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TOPICS IN RANDOM WALKS

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Abstract. We collect a few recent results on random walks, which are ubiquitous in probability theory. The topics covered are: persistence problems for stochastic processes, large fluctuations in multi-scale modeling for rest hematopoiesis, and fine properties of the elephant random walk.

1. General introduction

We present here some recent results on random walks. These results were presented in the session "Random walks" of the Journées MAS 2022, which consisted of four talks by Quentin Berger, Céline Bonnet, Lucile Laulin and Loïc de Raphélis. The talks covered a wide range of topics, reflecting the diversity in this field. The present article contains extended abstracts of three of these talks. We now give an outlook of their content.

In Section 2, Quentin Berger presents an overview on the question of persistence (or survival) for one-dimensional stochastic processes—indexed either by discrete of continuous time. This question has several applications, in particular in physics, and has been studied for a very long time: generally speaking, the main goal consists in estimating the asymptotic behavior of the probability that the process remains above a given barrier for a long time t. The main focus here is on the persistence problem for processes $(\zeta_t)_{t\geq 0}$ that are additive functionals of a Markov process, such as the integral of a function of a Markov process. The author reviews existing and recent results on the persistence probabilities of such processes: more specifically, he shows that, for a large class, the persistence probability decays like $t^{-\theta}$ as $t\to\infty$, for some exponent $\theta>0$. He also provides some ideas of the proof, the goal being to explain how the exponent θ appears and how it is related to the underlying Markov process and the associated integration function.

We now move to Section 3. Hematopoiesis is a biological phenomenon (process) of production of mature blood cells by cellular differentiation. It is based on amplification steps due to an interplay between renewal and differentiation in the successive cell types from stem cells to mature blood cells. Céline Bonnet presents here a published work, [12], in which she studies this mechanism with a stochastic point of view to explain unexpected fluctuations on the mature blood cell numbers, as surprisingly observed by biologists and medical doctors in a rest hematopoiesis. The author consider three cell types: stem cells, progenitors and mature blood

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cells. Each cell type is characterized by its own dynamics parameters: its division rate and by the renewal and differentiation probabilities at each division event. Céline Bonnet models the global population dynamics by a three-dimensional stochastic decomposable branching process. She shows that the amplification mechanism is given by the inverse of the small difference between the differentiation and renewal probabilities. Introducing a parameter K which scales simultaneously the size of the first component, the differentiation and renewal probabilities and the mature blood cell death rate, she describes the asymptotic behavior of the process for large K. Finally, the author shows that each cell type has its own size scale and its own time scale. Focusing on the third component, she proves that the mature blood cell population size, conveniently renormalized (in time and size), is expanded in a usual way inducing large fluctuations.

The Elephant Random Walk was introduced in 2004 as a variation of the simple random walk on $\mathbb{Z}$ with memory. Instead of choosing its next step independently of its previous movements, the elephant uniformly selects a time k from its past. For a given memory parameter p, it repeats what it did at that time with probability p, or does the opposite with probability $1-p$. This memory-dependent behavior results in three distinct regimes: diffusive, critical, and superdiffusive.

In Section 4, Lucile Laulin presents how this process can be studied from three different approaches: using a well-chosen martingale, establishing a connection with Polya-type urns, or considering it as a special case of step-reinforced random walk, which leads to a representation as random recursive trees with Bernoulli bond percolation.

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2. Persistence problems for (integrated) stochastic processes

Quentin Berger

The goal of this contribution is to give an overview of old and recent results on a very classical topic in probability theory and in theoretical physics: the question of persistence for (one-dimensional) stochastic processes, also known as survival problems. Given a stochastic process $(\zeta_t)_{t\geq 0}$ with $\zeta_0 = 0$, which can be either in discrete or continuous time, the problem consists in estimating the probability that the process remains above a given barrier (i.e., survives) for a long period of time. The most natural (and simple) example is when the barrier is fixed through time: one wants to estimate the probability

$$\mathbb{P}\big(\zeta_s > z \text{ for all } 0 \leq s \leq t\big) = \mathbb{P}\big(T_z > t\big) \tag{1}$$

as $t \to \infty$, for $z < 0$ fixed, where $T_z := \inf\{s \geq 0, \zeta_s \leq z\}$ is the hitting time of level z; one may also study the probability $\mathbb{P}(\zeta_s \geq 0 \text{ for all } 0 \leq s \leq t)$. In many cases of interest, the probability (1) decays polynomially, that is $\mathbb{P}(T_z > t) = t^{-\theta+o(1)}$ as $t \to \infty$, for some θ called the *persistence exponent*. Finding the value of θ is often a difficult problem which is at the center of what follows. As a second step, one also wants to understand the behavior of the process $(\zeta_t)_{t\geq 0}$ when conditioned on survival, i.e., on $\{T_z > t\}$, in the large t limit.

2.1. Some motivations from physics

Persistence problems appear naturally in many contexts, for instance in models of fluctuating interfaces, polymer chains, population dynamics, reaction-diffusions, etc. The process $(\zeta_t)_{t\geq 0}$ can either be a random walk or a Lévy process, a more general Markov chain or some more involved non-Markovian process, such as an additive functional of a Markov process. We refer to [13] and [3] for reviews in the physics and mathematical literature, respectively. My interest in this topic has been fueled by discussions with K. Chanard and F. Pétrélis who were interested in a sismology-related question; let me briefly describe their question as a motivation for the specific problem described below.

The one-dimensional model for earthquakes considered in [34] is a chain of sliders connected by springs, pulled by a force: the system alternates between periods where no slider moves (due to friction, the "stress" accumulates) and earthquake events when a slider moves and possibly makes other sliders move with it. One is led to consider the "stress" profile, which somehow describes in space the intensity of the constraint between tectonic plates (see Figure 1): physicists then argue that during an earthquake, the local displacement variation is a function of the stress.

In more mathematical terms, let us denote $(\sigma_x)_{x\in\mathbb{R}}$ the stress profile; it appears natural to model it by a stochastic process. For an earthquake that nucleates at $x=0$, the displacement ζ_t at position $t\geq 0$ is given by the accumulation of local displacements: in other words,

$$\zeta_t = z_0 + \int_0^t f(\sigma_x)\mathrm{d}x \tag{2}$$

for some function f. The earthquake then propagates until $\zeta_t = 0$ (i.e., there is no displacement anymore), that is up to distance $T=\inf\{t\geq 0, \zeta_t\leq 0\}$, see Figure 1. Additionally, an important physical quantity is the *moment* $M=\int_0^T \zeta_t \mathrm{d}t$ of the earthquake, closely related to its magnitude.

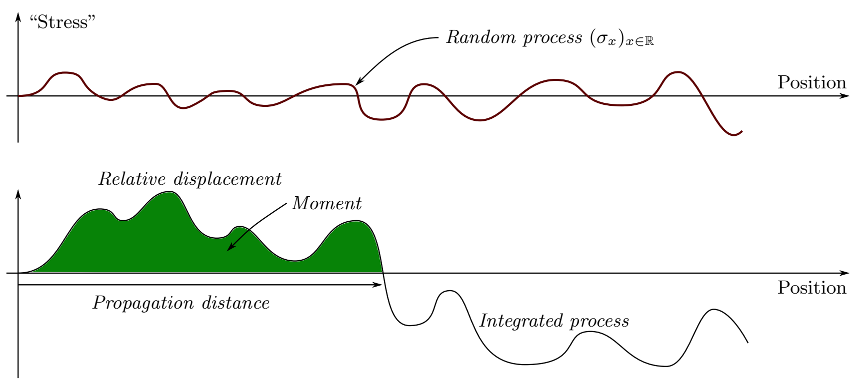

FIGURE 1. The stress profile $(\sigma_x)_{x\in\mathbb{R}}$ describes the intensity of the constraint in space. When an earthquake nucleates (say at $x=0$), the relative displacement ζ_t at position t is the integral of a function of the stress. The earthquake propagates up to distance $T=\inf\{t,\zeta_t\leq 0\}$ (there is no relative displacement for $t>T$) and the *moment* of the earthquake is $M:=\int_0^T \zeta_t\mathrm{d}t$.

We are therefore interested in estimating the probability of the rare events $\{T>t\}$ and $\{M>m\}$ as $t,m\to\infty$, which is directly related to the persistence problem for the process $(\zeta_t)_{t\geq 0}$, called an *additive functional* of $(\sigma_x)_{x\geq 0}$. One aims to obtain power-law asymptotics, of the type

$$\mathbb{P}(T>t) = t^{-\theta+o(1)} \quad \text{as } t\to\infty \qquad \text{and} \qquad \mathbb{P}(M>m) = m^{-\chi+o(1)} \quad \text{as } m\to\infty\,. \tag{3}$$

2.2. **Persistence problems for random walks and Lévy processes**

Before turning to persistence problems for additive functionals of Markov processes as (2), let us comment on the case of random walks and Lévy processes. For simplicity, we focus our exposition on the case of random walks: let $(U_i)_{i\geq 1}$ be i.i.d. real random variables and let $X_n := \sum_{i=1}^n U_i$, $n\geq 1$, be the associated random walk.

The symmetric case: Sparre Andersen's formula. Maybe the most striking result is in the symmetric case, where the persistence probability is known explicitly (at least when the increments are continuous). This is known as Sparre Andersen's fluctuation theorem, see [41]; we also refer to [17] for a short and elegant proof of the statement below.

Theorem 2.1 (Sparre Andersen, [41])**.** *Let* $(U_i)_{i\geq 0}$ *be i.i.d.* symmetric *real random variables. Then*

$$\mathbb{P}(X_k > 0 \ \ \textit{for all } 1 \leq k \leq n) \leq \frac{1}{4^n}\binom{2n}{n} \leq \mathbb{P}(X_k \geq 0 \ \ \textit{for all } 1 \leq k \leq n)\,.$$

In particular, if the law of U_i *has no atom, i.e., if* $\mathbb{P}(U_i = x) = 0$ *for all* $x \in \mathbb{R}$*, then the three terms are equal.*

This result is striking since it shows that, for symmetric and continuous variables, the survival probability *does not depend* on the specific law of the random walk; in particular no moment condition is needed. It also gives the persistence exponent $\theta = 1/2$, since $\frac{1}{4^n}\binom{2n}{n} \sim (\pi n)^{-1/2}$.

