérieures, et utilise un code alphanumérique basé sur le nom du ou des auteurs et l'année de publication, par exemple [dH07, Gia10, Com16].

Abstract

This manuscript offers an overview of some random polymer models, whose motivations range from physics to biology and chemistry, and which have attracted much attention from mathematicians over the past decades. For the models considered, I present a state of the art, I review recent results, and I put into perspective my contribution, obtained together with my collaborators. Some ideas of the mathematical tools and of the methods that are used, together with some comments on possible directions of research are also presented. A central question (in disordered systems in general) is that of the influence of disorder on the characteristics of the system: in our context, one wishes to understand whether the geometric properties of the polymer are affected by the presence of impurities in the medium. In particular, in the models presented here, a *localization* phenomenon may occur: the polymer adopts a specific shape according to the nature of the interaction, and is somehow "pinned" to surrounding impurities. Our goal is to describe the effect of disorder on this localization phenomenon, for different polymer models.

- The first part of this manuscript is dedicated to the copolymer, pinning and generalized Poland-Scheraga models: these models undergo a localization phase transition, and many works have focused on the question of disorder relevance, *i.e.* of knowing whether disorder modifies the critical properties of the system. Chapter 1 introduces the copolymer model, and presents my contributions [6, 19, 20]. Chapter 2 considers the pinning model, and describes my works [14, 17] (and [19, 20]). Chapter 3 introduces the generalized Poland Scheraga model for DNA, and presents the results obtained in [3, 10].
- The second part of the manuscript deals with the directed polymer model and last-passage percolation, that can be used to represent a polymer placed in a solvant with some impurities. Here, disorder has a localizing effect, and the polymer "stretches" to reach distant impurities: one then tries to give a quantitative description of this localization phenomenon. Chapter 4 introduces the directed polymer model, and presents my contribution [15]. Chapter 5 treats the directed polymer model in heavy-tail environment, and is based on my article [4]. Chapter 6 introduces the (entropy-controlled) last-passage percolation, and presents some of the results obtained in [2, 9].
- The third part focuses on the probabilistic objects at the center of the different models: random walks and renewal processes. Chapter 7 presents some of my works on renewal processes and their intersections [16, 17], as well as on (multivariate) random walks in the domain of attraction of stable laws [5, 7].

Keywords: Polymer models, disorder relevance, localization, phase transition, critical phenomena, copolymer model, pinning model, Poland-Scheraga model, directed polymer, last-passage percolation, random walks, renewal processes.

Remark: The manuscript contains two bibliographies: the first one, on page 1, lists my publications from the most recent to the oldest (preprints first), and uses numerical labels, like [1–26]; the second one, on page 81, lists external references, and uses an alphanumerical code from the name of the author(s) and year of publication, like [dH07, Gia10, Com16].

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Tout d'abord, je tiens à remercier Erwin Bolthausen, Bernard Derrida et Dima Ioffe d'avoir accepté d'être les rapporteurs de ce manuscrit, et d'avoir consacré de leur temps précieux à cette tâche. Je vous en suis extrêmement reconnaissant.

Erwin, Bernard, Giambattista, Yueyun, Zhan, Fabio et Yvan, je vous remercie sincèrement de faire partie de mon jury d'habilitation. Je suis très heureux et honoré de votre présence aujourd'hui.

Des remerciements particuliers sont dûs à mes différents mentors et collaborateurs, à qui cette HDR doit beaucoup. Je suis extrêmement chanceux d'avoir pu travailler avec vous tous, vos différentes manières d'aborder les maths m'ont beaucoup enrichi, et j'ai beaucoup apprécié ces nombreux moments (scientifiques ou non) partagés avec vous. Fabio, merci encore d'avoir guidé mes premiers pas dans le monde de la recherche, merci pour ta bienveillance à mon égard. Merci à Ken qui m'a accueilli a Los Angeles pendant deux ans, à Frank, qui m'a invité plusieurs fois à Leiden, j'ai eu un grand plaisir à travailler avec vous. Francesco, Nikos et Rongfeng, notre collaboration fut relativement courte, mais j'apprécie toujours échanger avec chacun des membres de votre trio magique. Julien, je suis très content que nous ayons pu "grandir" ensemble pendant notre thèse à Lyon, je suis sûr que nous continuerons sur cette voie. Giambattista, Hubert, c'est un privilège de pouvoir travailler régulièrement avec vous: séparément ou ensemble, nos discussions sont à chaque fois agréables et stimulantes. Finalement, Niccolò et Michele, je suis très content d'avoir pu travailler avec vous... et d'avoir réussi à finir nos papiers avant vos candidatures!

Si ces dernières années ont été très stimulantes d'un point de vue scientifique, c'est aussi grâce au dynamisme du LPSM. J'apprécie énormément l'ambiance bienveillante et détendue qui y règne. Merci à l'ensemble du laboratoire pour l'accueil qui m'y a été réservé, je sais le privilège que j'ai à évoluer dans cet environnement de travail exceptionnel, d'un point de vue scientifique et humain. Je tiens particulièrement à remercier l'ensemble de l'équipe administrative, à Jussieu et à Sophie Germain, pour leur efficacité et leur disponibilité. Parmi les moments du quotidien de la vie du labo, je pense notamment à ces discussions autour d'un café, dans le couloir, ou qui commencent autour d'un café et finissent dans le couloir... J'ai aussi une pensée particulière aux quelques personnes toujours motivées pour une balade au Jardin des Plantes, avec un prétexte qui varie au fil des saisons (bébés wallabies, mimosa ou cerisier en fleur, ou simplement une journée ensoleillée). Merci à Daphné, Thomas, Thierry, Camille, Cédric, Damien, Romain, Catherine, Nicolas, Nicolas, François, Lorenzo, Gilles, Idris (et toutes celles et ceux que j'oublie fatalement!)... de rendre si facile et agréable la vie au laboratoire. J'en profite pour adresser un clin d'œil à mes étudiants, partie intégrante et motivante de mon quotidien, dont certains m'ont dit un jour: "quand on parle aux gens du LPSM, ça donne envie de faire des probas, on a l'impression qu'ils sont tous heureux!".

Vous aurez compris que je trouve mon environnement de travail plutôt agréable, mais j'ai aussi

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Contents

List of publications

[1] Q. Berger and M. Salvi, "Scaling limit of sub-ballistic 1d random walk among biased conductances: a story of wells and walls," *preprint arXiv:1904.05283*, 2019.

[2] Q. Berger and N. Torri, "Beyond Hammersley's last-passage percolation: a discussion on possible new local and global constraints," *preprint arXiv:1802.04046*, 2018.

[3] Q. Berger, G. Giacomin, and M. Khatib, "Disorder and denaturation transition in the generalized Poland–Scheraga model," *Annales Henri Lebesgue*, 2019+.

[4] Q. Berger and N. Torri, "Directed polymers in heavy-tail random environment," *The Annals of Probability*, 2019+.

[5] Q. Berger, "Notes on random walks in the Cauchy domain of attraction," *Probability Theory and Related Fields*, 2019+.

[6] Q. Berger, G. Giacomin, and H. Lacoin, "Disorder and critical phenomena: the $\alpha{=}0$ copolymer model," *Probability Theory and Related Fields*, 2019+.

[7] Q. Berger, "Strong renewal theorems and local large deviations for multivariate random walks and renewals," *Electronic Journal of Probability*, vol. 24, no. 46, pp. 1–47, 2019.

[8] Q. Berger and M. Salvi, "Scaling of sub-ballistic 1D random walks among biased random conductances," *Markov Processes and Related Fields*, vol. 25, pp. 171–187, 2019.

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[10] Q. Berger, G. Giacomin, and M. Khatib, "DNA melting structures in the generalized Poland-Scheraga model," *ALEA, Lat. Am. J. Probab. Math. Stat.*, vol. 15, pp. 993–1025, 2018.

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[16] K. S. Alexander and Q. Berger, “Local asymptotics for the first intersection of two independent renewals,” *Electronic Journal of Probability*, vol. 21, p. 20 pp., 2016.

[17] K. S. Alexander and Q. Berger, “Local limit theorems and renewal theory with no moments,” *Electronic Journal of Probability*, vol. 21, p. 18 pp., 2016.

[18] Q. Berger, “Influence of disorder for the polymer pinning model,” *ESAIM: Proceedings and Surveys*, vol. 51, pp. 74–88, 2015.

[19] Q. Berger and J. Poisat, “On the critical curves of the pinning and copolymer models in correlated gaussian environment,” *Electronic Journal of Probability*, vol. 20, p. 35 pp., 2015.

[20] Q. Berger, F. Caravenna, J. Poisat, R. Sun, and N. Zygouras, “The critical curves of the random pinning and copolymer models at weak coupling,” *Communications in Mathematical Physics*, vol. 326, no. 2, pp. 507–530, 2014.

[21] Q. Berger, “Pinning model in random correlated environment: Appearance of an infinite disorder regime,” *Journal of Statistical Physics*, vol. 155, no. 3, pp. 544–570, 2014.

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[24] Q. Berger and H. Lacoin, “Sharp critical behavior for pinning models in a random correlated environment,” *Stochastic Processes and their Applications*, vol. 122, no. 4, pp. 1397 – 1436, 2012.

[25] Q. Berger and H. Lacoin, “The effect of disorder on the free-energy for the random walk pinning model: Smoothing of the phase transition and low temperature asymptotics,” *Journal of Statistical Physics*, vol. 142, no. 2, pp. 322–341, 2011.

[26] Q. Berger and F. L. Toninelli, “On the critical point of the random walk pinning model in dimension d=3,” *Electronic Journal of Probability*, vol. 15, pp. 654–683, 2010.

All articles [1–26] are available on my webpage www.lpsm.paris/pageperso/bergerq/

We stress that not all the above articles are described in this manuscript: we leave aside the papers [21–26], which were at the center of my Ph.D.; we mention only briefly in the Introduction the works [1, 8, 11, 13], whose subjects are a bit far from the core of the manuscript.

Introduction

Over the past 10 years, I have mostly focused my research on random polymers and related models. In this manuscript, I present an overview of the various models that I have studied since my Ph.D., that I put into context and perspective. One of the main question, that has been a guideline for me, is to understand the influence of *disorder* on physical systems. It can be stated as follows. First, one wants to know whether the presence of random impurities or inhomogeneities have an effect on the properties of a physical system: if it is the case then disorder is called *relevant*. Second, in the case where disorder is relevant, one wants to describe quantitatively the impact of the random impurities on the characteristics of the system, in particular on its phase transition.

Motivations from physics, chemistry and biology

Polymers are macromolecules, made from a large number of elementary units called monomers: the monomers may be all of the same type (forming a homogeneous polymer) or they may be of different types (forming a heterogeneous polymer, or copolymer). In biology and chemistry, examples of natural or synthetic polymers are extremely numerous: rubber, polyethylene (these are homopolymers) or DNA strands, proteins (these are copolymers). Polymers may have very complex structures and properties, and they have been studied in various domains, ranging from chemistry and biology to physics and mathematics.

Examples of polymer models. As mentioned above, my main line of research has been to investigate the role of inhomogeneities, or *disorder*, in some polymer models. The randomness may come from different factors: the composition of the polymer may be heterogenous (like for DNA or proteins), or the polymer may be placed in an environment with some impurities. Here are some examples of physical situations that are of interest for us, see Figures 1 to 5:

(a) a random copolymer lying at the interface between two solvants (the so-called copolymer model, cf. Chapter 1);

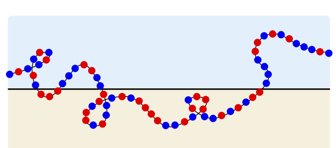


Figure 1 – A schematic view of a copolymer at the interface between two solvants.

(b) a protein in the vicinity of a cell, and sticking to its surface, which may be heterogeneous (one instance of the pinning model, cf. Chapter 2);

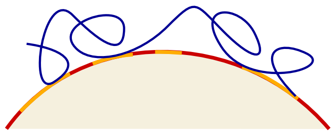


Figure 2 – A schematic view of a protein at the vicinicity of a cell.

(c) a DNA double strand going through a denaturation transition (the (generalized) Poland-Scheraga model, cf. Chapters 2 and 3);

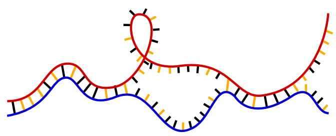


Figure 3 – A schematic view of a DNA double strand going through denaturation.

(d) a polymer placed in a solvant with some impurities (the directed polymer model, cf. Chapters 4 and 5);

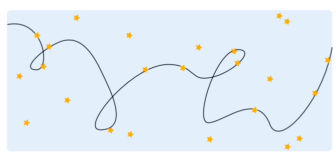


Figure 4 – A schematic view of a polymer in some heterogeneous solvant.

(e) a polymer whose monomers bear charges that repell or attract each other (the charged polymer, that I studied in [13]);

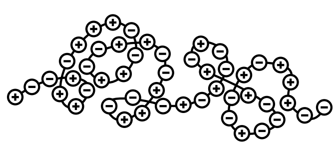


Figure 5 – A schematic view of a charged polymer, with charges '+' or '−'.

All these situations are relevant from a physics, chemistry or biology perspective, and are also very rich from a mathematical point of view.

Localization phenomena and influence of disorder

All the models described above can be defined properly and have been shown to undergo a phase transition. For a certain regime of temperature, the polymer is somehow "pinned": (a) the copolymer remains close to the interface, placing as many monomers as possible in their preferred solvant; (b) the protein sticks to the surface of the cell; (c) the two DNA strands are attached to each other; (d) the polymer reaches and sticks to the impurities present in the solvant; (e) the charged polymer folds onto itself, bonding as many attracting charges together as possible. More accurately, one speaks of a *localization* phenomenon (localization near an interface, or in the vicinity of some impurities, or in a small region of the space). On the other hand, when temperature reaches some critical value (for instance becomes strong enough to break many chemical bonds, or becomes low enough so that charges repel each other strongly), then the configuration of the polymer changes drastically, and becomes *delocalized*: the polymer (a)-(b) wanders away from the interface; (c)-(d) moves freely, as if it did not feel the interactions; (e) unfolds, avoiding interactions between its monomers. We do not develop on these informal descriptions: more precise statements will be given in the different chapters dedicated to the models.

But once one knows that a phase transition occurs, many questions remain: can one determine the critical temperature or at least give some estimates on its value? what is the behavior of the system when approaching this critical value? An important (and difficult!) question is to understand the role of disorder on the localization phenomenon.

Influence of disorder and Harris criterion

Understanding if and how disorder affects phase transitions has been a central question in the physical literature, and more recently in the mathematical literature (see [Bov06, Gia10] for an overview). The first question is that of disorder relevance: one wants to determine whether an arbitrarily small quantity of disorder affects the critical behavior of a physical system. Put otherwise, one wishes to know whether a disorder ω has any influence on the phase transition at all. One therefore needs to compare the disordered model with its homogeneous counterpart (*i.e.* taking the randomness $\omega \equiv 0$), and establish whether the characteristics of the phase transition differ.

In 1974, in a celebrated paper [Har74], the physicist A. B. Harris devised a criterion based on renormalization group arguments, to decide whether a system was sensitive to the introduction of disorder, providing predictions for the question of disorder relevance. This prediction is based on the critical behavior of the homogeneous model. More precisely, let $\xi(T)$ be the correlation length of the homogeneous model, *i.e.* the exponential rate of decay of the two-point correlation function associated to the model, and assume that there is some $\nu > 0$ such that $\xi(T) \propto |T - T_c|^{-\nu}$ as $T \to T_c$ (*i.e.* $\log \xi(T)/\log |T - T_c| \to -\nu$), where T_c is the critical temperature at which the phase transition occurs. The exponent ν is called the critical exponent of the correlation length. Then, Harris predicts that, if the system is $\mathtt{d}$-dimensional, an i.i.d. disorder should be irrelevant if $\nu > 2/\mathtt{d}$ and relevant if $\nu < 2/\mathtt{d}$. The case $\nu = 2/\mathtt{d}$, dubbed *marginal*, is left aside in Harris criterion, and should depend on the details of the model.

Putting this criterion to mathematical ground is an important challenge, and the copolymer and pinning models have been found perfect playgrounds for testing Harris prediction: there are a family of one-dimensional systems for which the homogeneous models are exactly solvable, with a critical exponent ν spanning values between 1 and ∞ (at least for the pinning model). The Harris criterion has been proven for the pinning model by a series of papers (over the past fifteen years), and the marginal case $\nu = 2$, after a long controversy among physicist, has also been treated completely. We refer to Chapter 2 for a more detailed discussion and relevant references.

But the question of the influence of disorder on phase transitions does not stop here: once disorder relevance is proven, an important issue remains to be able to describe (quantitatively) the critical behavior in presence of disorder. There are a number of important results in the physical literature (see e.g. [AW90], in the context of the Ising model), but this question is far from being fully understood, to put it mildly. Some predictions have been made in the physics literature for some of the models that we present in this manuscript (see Sections 1.4 and 2.3.2 for more details), but very few results have been proven.

Organization and overview of the manuscript

We now present an overview of the organization of the rest of the manuscript, and of the main questions addressed in the different chapters.

Part I. The first part deals with the copolymer, pinning, and DNA models, with the question of relevance/irrelevance of disorder as a guideline. These models could be put into the category of "pinning" models (in a wider sense): they exhibit the same type of localization/delocalization phenomenon, that could be dubbed as a depinning transition. Their common features is that they are based on renewal processes to represent the sequence of contact points with an interface. Because of their relative simplicity (in particular, their homogeneous counterparts are solvable), they have been used as test models for Harris' prediction.

Chapter 1 is dedicated to the copolymer model and presents the works [6, 19, 20]. Here, the question of disorder relevance has been settled, but some important open problems remain. One of them is to determine the behavior of the critical point in the weak-coupling limit: it has been shown to be universal (in some sense made clear in Section 1.3.2), but the explicit behavior is still mostly conjectural. One of my result, in collaboration with Julien Poisat, Francesco Caravenna, Rongfeng Sun and Nikos Zygouras [20], has been to answer this conjecture in the regime where the underlying renewal has a finite mean (see also [19]). Another important issue is to give sharp estimates on the critical behavior of the disordered model, for instance on its free energy. This appears to be very difficult, in particular since the critical point is not explicit. With Giambattista Giacomin and Hubert Lacoin [6], we considered a specific case where the critical point is known, which helped us obtain the sharp critical behavior of the free energy: we find that it has an infinite order phase transition, and we managed to obtain an explicit (stretch-)exponential behavior.

Chapter 2 focuses on the pinning model and presents the works [12, 14] (and [19, 20]). This model has been at the center of a very intense activity, the central question being that of disorder relevance. A series of recent papers has settle this question in terms of critical point shift, and my main contribution resides in a work with Hubert Lacoin [14], which gives a necessary and sufficient condition for disorder relevance, proving a conjecture by Derrida, Hakim and Vannimenus [DHV92] (it also gives sharp estimates for the critical point shift). Section 2.4 is dedicated to a different type of pinning model, in which the disorder sequence is given by a renewal process: it is based on an article with Kenneth S. Alexander [12].

Chapter 3 turns to the generalized Poland-Scheraga model, and presents the works [3, 10]. The Poland Scheraga model, introduced in [PS70], is a simplified model for DNA denaturation: when heated the two strands form loops, that are assumed to be symmetric, allowing no mismatches. When properly formulated, this is exactly the pinning model of Chapter 2. More recently, a generalized version of this model has been proposed by Garel and Orland [GO04], in which loops may be asymmetric, allowing for possible mismatches. Here, the phenomenology is much richer, already at the level of the homogeneous model: a localization transition is still present, but a condensation transition might occur. In an article with Giambattista Giacomin and Maha Khatib [10], we described precisely this condensation transition. For the disordered version of the model, several choices for the randomness are reasonable. In Section 3.3, we present results obtained in collaboration with Giambattista Giacomin and Maha Khatib [3], in the case where the disorder is i.i.d.: we confirm Harris' predictions in this case. In Section 3.4, we discuss ongoing work with my Ph.D. student Alexandre Legrand, considering a more natural choice of disorder from the point of view of DNA modeling.

Part II. The second part focuses on the directed polymer model, and another closely related model called last-passage percolation. It considers a directed random walk in dimension $1+d$ (one temporal dimension, d spatial or transverse dimensions), interacting with a disorder field. The main question addressed in this part is that of the localization of polymer trajectories inside "favorite corridors" where the disorder field is unusually attractive. Last-passage percolation is the zero temperature analogue of this model, and we introduce a generalization of it which appears as a natural tool when trying to deal with scaling limits of the directed polymer model.

Chapter 4 considers the directed polymer in random environment in dimension $d \geq 1$, and presents the work [15]. This model has been widely studied over the past decades, and a seminal work of Bolthausen [Bol89] proves that in dimension $d \geq 3$, there is a phase transition: polymer trajectories are diffusive at high temperature, whereas a localization phenomenon occurs at low temperature. In dimension $d = 1, 2$, it has been proven that localization holds at any positive temperature. In an article with Hubert Lacoin [15], we give the sharp high-temperature asymptotics of the free energy in dimension $d = 2$, which helps to quantify this localization phenomenon.

Chapter 5 deals with the directed polymer in a heavy-tail environment in dimension $d = 1$, and presents the work [4]. Here, we assume that disorder is i.i.d. with a power law decaying distribution function, with exponent $\alpha \in (0, 2)$ (hence disorder does not admit a second moment). In that case, Auffinger and Louidor [AL11] prove that the polymer has a transversal fluctuation exponent $\xi = 1$. To observe interesting behavior, an idea is to tune the inverse temperature β with the size of the system, *i.e.* take $\beta = \beta_n \to 0$ as $n \to +\infty$. Our main result is that, if $\alpha \in (1/2, 2)$, then by tuning properly β_n as a function of n, one can reach any transversal fluctuation exponent $\xi \in [1/2, 1]$: this generalizes the works of Auffinger and Louidor [AL11] (case $\xi = 1$) and of Dey and Zygouras [DZ16] (case $\xi = 1/2$), and answers an important conjecture. In the case $\alpha \in (0, 1/2)$, one can only reach transversal fluctuation exponents $\xi = 1/2$ and $\xi = 1$, and the transition between the two regimes is very abrupt.

Chapter 6 introduces the entropy-controlled last-passage percolation, based on the works [2, 9]. Hammersley's last-passage percolation considers n i.i.d. points in $[0, 1]^2$, and asks what is the maximal number of points that can be collected by an up-right path (or a 1-Lipschitz path after a 45° rotation). We introduce a generalization of this model, in which the up-right (or 1-Lipschitz) constraint is replaced by a global (entropy) constraint—it appears naturally if the paths are thought as scaling limits of random walks. Our main result is that the number of points that can be collected is of the same order as for standard last-passage percolation, *i.e.* $\sqrt{n}$. We explain how to apply these estimates to derive deeper results, in particular to show that the limiting variational problem found in Chapter 5 is well defined. We also present a more general version of last-passage percolation, and discuss its possible applications.

Part III. The last part of the manuscript (Chapter 7) is devoted to random walks and renewal processes, which are probabilistic objects of much interest on their own, and are at the center of the models above. We review some of the literature, and we present in Section 7.1 the results obtained with Kenneth S. Alexander on renewal processes with tail decay exponent $\alpha = 0$ (cf. [16]), and on the tail distribution of the intersection of two independent renewals (cf. [17]). Section 7.2 considers random walks in the domain of attraction of an α-stable law, $\alpha \in (0, 2)$, and in particular presents new results for (local) large deviation of a random walk in the Cauchy domain of attraction (*i.e.* when $\alpha = 1$) that I obtained in [5]. Finally, Section 7.3 gives some (new) results on the Green function of multivariate random walks, obtained in [7]—this was originally motivated by the generalized Poland Scheraga model of Chapter 3, which is based on bivariate renewal processes.

Other related works—not included in the manuscript

Let me mention here a few other works of mine that are related to the subject of this manuscript, but that I will not develop further.

• Together with Julien Poisat and Frank den Hollander [13], we studied the charged polymer model. This model is used to describe a self-interacting polymer, see Figure 5: charges are attached to the monomers, and each self-intersection contributes an energy that is equal to the product of the charges of the two monomers that meet. Very few results have been proven for the model with quenched disorder. For the annealed model (*i.e.* when disorder has been averaged), one can show that it undergoes a folding (or collapse) phase transition: our work [13] describes the phase diagram of this model in dimension $d \geq 2$—the dimension $d = 1$ has been considered in [CdHPP16].

• With Kenneth S. Alexander [11], we studied the first-passage percolation model (on $\mathbb{Z}^2$), which was designed as a model for the propagation of a fluid in a random porous medium. A first result, under very mild conditions, is the existence of a (convex) limit shape $\mathcal{B}$: if $\mathcal{B}_t$ is the "wet" region after time t, then a.s. $\frac{1}{t}\mathcal{B}_t$ converges to $\mathcal{B}$ as $n \to +\infty$. More recent results [DH14, AH16] prove the existence of coalescing semi-infinite geodesics for the model, but only in the directions where the limit shape has a differentiable boundary $\partial\mathcal{B}$ (loosely speaking). Our work with Kenneth S. Alexander considers an example where the limit shape has corners, and studies the question of the existence (and coalescence) of geodesics in the directions of these corners. Our finding is that there are some corners with no geodesics, and some corners with two non-coalescing geodesics—this shows that the question of geodesics in the direction of corners cannot have a universal answer.

• In two works with Michele Salvi [1, 8], we analyzed random walks among biased random conductances in dimension $d = 1$. Because of the bias, the random walk is transient, and it is ballistic under some conditions of integrability of the conductances. With Michele Salvi, we focused on the sub-ballistic case, which occurs when the conductances have a heavy-tail (at 0 or at $+\infty$). Sub-ballisticity arises because of a trapping mechanism: the random walk is slowed down by very large or very small conductances (or the combination of a large conductance followed by a small one, depending on the tail of the conductances), and a spends most of its time trapped near these abnormal conductances. This is another type of a localization effect. Our main results have been to quantify this slowdown, find the correct scaling of the random walk, and prove the convergence of the rescaled process to the inverse of an α-stable subordinator (this indicates an aging phenomenon).

Recurrent notation

Here are some notation we use throughout the manuscript:

- we use $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$, and $(x)^+ = \max(0, x)$, $(x)^- = \max(0, -x)$;
- we write $a_n = O(b_n)$ if $\limsup_{n\to+\infty} b_n/a_n < +\infty$;
- $a_n \asymp b_n$ if $a_n = O(b_n)$ and $b_n = O(a_n)$;
- $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to +\infty$;
- $a_n = o(b_n)$ or $a_n \ll b_n$ if $a_n/b_n \to 0$ as $n \to +\infty$;
- for a finite set A, $|A|$ denotes its cardinality;
- $\|\cdot\|_2$ denotes the euclidean distance in $\mathbb{R}^d$, and $\|\cdot\|_1$ the L^1 distance.
- $\mathbb{N} = \{1, 2, \dots\}$ is the set of integers, and we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- For $i, j \in \mathbb{Z}$ we denote $[\![i, j]\!] = \{i, i+1, \dots, j-1, j\}$; if $a, b \in \mathbb{R}$, we write $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$;
- $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$; if $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, then $\lfloor \mathbf{x} \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)$;
- i.i.d. means "independent and identically distributed", r.v. means "random variable", and *a.s.* means "almost surely";
- $\xrightarrow{(d)}$ stands for the convergence in distribution; $\overset{(d)}{=}$ stands for equality in distribution.

Also, we use generic constants C, C', c, c',... when their value are irrelevant (and may change from line to line), and we keep a subscript C_δ, c_ε,... when we want to stress the dependence of the constants on various parameters. We also often omit integer parts when it is not ambiguous, to lighten notations: for instance, for $t > 0$, $x \in \mathbb{R}$, we write $S_{tn} = xn^\xi$ in place of $S_{\lfloor tn \rfloor} = \lfloor xn^\xi \rfloor$ if $(S_n)_{n\geq 0}$ is an integer valued random walk (and $\xi > 0$ is a given exponent).

Part I

Influence of disorder for copolymer, pinning and DNA models

Chapter 1

The copolymer model

In this chapter, we present the copolymer model: in particular, we describe our contributions [19, 20] (in Section 1.3.2) and [6] (in Section 1.4).

1.1 Presentation of the model

A first model for a copolymer near a selective interface has been introduced by Garel, Huse, Leibler and Orland [GHLO89], to study the effect of disorder on the localization of a hydrophilic-hydrophobic copolymers placed near a water/oil interface. The interest of the mathematical community in this model grew with the seminal paper by Bolthausen and den Hollander [BdH97]. One considers a random walk path $(S_0, \dots, S_n)$, each step (S_i, S_{i+1}) being seen as a monomer. The path wanders near the interface between two solvants, and a random variable ω_i attached to the i^{th} monomer determines its prefered solvant. A natural definition for the Hamiltonian is then $\sum_{i=1}^n (\omega_i + \eta) \operatorname{sign}(\frac{S_{i-1}+S_i}{2})$, where η is an external field (or a bias in the disorder)—note that this definition relies solely on the lengths of the different excursions away from the interface and on their sign.

A more general definition of the model is based on a (one-dimensional) renewal process τ ($\tau_0 := 0$ and $(\tau_k - \tau_{k-1})_{k\geq 1}$ are i.i.d. $\mathbb{N}$-valued r.v.s, representing the length of the excursions), and on a sequence of i.i.d. symmetric r.v.s $\iota = (\iota_k)_{k\geq 1}$ with value in $\{-1, +1\}$, independent of τ (for $k \geq 1$, ι_k represents the sign of the k^{th} excursion). We denote by $\mathbf{P}$ the joint law of (τ, ι), and we also set $\varepsilon_i := \sum_{k\geq 1} \iota_k \mathbf{1}_{i\in(\tau_{k-1},\tau_k]}$, which is the sign of the i^{th} step of the walk (or of the i^{th} monomer). We refer to Figure 1.1 for an illustration.

We let $\omega = (\omega_i)_{i\geq 1}$ be a sequence of i.i.d. r.v.s whose law is denoted $\mathbb{P}$, and that the ω_i are centered and have unit variance ($\mathbb{E}[\omega_i] = 0$, $\mathbb{E}[\omega_i^2] = 1$). We denote $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_i}]$, and we suppose that $\beta_0 := \sup\{\beta : \lambda(\beta) < +\infty\} \in (0, +\infty]$. As far as the renewal process is concerned, we assume that there is some $\alpha \geq 0$ and some slowly varying function $\varphi(\cdot)$ such that for all $n \geq 1$

$$\mathbf{P}(\tau_1 = n) = \varphi(n) n^{-(1+\alpha)}. \tag{1.1}$$

(We recall that $\varphi(\cdot)$ is said to be slowly varying if for any $a > 0$, $\varphi(x)/\varphi(ax) \to 1$ as $x \to +\infty$, see [BGT89].) Assumption 1.1 is verified for instance if τ is the set of return times to 0 of S_{2n}, where S_n is the simple symmetric random walk on $\mathbb{Z}$ (one then has $\alpha = 1/2$).

Then, for a fixed realization of $\omega = (\omega_i)_{i\geq 1}$ (*quenched* disorder), and for $\beta \in [0, \beta_0)$ (the inverse temperature, or disorder strength) and $h \in \mathbb{R}$ (an external field), we define for $n \in \mathbb{N}$, the probability

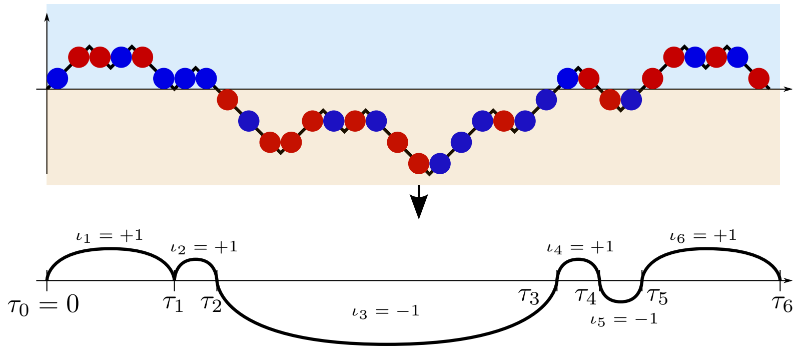


Figure 1.1 – On top, a representation of the (directed) random walk, with r.v.s associated to each step. The strength of the interaction depends on whether a monomer lies above or below the interface. On the bottom is the simplification of the model, with a renewal process representing the different excursions and an i.i.d. sequence of r.v.s representing their signs.

measures $\mathbf{P}^{\omega,\mathrm{cop}}_{n,\beta,h}$ by the Radon-Nikodym derivative with respect to the reference law $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}^{\omega,\mathrm{cop}}_{n,\beta,h}}{\mathrm{d}\mathbf{P}}(\tau) = \frac{1}{Z^{\omega,\mathrm{cop}}_{n,\beta,h}} \exp\Big(\sum_{i=1}^{n} (\beta\omega_i - \lambda(\beta) + h)\mathbf{1}_{\{\varepsilon_i=+1\}}\Big)\mathbf{1}_{\{n\in\tau\}}\,. \tag{1.2}$$

The quantity $\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h}$ is the (quenched) partition function of the model, used to renormalized $\mathbf{P}^{\omega,\mathrm{cop}}_{n,\beta,h}$ to a probability measure, and is equal to

$$\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n} (\beta\omega_i - \lambda(\beta) + h)\mathbf{1}_{\{\varepsilon_i=+1\}}\Big)\mathbf{1}_{\{n\in\tau\}}\Big]\,. \tag{1.3}$$

Notice that we placed the constraint $n \in \tau$ in (1.2)-(1.3), forcing the end-point of the polymer to return to the interface: this is essentially to simplify later exposition, but it is not a real issue, see Remark 1.2. Note that we substracted $\lambda(\beta)$ in the exponential, this is essentially for renormalization purposes, see (1.7). Also, one would have expected to find ε_i in the Gibbs weight: the choice $\mathbf{1}_{\{\varepsilon_i=+1\}} = \frac{1}{2}(\varepsilon_i + 1)$ simplifies some of the later analysis, without changing the measure $\mathbf{P}^{\omega,\mathrm{cop}}_{n,\beta,h}$ (up to a small change of parameters).

Remark 1.1. The choice of parameters in (1.2) is made in order to stress the parallel with the pinning model, see (2.1), and is the one used for instance in [10]. A reader familiar with the model may be aware of a different set of notation (see for instance [BdH97] or [CGT12]), where the Gibbs weight is $\exp\big(-2\bar\beta\sum_{i=1}^{n}(\bar\omega_i + \bar h)\mathbf{1}_{\{\varepsilon_i=+1\}}\big)$, but this is simply a change of parameters $\omega_i = -\bar\omega_i$, $\beta = 2\bar\beta$, $h = \lambda(2\bar\beta) - 2\bar\beta\bar h$.

1.2 Free energy and localization transition

A central physical quantity associated to the model is the *quenched* free energy (or energy per unit length) of the model, which is defined by

$$\mathtt{F}(\beta,h) := \lim_{n\to+\infty} \frac{1}{n}\log \mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log \mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P})\,. \tag{1.4}$$

The existence of the limit follows from standard super-additivity arguments (using the constraint $n \in \tau$). We refer to [dH07, Chap. 9] or [Gia07, Chap. 4] for details. Let us stress that the limit $\mathtt{F}(\beta, h)$ is almost surely constant, but that it does depend on the law $\mathbb{P}$, as well as on the law $\mathbf{P}$.

Remark 1.2. One could also work with the *free* model, that is with the indicator $\mathbf{1}_{\{n\in\tau\}}$ removed from (1.2)-(1.3). The free partition function is

$$\mathbf{Z}^{\omega,\mathrm{cop,free}}_{n,\beta,h} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n}(\beta\omega_i - \lambda(\beta) + h)\mathbf{1}_{\{\varepsilon_i=+1\}}\Big)\Big],$$

and one can show that for all n, β, h we have $\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} \le \mathbf{Z}^{\omega,\mathrm{cop,free}}_{n,\beta,h} \le Cn\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h}$, where C is a constant (that depends on ω), see [Gia07, (4.25)]. All together, we see that, at least at the level of the free energy, we can replace $\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h}$ by $\mathbf{Z}^{\omega,\mathrm{cop,free}}_{n,\beta,h}$ in (1.4) without changing the limit.

Localization transition and phase diagram. Another observation is that $\mathtt{F}(\beta, h) \ge 0$ for all β, h. Indeed, one can obtain a lower bound on the partition function by adding the indicator function that $\tau_1 = n$ and $\iota_1 = -1$ (so that $\mathbf{1}_{\{\varepsilon_i=+1\}} = 0$ for $1 \le i \le n$): we get $\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} \ge \mathbf{P}(\tau_1 = n, \iota_1 = -1)$. Then, taking the logarithm, dividing by n and letting $n \to +\infty$, we get that $\mathtt{F}(\beta, h) \ge 0$ for all β, h, thanks to the assumption (1.1).

We therefore have that $h \mapsto \mathtt{F}(\beta, h)$ is a non-negative, non-decreasing (and convex) function: it is then natural to define the (quenched) critical point

$$h_c(\beta) := \sup\big\{h : \mathtt{F}(\beta, h) = 0\big\} = \inf\big\{h : \mathtt{F}(\beta, h) > 0\big\}. \tag{1.5}$$

This critical point marks a transition in the properties of the polymer. Indeed, one notices that, if $h \mapsto \mathtt{F}(\beta, h)$ is differentiable (which is true for all except countably many h, since the function is convex), one can differentiate (1.4) and obtain by convexity arguments that

$$\frac{\partial}{\partial h}\mathtt{F}(\beta, h) = \lim_{n\to+\infty} \mathbf{E}^{\omega,\mathrm{cop}}_{n,\beta,h}\Big[\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\varepsilon_i=+1\}}\Big] \qquad \mathbb{P}\text{-a.s.} \tag{1.6}$$

This shows that the derivative of the free energy is related to the asymptotic proportion, *under the measure* $\mathbf{P}^{\omega,\mathrm{cop}}_{n,\beta,h}$, of monomers lying above the interface. Therefore, if $h < h_c(\beta)$, then $\partial_h\mathtt{F}(\beta, h) = 0$, and almost all monomers (*i.e.* a proportion asymptotic to 1) are placed below the interface. On the other hand, if $h > h_c(\beta)$, then $\partial_h\mathtt{F}(\beta, h) > 0$ (if the derivative exists), and a positive proportion of monomers lie above the interface—and a positive proportion of monomers lie below the interface if $\partial_h\mathtt{F}(\beta, h) < 1$. Hence, $h_c(\beta)$ marks a phase transition between a delocalized phase ($h < h_c(\beta)$) and a localized phase ($h > h_c(\beta)$). Some bounds on the critical point are easily obtained: we can show that $(h - \lambda(\beta))^+ \le \mathtt{F}(\beta, h) \le (h)^+$ for all β, h, which yields $0 \le h_c(\beta) \le \lambda(\beta)$. As discussed below, improving those bounds, and in particular obtaining the behavior of $h_c(\beta)$ as $\beta \downarrow 0$, has been the object of an intense activity. See Figure 1.2 for an overview of the phase diagram.