The case of general random walks or Lévy processes. The case of a general random walk (or Lévy process) can also be treated via fluctuation theory, see e.g. [20] and references therein. The so-called Wiener–Hopf factorization allows one to obtain the joint generating/characteristic function of the first ladder epoch $T = \min\{n \geq 1, S_n < 0\}$ and ladder height $H = S_T$, known as Spitzer–Baxter formula: for $\lambda \in [0,1]$ and $\mu \in \mathbb{C}$ with $\mathcal{R}\mathrm{e}(\mu) \geq 0$, we have[1]

$$1 - \mathbb{E}\left[\lambda^T \mathrm{e}^{\mu H}\right] = \exp\left(- \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \mathbb{E}\left[\mathrm{e}^{\mu S_n} \mathbf{1}_{\{S_n \leq 0\}}\right]\right). \tag{4}$$

Working with formula (4) and using Tauberian theorems, it can be shown that $n \mapsto \mathbb{P}(T > n)$ is regularly varying with index $-\rho$, $\rho \in (0,1)$, if and only if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{P}(S_k < 0) = \rho \in (0,1)$, the latter being known as Spitzer's condition (see e.g. [10]). As a particular case, if the random walk is centered and has finite variance, one has $\lim_{n\to\infty} \mathbb{P}(S_n > 0) = 1/2$ by the central limit theorem, so the persistence exponent is $\theta = 1/2$; one can actually show in that case that $\mathbb{P}(T > n) \sim c_0 n^{-1/2}$ for some constant c_0, see [21, XII §7, Thm. 1.a].

2.2.1. *Persistence problems for additive functionals of Markov processes: an overview*

We now turn to the problem motivated by Section 2.1: let $(X_s)_{s\geq 0}$ be a Markov process (in discrete or continuous time), and consider similarly to (2) the additive functional

$$\zeta_t := \sum_{s=1}^{t} f(X_s) \qquad \text{or} \qquad \zeta_t := \int_0^t f(X_s)\mathrm{d}s, \tag{5}$$

for some measurable function $f : \mathbb{R} \to \mathbb{R}$, depending on whether we are interested in the discrete or continuous time setting. We assume that f is sign-preserving, meaning that $xf(x) \geq 0$ (by convention we set $f(0) = 0$); note that a large part of the literature consider the case $f(x) = x$.

The general goal is again to estimate, for some fixed $z < 0$, the persistence probability $\mathbb{P}(T_z > t)$ as $t \to \infty$ where $T_z := \inf\{s > 0, \zeta_s \leq z\}$. We now present a quick overview of the literature, with some ideas of proof in the next sections.

Continuous time, part I: integrated Brownian motion and α-stable Lévy processes. The persistence problem for the integrated process $\zeta_t = \int_0^t X_s \mathrm{d}s$ has first been investigated in the case where $(X_s)_{s\geq 0}$ is a standard Brownian motion, starting with [22]. One finds the following asymptotic behavior[2]:

$$\mathbb{P}(T_z > t) \sim c_0 |z|^{1/6} t^{-1/4} \qquad \text{as} \quad \frac{t}{|z|^{2/3}} \to +\infty\,, \qquad \text{with } c_0 = \frac{3^{4/3}\Gamma(2/3)}{\pi 2^{13/12}\Gamma(3/4)}\,.$$

The results are more recent in the case where $(X_t)_{t\geq 0}$ is a strictly α-stable Lévy process, $\alpha \in (0,2]$. If $(X_t)_{t\geq 0}$ has a positivity parameter $1 - \varrho := \mathbb{P}(X_t > 0)$ (which is a constant that does not depend on t for strictly α-stable Lévy processes), then the persistence exponent is found to be $\theta = \frac{\varrho}{1+\alpha(1-\varrho)}$, see [36]. Note that one recovers

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[1] Let us also mention that a Wiener–Hopf factorization holds without taking transforms, see [2].

[2] Let us point out that Lachal [33] actually provides the explicit density of (T_z, X_{T_z}) for $z < 0$.

$\theta = 1/4$ for the Brownian motion since then $\alpha = 2$, $\varrho = 1/2$. Also, for spectrally positive α-stable processes with $\alpha \in (1,2)$, one has $\varrho = 1/\alpha$ so the persistence exponent is $\theta = \frac{\alpha-1}{2\alpha}$, as had been proven by [39].

Discrete time: integrated random walks. Let $(U_i)_{i\geq 1}$ be i.i.d. random variables and let $X_n := \sum_{i=1}^n U_i$ be the associated random walk and $\zeta_n = \sum_{k=1}^n X_k$ the integrated walk. If the random variables $(U_i)_{i\geq 1}$ are centered and have a finite variance, then we have[3] $\mathbb{P}(T_0 > n) \asymp n^{-1/4}$, recovering the persistence exponent $\theta = 1/4$ found in the case of the integrated Brownian motion. This has first been proved by [40] in the case of the simple random walk and then by [17] for general walks; finally the precise asymptotics $\mathbb{P}(T_0 > n) \sim c_0 n^{-1/4}$ as $n \to \infty$ was given in [18]. Let us mention that in the simple random walk case, the same persistent exponent $\theta = 1/4$ also holds for the additive functional $\zeta_n = \sum_{k=1}^n f(X_k)$ provided that f is symmetric, see [8]; we will come back to that in Section 2.3.

The case of a (centered) random walk in the domain of attraction of an α-stable law, $\alpha \in (0,2)$, remains mostly open. The results in the continuous setting (see the above paragraph) suggest that for assymptotically α-stable random walks with positivity parameter $1-\varrho$, the persistence exponent of $(\zeta_n)_{n\geq 0}$ should be $\theta = \frac{\varrho}{1+\alpha(1-\varrho)}$. We refer to [17, 43] for examples of asymptotically totally asymmetric α-stable random walks (skip-free centered random walks, to simplify), where the authors find the persistence exponent $\theta = \frac{\alpha-1}{2\alpha}$ as in the continuous case (see [39] and the above paragraph).

Continuous time, part II: homogeneous additive functionals of self-similar processes. Over the past 30 years, persistence problems have been studied for additive functionals as in (5), with a focus on the case of a homogeneous functional $f(x) = |x|^\gamma (c_+ \mathbf{1}_{\{x>0\}} - c_- \mathbf{1}_{\{x<0\}})$ for some $\gamma \in \mathbb{R}$ and on the case of self-similar Markov processes $(X_s)_{s\geq 0}$.

First, in the case of a Brownian motion, Isozaki and Kotani [26] proved that, for $\gamma > -1$,

$$\mathbb{P}(T_z > t) \sim c_0 |z|^{\nu\rho} t^{-\rho/2} \qquad \text{as} \quad \frac{t}{|z|^{2\nu}} \to +\infty\,, \qquad \text{with } \nu = \frac{1}{2+\gamma}\,,$$

where ρ is some asymmetry parameter that depends explicitly on γ and c_+/c_-; one has $\rho = 1/2$ if $c_+ = c_-$, recovering the persistence exponent $\theta = 1/4$ found in [25].

A more recent work [35] also considered the case of a *skew-Bessel* process $(X_s)_{s\geq 0}$ of dimension $\delta \in (0,2)$ and skewness parameter $\eta \in (-1,1)$; loosely speaking, it is a Bessel process[4] with an asymmetry η when it touches 0. We stress that $(X_s)_{s\geq 0}$ enjoys some self-similarity, and so does $(\zeta_s)_{s\geq 0}$ if the functional f is homogeneous. In that setting (for $\delta \in [1,2)$ and $\gamma > 0$) Profeta [35] proves, among many other things, that $\mathbb{P}(T_z > t) \sim c_0 |z|^{2\nu\theta} t^{-\theta}$ as $t \to \infty$, where θ is explicit and depends on δ, η, γ and c_+/c_-, and $\nu = 1/(2+\gamma)$ as above.

Continuous time, part III. The above-mentioned results deal with homogeneous additive functionals $(\zeta_t)_{t\geq 0}$ of some specific processes (Brownian motion or skew-Bessel process): this way, $(\zeta_t)_{t\geq 0}$ is built to enjoy scaling properties, making some exact computations possible. But the strategy developed in [26], based on an excursion decomposition of the underlying Markov process $(X_s)_{s\geq 0}$ together with a so-called Wiener–Hopf factorization of some auxiliary Lévy process, actually proves to be robust[5].

In that spirit, we attacked in [9] this problem from a general perspective, trying to make the argument as transparent as possible, clarifying also the role of the different parameters in the persistent exponent θ, which sometimes appears a bit mysterious (at least at first glance). In particular, in [9], we express the persistence exponent as $\theta = \beta\rho$, where $\beta \in (0,1]$ is the scaling exponent for the local time of $(X_s)_{s\geq 0}$ at level 0 and $1-\rho \in (0,1)$ is the (asymptotic) positivity parameter of some auxiliary Lévy process $(Z_t)_{t\geq 0}$ (we stress that ρ depends on the asymmetry of the function f and of the process $(X_s)_{s\geq 0}$). More precisely, under some relatively

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[3] We use the standard notation $a_n \asymp b_n$ if a_n/b_n is bounded away from 0 and infinity.

[4] A Bessel process $(Y_s)_{s\geq 0}$ of dimension $\delta \in (0,2)$ is a non-negative process, solution of the SDE $\mathrm{d}Y_s = \mathrm{d}B_s + \frac{\delta-1}{2Y_s}\mathrm{d}s$, with $(B_s)_{s\geq 0}$ a standard Brownian motion; see [37] as a reference. In particular $(Y_s)_{s\geq 0}$ is the absolute value of a Brownian motion when $\delta = 1$ and $(Y_s)_{s\geq 0}$ enjoys some scaling properties; note also that 0 is reflecting when $\delta \in (0,2)$.

[5] Somehow, the probabilist point of view has been abandonned later on for a more analytical approach.

weak condition on the process $(X_s)_{s\geq 0}$ and on the function f, we show in [9] that there exists a slowly varying function $\varsigma(\cdot)$ and some function $\mathcal{V}(\cdot)$ such that

$$\mathbb{P}(T_z > t) \sim c_0 \mathcal{V}(z)\, \varsigma(t)\, t^{-\beta\rho} \qquad \text{as} \quad t \to +\infty\,. \tag{6}$$

The intuition behind this result (and in particular behind the persistence exponent $\theta = \beta\rho$) and some of the ideas of the proofs are given in Sections 2.3–2.4 below.

2.3. **Discrete (symmetric) case: excursions and exchangeable Sparre Andersen**

As a warm up, let us consider the discrete setting where $(X_n)_{n\geq 0}$ is a (symmetric) birth and death chain, i.e., a Markov chain on $\mathbb{Z}$ with $|X_n - X_{n-1}| \leq 1$ for all n, with symmetric transition probabilities $p(x,y) = p(-x,-y)$ for $x, y \geq 0$. This includes for instance the simple random walk or Bessel-like random walks (which, when diffusively rescaled, converge to a symmetric skew-Bessel processes, see [1] and references therein). Let us show that is f is an odd function (with $xf(x) \geq 0$) the persistence exponent of $\zeta_n := \sum_{k=1}^n f(X_k)$ is $\theta = \beta/2$, where β is the scaling exponent for the local time $L_n = \sum_{i=1}^n \mathbf{1}_{\{X_i = 0\}}$; in particular $\theta = 1/4$ if $(X_k)_{k\geq 0}$ is the simple random walk (full details are given in [8]).

Excursion decomposition. A natural approach is to decompose the process $(X_k)_{k\geq 0}$ into excursions, on which the integrated process $(\zeta_n)_{n\geq 0}$ is monotone. Let $(\tau_j)_{j\geq 0}$ be the successive returns to 0 of $(X_k)_{k\geq 0}$, that is $\tau_0 = 0$ and iteratively $\tau_j := \min\big\{k > \tau_{j-1}, X_k = 0\big\}$ for $j \geq 1$. Then, we define

$$Y_j := \sum_{k=\tau_{j-1}+1}^{\tau_j} f(X_k)\,, \qquad \text{and} \qquad Z_k := \sum_{j=1}^{k} Y_j\,,$$

which are the contributions of the j-th excursion, resp. of the k first excursions, to ζ_n; note that we may also write $Z_k = \zeta_{\tau_k}$. By the (strong) Markov property the random variables $(Y_j)_{j\geq 1}$ are i.i.d., so $(Z_k)_{k\geq 0}$ is a random walk; note also that the increments $(Y_j)_{j\geq 1}$ are symmetric here, since the X_k are symmetric and f is odd.

All together, with $L_n = \sum_{i=1}^n \mathbf{1}_{\{X_i=0\}}$ the local time of $(X_k)_{k\geq 0}$ at 0, we get that

$$\zeta_n = Z_{L_n} + \hat{Y}_n\,, \tag{7}$$

where $\hat{Y}_n = \sum_{k=\tau_{L_n}+1}^{n} f(X_k)$ is the contribution of the last (uncomplete) excursion of $(X_k)_{k\geq 0}$. Then, either forgetting the contribution of $\hat{Y}_n$ or imposing $\hat{Y}_n$ to be non-negative, and recalling that ζ_k is monotone on each excursion, one easily gets that

$$\mathbb{P}\big(\zeta_k \geq 0 \text{ for all } 0 \leq k \leq n\big) \asymp \mathbb{P}\big(Z_k \geq 0 \text{ for all } 0 \leq k \leq L_n\big)\,. \tag{8}$$

Now, the last expression involves a persistence probability for a sum of i.i.d. symmetric random variables, $(Y_j)_{j\geq 1}$, but the number L_n of terms in the sum is random and more importantly is not independent of the random variables $(Y_j)_{j\geq 1}$.

Sparre Andersen for exchangeable and sign-invariant vectors. As mentioned above, we cannot a priori directly apply techniques used for random walks described in Section 2.2. However, Sparre Andersen's Theorem 2.1 remains valid assuming only that the random vector $(U_1, \ldots, U_n)$ is *exchangeable* in place of i.i.d., i.e., $(U_{\sigma(1)}, \ldots, U_{\sigma(n)})$ has the same distribution as $(U_1, \ldots, U_n)$ for any permutation σ, and *sign-invariant* as a substitute for symmetric, i.e., $(\varepsilon_1 U_1, \ldots, \varepsilon_n U_n)$ has the same distribution as $(U_1, \ldots, U_n)$ for any $(\varepsilon_i)_{1\leq i\leq n} \in \{-1,1\}^n$; this is only briefly outlined in [41] and we refer to [8] for an elementary proof.

Theorem 2.1'. *Let $(U_1, \dots, U_n)$ be an exchangeable and sign-invariant random vector and define $S_k = \sum_{i=1}^k U_i$ for $1 \le k \le n$. Then we have*

$$\mathbb{P}(S_k > 0 \text{ for all } 1 \le k \le n) \le \frac{1}{4^n}\binom{2n}{n} \le \mathbb{P}(S_k \ge 0 \text{ for all } 1 \le k \le n).$$

In particular, if the law of $(U_1, \dots, U_n)$ has no atom then the three terms are equal.

As an application, we can treat the right-hand side probability in (8), by noticing that conditionally on $\{L_n = m, \tau_m = x\}$, the excursion lengths $(\tau_j - \tau_{j-1})_{1\le j\le m}$ are simply conditioned to have a sum equal to x: this shows that $(Y_1, \dots, Y_n)$ are exchangeable and they are also sign-invariant since the sign of an excursion is independent from its length (by symmetry). Hence, conditioning by L_n, τ_{L_n} then applying Theorem 2.1', we get that

$$\mathbb{P}\big(Z_k > 0 \text{ for all } 0 \le k \le L_n\big) \asymp \mathbb{E}\left[\frac{1}{4^{L_N}}\binom{2L_n}{L_n}\right] \asymp \mathbb{E}\left[(1+L_n)^{-1/2}\right].$$

Therefore, if L_n is of order n^β for some $\beta \in (0, 1]$ (e.g. $\beta = 1/2$ for the simple random walk), then after a bit of technicality we end up with

$$\mathbb{P}\big(\zeta_k > 0 \text{ for all } 0 \le k \le n\big) \asymp n^{-\beta/2} \qquad \text{as} \quad n \to \infty.$$

Comments on the non-symmetric case. More generally, the excursion decomposition remains valid and in particular the relation (8) still holds. The last step then consists in showing that if one has the following two ingredients: (i) $\mathbb{P}(Z_k > 0 \text{ for all } k \le \ell) = \ell^{-\rho+o(1)}$ as $\ell \to \infty$, which is a persistence estimate for a random walk, tractable as discussed in Section 2.2; (ii) L_n is of order n^β for some $\beta \in (0, 1]$; then one can conclude that

$$\mathbb{P}\big(Z_k > 0 \text{ for all } 0 \le k \le L_n\big) \asymp \mathbb{E}\left[(1+L_n)^{-\rho}\right] \asymp n^{-\beta\rho}.$$

The difficulty here comes from the intricate relation between $(Z_k)_{k\ge0}$ and L_n and is overcome thanks to a Wiener–Hopf decomposition of the bivariate random walk $(\tau_k, Z_k)_{k\ge0}$; we refer to the discussion below, in the continuous setting. One should also be able to obtain the sharp persistence probability by controlling the last part of the integral, i.e., $\hat{Y}_n$ in (7).

2.4. Continuous case: excursion decomposition and some ideas of the proof

In the continuous setting, we consider a càdlàg strong Markov process $(X_s)_{s\ge0}$ in $\mathbb{R}$ and we assume that 0 is recurrent and regular for itself, i.e., $\eta_0 := \inf\{s > 0, X_s = 0\}$ satisfies $\mathbb{P}(\eta_0 = 0) = 1$. These assumptions allow one to have some excursion theory of $(X_t)_{t\ge0}$ outside 0. In particular, analogously to the discrete setting, one may define the local time L_t of $(X_t)_{t\ge0}$ at level 0 and its right-continuous inverse $\tau_t := \inf\{u \ge 0, L_t > u\}$.

Then, denoting $\xi_u := \inf\{\zeta_s, s \le u\}$ and $g_t := \sup\{s < t, X_s = 0\}$, we may rewrite the persistence probability $\mathbb{P}(T_z > t) = \mathbb{P}(\xi_t > z)$ as follows:

$$\mathbb{P}(\xi_t > z) = \mathbb{P}(\xi_{g_t} > z, \xi_{g_t} + \zeta_{g_t} - \xi_{g_t} + I_t > z) \sim \mathbb{P}(\xi_{g_t} > z, \zeta_{g_t} - \xi_{g_t} + I_t > 0), \tag{9}$$

where $I_t = \int_{g_t}^t f(X_s)\mathrm{d}s$ is the contribution of the last (uncomplete) excursion; we refer to Figure 2 for an illustration. The last equivalence in (9) has to be justified properly but comes from the fact that $z \le z - \xi_{g_t} \le 0$, with z much smaller than the typical fluctuations of $\zeta_{g_t} - \xi_{g_t} + I_t$.