Annealed and homogeneous model. Let us introduce here the annealed model, where one averages over the disorder: the annealed partition function is, for $\beta \in [0, \beta_0)$,

$$\mathbf{Z}^{\mathrm{a,cop}}_{n,\beta,h} := \mathbb{E}\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} = \mathbf{E}\Big[\exp\Big(h\sum_{i=1}^{n}\mathbf{1}_{\{\varepsilon_i=+1\}}\Big)\mathbf{1}_{\{n\in\tau\}}\Big]. \tag{1.7}$$

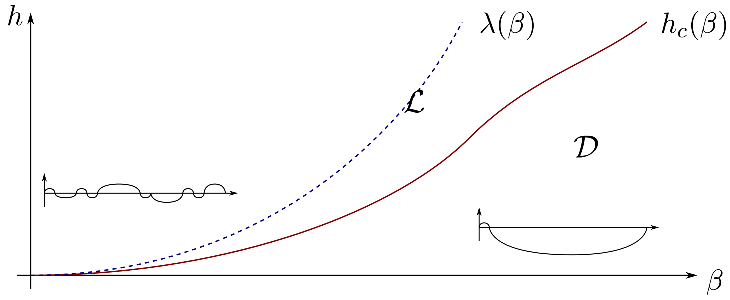


Figure 1.2 – Phase diagram for the copolymer model: the critical curve $\beta \mapsto h_c(\beta)$ is represented as a full line, and is bounded below by 0 and above by $\beta \mapsto \lambda(\beta)$. A typical realization under $\mathbf{P}^{\omega,\mathrm{cop}}_{n,\beta,h}$ is represented in both phases (it stays near the interface in the localized phase $\mathcal{L} := \{(\beta,h), \mathtt{F}(\beta,h) > 0\}$ and stays below the interface in the delocalized phase $\mathcal{D} := \{(\beta,h), \mathtt{F}(\beta,h) = 0\}$).

Notice that the second identity comes from exchanging the expectations with respect to $\mathbb{P}$ and $\mathbf{P}$, using that the ω_i's are i.i.d., with $\mathbb{E}[e^{\beta\omega_i}] = e^{\lambda(\beta)}$. Here, the reason we substracted $\lambda(\beta)$ in (1.2)-(1.3) becomes clear: it gets simplified in (1.7) so that the annealed model corresponds to the homogeneous model, *i.e.* the model with $\beta = 0$. From now on, we will write $\mathbf{Z}^{\mathrm{cop}}_{n,h}$ for $\mathbf{Z}^{\omega,\mathrm{cop}}_{n,0,h}$, the partition function of the homogeneous (and annealed) model. Noticing that $e^{n(h)^+} \geq \mathbf{Z}^{\mathrm{cop}}_{n,h} \geq \mathbf{P}(\tau_1 = n, \iota_1 = +1)e^{n(h)^+}$, by taking the logarithm, dividing by n and letting $n \to +\infty$, we get thanks to (1.1) that

$$\mathtt{F}(0,h) = \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{Z}^{\mathrm{cop}}_{n,h} = (h)^+ \qquad \text{for any } h \in \mathbb{R}. \tag{1.8}$$

Therefore, the homogeneous copolymer model has a phase transition of order 1: $\partial_h \mathtt{F}(0,h)$ is not continuous at $h_c^{\mathrm{hom}} = 0$ (the annealed critical point is $h_c^{\mathrm{a}}(\beta) = 0$), and the asymptotic density of monomers above the interface jumps from 0 to 1 when h goes from $h < 0$ to $h > 0$.

One can compare the free enegy $\mathtt{F}(\beta,h)$ to its annealed counterpart, using Jensen's inequality:

$$\mathtt{F}(\beta,h) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E} \log \mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} \leq \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h} = \lim_{n\to\infty} \frac{1}{n} \log \mathbf{Z}^{\mathrm{cop}}_{n,h} = \mathtt{F}(0,h)\,. \tag{1.9}$$

This shows in particular that $h_c(\beta) \geq 0 = h_c^{\mathrm{a}}(\beta)$.

1.3 Disorder relevance and the critical slope

As mentioned in the introduction, the problem of understanding whether a physical system is sensitive to the introduction of a small amount of disorder is central in the physical literature. Here, the question can be asked as follows: does the critical behavior of the model (for instance of its free energy) differ from that of its homogeneous counterpart, *as soon as* $\beta > 0$? Or can we take β sufficiently small so that the disordered and homogeneous models have the small critical properties. One wants in particular to determine whether the critical exponents of $\mathtt{F}(\beta,h)$ and $\mathtt{F}(0,h)$ are different, or whether the quenched and annealed critical point differ, *i.e.* $h_c(\beta) > 0$, which is another mark of disorder relevance.

1.3.1 Disorder relevance and smoothing of the phase transition

For the copolymer model, disorder has been found to be relevant for all $\alpha \geq 0$. First, in terms of critical exponents: in a celebrated work of Giacomin and Toninelli [GT06], it is shown that the quenched phase transition is of order at least 2, proving that it is *smoothened* by the presence of disorder—some improvement regarding the constants, have been obtained in [CdH13a]. Second, in terms of critical point shift: for $\alpha > 0$, $h_c(\beta)$ has been proven to be strictly positive first for large β in [Ton08a], and then for all $\beta > 0$ in [BGLT08]. We collect these results in the following theorem.

Theorem 1.1. *For all $0 < \beta < \beta_0$ and $u_0 > 0$ there exists a constant $C_{\beta,u_0} > 0$ such that for all $u \in (0, u_0)$*

$$\mathtt{F}(\beta, h_c(\beta) + u) \leq C_{\beta,u_0} \frac{1+\alpha}{2\beta^2} u^2 ,$$

and the constant C_{β,u_0} goes to 1 as $\beta \downarrow 0$ and $\beta^{-1}u_0 \downarrow 0$. Moreover, if $\alpha > 0$, then $h_c(\beta) > 0$ for all $\beta > 0$. (If $\alpha = 0$ then $h_c(\beta) = 0$ for all $\beta < \beta_0$.)

The ultimate goal here would be to obtain more precise bounds on the free energy close to the critical point. This appears so far untractable in general, but with Giambattista Giacomin and Hubert Lacoin [6], we managed to treat with unexpected precision the case $\alpha = 0$, in which $h_c(\beta) = 0$, see Section 1.4 below. On the other hand, much attention has been put on the value of $h_c(\beta)$ for $\alpha > 0$, and in particular on its behavior as $\beta \downarrow 0$, see Section 1.3.2.

1.3.2 About the critical slope

The first bound on $h_c(\beta)$ that we have mentioned above is $0 \leq h_c(\beta) \leq \lambda(\beta)$. Notice that, since we assumed that $\mathbb{E}[\omega_i] = 0$, $\mathbb{E}[\omega_i^2] = 1$, we have that $\lambda(\beta) \sim \frac{1}{2}\beta^2$ as $\beta \downarrow 0$. We therefore get that $h_c(\beta) = O(\beta^2)$, and a natural question is to know whether $\beta^{-2}h_c(\beta)$ converges to a limit.

Universality of the weak-coupling limit. Bolthausen and den Hollander [BdH97] (in the case of the simple random walk) and Caravenna and Giacomin [CG10] (in the general case with $\alpha \in (0,1)$) answer this question affirmatively, and they go one step further: not only the limit exists, but it is *universal*, in the sense that it does not depend on the specific disorder law $\mathbb{P}$ nor on the fine properties of $\mathbf{P}$, but only on α.

Theorem 1.2. *For every $\alpha \in (0,1)$, the limit $m_\alpha := \lim_{\beta\downarrow 0} \beta^{-2}h_c(\beta)$ exists and depends only on α.*

Remark 1.3. This is known in the literature as the "critical slope problem": as explained in Remark 1.1, the model was originally formulated with different parameters ($\bar\beta = \beta/2$, $\bar h = (\lambda(\beta)-h)/\beta$), and the critical point was $\bar h_c(\bar\beta) = (2\bar\beta)^{-1}(\lambda(2\bar\beta) - h_c(2\bar\beta))$. Hence, Theorem 1.2 gives that the *critical slope*, *i.e.* the slope at the origin of the critical curve $\bar\beta \mapsto \bar h_c(\bar\beta)$, the limit $\bar m_\alpha := \lim_{\bar\beta} \bar\beta^{-1}\bar h_c(\bar\beta)$, exists and is universal. We also have the relation $\bar m_\alpha = 1 - 2m_\alpha$, and for consistency with the literature we refer to $\bar m_\alpha$ as the critical slope.

The key ingredient in the proof of Theorem 1.2 is to look at the weak-coupling scaling limit of the system: one takes $\beta \downarrow 0$ and $h \downarrow 0$ simultaneously, and one tries to show that, in some sense, the discrete model converges to a continuous one (replacing the renewal process and the disorder ω by their scaling limits)—this will resonate with Section 3.4 below. The main result of [CG10] makes this precise: for any $\beta > 0, h \in \mathbb{R}$, we have that $\lim_{a\downarrow 0} \frac{1}{a^2}\mathtt{F}(a\beta, a^2 h) = \mathtt{F}^{\boldsymbol{\alpha}}(\beta, h)$, where $\mathtt{F}^{\boldsymbol{\alpha}}(\beta, h)$ is the free energy of a continuous model, called the α-*copolymer* model.

Looking for the value m_α. The question of the value of the critical slope has attracted much attention in the physical and mathematical literature, even before it was known that this value was universal, see for instance [GHLO89, Mon00, BG04, dH07, Gia07, BGLT08]. We mention in particular the result due to Bodineau and Giacomin [BG04], who showed that $0 \le \lambda(\beta) - h_c(\beta) \le \lambda\big(\frac{\beta}{1+\alpha}\big)$ for every $\beta > 0$. As a consequence, we get that the critical slope verify $\frac{1}{1+\alpha} \le \bar m_\alpha \le 1$ (or $0 \le m_\alpha \le \frac{\alpha}{2(1+\alpha)}$). Several improvement of this bound have been obtained. In particular [BGLT08] showed that $\bar m_\alpha < 1$ (implying the second part of Theorem 1.1). More recently, Bolthausen, den Hollander and Opoku [BdHO15] showed that $\bar m_\alpha > \frac{1}{1+\alpha}$, ruling out Monthus' conjecture [Mon00] that $\bar m_\alpha = \frac{1}{1+\alpha}$: they provide explicit upper and lower bounds, and proposed the following conjecture.

Conjecture 1.3. *For all $\alpha > 0$, the critical slope is $\bar m_\alpha = \frac{2+\alpha}{2(1+\alpha)}$; equivalently $m_\alpha = \frac{\alpha}{4(1+\alpha)}$.*

The case $\alpha > 1$ (or simply $\mathbf{E}[\tau_1] < +\infty$) is somehow a bit different: Theorem 1.2 does not hold in that case, simply because the scaling limit of the underlying renewal is trivial, and there is a priori no reason why universality should hold. We have studied this case in a work with Francesco Caravenna, Julien Poisat, Rongfeng Sun and Nikos Zygouras [20] published in *Communications in Mathematical Physics*: our main result is to show that Conjecture 1.3 holds in the case $\alpha > 1$, showing the universality of the slope as a biproduct.

Theorem 1.4 ([20], Theorem 1.4)**.** *If $\alpha > 1$, then the critical slope is $\bar m_\alpha = \frac{2+\alpha}{2(1+\alpha)}$. Equivalently, $\lim_{\beta\downarrow 0} \frac{h_c(\beta)}{\beta^2} = \frac{\alpha}{4(1+\alpha)}$.*

Our techniques are specific to the case of a finite mean, so we have no hope of adapting these methods to the case $\alpha \in (0,1)$.

Ideas of the proofs. Let us present briefly some ideas of the proofs, in order to explain how the constant $\frac{\alpha}{4(1+\alpha)}$ appears.

For the upper bound. Our idea came from Giacomin's book [Gia07, Chap. 6]: in its Theorem 6.3, it shows that if $\mathbf{E}[\tau_1] < +\infty$ then $\lim_{\beta\downarrow 0} \frac{1}{\beta^2}\mathtt{F}(\beta,\lambda(\beta)) = \frac18$ (recall that we have a different parametrization here). This idea can be used to prove a lower bound on the weak coupling limit of the free energy for a wider range of h. Lemma 5.1 in [20] gives that, if $\mathbf{E}[\tau_1] < +\infty$, we have $\liminf_{\beta\downarrow 0} \frac{1}{\beta^2}\mathtt{F}(\beta, a\beta^2) \ge \frac12 (a-\frac14)^+$ for any $a\in\mathbb{R}$.

This gives a first upper bound on $h_c(\beta)$: we necessarily have that $\limsup_{\beta\downarrow 0}\frac{1}{\beta^2} h_c(\beta) \le 1/4$, since $\mathtt{F}(\beta, a\beta^2)$ is asymptotically positive for any $a > 1/4$. However, the smoothing inequality of Theorem 1.1 enables us to improve this inequality. Doing as if $h_c(\beta) \sim m_\alpha \beta^2$, we get from Theorem 1.1 (using that the constant $C_{\beta, c\beta^2}$ goes to 1) that $\limsup_{\beta\downarrow 0}\frac{1}{\beta^2}\mathtt{F}(\beta,(m_\alpha+b)\beta^2) \le \frac{1+\alpha}{2} b^2$ for all $b\ge 0$. Combining this with the lower bound above, we get that $\frac{1+\alpha}{2}b^2 \ge \frac12 (m_\alpha + b - \frac14)$, or equivalently $m_\alpha \le \frac14 + (1+\alpha) b^2 - b$. Since this must be valid for all $b\ge 0$, we get that $m_\alpha \le \frac{\alpha}{4(1+\alpha)}$. We refer to Figure 1.3 for a graphical illustration of that fact.

In view of Figure 1.3, one could ask the question of the value of the limit $\lim_{\beta\downarrow 0}\frac{1}{\beta^2}\mathtt{F}(\beta, x\beta^2)$: does it match with its lower or with its upper bound? Well, since Theorem 1.4 gives the value for m_α, we already have that the limit is equal to $\frac{1}{8(1+\alpha)}$ at $x = \frac{\alpha+2}{4(1+\alpha)}$, where the lower and upper bounds meet. Moreover, [Gia07, Thm. 6.3] gives that the limit is equal to $\frac18$ at $x=\frac12$. It is an

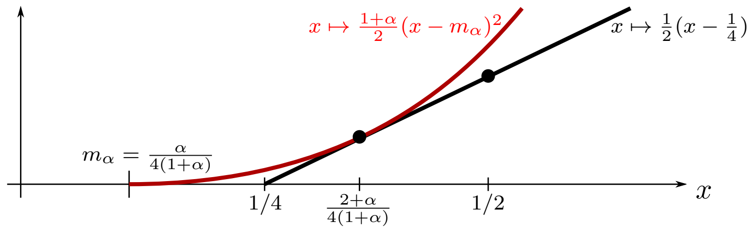


Figure 1.3 – Representation of the weak-coupling limit bounds on the free energy, in the case $\alpha > 1$. We plotted the functions $x \mapsto \frac12(x-\frac14)$ and $x \mapsto \frac{1+\alpha}{2}(x-m_\alpha)^2$ which are respective lower and upper bounds on $\lim_{\beta\downarrow 0}\frac{1}{\beta^2}\mathtt{F}(\beta, x\beta^2)$. The linear (black) lower bound directly gives that $m_\alpha \le 1/4$, but together with the quadratic (red) upper bound it tells that m_α cannot be greater than $\frac{\alpha}{4(1+\alpha)}$.

exercise to show that $x \mapsto \mathtt{F}(\beta, x\beta^2)$ is convex, so the limit is convex: we therefore get that, if $\alpha > 1$,

$$\lim_{\beta\downarrow 0}\frac{1}{\beta^2}\mathtt{F}(\beta, x\beta^2) = \frac12\big(x-\frac14\big) \qquad \text{for any } x \in \big[\tfrac{2+\alpha}{4(1+\alpha)}, \tfrac12\big]\,. \tag{1.10}$$

The question for $x \ge \frac12$ or $x \le \frac{\alpha+2}{4(1+\alpha)}$ remains, and does not follow from any easy argument I can think of. The idea in [Gia07, Chap. 6] gives as an upper bound $(x-\frac12)_+ + \frac18$, and it shouldn't be excluded that this is the correct value for $x > 1/2$.

For the lower bound. The idea is to use a fractional moment estimate, together with a coarse-graining procedure: this method has been developed in the context of the pinning model in [DGLT09], and refined in subsequent articles [GLT10b, GLT11] (and also in [14, 20]). The idea is to find some $\zeta < 1$ such that $\limsup_{n\to+\infty}\frac1n \log \mathbb{E}[(\mathbf{Z}_{n,\beta,c\beta^2})^\zeta] = 0$ for any $c < m_\alpha = \frac{\alpha}{4(1+\alpha)}$ and β small enough: by Jensen's inequality

$$\mathtt{F}(\beta, c\beta^2) = \lim_{n\to+\infty}\frac1n \mathbb{E}\log \mathbf{Z}_{n,\beta,c\beta^2} \le \lim_{n\to+\infty}\frac{1}{n\zeta}\log \mathbb{E}[(\mathbf{Z}_{n,\beta,c\beta^2})^\zeta] = 0\,,$$

proving that $h_c(\beta) \ge c\beta^2$. We estimate $\mathbb{E}[(\mathbf{Z}_{n,\beta,c\beta^2})^\zeta]$ via a change of measure argument. Using Hölder's inequality, we get that

$$\mathbb{E}[(\mathbf{Z}_{t/\beta^2,\beta,c\beta^2})^\zeta] \le \widetilde{\mathbb{E}}[\mathbf{Z}_{t/\beta^2,\beta,c\beta^2}]^\zeta \widetilde{\mathbb{E}}\Big[\Big(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\Big)^{\frac{1}{1-\zeta}}\Big]^{1-\zeta}, \tag{1.11}$$

where we chose $\widetilde{\mathbb{P}}$ to be the law of ω tilted by $-\frac{1-\zeta}{2}\beta$ (this tuning of parameter has been optimized). After some calculations, this enables us to show that for any $t > 0$ and $c > 0$,

$$\limsup_{\beta\downarrow 0}\mathbb{E}[(\mathbf{Z}_{t/\beta^2,\beta,c\beta^2})^\zeta] \le \frac{1}{\mathbf{E}[\tau_1]^\zeta}\exp\Big(\frac{t\zeta}{2}\big(c-\frac14(1-\zeta)\big)\Big)\,. \tag{1.12}$$

For this to be very small, one needs to have $c < \frac14(1-\zeta)$. Then, we employ a coarse-graining procedure to get an upper bound on $\mathbb{E}[(\mathbf{Z}_{n,\beta,c\beta^2})^\zeta]$ for $n \gg 1/\beta^2$: we split the system into blocks of size t/β^2, and somehow "glue" the estimates on different blocks together—this can work only for $\zeta > \frac{1}{1+\alpha}$. All together, we can show that $\mathtt{F}(\beta, c\beta^2) = 0$ for β small enough, for any $c < \frac14(1-\zeta)$ with $\zeta > \frac{1}{1+\alpha}$, in other words $\mathtt{F}(\beta, c\beta^2) = 0$ for any $c < \frac{\alpha}{4(1+\alpha)}$.

1.3.3 A word on the case of a correlated disorder.

Together with Julien Poisat, we also explored the case of a correlated Gaussian disorder in the article [19], published in *Electronic Journal of Probability*. We consider $\varpi = (\varpi_i)_{i\in\mathbb{Z}}$ a centered and unitary Gaussian sequence, with covariance function $\rho_i := \mathbb{E}[\varpi_0\varpi_i]$ for $i \in \mathbb{Z}$, with $\rho_{-i} = \rho_i$. We assume that the correlations are summable, *i.e.* $\sum_{i\in\mathbb{Z}} |\rho_i| < +\infty$, and we set $\Upsilon_\infty = \sum_{i\in\mathbb{Z}} \rho_i$. The partition function of the model is

$$\mathbf{Z}_{n,\beta,h}^{\varpi,\mathrm{cop}} = \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n}(\beta\varpi_i + h)\mathbf{1}_{\{\varepsilon_i=+1\}}\Big)\mathbf{1}_{\{n\in\tau\}}\Big], \tag{1.13}$$

and note that we did not substract $\lambda(\beta)$ in the Hamiltonian as done in (1.3): there is no reason to do so since it will not be canceled out in the annealed model as it is in (1.7).

The free energy $\mathtt{F}^{(\rho)}(\beta,h) := \lim_{n\to+\infty}\frac{1}{n}\log \mathbf{Z}_{n,\beta,h}^{\varpi,\mathrm{cop}}$ still exists a.s. and in L^1, and a localization transition still occurs at some critical point $h_c^{(\rho)}(\beta)$. Moreover, the annealed free energy exists, and it is explicit if all covariances are non-negative, see [19, Prop. 1.5]. As for the correlated pinning model, cf. [21], we are able to obtain a smoothing inequality: for all $\beta > 0$ and any $u \geq 0$, we have

$$\mathtt{F}^{(\rho)}(\beta, h_c(\beta)) \leq \frac{1+\alpha}{2\beta^2\Upsilon_\infty}u^2\,. \tag{1.14}$$

(The constant is explicit because we are working with Gaussian variables, which makes calculations explicit.) This shows disorder relevance in terms of critical exponents.

As far as the slope of the critical curve is concerned, there is no universality result in the case $\alpha \in (0,1)$, even if we believe that an analogous of Theorem 1.2 should hold. In the case $\alpha > 1$, we were able in [19] to obtain the sharp asymptotic behavior of $h_c^{(\rho)}(\beta)$ as $\beta \downarrow 0$, *i.e.* the critical slope. Interestingly, it is expressed as the minimum between two quantities (it appears as an optimization between two localization strategies), and each of them may be the correct one, depending on the properties of the covariance sequence $(\rho_i)_{i\in\mathbb{Z}}$ (in particular if it has some negative terms) and of the underlying renewal. In particular, the critical slope is *not* universal in that case.

Theorem 1.5 ([19], Theorem 1.9)**.** *If* $\mathbf{E}[\tau_1] < +\infty$*, then we have*

$$\lim_{\beta\downarrow 0}\frac{1}{\beta^2}h_c^{(\rho)}(\beta) = \min\Big\{-\frac{\Upsilon_\infty}{2(1+\alpha)}; -\frac{\Upsilon_\infty}{4(1+\alpha)} - \frac{1}{4}\mathbf{C}^{\mathrm{cop}}\Big\}\,, \tag{1.15}$$

with $\mathbf{C}^{\mathrm{cop}} = \frac{1}{\mathbf{E}[\tau_1]}\sum_{i\in\mathbb{Z}}\sum_{k\geq|i|}\mathbf{P}(\tau_1 > k)$.

One recovers the result of Theorem 1.4 in the case where $\rho_i = 0$ for all $i \neq 0$. Indeed, one then has $\mathbf{C}^{\mathrm{cop}} = \Upsilon_\infty = 1$ so the minimum is attained by the second term and is equal to $-\frac{2+\alpha}{4(1+\alpha)} = \frac{\alpha}{4(1+\alpha)} - \frac{1}{2}$ (recall that in (1.2)-(1.3) we substracted $\lambda(\beta) \sim \frac{1}{2}\beta^2$). Let us also mention that in (1.15), the first term in the minimum corresponds to Monthus' conjecture (mentioned in Section 1.3.2), whereas the second term corresponds to Theorem 1.4. The critical slope is the "best" (*i.e.* the smallest) of these two terms.

1.4 Critical behavior in the $\alpha = 0$ case

Theorem 1.1 establishes that for the copolymer model, disorder is relevant whatever the value of $\alpha \geq 0$ is. The main question is now to describe the critical behavior of the model, and in particular to get sharp estimates on the free energy as $h \downarrow h_c(\beta)$. This is a difficult issue, and the only mathematical results we are aware of are in the context of the pinning of a $(1+d)$-dimensional free field on a disordered surface, see [GL18] for $d \geq 3$ and [Lac19] for $d = 2$. We also mention the case of the Derrida-Retaux model [DR14], which can be seen as a toy-model for the disordered pinning model, where some sharp predictions can be made, and mathematical results are at reach [HMP18] (see Section 2.3.2 for more details). In all the models cited above, disorder is relevant, but one of the key ingredient to be able to describe the critical behavior is the fact that the quenched critical point is known exactly.

As far as the copolymer model is concerned, some predictions have been made thanks to a *strong disorder* renormalization group approach, in [Mon00, IM05]. In an article in collaboration with Giambattista Giacomin and Hubert Lacoin [6], to appear in *Probability Theory and Related Fields*, we consider the copolymer model in the case $\alpha = 0$: our idea was that in that case too, the critical point was known exactly ($h_c(\beta) = 0$), so there was hope to derive sharp asymptotics for the free energy. In fact, we are able to give matching upper and lower bounds, to a level of precision that we were not expecting. We find, as suggested in [Mon00, IM05], that the phase transition is of infinite order, but our result is much finer and we show in particular that the free energy vanishes faster than exponentially.

Assume that $\alpha = 0$ in (1.1), *i.e.* that $\mathbf{P}(\tau_1 = n) = \varphi(n)n^{-1}$. Then, let $\tilde\varphi(n) := \sum_{k>n} \varphi(n)n^{-1}$, which goes to 0 as a slowly varying function, and which verifies $\tilde\varphi(n)/\varphi(n) \to +\infty$ as $n \to +\infty$, see [BGT89, Prop. 1.5.9a]. Note that in [6], we additionally assume that $\mathbf{P}(\tau_1 < +\infty) = 1$, but this assumption is in fact not necessary.

Theorem 1.6 ([6], Theorem 1.2)**.** *If $\alpha = 0$ in (1.1), then $h_c(\beta) = 0$ for all $\beta < \beta_0 := \sup\{\beta; \lambda(\beta) < +\infty\}$. Moreover, for all $\beta \in (0, \beta_0)$, we have as $h \downarrow 0$*

$$\mathtt{F}(\beta, h) \leq \exp\Big(- (1 + o(1))\, q_1(\beta) \frac{1}{h} \log\Big(\frac{\tilde\varphi(1/h)}{\varphi(1/h)}\Big)\Big),$$

where $q_1(\beta) = \beta\lambda'(\beta) - \lambda(\beta)$.

Since $\tilde\varphi(1/h)/\varphi(1/h) \to +\infty$, we get that $\mathtt{F}(\beta, h)$ decays much faster that exponentially in $1/h$. Additionally, we get sharper results for some specific choices of the slowly varying function $\varphi(\cdot)$. We consider the cases

$$(i) \quad \varphi(x) = (1+o(1))\, \frac{c_\varphi}{\log x (\log\log x)^{\upsilon}} \quad \text{as } x \to +\infty, \qquad \text{(sub-logarithmic)}$$

$$(ii) \quad \varphi(x) = (1+o(1))\, \frac{c_\varphi}{(\log x)^{\upsilon}} \quad \text{as } x \to +\infty, \qquad \text{(logarithmic)}$$

$$(iii) \quad \varphi(x) = \exp\big(- (1+o(1))\, (\log x)^{1/\upsilon}\big) \quad \text{as } x \to +\infty, \qquad \text{(super-logarithmic)}$$

for some $\upsilon > 1$. We obtain (almost) matching upper and lower bounds for the free energy in all three cases (i)–(iii), only the constant being non-optimal in cases (i)-(ii).

Theorem 1.7 ([6], Theorem 1.4)**.** *Fix $\beta\in(0,\beta_0)$, and set $q_1(\beta)=\beta\lambda'(\beta)-\lambda(\beta)$, $q_2(\beta)=\lambda(2\beta)-2\lambda(\beta)$. Then as $h\downarrow 0$*

(i) in the sub-logarithmic case,

$$(1+o(1))\,\upsilon\,q_1(\beta)\tfrac{1}{h}\log\log\left(\tfrac{1}{h}\right)\leq-\log\mathtt{F}(\beta,h)\leq(1+o(1))\,(\upsilon+1)q_2(\beta)\tfrac{1}{h}\log\log\left(\tfrac{1}{h}\right);$$

(ii) in the logarithmic case,

$$(1+o(1))\,(\upsilon-1)\,q_1(\beta)\tfrac{1}{h}\log\left(\tfrac{1}{h}\right)\leq-\log\mathtt{F}(\beta,h)\leq(1+o(1))\,(\upsilon+\tfrac{5}{2})q_1(\beta)\tfrac{1}{h}\log\left(\tfrac{1}{h}\right);$$

(iii) in the super-logarithmic case,

$$-\log\mathtt{F}(\beta,h)=(1+o(1))\left(\tfrac{h}{q_1(\beta)}\right)^{-\upsilon/(\upsilon-1)}.$$

We now present briefly how the proofs work.

Ideas for the upper bound: change of measure argument. We apply Jensen's inequality as in (1.9), with a twist: instead of applying it directly, we use a function $f(\omega)$ (chosen in a moment), and write

$$\mathbb{E}[\log\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h}]=\mathbb{E}[\log(f(\omega)\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h})]-\mathbb{E}[\log f(\omega)]\leq\log\mathbb{E}[f(\omega)\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h}]-\mathbb{E}[\log f(\omega)]\,. \tag{1.16}$$

This procedure amounts to a change of measure as done in (1.11), but directly at the level of the log-partition function: $f(\omega)$ can be seen as a probability density, and $\mathbb{E}[f(\omega)\mathbf{Z}^{\omega,\mathrm{cop}}_{n,\beta,h}]$ as the expectation of the partition function under a new measure.

Now, all the difficulty resides in the choice of the function $f(\omega)$. We set $k=k(h)$ properly, and we choose some $f(\omega)$ which penalizes environments which have long stretches (*i.e.* longer than k) where ω assumes unusually large values (*i.e.* with empirical mean larger than $c\lambda'(\beta)$). Then, one is able to estimate $-\mathbb{E}[\log f(\omega)]$: this gives the main contribution in the upper bound of Theorem 1.6. It then remains to show that $\frac{1}{n}\log\mathbb{E}[f(\omega)\mathbf{Z}^{\omega,\mathrm{cop}}_{n,h,\beta}]\to 0$, which is a bit more technical—we do not go into much detail here. The key ingredient is that, thanks to our choice of $f(\omega)$, one can actually bound $\mathbb{E}[f(\omega)\mathbf{Z}^{\omega,\mathrm{cop}}_{n,h,\beta}]$ by an explicit partition function, where an excursion of length ℓ receive: a reward if $\ell\leq k$; a penalty if $\ell>k$. This is summarized in Equation (4.15) in [6].

In the logarithmic and super-logarithmic cases, we are able to improve this bound. The idea is that the change of measure argument (1.16) gives a good bound for systems of length $n\approx e^{1/h}$, but beyond that scale the bound becomes non-optimal. We therefore apply the change of measure only to sub-blocks of length $e^{1/h}$ (we avoid penalizing regions that will not be visited), and we use a coarse-graining argument to "glue" these estimates together.

Ideas for the lower bound: rare-stretch strategy. We use here a method which is by now standard: a lower bound on the partition function is obtained by restricting it to trajectories visiting only "favorable" regions in the environment. This idea is already present in [BG04], and is a key tool in the proof of the smoothing inequality of Theorem 1.1, see [GT06]. We do not give further details.

Chapter 2

The polymer pinning model

This chapter is dedicated to the pinning model: in particular, we describe some of our contributions, [14] and [19, 20] (in Section 2.3), and [12] (in Section 2.4).

2.1 Presentation of the model and physical motivations

The pinning model has been used in many different contexts: one may trace it back to Poland and Scheraga [PS70] as a model for DNA denaturation, and to [Fis84] as a wetting model.

Pinning a polymer on a line of defects. Take $(S_i)_{i\geq 0}$ a Markov chain on $\mathbb{Z}^d$ (for some $d \geq 1$), started from $S_0 = 0$, and denote its law $\mathbf{P}$. For $n \in \mathbb{N}$, we consider the directed trajectory $(i, S_i)_{1\leq i\leq n}$: it represents a directed (or stretched) polymer. This polymer interacts with the line $\mathbb{N} \times \{0\}$ (the defect line) when it touches it, *i.e.* when $S_i = 0$. Since interactions occur only when S_i returns to 0, we consider directly the set of return times $\tau = \{i, S_i = 0\}$, which is a renewal process: $\tau_0 = 0$, and $(\tau_k - \tau_{k-1})_{k\geq 1}$ are i.i.d. $\mathbb{N}$-valued r.v.s.

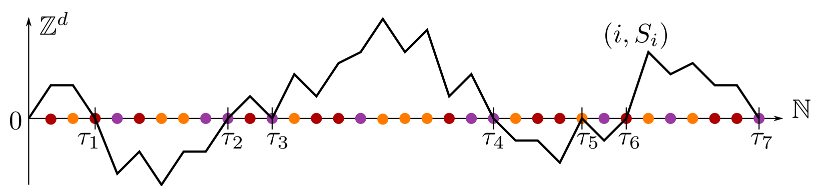


Figure 2.1 – The polymer trajectory is represented by a directed Markov chain (i, S_i). The interactions occur along the defect line, at the sites where S_i returns to 0, *i.e.* at times τ_1, τ_2,... The defect line is inhomogeneous, represented by random variables $(\omega_i)_{i\geq 0}$ being attached to the different sites.

We consider a sequence $\omega = (\omega_i)_{i\geq 0}$ of i.i.d. r.v.s, whose law is denoted $\mathbb{P}$: the ω_i's represent the inhomogeneities along the defect line. As above, we assume that $\mathbb{E}[\omega_i] = 0$, $\mathbb{E}[\omega_i^2] = 1$ and that $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_i}] < +\infty$ for all $0 \leq \beta < \beta_0 \in (0, +\infty]$. For a fixed realization of ω (quenched disorder), and for $\beta \in [0, \beta_0)$, $h \in \mathbb{R}$, we define the *polymer measures* $\mathbf{P}^{\omega,\mathrm{pin}}_{n,\beta,h}$ for $n \in \mathbb{N}$ by the following Radon-Nikodym derivative with respect to the reference law $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}^{\omega,\mathrm{pin}}_{n,\beta,h}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{\mathbf{Z}^{\omega,\mathrm{pin}}_{n,\beta,h}} \exp\Big(\sum_{i=1}^{n} (\beta\omega_i - \lambda(\beta) + h)\mathbf{1}_{\{i\in\tau\}}\Big) \mathbf{1}_{\{n\in\tau\}}\,. \tag{2.1}$$

The quantity $\mathbf{Z}_{n,\beta,h}^{\omega,\mathrm{pin}}$ is the partition function of the model, and is equal to

$$\mathbf{Z}_{n,\beta,h}^{\omega,\mathrm{pin}} := \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n}(\beta\omega_i - \lambda(\beta) + h)\mathbf{1}_{\{i\in\tau\}}\Big)\mathbf{1}_{\{n\in\tau\}}\Big]. \tag{2.2}$$

The measure $\mathbf{P}_{n,\beta,h}^{\omega,\mathrm{pin}}$ corresponds to giving a "reward" $\beta\omega_i - \lambda(\beta) + h$ (or a "penalty", depending on its sign) if the polymer touches the defect line at site i—note the analogy with the copolymer model (1.2)-(1.3), $\mathbf{1}_{\{\varepsilon_i=+1\}}$ being replaced by $\mathbf{1}_{\{i\in\tau\}}$. We again added in (2.1)-(2.2) the indicator function that $n\in\tau$, forcing the end-point to be pinned down: Remark 1.2 still holds here.

Similarly to (1.1), we assume that there is some $\alpha \geq 0$ and some slowly varying function $\varphi(\cdot)$ such that for all $n \geq 1$

$$\mathbf{P}(\tau_1 = n) = \varphi(n) n^{-(1+\alpha)}. \tag{2.3}$$

This is verified for instance if $\tau = \{n, S_{2n} = 0\}$ with S_n the simple random walk on $\mathbb{Z}^d$: one has $\alpha = 1/2$ and $\varphi(n) \to 1/2\sqrt{\pi}$ if $d=1$ (see e.g. [Fel66, Ch. III]); $\alpha = 0$ and $\varphi(n) \sim \pi/(\log n)^2$ if $d=2$ (cf. [JP72, Thm. 4]); $\alpha = d/2-1$ and $\varphi(n) \to c_d$ if $d \geq 3$ (cf. [DK11, Thm. 4]). We also assume that $\mathbf{P}(\tau_1 < +\infty) = 1$, *i.e.* that τ is *persistent*: if $\mathbf{P}(\tau_1 < +\infty) < 1$, one may reduce to the persistent case by in a change of variable $h \to h - \log \mathbf{P}(\tau_1 < +\infty)$, see [Gia07, Chap. 1].

The Poland Scheraga model for DNA denaturation. Poland and Scheraga, in [PS70], introduced a simplified model to describe the DNA denaturation (or melting) transition. This phenomenon is extremely complex, and in order to simplify the model, one forgets about the helix configuration of DNA, and considers that when heated symmetric "loops" are formed in the DNA double strand, see Figure 2.2. More formally, the sequence of contact points is given by a renewal process $\tau = (\tau_k)_{k\geq 0}$, whose law is denoted by $\mathbf{P}$. The size of the k^{th} loop is given by $\tau_k - \tau_{k-1}$, and the i^{th} monomer is a contact point if $i \in \tau$ (the contact points are the only places where interactions occur). Considering $\omega = (\omega_i)_{i\geq 0}$ a sequence of r.v.s that are attached to the monomers and represent the inhomogeneities along DNA, one uses the definition (2.1) to describe this situation. The exponent α in (2.3) is sometimes referred to as the loop exponent: it quantifies the entropic cost of having a loop of size n in the Poland Scheraga model.

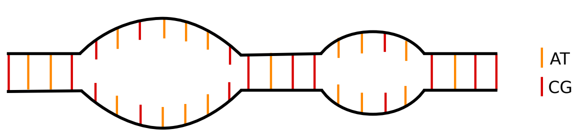


Figure 2.2 – Schematic view of DNA denaturation: symmetric loops are formed in a DNA double strand.

An effective model for wetting of interfaces in 2D models. Another context in which the pinning model has been used is the wetting of a $+/-$ interface in the Ising model, see [Fis84] (or [IV18] for an overview of the recent results). Consider the Ising model on $\mathbb{Z}^2$, on a large square of size n, with '+' boundary conditions on three sides, and '$-$' boundary condition on the last side. Then, at low temperature there are two main phases (a '+' and a '$-$' phase), and there is an interface between them, going from one corner to the other. A good approximation at low temperature is to forget the "overhangs", and use a random walk conditioned to stay non-negative and to come back to 0 as an effective interface model. We refer to Figure 2.3 for an illustration. The situation becomes even more interesting if the spins at the base have an additional random magnetic field δ_i:

the interface will be "penalized" if it touches the base at a site where $\delta_i > 0$ and "rewarded" if it touches the base at a site with $\delta_i < 0$ (this is our line of defect). One recovers the model described in (2.1), with τ the set of return times of a random walk conditioned to stay non-negative.

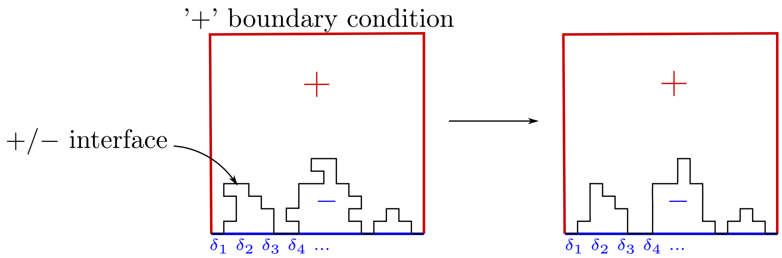


Figure 2.3 – Schematic view of the $+/-$ interface in the Ising model with '+' boundary condition on three sides and '−' on the last side of the square. On the right is the *effective* interface, which is modeled by a random walk conditioned to stay non-negative and to return to 0.