Then, analogously to the discrete setting, we define the Lévy process

$$Z_t := \zeta_{\tau_t} = \sum_{s\le t} \int_{\tau_{s-}}^{\tau_s} f(X_u)\mathrm{d}u.$$

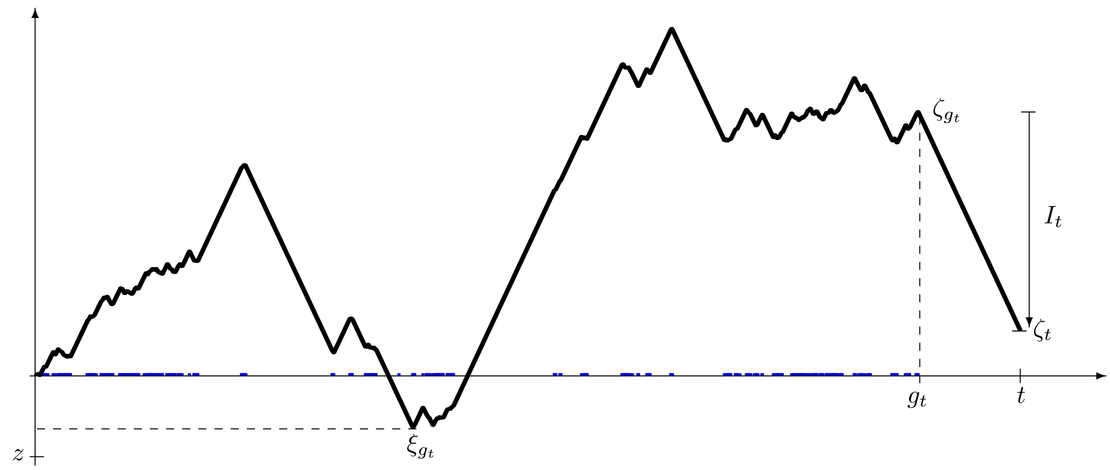


FIGURE 2. Representation of a trajectory of $(\zeta_s)_{s\geq 0}$ and of its decomposition into three parts: ξ_{g_t}, $\zeta_{g_t}-\xi_{g_t}$ and I_t. The blue dots represent the returns to 0 of $(X_t)_{t\geq 0}$.

Let us note that, while τ_t and Z_t are not independent, $(\tau_t, Z_t)_{t\geq 0}$ defines a Lévy process. Then, provided that $f(x)$ preserves the sign of x and that $(X_t)_{t\geq 0}$ cannot cross 0 without touching 0, we get that $(\zeta_t)_{t\geq 0}$ is monotone on excursions of $(X_t)_{t\geq 0}$, i.e., on intervals $[\tau_{s-}, \tau_s]$, so we have that $\xi_{\tau_t} = \inf\{\zeta_s, s\leq \tau_t\} = \inf\{Z_s, s\leq t\}$. Note also that $\zeta_{g_t} = Z_{L_t}$ and $\xi_{g_t} = \inf\{Z_s, s\leq L_t\}$.

A (standard) trick to gain independence. Let e be an exponential random variable of parameter $q>0$, independent of $(X_t)_{t\geq 0}$. Then, one may consider the persistence probability $\mathbb{P}(T_z > e)$, which corresponds to taking the Laplace transform of $t\mapsto \mathbb{P}(T_z>t)$. One can then easily relate the behavior of $\mathbb{P}(T_z>e)$ as $q\downarrow 0$ to that of $\mathbb{P}(T_z>t)$ as $t\uparrow\infty$ by Tauberian theorems.

It turns out that this trick allows us to gain independence: Proposition 4.2 in [9] shows that $(X_t)_{t<g_e}$ and $(X_{g_e+s})_{s\leq e-g_e}$ are independent, so in particular $\xi_{g_e}, \zeta_{g_e}-\xi_{g_e}$ are independent of I_e. One also gain independence thanks to a Wiener–Hopf factorization of the Lévy process (τ_t, Z_t), from which one may obtain a formula for the joint Laplace transform of $(\xi_{g_e}, \zeta_{g_e}-\xi_{g_e})$ (see [9, Cor. 4.6]): one deduces in particular that ξ_{g_e} and $\zeta_{g_e}-\xi_{g_e}$ are also independent.

Going back to (9) and replacing t by the exponential random variable e of parameter q, this gives that

$$\mathbb{P}(T_z>e)\sim \mathbb{P}(\xi_{g_e}>z)\mathbb{P}(\zeta_{g_e}-\xi_{g_e}+I_e>0)\,,$$

with $\zeta_{g_e}-\xi_{g_e}$ and I_e independent. It then remains to study the probability $\mathbb{P}(\xi_{g_e}>z)$ as $q\downarrow 0$ and show that $\mathbb{P}(\zeta_{g_e}-\xi_{g_e}+I_e>0)$ converges to a constant $c_1\in(0,1]$.

The behavior of $\mathbb{P}(\xi_{g_e}>z)$ can be deduced from the computation of the Laplace transform of ξ_{g_e}, which is expressed in terms of (τ_t, Z_t), see [9, Cor. 2.7]. For the term $\mathbb{P}(\zeta_{g_e}-\xi_{g_e}+I_e>0)$, one needs some assumption. In practice the study is divided into two cases: (i) if $(X_t)_{t\geq 0}$ is positive recurrent, then I_e actually remains tight and $\mathbb{P}(\zeta_{g_e}-\xi_{g_e}+I_e>0)$ goes to 1; (ii) if there is a scaling sequence $(c_q)_{q>0}$ such that I_e/c_q and $(\zeta_{g_e}-\xi_{g_e})/c_q$ converge in distribution as $q\downarrow 0$ to I and W respectively (that are independent), then $\mathbb{P}(\zeta_{g_e}-\xi_{g_e}+I_e>0)$ converges to $\mathbb{P}(I+W>0)\in(0,1)$.

A Wiener–Hopf factorization. A key tool in the analysis in [9] is a Wiener–Hopf factorization for the bivariate process (τ_t, Z_t). In practice, we introduce the infimum process $S_t = \inf_{s\in[0,t]} Z_s$ and the last time $G_t := \sup\{u<t, Z_u = S_u\}$ when the process is equal to its infimum. Then, Theorem 4.3 in [9] shows that if e is an exponential random variable of parameter $q>0$ independent of $(X_t)_{t\geq 0}$, the triplets (G_e, τ_{G_e}, S_e) and $(e-G_e, \tau_e-\tau_{G_e}, Z_e-S_e)$ are independent, with an infinitely divisible distribution that is expressed explicitly in terms of the distribution of (τ_t, Z_t). This also allows one to obtain various (Spitzer–Baxter) formulas for joint Laplace transforms, in particular for $(\xi_{g_t}, \zeta_{g_t} - \xi_{g_t}) = (S_{L_t}, Z_{L_t} - S_{L_t})$, in terms of the distribution of (τ_t, Z_t).

Some results. Let us now briefly describe the results of [9], separated into two parts.

(i) If $(X_t)_{t\geq 0}$ is positive recurrent. Then $t \mapsto \mathbb{P}(T_z > t)$ is regularly varying with exponent $-\rho$, $\rho \in (0,1)$, if and only if Spitzer's condition holds: $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}(\zeta_s < 0)\mathrm{d}s = \rho \in (0,1)$.

(ii) If there are scaling sequences $(a_t)_{t\geq 0}$ $(b_t)_{t\geq 0}$ such that $(\tau_t/b_t, Z_t/a_t)$ converges in distribution to a (β,α)-stable law with $\beta \in (0,1)$ and $\alpha \in (0,2]$, and such that I_t/c_t converges in distribution, with $c_t = a_{(b^{-1})_t}$. Then $t \mapsto \mathbb{P}(T_z > t)$ is regularly varying with exponent $-\beta\rho$, with $1-\rho = \mathbb{P}(\mathcal{Z} > 0)$ where $\mathcal{Z}$ is the limiting α-stable law.

The assumption (ii) is usually not simple to verify, since it is about the process $(\tau_t, Z_t)_{t\geq 0}$ rather than simply on $(X_t)_{t\geq 0}$. The context of generalized one-dimensional diffusions (based on a time-and-scale changed Brownian motion) is considered in [9]: some conditions are given on their *speed function* and *scale measure* to make sure that assumption (ii) is satisfied. This framework allows one to treat a wide variety of processes $(X_t)_{t\geq 0}$ that are skew-Bessel processes only asymptotically and of functions f that are only asymptotically powers (we refer to [9, §2.5] for a series of examples).

Going further: scaling and heuristics for the tail asymptotic of the moment M. The condition (ii) above implies that L_t is of order $(b^{-1})_t = t^{\beta+o(1)}$ and Z_t has fluctuations of order $a_t = t^{1/\alpha+o(1)}$. One can actually show (thanks to the Wiener–Hopf factorization) that $\zeta_{g_t}/c_t = Z_{L_t}/c_t$ converges in distribution, with $c_t = a_{(b^{-1})_t} = t^{\beta/\alpha+o(1)}$.

In view of our physical motivation, this should enable us to obtain the tail asymptotic of the *moment* $M = \int_0^{T_z} \zeta_s \mathrm{d}s$. By the scaling described above, we expect that $M \asymp T_z c_{T_z} = T_z^{\frac{\alpha+\beta}{\alpha}+o(1)}$, so that using the asymptotics (6) we should get that

$$\mathbb{P}(M > m) \asymp \mathbb{P}\big(T_z\, c_{T_z} > m\big) \asymp m^{-\frac{\alpha}{\alpha+\beta}\beta\rho} \qquad \text{as } m \to \infty\,,$$

leading to a power-decay with exponent $\chi = \frac{\alpha\beta}{\alpha+\beta}\rho$ in (3).

2.5. Conclusion: two (challenging) open problems

Naturally, the excursion decomposition approach strongly requires that the integrated process has the Markov property and cannot jump over 0, and several questions remain open when this is not the case. As a conclusion, let us simply mention two important models that have attracted a lot of attention where the persistence exponent θ for the integrated process $\zeta_t = \int_0^t X_s \mathrm{d}s$ remains unknown: when $(X_s)_{s\geq 0}$ is a fractional Brownian motion and when $(X_s)_{s\geq 0}$ is an integrated Brownian motion. We refer the reader to the review [3] for further discussion on these problems and on their relation with the physics literature.

3. Large fluctuations in multi-scale modeling for rest hematopoiesis

Céline Bonnet

3.1. Introduction

Here is a summary of my talk presenting the article [12], written jointly with S. Méléard. I will not repeat the proofs. My intention is rather to present our work as an example of random walk applications in biology. Indeed, we have studied a stochastic model of rest hematopoiesis, the blood cells production process. Blood cells are produced by differentiation stages from stem cells with amplification of the amount of cells. I present and explain here only the main goal of the article: to analyze the effects of amplification on fluctuations in blood cell counts, which biological observations have shown to be higher than the expected standard fluctuations (cf. [42]).

We introduce a multi-type branching process on three types of cells (cf. [23, Sec. 12], [4, Sec. 6.9.1]), with a different size scale for each type, and a different time scale for the birth (or death) rate of each of the three types. The dynamics of stem cells (type 1) and progenitors (type 2) result in two distinct events, renewal and differentiation. Each cell of type 1 and 2 divides into two cells at a constant rate, depending on its type. These two new cells are either of the same type as the mother cell (renewal) or of the "next" cell type (differentiation). Blood cells (type 3) do not divide and can only die at a constant rate.

In the model, we introduce as unique scaling parameter K the size of the type 1 cell population. The parameter K will also scale the quantities leading to amplification, i.e., the small difference between the differentiation and renewal probabilities at step 2 and the death rate at step 3. We will see that the amplification from type 1 cells to type 3 cells is proportional to these two quantities, which will play a main role in our analysis.

Indeed, those multi-scale assumptions, biologically inspired, allows us to compare the size and time scales of each cell type population sizes: they strongly differ from type 1 to type 3 with increasingly slow time scales and large size scales. Usually, slow and fast components appear naturally, as associated which different species behaviors (see [29]), contrary to our case, where the different time scales are deduced from a fine study of the cell differentiation dynamics.

Our approach is inspired by [28] in which a general theorem for convergence and fluctuations of multiscale processes is obtained but the latter cannot explain the asymptotics of our model. Indeed, in their result, the fluctuations around the deterministic behavior of the slow component are Gaussian, which will not be the case of the type 3 cells dynamics previously described.

Notation. As in [32], we will denote by $l_m(\mathbb{R}_+)$ the space of measures μ on $[0,\infty)\times\mathbb{R}_+$ such that $\mu([0,t]\times\mathbb{R}_+)=t$, for each $t\geq 0$.

3.2. Model and assumptions

We consider a jump process in dimension 3 with exponential inter-arrival times to modeling blood cells production dynamics.

Cells of type 1 evolve according to a critical linear birth and death process. Birth events correspond to renewal division events, occurring at rate $\frac{\tau_1}{2}>0$, while death events correspond to differentiation events occurring at the same rate (a cell of type 1 divides in two cells of type 2). Cells of type 2 divide at rate $\tau_2>0$ in two cells of the same type (renewal event) with probability p_2^R and in two cells of type 3 (differentiation event) with probability $p_2^D=1-p_2^R\in(1/2,1)$. Cells of type 3 die at rate $d_3>0$.

We can summarize the dynamics as follows. If $N=(N_1,N_2,N_3)$ denotes the vector of sub-population sizes, the transitions of the hematopoietic process are given by

$$\begin{array}{rcll}
N_1 & \longrightarrow & N_1+1 & \text{at rate } (\tau_1/2)\,N_1\\
(N_1,N_2) & \longrightarrow & (N_1-1,N_2+2) & \text{at rate } (\tau_1/2)\,N_1\\
N_2 & \longrightarrow & N_2+1 & \text{at rate } \tau_2\,p_2^R\,N_2\\
(N_2,N_3) & \longrightarrow & (N_2-1,N_3+2) & \text{at rate } \tau_2\,p_2^D\,N_2\\
N_3 & \longrightarrow & N_3-1 & \text{at rate } d_3\,N_3.
\end{array}$$

Here, we have assumed that each division is symmetric, so that

$$p_2^D+p_2^R=1. \tag{10}$$

We could have included asymmetric division without changing the results of our study. Indeed it doesn't change the main characteristics of the dynamics.

In the model, we don't consider mortality rates for type 1 and type 2 cells and we assume that cell loss is only due to differentiation in the next cell type. Indeed the hematopoietic stem cell and progenitor death rates, at steady-state, have been biologically estimated and are so small that they can be neglected (cf. [19]).

As explained in the introduction, we define the scaling parameter K as the size of the type 1 cells population. K is assumed to be large and to scale $p_2^D - p_2^R$ and d_3, in a way which is defined now. More precisely, inspired by biological observations [14] and [11], we assume that

$$\text{the size of the type 1 cells population is of order } K, \tag{11}$$

and there exists a couple of positive parameters $(\gamma_2, \gamma_3) \in (0,1)$ such that

$$p_2^D - p_2^R = K^{-\gamma_2} \quad \text{and} \quad d_3 = \tau_3 K^{-\gamma_3} \quad \text{with } \tau_3 > 0. \tag{12}$$

Let us note that (10) and (12) make the probabilities p_2^R and p_2^D depend on K,

$$p_2^D = 1 - p_2^R = 1/2 + K^{-\gamma_2}/2.$$

Therefore the dynamics of this cell type is close to a critical process, in the sense that the renewal and differentiation rates are close.

Assumption (12) introduces the different time and size scales playing a role for the multi-scale population process describing the dynamics of each cell type size. From now on, since the dynamics depend on K, we will denote by N^K, the population process N previously defined.

In the article we have studied the most interested case, i.e,

$$\gamma_2 < \gamma_3 < 1. \tag{13}$$

3.3. Simulations and main result

In the article [12], we finely describe the process N^K dynamics, when K goes to infinity, using appropriate renormalizations. Here I describe the dynamics using simulations and explain the surprising fluctuations dynamics in the third component. Then I comment the result regarding biological observations.

3.4. Simulations

We take as initial condition

$$N^K(0) = (K, 0, 0)$$

and choose $K = 2000$ cells of type 1, $\gamma_2 = 0.55$, $\gamma_3 = 0.8$. Hence $K^{\gamma_2} \sim 60$ and $K^{\gamma_3} \sim 400$. The others parameters are equal to 1.

Figure 3 shows the simulation of a trajectory of the process $(N^K(t),\ t \in [0,T])$ for $T \sim 1$, decomposed on the three cell types. Figure 4 shows the simulation of a trajectory of the process $(N^K(t),\ t \in [0,T])$ for $T \sim K^{\gamma_2}$, and Figure 5 shows the simulation of a trajectory of the process $(N^K(t),\ t \in [0,T])$ for $T \sim K^{\gamma_3}$. The horizontal orange line gives the order of magnitude for size of each cell type (K, resp. $K^{1+\gamma_2}$, $K^{1+\gamma_2+\gamma_3}$, see Remark 3.1).

Remark 3.1. A simple reflection on the expectation of the process allows us to identify the different size scales. Indeed, the function $t \mapsto n(t) = \mathbb{E}N^K(t) = (n_1(t), n_2(t), n_3(t))$ satisfies the following system of equations, for all $t \leq T$,

$$\begin{cases} n_1(t) &= \mathbb{E}N_1^K(0), \\ \frac{d}{dt} n_2(t) &= \tau_1\, n_1(t) - \tau_2 K^{-\gamma_2}\, n_2(t), \\ \frac{d}{dt} n_3(t) &= 2\tau_2 p_2^D\, n_2(t) - \tau_3 K^{-\gamma_3}\, n_3(t). \end{cases}$$

By assumption (11),

$$\mathbb{E}N_1^K(0) \sim K.$$

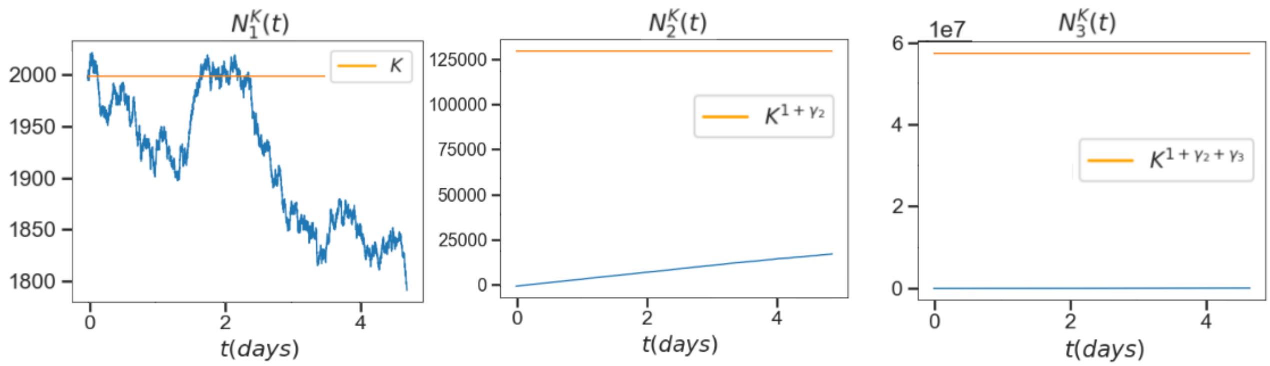


FIGURE 3. A trajectory of the N^K process for $t \in [0, T]$ with $T = O(1)$

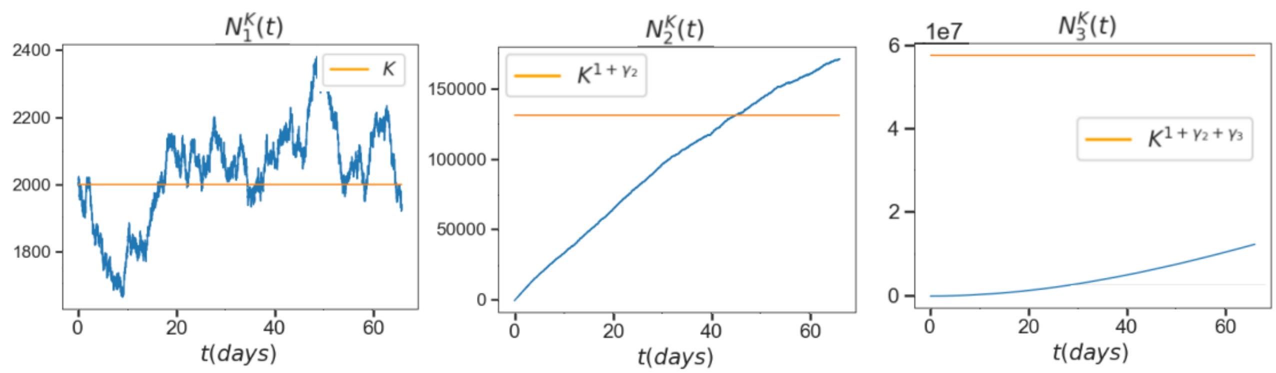


FIGURE 4. A trajectory of the N^K process for $t \in [0, T]$ with $T = O(K^{\gamma_2})$.