2.2 About the localization transition

The question is then to know whether, under the measure $\mathbf{P}^{\omega}_{n,\beta,h}$ the polymer trajectories are pinned to the defect line (*localized* phase, or *dry* phase in the wetting model), or wander away from it (*delocalized* phase, or *wet* phase in the wetting model). When tuning the parameters, the system undergoes a depinning (or denaturation, or wetting) phase transition: this can be seen through the *quenched* free energy, defined by

$$\mathtt{F}(\beta,h) := \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{Z}^{\omega,\mathrm{pin}}_{n,\beta,h} = \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log \mathbf{Z}^{\omega,\mathrm{pin}}_{n,\beta,h}, \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P})\,. \tag{2.4}$$

The existence of the limit again follows from super-additivity arguments, and we refer to [Gia07, Chap. 4] for details. Also here, we have that $h \mapsto \mathtt{F}(\beta,h)$ is non-negative, non-decreasing and convex. We define the (quenched) critical point

$$h_c(\beta) := \sup\big\{h : \mathtt{F}(\beta,h) = 0\big\} = \inf\big\{h : \mathtt{F}(\beta,h) > 0\big\}\,. \tag{2.5}$$

Analogously to (1.6), the derivative of the free energy $\partial_h \mathtt{F}(\beta,h)$, when it exists (for all but at most countably many h), is equal to $\lim_{n\to+\infty} \mathbf{E}^{\omega,\mathrm{pin}}_{n,\beta,h}\big[\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{i\in\tau}\big]$ a.s., the limiting density of contacts under $\mathbf{P}^{\omega,\mathrm{pin}}_{n,\beta,h}$. Hence, the critical point $h_c(\beta)$ marks the transition between a delocalized phase ($h < h_c(\beta)$, null density of contacts) and a localized phase ($h > h_c(\beta)$, positive density of contacts).

2.2.1 The annealed and homogeneous models.

As for the copolymer model, we define the annealed partition function, for $\beta \in [0,\beta_0)$, $\mathbf{Z}^{\mathrm{a,pin}}_{n,\beta,h} := \mathbb{E}\mathbf{Z}^{\omega,\mathrm{pin}}_{n,\beta,h}$, which is the partition function of the homogeneous model (*i.e.* with $\beta = 0$) with parameter h; we write it $\mathbf{Z}^{\mathrm{pin}}_{n,h}$ for short. The homogeneous free energy is $\mathtt{F}(0,h) := \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{Z}^{\mathrm{pin}}_{n,h}$, and Jensen's inequality gives that $\mathbb{E} \log \mathbf{Z}^{\omega,\mathrm{pin}}_{n,\beta,h} \le \log \mathbf{Z}^{\mathrm{a,pin}}_{n,\beta,h} = \log \mathbf{Z}^{\mathrm{pin}}_{n,h}$, so we obtain that $\mathtt{F}(\beta,h) \le \mathtt{F}(0,h)$ and $h_c(\beta) \ge h_c(0)$.

The homogeneous model is exactly solvable, but on the contrary to the copolymer model, it has a rich behavior and some work is needed to solve it. We collect the results on the homogeneous phase transition, which are detailed in [Gia07, Chap. 2].

Theorem 2.1. *Assume that (2.3) holds, and that $\mathbf{P}(\tau_1 < +\infty) = 1$. The homogeneous free energy $\mathtt{F} = \mathtt{F}(0,h)$ is characterized by the relation $\sum_{n\in\mathbb{N}} e^{-\mathtt{F}n}\mathbf{P}(\tau_1 = n) = e^{-h}$. From this, we derive that $h_c(0) = 0$, and that there exists a slowly varying function $\hat\varphi(\cdot)$ such that, as $h \downarrow 0$,*

$$\mathtt{F}(0,h) \sim \hat\varphi(1/h) h^{\nu} \qquad \textit{with } \nu = \max(1, \tfrac{1}{\alpha})\,.$$

2.2.2 A comment on the assumption $\lambda(\beta) < +\infty$.

In the definitions (2.1)-(2.2), we substracted $\lambda(\beta)$ in the Hamiltonian: this choice is made for renormalization purposes, so that the annealed model is exactly the homogeneous one. However, this restricts us to $\beta \le \beta_0 := \sup\{\beta; \lambda(\beta) < +\infty\}$. One could perfectly define the model for all $\beta \ge 0$, with partition function given by $\widetilde{\mathbf{Z}}^{\omega,\mathrm{pin}}_{n,\beta,h} := \mathbf{E}\big[\exp\big(\sum_{i=1}^n (\beta\omega_i + h)\mathbf{1}_{\{i\in\tau\}}\big)\mathbf{1}_{\{n\in\tau\}}\big]$. Then, the free energy $\widetilde{\mathtt{F}}(\beta,h) := \lim_{n\to+\infty}\frac{1}{n}\log\widetilde{\mathbf{Z}}^{\omega,\mathrm{pin}}_{n,\beta,h}$ exists and is finite (and is a.s. constant) as soon as $\mathbb{E}[|\omega_i|] < +\infty$. The critical point $\tilde h_c(\beta) = \inf\{h : \tilde{\mathtt{F}}(\beta,h) > 0\}$ is again well-defined. The only problem when $\lambda(\beta) = +\infty$ is that the annealed model is degenerated ($\mathbb{E}[\widetilde{\mathbf{Z}}^{\omega,\mathrm{pin}}_{n,\beta,h}] = +\infty$): a natural question is then to know whether a localization phase transition remains, *i.e.* if one has $\tilde h_c(\beta) > -\infty$.

Inspired by discussions with Hubert Lacoin, I gave this problem to a Master 1 student from ENS Lyon, Vincent Lerouvillois (in 2015). It turns out that it is not so difficult to show that $h_c(\beta) > -\infty$ if $\beta < (1+\alpha)\beta_0$, and that $h_c(\beta) = -\infty$ if $\beta > (1+\alpha)\beta_0$. The answer at $\beta = \beta_\alpha := (1+\alpha)\beta_0$ is more delicate, and depends on the finer properties of the distribution of ω and of τ_1. The results in the case $\beta_0 > 0$ can be summarized as follows: if $\lambda(\beta_0) < +\infty$ then $h_c(\beta_\alpha) > -\infty$, whereas if $\lambda(\beta_0) = +\infty$, there is a distribution for τ_1 for which we have $h_c(\beta_\alpha) = -\infty$. Obtaining a necessary and sufficient condition for having $h_c(\beta_\alpha) > -\infty$ is a difficult but interesting question, that resonates with some results on the Derrida-Retaux model, see Section 2.3.2. For instance, in analogy with [CEGM84], we can conjecture that in the case where $\lim_{n\to+\infty}\varphi(n) = c$, we have $h_c(\beta_\alpha) > -\infty$ if and only if $\mathbb{E}[\omega_i e^{\beta_0\omega_i}] < +\infty$ (see Section 2.3.2 for more details).

2.3 The question of disorder relevance

As mentioned in the introduction, one of our main goal is to understand if the characteristics of the phase transition are sensitive to the introduction of a small amount of disorder. The question can be asked in terms of critical exponent (is the critical exponent of the disordered model equal to that of the homogeneous one?) as well as in terms of critical point (is the quenched critical point equal to its annealed counterpart, *i.e.* do we have $h_c(\beta) = 0$ for β small enough?).

Because the pinning model is relatively simple but still exhibits a very rich behavior, it has been a perfect framework to test Harris criterion. The homogeneous correlation length critical exponent is $\nu = \max(1, \frac{1}{\alpha})$ (Giacomin [Gia08] shows that the correlation length is asymptotic to $1/\mathtt{F}(0,u)$), which covers the whole range $[1, +\infty]$, as α varies. Harris' predictions tell that disorder should be irrelevant if $\nu > 2$ (*i.e.* if $\alpha < 1/2$), and relevant if $\nu < 2$ (*i.e.* if $\alpha > 1/2$). Over the past decades, the question of disorder relevance for the pinning model has attracted much attention both from the physics community [FLNO86, DHV92, BM93, CH97, DR14, TC01, Mon06] and from the mathematical community [GT06, Ale08, Ton08b, Ton08a, AZ09, DGLT09, GT09, AZ10, GLT10b, Lac10b, GLT11, CdH13a, CdH13b, CTT17] and [14, 19, 20] (to cite a few). As we will see below, the answer is by now complete: Harris criterion has been confirmed rigorously, and the marginal case $\alpha = 1/2$ (corresponding to the simple random walk on $\mathbb{Z}$) has also been settled, after a long controversy in the physics literature. One of my main contribution, in an article in collaboration with Hubert Lacoin [14] published in *Journal de l'Institut Mathématique de Jussieu*, has been to prove a necessary and sufficient condition for disorder relevance, giving the final answer to the question (and proving a conjecture of [DHV92]).

Remark 2.1. We have that $\beta \mapsto h_c(\beta)$ is non-decreasing (see [GLT11, Prop. 6.1]): in a sense, *disorder relevance is non-decreasing in* β, and there is some $\beta_c \in [0, +\infty]$ such that $h_c(\beta) = 0$ if $\beta \le \beta_c$ and $h_c(\beta) > 0$ if $\beta > \beta_c$. The question of disorder relevance is therefore that of determining whether $\beta_c = 0$ (relevant case) or $\beta_c > 0$ (irrelevant case).

2.3.1 Summary of the results.

First of all, let us present the necessary and sufficent condition for disorder relevance (in terms of critical point shift), obtained in [14]. Two contradicting predictions were made in [FLNO86] and in [DHV92] in the case where $\alpha = 1/2$ and $\varphi(n)$ converges to a constant: the article [GLT10b] settles in favor of [DHV92], and in [14] we give the complete picture in the whole $\alpha = 1/2$ case.

Theorem 2.2 ([14], Theorem 2.2)**.** *Let τ and τ' be two independent copies of a renewal process with inter-arrival law $\mathbf{P}(\tau_1 = n) = \varphi(n) n^{-(1+\alpha)}$. Then, disorder is relevant, in the sense that $h_c(\beta) > 0$ for all $\beta > 0$, if and only if $\tau \cap \tau'$ is persistent (i.e. $|\tau \cap \tau'| = +\infty$ a.s.).*

Let us stress that the intersection of two renewal processes is a renewal process—some properties of intersections of renewal processes are studied in a work with Kenneth S. Alexander [16], see Chapter 7. In particular, $|\tau \cap \tau'|$ is a geometric r.v., with parameter $\mathbf{E}[|\tau \cap \tau'|]^{-1}$, and it is terminating if and only if $\mathbf{E}[|\tau \cap \tau'|] < +\infty$. Then, one can compute $\mathbf{E}[|\tau \cap \tau'|] = \sum_{n \ge 1} \mathbf{P}(n \in \tau \cap \tau') = \sum_{n \ge 1} \mathbf{P}(n \in \tau)^2$. If $\alpha \in (0,1)$, a result of Doney [Don97] gives that $\mathbf{P}(n \in \tau) \sim c_\alpha \varphi(n)^{-1} n^{-(1-\alpha)}$, for some constant c_α (see (7.5)). We get that $\mathbf{E}[|\tau \cap \tau'|]$ is finite if $\alpha < 1/2$ and infinite if $\alpha > 1/2$. In the case $\alpha = 1/2$, $\tau \cap \tau'$ is persistent if and only if $\sum_{n \ge 1} \frac{1}{n \varphi(n)^2} = +\infty$.

Free energy critical exponents. A fondamental result, obtained by Giacomin and Toninelli [GT06], shows disorder relevance for $\alpha > 1/2$ in terms of critical exponents.

Theorem 2.3. *There exists a constant $C > 0$ such that for all $0 < \beta < \beta_0$ and all $u \in (0,1)$*

$$\mathtt{F}(\beta, h_c(\beta) + u) \le \tfrac{C}{\beta^2}\, u^2 \,.$$

This is known as the smoothing phenomenon (see also Theorem 1.1): it proves that, in presence of disorder, the phase transition is of order at least two ($\partial_h \mathtt{F}(\beta, h)$ is continuous). When $\alpha > 1/2$, the homogeneous critical exponent is $\nu = \max(1/\alpha, 1) < 2$, and this proves disorder relevance.

In the case where $\tau \cap \tau'$ is terminating ($\alpha < 1/2$ or $\alpha = 1/2$ and $\sum_{n \ge 1} \frac{1}{n \varphi(n)^2} < +\infty$), results on disorder irrrelevance have been obtained in [Ale08], see also [Ton08b, Lac10b] for shorter proofs. These results prove at the same time that $h_c(\beta) = 0$ for β small enough, and that the critical exponent of the disordered model is also $\nu = 1/\alpha$.

Theorem 2.4. *Assume that $|\tau \cap \tau'| < +\infty$, where τ, τ' are two independent copies of a renewal process satisfying (2.3). There is some $\beta_1 > 0$ such that for all $\beta \le \beta_1$ we have that $h_c(\beta) = 0$, and*

$$\lim_{h \downarrow 0} \frac{\log \mathtt{F}(\beta, h)}{\log h} = \frac{1}{\alpha} \,.$$

Critical point shift. In the case $\alpha > 1/2$, the shift of the critical point (*i.e.* the fact that $h_c(\beta) > h_c^{\mathrm{a}}(\beta) = 0$), has been proven in [AZ09, DGLT09], together with the correct order for $h_c(\beta)$. In the case $\alpha = 1/2$, [GLT10b, GLT11] prove a critical point shift in the case where $\varphi(n)$ is asymptotically equivalent to $(\log n)^\kappa$ with $\kappa < 1/2$ (falling relatively close to the condition in Theorem 2.2), and sub-optimal bounds on the critical point shift are provided.
More recently, some works have managed to obtain sharp asymptotics for the critical point shift:

(i) in the case $\alpha > 1$, this is in an article by myself, in collaboration with Francesaco Caravenna, Julien Poisat, Rongfeng Sun and Nikos Zygouras [20] (and is analogous to Theorem 1.4);

(ii) in the case $\alpha \in (1/2, 1)$, this is in a work by Caravenna, Torri and Toninelli [CTT17];

(iii) in the case $\alpha = 1/2$, this is in the work with Hubert Lacoin [14]—we present only the case where $\varphi(n)$ converges to a constant, but more general results hold (see [14, Prop. 6.1 and 7.1]).

Theorem 2.5. *(i) If $\alpha > 1$, we have $\lim_{\beta\downarrow 0} \frac{h_c(\beta)}{\beta^2} = \frac{1}{\mathbf{E}[\tau_1]} \frac{\alpha}{2(1+\alpha)}$.*

(ii) If $\alpha \in (\frac12, 1)$, we have $\lim_{\beta\downarrow 0} \frac{h_c(\beta)}{\psi(1/\beta)\beta^{\frac{2\alpha}{2\alpha-1}}} = \mathtt{c}_\alpha$, for some slowly varying $\psi(\cdot)$ (depending explicitly on $\varphi(\cdot)$ and α) and $\mathtt{c}_\alpha$ a universal *constant depending only on α (but not on $\mathbb{P}$ or $\mathbf{P}$).*

(iii) If $\alpha = 1/2$ and $\lim_{n\to+\infty} \varphi(n) = c_\varphi$, we have $\lim_{\beta\downarrow 0} \beta^2 \log h_c(\beta) = -2\,(\pi c_\varphi)^2$.

In particular, in the case of a simple random walk (taking also into account parity issues), we have that $\lim_{\beta\downarrow 0} \beta^2 \log h_c(\beta) = -\frac{\pi}{2}$.

In a work with Julien Poisat [19], we obtain a result in the case of a correlated disorder, the analogous to Theorem 1.5 for the pinning model. Our main result is that, if $\mathbf{E}[\tau_1] < +\infty$, the critical point shift verifies $\lim_{\beta\downarrow 0} \frac{h_c(\beta)}{\beta^2} = \frac{1}{\mathbf{E}[\tau_1]} \frac{\alpha \Upsilon_\infty}{2(1+\alpha)}$ (with the same notations as in Section 1.3.3).

Some heuristics for the proof. We focus on the case of a Gaussian disorder $\omega_i \sim \mathcal{N}(0,1)$ for simplicity of the exposition (in particular $\lambda(\beta) = \frac12 \beta^2$). Let us compute the second moment of the partition function at the annealed critical point $h_c^{\mathrm{a}}(\beta) = 0$: using a replica trick, we have

$$\mathbb{E}\big[(\mathbf{Z}_{n,\beta,0}^{\omega,\mathrm{pin}})^2\big] = \mathbf{E}^{\otimes 2}\Big[\exp\Big(\beta^2 \sum_{i=1}^n \mathbf{1}_{\{i\in\tau\cap\tau'\}}\Big)\mathbf{1}_{\{n\in\tau\cap\tau'\}}\Big] . \tag{2.6}$$

Here, we realize that this is the partition of a homogeneous pinning model, with underlying renewal $\tau\cap\tau'$. A key quantity to consider is the mean overlap fraction of τ and τ': $D_n := \sum_{i=1}^n \mathbf{P}(i\in\tau)^2$. If $\sup_{n\in\mathbb{N}} D_n < +\infty$, the intersection renewal $\tau\cap\tau'$ is terminating, and one can take β small enough so that $\sup_{n\in\mathbb{N}} \mathbb{E}[(\mathbf{Z}_{n,\beta,0}^{\omega,\mathrm{pin}})^2] < +\infty$: this should be a sign of disorder irrelevance, since $\mathbf{Z}_{n,\beta,h}^{\omega,\mathrm{pin}}$ should remain concentrated around its mean, so the quenched and annealed free energy should remain close to each other. On the other hand, if $D_n \to +\infty$ then $\mathbb{E}[(\mathbf{Z}_{n,\beta,0}^{\omega,\mathrm{pin}})^2] \to +\infty$, and it should be a sign that the quenched and annealed critical point differ.

Upper bound on the critical point shift. One may obtain an upper bound on the critical point shift by quantifying the length scale up to which the quenched and annealed partition functions are close. This idea is present in [Lac10b] and is exploited in [14]: one can prove that as long as $\mathbb{E}[(\mathbf{Z}_{n,\beta,0}^{\omega,\mathrm{pin}})^2]$ is of order 1, the measure $\mathbf{P}_{n,\beta,0}^{\omega}$ is "close" to $\mathbf{P}$. Fix some constant $C > 1$, and set

$$n_\beta := \sup\big\{n : \mathbb{E}\big[(\mathbf{Z}_{n,\beta,0}^{\omega,\mathrm{pin}})^2\big] \le C\big\} . \tag{2.7}$$

Then, one can relate n_β to an upper bound on the critical point shift—after some calculation, we get $h_c(\beta) \le n_\beta^{-\alpha\wedge 1+o(1)}$. Let us consider the case $\alpha = 1/2$ with $\varphi(n) \to 1$ as $n \to \infty$ to avoid keeping track of the slowly varying functions: $\mathbf{P}(\tau_1 = n) \sim n^{-3/2}$ and $\mathbf{P}(n \in \tau) \sim (2\pi)^{-1} n^{-1/2}$, see (7.5). We then have that $D_n \sim \frac{1}{(2\pi)^2} \log n$ and we are able to show that $\log n_\beta = (1+o(1))\,(2\pi)^2/\beta^2$ (in [CSZ18], the authors make the $o(1)$ explicit). This leads to the upper bound in Theorem 2.5.

Lower bound on the critical point shift. The idea is similar to that presented in Section 1.3.2: we use a fractional moment estimate combined with a coarse-graining argument. Let us explain briefly the change of measure we introduce in the case $\alpha = 1/2$, in order to estimate the fractional moment $\mathbb{E}[(\mathbf{Z}_{\ell,\beta,h}^{\omega,\mathrm{pin}})^\zeta]$. It is based on a functional of the environment ω which quantifies its "positive correlations": we define the q-body interaction as

$$X_\ell(\omega) := \sum_{1\le i_0<\cdots<i_{q_\ell}\le \ell} U(i_0,\dots,i_{q_\ell})\omega_{i_0}\cdots\omega_{i_{q_\ell}} \tag{2.8}$$

with $U(i_0,i_1,\dots,i_{q_\ell}) = \prod_{k=1}^{q_\ell} \mathbf{P}(i_k - i_{k-1} \in \tau)$. The change of measure then penalizes environment where $X(\omega)$ is large, which are the one contributing most to $\mathbf{Z}_{\ell,\beta,h}^{\omega,\mathrm{pin}}$.

This choice is motivated by the (Wick) expansion of $\mathbf{E}\big[\exp\big(\sum_{i=a}^b \beta\omega_i \mathbf{1}_{\{i\in\tau\}}\big)\mathbf{1}_{\{b\in\tau\}} | a\in\tau\big]$, with $a<b$ thought as the entrance and exit points in a block of the coarse-graining decomposition—we dropped the renormalization $\lambda(\beta)$. The $q+1$-th term in the expansion is

$$\sum_{a=i_0<i_1<\cdots<i_q=b} \omega_{i_0}\mathbf{P}(i_1-i_0\in\tau)\omega_{i_1}\cdots\mathbf{P}(i_q-i_{q-1}\in\tau)\omega_{i_q}\,.$$

Summing over the entrance and exit points $a,b \in [\![1,\ell]\!]$ (so the functional is somehow homogeneous with respect to entrance and exit points of a coarse-grained block), we arrive at the choice (2.8).

The idea for this change of measure was already present in [GLT11], and it was noted that the lower bound on the critical point shift gets sharper as q gets larger: it is due to the fact that the main contribution to $\mathbb{E}\mathbf{Z}_{n,\beta,h}^{\omega,\mathrm{pin}}$ comes from the high terms in the expansion. The novelty of our method is that we are able to deal with a q-body interaction with q growing as the size ℓ of a coarse-grained block increases—for instance, if $\lim_{n\to+\infty}\varphi(n) = c$, we take $q = \log\log \ell$.

2.3.2 A conjecture about the critical behavior: Derrida-Retaux's toy model

All the above results leave aside one of the most important question: can we obtain the critical behavior of the quenched model explicitly, in the case of a relevant disorder? In [DR14], Derrida and Retaux propose a toy-model for the pinning model, which is also related to other models in physics, see [CEGM84]. The model is defined via a max-type recursion:

$$X_{n+1} \overset{(d)}{=} \big(X_n^{(1)} + X_n^{(2)} - 1\big)_+\,, \tag{2.9}$$

where $X_n^{(1)}, X_n^{(2)}$ are independent copies of X_n. This arises as a toy-version of the pinning model on a hierarchical (diamond) lattice introduced in [DHV92], which has served as a simpler pinning model to test Harris criterion—we refer to [GLT10a, Lac10a]. In this hierarchical model, the partition function satisfies the recursion $\mathbf{Z}_{n+1} = \frac{\mathbf{Z}_n^{(1)}\mathbf{Z}_n^{(2)}+B-1}{B} \approx \frac{1}{B}\max(\mathbf{Z}_n^{(1)}\mathbf{Z}_n^{(2)}, B-1)$ (B is a parameter tuning the geometry of the diamond lattice): the second approximation is the one made in [DR14] to simplify the model—one recovers the above max-type recursion by taking the logarithm, up to a parameter change.

The max-type recursion (2.9) model has a free energy $\mathtt{F} := \lim_{n\to+\infty} 2^{-n}\mathbb{E}[X_n]$, which is non-negative and depends on the distribution of X_0. A simple choice is to take $\mathbb{P}_{X_0} = (1-p)\delta_0 + p\mathbb{P}_Y$, where $p \in [0,1]$ is a parameter and Y is a positive r.v. (for the analogy with the pinning model, p corresponds to the homogeneous parameter h and Y to the r.v. ω). With that choice, the free energy depends on the parameter p, and we denote it $\mathtt{F}_Y(p)$: there is a critical value p_c such that $\mathtt{F}_Y(p) = 0$ for $p < p_c$ and $\mathtt{F}_Y(p) > 0$ for $p > p_c$. In [CEGM84], the authors compute explicitly the critical point in the case where Y is $\mathbb{N}$-valued: they obtain $p_c = (1 + \mathbb{E}[(Y-1)2^Y])^{-1}$. As a consequence, for a general distribution for Y, $p_c > 0$ if and only if $\mathbb{E}[Y2^Y] < +\infty$ (for the analogy with the pinning model, one can ask about conditions for having $h_c(\beta) > -\infty$, see Section 2.2.2).

The main result of [DR14] is about the critical behavior of the free energy: their conjecture is that, if $p_c > 0$, we have $\mathtt{F}_Y(p) = \exp\big((1+o(1))\frac{K}{\sqrt{p-p_c}}\big)$ as $p \downarrow p_c$, *i.e.* the phase transition is of infinite order (of the Berezinskii-Kosterlitz-Thouless type). The analogy with the pinning model suggests that in the disorder relevant regime, *i.e.* if $h_c(\beta) > 0$, then $\mathtt{F}(\beta, h_c(\beta)+u) = \exp\big((1+o(1))K'/\sqrt{u}\big)$ as $u \downarrow 0$. There are now some mathematical results on the Derrida-Retaux model: we mention [HS18] in the case $p_c = 0$, [HMP18] where the conjecture is proven for an exactly solvable (continuous) version of the model, or [CDD+19] where a weak version of the conjecture is proven.

2.4 Pinning a renewal on a quenched renewal

This section is devoted to one of my work in collaboration with Kenneth S. Alexander [12], published in *Electronic Journal of Probability*: it deals with a pinning model where interactions occur only when the renewal τ intersects a quenched renewal sequence σ (of distribution denoted by $\widetilde{\mathbf{P}}$, independent of τ). The inter-arrival distribution of σ is assumed to satisfy

$$\widetilde{\mathbf{P}}(\sigma_1 = n) = \tilde{\varphi}(n) n^{-(1+\tilde{\alpha})}\,, \tag{2.10}$$

for some $\tilde{\alpha} > 0$ and some slowly varying function $\tilde{\varphi}(\cdot)$. We also suppose that $\widetilde{\mathbf{P}}(\sigma_1 < +\infty) = 1$. The renewal τ is still assumed to satisfy (2.3), but we do not assume that τ and σ have the same law. For the simplicity of the exposition and to avoid many subcases, we assume that $\alpha, \tilde{\alpha} \in (0,1)$.

For a fixed realization of σ (quenched disorder), $m \in \mathbb{N}$, and parameter $\beta \geq 0$, we define the partition function of the model by

$$\mathbf{Z}^{\sigma,\mathrm{pin}}_{m,\beta} := \mathbf{E}\Big[\exp\Big(\beta\sum_{i=1}^{m} \mathbf{1}_{\{\sigma_i\in\tau\}}\Big)\mathbf{1}_{\{\sigma_m\in\tau\}}\Big] = \mathbf{E}\Big[\exp\Big(\beta\big|\tau\cap\sigma\cap[0,\sigma_m]\big|\Big)\mathbf{1}_{\{\sigma_m\in\tau\}}\Big]\,. \tag{2.11}$$

(A pinning measure $\mathbf{P}^{\sigma,\mathrm{pin}}_{n,\beta}$ analogous to (2.1) is defined implicitly by (2.11).) Our original motivation for this problem came from a paper by Cheliotis and den Hollander [CdH13b], where the authors show that the critical point of the pinning model can be expressed as the free energy of a model of the type (2.11) (in which τ and σ have the same law), with an additional source of disorder: we refer to the introduction of [12] (see Equation (1.2)) for more details. Let us make a few remarks on this model.

1. The size of the system is σ_m, and not m. Since $\tilde{\alpha} \in (0,1)$, disorder is *sparse*: it is non-zero only at renewal points of σ, and these points have a limiting density equal to 0, $\frac{1}{n}|\sigma\cap[0,n]| \to 0$

a.s. as $n \to +\infty$. The question here is to know whether such sparse disorder is able to pin down the renewal τ for arbitrarily small β: in order to create a positive energy, τ needs to be able to visit order-m renewal points of σ, so we need to consider a system of length σ_m.

2. This model is analogous to the random walk pinning model, where a random walk X is pinned onto the quenched trajectory of another random walk Y, and where each intersection between the two random walks is rewarded by a constant parameter β: the partition function is $\mathbf{E}^X[\exp(\beta \sum_{i=1}^n \mathbf{1}_{\{X_i=Y_i\}})\mathbf{1}_{\{X_n=Y_n\}}]$, with X, Y two independent random walks on $\mathbb{Z}^d$ with the same distribution, the trajectory of Y being quenched (see [BGdH11, BS10, BS11] and [26] for more on this model). Our model is therefore similar to the random walk pinning model, with $\mathbf{1}_{\{X_i=Y_i\}}$ replaced by $\mathbf{1}_{\{\sigma_i \in \tau\}}$. In our case however, we allow the two renewals to have different distributions.

3. We do not give any homogeneous reward h to each renewal point in τ: this would completely overcome the pinning effect of σ in the case where τ has many more renewal points than σ. However, the model with an additional external field h is interesting (as a pinning model with a correlated disorder given by a renewal sequence, $\omega_i := \mathbf{1}_{\{i\in\sigma\}}$), and is studied in [CCP19].

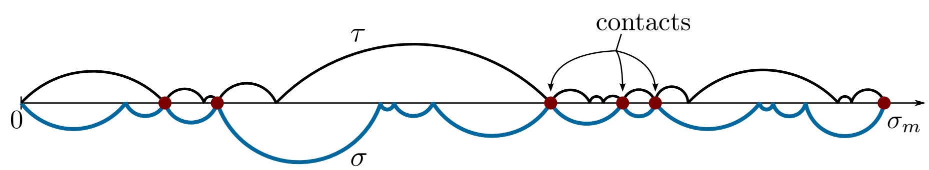


Figure 2.4 – Pinning of the renewal τ by the quenched renewal σ: we consider a system of length σ_m (here $m = 13$), and to a trajectory τ, we give a reward β for each intersection point between τ and σ.

Free energy and localization. We may define the free energy of the model, which exists and is $\widetilde{\mathbf{P}}$-a.s. constant (by super-additivity arguments),

$$\mathtt{F}(\beta) := \lim_{m\to+\infty} \frac{1}{m} \log \mathbf{Z}_{m,\beta}^{\sigma,\mathrm{pin}} = \lim_{m\to+\infty} \frac{1}{m} \widetilde{\mathbf{E}} \log \mathbf{Z}_{m,\beta}^{\sigma,\mathrm{pin}} \qquad \widetilde{\mathbf{P}}\text{-a.s. and in } L^1. \tag{2.12}$$

Also here, the positivity of the free energy indicates a "pinning" of τ, *i.e.* the fact that τ visits a positive fraction of the renewal points of σ: we have $\partial_\beta \mathtt{F}(\beta) = \lim_{m\to+\infty} \mathbf{E}_{m,\beta}^{\sigma,\mathrm{pin}}\big[\frac{1}{m}\sum_{i=1}^m \mathbf{1}_{\{\sigma_i\in\tau\}}\big]$, when the limit exists. Hence, the quenched critical point $\beta_c := \inf\{\beta\colon \mathtt{F}(\beta) > 0\}$ marks the transition between a delocalized phase ($\beta < \beta_c$) and a localized phase ($\beta > \beta_c$). One of the question we are answering in [12] is to determine under which conditions (on the distribution of τ and σ) we have $\beta_c = 0$, that is to know if an arbitrarily small pinning energy is enough localize τ.

Theorem 2.6 ([12], Theorem 2.1)**.** *We have $\beta_c = 0$ if and only if $\tau\cap\sigma$ is persistent, i.e. $|\tau\cap\sigma| = +\infty$.*

Note that $|\tau \cap \tau'| = +\infty$ if and only if $\sum_{n\geq 0} \mathbf{P}(n \in \tau)\widetilde{\mathbf{P}}(n \in \sigma) = +\infty$. In the case where $\alpha, \tilde\alpha \in (0,1)$, we have thanks to Doney's renewal theorem (7.5) that

$$|\tau\cap\sigma| = +\infty \ \text{ a.s.} \qquad \Longleftrightarrow \qquad \sum_{n\geq 1} \frac{1}{\varphi(n)\tilde\varphi(n)} n^{-(2-\alpha-\tilde\alpha)} = +\infty\,.$$

In particular, if $\alpha + \tilde\alpha > 1$ then $\tau \cap \tau'$ is persistent, if $\alpha + \tilde\alpha < 1$ it is terminating, and in the case $\alpha + \tilde\alpha = 1$ it can be either persistent or terminating depending on the finiteness of $\sum_{n\geq 1} \frac{1}{n\varphi(n)\tilde\varphi(n)}$.

Quenched vs annealed critical points. Define the annealed partition function $\widetilde{\mathbf{E}}\mathbf{Z}_{m,\beta}^{\sigma,\mathrm{pin}}$, the annealed free energy $\mathtt{F}^{\mathrm{a}}(\beta) = \lim_{m\to+\infty} \frac{1}{m} \log \widetilde{\mathbf{E}}\mathbf{Z}_{m,\beta}^{\sigma,\mathrm{pin}}$ and its critical point $\beta_c^{\mathrm{a}} = \sup\{\beta \colon \mathtt{F}^{\mathrm{a}}(\beta) > 0\}$.

The annealed partition function looks like a homogeneous pinning model with underlying renewal $\tau\cap\sigma$, but it is not exactly the same: the size of the system is σ_m, which is random. However, we are able to solve the annealed model: the annealed critical point is $\beta_c^{\mathrm{a}} := \tilde{\mathbf{E}}\mathbf{E}[|\tau\cap\sigma|]^{-1}$, and it is equal to 0 if and only if $\tau\cap\sigma$ is persistent. For $\alpha,\tilde\alpha \in (0,1)$, the annealed critical exponent is found to be $\nu = \max(1, 1/|\alpha^*|)$ with $\alpha^* = (1-\alpha-\tilde\alpha)/\tilde\alpha \in (-1,+\infty)$. Note that this is not the critical exponent for the homogeneous pinning model with underlying renewal $\tau\cap\sigma$ (which is $\max(1, 1/|1-\alpha-\tilde\alpha|)$).

Jensen's bound gives that $\mathtt{F}(\beta) \leq \mathtt{F}^{\mathrm{a}}(\beta)$, so that $\beta_c \geq \beta_c^{\mathrm{a}}$. Theorem 2.6 above tells that if $|\tau\cap\sigma| = +\infty$ then we have $\beta_c = \beta_c^{\mathrm{a}} = 0$. A natural question (related to that of disorder relevance), is whether one has the equality $\beta_c = \beta_c^{\mathrm{a}}$ beyond the condition $|\tau\cap\sigma| = +\infty$.

Theorem 2.7 ([12], Theorems 2.1 and 2.2)**.** *Note that $\alpha+\tilde\alpha < 1$ if and only if $\alpha^* > 0$.*

(i) if $\alpha+\tilde\alpha \geq 1$, then $\beta_c = \beta_c^{\mathrm{a}}$, and moreover $\frac{\log \mathtt{F}(\beta)}{\log(\beta-\beta_c^{\mathrm{a}})} \to \frac{1}{\alpha^*}$ as $\beta \downarrow \beta_c^{\mathrm{a}}$;

(ii) if $\alpha+\tilde\alpha < 1$ and $\alpha^* > 1/2$, then $\beta_c > \beta_c^{\mathrm{a}}$;

(iii) for any $\alpha,\tilde\alpha$ such that $\alpha+\tilde\alpha < 1$ and any $\varphi(\cdot),\tilde\varphi(\cdot)$, there exist distributions verifying $\mathbf{P}(\tau_1 = n) \sim \varphi(n) n^{-(1+\alpha)}$ and $\widetilde{\mathbf{P}}(\sigma_1 = n) \sim \tilde\varphi(n) n^{-(1+\tilde\alpha)}$ and for which we have $\beta_c > \beta_c^{\mathrm{a}}$.

Let us comment the three points in the theorem.

(i) The first item in Theorem 2.7 shows disorder irrelevance if $\alpha+\tilde\alpha \geq 1$, both in terms of critical points and of critical exponents. Note that (i) includes the result of Theorem 2.6, since having $|\tau\cap\sigma| = +\infty$ implies that $\alpha+\tilde\alpha \geq 1$. But Theorem 2.7 goes slightly beyond Theorem 2.6: one may have $\alpha+\tilde\alpha = 1$ and $|\tau\cap\sigma| < +\infty$: in that case we have $\beta_c^{\mathrm{a}} > 0$ but still $\beta_c = \beta_c^{\mathrm{a}}$.

(ii) The condition $\alpha^* > 1/2$ seems reminiscent of Harris criterion for disorder relevance: it corresponds to the case where the annealed model has a critical exponent $\max(1, 1/|\alpha|^*) < 2$. Harris criterion suggests that disorder should be irrelevant if $\alpha^* < 1/2$, *i.e.* that disorder, *provided its strength is small enough*, does change the critical properties of the system. The issue here is that one cannot tune the strength of the disorder: a quenched renewal σ is given, and there is no extra parameter to play with to get an arbitrarily small disorder strength. Having $\alpha^* > 1/2$ makes it easier for us to prove $\beta_c > \beta_c^{\mathrm{a}}$ since a quenched σ (hence with a positive strength) is necessarily relevant according to Harris' prediction.

We may conjecture that $\beta_c > \beta_c^{\mathrm{a}}$ as soon as $\alpha^* > 0$. This is based on a analogy with the random walk pinning model: the exponent $1+\alpha^*$ appears when considering the probability $\widetilde{\mathbf{P}}\mathbf{P}(\sigma_m \in \tau) = m^{-(1+\alpha^*)+o(1)}$, and is the analogous of the exponent ρ such that $\mathbf{P}^Y\mathbf{P}^X(X_n = Y_n) = n^{-\rho+o(1)}$ in the context of the random walk pinning model. In [BGdH11], the authors conjecture that the quenched and annealed critical points differ as soon as $\rho > 1$ (which corresponds to $\alpha^* > 0$ in our case). We mention that in [BGdH11], the inequality $\beta_c > \beta_c^{\mathrm{a}}$ is proven for $\rho > 2$: we prove the critical point shift for $\alpha^* > 1/2$, which is an improvement (it would correspond to the case $\rho > 3/2$).

(iii) The last item in Theorem 2.7 goes in the direction of proving the aforementioned conjecture that $\beta_c > \beta_c^{\mathrm{a}}$ as soon as $\alpha^* > 0$: for any $\alpha^* > 0$, it provides examples of distributions for τ,σ for which $\beta_c > \beta_c^{\mathrm{a}}$. Hence, if the condition for a critical point shift depends only on asymptotic properties of $\widetilde{\mathbf{P}},\mathbf{P}$, then this condition must be $\alpha^* > 0$. An important feature of our model that allows us to derive (iii) is that the distributions of τ and σ are allowed to be different: this gives us some flexibility—it enables us to tune explicitly $\widetilde{\mathbf{P}},\mathbf{P}$ in order to make some properties of the intersection $\tau\cap\sigma$ hold.

Chapter 3

The generalized Poland-Scheraga (gPS) model for DNA denaturation

In this chapter, we introduce the generalized Poland Scheraga (gPS) model, and we present our contributions [10] (in Section 3.2) and [3] (in Section 3.3), together with some work in progress with my Ph.D. student Alexandre Legrand (in Section 3.4).

3.1 Presentation of the model

The Poland Scheraga model, thanks to its relative simplicity, plays a central role in the study of DNA denaturation, even though some aspects of it are oversimplified. In particular, loops are assumed to be symmetric, tolerating no *mismatch* (see Figure 2.2). In an attempt to overcome this limitation, Garel and Orland [GO04] (see also [NG06]) introduced a few years ago a generalization of the model, allowing loops to be asymmetric, and the two strands to be of different lengths. See Figure 3.1 for an illustration, in comparison with Figure 2.2.

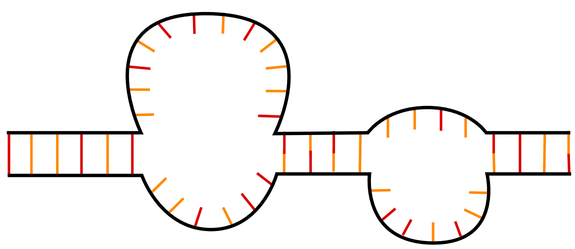


Figure 3.1 – Schematic view of DNA denaturation: loops are formed in the DNA double strand, but they may be asymmetric: mismatches are allowed, and the two strands may be of different lengths.