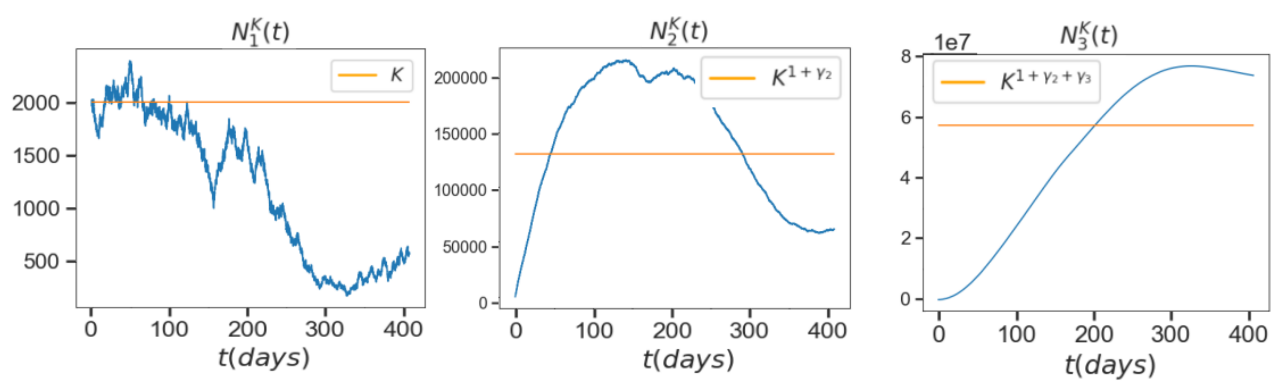


FIGURE 5. A trajectory of the N^K process for $t \in [0, T]$ with $T = O(K^{\gamma_3})$.

Therefore there is a unique equilibrium given for all $t \geq 0$ by

$$\begin{aligned} n_1^* &= \mathbb{E}N_1^K(0) \sim K, \\ n_2^* &= \frac{\tau_1 \, n_1^*}{\tau_2} K^{\gamma_2} \sim K^{1+\gamma_2}, \\ n_3^* &= \frac{2p_2^D \tau_2 \, n_2^*}{\tau_3} K^{\gamma_3} \sim K^{1+\gamma_2+\gamma_3}. \end{aligned}$$

We observe in Figure 3 that at a time scale of order 1, the two last components of the process N^K are far from their equilibrium size. We observe in Figure 4 that the two first components of $(N^K(t),\ t \in [0,T])$ for $T \sim K^{\gamma_2}$ evolve around their equilibrium size, which is not the case of the third one. In Figure 5, the process is considered on a longer period of time, $T \sim K^{\gamma_3}$ and one sees that the third component hits a neighborhood of its equilibrium. Furthermore, in Figure 6, we can observe the fluctuations of the components of N^K around their equilibrium. Indeed, Figure 6 shows the simulation of a trajectory of the process $(N^K(t),\ t \in [0,T])$ over a long period of time (of order K^{γ_3}), starting from the "equilibrium values" $(K, K^{1+\gamma_2}, K^{1+\gamma_2+\gamma_3})$. The horizontal orange line represents this information. We note that they get smoother from cell type 1 to cell type 3 and that the amplitude of the waves get longer.

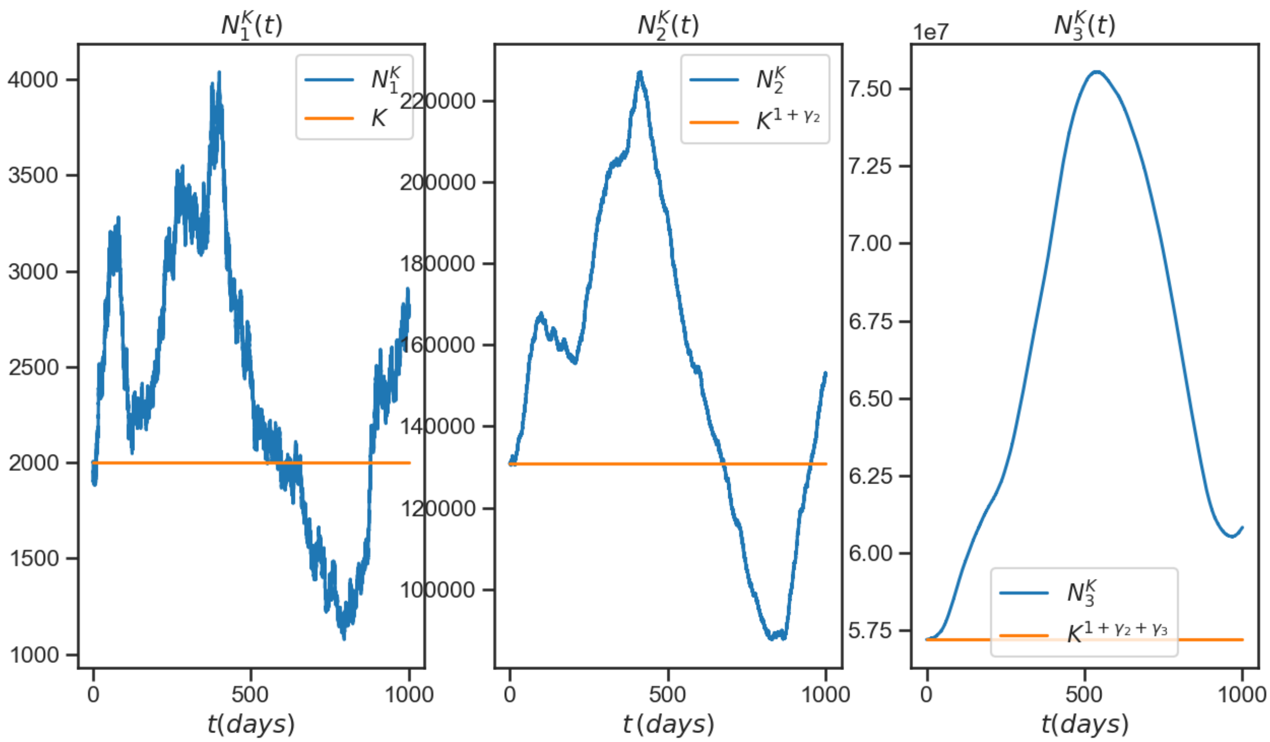


FIGURE 6. A trajectory of the process $(N^K(t),\ t \in [0,T])$ for $T = 1000$ (days) and starting from $(K, K^{1+\gamma_2},\ K^{1+\gamma_2+\gamma_3})$.

3.5. Large fluctuations result

As illustrated with simulations, a size renormalization of the stochastic process is not enough to understand the dynamics of the model. We need to change the time scale. In order to catch the long time dynamics of the third component we will study the process N^K on the time scale K^{γ_3}. To this end, let us introduce the jump process Z^K defined for all $t \geq 0$ by

$$Z^K(t) = \left(\frac{N_1^K(t\,K^{\gamma_3})}{K}, \frac{N_2^K(t\,K^{\gamma_3})}{K^{1+\gamma_2}}, \frac{N_3^K(t\,K^{\gamma_3})}{K^{1+\gamma_2+\gamma_3}} \right). \tag{14}$$

We can show that Z^K converges in some sense, when K tends to infinity, to a deterministic and continuous function $z : t \in \mathbb{R}_+ \to (z_1(t), z_2(t), z_3(t)) \in \mathbb{R}_+^3$ (for details see Theorem 2 in [12]).

Then introducing the following two processes, we can give a more precise description of the asymptotics behavior of the third component, for all $t \geq 0$,

$$\begin{cases} V_1^K(t) & := K^{(1-\gamma_3)/2}\big(Z_1^K(t) - z_1(t)\big), \\ V_3^K(t) & := K^{(1-\gamma_3)/2}\big(Z_3^K(t) - z_3(t)\big). \end{cases}$$

Indeed, we state the following Theorem.

Theorem 3.2. *We assume that*

$$\sup_K \mathbb{E} V_1^K(0)^4 < +\infty \quad ; \quad \sup_K \mathbb{E} Z_2^K(0)^2 < +\infty, \tag{15}$$

and that there exists $V_0 = (V_0^{(1)}, V_0^{(3)})$ *a* $\mathbb{R}^2$*-valued random vector such that the sequence* $(V_1^K(0), V_3^K(0))_{K\in\mathbb{N}^*}$ *converges in law to* V_0 *and such that*

$$\sup_K \mathbb{E}|V_3^K(0)| < +\infty. \tag{16}$$

Then for all $T > 0$, *the sequence* $(V_1^K, V_3^K)_{K\in\mathbb{N}^*}$ *converges in law in* $\mathbb{D}([0,T], \mathbb{R}^2)$ *to* (V_1, V_3) *such that for all* t,

$$\begin{aligned} V_1(t) &= V_0^{(1)} + \sqrt{\tau_1\, z_1(0)}\, W_1(t), \\ V_3(t) &= V_0^{(3)} + \tau_1 \int_0^t V_1(s) ds - \tau_3 \int_0^t V_3(s)\, ds, \end{aligned}$$

where W_1 *is a standard Brownian motion.*

Let us interpret this result in terms of fluctuations. Assuming that the initial vector V_0 is equal to zero, we obtain that for any t and large K,

$$N_3^K(t) \sim K^{1+\gamma_2+\gamma_3}\, z_3(t\, K^{-\gamma_3}) + K^{(1+2\gamma_2+3\gamma_3)/2}\, V_3(t\, K^{-\gamma_3}) \tag{17}$$

where for all t,

$$V_3(t) = \tau_1 \sqrt{\tau_1 z_1(0)} \int_0^t W_1(s)\, ds - \tau_3 \int_0^t V_3(s)\, ds$$

and W_1 is a standard Brownian motion.