The mathematical formulation has been devised by Giacomin and Khatib [GK17]. Let $\boldsymbol{\tau} = (\boldsymbol{\tau}_k)_{k\geq 0}$ be a bivariate renewal process (its law is denoted by $\mathbf{P}$): $\boldsymbol{\tau}_0 = (0,0)$, and $(\boldsymbol{\tau}_k - \boldsymbol{\tau}_{k-1})_{k\geq 1}$ is a sequence of i.i.d. $\mathbb{N}^2$-valued r.v.s. A configuration for $\boldsymbol{\tau}$ translates into a polymer configuration: having $\boldsymbol{\tau}_k - \boldsymbol{\tau}_{k-1} = (a,b)$ correspond to the k^{th} loop being formed by a piece of length a in the first strand and a piece of length b in the second strand, see Figure 3.2. One can also interpret the event $(i,j) \in \boldsymbol{\tau} = \{\boldsymbol{\tau}_0, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \dots\}$ as the fact that the i^{th} monomer from the first strand is paired with the j^{th} monomer of the second strand.

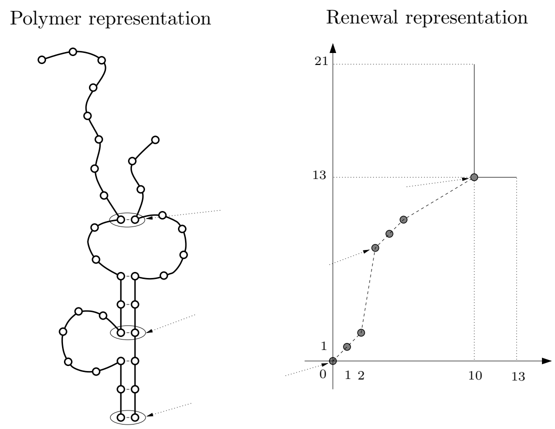


Figure 3.2 – A polymer configuration and its corresponding bivariate renewal representation: there are two strands, one with length 14, one with length 22, and the contact points are $(1,1)$, $(2,2)$, $(3,3)$, $(4,9)$, $(5,10)$, $(6,11)$, $(10,13)$. There are two free ends, of respective length 3 and 8.

In the physics literature, a loop of total length ℓ is associated to an entropic factor ℓ^{-c}, for a constant $c > 2$. This corresponds to assuming that there is some $\alpha \geq 0$ and some slowly varying function $\varphi(\cdot)$ such that for all $n, m \in \mathbb{N}$

$$\mathbf{P}\big(\boldsymbol{\tau}_1 = (n,m)\big) = \mathrm{K}(n+m) \quad \text{with } \mathrm{K}(\ell) := \varphi(\ell)\ell^{-(2+\alpha)} . \tag{3.1}$$

We also assume that $\sum_{n,m\in\mathbb{N}} \mathrm{K}(n+m) = 1$ so that $\boldsymbol{\tau}$ is persistent. Note that the role of the coordinates $\tau^{(1)}, \tau^{(2)}$ of $\boldsymbol{\tau}$ is symmetric, and that they are one-dimensional renewal processes, with inter-arrival distribution that satisfy $\mathbf{P}(\tau_1^{(r)} = n) \sim \frac{1}{1+\alpha}\varphi(n)n^{-(1+\alpha)}$ as $n \to +\infty$, for $r = 1, 2$.

Then, we take $\omega := (\omega_{i,j})_{(i,j)\in\mathbb{N}^2}$ a (ergodic) field of r.v.s, whose law is denoted $\mathbb{P}$: $\omega_{i,j}$ represents the interaction between the i^{th} monomer from the first strand and the j^{th} monomer of the second strand. We assume that $\mathbb{E}\omega_{i,j} = 0$, $\mathbb{E}[\omega_{i,j}^2] = 1$ and that $\lambda(\beta) := \mathbb{E}[e^{\beta\omega_{i,j}}] < +\infty$ for all $0 \leq \beta < \beta_0 \in (0, +\infty]$. For a fixed realization of ω, for $\beta \in [0, \beta_0)$ and $h \in \mathbb{R}$, we define for $n, m \in \mathbb{N}$ (the respective lengths of the strands) the partition function of the model as

$$\mathbf{Z}_{n,m,\beta,h}^{\omega,\mathrm{gPS}} = \mathbf{E}\Big[\exp\Big(\sum_{i=1}^{n}\sum_{j=1}^{m}(\beta\omega_{i,j} - \lambda(\beta) + h)\mathbf{1}_{\{(i,j)\in\boldsymbol{\tau}\}}\Big)\mathbf{1}_{\{(n,m)\in\boldsymbol{\tau}\}}\Big] . \tag{3.2}$$

There is a probability measure $\mathbf{P}_{n,m,\beta,h}^{\omega,\mathrm{gPS}}$ associated to (3.2). We put $\mathbf{1}_{\{(n,m)\in\boldsymbol{\tau}\}}$ in (3.2), constraining the two endpoints of the polymer to meet (in opposition to the illustration of Figure 3.2)—we call $\mathbf{Z}_{n,m,\beta,h}^{\omega,\mathrm{gPS}}$ the *constrained* partition function.

In the physical literature, the entropy of free ends is considered to be different from the entropy of loops: a free end of length ℓ has an entropy $\mathrm{K}_f(\ell) = \bar{\varphi}(\ell)\ell^{-\bar{\alpha}}$, for some slowly varying $\bar{\varphi}(\cdot)$ and some $\bar{\alpha} \in \mathbb{R}$. Note that $\mathrm{K}_f(\cdot)$ is not necessarily a probability distribution, and in fact we fix $\mathrm{K}_f(0) = 1$ (for normalization purposes). Then, we define the *free* partition function as

$$\mathbf{Z}_{n,m,\beta,h}^{\omega,\mathrm{gPS,free}} := \sum_{i=0}^{n}\sum_{j=0}^{m} \mathbf{Z}_{n-i,m-j,\beta,h}^{\omega,\mathrm{gPS}} \mathrm{K}_f(i)\mathrm{K}_f(j) , \tag{3.3}$$

and we denote $\mathbf{P}_{n,m,\beta,h}^{\omega,\mathrm{gPS,free}}$ the probability measure associated to it.

Free energy and denaturation transition. In the gPS model, there is an extra parameter $\gamma > 0$ which accounts for the asymptotic strand length ratio. By symmetry we may take $\gamma \geq 1$. We define

$$\mathtt{F}_\gamma(\beta,h) := \lim_{\substack{n,m\to+\infty \\ m/n\to\gamma}} \frac{1}{n}\log \mathbf{Z}^{\omega,\mathrm{gPS}}_{n,m,\beta,h} = \lim_{\substack{n,m\to+\infty \\ m/n\to\gamma}} \frac{1}{n}\mathbb{E}\log \mathbf{Z}^{\omega,\mathrm{gPS}}_{n,m,\beta,h} \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P})\,. \tag{3.4}$$

The existence of the limit is shown in [3] in the case of an i.i.d. disorder $(\omega_{i,j})_{i,j\in\mathbb{N}}$, but the proof works in full generality—we may also replace $\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,m,\beta,h}$ by $\mathbf{Z}^{\omega,\mathrm{gPS,free}}_{n,m,\beta,h}$. We have once again $h\mapsto \mathtt{F}(\beta,h)$ is non-negative, non-decreasing and convex. We define the critical point $h_c(\beta) := \inf\{\mathtt{F}_\gamma(\beta,h)>0\}$ (its dependence on γ is implicit): it marks the transition between a delocalized phase ($h<h_c(\beta)$, null density of contacts) and a localized phase ($h>h_c(\beta)$, positive density of contact)—indeed, $\partial_h \mathtt{F}_\gamma(\beta,h)$ (when it exists) is also here equal to the limiting density of contacts between the two strands, under $\mathbf{P}^{\omega,\mathrm{gPS}}_{n,m,\beta,h}$. This phase transition, from the DNA point of view, corresponds to the denaturation (or melting) transition.

However, in the homogeneous model, there might be another point of non-analyticity of the free energy, corresponding to another phase transition, see Section 3.2 below. In the disordered model, the question of determining whether this second transition survives is more difficult (there is no easy characterization of what the critical point should be), and has been left aside for now.

3.2 The homogeneous model: denaturation and condensation transitions

Let $\mathbf{Z}^{\mathrm{gPS}}_{n,m,h}$ (resp. $\mathbf{Z}^{\mathrm{gPS,free}}_{n,m,h}$) be the partition function (resp. *free* partition function) of the homogeneous *i.e.* $\beta=0$ model, and $\mathbf{P}^{\mathrm{gPS}}_{n,m,h}$ (resp. $\mathbf{P}^{\mathrm{gPS,free}}_{n,m,h}$) the corresponding polymer measures.

We start by stating the first result of [GK17], which describes the localization transition (in analogy with Theorem 2.1), but points out the fact that a second phase transition might exist. We denote $h_c = h_c(0)$.

Theorem 3.1. *For any $\gamma\geq 1$, we have that $h_c=0$. Moreover, there exist a slowly varying function $\hat\varphi(\cdot)$ and a constant $c_{\alpha,\gamma}$ such that $\mathtt{F}_\gamma(0,h)\sim c_{\alpha.\gamma}\hat\varphi(1/h)h^\nu$ as $h\downarrow 0$, with $\nu=\max\big(1,\frac{1}{\alpha}\big)$.*

Additionally, there exists some $h_{c,\gamma}\in(0,+\infty]$ such that $h\mapsto \mathtt{F}(0,h)$ is analytic on $(-\infty,0)\cup(0,h_{c,\gamma})$. If $h_{c,\gamma}<+\infty$, then $h_{c,\gamma}$ is another non-analicity point of $\mathtt{F}(0,h)$.

We stress that $h_{c,1}=+\infty$, and that if $h_{c,\gamma}<+\infty$, then $h_{c,\gamma}$ may not be the only non-analyticity point in the localized phase, see [GK17, Sec. 3.4] for more details. The point $h_{c,\gamma}$ marks another phase transition, a *condensation* phase transition, inside the localized phase. In an article with Giambattista Giacomin and Maha Khatib [10] published in *ALEA: Lat. Am. J. Probab. Math. Stat.*, we manage to give the path properties of the system in this "condensed" phase.

Take $h>0$, and for non-negative λ_1,λ_2, rewrite the constrained partition function ($m\geq n$) as

$$\mathbf{Z}^{\mathrm{gPS}}_{n,m,h} = e^{\lambda_1 n+\lambda_2 m}\sum_{\ell=1}^{n}\sum_{\substack{i_1,\dots,i_\ell\in\mathbb{N}\\ i_1+\dots+i_\ell=n}}\sum_{\substack{j_1,\dots,j_\ell\in\mathbb{N}\\ j_1+\dots+j_\ell=m}}\prod_{k=1}^{\ell} e^{h-\lambda_1 i_k-\lambda_2 j_k}\mathrm{K}(i_k+j_k)\,. \tag{3.5}$$

For $h>0$, we consider the family of parameters $\mathcal{C}_h = \big\{(\lambda_1,\lambda_2)\colon \sum_{n,m\in\mathbb{N}} e^{h-\lambda_1 n-\lambda_2 n}\mathrm{K}(n+m)=1\big\}$: each $(\lambda_1,\lambda_2)\in\mathcal{C}_h$ defines an inter-arrival probability distribution $\widehat{\mathbf{P}}_{\lambda_1,\lambda_2}$. Then, for all $(\lambda_1,\lambda_2)\in\mathcal{C}_h$, we have $\mathbf{Z}^{\mathrm{gPS}}_{n,m,h} = e^{\lambda_1 n+\lambda_2 m}\widehat{\mathbf{P}}_{\lambda_1,\lambda_2}((n,m)\in\hat{\boldsymbol{\tau}})$. Then, if one is able to choose λ_1,λ_2 so that the last probability if $e^{-o(n)}$, the free energy will be $\mathtt{F}_\gamma(0,h)=\lambda_1+\gamma\lambda_2$. Two situations may occur: (1) the optimal (λ_1,λ_2) is in $(0,\infty)^2$, and $\widehat{\mathbf{P}}_{\lambda_1,\lambda_2}$ has exponential tails in both directions—this is dubbed as the Cramér regime; (2) $\lambda_2=0,\lambda_1>0$, and $\widehat{\mathbf{P}}_{\lambda_1,\lambda_2}$ has exponential tail only in the horizontal direction—this is dubbed as the non-Cramér regime.

Let $\mathtt{N}(h)$ be the unique solution of $\sum_{n,m\in\mathbb{N}} e^{h-n\mathtt{N}(h)}\mathrm{K}(n+m)=1$ (*i.e.* $\lambda_2=0$), and let $\widehat{\mathbf{P}}_h(\cdot)$ be the corresponding probability distribution. In [GK17], it is shown that $\mathtt{F}(0,h)=\mathtt{N}(h)$ if and only if $\gamma>\gamma_c(h)$, where $\gamma_c(h)=\widehat{\mathbf{E}}_h[\hat\tau^{(2)}]/\widehat{\mathbf{E}}_h[\hat\tau^{(1)}]$ (we mention that $h\mapsto\gamma_c(h)$ may not be monotonous, and depends heavily on the distribution $\mathbf{P}$). In [10], we derive the path properties in the case $\gamma>\gamma_c(h)$. In order to state our results, define $\kappa_n := |\{k,\tau_k^{(1)}\le n\}|$. Let $L_n:=\max_{j\le\kappa_n}\{\tau_j^{(2)}-\tau_{j-1}^{(2)}\}$ be the size of the largest jump (in the second coordinate), attained for some j_n, and let $\ell_n=\max_{j\le\kappa_n,j\neq j_n}\{\tau_j^{(2)}-\tau_{j-1}^{(2)}\}$ be the size of the second largest jump. In the free case, we also need $V_n:=m-\tau^{(2)}_{\kappa_n}$ the length of the unbound part of the second strand. The main result of [10] can be summarized (and simplified) as follows.

Theorem 3.2 ([10], Theorems 1.1 and 1.2)**.** *Let h be such that $\gamma_c:=\gamma_c(h)<\gamma$. For any $\varepsilon>0$, we have as $n,m\to+\infty$, $\frac{m}{n}\to\gamma$:*

(i) in the constrained *case,*

$$\mathbf{P}^{\mathrm{gPS}}_{n,m,h}\Big(\frac{L_n}{n}-(\gamma-\gamma_c)\in[1-\varepsilon,1+\varepsilon]\,;\,\ell_n\le n^{\frac{1+\varepsilon}{(1+\alpha)\wedge 2}}\Big)\to 1;$$

(ii) in the free *case,*

$$\text{if } \bar\alpha<1+\alpha,\qquad \mathbf{P}^{\mathrm{gPS,free}}_{n,m,h}\Big(\frac{V_n}{n}-(\gamma-\gamma_c)\in[1-\varepsilon,1+\varepsilon]\,;\,L_n\le n^{\frac{1+\varepsilon}{(1+\alpha)\wedge 2}}\Big)\to 1\,;$$
$$\text{if } \bar\alpha>1+\alpha,\qquad \mathbf{P}^{\mathrm{gPS,free}}_{n,m,h}\Big(\frac{L_n}{n}-(\gamma-\gamma_c)\in[1-\varepsilon,1+\varepsilon]\,;\,\ell_n\le n^{\frac{1+\varepsilon}{(1+\alpha)\wedge 2}}\,;\,V_n\le\log n\Big)\to 1\,.$$

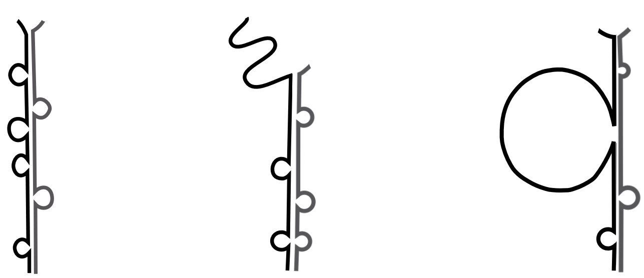


Figure 3.3 – Schematic view of path trajectories in the (free) gPS model. On the left, in the Cramér regime $\gamma_c(h)>\gamma$: all loops are $O(\log n)$, as proven in [GK17]. In the middle and on the right, in the non-Cramér regime $\gamma_c(h)<\gamma$. The middle configuration corresponds to the case $\bar\alpha<1+\alpha$ where there is an unbound free end of length $\approx(\gamma-\gamma_c)n$. The configuration on the right corresponds to the case $\bar\alpha>1+\alpha$ where a loop of length $\approx(\gamma-\gamma_c)n$ is formed. In the constrained case, the unbound free end is of course absent, so the middle configuration does not exist.

In loose terms, $\gamma_c(h)$ is the maximal length ratio that can be "absorbed" in an homogeneous way along the double strand. When $\gamma > \gamma_c$, then the excess length $(\gamma - \gamma_c)n$ is placed either in a macrocsopic loop (when $\bar{\alpha} > 1 + \alpha$) or an unbound free end (when $\bar{\alpha} < 1 + \alpha$), see Figure 3.3.

In order to prove Theorem 3.2, one needs to obtain sharp asymptotics on the partition function $\mathbf{Z}^{\mathrm{gPS}}_{n,m,h} = e^{n\mathtt{N}(h)}\widehat{\mathbf{P}}_h((n,m) \in \boldsymbol{\tau})$, and in particular on the renewal mass function $\widehat{\mathbf{P}}_h((n,m) \in \boldsymbol{\tau})$. The difficulty here is that $\widehat{\mathbf{P}}_h$ has an exponential tail in the first direction, and a polynomial tail (with decay exponent $2+\alpha$) in the second direction. In the non-Cramér regime, (n,m) is not along the "natural" direction $\gamma_c = \widehat{\mathbf{E}}_h[\hat{\tau}^{(2)}]/\widehat{\mathbf{E}}_h[\hat{\tau}^{(1)}]$: the best strategy for $\hat{\tau}$ to reach (n,m) is to absorb the deviation in one big-jump, of length $\approx (\gamma - \gamma_c)n$ (of course, one needs to make this precise). Theorem 3.1 can therefore be seen as a renewal theorem for the specific bivariate renewal with inter-arrival law $\widehat{\mathbf{P}}_h$. This was what inspired my work [7], which aimed at understanding multivariate renewal processes with different tails in the different coordinates: a goal was in particular to obtain renewal theorems away and along the favorite direction. We refer to Section 7.3 for an overview—even though results are presented there only in the case where the tails are the same in all directions.

3.3 Disorder relevance in the i.i.d. case

In a joint work with Giambattista Giacomin and Maha Khatib [3], accepted for publication in *Annales Henri Lebesgue*, we considered the disordered version of the model (3.2), in the case where $(\omega_{i,j})_{i,j\in\mathbb{N}}$ is a field of i.i.d. r.v.s, and we focus on the localization transition. This is a natural choice if one thinks of the model as a pinning of a bivariate renewal on a disordered surface, or as a directed (stretched) polymer in random i.i.d. environment. The first remark is that, in the i.i.d. case, we have $\mathbb{E}[\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,m,\beta,h}] = \mathbf{Z}^{\mathrm{gPS}}_{n,m,h}$, so that the annealed model is the homogeneous one. In particular, the annealed critical point is $h_c^{\mathrm{a}}(\beta) = 0$ for any $\gamma \geq 1$, and Jensen's inequality gives that $\mathtt{F}_\gamma(\beta,h) \leq \mathtt{F}_\gamma(0,h)$ so that $h_c(\beta) \geq h_c^{\mathrm{a}}(\beta) = 0$.

In view of Theorem 3.1, the critical exponent of the homogeneous model is $\nu = \max(1, 1/\alpha)$. Here, the dimension of the disorder is $d = 2$, so if we apply apply Harris criterion, disorder should be irrelevant if $\nu < 1$, *i.e.* if $\alpha < 1$. The case $\alpha > 1$ corresponds to a homogeneous critical exponent $\nu = 1$ which is marginal, but this is actually not the case—the marginal case is $\alpha = 1$. The main results of [3] confirm these predictions.

Theorem 3.3 ([3], Theorems 1.3 and 1.4)**.**

- *If $|\boldsymbol{\tau} \cap \boldsymbol{\tau}'| < +\infty$ with $\boldsymbol{\tau}, \boldsymbol{\tau}'$ two independent bivariate renewals (a sufficient condition is $\alpha < 1$, see Section 7.3.3), then disorder is irrelevant: there exists some $\beta_1 > 0$ such that for all $\beta \in [0,\beta_1)$ we have $h_c(\beta) = 0$ and $\lim_{h\downarrow 0} \frac{\log \mathtt{F}(\beta,h)}{\log h} = \frac{1}{\alpha}$.*
- *If $\alpha > 1$, then disorder is relevant: for every $\beta > 0$ we have $h_c(\beta) > 0$. More precisely, for every $\varepsilon > 0$, there is some β_ε such that $\beta^{q_\alpha+\varepsilon} \leq h_c(\beta) \leq \beta^{q_\alpha-\varepsilon}$ for $\beta \in (0,\beta_\varepsilon)$, with $q_\alpha = \frac{2\alpha}{\alpha-1} \wedge 4$.*

In [3], we leave aside the case $\alpha = 1$ with $|\boldsymbol{\tau} \cap \boldsymbol{\tau}'| = +\infty$ to avoid too many technicalities—this case is already delicate at the level of the bivariate renewal, see Section 7.3. We believe that, as in Theorem 2.2, the necessary and sufficient condition for disorder relevance should be the persistence of $\boldsymbol{\tau} \cap \boldsymbol{\tau}'$. In terms of critical exponent, we would have liked to show that the critical exponent is modified when $\alpha > 1$. However, we should not expect a general result stating that the phase transition is of order at least 2, as in Theorems 1.1 and 2.3: indeed, when $|\boldsymbol{\tau} \cap \boldsymbol{\tau}'| < +\infty$, the critical exponent of the free energy is $1/\alpha$, and can be equal to 1! Our best hope is therefore to obtain a smoothing inequality where the exponent depends on α, see [3, Conj. 1.5].

Second moment and intersection of bivariate renewals. Let us consider the case of a Gaussian disorder and take $m = n$, for the simplicity of the exposition. As in (2.6), we obtain that the second moment of the partition function at $h = h_c^{\mathrm{a}}(\beta) = 0$ is

$$\mathbf{E}\big[(\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,n,\beta,0})^2\big] = \mathbf{E}^{\otimes 2}\Big[\exp\Big(\beta^2 \sum_{i,j=1}^{n} \mathbf{1}_{\{(i,j)\in\boldsymbol{\tau}\cap\boldsymbol{\tau}'\}}\Big)\mathbf{1}_{\{(n,n)\in\boldsymbol{\tau}\cap\boldsymbol{\tau}'\}}\Big]\,, \tag{3.6}$$

where $\boldsymbol{\tau},\boldsymbol{\tau}'$ are two independent renewal processes. Note that $\boldsymbol{\tau}\cap\boldsymbol{\tau}'$ is a renewal process, and that (3.6) is the partition function of a homogeneous (bivariate) pinning model. In particular, we get that: if $|\boldsymbol{\tau}\cap\boldsymbol{\tau}'| < +\infty$ then $\sup_{n\in\mathbb{N}} \mathbf{E}[(\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,n,\beta,0})^2] < +\infty$ for β sufficiently small; if $|\boldsymbol{\tau}\cap\boldsymbol{\tau}'| = +\infty$ then $\mathbf{E}[(\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,n,\beta,0})^2] \to +\infty$ for all $\beta > 0$. As for the pinning model, this is an indication that disorder is relevant if and only if $\boldsymbol{\tau}\cap\boldsymbol{\tau}'$ is persistent.

One therefore needs to study $D_n := \sum_{i,j=1}^{n} \mathbf{P}((i,j)\in\boldsymbol{\tau})^2$, the mean overlap of two independent copies $\boldsymbol{\tau},\boldsymbol{\tau}'$, up to length n. Good estimates on the renewal mass function are necessary, and this is one of scope for the article [7], see Section 7.3—in particular, the case $\alpha\in[1,2)$ was missing in the literature, and the case $\alpha = 1$ is quite delicate. The consequences for D_n can be summarized as follows, see Section 7.3.3: if $\alpha < 1$ then $\sup_{n\in\mathbb{N}} D_n < +\infty$; if $\alpha > 1$ then $D_n = n^{\frac{\alpha-1}{\alpha}\wedge\frac12 + o(1)}$.

3.4 A more realistic disordered model

Going back to the DNA interpretation of the model, the choice of an i.i.d. field $(\omega_{i,j})_{i,j\geq 1}$ does not appear so natural. One would rather consider two sequences, $\hat\omega = (\hat\omega_i)_{i\geq 1}$ and $\bar\omega = (\bar\omega_j)_{j\geq 1}$, attached to the two strands, the interaction between the i^{th} monomer of the first strand and the j^{th} of the second strand being given by $\omega_{i,j} = \hat\omega_i\bar\omega_j$. Together with my Ph.D. student Alexandre Legrand, we are currently working this case: this section presents some of the ideas we are developing.

Assume that $\hat\omega,\bar\omega$ are two independent sequences of independent r.v.s, with the same zero mean and unit variance distribution, and set $\omega_{i,j} = \hat\omega_i\bar\omega_j$ for $i,j\in\mathbb{N}$. Not that $\mathbb{E}[\omega_{i,j}] = 0$, $\mathbb{E}[(\omega_{i,j})^2] = 1$. We assume that $\mathbb{E}[e^{\beta\hat\omega_i^2}] < +\infty$ for $\beta\in[0,\beta_0)$ with $\beta_0 > 0$, which ensures that $\lambda(\beta) = \log\mathbb{E}[e^{\beta\omega_{i,j}}] < +\infty$ for $\beta \leq \beta_0/2$ (since $\omega_{i,j} \leq \frac12(\hat\omega_i^2 + \bar\omega_j^2)$). As typical examples, we have in mind the cases $\hat\omega_i \sim \mathcal{N}(0,1)$ and $\hat\omega_i \in \{-1,+1\}$. The r.v.s $(\omega_{i,j})_{i,j\in\mathbb{N}}$ are not independent, but they are independent if they are not on the same line or column: $\omega_{i,j}$ is independent of $\omega_{k,l}$ if $i\neq k$ and $j\neq l$. As a consequence, we still have that $\mathbb{E}[\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,m,\beta,h}] = \mathbf{Z}^{\mathrm{gPS}}_{n,m,h}$: indeed, for any fixed $\boldsymbol{\tau}$, the r.v.s $(\omega_{i,j}\,;\,(i,j)\in\boldsymbol{\tau})$ are independent. Once again, Jensen's inequality gives that $\mathtt{F}_\gamma(\beta,h) \leq \mathtt{F}_\gamma(0,h)$, so that $h_c(\beta) \geq h_c^{\mathrm{a}}(\beta) = 0$. Our goal is then to understand the role of disorder on the localization phase transition. Here, the second moment of the partition function is much more complicated than (3.6), because of the correlations in $(\omega_{i,j})_{i,j\in\mathbb{N}}$. However, Alexandre Legrand managed to find that:

(i) if $m_4 := \mathbb{E}[\hat\omega_i^4] > 1$, then $\mathbf{E}\big[(\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,n,\beta,0})^2\big] \to +\infty$ for all $\beta > 0$ if and only if $|\tau^{(1)}\cap\tau'^{(1)}| = +\infty$;

(ii) if $m_4 := \mathbb{E}[\hat\omega_i^4] = 1$ (that is $\hat\omega_i^2 = 1$ a.s., or $\hat\omega_i\in\{-1,+1\}$), then $\mathbf{E}\big[(\mathbf{Z}^{\omega,\mathrm{gPS}}_{n,n,\beta,0})^2\big] \to +\infty$ for all $\beta > 0$ if and only if $|\boldsymbol{\tau}\cap\boldsymbol{\tau}'| = +\infty$.

The condition for disorder relevance hence seems to depend more on the distribution of the disorder—well, the case $\mathbb{E}[\hat\omega_i^4] = 1$ is degenerate. The case $m_4 > 1$ involves the intersection of the two unidimensional renewals $\tau^{(1)}, \tau'^{(1)}$: the intersection is terminating if $\alpha < 1/2$ and persistent if $\alpha > 1/2$ (recall the discussion below Theorem 2.2). One should therefore find back the Harris criterion of Chapter 2 (this makes sense, since disorder is fundamentally unidimensional here).

The weak-coupling scaling limit approach. Another recent approach to disorder relevance, initiated by Caravenna, Sun and Zygouras [CSZ17a, CSZ17b, CSZ16], has been to consider the weak-coupling limit of the model, that is to take $\beta_n, h_n \downarrow 0$ as $n \to +\infty$. One wishes to understand whether it is possible to tune β_n, h_n in such a way that the discrete model (in fact its partition function) converges to a non-trivial, *i.e.* disordered, continuous version of the model. If it is the case, disorder is relevant, in the sense that it "survives" in the scaling limit. This idea of considering the intermediate disorder regime had been used in [AKQ14b], and has been fruitful to obtain universality results.

We present the approach of [CSZ17a] in our setting (with $h = 0$, $m = n$ for simplicity). We develop $\mathbf{Z}_{n,\beta_n}^{\omega,\mathrm{free}} := \mathbf{Z}_{n,n,\beta_n,0}^{\omega,\mathrm{gPS},\mathrm{free}}$, by writing $e^{(\beta_n\omega_{i,j}-\lambda(\beta_n))\mathbf{1}_{\{(i,j)\in\tau\}}} = 1 + \xi_{i,j}^{(n)}\mathbf{1}_{\{(i,j)\in\boldsymbol{\tau}\}}$ with $\xi_{i,j}^{(n)} := (e^{(\beta_n\omega_{i,j}-\lambda(\beta_n))}-1)$, and by expanding the product over $(i,j)\in[\![1,n]\!]^2$: we have, with the convention $i_0 = 0, j_0 = 0$

$$\mathbf{Z}_{n,n,\beta_n}^{\omega,\mathrm{free}} = 1 + \sum_{k=1}^{+\infty} \sum_{\substack{0<i_1<\cdots<i_k\leq n\\ 0<j_1<\cdots<j_k\leq n}} \prod_{l=1}^{k} \xi_{i_l,j_l}^{(n)} \mathbf{P}\big((i_l - i_{l-1}, j_l - j_{l-1}) \in \boldsymbol{\tau}\big) . \tag{3.7}$$

Then, one needs to show that each term converges as $n \to +\infty$ (and that the sum converges), provided that β_n has been tuned properly.

Let us focus on the term $k = 1$, $\sum_{(i,j)\in[\![1,n]\!]^2} \xi_{i,j}^{(n)}\mathbf{P}((i,j)\in\boldsymbol{\tau})$. The idea in [CSZ17a] is to use a Lindeberg principle, in order to replace one by one all $\xi_{i,j}^{(n)}$ by i.i.d. Gaussian $\mathcal{N}(0,\beta_n^2)$ r.v.s, keeping the difference in L^2 norm under control. Here, correlations prevent us from using this method, and in fact the main contribution does not come from the second order in the small-β_n expansion of $\xi_{i,j}^{(n)} = e^{\beta_n\omega_{i,j}-\lambda(\beta_n)} - 1$, but from the 4$^{\text{th}}$ order. We use a more direct approach: as a first step, we show the convergence of the field $(\xi_{i,j}^{(n)})_{(i,j)\in[\![1,n]\!]^2}$, in the following sense (we omit the integer parts to lighten notations).

Proposition 3.4. *Let $\beta_n \to 0$ be such that $n^{1/4}\beta_n \to +\infty$, and let $\xi_{i,j}^{(n)} := e^{\beta_n\omega_{i,j}-\lambda(\beta_n)} - 1$. Then, we have the following convergence in distribution (in the space of continuous functions from $[0,1]^2$ to $\mathbb{R}$, with the $\|\cdot\|_\infty$ norm):*

$$\Big(\frac{1}{n^{3/2}\beta_n^2}\sum_{i=1}^{xn}\sum_{j=1}^{yn}\xi_{i,j}^{(n)}\Big)_{(x,y)\in[0,1]^2} \xrightarrow{(d)} \Big(\tfrac12\sqrt{m_4-1}\,\mathcal{M}(x,y)\Big)_{(x,y)\in[0,1]^2} ,$$

where $\mathcal{M}$ is a centered Gaussian field (represented in Figure 3.4) with covariance given by

$$K\big((x,y),(x',y')\big) := (x\wedge x')(y\wedge y')(x\vee x' + y\vee y') . \tag{3.8}$$

Now, when $\alpha\in(0,1)$, for any $x,y\in\mathbb{R}_+^*$ we have $\mathbf{P}((xn,yn)\in\boldsymbol{\tau}) \sim \psi(x,y)\varphi(n)^{-1}n^{-(2-\alpha)}$ as $n\to+\infty$, for some symmetric and radial function $\psi(\cdot,\cdot)$, see [Wil68] or Theorem 7.11 below. We

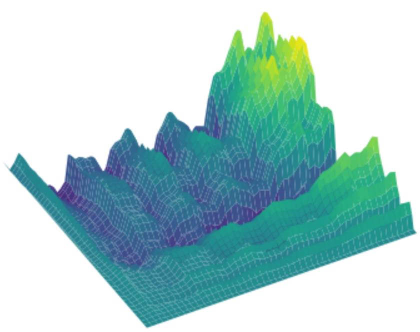


Figure 3.4 – A realization of the (non-isotropic) Gaussian Field $\mathcal{M}$, with covariance given by (3.8).

can therefore expect that, choosing $\beta_n := \hat\beta n^{-\frac{2\alpha-1}{4}} \varphi(n)^{1/2}$ so that $\varphi(n)^{-1} n^{-(2-\alpha)} = \hat\beta^2 n^{-3/2} \beta_n^{-2}$,

$$\sum_{(i,j)\in[\![1,n]\!]^2} \xi_{i,j}^{(n)} \mathbf{P}((i,j)\in\boldsymbol{\tau}) = (1+o(1))\,\varphi(n)^{-1} n^{-(2-\alpha)} \sum_{(i,j)\in[\![1,n]\!]^2} \xi_{i,j}^{(n)} \psi\big(\tfrac{i}{n},\tfrac{j}{n}\big)$$
$$\xrightarrow{n\to+\infty} \frac{\hat\beta^2}{2}\sqrt{m_4-1} \int_{[0,1]^2} \psi(x,y)\,\mathrm{d}\mathcal{M}(x,y)\,,$$

where the last convergence holds for $\alpha\in(1/2,1)$ thanks to Proposition 3.4. Note that the condition $\alpha>1/2$ is crucial here, since it ensures that $\beta_n\to 0$. Let us also stress that the 4^{th} moment of $\hat\omega_i$ appears in the limit: if $m_4=1$ then the limit is equal to 0, which confirms the observation made above that the case $m_4=1$ is somehow degenerate. There are many technicalities involved (for example the well-posedness of the integral), but together with Alexandre Legrand, our goal is to show that for $\alpha\in(1/2,1)$, setting $\beta_n := \hat\beta n^{-\frac{2\alpha-1}{4}}\varphi(n)^{1/2}$, the partition function $\mathbf{Z}_{n,\beta_n}^{\omega,\text{free}}$ converges in distribution to a non-trivial random variable $\mathcal{Z}_{\hat\beta}^{\mathcal{M}}$, expressed as a sum of (stochastic) iterated integrals with respect to the Gaussian field $\mathcal{M}$. This would prove disorder relevance for $\alpha\in(1/2,1)$ (in the sense of Caravenna-Sun-Zygouras [CSZ16]), and it would also open the way to the construction of scaling limits of other models with correlated disorder.

Part II

Directed polymers and Last-Passage Percolation

Chapter 4

Directed polymers and the localization phenomenon

This chapter reviews some results on the directed polymer model, and describes our contribution [15] (in Section 4.2).

4.1 Presentation of the model

The directed polymer model has been introduced by Huse and Henley (in dimension 1+1) in [HH85], as an effective model for an interface in the Ising model with impurities. It has then been generalized to arbitrary dimension $1+d$ with $d \geq 1$ (one temporal dimension, d spatial dimensions), as a model for a stretched polymer interacting with an heterogeneous solvant.

Let $S = (S_n)_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}^d$, started from the origin, and whose law is denoted $\mathbf{P}$: $S_0 = 0$, and $(S_i - S_{i-1})_{i\geq 1}$ are i.i.d. r.v.s, uniform in the set $\{\pm e_i\}_{1\leq i\leq d}$ (e_i is the i^{th} vector of the canonical basis of $\mathbb{Z}^d$). A trajectory of a directed polymer of length $n \in \mathbb{N}$ is represented by the n-step trajectory of the directed random walk $((i, S_i))_{0\leq i\leq n}$, the i^{th} monomer sitting on the site $(i, S_i) \in \mathbb{N}\times\mathbb{Z}^d$ (see Figure 4.1). The random environment is represented by a field $\omega = (\omega_{i,x})_{i\in\mathbb{N},x\in\mathbb{Z}^d}$ of i.i.d. r.v.s, whose law is denoted by $\mathbb{P}$. We assume (for now, see Chapter 5) that $\mathbb{E}[\omega_{1,1}] = 0$, $\mathbb{E}[(\omega_{1,1})^2] = 1$, and that there is some $\beta_0 \in (0, +\infty]$ such that $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{1,1}}] < +\infty$ for all $\beta \in (-\beta_0, \beta_0)$.

For a fixed realization of ω (quenched disorder), and any $\beta \geq 0$ (the inverse temperature), we define for $n \in \mathbb{N}$ the *polymer measure* $\mathbf{P}^{\omega}_{n,\beta}$ by

$$\frac{\mathrm{d}\mathbf{P}^{\omega}_{n,\beta}}{\mathrm{d}\mathbf{P}}(S) := \frac{1}{\mathbf{Z}^{\omega}_{n,\beta}} \exp\Big(\beta \sum_{i=1}^{n} \omega_{i,S_i}\Big)\,, \tag{4.1}$$

with $\mathbf{Z}^{\omega}_{n,\beta} := \mathbf{E}\big[\exp\big(\beta\sum_{i=1}^{n}\omega_{i,S_i}\big)\big]$ the partition function of the model. Note that we have not substracted $\lambda(\beta)$ in the Hamiltonian in (4.1) as we did in Chapters 1-2-3: this is for consistency with most of the literature (and in particular with [15]), and it makes the definition (4.1) valid even when $\lambda(\beta) = +\infty$ (as in Chapter 5). The measure $\mathbf{P}^{\omega}_{n.\beta}$ corresponds to giving a reward (or penalty) $\sum_{i=1}^{n}\omega_{i,S_i}$ to a trajectory S, the coupling being tuned by the parameter β: random walk trajectories

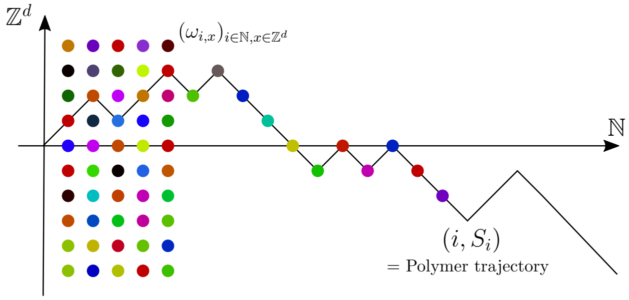


Figure 4.1 – Schematic view of the directed polymer model: the polymer is represented by the directed random walk $((i,S_i))_{1\le i\le n}$; the inhomogeneous solvant is represented via the field $(\omega_{i,x})_{i\in\mathbb{N},x\in\mathbb{Z}^d}$ of i.i.d. r.v.s; the intensity of the interactions is tuned by the inverse temperature β.

that collect a large sum of $\omega_{i,x}$ are favored with respect to those collecting small or negative sums of $\omega_{i,x}$. We refer to Figure 4.1 for an illustration of the model.

The main goal is then to determine (for typical ω) whether trajectories, under the measure $\mathbf{P}^\omega_{n,\beta}$, still have a diffusive behavior as when $\beta = 0$, or if they are super-diffusive and somehow localized in some corridors of favorable environment. In particular, one wishes to derive the transversal (or wandering) exponent ξ, which describes the fluctuations of the end-point of the polymer, that is $\mathbf{E}^\omega_{n,\beta}[(S_n)^2] \approx n^{2\xi}$ as $n \to +\infty$. Another quantity of interest is the fluctuation exponent χ, which describes the fluctuations of $\log \mathbf{Z}^\omega_{n,\beta}$, *i.e.* $\mathbb{V}\mathrm{ar}(\log \mathbf{Z}^\omega_{n,\beta}) \approx n^{2\chi}$ as $n \to +\infty$. The exponents ξ and χ are expected to verify the relation $\chi = 2\xi - 1$.

Let us cite here the monograph of Comets [Com16] for a recent overview of the model and of the results. Let us also mention that the directed polymer model is related to many interesting problems: particles in a random potential, last-passage percolation, random growth models, and it is related to the Kardar-Parisi-Zhang (KPZ) equation. Note that last-passage percolation (a version of which is at the center of Chapter 6) is seen as the zero-temperature version of the directed polymer model: taking $\beta = +\infty$ in (4.1), the partition function $\mathbf{Z}^\omega_{n,\beta}$ becomes

$$L^\omega_n = \max\Big\{ \sum_{i=1}^n \omega_{i,s_i} \ ; s_0 = 0, \|s_i - s_{i-1}\|_1 = 1 \ \forall i \in [\![1,n]\!] \Big\}, \tag{4.2}$$

and the measure $\mathbf{P}^\omega_{n,\beta=+\infty}$ concentrates on trajectories for which the maximum is attained.

Remark 4.1. We mention here that the simple random walk $(S_n)_{n\ge 0}$ may be replaced by a more general walk. For instance the steps $(S_i - S_{i-1})$ may have a stretch-exponential distribution, *i.e.* $\mathbf{P}(S_1 = x) = e^{-\|x\|_1^a}$ for some $a > 0$, see [CFNY15]; or $(S_n)_{n\ge0}$ may be in the domain of attraction of an α-stable law, see [Com07, Wei16, Wei18]. All the results we state below are presented in the case of the simple random walk (in particular for the constants in Theorem 4.2), but they hold with more generality, see [Bat18] for an overview of the results with a general reference random walk.

4.2 Localization phenomenon, weak vs. (very) strong disorder

The answer to the question of the localization of trajectories depends on the dimension d. Let us define the (quenched) free energy as

$$\mathtt{F}(\beta) := \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{Z}^\omega_{n,\beta} = \lim_{n\to+\infty} \frac{1}{n} \mathbb{E} \log \mathbf{Z}^\omega_{n,\beta} \qquad \mathbb{P}\text{-a.s. and in } L^1(\mathbb{P})\,.$$

The fact that it exists, that it is constant $\mathbb{P}$-a.s. and that the convergence holds in $L^1(\mathbb{P})$ (in fact in $L^p(\mathbb{P})$ for all $p \in [1, +\infty)$) is proven for example in [Com16, Thm. 2.1]. An easy upper bound on $\mathtt{F}(\beta)$ is given by Jensen's inequality: since $\mathbb{E}[\log \mathbf{Z}_{n,\beta}^\omega] \le \log \mathbb{E}[\mathbf{Z}_{n,\beta}^\omega] = n\lambda(\beta)$, we get that $\mathtt{F}(\beta) \le \lambda(\beta)$.

Having $\mathtt{F}(\beta) < \lambda(\beta)$ is a sign that trajectories under $\mathbf{P}_{n,\beta}^\omega$ are localized around favorite corridors (with favorable regions of ω): the latter is referred to as the *very strong* disorder regime. The difference $\Delta\mathtt{F}(\beta) := \lambda(\beta) - \mathtt{F}(\beta)$ has indeed been shown to be linked to the limiting overlap fraction of two replicas under $\mathbf{P}_{n,\beta}^\omega$, see [CSY03, CH06]: more precisely, [CH06] shows (in our context) that

$$\Delta\mathtt{F}(\beta) = \lambda(\beta) \times \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (\mathbf{P}_{k-1,\beta}^\omega)^{\otimes 2}\big(S_k = S_k'\big) \qquad \mathbb{P}\text{-a.s.}, \tag{4.3}$$

with $(S_k)_{k\ge1}$, $(S_k')_{k\ge1}$ two independent copies of the random walk. Hence, if $\Delta\mathtt{F}(\beta) > 0$, under $\mathbf{P}_{n,\beta}^\omega$, two replicas overlap a positive fraction of the time: this is a localization phenomenon, and having estimates on $\Delta\mathtt{F}(\beta)$ quantifies it. Other statements regarding the localization phenomenon hold (for instance, making the localization along a favorite corridor more precise), and we refer to [Com16, Chap. 5] for an overview. It is also expected that trajectories are *super-diffusive* for β's such that $\Delta\mathtt{F}(\beta) > 0$, *i.e.* that the wandering exponent is $\xi > 1/2$. In dimension $d = 1$ is is widely expected that $\xi = 2/3$, $\chi = 1/3$ (the so-called KPZ scaling), and this has been proven only for an integrable version of the model [Sep12] (or [Joh00, SV10, BQS11] in related settings). However, much is still open, and in dimension $d \ge 2$ the exponents ξ, χ remain mysterious.

Otherwise, the diffusive behavior of trajectories has been put to rigorous ground in some regimes: for instance, Bolthausen [Bol89] showed that in dimension $d \ge 3$, trajectories are diffusive if β is small enough. Define $\mathbf{W}_{n,\beta}^\omega := e^{-\lambda(\beta)n} \mathbf{Z}_{n,\beta}^\omega$ the renormalized partition function, which is easily proven to be a positive martingale. Then having $\lim_{n\to+\infty} \mathbf{W}_{n,\beta}^\omega > 0$ $\mathbb{P}$-a.s. is dubbed as *weak* disorder (note that it implies $\mathtt{F}(\beta) = \lambda(\beta)$), and it is shown in [CY06] that in this regime trajectories are diffusive, see Theorem 4.1 below.

On the other hand, having $\lim_{n\to+\infty} \mathbf{W}_{n,\beta}^\omega = 0$ is referred to as *strong* disorder, but it is weaker than having $\mathtt{F}(\beta) < \lambda(\beta)$ (*i.e. very strong* disorder): a long-standing conjecture is that *strong* disorder and *very strong* disorder regimes coincide, in the sense that

$$\beta_c^{\mathrm{str}} := \sup\big\{\beta ; \lim_{n\to+\infty} \mathbf{W}_{n,\beta}^\omega > 0 \text{ a.s.}\big\} \overset{\text{(conj.)}}{=} \beta_c^{\text{v-str}} := \sup\big\{\beta ; \mathtt{F}(\beta) = \lambda(\beta)\big\}\,. \tag{4.4}$$

The existence of such critical points follows from monotonicity properties of (i) $\beta \mapsto \mathbb{E}[(\mathbf{W}_{n,\beta}^\omega)^\delta]$ for any $\delta \in (0,1)$, showing the existence of β_c^{str} (see [Com16, Prop. 3.1] and its proof); (ii) of $\beta \mapsto \Delta\mathtt{F}(\beta)$, showing the existence of $\beta_c^{\text{v-str}}$ (see [Com16, Thm. 2.3]). Note that we have the inequality $\beta_c^{\mathrm{str}} \le \beta_c^{\text{v-str}}$.

4.2.1 The case of dimension $d \ge 3$

In dimension $d \ge 3$, Bolthausen [Bol89] shows that $\lim_{n\to+\infty} \mathbf{W}_{n,\beta}^\omega > 0$ a.s. for sufficiently small β. Put otherwise, $\beta_c^{\text{v-str}} \ge \beta_c^{\mathrm{str}} > 0$: in words of Part I, disorder is irrelevant, cf. Remark 2.1. Comets and Yoshida [CY06] prove that under $\mathbf{P}_{n,\beta}^\omega$, trajectories are diffusive in the weak disorder regime.

Theorem 4.1. *Assume that $d \ge 3$ and that $\beta < \beta_c^{str}$. Then for all continuous and bounded function F on the path space, $\mathbf{E}_{n,\beta}^\omega\big[F\big(S^{(n)}\big)\big]$ converges in $\mathbb{P}$-probability to $\mathbf{E}[F(\mathbf{B})]$, where $S^{(n)}$ is the rescaled path $(S_{nt}/\sqrt{n})_{t\ge0}$, and $\mathbf{B}$ is a Brownian motion with diffusion matrix $d^{-1} I_d$.*

This result can be improved in the L^2 region, *i.e.* when the martingale $\mathbf{W}_{n,\beta}^{\omega}$ is bounded in L^2, we refer to [Com16, Chap. 3]. Let us mention that the case of an underlying one-dimensional random walk $(S_n)_{n\geq 0}$ in the domain of attraction of an α-stable law with $\alpha\in(0,2]$ has been considered in [Com07, Wei16]: it is proven that if $\alpha\in(0,1)$ then $\beta_c^{\text{v-str}}\geq\beta_c^{\text{str}}>0$, and a result analogous to Theorem 4.1 is proven in [Wei16]. (The case $\alpha=1$ is marginal and considered in [Wei18].)

4.2.2 The case of dimension $d=1,2$

In dimensions $d=1,2$, it has been shown that $\Delta\mathtt{F}(\beta)>0$ for all $\beta>0$, *i.e.* $\beta_c^{\text{str}}=\beta_c^{\text{v-str}}=0$: in words of Part I, disorder is relevant, cf. Remark 2.1. This has been proven in [CV06] in dimension $d=1$, and in [Lac10c] in dimension $d=2$. Moreover, [Lac10c] provides (almost) sharp estimates on $\Delta\mathtt{F}(\beta)$ in the high temperature limit $\beta\downarrow 0$ (hence quantifying the mean overlap fraction of two replicas). These estimates have been refined more recently, and the asymptotic behavior of $\Delta\mathtt{F}(\beta)$ as $\beta\downarrow 0$ is now known: in dimension $d=1$ this has been proven by Nakashima [Nak16]; in dimension $d=2$ (which is the marginal dimension), this is one of my results, in collaboration with Hubert Lacoin [15], published in *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques.*

Theorem 4.2. *In dimension $d=1$,*

$$\Delta\mathtt{F}(\beta)\sim\tfrac16\beta^4\qquad \text{as } \beta\downarrow 0. \tag{4.5}$$

In dimension $d=2$,

$$\Delta\mathtt{F}(\beta)=\exp\big(-(1+o(1))\,\pi\beta^{-2}\big)\qquad \text{as } \beta\downarrow 0. \tag{4.6}$$

We mention that, in the case where the reference random walk $(S_n)_{n\geq 0}$ is one-dimensional and in the domain of attraction of an α-stable law with $\alpha\in(0,2]$, Wei [Wei16, Wei18] shows that if $\alpha\in(1,2]$ then $\beta_c^{\text{v-str}}=0$, and $\Delta\mathtt{F}(\beta)\asymp\psi(1/\beta)\beta^{\frac{2\alpha}{2\alpha-1}}$ for some specific slowly varying function ψ (in analogy with Theorem 2.5). In the marginal case $\alpha=1$, the estimate is similar to (4.6). Note that determining whether $\beta_c^{\text{v-str}}>0$ or $\beta_c^{\text{v-str}}=0$ is reminiscent of the question of disorder relevance/irrelevance of Chapter 2: in these terms, the α-stable directed polymer is disorder relevant if $\alpha\in(1,2]$ and irrelevant if $\alpha\in(0,1)$. In the marginal case $\alpha=1$, it is shown in [Wei18] (under some additional technical assumption), that, in analogy with Theorem 2.2, disorder is relevant if and only if $\sum_{i=1}^{+\infty}\mathbf{1}_{\{S_i=S_i'\}}=+\infty$ a.s., with S,S' two independent copies of the reference random walk.

Relation with the intermediate disorder regime. Let us stress that the critical behaviors found in Theorem 4.2 are very much related to the intermediate disorder scaling limit of the directed polymer model, where one takes $\beta_n\downarrow 0$ as $n\to+\infty$ in such a way to obtain a non-trivial (*disordered*) scaling limit. This has been investigated by Alberts, Khanin and Quastel [AKQ14b] in dimension $d=1$, and by Caravenna, Sun and Zygouras [CSZ17b] in dimension $d=2$. In dimension $d=1$, the authors in [AKQ14b] pick $\beta_n=\beta n^{-1/4}$ (with $\beta\in(0,+\infty)$), and prove that $\mathbf{W}_{n,\beta_n}^{\omega}$ converges in distribution to a random variable $\mathcal{Z}_{\sqrt2\beta}$, which is a solution to the stochastic heat equation. The scaling $n^{-1/4}$ is related to the β^4 behavior in (4.5), and the constant $1/6$ is the free energy of the continuum directed polymer of [AKQ14a], *i.e.* $\lim_{\beta\to+\infty}\frac1\beta\log\mathcal{Z}_{\sqrt2\beta}$. In dimension $d=2$, the authors in [CSZ17b] pick $\beta_n=\beta(\frac1\pi\log n)^{-1/2}$, and show that $\mathbf{W}_{n,\beta_n}^{\omega}$ converges in distribution to: a log-normal distribution if $\beta<1$; zero if $\beta\geq 1$. Note the relation between the threshold scaling $\beta_n=(\frac1\pi\log n)^{-1/2}$ and the $\exp(\pi\beta^{-2})$ behavior in (4.6).

Chapter 5

Directed polymers in heavy-tail random environment

In this chapter, we study the directed polymer model of the previous chapter, cf. (4.1), but in the case where the environment ω has a heavy-tail: in particular, we present the results obtained in [4].

5.1 Presentation of the setting

Most of the results presented in the previous chapter are valid under the assumption that the disorder ω has some exponential moment, *i.e.* $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{1,1}}] < +\infty$ for $\beta \in (-\beta_0, \beta_0)$, $\beta_0 > 0$. In this chapter, we consider the case where the i.i.d. random variables (or weights) $\omega_{i,x}$ have a heavy tail: we suppose that there exists an exponent $\alpha > 0$ such that

$$\mathbb{P}\big(\omega > t\big) \sim t^{-\alpha} \qquad \text{as } t \to +\infty \tag{5.1}$$

(with a slight abuse of notation, we denote by ω a generic r.v. with the common law of the $\omega_{i,x}$). In particular $\lambda(\beta) = +\infty$ for all $\beta > 0$. For simplicity, we assume that $\omega \geq 0$ a.s.—this does not hide anything deep. Let us stress that in the literature (and in [4]), one usually assumes that $\mathbb{P}\big(\omega > t\big) \sim L(t)t^{-\alpha}$ for some slowly varying function $L(\cdot)$: in (5.1) we got rid of the slowly varying function, in order to simplify notations and to clarify the exposition.

We focus on the directed polymer in dimension $1+1$, see Section 5.4 for comments on higher dimensions. Our main goal is to describe trajectories under $\mathbf{P}^{\omega}_{n,\beta}$, and in particular to find the transversal or wandering exponent ξ ($\mathbf{E}^{\omega}_{n,\beta}[(S_n)^2] \approx n^{2\xi}$) and the fluctuation exponent χ ($\mathbb{V}\mathrm{ar}(\log \mathbf{Z}^{\omega}_{n,\beta}) \approx n^{2\chi}$). We recall that in dimension 1, when $\lambda(\beta) < +\infty$ it is expected that $\xi = 2/3, \chi = 1/3$—this has been shown for some special, integrable version of the model [Sep12]. In the case of a heavy-tail environment with exponent $\alpha > 0$ (5.1), the $\xi = 2/3, \chi = 1/3$ picture is expected to be modified, depending on the value of α. According to the heuristics (and terminology) of [BBP07, GLDBR15], three regimes should occur (they are still mostly conjectural), with different paths behaviors:

(a) if $\alpha > 5$, there should be a *collective* optimization and we should have $\xi = 2/3, \chi = 1/3$, as in the finite exponential moment case (which can be thought as a case $\alpha = +\infty$);

(b) if $\alpha \in (2,5)$, the optimization strategy should be *elitist*: most of the total weigth collected should be via a small fraction of the points visited by the path, and we should have $\xi = \frac{1+\alpha}{2\alpha-1}$, $\chi = 2\xi - 1 = \frac{3}{2\alpha-1}$;

(c) if $\alpha \in (0,2)$, the strategy is *individual*: the polymer targets few exceptional points, and we have $\xi = 1$, $\chi = \frac{2}{\alpha} > 1 = 2\xi - 1$. (This has been proven in [HM07, AL11].)

The idea behind the exponent $\xi = \frac{\alpha+1}{2\alpha-1}$ in case (b) is based on a Flory argument. For trajectories to reach transversal fluctuations of order n^ξ, one may consider the strategy of targeting the largest weight $\omega_{i,x}$ in the box $[\![\frac{n}{2}, n]\!] \times [\![n^\xi, 2n^\xi]\!]$: this largest weight is of order $(n^{1+\xi})^{1/\alpha}$ (it is the maximum of $\frac{1}{2}n^{1+\xi}$ i.i.d. variables with tail given by (5.1)); the entropic cost of targeting this maximal weight is of order $n^{2\xi-1}$ (from the deviation probability of the random walk, $-\log \mathbf{P}(S_n \geq n^\xi) \asymp n^{2\xi-1}$ for $\xi \in [1/2, 1]$). Finding the correct balance between energy and entropy leads to choosing ξ such that $\frac{1+\xi}{\alpha} = 2\xi - 1$, *i.e.* $\xi = \frac{1+\alpha}{2\alpha-1}$. This does not work if $\alpha < 2$ since it would then give $\xi > 1$, which is not possible for the nearest-neighbor random walk: this is the reason why $\xi = 1$ and $\chi = 2/\alpha$ in case (c) (the fluctuations are driven by the largest weight, of order $(n^2)^{1/\alpha}$). On the other hand, if $\alpha > 5$ then $\frac{1+\alpha}{2\alpha-1} < \frac{2}{3}$, and using a collective optimization as in the $\lambda(\beta) < +\infty$ case would lead to a transversal exponent $\xi = 2/3$, outperforming the above strategy of only targeting large weights.

5.1.1 The intermediate disorder picture

A recent and fruitful approach to proving universality results for the directed polymer model has been to consider its weak-coupling limit, *i.e.* to take β go to 0 as $n \to +\infty$. As mentioned in Section 4.2.2, in the case where ω has an exponential moment, Alberts, Khanin and Quastel [AKQ14b] show that taking $\beta_n = \beta n^{-\gamma}$ with $\gamma = 1/4$, $\beta \in (0, +\infty)$, $\log \mathbf{Z}^\omega_{n,\beta_n} - n\lambda(\beta_n)$ converges in distribution to $\log \mathcal{Z}_{\sqrt{2}\beta}$. This has been extended to the case of a heavy-tail distribution (5.1) by Dey and Zygouras [DZ16]: one needs to take $\beta_n = \beta n^{-\gamma}$ with $\gamma = \frac{1}{4}$ if $\alpha > 6$ and $\gamma = \frac{3}{2\alpha}$ if $\alpha \in (\frac{1}{2}, 6)$, so that $\log \mathbf{Z}^\omega_{n,\beta_n} - n\bar\lambda(\beta_n)$ converges in distribution to a non-trivial limit (the centering $\bar\lambda(\cdot)$ is a truncated version of the log-moment generating function $\lambda(\cdot)$; the limit is $\log \mathcal{Z}_{\sqrt{2}\beta}$ if $\alpha > 6$, Gaussian if $\alpha \in (2,6)$ and α-stable if $\alpha \in (\frac{1}{2}, 2)$.

Note that the intermediate disorder regime can be thought as the exact β_n-window for which disorder "kicks in": the choice of β_n is precisely the correct scaling for which trajectories still have a diffusive scaling under $\mathbf{P}^\omega_{n,\beta_n}$ (*i.e.* $\xi = 1/2$), but have a non-Brownian limit. In other words, disorder is strong enough to have some effect (taking $\beta_n \downarrow 0$ faster would yield a Brownian scaling limit), but weak enough for the trajectories to be diffusive (taking $\beta_n \downarrow 0$ slower would yield super-diffusive trajectories). As suggested in [DZ16], this is part of a larger picture: setting $\beta_n := \beta n^{-\gamma}$ with $\gamma \geq 0$, the transversal fluctuation exponent ξ should depend on α, γ as follows

$$\xi = \begin{cases} \frac{2}{3}(1-\gamma) & \text{for } \alpha \geq \frac{5-2\gamma}{1-\gamma},\ 0 \leq \gamma \leq \frac{1}{4}\,, \\ \frac{1+\alpha(1-\gamma)}{2\alpha-1} & \text{for } \alpha \leq \frac{5-2\gamma}{1-\gamma},\ \frac{2}{\alpha} - 1 \leq \gamma \leq \frac{3}{2\alpha}\,. \end{cases} \tag{5.2}$$

Outside of these regions, one should have $\xi = 1/2$ (γ large) or $\xi = 1$ ($\alpha \in (0,2)$, $\gamma < \frac{2}{\alpha} - 1$). Hence, by tuning properly β_n, one should be able to make the transversal exponent ξ interpolate between $1/2$ (for γ larger than for the intermediate disorder scaling) and 1, $\frac{1+\alpha}{2\alpha-1}$ or $\frac{2}{3}$ (for $\gamma = 0$) depending on whether $\alpha \in (0,2)$, $(2,5)$ or $(6, +\infty]$. This is summarized in Figure 5.1 below.

This picture is far from being settled, and only the border cases where $\xi = 1/2$ (region **B** in Figure 5.1) or $\xi = 1$ (region **A** in Figure 5.1) had been proven. Dey and Zygouras [DZ16] proved that $\xi = 1/2$ in the cases $\alpha > 6, \gamma \geq \frac{1}{4}$ and $\alpha \in (\frac{1}{2}, 6), \gamma \geq \frac{3}{2\alpha}$; Auffinger and Louidor [AL11] proved that $\xi = 1$ for $\alpha \in (0,2)$ and $\gamma \leq \frac{2}{\alpha} - 1$. We also mention that in a special integrable version of the model (in which $\alpha = +\infty$), [MFSV14] proves that $\xi = \frac{2}{3}(1-\gamma)$ for $\gamma \in [0, \frac{1}{4}]$.

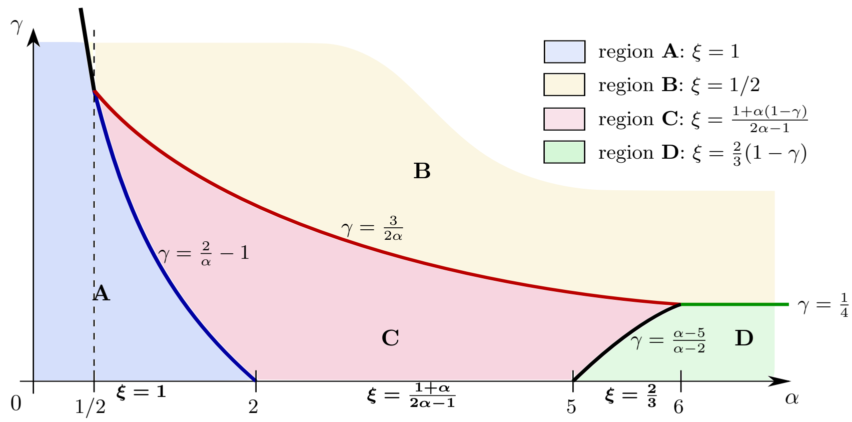


Figure 5.1 – Overview of the conjectured value for the transversal exponent ξ, depending on the disorder tail exponent $\alpha > 0$ ($\mathbb{P}(\omega > t) \sim t^{-\alpha}$) and on the weak-coupling decay exponent $\gamma \geq 0$ ($\beta_n \sim n^{-\gamma}$)—the case $\gamma = 0$ corresponds to a fixed temperature $\beta > 0$, and ξ should be as described in points (a),(b),(c) above. Four regions are identified. In region $\mathbf{A}$, we have $\xi = 1$ (this is proven in [AL11]). In region $\mathbf{B}$, we have $\xi = 1/2$ (this is proven in [DZ16] for $\alpha > \frac{1}{2}$). In region $\mathbf{C}$, we should have $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha-1}$ (we prove it in the sub-region $\alpha < 2$ in [4]). In region $\mathbf{D}$, we should have $\xi = \frac{2}{3}(1-\gamma)$. In regions $\mathbf{C}$ and $\mathbf{D}$, the KPZ relation $\chi = 2\xi - 1$ should hold.

Our main contribution, with Niccolò Torri, is the article [4], accepted in *The Annals of Probability*: we complete the picture in the case $\alpha \in (0, 2)$. More precisely,

(i) if $\alpha \in (0, \frac{1}{2})$, we prove that there is a sharp transition on the line $\gamma = \frac{2}{\alpha} - 1$, from order-$\sqrt{n}$ fluctuations for $\beta_n = \beta n^{-(\frac{2}{\alpha}-1)}$ with $\beta < \hat\beta_c$, to order-n fluctuations for $\beta_n = \beta n^{-(\frac{2}{\alpha}-1)}$ with $\beta > \hat\beta_c$, see Section 5.2 for more precise statements.

(ii) if $\alpha \in (\frac{1}{2}, 2)$, we find the correct transversal fluctuations $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha-1}$ when $\frac{2}{\alpha} - 1 < \gamma < \frac{3}{2\alpha}$ (a sub-region of region $\mathbf{C}$ in Figure 5.1), see Section 5.3 for more precise statements;

5.1.2 A few preliminary definitions: rescaled paths and rescaled weigths

Since we will consider the scaling limit of the model, it is natural to define the set of rescaled paths, and their continuous "entropy" and "energy". Recall that we wish to rescale paths by n in the temporal direction and by n^ξ in the transversal direction: the rescaled paths will be in the set $\mathscr{D} := \{s : [0,1] \to \mathbb{R}\,; s(0) = 0, s \text{ is continuous and a.e. differentiable}\}$. We define the (continuum) entropy of a path $s \in \mathscr{D}$ by

$$\mathrm{Ent}(s) = \frac{1}{2}\int_0^1 \big(s'(t)\big)^2 dt\,, \tag{5.3}$$

which derives from the rate function of the moderate deviation of the simple random walk, *i.e.* $\mathbf{P}(S_{tn} = xn^\xi) = \exp\big(-(1+o(1))\,\frac{x^2}{2t}\,n^{2\xi-1}\big)$ if $\xi \in (1/2, 1)$.

As far as the (rescaled) disorder field is concerned, we define $\mathcal{P}_\alpha := \{(t_i, x_i, w_i)\}_{i\geq 1}$ a Poisson Point Process (PPP) on $[0,1] \times \mathbb{R} \times \mathbb{R}_+$ of intensity $\mu(\mathrm{d}t\mathrm{d}x\mathrm{d}w) = \frac{\alpha}{2} w^{-\alpha-1}\mathbf{1}_{\{w>0\}}\mathrm{d}t\mathrm{d}x\mathrm{d}w$. The PPP $\mathcal{P}_\alpha$ can be shown to be the limit in law of $(n^{-\frac{1+\xi}{\alpha}}\omega_{\lfloor tn\rfloor, \lfloor xn^\xi\rfloor})_{(t,x)\in[0,1]\times\mathbb{R}}$ (for example via order statistics), and we refer to Figure 5.2 for a realization of $\mathcal{P}_\alpha$ for different values of α. For a quenched realization of $\mathcal{P}_\alpha$, the energy of a continuous path $s \in \mathscr{D}$ is then defined by

$$\pi(s) = \pi^{(\mathcal{P}_\alpha)}(s) := \sum_{(t,x,w)\in\mathcal{P}_\alpha} w\, \mathbf{1}_{\{s(t)=x\}}, \tag{5.4}$$

which is the total weight in $\mathcal{P}_\alpha$ collected by s.

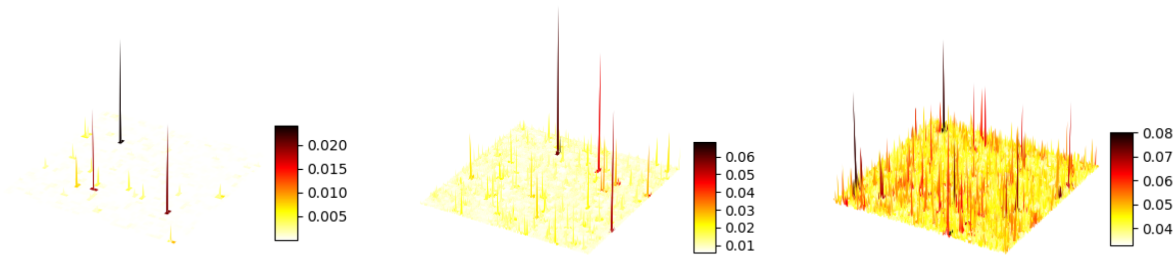


Figure 5.2 – An illustration of the Poisson Point Process $\mathcal{P}_\alpha$ described above (on $[0,1]\times[0,1]$), with different values of α: from left to right, we have $\alpha = 1$, $\alpha = 2$, $\alpha = 3$. The height of the pikes (and their color) represents the value of the weight at that location. Notice that as α becomes larger, the small weights have a higher density (the weight density is $w^{-(1+\alpha)}$, so the tail at 0 gets lighter).

5.2 Main results I: the case $\alpha \in (0, 1/2)$

In the case $\alpha \in (0, \frac12)$, we find that there is no intermediate transversal fluctuations possible. Let us first state the main result of Auffinger and Louidor [AL11].

Theorem 5.1. *Assume that $\alpha \in (0,2)$ in (5.1), and that $\lim_{n\to+\infty} \beta_n n^{\frac{2}{\alpha}-1} = \beta \in [0,+\infty]$. Then*

$$\frac{1}{\beta_n n^{2/\alpha}} \log \mathbf{Z}^\omega_{n,\beta_n} \xrightarrow[n\to+\infty]{(d)} \widehat{\mathcal{T}}_\beta := \sup_{s\in \mathrm{Lip}_1} \left\{ \pi(s) - \tfrac{1}{\beta}\widehat{\mathrm{Ent}}(s) \right\}, \tag{5.5}$$

with Lip_1 the set of 1-Lipschitz functions from $[0,1]$ to $\mathbb{R}$, and $\widehat{\mathrm{Ent}}(s) := \int_0^1 \mathbf{e}(s'(t))\mathrm{d}t$ where $\mathbf{e}(x) = \frac12(1+x)\log(1+x) + \frac12(1-x)\log(1-x)$ derives from a large deviation principle for the simple random walk (and $\widehat{\mathcal{T}}_\beta = 0$ for $\beta = 0$ by convention).

Moreover, $\widehat{\mathcal{T}}_\beta \le \widehat{\mathcal{T}}_\infty = \sup_{s\in\mathrm{Lip}_1} \pi(s) < +\infty$ a.s., and there exists some $\hat\beta_c = \hat\beta_c(\mathcal{P}_\alpha)$ such that $\widehat{\mathcal{T}}_\beta = 0$ for $\beta \le \hat\beta_c$ and $\widehat{\mathcal{T}}_\beta > 0$ for $\beta > \hat\beta_c$. We have $\hat\beta_c > 0$ a.s. if $\alpha < \frac12$ and $\hat\beta_c = 0$ a.s. if $\alpha \ge \frac12$.

The fact that $\widehat{\mathcal{T}}_\infty < +\infty$ a.s. is proven in [HM07], and the fact that $\hat\beta_c > 0$ a.s. for all $\alpha < \frac12$ is shown in [Tor16]. In the case $\alpha \in (0,\frac12)$, we therefore get that there is a non-trivial phase transition, at some *random* critical value $\hat\beta_c$. By an extended version of the Skorokhod representation theorem, we can upgrade the convergence in Theorem 5.1 to an almost sure one (and define ω and $\mathcal{P}_\alpha$ on the same space): it makes sense to consider the events $\widehat{\mathcal{T}}_\beta > 0$ or $\widehat{\mathcal{T}}_\beta = 0$ even at the discrete level.

A by-product of Theorem 5.1 (more precisely of [AL11, Thm. 2.1] and [Tor16, Thm. 1.8]) is that, on the event $\{\widehat{\mathcal{T}}_\beta > 0\}$, transversal fluctuations are of order n. One of our main results in [4] is to show that on the event $\{\widehat{\mathcal{T}}_\beta = 0\}$, transversal fluctuations are of order $\sqrt{n}$ (and we also determine the scaling limit).

Theorem 5.2 ([4], Theorem 2.10)**.** *Assume that* $\alpha \in (0, \frac{1}{2})$ *in* (5.1)*, and that* $\lim_{n\to+\infty} \beta_n n^{\frac{2}{\alpha}-1} = \beta \in [0, +\infty]$*. Then, for every* $\varepsilon, \delta > 0$*, there exists some* $\nu > 0$ *such that, for* n *large enough*

$$\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta}\big(\max_{1\le i\le n} |S_i| < \delta n \big) \le e^{-\nu n} \,\Big|\, \widehat{\mathcal{T}}_\beta > 0 \Big) \ge 1 - \varepsilon\,;$$

there is some constant $c > 0$ *such that for any* $A > 0$ *and* $\varepsilon > 0$*, for* n *large enough*

$$\mathbb{P}\Big(\mathbf{P}^{\omega}_{n,\beta}\big(\max_{1\le i\le n} |S_i| > A\sqrt{n} \big) \le e^{-cA^2} \,\Big|\, \widehat{\mathcal{T}}_\beta = 0 \Big) \ge 1 - \varepsilon\,.$$

Moreover, conditionally on $\{T_\beta = 0\}$ *(in particular for* $\beta \le \hat\beta_c$*), we have as* $n \to +\infty$

$$\frac{\sqrt{n}}{\beta_n n^{3/2\alpha}} \log \mathbf{Z}^{\omega}_{n,\beta_n} \xrightarrow[n\to+\infty]{(d)} 2\mathcal{W}^{(\alpha)}_0 := 2\int_{[0,1]\times\mathbb{R}\times\mathbb{R}_+} w\rho(t,x)\mathcal{P}_\alpha(\mathrm{d}t, \mathrm{d}x, \mathrm{d}w)\,, \tag{5.6}$$

with $\mathcal{P}_\alpha$ *a realization of the PPP defined above in Section 5.1.2, and* $\rho(t,x) = (2\pi t)^{-1/2} e^{-\frac{x^2}{2t}}$ *the Gaussian Heat kernel. (It is shown in [DZ16] that* $\mathcal{W}^{(\alpha)}_0$ *has an* α*-stable distribution.)*

5.3 Main results II: the case $\alpha \in (1/2, 2)$

For the sake of clarity, we first state our results in the case when $\beta_n = \beta n^{-\gamma}$, $\beta \in (0, +\infty)$. In Section 5.3.2 below, we treat the general case of a sequence $\beta_n \downarrow 0$, uncovering new regimes, in particular when β_n is of the type $(\log n)^\zeta n^{-\frac{3}{2\alpha}}$ (*i.e.* when "close" to the line $\gamma = \frac{3}{2\alpha}$ in Figure 5.1).

5.3.1 Statement of the results

Let us start by stating the main result of Dey and Zygouras [DZ16] in the case $\alpha \in (\frac{1}{2}, 2)$, which focuses on region **B** in Figure 5.1, *i.e.* $\gamma \ge \frac{3}{2\alpha}$ (and for which $\xi = 1/2$).

Theorem 5.3. *Assume that* $\alpha \in (\frac{1}{2}, 2)$*, and that* $\lim_{n\to+\infty} \beta_n n^{\frac{3}{2\alpha}} = \beta \in [0, +\infty)$*. Then*

$$\frac{\sqrt{n}}{\beta_n n^{3/2\alpha}} \Big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\beta_n \mathbb{E}\big[\omega \mathbf{1}_{\{\omega \le n^{3/2\alpha}\}}\big] \mathbf{1}_{\alpha\ge1} \Big) \xrightarrow[n\to+\infty]{(d)} 2\mathcal{W}^{(\alpha)}_\beta\,. \tag{5.7}$$

Here, $\mathcal{W}^{(\alpha)}_\beta$ *is some specific* α*-stable r.v. (defined in [DZ16, p. 4011]). Moreover, trajectories have a diffusive scaling under* $\mathbf{P}^{\omega}_{n,\beta_n}$*, i.e.* $\xi = 1/2$*.*

This also includes the case $\gamma > \frac{3}{2\alpha}$, which corresponds to having $\beta = 0$ in Theorem 5.3. On the other hand, Theorem 5.1 above treats the case $\gamma \le \frac{2}{\alpha} - 1$: for $\alpha \in (\frac{1}{2}, 2)$ we have that $\widehat{\mathcal{T}}_\beta > 0$ a.s. for all $\beta \in (0, +\infty]$, which implies that $\xi = 1$.

With Niccolò Torri, our main result in [4] solves the remaining case $\gamma \in (\frac{2}{\alpha} - 1, \frac{3}{2\alpha})$, finishes to establish the picture presented in Figure 5.1 in the region $\alpha \in (0, 2)$.

Theorem 5.4 ([4], Theorem 2.4)**.** *Assume that* $\alpha \in (\frac{1}{2}, 2)$*, that* $\gamma \in (\frac{2}{\alpha} - 1, \frac{3}{2\alpha})$*, and suppose that* $\lim_{n\to+\infty} \beta_n n^\gamma = \beta$ *with* $\beta \in (0, +\infty)$*. Then, defining* $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha-1}$ *as in* (5.2)*, we have*

$$\frac{1}{n^{2\xi-1}} \Big(\log \mathbf{Z}^{\omega}_{n,\beta_n} - n\beta_n \mathbb{E}[\omega] \mathbf{1}_{\{\alpha\ge 3/2\}} \Big) \xrightarrow[n\to+\infty]{(d)} \beta^{\frac{2\alpha}{2\alpha-1}} \mathcal{T}\,, \qquad \mathcal{T} := \sup_{s\in\mathscr{D}} \big\{ \pi(s) - \mathrm{Ent}(s) \big\}\,, \tag{5.8}$$

with $\mathcal{T} \in (0, +\infty)$ *a.s. Moreover, under* $\mathbf{P}^{\omega}_{n,\beta_n}$*, trajectories have transversal fluctuation exponent* ξ*.*

Let us stress that the main difficulty in proving the above theorem is not to control the contribution of the few largest weights in the region $[\![0,n]\!] \times [\![-An^\xi, An^\xi]\!]$–this is undoubtedly technical, but the scheme is clear once we have proven that other weigths do not contribute to the partition function. The real crux of the proof is in controlling the contribution of other weights: one needs to show that collecting many intermediate weights (*i.e.* much smaller than the maximal one) is not a good strategy, in the sense that it has an entropic cost that largely overcomes the energetic gain from these weights. Note that the centering $n\beta_n \mathbb{E}[\omega]\mathbf{1}_{\{\alpha\geq 3/2\}}$ is here to account for the contribution of the order-1 weights (and would be slightly different if ω were not non-negative).