The order of magnitude appearing in the fluctuation second order term in (17) summarizes the cumulative effects of the third dynamics driven by the fluctuations of the first level. That can explain the exceptionally large fluctuations observed for the size of blood cells populations, in hematopoietic systems.

Let us explain why the observed fluctuation scale of the third component is surprising. We have seen that the size of the population process of the third type is of order of magnitude $K^{1+\gamma_2+\gamma_3}$. In the usual setting, the Central Limit Theorem would lead to fluctuations of order $K^{(1+\gamma_2+\gamma_3)/2}$. Using Theorem 3.2, we see in (17) that they are of order $K^{(1+2\gamma_2+3\gamma_3)/2} \gg K^{(1+\gamma_2+\gamma_3)/2}$.

Come back to the simulation of Figure 6, we see that the order of magnitude given by (17) better describe the dynamics of N^K than classical CLT results. Indeed, following [28] (and classical CLT results), the order of magnitude of the population size fluctuations should be given by the square of the population size, i.e., $K^{(1+\gamma_2+\gamma_3)/2} \sim 7600$ cells for type 3. Theorem 3.2 gives these orders of magnitude, with about $K^{(1+3\gamma_2)/2} \sim 24\,000$ cells of type 2 and $K^{(1+2\gamma_2+3\gamma_3)/2} \sim 2.7\ 10^7$ cells of type 3.

4. The elephant random walk

Lucile Laulin

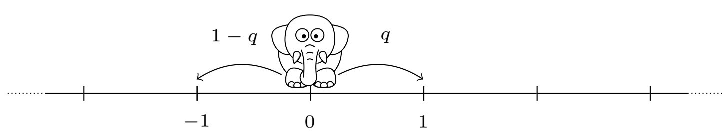
4.1. The elephant random walk

The **elephant random walk (ERW)** is a one-dimensional discrete-time random walk on integers, which has a complete memory of its whole history. It was introduced in 2004 by Schütz and Trimper [38] in order to investigate the long-term memory effects in non-Markovian random walks and was referred to as the ERW in allusion to the famous saying that elephants can remember where they have been. It appears to be a time-inhomogeneous Markov chain.

One of the natural questions regarding the ERW concerns the influence some the memory parameter p on the asymptotic behavior of the ERW. Depending on the value of p with respect to $3/4$, the behavior of the ERW is quite different and we observe three regimes. More precisely, a strong law of large numbers (LLN) and a central limit theorem (CLT) for the position S_n, properly normalized, were established in the diffusive regime $p < 3/4$ and the critical regime $p = 3/4$. The main change between the two regimes is the rate of the normalization.

The superdiffusive regime $p > 3/4$ turns out to be harder to deal with. Both Coletti et al. [16] and Bercu [7] proved a type of law of large numbers: the position of the ERW, properly normalized at some superdiffusive scale, converges almost surely to a random variable L_q which is not Gaussian. After that, Kubota and Takei [30] an a result analogous to a central limit theorem and showed that the fluctuation of the ERW around its limit in the superdiffusive regime is Gaussian.

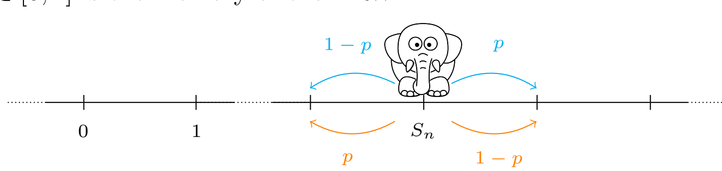
The one-dimensional ERW is defined as follows. The elephant starts at the origin at time zero, $S_0 = 0$. At time $n = 1$, it moves to the right with probability q or to the left with probability $1-q$ where q lies between zero and one. Hence, the position of the elephant at time $n = 1$ is given by $S_1 = X_1$ where X_1 has a Rademacher $\mathcal{R}(q)$ distribution.



Afterwards, at any time $n \geq 1$, the elephant chooses uniformly at random an integer k among the previous times $1, \dots, n$, and we define

$$X_{n+1} = \begin{cases} +X_k & \text{with probability} \quad p, \\ -X_k & \text{with probability} \quad 1-p, \end{cases}$$

where the parameter $p \in [0, 1]$ is the memory of the ERW.



Then, the position of the ERW is given by

$$S_{n+1} = S_n + X_{n+1}.$$

The main results regarding the behavior of the ERW are given here.

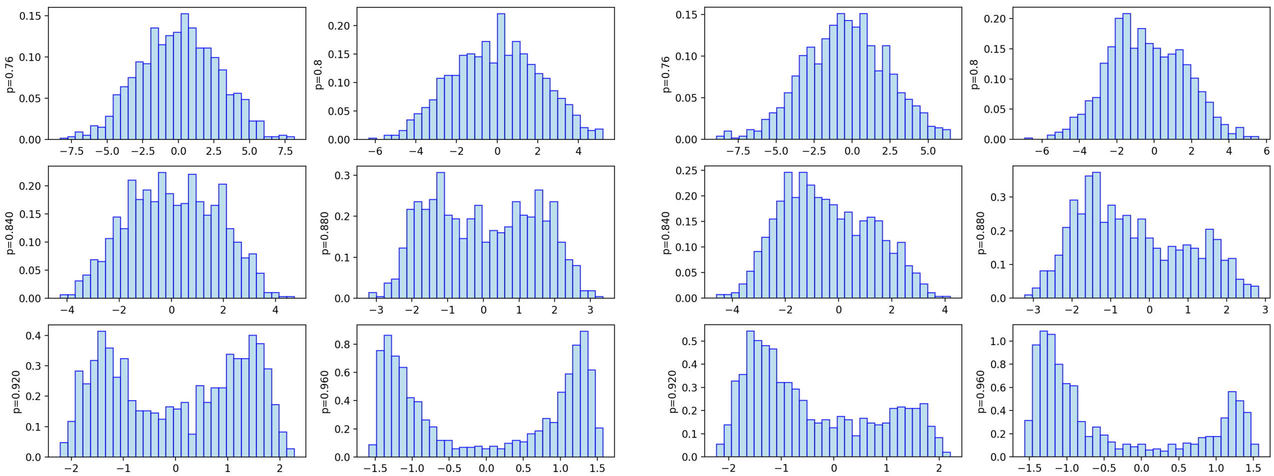


Figure 7. Histograms of L values when $q = 0.5$ (left) or $q = 0.3$ (right) depending on the value of p.

	Diffusive	**Critical**	**Superdiffusive**
LLN	$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} 0$	$\frac{S_n}{\sqrt{n}\log n} \xrightarrow[n\to\infty]{\text{a.s.}} 0$	$\frac{S_n}{n^{2p-1}} \xrightarrow[n\to\infty]{\text{a.s. } / \ \mathbb{L}^4} L_q$
CLT	$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3-4p}\Big)$	$\frac{S_n}{\sqrt{n \log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$	$\frac{S_n - n^{2p-1} - L_q}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{4p-3}\Big)$

There are various ways to study the asymptotic behavior of the ERW. Baur and Bertoin [6] used the connection to Pólya-type urns as well as two functional limit theorems for multitype branching processes due to Janson [27]. Bercu [7] and Coletti et al. [16] used martingales to obtain the almost sure convergences and asymptotic normality, among other results. Kürsten [31] and Businger [15] used a relation between the ERW and random trees constructed with Bernoulli percolation, which ensures that one remembers all of the past information. In this note, we will give more details on this three approaches.

4.2. **The martingale approach**

Martingales were first used by Coletti et al. [16] in order to obtain the law of large numbers and the central limit theorem. Afterwards, Bercu [7] used a more general martingale theory to obtain the law of iterated logarithm and the quadratic strong law in the diffusive and critical regimes, as well as the convergence in $\mathbb{L}^4$ in the superdiffusive regime, and also retrieved the previous results.

In order to understand well how the elephant moves, it is straightforward to see that for any time $n \geq 1$,

$$X_{n+1} = \alpha_{n+1} X_{\beta_{n+1}}$$

where α_{n+1} and β_{n+1} are two independent discrete random variables such that α_{n+1} has a Rademacher $\mathcal{R}(p)$ distribution while β_{n+1} is uniformly distributed over the integers $\{1, \ldots, n\}$. Moreover, α_{n+1} is independent of $X_1, \ldots, X_n$.

Let $(\mathcal{F}_n)$ be the increasing sequence of σ-algebras, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.We deduce that

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \Big(1 + \frac{a}{n}\Big) S_n = \gamma_n S_n \quad \text{where} \quad a = 2p-1.$$

Then, the process

$$M_n = a_n S_n \quad \text{where} \quad a_1 = 1,\, a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(a+1)\Gamma(n)}{\Gamma(n+a)}$$

is a locally bounded square-integrable martingale. Indeed,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = a_{n+1}\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = a_{n+1}\gamma_n S_n = a_n S_n = M_n$$

and $\mathbb{E}[M_n^2] \leq (na_n)^2$. The process (M_n) can be rewritten as

$$M_n = \sum_{k=1}^{n} a_k \varepsilon_k \quad \text{where} \quad \varepsilon_k = S_k - \gamma_{k-1} S_{k-1}.$$

The asymptotic behavior of (M_n) is directly given by the one of its quadratic variation and it is possible to show that

$$\langle M \rangle_n = v_n + \mathrm{o}(v_n) \quad \text{where} \quad v_n = \sum_{k=1}^{n} a_k^2.$$

such that the asymptotic behavior of $\langle M \rangle_n$ is closely related to the one of (v_n).