This took us quite some time with Niccolò to solve this issue, but we finally came up with a new tool, that enabled us to control the maximal (total) weight a path with a fixed entropy can collect—or the minimal entropy required for a path to collect a given weight. This is a generalized version of last-passage percolation that we call Entropy-Controlled Last Passage Percolation (E-LPP), and which is the object of Chapter 6 (and of [9]). Let us stress that the difficulty is already encapsulated at the level of the continuum variational problem, in the fact that $\mathcal{T} < +\infty$ a.s. (see Theorem 6.3 below, stated as a conjecture in [DZ16]). This is developed in Section 6.2.

5.3.2 Other regimes for more general weak-coupling sequences $\beta_n \downarrow 0$

So far, we considered only the case where $\beta_n = \beta n^{-\gamma}$. In [4] however, we treat general sequences $\beta_n \downarrow 0$: transversal fluctuations are simply not given by n^ξ, but by a sequence h_n (depending on β_n). The energy/entropy balance argument sketched above remains valid: in the box $[\![\frac12 n, n]\!] \times [\![h_n, 2h_n]\!]$, the largest weight brings an energy gain roughly $\beta_n (nh_n)^{1/\alpha}$, and the entropic cost of targeting it is roughly h_n^2/n. The energy/entropy balance $\beta_n(nh_n)^{1/\alpha} = h_n^2/n$ therefore leads us to define

$$h_n := (\beta_n)^{\frac{\alpha}{2\alpha-1}} n^{\frac{1+\alpha}{2\alpha-1}} . \tag{5.9}$$

In the case $\beta_n = \beta n^{-\gamma}$, one finds back that $h_n = \beta^{\frac{\alpha}{2\alpha-1}} n^\xi$ with $\xi = \frac{1+\alpha(1-\gamma)}{2\alpha-1}$ as in (5.2). In [4], we realized that, with a general choice of weak-coupling sequence, other regimes for the weak-coupling scaling limit exist: in particular, there is some transition *close to* the line $\gamma = \frac{3}{2\alpha}$ in Figure 5.1. More precisely, we identify three regimes, depending on whether β_n is much larger, of the order, or much smaller than the threshold scaling $(\log n)^{\frac{2\alpha-1}{2\alpha}} n^{-\frac{3}{2\alpha}}$, for which one has $h_n = \sqrt{n\log n}$ in (5.9).

For the simplicity of the statements, we define the renormalized partition function $\bar{\mathbf{Z}}^\omega_{n,\beta_n} := e^{-C_n}\mathbf{Z}^\omega_{n,\beta_n}$, where the centering comes from the contribution of order-1 weights, and is defined by $C_n := n\beta_n\mathbb{E}[\omega]$ if $\alpha \in (1,2)$, $C_n := n\beta_n \mathbb{E}[\omega \mathbf{1}_{\{\omega \leq \beta_n^{-1}\}}]$ if $\alpha = 1$, and $C_n = 0$ if $\alpha \in (0,1)$.

(1) If $(\log n)^{\frac{2\alpha-1}{2\alpha}} n^{-\frac{3}{2\alpha}} \ll \beta_n \ll n^{-(\frac{2}{\alpha}-1)}$**.** In that case, we have $\sqrt{n\log n} \ll h_n \ll n$ in (5.9), and essentially, Theorem 5.4 remains valid.

Theorem 5.5 ([4], Theorem 2.4)**.** *Let* $\alpha \in (\frac12, 2)$*, and assume* $\lim_{n\to+\infty} \beta_n (\log n)^{-\frac{2\alpha-1}{2\alpha}} n^{\frac{3}{2\alpha}} = +\infty$ *and* $\lim_{n\to+\infty} \beta_n n^{\frac{2}{\alpha}-1} = 0$*. Recall the definition* (5.9) *of* h_n*. Then we have that*

$$\frac{n}{h_n^2} \log \bar{\mathbf{Z}}^\omega_{n,\beta_n} \xrightarrow[n\to+\infty]{(d)} \mathcal{T} := \sup_{s\in\mathscr{D}} \big\{ \pi(s) - \mathrm{Ent}(s) \big\} ,$$

with $\mathcal{T} \in (0,+\infty)$ *a.s. Moreover, under* $\mathbf{P}^\omega_{n,\beta_n}$*, trajectories have transversal flucutations of order* h_n*.*

Note that the factor $\beta^{\frac{2\alpha}{2\alpha-1}}$ we found in (5.8) does not appear here, simply because it is included in the renormalization n/h_n^2, recall (5.9)—the choice for h_n is the exact balance between the scaling of the energy reward and of the entropic cost. Note also that $h_n^2/n \gg \log n$ in that regime.

(2) If $\beta_n = \beta(\log n)^{\frac{2\alpha-1}{2\alpha}} n^{-\frac{3}{2\alpha}}$. In that case, we have $h_n = \beta^{\frac{\alpha}{2\alpha-1}}\sqrt{n\log n}$ in (5.9), and $h_n^2/n = \beta^{\frac{2\alpha}{2\alpha-1}}\log n$. We find that the limiting variational problem in Theorem 5.4 is modified. This is due to the fact that targeting a specific point at scale h_n has an extra entropic cost $\frac12 \log n$ (per targeted point), because of the local limit theorem in the moderate deviation regime (see e.g. [Sto67, Thm. 3])

$$\mathbf{P}\big(S_{tn} = xh_n\big) = \frac{1}{\sqrt{tn}} e^{-(1+o(1))\frac{h_n^2}{n}\frac{x^2}{2t}} = \exp\Big(\log n\big(-\beta^{\frac{2\alpha}{2\alpha-1}}\tfrac{x^2}{2t} - \tfrac12 + o(1)\big)\Big). \tag{5.10}$$

We show the following, which tells that there is a transition when β_n is of the order $(\log n)^{\frac{2\alpha-1}{2\alpha}} n^{-\frac{3}{2\alpha}}$.

Theorem 5.6 ([4], Theorems 2.5 and 2.6)**.** *Let $\alpha\in(\frac12,2)$, and consider a sequence $\beta_n\downarrow 0$ such that $\lim_{n\to+\infty}\beta_n(\log n)^{-\frac{2\alpha-1}{2\alpha}} n^{\frac{3}{2\alpha}} = \beta\in(0,+\infty)$. Then, we have*

$$\frac{1}{\log n}\log\big(\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} - 1\big) \xrightarrow[n\to+\infty]{(d)} \widetilde{\mathcal{T}}^{[\geq 1]}_{\beta} := \sup_{s\in\mathscr{D}; N(s)\geq 1}\Big\{\beta^{\frac{2\alpha}{2\alpha-1}}\big[\pi(s) - \mathrm{Ent}(s)\big] - \tfrac12 N(s)\Big\},$$

where $N(s) = \sum_{(t,x,w)\in\mathcal{P}_\alpha}\mathbf{1}_{\{s(t)=x\}}$ is the number of points in $\mathcal{P}_\alpha$ visited by the path s. Moreover, under $\mathbf{P}^{\omega}_{n,\beta_n}$, trajectories have transversal fluctuation of order $\sqrt{n\log n}$.

We have that $\beta\mapsto\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}$ is continuous and increasing, with $\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}\in(-\frac12,+\infty)$ a.s.: there exists some $\tilde\beta_c = \tilde\beta_c(\mathcal{P}_\alpha)\in(0,+\infty)$ such that $\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}\in(0,+\infty)$ for $\beta>\tilde\beta_c$ and $\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}\in(-\frac12,0)$ for $\tilde\beta<\beta_c$.

Put differently, for $\beta_n = \beta(\log n)^{\frac{2\alpha-1}{2\alpha}} n^{-\frac{3}{2\alpha}}$, we get that $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = 1 + n^{\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}+o(1)}$, with $\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}$ a *random* exponent, that spans from $-1/2$ to $+\infty$ as β goes from 0 to $+\infty$ (the superscript $[\geq 1]$ refers to the fact that the variational problem is restricted to trajectories that visit at least one point of $\mathcal{P}_\alpha$). Hence there is some transition in the behavior of $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}$: for $\beta<\tilde\beta_c$ we have that $\lim_{n\to+\infty}\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = 1$ (the convergence is at polynomial speed, with an exponent $\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}\in(-\frac12,0)$); whereas for $\beta>\tilde\beta_c$ we have $\lim_{n\to+\infty}\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = +\infty$ (at polynomial speed, with an exponent $\widetilde{\mathcal{T}}^{[\geq 1]}_{\beta}\in(0,+\infty)$). This is in fact presented as two theorems in [4] (Theorems 2.7 and 2.8), to highlight this change of behavior at the scale $\beta_n\asymp(\log n)^{\frac{2\alpha-1}{2\alpha}} n^{-\frac{3}{2\alpha}}$, and can be thought as a tipping point between a regime where $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n}$ goes to 1 (and disorder has an effect only at the second order), and a regime where $\bar{\mathbf{Z}}_{n,\beta_n}$ goes to $+\infty$ (and disorder governs the rate of growth).

(3) If $n^{-\frac{3}{2\alpha}} \ll \beta_n \ll (\log n)^{\frac{2\alpha-1}{2\alpha}} n^{-\frac{3}{2\alpha}}$. In that case we have that $\sqrt{n}\ll h_n\ll\sqrt{n\log n}$ in (5.9), and $1\ll h_n^2/n\ll\log n$. We prove the following theorem.

Theorem 5.7 ([4], Theorem 2.7)**.** *Let $\alpha\in(\frac12,2)$, and assume $\lim_{n\to+\infty}\beta_n(\log n)^{-\frac{2\alpha-1}{2\alpha}} n^{\frac{3}{2\alpha}} = 0$ and $\lim_{n\to+\infty}\beta_n n^{\frac{3}{2\alpha}} = +\infty$. Recall the definition (5.9) of h_n. Then we have that*

$$\frac{n}{h_n^2}\log\Big(\sqrt{n}\big(\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} - 1\big)\Big) \xrightarrow[n\to+\infty]{(d)} W := \sup_{(t,x,w)\in\mathcal{P}_\alpha}\Big\{w - \frac{x^2}{2t}\Big\},$$

with $W\in(0,+\infty)$ a.s. Moreover, under $\mathbf{P}^{\omega}_{n,\beta_n}$, trajectories have transversal fluctuations h_n.

Loosely speaking, this means that $\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} = 1 + \frac{1}{\sqrt{n}} e^{W \frac{h_n^2}{n}(1+o(1))}$, and we stress that $e^{W\frac{h_n^2}{n}}$ goes to infinity slower than any power of n. This can be interpreted as an intermediary regime between Theorem 5.3 (where $\sqrt{n}(\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} - 1)$ converges in distribution) and Theorem 5.6 (where $\sqrt{n}(\bar{\mathbf{Z}}^{\omega}_{n,\beta_n} - 1)$ goes to $+\infty$ as a power of n).

5.4 Comments and open questions

Higher dimensions. Similarly to [AL11] and [HM07], our methods work in any dimension $1+d$. The energy-entropy balance argument would then give that the transversal fluctuations h_n are given by the relation $\beta_n(nh_n^d)^{1/\alpha} = h_n^2/n$ in place of (5.9). Hence, choosing $\beta_n = n^{-\gamma}$ with $\gamma \geq 0$, we should therefore find, for $\alpha \in (0, 1+d)$, a similar picture to Figure 5.1:

Case $\alpha \in (0, \frac{d}{2})$

$\gamma < \frac{1+d}{\alpha} - 1$	$\gamma > \frac{1+d}{\alpha} - 1$
$\xi = 1$	$\xi = 1/2$

Case $\alpha \in (\frac{d}{2}, 1+d)$

$\gamma \leq \frac{1+d}{\alpha} - 1$	$\frac{1+d}{\alpha} - 1 < \gamma < \frac{2+d}{2\alpha}$	$\gamma \geq \frac{2+d}{2\alpha}$
$\xi = 1$	$\xi = \frac{1+(1-\gamma)\alpha}{2\alpha - d} \in (1/2, 1)$	$\xi = 1/2$

We mention that results for the Entropy-controlled LPP are easily generalized to higher dimensions, see Chapter 6: one should be able to repeat the same scheme of proof, without any major difficulty.

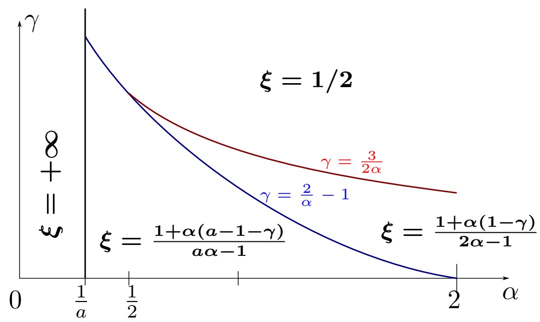
Unbounded jumps. We should also be able to deal with random walks with unbounded jumps, in particular when the increments have a stretch-exponential tail, as in [CFNY15], cf. Remark 4.1. If $(S_n)_{n\geq 0}$ is a random walk on $\mathbb{Z}$ with centered and unit variance increments verifying $\mathbf{P}(S_1 = k) \sim e^{-|k|^a}$ as $k \to \pm\infty$ for some $a > 0$, then the large deviations for the random walk become

$$- \log \mathbf{P}(S_{tn} \geq xn^{\xi}) \sim \begin{cases} \frac{x^2}{2t}\, n^{2\xi - 1} & \text{if } a > 1, \xi < 1 \text{ or } a \in (0,1), \xi < \frac{1}{2-a}\,, \\ c_a \frac{x^a}{t^{a-1}}\, n^{a(\xi-1)+1} & \text{if } a > 1, \xi > 1\,, \\ x^a\, n^{a\xi} & \text{if } a \in (0,1), \xi > \frac{1}{2-a}\,, \end{cases} \tag{5.11}$$

as $n \to +\infty$, for $t > 0$ and $x \in \mathbb{R}$ fixed. (We focus on the case of dimension $d = 1$ for simplicity.) The first line is the so-called Cramér regime, the second line corresponds to making tn jumps of size $\frac{x}{t} n^{\xi - 1}$, and the third line comes from the so-called big-jump principle—we refer to [BB08] and references therein. Hence, one may use the energy-entropy balance to guess what the transversal fluctuations should be: taking $\beta_n = n^{-\gamma}$, there is an energy-entropy balance for transversal fluctuations n^{ξ} with ξ verifying $\frac{1+\xi}{\alpha} - \gamma = 2\xi - 1$, $a(\xi-1)+1$, or $a\xi$, depending on the different cases in (5.11). Figure 5.3 gives an overview of the different regimes, in analogy with Figure 5.1.

Then, when considering the scaling limit of the partition function, one expects to obtain a variational problem of the type $\mathcal{T} := \sup_s\{\pi(s) - \mathrm{Ent}(s)\}$, where the entropy of a path $s : [0,1] \to \mathbb{R}$ comes from the large deviation principle of (5.11). In fact, new definitions for the entropy of paths need to be introduced:

- if $a > 1$, $\xi < 1$ or if $a \in (0,1)$, $\xi < \frac{1}{2-a}$, then the first line of (5.11) suggests that (5.3) is still the correct definition for the (rescaled) entropy of a (rescaled) continuous path $s : [0,1] \to \mathbb{R}$;



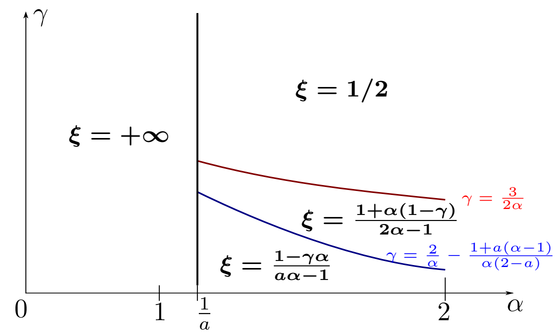


Figure 5.3 – Diagrams presenting the predicted transversal fluctuation exponent ξ, depending on α and γ. The case $a > 1$ is presented on the left, and we identify four regions: $\alpha \le 1/a$, for which $\xi = +\infty$; $\gamma > \max(\frac{2}{\alpha} - 1, \frac{3}{2\alpha})$, for which $\xi = 1/2$; $\gamma \in (\frac{2}{\alpha} - 1, \frac{3}{2\alpha})$, for which $\xi \in (\frac12, 1)$; and $\gamma < \frac{2}{\alpha} - 1$, for which $\xi > 1$. The case $a \in (0,1)$ is presented on the right, and we also identify four regions: $\alpha \le 1/a$, for which $\xi = +\infty$; $\gamma > \frac{3}{2\alpha}$, for which $\xi = 1/2$; $\gamma \in (\frac{2}{\alpha} - \frac{1+a(\alpha-1)}{\alpha(2-a)}, \frac{3}{2\alpha})$, for which $\xi \in (\frac12, \frac{1}{2-a})$; and $\gamma < \frac{2}{\alpha} - \frac{1+a(\alpha-1)}{\alpha(2-a)}$, for which $\xi > \frac{1}{2-a}$ (and which corresponds to a *one-jump* strategy of trajectories).

- if $a > 1$ and $\xi > 1$, then the second line of (5.11) suggests to define the (rescaled) entropy of a (rescaled) path $s : [0,1] \to \mathbb{R}$ as

$$\mathrm{Ent}(s) := c_a \int_0^1 |s'(t)|^a \mathrm{d}t\,; \tag{5.12}$$

- if $a \in (0,1)$ and $\xi > \frac{1}{2-a}$, then the third line of (5.11), based on a big-jump principle, suggests that the (rescaled) paths $s : [0,1] \to \mathbb{R}$ are not necessarily continuous, and that the (rescaled) entropy of s should be defined as

$$\mathrm{Ent}(s) := \sup_{0=t_0<t_1<\cdots<t_k\le 1} \sum_{i=1}^{k} \big|s(t_k) - s(t_{k-1})\big|^a\,, \tag{5.13}$$

  where the supremum is taken over all subdivisions of $[0,1]$—this is a generalization of the total (or of the quadratic) variation of s.

We stress that in [2] (see also Section 6.3), a general Entropy-controlled LPP is defined: this should enable us to consider the variational problem $\mathcal{T} := \sup_s\{\pi(s) - \mathrm{Ent}(s)\}$ also in the cases of an entropy given by (5.12) for which one can show that $\mathcal{T} < +\infty$ a.s. for $\alpha \in (\frac{1}{a}, 2)$; or given by (5.13) for which one can show that $\mathcal{T} < +\infty$ a.s. for $\alpha \in (\frac{1}{a}, \frac{1}{a} + 1)$. We refer to Section 6.3 for more details. Many technicalities remain, and it is not excluded that there are some serious issues, but at least all the tools seems to be at hand for this problem.

Non-directed paths. We stress that a notion of entropy also makes sense in the context of a non-directed random walk. Take $(S_n)_{n\ge0}$ a simple symmetric random walk on $\mathbb{Z}^d$ with $d = 2$ (for simplicity of the exposition), and consider the partition function $\widehat{\mathbf{Z}}^{\omega}_{n,\beta_n} = \mathbf{E}[\exp(\beta_n \sum_{x\in\mathbb{Z}^2} \omega_x \mathbf{1}_{\{x\in R_n\}})]$, where $R_n = \{S_0, \ldots, S_n\}$ is the range of the random walk, and $(\omega_x)_{x\ge1}$ is a field of i.i.d. r.v.s. This corresponds to having interactions between the polymer and the environment only at the site that are visited by the random walk—this model has been considered only very recently, in [Hua19]. In the case where the disorder $(\omega_x)_{x\in\mathbb{Z}^2}$ has a heavy-tail as in (5.1), then one could apply the energy-entropy argument: one would get that the transversal exponent ξ should be given by (5.2), at least when $\alpha \in (0,2)$. One then needs to consider, for fixed $x_1,\dots,x_k \in \mathbb{R}^2$ and $\xi \in (1/2,1)$, the (large deviation) probability for the random walk to visit the points $x_i n^\xi$ (in that order), up to time n, that is

$$\sup_{0=t_0\le t_1<\cdots<t_k\le 1} \mathbf{P}\big(S_{t_i n} = x_i n^\xi \text{ for all } 1\le i\le k\big) = \exp\Big((1+o(1))\,\frac12\Big(\sum_{i=1}^k \|x_i-x_{i-1}\|_2\Big)^2 n^{2\xi-1}\Big)\,.$$

Here, we used for the second line that $\mathbf{P}(S_{tn}=xn^\xi) = e^{-(1+o(1))\frac{\|x\|_2^2}{2t}n^{2\xi-1}}$, together with the fact that $\sum_{i=1}^k \frac{\|x_i-x_{i-1}\|_2^2}{2(t_i-t_{i-1})}$ is minimized for the subdivision $t_i-t_{i-1} = \|x_i-x_{i-1}\|_2/(\sum_{i=1}^k \|x_i-x_{i-1}\|_2)$. Here again, this gives rise to a natural definition for a path entropy of a (continous) curve $s:[0,1]\to\mathbb{R}^2$, defined as $\mathrm{Ent}(s) = \frac12\big(\int_0^1 \|s'(t)\|_2 \mathrm{d}t\big)^2$, *i.e.* the length of the curve squared (up to a factor 1/2). The corresponding Entropy-controlled LPP is naturally defined, and in [2, Sec. 4], it is shown that the non-directed variational problem $\mathcal{T}^{\mathrm{non-dir}} := \sup_s\{\pi(s)-\mathrm{Ent}(s)\}$ is well-defined and finite a.s. if $\alpha\in(1,2)$ (the definition of the Poison Point Process $\mathcal{P}_\alpha$ and of the energy $\pi(s)$ of a path are naturally generalized to the non-directed setting). Again, many technicalities remain (and some of them may be substantial), but it seems that all the arguments needed for the proof are at hand.

Toward the case $\alpha\in(2,5)$? Maybe the most interesting open problem that remains is to extend all the results, and possibly our methods, to the case $\alpha>2$, more specifically to the case $\alpha\in(2,5)$ (region **C** in Figure 5.1), in which trajectories should adopt an *elitist* strategy, collecting most of the total energy via a small fraction of the points visited by the path, see [GLDBR15]. One of the main difficulty is to find the correct centering term for $\log \mathbf{Z}^\omega_{n,\beta_n}$. Another important difficulty is that the variational problem $\mathcal{T}$ defined in (5.8) is almost surely infinite, because of the contribution of many small weights (Figure 5.2 illustrates how small weights may accumulate in the case $\alpha>2$). The difficulties are therefore substantial, and there is no apparent reason why the scaling limit of $\log \mathbf{Z}^\omega_{n,\beta_n}$ could be expressed as a variational problem. As a first step, we believe it should be fruitful to understand the case $\alpha=2$, in the "simpler" context of the last-passage percolation: we refer to Section 6.2.1 for further discussion on this.

Chapter 6

Entropy-controlled Last-Passage Percolation and applications

Recall that Last-Passage Percolation (LPP) can seen as a zero-temperature version of the directed polymer model, cf. (4.2). Here, we focus on some continuous-space version of it, in the sense that the weights are placed (randomly) in $\mathbb{R}^2$. The main focus of this chapter is the Entropy-controlled Last-Passage Percolation introduced in [9], but we also describe further generalizations, from [2].

6.1 Last-Passage Percolation and Entropy-controlled Last-Passage Percolation

Reminder of Hammersley's LPP. In order to study the length of the longest increasing subsequence of a (uniform) random permutation of $\{1, \dots, m\}$, and its behavior as $m \to +\infty$ (known as Ulam's problem), Hammersley [Ham72] introduced the following problem. Take m points $\{Z_i\}_{1\leq i\leq m}$ uniformly on the square $[0,1]^2$: sorting the points in increasing abscissa, the relative order of their ordinates is a (uniform) random permutation. Then, the longest increasing subsequence of the permutation is simply the length of the longest *increasing* chain of points $Z_{i_1} \prec \cdots \prec Z_{i_k}$ ($z = (x, y) \prec z' = (x', y')$ if $x < x'$ and $y < y'$). We denote

$$\mathcal{L}_m := \max\left\{k \; ; \; \exists\, (i_1, \dots, i_k) \text{ such that } Z_{i_1} \prec \cdots \prec Z_{i_k}\right\}, \tag{6.1}$$

referred to as the LPP problem with m uniform points in $[0,1]^2$. One then wants to study the asymptotic properties of $\mathcal{L}_m$. The key idea of Hammersley was to replace the random m points by a Poisson Point Process (PPP) of intensity m, and then use scaling arguments to see that $\mathcal{L}_m$ is equal in law to the LPP problem in $[0, \sqrt{m}]^2$ with points given by a PPP of intensity 1. The latter has a super-additive property: it enabled Hammersley to show that $\mathcal{L}_m/\sqrt{m}$ converges in probability to some constant $\mathtt{c}$. He believed that the constant was equal to 2, which was then confirmed by later works [LS77, VK77]. More recently, and quite remarkably, this model has been found to be exactly solvable by Baik, Deift and Johansson [BDJ99]: they identify the fluctuations of $\mathcal{L}_m - 2\sqrt{m}$, and prove that the model is in the so-called KPZ universality class.

Theorem 6.1. *The centered and normalized LPP* $\frac{\mathcal{L}_m - 2\sqrt{m}}{(\sqrt{m})^{1/3}}$ *converges in distribution to the Tracy-Widom GUE distribution.*

Additionnally, Johansson [Joh00] identifies the transversal fluctuations of a maximizing path: for the LPP problem in $[0,n]^2$ with points given by a PPP of intensity 1, transversal fluctuations away from the diagonal are found to be of order $n^{2/3}$.

6.1.1 Entropy-controlled LPP.

The LPP (6.1) can be thought as the problem of finding the maximal number of points in $\{Z_i\}_{1\le i\le m}$ that an up-right path going from $(0,0)$ to $(1,1)$ can collect. In order to be able to interpret a path as a possible scaling for the random walk, we perform a 45° rotation: the set of up-right paths is transformed into the set of 1-Lipschitz functions. However, this is not satisfactory for us, since we wish to consider random walk paths at a scale n^{ξ}: their scaling limit will not be 1-Lipschitz (except when $\xi = 1$). We therefore develop a notion of Entropy-controlled LPP, in order to replace the 1-Lipschitz constraint by a *gobal* entropy constraint. This model is introduced and studied in the article [9] in collaboration with Niccolò Torri, published in *The Annals of Applied Probability*.

Recall the definition (5.3) of the entropy of a continuous path $s : [0,1] \to \mathbb{R}$. For a finite (directed) set $\Delta = \{(t_i,x_i), 1\le i\le k\} \subset \mathbb{R}_+ \times \mathbb{R}$ with $t_1 < \cdots < t_k$, we define its entropy as

$$\mathrm{Ent}(\Delta) := \frac{1}{2}\sum_{i=1}^{k} \frac{|x_i - x_{i-1}|^2}{t_i - t_{i-1}}\,, \tag{6.2}$$

with the convention that $t_0 = 0$, $x_0 = 0$. This corresponds to the entropy (5.3) of the linear interpolation of the points of Δ (Δ the set of points that the path has to go through).

Then, we consider the domain $\Lambda := [0,1] \times [-\frac12, \frac12]$ (of volume 1), and for $m \in \mathbb{N}$, we let $\Upsilon_m := \{Z_i\}_{1\le i\le m}$ be a set of m points drawn at random, independently and uniformly in Λ (Υ_m may be thought as a random environment, with law denoted by $\mathbb{P}$). We define the Entropy-controlled Last-Passage Percolation (E-LPP) in Λ, with set of points Υ_m and entropy constraint $B > 0$, by

$$\mathcal{L}_m^{(B)} := \max\big\{ |\Delta| \,;\, \Delta \subset \Upsilon_m,\, \mathrm{Ent}(\Delta) \le B \big\}\,. \tag{6.3}$$

This is the maximal number of points that can be collected via a path with entropy smaller than B. We refer to Figure 6.1 for an illustration of the E-LPP (and a comparison with standard Hammersley's LPP). One of our main result in [9] is to show that as $m \to +\infty$, $\mathcal{L}_m^{(B)}$ is of the order of $\sqrt{m}$ (with an explicit dependence on B).

Theorem 6.2 ([9], Theorem 2.1)**.** *Define $K_m = K_m(B) := \min\big(B^{1/4}\sqrt{m}, m\big)$. There is a constant $c > 0$ such that for any $B > 0$ and any $m \in \mathbb{N}$, we have*

$$\mathbb{P}\Big(\mathcal{L}_m^{(B)} \ge cK_m\Big) \le 2^{-K_m} \quad \text{and} \quad \mathbb{P}\Big(\mathcal{L}_m^{(B)} \le \tfrac{1}{c}K_m\Big) \le 2^{-K_m}\,. \tag{6.4}$$

If we fix B, we get that $\mathcal{L}_m^{(B)}$ is of the order of $B^{1/4}\sqrt{m}$ as $m \to +\infty$, *i.e.* of the same order as for Hammersley's LPP—note that a 1-Lipschitz path $s : [0,1] \to \mathbb{R}$ has $\mathrm{Ent}(s) \le 1/2$, so that the E-LPP gives an upper bound on standard LPP. Another important feature of this theorem is that it allows to take B depending on m, which is essential in our applications to the directed polymer in heavy-tail random environment (for the clarity of the exposition, we also left aside the dependence on the domain, that one could choose to be $\Lambda_{t,x} = [0,t] \times [-x,x]$).

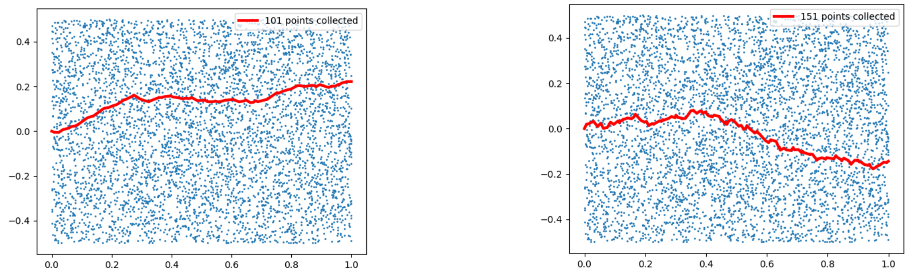


Figure 6.1 – In the box $\Lambda = [0,1]\times[-\frac{1}{2},\frac{1}{2}]$, $m = 5000$ points are drawn uniformly and independently (we give $\sqrt{m} \approx 70.71$). On the left, an optimal path for Hammersley's LPP, *i.e.* with a 1-Lipschitz constraint, which collects 101 points—note that $\mathcal{L}_m/\sqrt{m} \approx \sqrt{2}$, which is the correct constant for standard LPP (where an additional $\sqrt{2}$ comes from the length of the diagonal). On the right, a (near)-optimal path for E-LPP with constraint $\mathrm{Ent}(s) \le B = 1$, which collects 151 points—we get $\mathcal{L}_m^{(1)}/\sqrt{m} \approx 2.13$. (We used a simulated annealing procedure, hence the *near*-optimality.)

Let us mention that the proof is simple and robust enough to be adapted to other definitions of entropy: we present more general results in Section 6.3 below (and we refer to the discussions in Section 5.4 for motivations). Our E-LPP is also easily generalized to the case of higher dimensions, but for simplicity, we stick to the case of dimension $1+1$.

About the convergence of E-LPP. In the case where $\widetilde{\Upsilon}_m$ is a PPP on $[0,1]\times\mathbb{R}$ of intensity m, define the *constrained* Poissonian E-LPP, $\widetilde{\mathcal{L}}_m^{(B),c} := \sup\{|\Delta| \;;\; \Delta \subset \tilde{\Upsilon}_m, \mathrm{Ent}\big(\Delta^c\big) \le B\}$, where $\Delta^c = \Delta \cup \{(1,0)\}$, *i.e.* we add the condition that the path returns to 0 at time 1 (and we also drop the condition that the path is limited to the box Λ). Then, scaling and sub-additive arguments allows us to show in [2] that $\widetilde{\mathcal{L}}_m^{(B)}/\sqrt{m}$ converges a.s. to a (universal) constant $\mathbf{c}$ times $B^{1/4}$, suggesting that the E-LPP (6.3), renormalized by $\sqrt{m}$, also converges to a constant. However, we have no conjecture on the value of the constant $\mathbf{c}$, and because the constraint is *global* (in opposition to Hammersley's LPP) this appears as a difficult problem: numerical simulations suggest that $\mathbf{c} \approx 2.17$.

6.2 (Entropy-controlled) Last-Passage Percolation with heavy-tail weights

In this section, we show how one can use Theorem 6.2 to get that the variational problem in Theorem 5.4 is well-defined for $\alpha \in (\frac{1}{2}, 2)$. Recall that $\mathcal{P}_\alpha$ is a PPP on $[0,1]\times\mathbb{R}\times\mathbb{R}_+$ with intensity $\mu(\mathrm{d}t\mathrm{d}x\mathrm{d}w) := \frac{\alpha}{2} w^{-(1+\alpha)}\mathrm{d}t\mathrm{d}x\mathrm{d}w$, for some $\alpha > 0$. Recall also that for a path $s : [0,1] \to \mathbb{R}$, we defined $\pi(s) := \sum_{(t,x,w)} w \mathbf{1}_{\{s(t)=x\}}$ the total weight collected by s. Finally, recall the definition

$$\mathcal{T} := \sup_{s\in\mathscr{D}} \big\{\pi(s) - \mathrm{Ent}(s)\big\}\,, \tag{6.5}$$

with $\mathrm{Ent}(s) := \frac{1}{2}\int_0^1 (s'(t))^2 \mathrm{d}t$ defined in (5.3). One of the first (and main) application of our E-LPP has been to prove the following theorem, which answers a conjecture of [DZ16].

Theorem 6.3 ([9], Theorem 2.4)**.** *For $\alpha \in (\frac{1}{2}, 2)$, we have that $\mathcal{T} \in (0,+\infty)$ a.s. Moreover, the supremum in (6.5) is attained by some unique continuous path s^*.*

Idea of the proof. We focus on the proof that $\mathcal{T} < +\infty$ a.s. for $\alpha \in (\frac12, 2)$.

First, we decompose the variational problem according to the value of the path entropy. Write $\mathcal{T} = \sup_B \big\{ \sup_{s,\mathrm{Ent}(s)\le B} \pi(s) - B \big\}$, and split this according to the value of $B \in (2^k, 2^{k+1}]$, $k \ge 0$:

$$\mathcal{T} \le \mathcal{H}_1 \vee \sup_{k\ge 0} \big\{ \mathcal{H}_{2^{k+1}} - 2^k \big\} \qquad \text{with } \mathcal{H}_B := \sup_{s,\mathrm{Ent}(s)\le B} \pi(s)\,. \tag{6.6}$$

We are therefore reduced to having good tail estimates on $\mathcal{H}_B$, and in particular, we need to control $\mathbb{P}(\mathcal{H}_{2^{k+1}} > v2^k)$ for $v > 0$ and any k. We may use some scaling relations for $\mathcal{P}_\alpha$. Consider the function $\varphi(t,x,w) = (t, x/b, w)$ which scales space by a factor $1/b$ (and hence the entropy by a factor $1/b^2$), and $\psi(t,x,w) = (t,x,b^{\frac{1}{\alpha}} w)$ which scales the weights by a factor $b^{1/\alpha}$: we have that $\varphi(\mathcal{P}_\alpha) \overset{(d)}{=} \psi(\mathcal{P}_\alpha)$. Applying this scaling with $b = \sqrt{B}$, we get that $\mathcal{H}_B \overset{(d)}{=} B^{\frac{1}{2\alpha}} \mathcal{H}_1$. We therefore obtain that $\mathbb{P}\big(\mathcal{H}_{2^{k+1}} > v2^k\big) = \mathbb{P}\big(\mathcal{H}_1 > c_\alpha v 2^{k(1-\frac{1}{2\alpha})}\big)$, and we realize that if $1 - \frac{1}{2\alpha} < 0$ (*i.e.* $\alpha < \frac12$) then this probability goes to 1 as $k \to +\infty$. This explains why one needs to have $\alpha > 1/2$.

On the other hand, if $\alpha > \frac12$, then $2^{k(1-\frac{1}{2\alpha})} \to +\infty$ as $k \to +\infty$, and we need to get a bound on the tail of $\mathcal{H}_1$: we have reduced our problem to a question of E-LPP in heavy-tail random environment. In [9, Lem. 4.1], we prove that $\mathbb{P}(\mathcal{H}_1 > v) \le v^{-a}$ as $v \to +\infty$, for $a < \alpha < 2$. Here, we simply give an idea of why $\mathcal{H}_1 < +\infty$ a.s. The key idea is to "slice up" the field $\mathcal{P}_\alpha$ into stripes with weights of more or less the same value: define $\mathcal{P}_k := \{(t,x,w) \in \mathcal{P}_\alpha, 2^k w \in (\frac12, 1]\}$ and write

$$\pi(s) := \sum_{(t,x,w)\in\mathcal{P}_\alpha} w \mathbf{1}_{\{s(t)=x\}} \le \sum_{(t,x,w)\in\mathcal{P}_\alpha, w>1} w \mathbf{1}_{\{s(t)=x\}} + \sum_{k=1}^{+\infty} 2^{-k} \sum_{(t,x,w)\in\mathcal{P}_k} \mathbf{1}_{\{s(t)=x\}}\,. \tag{6.7}$$

Notice that since the paths in $\mathcal{H}_1$ have an entropy bounded by 1, they are confined in the box $\Lambda' = [0,1]\times[-2,2]$. Hence, the first term in (6.7) is bounded by $\sum_{(t,x,w)\in\mathcal{P}_\alpha, w>1} w$, which is finite since there are finitely many points in $\mathcal{P}_\alpha$ that lie in $[0,1]\times[-2,2]\times(1,+\infty)$. For the second term in (6.7), notice that $|\mathcal{P}_k| \approx 2^{k\alpha}$, because the weight density is proportional to $w^{-(1+\alpha)}$; actually, setting $m_k := 2^{k\alpha}$, one has that a.s. $\frac12 m_k \le |\mathcal{P}_k| \le 2m_k$ for all k sufficiently large. Taking the supremum in (6.7) over paths $s : [0,1] \to \mathbb{R}$ with $\mathrm{Ent}(s) \le 1$, we therefore get

$$\mathcal{H}_1 := \sup_{s,\mathrm{Ent}(s)\le 1} \pi(s) \le C(\mathcal{P}_\alpha) + \sum_{k=1}^{+\infty} 2^{-k} \mathcal{L}_{2m_k}\,. \tag{6.8}$$

Here, for simplicity, we set $\mathcal{L}_m$ the E-LPP problem of (6.3) with $B = 1$, with m points taken in the domain $\Lambda' = [0,1]\times[-2,2]$ instead of $\Lambda = [0,1]\times[-\frac12,\frac12]$ (this does not change the conclusion of Theorem 6.2). Now, Theorem 6.2 gives that $\mathcal{L}_{2m_k} \le C\sqrt{m_k} \le C' 2^{k\frac{\alpha}{2}}$ (a.s., for k sufficiently large): the condition $\alpha < 2$ appears crucial to show that $\sum_{k=1}^{+\infty} 2^{-k}\mathcal{L}_{m_k} \le C' \sum_{k=1}^{+\infty} 2^{k(\frac{\alpha}{2}-1)}$ is finite.

6.2.1 About the case $\alpha = 2$

Let us now consider the case $\alpha \ge 2$. Analogously to (6.7), we write $\pi(s) = \pi_0(s) + \sum_{k=1}^{+\infty} \pi_k(s)$, where $\pi_0(s) := \sum_{(t,x,w)\in\mathcal{P}_\alpha, w>1} w\mathbf{1}_{\{s(t)=x\}}$ (it is bounded by a constant), and $\pi_k(s) := \sum_{(t,x,w)\in\mathcal{P}_k} w\mathbf{1}_{\{s(t)=x\}}$ is the total weight collected by s in $\mathcal{P}_k$, the k^{th} level of the field. Above, we have seen that $\sup_{s,\mathrm{Ent}(s)\le 1} \pi_k(s)$ is of the order of $2^{k(\frac{\alpha}{2}-1)}$, which is summable if $\alpha < 2$, and diverges if $\alpha > 2$. In the case $\alpha > 2$, using that $\pi(s) \ge \pi_k(s)$, we directly get that $\mathcal{H}_1$ (hence $\mathcal{T}$) is infinite a.s.

Let us focus on the case $\alpha = 2$, where $\sup_{s,\mathrm{Ent}(s)\le 1} \pi_k(s)$ is of constant order. The idea of the upper bound $\sup_s \pi(s) \le \sup_s \pi_0(s) + \sum_{k=1}^{+\infty} \sup_s \pi(s)$ is that it "decouples" the different weight scales $\mathcal{P}_k$: it gives and infinite upper bound, but it is not excluded that $\mathcal{H}_1$ remains finite—for instance, collecting many weights in $\mathcal{P}_k$ may make it very difficult to collect weights at levels $\mathcal{P}_{k'}$, $k' > k$. The first question is therefore to know whether $\mathcal{H}_1$ is finite or infinite in the case $\alpha = 2$. We now state a conjecture for the 1-Lipschitz LPP (for which we have a better understanding, and where simulations can be run more easily) and for a simplified Poisson field, for the ease of the presentation.

For $k \in \mathbb{N}$, let $\mathcal{P}_k$ be a PPP on $[0,1] \times \mathbb{R}$ of intensity 2^{2k}, whose points carry a weight 2^{-k} (it is thought as the k^{th} level of our field). Then we define the n-level LPP (with weight field $(\mathcal{P}_k)_{k\ge 1}$)

$$\Pi_n := \sup_{s\in \mathrm{Lip}_1} \Big\{ \sum_{k=1}^{n} \pi_k(s) \Big\} \qquad \text{with } \pi_k(s) := 2^{-k} \sum_{(t,x)\in \mathcal{P}_k} \mathbf{1}_{\{s(t)=x\}} , \tag{6.9}$$

which should have the same type of behavior (as $n \to +\infty$) as $\mathcal{H}_1$ in the case $\alpha = 2$. We propose the following conjecture.

"Conjecture" 6.4. *The sequence $(\Pi_n/\sqrt{n})_{n\ge 1}$ converges a.s. to a constant. Moreover, for $n \in \mathbb{N}$ let s_n^* be a maximizer of Π_n: then a.s. there is some continuous path $s^* : [0,1] \to \mathbb{R}$ such that $\lim_{n\to+\infty} \|s_n^* - s^*\|_\infty = 0$.*

The first part of the conjecture is supported by some work in progress with Niccolò Torri and Nikos Zygouras (at least for the fact that Π_n is of order $\sqrt{n}$), and the second part of the conjecture is based on discussions with Christophe Garban and on some numerical simulations, see Figure 6.2 below.

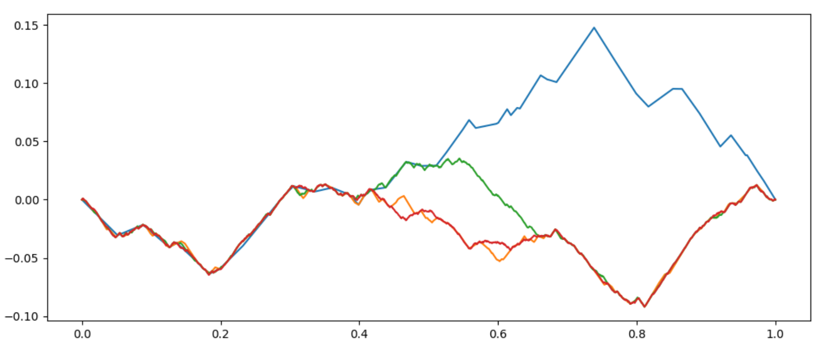


Figure 6.2 – Simulations for Π_n defined in (6.9): the maximizing paths for Π_n are plotted, for $n = 6$ (in blue, $\Pi_6 \approx 7.25$), $n = 8$ (in orange, $\Pi_8 \approx 8.72$), $n = 10$ (in green, $\Pi_{10} \approx 9.97$), $n = 11$ (in red, $\Pi_{11} \approx 10.6$).

6.3 Generalizations: Last-Passage Percolation with constraints

In the definition (6.3), we realize that we can replace the constraint $\mathrm{Ent}(s) \le B$ by any constraint on the path, such as 1-Lipschitz, entropy, or even convexity constraints. We propose below two natural examples: one where we generalize the definition of the entropy of a path; one where we replace the 1-Lipschitz condition by a Hölder constraint. This section is based on a work with Niccolò Torri [2].

6.3.1 More general definitions of path entropy

As mentioned in Section 5.4, if one considers more general random walks for the directed polymer model (in particular with unbounded jumps), different type of entropy may arise. For this reason, and in analogy with (6.2), for a finite (directed) set $\Delta = \{(t_i, x_i), 1 \le i \le k\}$ with $0 < t_1 < \cdots < t_k < t$ (by convention $(t_0, x_0) = (0,0)$), we define the (a,b)-entropy of Δ by

$$\mathrm{Ent}_{a,b}(\Delta) := \sum_{i=1}^{k} \frac{|x_i - x_{i-1}|^a}{(t_i - t_{i-1})^b}\,, \qquad a > b \ge 0\,. \tag{6.10}$$

The parameters a, b are inherited from the large deviation principle: for the simple random walk, we get $a = 2, b = 1$ (we dropped the constant $\frac{1}{2}$); if $\mathbf{P}(S_1 = k) \sim e^{-|k|^a}$ as $k \to \pm\infty$ for some $a > 0$, then we may get $a > 1, b = a - 1$ or $a \in (0,1), b = 0$, see (5.12)-(5.13). In the case $b > 0$, $a = b + 1$, the entropy (6.10) can be extended to continuous paths (a.e. differentiable) by $\int_0^1 |s'(t)|^a \mathrm{d}t$, whereas in the case $b = 0$, $a > 0$, it can be extended to non-necessarily continuous paths by $\sup_{0 = t_0 < t_1 < \cdots < t_k \le 1} \sum_{i=1}^{k} |s(t_i) - s(t_{i-1})|^a$ (the supremum is over all subdivisions of $[0,t]$).

Then, we define, analogously to (6.3) (we keep the same notations for Λ and Υ_m)

$$\mathcal{L}_m^{(B),a,b} := \max\big\{ |\Delta| \ ;\ \Delta \subset \Upsilon_m,\, \mathrm{Ent}_{a,b}(\Delta) \le B \big\}\,. \tag{6.11}$$

As for Theorem 6.2, we are able to show that $\mathcal{L}_m^{(B),a,b}$ is of the order m^κ as $m \to +\infty$, with $\kappa = \frac{a}{a+b+1}$.

Theorem 6.5 ([2], Theorem 2.3)**.** *Define $\kappa = \frac{a}{a+b+1}$ and $K_m := K_m(B,a,b) := \min\big(B^{\kappa/a} m^\kappa, m\big)$. There is a constant $c_{a,b} > 0$ such that for any $B > 0$ and any $m \in \mathbb{N}$, we have*

$$\mathbb{P}\Big(\mathcal{L}_m^{(B),a,b} \ge c_{a,b} K_m\Big) \le 2^{-K_m} \quad \text{and} \quad \mathbb{P}\Big(\mathcal{L}_m^{(B),a,b} \le \tfrac{1}{c_{a,b}} K_m\Big) \le 2^{-K_m}\,. \tag{6.12}$$

Notice that when $b > 0$, $a = b + 1$, we still have $\kappa = 1/2$ as in Theorem 6.2; on the other hand, when $b = 0$, $a > 0$, we obtain that $\kappa = \frac{a}{a+1}$, which is smaller than $1/2$ if $a \in (0,1)$.

An energy-entropy variational problem. Similarly to (6.5), we define an energy-entropy variational problem for a path $s : [0,1] \to \mathbb{R}$ in the heavy-tail field $\mathcal{P}_\alpha$ (recall the definitions of Section 6.2): for $a > 0$, define

$$\mathcal{T}_a := \sup_{s:[0,1]\to\mathbb{R}, s(0)=0} \big\{ \pi(s) - \mathrm{Ent}_a(s) \big\}\,, \tag{6.13}$$

with $\mathrm{Ent}_a(s) := \int_0^t |s'(t)|^a \mathrm{d}t$ if $a > 1$ and $\mathrm{Ent}_a(s) := \sup\limits_{0 < t_1 < \cdots < t_k < 1} \sum_{i=1}^{k} |s(t_i) - s(t_{i-1})|^a$ if $a \in (0,1)$.

In view of Section 5.4 (see in particular (5.11) and (5.12)-(5.13)), this should arise as the natural scaling limit of a directed polymer model in heavy-tail environment, with underlying random walk with unbounded jumps $\mathbf{P}(S_1 = k) \sim e^{-|k|^a}$. Then, Theorem 6.5 allows us to adapt the proof of Theorem 6.3 to get the following:

$$\mathcal{T}_a \in (0, +\infty) \quad a.s. \qquad \text{for} \begin{cases} \alpha \in (\frac{1}{a}, 2) & \text{if } a > 1; \\ \alpha \in (\frac{1}{a}, \frac{1}{a} + 1) & \text{if } a \in (0,1)\,. \end{cases} \tag{6.14}$$

Let us explain in a few words how to proceed to show this. Recall the comments below Theorem 6.3, and in particular (6.6): we may reduce to the study of $\mathcal{H}_B^a := \sup_{s, \mathrm{Ent}_a(s) \le B} \pi(s)$. Then, the same scaling argument (scaling space by $1/b$, hence the entropy by $1/b^a$, and weights by $b^{1/\alpha}$) shows that $\mathcal{H}_B^a \overset{(d)}{=} B^{\frac{1}{a\alpha}}\mathcal{H}_1^a$. It gives that $\mathbb{P}(\mathcal{H}_{2^{k+1}}^a > v2^k) = \mathbb{P}(\mathcal{H}_1^a > c_\alpha v 2^{k(1-\frac{1}{a\alpha})})$, and this explains why one needs that $a\alpha > 1$ in order to avoid this probability to go to 1 as $k \to +\infty$.

On the other hand, one needs to show that $\mathcal{H}_1^a < +\infty$ a.s. (more precisely one needs a bound on the tail of $\mathcal{H}_1^a$). It follows the same line of proof as the one sketched in Section 6.2: analogously to (6.8), we have that

$$\mathcal{H}_1^a := \sup_{s,\mathrm{Ent}_a(s)\le 1} \pi(s) \le C + \sum_{k=1}^{+\infty} 2^{-k}\mathcal{L}_{2m_k}^a \,,$$

where we recall that $m_k := 2^{k\alpha} \approx \mathcal{P}_k$. For simplicity we denoted $\mathcal{L}_m^a$ the E-LPP problem of (6.11) with $B=1$, and $b = a-1$ if $a>1$; $b=0$ if $a\in(0,1)$ (in the box $\Lambda' = [0,1]\times[-2,2]$ instead of Λ). Now, Theorem 6.5 gives that $\mathcal{L}_{2m_k}^a \le C(m_k)^\kappa \le C' 2^{k\kappa\alpha}$ with $\kappa = 1/2$ if $a>1$ and $\kappa = \frac{a}{a+1}$ if $a\in(0,1)$. We therefore get that $\sum_{k=1}^{+\infty} 2^{-k}\mathcal{L}_{m_k}^a$ is finite if $\kappa\alpha<1$: this corresponds to having $\alpha<2$ when $a>1$ and $\alpha < \frac{1}{a}+1$ when $a\in(0,1)$.

A word on the non-directed case. In Section 5.4, we comment on the case of a non-directed random walk in heavy-tail environment: we noticed that the moderate deviation for the simple random walk gives rise to the entropy $\mathrm{Ent}(s) = \frac12 \ell(s)^2$ for continuous curves $s:[0,1]\to\mathbb{R}^2$, $s(0)=0$, where $\ell(s) := \int_0^1 \|s'(t)\|_2 \mathrm{d}t$ is the length of the curve s. In [2], we define a more general entropy for non-directed paths, and we obtain the correct order for the corresponding E-LPP problem. To state briefly our result, let Υ_m be a (random) set of m independent points drawn uniformly in the unit disk $\mathcal{D}_1 := \{x\in\mathbb{R}^2, \|x\|_2\le 1\}$, and define, for $L>0$

$$\mathcal{L}_m^{(L),\text{non-dir}} := \sup_{s:[0,1]\to\mathbb{R}^2, s(0)=0} \Big\{ \sum_{x\in\Upsilon_m} \mathbf{1}_{\{s(t)=x\}} \;;\; \ell(s)\le L \Big\} .$$

Then we show that $\mathcal{L}_m^{(L),\text{non-dir}}$ is of the order of $L\sqrt{m}$. In other word, a path with a length smaller than L (*i.e.* with entropy smaller than $\frac12 L^2$) collects at most $L\sqrt{m}$ points of Υ_m.

This result allows us to adapt the method used for Theorem 6.3, and enables us to show that for $a>1$ the (non-directed) variational problem

$$\mathcal{T}_a^{\text{non-dir}} := \sup_{s:[0,1]\to\mathbb{R}^2, s(0)=0} \big\{ \pi(s) - \ell(s)^a \big\} . \tag{6.15}$$

is finite a.s. for $\alpha \in (\frac{2}{a}, 2)$, see [4, Prop. 4.1]—the definitions of the heavy-tail PPP $\mathcal{P}_\alpha$ and of $\pi(s)$ are naturally generalized to the non-directed setting.

6.3.2 The case of a Hölder constraint

In this last subsection, we give some results on another variation of the LPP: we study in [2] what is the effect of replacing the 1-Lipschitz constraint of Hammersley's LPP by a γ-Hölder one. For a finite (directed) set $\Delta = \{(t_i,x_i), 1\le i\le k\}$ with $t_1<\cdots<t_k$ (by convention $(t_0,x_0)=(0,0)$), we define its *local* γ-Hölder norm

$$\mathrm{H}_\gamma(\Delta) = \sup_{1\le i\le k} \frac{|x_i - x_{i-1}|}{(t_i-t_{i-1})^\gamma} , \qquad \gamma \ge 0 . \tag{6.16}$$

Notice here that this definition is not the γ-Hölder norm of the linear interpolation of the points of Δ: it considers only consecutive points (for instance the case $\gamma > 1$ is not degenerate). Let Υ_m be a set of m points taken uniformly at random in $\Lambda = [0,1] \times [-\frac{1}{2}, \frac{1}{2}]$, and define the γ-Hölder LPP

$$\mathcal{L}_m^\gamma := \max\left\{|\Delta| \; ; \; \Delta \subset \Upsilon_m, \, \mathrm{H}_\gamma(\Delta) \le 1\right\}. \tag{6.17}$$

In [2], we show that $\mathcal{L}_m^\gamma$ is of order $m^{1/(1+\gamma)}$ as $m \to +\infty$.

Theorem 6.6 ([2], Theorem 2.1)**.** *Define $\kappa = \frac{1}{1+\gamma}$, and $K_m := m^\kappa$. There is a constant $c_\gamma > 0$ such that for any $n \in \mathbb{N}$, we have*

$$\mathbb{P}\Big(\mathcal{L}_m^\gamma(t,x) \ge c_\gamma K_m\Big) \le 2^{-K_m} \quad \text{and} \quad \mathbb{P}\Big(\mathcal{L}_m^\gamma(t,x) \le \tfrac{1}{c_\gamma} K_m\Big) \le 2^{-K_m}. \tag{6.18}$$

As for the E-LPP of Section 6.1.1, we may take $\widetilde{\Upsilon}_m$ a PPP on $[0,1] \times \mathbb{R}$ of intensity m, and define the constrained Poissonian γ-Hölder LPP as $\widetilde{\mathcal{L}}_m^{\gamma,c} := \sup\{\Delta \; ; \; \Delta \subset \widetilde{\Upsilon}_m, \mathrm{H}_\gamma(\Delta^c) \le 1\}$ (recall $\Delta^c = \Delta \cup \{(1,0)\}$). Again, scaling and sub-additive arguments lead to showing that $\widetilde{\mathcal{L}}_m^{\gamma,c}/m^\kappa$ converges a.s. to a constant $\mathbf{c}_\gamma$. We refer to [2, App. A.1] for a discussion on the value of the constant (and more numerical simulations), and to Figure 6.3 for an illustration of the Poissonian γ-Hölder LPP.

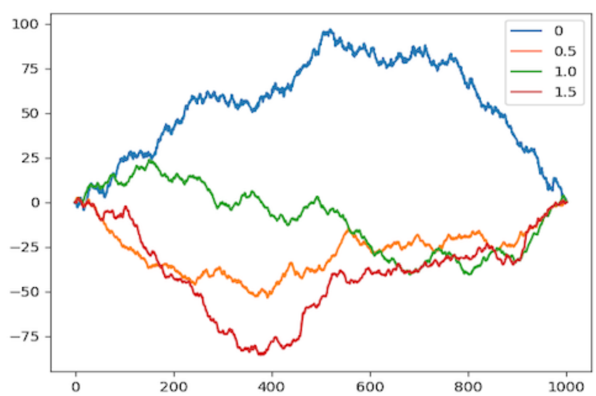


Figure 6.3 – Maximizing paths for the constrained Poissonian γ-Hölder LPP. We consider a PPP of intensity 1 on $[0, 1000] \times [-100, 100]$, and we find optimal paths for the γ-Hölder LPP for different values of γ ($\gamma = 0$, $\gamma = 0.5$, $\gamma = 1$ and $\gamma = 1.5$).

Part III

Probabilistic tools: random walks and renewal processes

Chapter 7

Some (recent and not so recent) results on renewal processes and random walks

In this chapter, we review some results on univariate and multivariate renewal processes and random walks, that are the main probabilistic objects at the center of the different models of parts I and II. In particular, we present our results [5, 7, 16, 17]. We consider processes with increments in $\mathbb{Z}^d$ ($d \geq 1$), but (most of) the results have corresponding statements for processes in $\mathbb{R}^d$.

7.1 Renewal processes and intersection of renewal processes

Let $\tau = \{\tau_0, \tau_1, \tau_2, \ldots\}$ be a one-dimensional renewal process whose law is denoted by $\mathbf{P}$: $\tau_0 = 0$, and $(\tau_i - \tau_{i-1})_{i \geq 1}$ are i.i.d. $\mathbb{N}$-valued r.v.s—in other words, $(\tau_k)_{k \geq 0}$ is a random walk with positive increments. One of the main question for renewal processes is to estimate the renewal mass function $\mathbf{P}(n \in \tau)$, depending on the properties of the inter-arrival distribution $\mathbf{P}(\tau_1 = n)$. As in Chapters 1 and 2, we assume that there exists a slowly varying function $\varphi(\cdot)$ and $\alpha \geq 0$ such that

$$\mathbf{P}(\tau_1 = n) = \varphi(n)\, n^{-(1+\alpha)} \,. \tag{7.1}$$

We say that τ is *persistent* if $\mathbf{P}(\tau_1 < +\infty) = 1$ (there are infinitely many renewal points), and *terminating* if $\mathbf{P}(\tau_1 < +\infty) < 1$ (there is only a geometric number of renewal points). In the literature the terminology recurrent and transient renewals is used, but we keep these terms here for the standard recurrence/transience of random walks on $\mathbb{Z}$ or $\mathbb{Z}^d$.

7.1.1 Renewal theorems

The asymptotics of the renewal mass function $\mathbf{P}(n \in \tau)$ have been extensively studied in the literature, see for instance [GL62, Wil68, Eri70, Don97, Nag12]. Let us review the results—under assumption (7.1).

First, if τ is terminating, then we have (see [Gia07, App. A.5])

$$\mathbf{P}(n \in \tau) \sim \frac{\mathbf{P}(\tau_1 = n)}{\mathbf{P}(\tau_1 = +\infty)^2} \quad \text{as } n \to +\infty \,. \tag{7.2}$$

In the case where τ is persistent, we have the following cases:

- if $\mathbf{E}[\tau_1] < +\infty$, then the renewal theorem gives that

$$\lim_{n\to\infty} \mathbf{P}(n \in \tau) = \mathbf{E}[\tau_1]^{-1}\,; \tag{7.3}$$

- if $\alpha = 1$, $\mathbf{E}[\tau_1] = +\infty$, then Erickson [Eri70] proved that, setting $\mu(n) := \mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le n\}}]$,

$$\mathbf{P}(n \in \tau) \sim \mu(n)^{-1} \qquad \text{as } n \to +\infty\,; \tag{7.4}$$

- if $\alpha \in (0,1)$, then Doney [Don97, Thm. B] showed that

$$\mathbf{P}(n \in \tau) \sim \tfrac{\alpha \sin(\pi\alpha)}{\pi}\, n^{-(1-\alpha)}\, \varphi(n)^{-1} \qquad \text{as } n \to +\infty\,; \tag{7.5}$$

- if $\alpha = 0$, then Nagaev [Nag12] proved

$$\mathbf{P}(n \in \tau) \sim \frac{\mathbf{P}(\tau_1 = n)}{\mathbf{P}(\tau_1 > n)^2} \qquad \text{as } n \to +\infty\,. \tag{7.6}$$

We stress here that the condition (7.1) is not the optimal one for obtaining renewal theorems with infinite mean. For instance, having $\mathbf{P}(\tau_1 > n) \sim \alpha^{-1}\varphi(n) n^{-\alpha}$ with $\alpha \in (0,1)$ ensures that $(\tau_k)_{k\ge 0}$ is in the domain of attraction of an α-stable: in that case, Garsia and Lamperti [GL62] showed that (7.5) is valid for all $\alpha \in (\frac12, 1)$ (and (7.4) also holds in the case $\alpha = 1$ with infinite mean). When $\alpha \in (0, \frac12]$, additional conditions are needed: (7.1) is sufficent, and a necessary and sufficient condition for the strong renewal theorem (7.5) to hold was recently established in [CD16].

About the case $\alpha = 0$. With Kenneth S. Alexander, we studied the case where $\alpha = 0$ in (7.1), in the article [17] published in *Electronic Journal of Probability*. This appears for example when $\tau = \{n, S_{2n} = 0\}$, with $(S_n)_{n\ge 0}$ the simple random walk on $\mathbb{Z}^2$, see [JP72]. This case had received very little attention, partly because τ_1 has no moment and $(\tau_k)_{k\ge 0}$ is not in the domain of attraction of stable laws (the "0-stable subordinator" is the Dickman subordinator, we refer to [CSZ18] for recent results). One of our main result in [17] is to derive local large deviations for $(\tau_k)_{k\ge 1}$, from which one may deduce (7.6) easily.

Theorem 7.1 ([17], Theorem 1.1)**.** *Assume that* (7.1) *holds with $\alpha = 0$, and that τ is persistent. Then uniformly for k such that $k\varphi(n) \to 0$, we have*

$$\mathbf{P}(\tau_k = n) \sim k\mathbf{P}(\tau_1 = n)\mathbf{P}(\tau_1 \le n)^k \qquad \textit{as } n \to +\infty. \tag{7.7}$$

This theorem shows that the local large deviation probability comes from a large-jump strategy— our proof is probabilistic and shows that other trajectories do not contribute to the probability, via some adapted Fuk-Nagaev inequalities. We refer to Section 7.2 for related results on local large deviations for random walks in the domain of attraction of an α-stable law, see e.g. Theorem 7.8.

The strength of this results comes from the fact that it is valid for a wide range of k: for $k \ll \mathbf{P}(\tau_1 > n)^{-1}$ we obtain a "big-jump" type asymptotics, $\mathbf{P}(\tau_k = n) \sim k\mathbf{P}(\tau_1 = n)$, but (7.7) is also valid in a range of k for which $\mathbf{P}(\tau_1 \le n)^k \to 0$. Indeed, when $\alpha = 0$, we have that $\mathbf{P}(\tau_1 > n)/\varphi(n) \to +\infty$ as $n \to +\infty$, cf. [BGT89, Prop. 1.5.9a]. This allows us to derive (7.6) quite easily: choose $\theta_n \to +\infty$ such that $\mathbf{P}(\tau_1 > n)^{-1} \ll \theta_n \ll \varphi(n)^{-1}$, then write

$$\mathbf{P}(n \in \tau) = \sum_{k=1}^{+\infty} \mathbf{P}(\tau_k = n) = \sum_{n=1}^{\theta_n} \mathbf{P}(\tau_k = n) + \sum_{k=\theta_n+1}^{+\infty} \mathbf{P}(\tau_k = n). \tag{7.8}$$

For the first term, we may use Theorem 7.1 to get that it is asymptotic to $\mathbf{P}(\tau_1 = n)/\mathbf{P}(\tau_1 > n)^2$. For the second term, we need to use the following (uniform) upper bound, proven in [17]: there is some constant $c > 0$ such that $\mathbf{P}(\tau_k = n) \le ck\mathbf{P}(\tau_1 = n)\mathbf{P}(\tau_1 \le n)^k$ for all $k \le n$. This enables us to show that the second sum in (7.8) is of the order of $\mathbf{P}(\tau_1 = n)/\theta_n^2$, which is negligible compared to the first term.

7.1.2 "Reverse" renewal theorems.

In [17], we also introduce what we call *reverse renewal theorems*. Indeed, verifying that (7.1) holds is sometimes difficult, and one has often an easier access to $\mathbf{P}(n \in \tau)$. This is for instance the case when τ is a set of return times to 0 of a random walk in the domain of attraction of an α-stable law, $\tau = \{n, S_n = 0\}$: a standard local limit theorem gives the behavior of $\mathbf{P}(S_n = 0) = \mathbf{P}(n \in \tau)$, but obtaining the asymptotics of $\mathbf{P}(\tau_1 = n)$ is much harder, see [Kes63]. We now present some results where one is able to infer something on the behavior of $\mathbf{P}(\tau_1 = n)$ from that of $\mathbf{P}(n \in \tau)$.

In the persistent case, define

$$U_n := \sum_{k=0}^{n} \mathbf{P}(k \in \tau), \tag{7.9}$$

and set $U_\infty := \sum_{k=0}^{\infty} \mathbf{P}(k \in \tau) = \mathbf{P}(\tau_1 = +\infty)^{-1} < +\infty$ in the terminating case. Note that, if U_n is regularly varying with exponent $\alpha \in [0, 1)$, then Tauberian theorems (see [BGT89, Thm. 8.7.3]) give that $\mathbf{P}(\tau_1 > n) \sim \frac{\sin(\pi\alpha)}{\pi\alpha}(U_n)^{-1}$. In particular, if U_n is slowly varying then $\mathbf{P}(\tau_1 > n) \sim 1/U_n$.

Theorem 7.2 ([17], Theorem 1.3)**.** *Suppose that $\mathbf{P}(n \in \tau)$ is regularly varying of exponent -1 (hence U_n is slowly varying). Then there is $\varepsilon_n \to 0$ such that*

$$\frac{1}{\varepsilon_n n} \sum_{k=(1-\varepsilon_n)n}^{n} \mathbf{P}(\tau_1 = k) \sim \mathbf{P}(\tau_1 > n)^2 \mathbf{P}(n \in \tau)\,. \tag{7.10}$$

If in addition $\mathbf{P}(\tau_1 = n)$ is regularly varying, then

$$\mathbf{P}(\tau_1 = n) \sim \mathbf{P}(\tau_1 > n)^2 \mathbf{P}(n \in \tau). \tag{7.11}$$

Hence, in order to obtain a reverse renewal theorem of the type (7.11),Theorem 7.2 tells that one only has to show that $\mathbf{P}(\tau_1 = k)$ is approximately constant on the interval $[(1-\varepsilon_n)n, n]$. In the case where τ is terminating, we are able to prove a complete reverse renewal theorem.

Theorem 7.3 ([17], Theorem 1.4)**.** *If $\mathbf{P}(n \in \tau)$ is regularly varying and τ is transient, then*

$$\mathbf{P}(\tau_1 = n) \sim \mathbf{P}(\tau_1 = \infty)^2 \mathbf{P}(n \in \tau)\,.$$

Our proof in [17] is quite short, and of probabilistic nature. An easy corollary of Theorem 7.3 is the following. Let $(S_n)_{n \ge 0}$ be the simple symmetric random walk on $\mathbb{Z}^d$, with $d \ge 3$, and let $\tau = \{n, S_{2n} = 0\}$. The local limit theorem gives that $\mathbf{P}(n \in \tau) = \mathbf{P}(S_{2n} = 0) \sim (2\pi n)^{-d/2}$: for $d \ge 3$, the random walk is transient, and we directly get from Theorem 7.3 that $\mathbf{P}(\tau_1 = n) \sim c_d n^{-d/2}$, with $c_d = (2\pi)^{-d/2}\mathbf{P}(S_n \ne 0, \forall\ n \ge 1)^2$—this is very natural but was apparently only proven in 2011 in [DK11].

7.1.3 About the intersection of two independent renewal processes

Let us now consider two independent renewal processes τ and σ, and their intersection $\rho = \tau \cap \sigma$, which is also a renewal process. In the article [16], in collaboration with Kenneth S. Alexander and published in *Electronic Journal of Probability*, we study the asymptotic behavior of $\mathbf{P}(\rho_1 = n)$ (this was surprisingly absent from the literature). Intersection of renewals appear in various contexts, but our original motivation came from the pinning model of Chapter 2, where it appears naturally when computing the second moment of the partition function; also from the model in Section 2.4.

For the simplicity of the exposition, we assume here that τ and σ have the same distribution, which verify (7.1)—in [16], we consider the case where they have different distributions, as in Section 2.4, see (2.10). From (7.2) and (7.3)–(7.6), we can obtain the behavior of the renewal mass function of ρ. Our strategy is to derive the asymptotic behavior of $\mathbf{P}(\rho_1 = n)$ from that of $\mathbf{P}(n \in \rho)$.

In the case where τ is terminating, we get that as $n \to +\infty$

$$\mathbf{P}(n \in \rho) \sim \mathbf{P}(\tau_1 = +\infty)^{-4}\, \varphi(n)^2 n^{-2(1+\alpha)} \,. \tag{7.12}$$

In the case where τ is persistent, we get that as $n \to +\infty$ (recall that $\mu(n) := \mathbf{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le n\}}]$, which goes to $\mathbf{E}[\tau_1]$ in the finite mean case)

$$\mathbf{P}(n \in \rho) = \mathbf{P}(n \in \tau)^2 \sim \begin{cases} \mu(n)^{-2} & \text{if } \alpha \ge 1; \\ c_\alpha n^{-2(1-\alpha)} \varphi(n)^{-2} & \text{if } \alpha \in (0,1), \text{with } c_\alpha = \frac{\alpha^2 \sin(\pi\alpha)^2}{\pi^2}; \\ \varphi(n)^2 n^{-2} \mathbf{P}(\tau_1 > n)^{-4} & \text{if } \alpha = 0\,. \end{cases} \tag{7.13}$$

Hence, we find that ρ is terminating if either τ is terminating, or if τ is persistent with $\alpha < 1/2$ or $\alpha = 1/2$ and $\sum_n \frac{1}{\varphi(n)^2 n} < +\infty$.

Cas of a terminating ρ. Suppose that ρ is terminating, *i.e.* $\mathbf{E}[|\rho|] = \sum_{n \ge 0} \mathbf{P}(n \in \rho) < +\infty$. Here, a direct application of Theorem 7.3 gives the following.

Theorem 7.4 ([16], Theorem 1.2)**.** *In the case where ρ is terminating, we have*

$$\mathbf{P}(\rho_1 = n) \sim \mathbf{E}[|\rho|]^{-2} \mathbf{P}(n \in \rho) \qquad \text{as } n \to +\infty\,,$$

with $\mathbf{P}(n \in \rho)$ given by (7.12) if τ is terminating, or by (7.13) if τ is persistent.

Case of a persistent ρ. Here, we need to have τ persistent, and $\alpha \ge 1/2$. A first step toward finding the asymptotic behavior of $\mathbf{P}(\rho_1 = n)$ is to obtain that of $\mathbf{P}(\rho_1 > n)$. This is somehow easy in the case $\alpha \in [1/2, 1)$: thanks to (7.13) we get the explicit behavior of $U_n^* := \sum_{k=0}^n \mathbf{P}(n \in \rho)$, which is regularly varying with exponent $\alpha^* := 2\alpha - 1 \in [0,1)$; then we can use [BGT89, Thm. 8.7.3] to get that $\mathbf{P}(\rho_1 > n) \sim \frac{\sin(\pi\alpha^*)}{\pi\alpha^*} (U_n^*)^{-1}$. In the case $\alpha \ge 1$, this is more difficult and it did not appear in the literature: in [16, Thm. 1.3], we show that $\mathbf{P}(\rho_1 > n) \sim 2\mu(n) \mathbf{P}(\tau_1 > n)$. The idea is that in order to have no intersection, one of the two renewals has to make a jump larger than n, but has the liberty to "wait" a few number of steps before doing that jump: this explains the prefactor $\mu(n)$ which counts that number.

Once we have the behavior of $\mathbf{P}(\rho_1 > n)$, the key tool is [16, Lem. 1.5], which shows that under (7.1), $\mathbf{P}(\rho_1 = n)$ is approximately constant over intervals of length $o(n)$. In the case where

$\mathbf{P}(\rho_1 > n)$ is regularly varying with exponent $\alpha^* > 0$, this is enough to conclude, whereas when $\mathbf{P}(\rho_1 > n)$ is slowly varying (equivalently when U_n^* is slowly varying), we need to use the reverse renewal Theorem 7.2. Overall, we obtain the asymptotics of $\mathbf{P}(\rho_1 = n)$.

Theorem 7.5 ([16], Theorem 1.5). *In the case where ρ is persistent, we have:*

(i) if $\alpha \geq 1$, then

$$\mathbf{P}(\rho_1 = n) \sim 2\mu(n)\mathbf{P}(\tau_1 = n) \qquad \text{as } n \to +\infty\,;$$

(ii) if $\alpha \in (\frac{1}{2}, 1)$, then setting $\alpha^* = 2\alpha - 1 \in (0,1)$, we have

$$\mathbf{P}(\rho_1 = n) \sim c_\alpha^* \varphi(n)^2 n^{-(1+\alpha^*)}, \quad c_\alpha^* := \tfrac{\pi\alpha^* \sin(\pi\alpha^*)}{\alpha^2 \sin(\pi\alpha)^2} \qquad \text{as } n \to +\infty\,;$$

(iii) if $\alpha = 1/2$ and $\sum_{n\geq 1} \frac{1}{\varphi(n)^2 n} = +\infty$, then we have

$$\mathbf{P}(\rho_1 = n) \sim (2\pi)^2 \Big(\sum_{k=1}^{n} \frac{1}{\varphi(k)^2 k} \Big)^{-2} \varphi(n)^{-2} n^{-1} \qquad \text{as } n \to +\infty\,.$$

As a nice application of item (iii), consider the case where $\tau = \{n, S_{2n} = 0\}$, with $(S_n)_{n\geq 0}$ the symmetric simple random walk on $\mathbb{Z}$. We have $\alpha = 1/2$ and $\lim_{n\to+\infty} \varphi(n) = 1/2\sqrt{\pi}$. Then, $\rho_1/2$ is the first simultaneous return time to 0 of two independent simple random walks on $\mathbb{Z}$: by a rotation of $\pi/4$, it has the same law as the first return time to 0 of the simple symmetric random walk on $\mathbb{Z}^2$. Item (iii) above gives that

$$\mathbf{P}(\rho_1 = n) \sim \frac{\pi}{n(\log n)^2} \qquad \text{as } n \to +\infty\,,$$

which recovers a classical result of Jain and Pruitt [JP72].

7.2 Random walks in the domain of attraction of an α-stable law

We now turn to the study of random walks on $\mathbb{Z}$: let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. $\mathbb{Z}$-valued r.v.s, with law denoted $\mathbf{P}$. This includes the case of renewal processes if $X_i \in \mathbb{N}$. We let $S_n := \sum_{i=1}^n X_i$, and we consider the case where $(S_n)_{n\geq 0}$ is in the domain of attraction of an α-stable distribution, with $\alpha \in (0,2)$: there are sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $(S_n - b_n)/a_n$ converges in distribution to a non-trivial α-stable distribution. From [Fel71, IX.8], a condition which is equivalent is that there is some slowly varying function $L(\cdot)$, such that as $x \to +\infty$, $\mathbf{P}(|X_1| > x) \sim L(x)x^{-\alpha}$ and

$$\mathbf{P}(X_1 > x) \sim pL(x)\,x^{-\alpha}\,, \qquad \mathbf{P}(X_1 < -x) \sim qL(x)\,x^{-\alpha}, \tag{7.14}$$

with $p+q = 1$. If $p = 0$ (or $q = 0$), then we interpret this as $o(L(x)x^{-\alpha})$. A renewal process verifying (7.1) with $\alpha \in (0,2)$ fits into this framework, with $p = 1$, $q = 0$, and $L(n) = \alpha^{-1}\varphi(n)$.

The scaling sequence a_n is defined up to asymptotic equivalence by

$$L(a_n)(a_n)^{-\alpha} \sim \tfrac{1}{n} \quad \text{as } n \to \infty\,. \tag{7.15}$$

The centering sequence b_n is defined by (see e.g. [Fel71, IX.8], in particular p. 315 (8.15))

$$\begin{aligned} &b_n \equiv 0 \quad \text{if } \alpha \in (0,1)\,; \qquad\qquad b_n = n\mathbf{E}[X_1] \quad \text{if } \alpha > 1\,; \\ &b_n = n\mu(a_n) \quad \text{with } \mu(x) = \mathbf{E}\big[X_1 \mathbf{1}_{\{|X_1|\leq x\}}\big] \quad \text{if } \alpha = 1\,. \end{aligned} \tag{7.16}$$

In this section, we review some results on $(S_n)_{n\geq 0}$, such as large deviations and local large deviations. The case $\alpha = 1$ had often been left aside in the literature because of additional technicalities: it is the object of my article [5], accepted for publication in *Probability Theory and Related Fields*, which proves many natural results that were surprisingly absent from the literature.

As an illustration of the subtleties that may appear in the case $\alpha = 1$, let us mention the following example. Assume that $\mathbf{P}(X_1 = 1) = \frac{1}{2}$ and that $\mathbf{P}(X_1 < -x) \sim (\log x)^{-2} x^{-1}$, with $\mathbf{E}[X_1] = 0$; we have $\mu(n) = -\mathbf{E}[X_1 \mathbf{1}_{\{X_1 < -n\}}] \sim 1/\log n$ as $n \to +\infty$. We get that $a_n \sim n/(\log n)^2$, and $b_n \sim n/\log n$. Since $(S_n - b_n)/a_n$ converges in distribution, we get that $S_n/b_n \to 1$ in probability. On the other hand, having $\mathbf{E}[X_1] = 0$ proves that the random walk $(S_n)_{n\geq 1}$ is recurrent on $\mathbb{Z}$. It therefore gives an example of a random walk on $\mathbb{Z}$ which is recurrent, but goes to $+\infty$ in probability!

7.2.1 Large deviations.

The first natural question is that of estimating the large deviation probabilities $\mathbf{P}(S_n - b_n \geq x)$, with $|x|/a_n \to +\infty$. This is given by a one-jump strategy, as standard in the case $\alpha \in (0,1)\cup(1,2)$, but appeared to be missing (in full generality) in the case $\alpha = 1$. This is one of the results in [5] which has the most applications.