Thanks to asymptotic equivalent for the Gamma function, we have that $a_n = O(n^{-a})$ and we obtain three different regimes for the elephant's behavior :

- The diffusive regime where $a < 1/2$ (or $p < 3/4$) and $v_n = O(n^{1-2a})$,
- The critical regime where $a = 1/2$ (or $p = 3/4$) and $v_n = O(\log n)$,
- The superdiffusive regime where $a > 1/2$ (or $p > 3/4$) and $v_n = O(1)$.

The strategy here to obtain asymptotic results for the ERW relies on the theory of martingales.

4.3. **The Pólya-type urns approach**

This approach was first introduced by Baur and Bertoin [6] in order to obtain functional convergences for the elephant random walk thanks to the work of Janson [27]. The method uses a connection to Pólya-type urns that was already known before in the literature. A bit more precisely, given what is known from the theory of urns, it implies that the asymptotic behavior of such models is determined by the spectral decomposition of the (mean) replacement matrix of the corresponding urn.

Let (U_n) be discrete-time urn with balls of two colors, red and blue. The composition of the urn at time $n \in \mathbb{N}$ is given by a vector $U_n = (R_n, B_n)$ where R_n stands for the number of red balls and B_n for the number of blue balls at time n. The starting composition of the urn is $(1, 0)$ with probability q or $(0, 1)$ with probability $1 - q$. Then, the urn is implemented as follows. At any time $n \geq 2$ a ball is drawn uniformly at random, its color observed, then it is returned to the urn together with a ball of the same color with probability p, or with a ball of the other color with probability $1 - p$.

The connection to the ERW model is straightforward. Let (S_n) denotes the ERW started from $S_0 = 0$ and such that $S_1 = R_1 - B_1$, then for every $n \geq 1$

$$(S_n)_{n\geq 1} \stackrel{\mathcal{L}}{=} (R_n - B_n)_{n\geq 1}$$

where $\stackrel{\mathcal{L}}{=}$ refers to equality in law. In other words, the difference between the number of red and blue balls in the urn behaves like an ERW with first step equals to $R_1 - B_1$.

To study this process, we are interested in the spectral decomposition of the mean replacement matrix A, given by

$$A = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

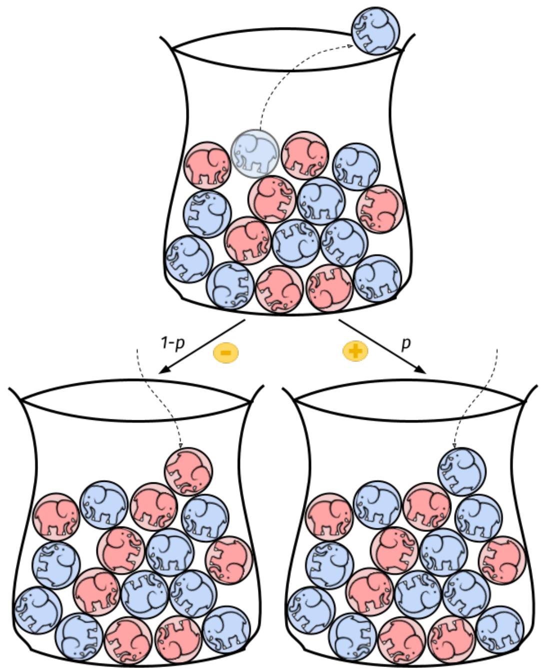


Figure 8. The Pólya urn representing the ERW.

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2p - 1 = a$ and the corresponding unit vectors in $\mathbb{L}^1$ are

$$v_1^T = \frac{1}{2}(1,\ 1), \quad v_2^T = \frac{1}{2}(1,\ -1).$$

It is well-known that the asymptotics of the urn depends on the ratio λ_2/λ_1 with respect to $1/2$. This is coherent and yet another good explanation to why the transition between the regimes for the ERW occurs at $a = 1/2$ which, as expected, is equivalent to $p = 3/4$.

4.4. The random recursive tree approach

Let (Z_n) be a sequence of i.i.d. $\mathcal{R}(\frac{1}{2})$ random variables and (ε_n) a sequence of i.i.d. Bernoulli random variables with parameter $0 < (1-a) < 1$. Then, set $\hat{X}_1 = Z_1$ and, for $n \geq 1$,

$$\hat{X}_{n+1} = \begin{cases} Z_{\sigma(n)+1} & \text{if } \varepsilon_{n+1} = 1, \\ \hat{X}_{\mathcal{U}(n)} & \text{if } \varepsilon_{n+1} = 0, \end{cases}$$

where $\mathcal{U}(n)$ stands for the uniform distribution on $\{1, \ldots, n\}$ and $\sigma(n) = \sum_{j=1}^{n} \varepsilon_j$ is counting the number of innovations (i.e., the times j where $\varepsilon_j = 1$). Kürsten [31] explained that the sequence

$$\hat{S}_n = \hat{X}_1 + \ldots + \hat{X}_n$$

is the elephant random walk with memory parameter $p = (a+1)/2$. This approach can also be seen as a sequence of random recursive trees on which a Bernoulli percolation of parameter a has been performed.

More precisely, we first recall that random recursive trees are rooted trees with increasing labels along branches that are build in a recursive manner in the following way. We denote by $\mathcal{T}_1$ the tree with a single node with label 1. Then $\mathcal{T}_{n-1}$ stands for the tree with $n-1$ nodes and $\mathcal{T}_n$ is build by choosing uniformly at random one of the nodes of $\mathcal{T}_{n-1}$ and adding the n-th node (the node with label n) to the chosen node. Then, each edge deleted with probability $0 < 1 - a < 1$, independently of the other edges, such that we have performed a Bernoulli bond percolation on the tree. This is the same process as the one were every time a new node is attached to an old one, the edge is in fact kept with probability p or deleted with probability $1 - p$.

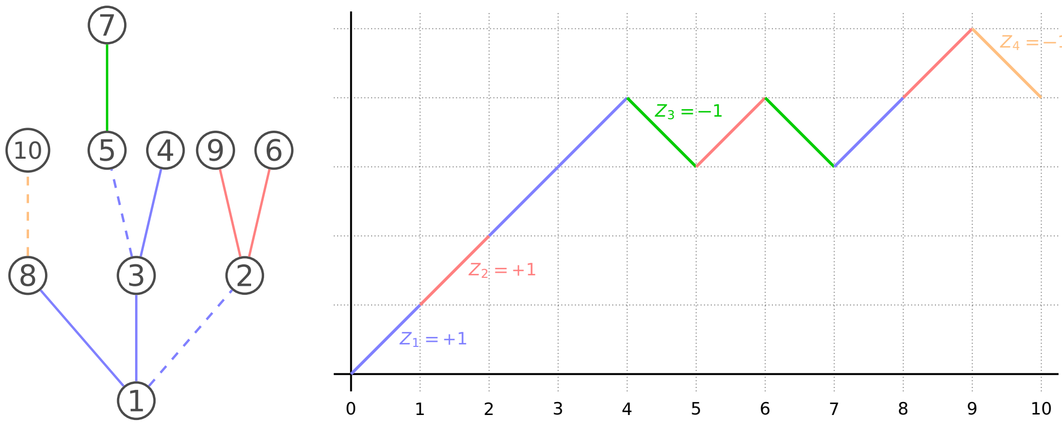


FIGURE 9. On the left, some RRT of size 10 after the Bernoulli bond percolation has been performed. The deleted edges are represented with dashed lines. On the right, the ERW deduced from the the RRT.

Then, for the ERW: the first step corresponds to the node with label 1 (the root), the second step to the node with label 2 and so on. We assign the spin Z_1 to the root. Then, the tree is built recursively as described above. For building $\mathcal{T}_n$, we pick one of the nodes of $\mathcal{T}_{n-1}$ and connect the n-th node to the chosen node. With probability $1 - p$ the edge connecting the new node to the existing node is deleted and, when this happens, we assign the spin $Z_{\sigma(n)+1}$ to the new node. Otherwise, when the edge is kept (with probability p), the new node takes the same spin as the node it is attached to.

In that setting, we denote the *i-th* cluster at time n $c_n(i) = \{j \leq n,\ \hat{X}_j = Z_i\}$ in the way that

$$\hat{S}_n = \sum_{i=1}^{\infty} |c_n(i)| Z_i$$

where the size clusters are independant of the sequence (Z_i).

We denote by τ_i the first instant at which the i-*th* cluster is not empty, $\tau_i = \inf\{j \geq 1,\ \hat{X}_j = Z_i\} = \inf\{j \geq 1,\ \sigma(j) = i - 1\}$ with $\tau_1 = 1$. It has been proved by Baur and Bertoin [5] that

$$\lim_{n \to \infty} \frac{|c_n(i)|}{n^a} = C_i \quad \text{a.s.}$$

where that C_1 has a Mittag-Leffler distribution with parameter a and C_i a random variable with the same law as $(\beta_{\tau_i})^a \cdot C_1$, where β_i denotes a beta variable with parameter $(1, i - 1)$ and is further independent of β_1. Consequently, it is possible to obtain the following decomposition of the random variable L_q appearing in the superdiffusive regime

$$L_q = \sum_{i=1}^{\infty} C_i Z_i = C_1 \cdot \sum_{i=1}^{\infty} (\beta_{\tau_i})^a Z_i.$$

Then, it is possible to show that the random variable L is continuous, $\mathbb{P}(L = 0) = 0$, via well-known properties of Rademacher series with infinitely many nonzero random coefficients. However, this is not enough to conclude that the random variable L_q is absolutely continuous and that it admits a density.

This was still an open problem until the recent work [24], where thanks to the connection with Pólya urns, questions such as the existence of a density, an explicit formula for the moments, the moment problem, the finiteness of the moment-generating function have been solved.

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