Theorem 7.6 ([5], Theorem 2.1)**.** *With the notations above, we have*

$$\begin{aligned} \mathbf{P}(S_n - b_n > x) &\sim npL(x)x^{-\alpha}, \qquad && \text{as } x/a_n \to +\infty\,, \\ \mathbf{P}(S_n - b_n < -x) &\sim nqL(x)x^{-\alpha}, \qquad && \text{as } x/a_n \to +\infty\,. \end{aligned}$$

One possible application of this result concerns the first ascending and descending ladder epochs, $T_+ := \min\{n, S_n > 0\}$ and $T_- := \min\{n, S_n < 0\}$, cf. [5, § 3]. A central tool for finding the asymptotic behavior of $\mathbf{P}(T_\pm > n)$ is the Wiener-Hopf factorization (see e.g. [Fel71, XII.7]), which relates the generating function of $\mathbf{P}(T_- > n)$ to that of $n^{-1}\mathbf{P}(S_n \geq 0)$. Therefore, knowing the asymptotics of $\mathbf{P}(S_n > 0)$ and $\mathbf{P}(S_n < 0)$ gives access to the asymptotics of $\mathbf{P}(T_\pm > n)$.

As an example, consider the case where $\mathbf{P}(X_1 > x) \sim px^{-1}$ and $\mathbf{P}(X_1 < -x) \sim qx^{-1}$ with $p + q = 1$; we find that $\mu(n) \sim (p-q)\log n$. A consequence of [5, Thm. 3.2] is that there are some slowly varying functions $\bar{L}(\cdot), \hat{L}(\cdot)$ such that as $n \to +\infty$

$$\mathbf{P}(T_- > n) \sim \begin{cases} (\log n)^{-\frac{q}{p-q}}\, \bar{L}(\log n) & \text{if } p > q\,; \\ \frac{1}{n}(\log n)^{-1+\frac{p}{q-p}}\, \hat{L}(\log n) & \text{if } p < q\,. \end{cases} \tag{7.17}$$

Let us mention that if (7.14) holds with $\alpha > 2$, then $(S_n)_{n\geq 0}$ is in the Normal domain of attraction. In that case, we have some sort of crossover between a Cramér-type and a big-jump large deviation, see [Nag79, Thm. 1.9]: for $n \to +\infty$ and $x \geq \sqrt{n}$, we have

$$\mathbf{P}\big(S_n - n\mathbf{E}[X_1] > x\big) = (1+o(1))\,\mathbf{P}\big(Z > \tfrac{x}{\sigma\sqrt{n}}\big) + (1+o(1))\, pnL(x)x^{-\alpha}\,, \tag{7.18}$$

where $Z \sim \mathcal{N}(0,1)$. Hence, if $p > 0$, there is a crossover at $\sigma\sqrt{2-\alpha}\sqrt{n\log n}$, in the sense that the main term in (7.18) is the first one if $x \leq a\sqrt{n\log n}$ with $a < \sigma\sqrt{2-\alpha}$; the second one if $x \geq a\sqrt{n\log n}$ with $a > \sigma\sqrt{2-\alpha}$. (One can do more involved calculations, and find that the crossover occurs at $\sigma\sqrt{2-\alpha}\sqrt{n\log n} + \frac{\sigma}{\sqrt{2-\alpha}}\big(\log\log n + \log L(\sqrt{n\log n}) + O(1)\big)\sqrt{\frac{n}{\log n}}$.)

7.2.2 Local large deviations

The next question is that of the behavior of the local large deviation $\mathbf{P}(S_n - b_n = x)$, as $|x|/a_n \to +\infty$ (we do as if b_n is an integer). Such estimates appear central in the proof of renewal theorems, see e.g. [Don97, CD16]. In the case where x is of the order of a_n, then one can use Gnedenko's local limit theorem, cf. [GK54, Ch. 9, §50],

$$\sup_{x\in\mathbb{Z}} \Big| a_n \mathbf{P}\big(S_n - b_n = x\big) - g\big(\tfrac{x}{a_n}\big)\Big| \xrightarrow{n\to+\infty} 0\,, \tag{7.19}$$

where $g(\cdot)$ is the density of the limiting α-stable law. In the case $|x|/a_n \to +\infty$, recent results by Caravenna and Doney [CD16] give a very general bound in the case $\alpha \in (0,1)\cup(1,2)$. The case $\alpha = 1$ was again left aside in the literature, and we completed this gap in [5]. The general result is seen as a local version of Theorem 7.6 and can be stated as follows (we consider only the case $x/a_n \to +\infty$, the case $x/a_n \to -\infty$ being symmetric).

Theorem 7.7 ([5], Theorem 2.3)**.** *There exists a constant $C > 0$ such that for any $x \geq a_n$*

$$\mathbf{P}(S_n - b_n = x) \leq \frac{C}{a_n} nL(x)x^{-\alpha}\,. \tag{7.20}$$

In the case $p = 0$, we get that $\mathbf{P}(S_n - b_n = x) = o\big(\frac{1}{a_n} nL(x)x^{-\alpha}\big)$ as $\frac{x}{a_n} \to +\infty$.

Improved local large deviation. In order to improve the estimate in Theorem 7.7, some additional assumption on the (left or right) tail of X_1 is needed. For instance, Doney's condition [Don97] (Doney considered only the case $\alpha \in (0,1)$) is that there is a constant $C > 0$ such that

$$\mathbf{P}(X_1 = x) \leq CL(x)(1+x)^{-(1+\alpha)} \qquad \text{for all } x \in \mathbb{N}\,, \tag{7.21}$$

which is a (weak) "local" version (7.14). One can make another (stronger) assumption, analogous to Eq. (1.3) in [Don97]:

$$\mathbf{P}(X_1 = x) \sim p\alpha L(x)x^{-(1+\alpha)} \qquad \text{as } x \to +\infty\,. \tag{7.22}$$

If $p = 0$, this is interpreted as $o(L(x)x^{-(1+\alpha)})$. In [5], we prove the following result, which extends Doney's [Don97, Thm. A] to the whole range $\alpha \in (0,2)$—I was unable to find a reference for this result in the case $\alpha \in (1,2)$, let alone the case $\alpha = 1$.

Theorem 7.8 ([5], Theorem 2.4)**.**
If (7.21) holds, then there is a constant $C' > 0$ such that for any $x \geq a_n$,

$$\mathbf{P}(S_n - b_n = x) \leq C' nL(x)x^{-(1+\alpha)}\,. \tag{7.23}$$

If (7.22) holds, then as $n \to +\infty$, $x/a_n \to +\infty$

$$\mathbf{P}(S_n - b_n = x) \sim np\alpha L(x)x^{-(1+\alpha)}\,. \tag{7.24}$$

We considered only the right tail here, but similar assumptions can be made on the left tail, with similar consequences.

These results are useful when trying to find the asymptotic behavior of the Green function $G(x) := \sum_{k=1}^{+\infty} \mathbf{P}(S_k = x)$, as $x \to +\infty$. In the case $\alpha \in (0,1)$, this may be found in [Wil68, CD16]; in the case of a finite positive mean this is in [Fel71, XI.9]; in the case of a finite negative mean I proved it in [5] (even if there must be another reference I am not aware of). In the case $\alpha = 1$ with infinite mean, asymptotics of $G(x)$ as $x \to +\infty$ are given in [5], under some local conditions of the type (7.22)-(7.21).

7.3 Multivariate random walks and renewals

Let us now consider d-dimensional random walks, $d \geq 1$. Let $(\mathbf{X}_j)_{j\geq 0}$ be an i.i.d. sequence of $\mathbb{Z}^d$-valued r.v.s, and let $\mathbf{S}_n := \sum_{j=1}^n \mathbf{X}_j$. If $\mathbf{X}_1$ takes its values in $\mathbb{N}^d$, then $\mathbf{S}_n$ is called a *multivariate renewal process*, and we interpret $\mathbf{S} = \{\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \ldots\}$ as a random subset of $\mathbb{N}^d$ (with a slight abuse of notations): this is motivated by the study of the generalized Poland-Scheraga model of Chapter 3. One of our goal is to give the asymptotic behavior of the Green function $G(\mathbf{x}) := \sum_{n=1}^{+\infty} \mathbf{P}(\mathbf{S}_n = \mathbf{x})$ as $\|\mathbf{x}\|_1 \to +\infty$; if $(\mathbf{S}_n)_{n\geq 1}$ is a renewal process, $G(\mathbf{x})$ is the renewal mass function. The literature is quite vast on the matter, but many questions had remained open: this is the core subject of my article [7], published in Electronic Journal of Probability.

We assume that $(\mathbf{S}_n)_{n\geq 0}$ is in the domain of attraction of a d-dimensional α-stable law, $\alpha \in (0,2)$: there is some sequence $\mathbf{b}_n = (b_n^{(1)}, \ldots, b_n^{(d)})$ and some sequence a_n such that $\frac{1}{a_n}(\mathbf{S}_n - \mathbf{b}_n)$ converges in distribution to a non-trivial multivariate stable law $\mathbf{Z}$, whose density is denoted $g(\cdot)$ (multivariate domains of attraction are studied in [Rva61]). In [7], we even allow for different scaling sequences along the different coordinates (with an index α_i for each coordinate): this is known as *generalized* domains of attractions, see e.g. [MS01]—we also allow for random walks in the Normal domain of attraction. Here, for the clarity and the simplicity of the exposition, we restrict ourselves to the "balanced" α-stable case, $\alpha \in (0,2)$: in particular, each coordinate $S_n^{(i)}$ is in the domain of attraction of a univariate α-stable distribution, and the scaling sequence a_n is taken to be the same for all coordinates (but we do not assume that all coordinates have the same law). The bivariate renewal $\boldsymbol{\tau}$ considered in Chapter 3 fits into this framework (if $\alpha \in (0,2)$), see (3.1). On the other hand, in Section 3.2, a bivariate renewal with exponential tail in the first coordinate and polynomial tail in the second one is studied (in a large deviation regime only): it is not "balanced", and this was one of the motivations for considering the more general setting of generalized domains of attraction in [7].

Some notation. There is some slowly varying $L(\cdot)$, and constants $p_i, q_i \geq 0$ (with $p_i + q_i > 0$) such that for all $i \in \{1, \ldots, d\}$ we have

$$\mathbf{P}(X_1^{(i)} > x) \sim p_i L(x) x^{-\alpha} \quad \text{and} \quad \mathbf{P}(X_1^{(i)}) \sim q_i L(x) x^{-\alpha} \quad \text{as } x \to +\infty\,. \tag{7.25}$$

This is equivalent to the i^{th} coordinate $(S_n^{(i)})_{n\geq 1}$ being in the domain of attraction of an α-stable law, $\alpha \in (0,2)$. However, for having that $(\mathbf{S}_n)_{n\geq 0}$ is in the domain of attraction of a multivariate α-stable, a stronger condition than (7.25) is needed: the condition is related to the *multivariate* regular variation of the distribution of $\mathbf{X}_1$, see e.g. [Mee91].

The scaling sequence a_n is defined as in (7.15), up to asymptotic equivalence, by

$$L(a_n)(a_n)^{-\alpha} \sim \tfrac{1}{n} \quad \text{as } n \to +\infty\,. \tag{7.26}$$

As far as the recentering sequence $\mathbf{b}_n = (b_n^{(1)}, \ldots, b_n^{(d)})$ is concerned, we set as in (7.16)

$$\mathbf{b}_n \equiv \mathbf{0} \quad \text{if } \alpha \in (0,1); \quad \mathbf{b}_n := n\boldsymbol{\mu} \quad \text{if } \alpha > 1; \quad \mathbf{b}_n = n\boldsymbol{\mu}(a_n) \quad \text{if } \alpha = 1. \tag{7.27}$$

Here, we defined $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_d)$ with $\mu_i := \mathbf{E}[X_1^{(i)}]$ when $\sum_x L(x) x^{-1} < +\infty$; and $\boldsymbol{\mu}(a_n) := (\mu_1(a_n), \ldots, \mu_d(a_n))$ with $\mu_i(x) := \mathbf{E}[X_1^{(i)} \mathbf{1}_{\{|X_1^{(i)}| \leq x\}}]$. In the following, we do as if $\mathbf{b}_n \in \mathbb{Z}^d$.

7.3.1 Local large deviations

In order to prove renewal theorems, *i.e.* find sharp asymptotics of the Green function $G(\mathbf{x})$, a method is to use local large deviations estimates. Indeed, one divides the sum into

$$G(\mathbf{x}) := \sum_{n\geq 1} \mathbf{P}(\mathbf{S}_n = \mathbf{x}) = \sum_{n<n_0-m_0} \mathbf{P}(\mathbf{S}_n = \mathbf{x}) + \sum_{n=n_0-m_0}^{n_0+m_0} \mathbf{P}(\mathbf{S}_n = \mathbf{x}) + \sum_{n>n_0+m_0} \mathbf{P}(\mathbf{S}_n = \mathbf{x}) \,. \tag{7.28}$$

Here, $n_0 = n_0(\mathbf{x})$ and m_0 are chosen so that $\mathbf{x} - \mathbf{b}_n = O(a_n)$ for all $n \in [n_0 - m_0, n_0 + m_0]$, *i.e.* so that $\mathbf{P}(\mathbf{S}_n = \mathbf{x})$ falls into the range of the local limit theorem [Rva61]

$$\sup_{\mathbf{x}\in\mathbb{Z}^d} \Big| (a_n)^d \mathbf{P}\big(\mathbf{S}_n = \mathbf{x}\big) - g\big(\tfrac{\mathbf{x}-\mathbf{b}_n}{a_n}\big) \Big| \xrightarrow{n\to+\infty} 0 \,. \tag{7.29}$$

The local limit theorem (7.29) enables us to deal with the middle sum in (7.28), but for the first and the last sum, one needs some estimates on $\mathbf{P}(S_n = \mathbf{x})$ when $\|\mathbf{x} - b_n\|_1/a_n \to +\infty$, in analogy with Theorems 7.7- 7.8 in the univariate setting.

For a (simple) large deviations estimate, we can use univariate large deviations (see Theorem 7.6): there is a constant C such that for any $\mathbf{x} = (x_1, \ldots, x_d) \geq \mathbf{0}$,

$$\mathbf{P}\big(\mathbf{S}_n - \mathbf{b}_n \geq \mathbf{x}\big) \leq C \min_{i\in\{1,\ldots d\}} \big\{ nL\big(x_i\big)\, x_i^{-\alpha} \big\}, \tag{7.30}$$

where the inequality $\mathbf{S}_n - \mathbf{b}_n \geq \mathbf{x}$ is componentwise. A local version of (7.30) can be proven, analogously to Theorem 7.7.

Theorem 7.9 ([7], Theorem 2.1)**.** *There is a constant C' such that, for $i \in \{1, \ldots, d\}$, for any $\mathbf{x}$ with $x_i \geq a_n$,*

$$\mathbf{P}\big(\mathbf{S}_n - \mathbf{b}_n = \mathbf{x}\big) \leq C'(a_n)^{-d}\, nL(x_i)\, x_i^{-\alpha} \,. \tag{7.31}$$

The bound is changed to $o(n(a_n)^{-d} L(x_i) x_i^{-\alpha})$ if $p_i = 0$ in (7.25).

Improved local large deviations. In order to improve the bound in Theorem 7.9, analogously to (7.21), we need an additional local assumption on the tail of $\mathbf{X}_1$. Williamson [Wil68, Eq. (3.10)] considers the condition that there is some constant $C > 0$ such that for all $\mathbf{x} \in \mathbb{Z}^d$

$$\mathbf{P}(\mathbf{X}_1 = \mathbf{x}) \leq CL(\|\mathbf{x}\|_1) \|\mathbf{x}\|_1^{-(d+\alpha)} \,. \tag{7.32}$$

We mention that the bivariate renewal $\boldsymbol{\tau}$ of Chapter 3 satisfies this assumption, see (3.1). Under (7.32), one gets improved local large deviations.

Theorem 7.10 ([7], Theorem 2.4)**.** *Suppose that (7.32) holds. There is a constant C', such that for any $\mathbf{x}$ with $\|\mathbf{x}\|_1 \geq a_n$, we have*

$$\mathbf{P}\big(\mathbf{S}_n - \mathbf{b}_n = \mathbf{x}\big) \leq C' n\, L(\|\mathbf{x}\|_1)\, \|\mathbf{x}\|_1^{-(d+\alpha)} \,.$$

Let us mention that Williamson's condition (7.32) is not completely satisfying: for instance, it does not allow the different components to be independent, or to have different tails. In [7], we devise a more general condition which enables us to obtain "improved" local large deviations (different than Theorem 7.10) in the case of generalized domains of attractions. We do not go into the details, but Assumption 2.2 in [7] is quite intricate, and it implies that Doney's condition (7.21) holds for every component (it is essentially Doney's condition, with an additional twist to ensure some summability over other coordinates). Note that for all renewal theorems that we present below, we actually use Assumption 2.2 of [7] instead of (7.32).

7.3.2 Renewal theorems

In view of (7.28), we realize that the main contribution to $G(\mathbf{x})$ should come from some range $n = n_0 + O(m_0)$, where n_0 is the typical number of jumps needed for the random walk to reach $\mathbf{x}$. There may however be two cases. A first case is when $\mathbf{x}$ is "reachable" by a typical random walk, in the sense that there exists some n_0 such that $\|\mathbf{x} - \mathbf{b}_{n_0}\|_1 = O(a_{n_0})$: we then say that $\mathbf{x}$ belongs to the *favorite* direction or scaling. A second case is when $\mathbf{x}$ cannot be reached easily by the random walk, in the sense that there is no n such that $\|\mathbf{x} - \mathbf{b}_n\|_1 = O(a_n)$: we say that $\mathbf{x}$ is away from the favorite direction—in other words, reaching $\mathbf{x}$ is in the large deviation regime for the random walk, whatever n is. In [7], we obtain the sharp asymptotics of $G(\mathbf{x})$ in the favorite direction or scaling (extending some results in the literature), as well as good upper bounds on $G(\mathbf{x})$ when $\mathbf{x}$ is away from it. We distinguish three different cases, that we treat separately:

I. *Centered*: $\mathbf{b}_n \equiv \mathbf{0}$ (for instance if $\alpha \in (0,1)$, or if $\alpha > 1$ and $\boldsymbol{\mu} = 0$). The *favorite scaling* are the points of the type $n\mathbf{t}$ with $t \in (\mathbb{R}^*)^d$.

II. *Non-zero mean*: $\alpha > 1$ and $\boldsymbol{\mu} \neq \mathbf{0}$. The *favorite direction* are the points of the type $n\boldsymbol{\mu}+O(a_n)$.

III. *Marginal*: $\alpha = 1$, with $\mathbf{b}_n \not\equiv \mathbf{0}$. The favorite direction are the points of the type $\mathbf{b}_n + O(a_n)$.

We review the results in cases I-II-III: each time, we give the sharp behavior in the favorite direction or scaling, as well as some some general bounds which become good when moving away from it (in dimension $d = 2$ for simplicity). To lighten notations, we omit the integer parts and always assume that $\mathbf{x} \in \mathbb{Z}^d$.

Case I. *Centered.* When $\mathbf{b}_n \equiv \mathbf{0}$, the random walk does not drift away. We consider the case $\alpha < \min(2, d)$, which ensures that the random walk $(\mathbf{S}_n)_{n\geq 0}$ is transient—we have that $\mathbf{P}(\mathbf{S}_n = \mathbf{0}) \asymp a_n^{-d}$ thanks to the local limit theorem (7.29), and this is summable if $d/\alpha > 1$.

Theorem 7.11 ([7], Theorems 3.1 and 4.1)**.** *Suppose that $\mathbf{b}_n \equiv \mathbf{0}$ and that $\alpha < \min(2, d)$. Assume additionally either that $\alpha > d/2$ or that (7.32) holds.*
(i) Favorite scaling. Let $\mathbf{x}_n = \mathbf{x}_n(\mathbf{t}) := n\mathbf{t}$, for some $\mathbf{t} = (t_1, \ldots, t_d) \in (\mathbb{R}^*)^d$. Then, as $n \to +\infty$

$$G(\mathbf{x}_n) \sim \mathtt{C}_{\mathbf{t}}\, n^{\alpha-d} L(n)^{-1}, \qquad \text{with } \ \mathtt{C}_{\mathbf{t}} = \alpha \int_0^\infty u^{d-1-\alpha} g(u\mathbf{t}) \mathrm{d}u\,, \tag{7.33}$$

(ii) General bound. Assume that $d = 2$ for the simplicity of the statement. If (7.32) holds, for any $\delta > 0$ there is a constant C_δ such that for any sequence $t_n \geq 1$, letting $\mathbf{x}_n = (n, nt_n)$

$$G(\mathbf{x}_n) \leq C_\delta n^{\alpha-2} L(n)^{-1} \times (t_n)^{-\theta+\delta} \quad \text{with } \theta := (1+\alpha)\, \tfrac{2-\alpha}{2+\alpha}\,. \tag{7.34}$$

If $(\mathbf{S}_n)_{n\geq 0}$ is a renewal process, θ can be replaced by $1 + \alpha$.

Note that the general bound (7.34) improves (7.33) in the case where $t_n \to +\infty$. We stress that (7.33) was obtained in [Wil68] under the conditions of Theorem 7.11. In [7] however, our Assumption 2.2 is weaker than (7.32), and we also treat the case of generalized domains of attraction.

Case II. *Non-zero mean.* When $\alpha \in (1, 2)$ with $\boldsymbol{\mu} \neq \mathbf{0}$, the random walk drifts in the direction of $\boldsymbol{\mu}$. We obtain in [7] a strong renewal theorem when $\mathbf{x}$ is along that favorite direction: this appears to be a new result in the case $\alpha \in (1, 2)$—it was only proven for random walks with finite variance (see e.g. [Nag80]).

Theorem 7.12 ([7], Theorems 3.3 and 4.2)**.** *Suppose that $\alpha \in (1,2)$ and $\boldsymbol{\mu} \neq \mathbf{0}$. Assume either that $\alpha > d/2$ or that (7.32) holds.*
(i) Favorite direction. Let $\mathbf{x}_n = \mathbf{x}_n(\mathbf{t}) := n\boldsymbol{\mu} + a_n \mathbf{t}$, for some $\mathbf{t} \in \mathbb{R}^d$. Then, as $n \to +\infty$

$$G\big(\mathbf{x}_n\big) \sim \mathtt{C}'_{\mathbf{t}}\, (a_n)^{-(d-1)} \qquad \text{with } \ \mathtt{C}'_{\mathbf{t}} = \int_{-\infty}^{+\infty} g\big(\mathbf{t} + u\boldsymbol{\mu}\big) du\,. \tag{7.35}$$

(ii) General bound. Assume that $d=2$, and that $\mu_1, \mu_2 \neq 0$. If (7.32) holds, for any $\delta > 0$ there is a constant C_δ such that for any sequence $t_n \geq 1$, letting $\mathbf{x}_n = (\mu_1 n, \mu_2 n + t_n a_n)$

$$G\big(\mathbf{x}_n\big) \leq C_\delta\, (a_n)^{-1} \times (t_n)^{-(1+\alpha)+\delta}\,. \tag{7.36}$$

Again, (7.36) improves (7.35) when $t_n \to +\infty$. (A different general bound is found in the case where $\mu_1 = 0$ or $\mu_2 = 0$, see [7, Thm. 4.3].) This result is useful for instance when studying the intersection of two independent (multivariate) renewal process, which appears when computing the second moment of the partition function of the generalized Poland-Scheraga model, see Section 3.3. Indeed, (7.35) estimates the contribution from points in the favorite direction, and (7.36) allows to control the contribution from points away from it, we refer to Section 7.3.3 below for details.

Case III: $\alpha = 1$, $\mathbf{b}_n \not\equiv \mathbf{0}$. This case was systematically left aside in the literature, and is studied in detail in [7]. For the simplicity of exposition, we state only results in the symmetric case, *i.e.* when all the coordinates have the same distribution. In particular, we have $p_i \equiv p$ and $q_i \equiv q$ in (7.25), and also $\mu_i(x) \equiv \mu(x)$, $\mathbf{b}_n = b_n \mathbf{1}$ with $b_n := n\mu(a_n)$ (we use the notation $\mathbf{1} = (1, \ldots, 1)$).

We assume that either $\sum_{i\geq 1} L(i) i^{-1} < +\infty$ and $\mu := \lim_{n\to+\infty} \mu(n) \neq 0$, or that $\mu(n) \to +\infty$ with $p > q$ so that $\mu(n) \sim (p-q) \sum_{i=1}^n L(i) i^{-1}$. It includes in particular the case of the bivariate renewal process $\boldsymbol{\tau}$ considered in Chapter 3 with $\alpha = 1$, cf (3.1). In all cases, one can show that $b_n \gg a_n$, and the random walks drifts in the direction $\mathbf{b}_n = b_n \mathbf{1}$. We denote $k_n := n/\mu(n)$, which is the typical number of steps to reach distance n, *i.e.* such that $b_{k_n} \sim n$ as $n \to +\infty$.

Theorem 7.13 ([7], Theorems 3.4 and 4.2)**.** *Assume that $\alpha = 1$ with $\mathbf{X}_1$ symmetric, and that either $\mu \in \mathbb{R}^*$ or $\mu(n) \sim (p-q) \sum_{i=1}^n L(i) i^{-1} \to +\infty$ with $p > q$. Suppose that (7.32) holds.*
(i) Favorite direction. Let $\mathbf{x}_n := n\mathbf{1} + a_{k_n} \mathbf{t}$ for some $\mathbf{t} \in \mathbb{R}^d$ ($k_n = n/\mu(n)$). Then as $n \to +\infty$

$$G\big(\mathbf{x}_n\big) \sim \mathtt{C}''_{\mathbf{t}}\, \mu(n)^{-1} (a_{k_n})^{-(d-1)} \quad \text{with } \ \mathtt{C}''_{\mathbf{t}} = \int_{-\infty}^{\infty} g\big(\mathbf{t} + u\mathbf{1}\big) \mathrm{d}u\,. \tag{7.37}$$

(ii) General bound. Assume that $d = 2$ for simplicity. For any $\delta > 0$ there is some C_δ such that for any sequence $t_n \geq 1$, letting $\mathbf{x}_n = (n, n + a_{k_n} t_n)$

$$G\big(\mathbf{x}_n\big) \leq C_\delta\, \mu(n)^{-1} (a_{k_n})^{-1} \times (t_n)^{-2+\delta}\,. \tag{7.38}$$

Note that in the finite mean case, one recovers the same results as in Theorem 7.12. We now explain where the scaling $\mu(n)^{-1}(a_{k_n})^{-(d-1)}$ comes from: in $G(n\mathbf{1}) = \sum_{k=1}^{+\infty} \mathbf{P}(\mathbf{X}_1 = n\mathbf{1})$, the main contribution comes from the terms $k = k_n + O(m_n)$, where $m_n = a_{k_n}/\mu(a_{k_n})$. Indeed, we have $b_{k_n + m_n} - b_{k_n} \asymp m_n \mu(a_{k_n}) \asymp a_{k_n}$, so $k = k_n + O(m_n)$ is the exact range of k for which we have $n = b_k + O(a_k)$, see [7, § 8.1]. Therefore, the main contribution to the sum consists of $a_{k_n}/\mu(a_{k_n})$ terms, all of order $(a_{k_n})^{-d}$ by the local limit theorem (7.29): it gives that $G(n\mathbf{1})$ is of the order of $(a_{k_n})^{1-d}/\mu(a_{k_n})$, which is the correct order in (7.37) since $\mu(a_{k_n}) \sim \mu(b_{k_n}) \sim \mu(n)$, see [5, Lem. 4.3].

As an illustration of Theorem 7.13 let us consider the case of a d-dimensional renewal process similar to that of Chapter 3, with $\mathbf{P}(\mathbf{X}_1 = \mathbf{x}) \sim (\log \|\mathbf{x}\|_1)^a \|\mathbf{x}\|_1^{-(1+d)}$ as $\|\mathbf{x}\|_1 \to +\infty$, for some $a \in \mathbb{R}$. We have $a_n \sim c' n (\log n)^a$ and $\mu(n) \sim c''(\log n)^{1+a}$ if $a > -1$; $\lim_{n\to+\infty} \mu(n) < +\infty$ if $a < -1$. We therefore get from Theorem 7.13 that for $\mathbf{t} \in \mathbb{R}^d$, as $n \to +\infty$,

$$\begin{aligned} &\text{if } a > -1, \qquad G\big(n\mathbf{1} + \tfrac{n}{\log n}\mathbf{t}\big) \sim \mathsf{C}''_{\mathbf{t}}\, (\log n)^{d-2-a} n^{-(d-1)}\,; \\ &\text{if } a < -1, \qquad G\big(n\mathbf{1} + \tfrac{n}{(\log n)^a}\mathbf{t}\big) \sim \mathsf{C}''_{\mathbf{t}}\, (\log n)^{-a(d-1)} n^{-(d-1)}\,. \end{aligned} \tag{7.39}$$

The bound (7.38) improves this when going away from the favorite direction: this is useful when studying the intersection of two bivariate renewals, see Section 7.3.3 below.

7.3.3 Intersection of two independent bivariate renewals

Let us now comment on the implication of the above renewal theorems on the intersection of two independent bivariate renewals. We consider $\boldsymbol{\tau}, \boldsymbol{\tau}'$ two independent copies of a renewal process satisfying $\mathbf{P}(\boldsymbol{\tau} = (n,m)) = \varphi(n+m)(n+m)^{-(2+\alpha)}$, as in Chapter 3, cf. (3.1). Then, we wish to study $D_n := \sum_{i,j=1}^n \mathbf{P}((i,j) \in \boldsymbol{\tau})^2$, the mean overlap of $\boldsymbol{\tau}, \boldsymbol{\tau}'$ up to length n, as needed in Section 3.3. Let us rather consider here $\widehat{D}_n := \sum_{i=1}^n \sum_{r=1}^n \mathbf{P}((i, i+r) \in \boldsymbol{\tau})^2$, which has the same behavior as D_n.

- If $\alpha \in (0,1)$, we use Theorem 7.11 to get that

$$\widehat{D}_n \le C \sum_{i=1}^n \sum_{r=0}^i i^{2\alpha-4} L(i)^{-2} + C \sum_{i=1}^n \sum_{r=i}^{+\infty} i^{2\alpha-4} L(i)^{-2} \Big(\frac{r}{i}\Big)^{-2} \le C' \sum_{i=1}^n i^{2\alpha-3} L(i)^{-2}\,.$$

To bound $\mathbf{P}((i, i+r) \in \boldsymbol{\tau})$, we used (7.33) in the first sum and (7.34) in the second sum. Since $\alpha < 1$, we therefore get that $\sup_{n\in\mathbb{N}} \widehat{D}_n < +\infty$ and $\boldsymbol{\tau} \cap \boldsymbol{\tau}'$ is terminating.

- If $\alpha \in (1,2)$, we use Theorem 7.12 to get that

$$\widehat{D}_n \le \sum_{i=1}^n \sum_{r=0}^{a_i} \frac{1}{(a_i)^2} + C \sum_{i=1}^n \sum_{r=a_i}^{+\infty} \frac{1}{(a_i)^2} \Big(\frac{r}{a_i}\Big)^{-2} \le C' \sum_{i=1}^n \frac{1}{a_i}\,,$$

and a corresponding lower bound holds. To bound $\mathbf{P}((i, i+r) \in \boldsymbol{\tau})$, we used (7.35) in the first sum, and (7.36) in the second sum: the main contribution to $\widehat{D}_n$ comes from the terms $r = O(a_i)$. Since a_i is regularly varying with exponent $1/\alpha < 1$, this gives that $\widehat{D}_n = n^{\frac{\alpha-1}{\alpha}+o(1)}$ as $n \to +\infty$, and $\boldsymbol{\tau} \cap \boldsymbol{\tau}'$ is persistent. A similar idea works in the case $\alpha \ge 2$: the normalizing sequence is then $a_n = n^{\frac12+o(1)}$, and we get that $\widehat{D}_n = n^{\frac12+o(1)}$ as $n \to +\infty$, cf. [3, Prop. A.3].

- If $\alpha = 1$, then one uses Theorem 7.13 (both (7.37) and (7.38)), to get that

$$\widehat{D}_n \le \sum_{i=1}^n \sum_{r=0}^{a_{k_i}} \frac{1}{\mu(i)^2 (a_{k_i})^2} + C \sum_{i=1}^n \sum_{r=a_{k_i}}^{+\infty} \frac{1}{\mu(i)^2 (a_{k_i})^2} \Big(\frac{r}{a_{k_i}}\Big)^{-2} \le C' \sum_{i=1}^n \frac{1}{\mu(i)^2 a_{k_i}}\,.$$

A corresponding lower bound holds. Therefore, we get that $\boldsymbol{\tau} \cap \boldsymbol{\tau}'$ is terminating if and only if $\sum_{i\ge1} \frac{1}{\mu(i)^2 a_{k_i}} < +\infty$, which is equivalent to $\sum_{i\ge1} \frac{1}{i\varphi(i)\mu(i)} < +\infty$, see [3, Rem. A.7]. As an example, if $\varphi(n) \sim (\log n)^a$ for some $a \in \mathbb{R}$, then we get that $|\boldsymbol{\tau} \cap \boldsymbol{\tau}'| < +\infty$ a.s. if and only if $a > 0$ (recall $\mu(n) \sim c(\log n)^{1+a}$ if $a > -1$).

A few perspectives of research

Throughout this manuscript, I alluded to several ongoing works, and mentioned some interesting open problems. Let me highlight again here, briefly, a few problems that I consider as appealing directions of research.

The critical behavior of the disorder relevant pinning model

The main (and most important) question that remains for the disordered pinning model of Chapter 2 is to describe the critical behavior of the system, in case of a relevant disorder. The Derrida-Retaux model allows us to make some predictions, namely that $\mathtt{F}(\beta, h_c(\beta)+u) = \exp(-(1+o(1))\frac{K}{\sqrt{u}})$, recall Section 2.3.2. The intense activity and great progress around Derrida-Retaux's model, cf. [HS18, HMP18, CDD$^+$19], brings new light on the mechanisms at stake, and gives hope for some progress on the pinning model. In particular, finding any strategy to improve the "smoothing inequality" of Theorem 2.3 in the relevant disorder regime would be a great achievement. One related question that I mention in Section 2.2.2 is that of determining a necessary and sufficient condition for the existence of a phase transition for the pinning model, *i.e.* of knowing whether $h_c(\beta) > -\infty$ or not.

Scaling limits of correlated disordered systems

In Chapter 3, I outline some ongoing work with my Ph.D. student Alexandre Legrand: the study of the gPS model has led us to consider the weak-coupling scaling limit of a disordered system with long-range correlations. There are some technicalities that need to be overcome, but we have good hope of proving the convergence of the gPS model at weak-coupling to a *disordered* continuum model, with underlying randomness given by a correlated Gaussian field (represented in Figure 3.4).

Our next goal is to consider the pinning model (and the copolymer and directed polymer models) with disorder given by a correlated Gaussian sequence $(\varpi_i)_{i\geq 1}$ as in Section 1.3.3, with correlation function $\rho_i := \mathbb{E}[\varpi_0 \varpi_i] \sim i^{-a}$ as $i \to +\infty$, for some $a > 0$. For the question of disorder relevance with a correlated disorder of that type, one turns to the predictions made by Weinrib and Halperin [WH83], that extend those made by Harris [Har74] in the case of an i.i.d. disorder. The predictions are that disorder should be irrelevant if $\nu > 2/\min(a, \mathtt{d})$ and relevant if $\nu < 2/\min(a, \mathtt{d})$, where ν is the correlation length critical exponent of the homogeneous system, and $\mathtt{d}$ is the dimension of the system (for the pinning model, $\mathtt{d} = 1$ and $\nu = \max(1, \frac{1}{\alpha})$): in particular, Harris criterion should remain valid when $a > \mathtt{d}$, and should be modified when $a < \mathtt{d}$. During my Ph.D., I showed that for the pinning model, disorder is *always* relevant when $a < 1$, contradicting those predictions, cf. [22]—this is due to the appearance of very large favorable regions whose size do not depend in general only on the two-point correlation function, see [21]. However, if one scales $\beta_n, h_n \downarrow 0$ in the appropriate manner, *i.e.* taking the intermediate disorder limit of the correlated pinning model, we believe that one could show that the scaling limit is non-trivial if $\alpha \in (\frac{a\wedge 1}{2}, 1)$, and trivial if $\alpha < \frac{a\wedge 1}{2}$. This would prove disorder relevance/irrelevance in the sense of Caravenna-Sun-Zygouras [CSZ16], and go in the direction of Weinrib-Halperin's predictions. If $a \in (0,1)$ and $\alpha \in (\frac{a}{2}, 1)$, one should also find that the weak-coupling scaling limit of the model is a disordered continuum model, with underlying randomness given by a fractional Brownian motion (with Hurst exponent $1 - \frac{a}{2}$).

More general polymers in heavy-tail random environment

As mentioned in Section 5.4, the results of Chapter 5 should extend to the case of more general underlying random walks. For instance, one wishes to consider random walks with unbounded jumps, with $\mathbf{P}(S_1 = k) \sim e^{-|k|^a}$ for some $a > 0$. The phase diagram is then expected to be the one described in Figure 5.3. The corresponding Entropy-controlled LPP is well defined, cf. Section 6.3.1, so all the technical tools to mimic the proofs of [4] seem to be at hand—there are however some complications, due to the fact that the scaling limits of random walk paths are not necessarily continuous. Another interesting problem is to consider a non-directed version of the model, where the polymer is given by a (non-directed) random walk, as mentioned in Section 5.4, see also [Hua19]: here again, the corresponding Entropy-controlled LPP is well defined, so aside from technical considerations, we should also be able to treat this case. Moreover, the non-directed polymer model in random environment of [Hua19] raises many questions, as that of the super-diffusivity and scaling limit of the trajectories, the dependence on the dimension being different than for the directed polymer model of Chapter 4 (disorder should be relevant if $d \leq 4$ and irrelevant if $d \geq 5$). We are currently investigating these questions, together with Niccolò Torri.

Heavy-tail last-passage percolation

In Chapter 5, we consider the directed polymer model in heavy-tail random environment, with tail decay exponent $\alpha \in (0,2)$. Our main objective is now to make some progress on the case $\alpha \geq 2$, and specifically on the case $\alpha \in (2,5)$. However, this question appears really difficult: when $\alpha > 2$ the main contribution to the partition function comes from small weights, and larges weights are believed to be driving the fluctuations. It is not clear what the correct recentering should be, and we also do not have any real idea on what the limiting object should be. This is why, as a first step, together with Niccolò Torri and Nikos Zygouras we are focusing on the case $\alpha = 2$, in the "simpler" setting of last-passage percolation, see Section 6.2.1. This case appears to be already very rich, and it would be a great achievement to be able to prove Conjecture 6.4—it would also possibly give some insight on the case $\alpha > 2$.

Additionally, the methods developed should enable us to make progress on a long-standing conjecture for last-passage percolation. Define L_n as in (4.2), $L_n := \max_s \sum_{i=1}^n \omega_{i,s_i}$, where $(\omega_{i,j})_{i\in\mathbb{N},j\in\mathbb{Z}}$ is a field of i.i.d. r.v.s. It is known that $\frac{1}{n}L_n$ converges a.s. as $n \to +\infty$, and that the limit is finite if $\int_0^\infty \mathbb{P}(\omega > t)^{1/2}\mathrm{d}t < +\infty$, and infinite if $\mathbb{E}[\omega^2] = +\infty$, see [Mar06] for a survey. A necessary and sufficient condition for $\lim_{n\to+\infty} \frac{1}{n}L_n$ to be finite is believed to be that $\mathbb{E}[\omega^2] < +\infty$, but no progress has been made over the past fifteen years—it is plausible that understanding the behavior of last-passage percolation in heavy-tail (Poissonian) environment with $\alpha = 2$ would be useful to attack that problem.

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