Random Interfaces and Pinning Models

Lecture Notes, Quentin Berger

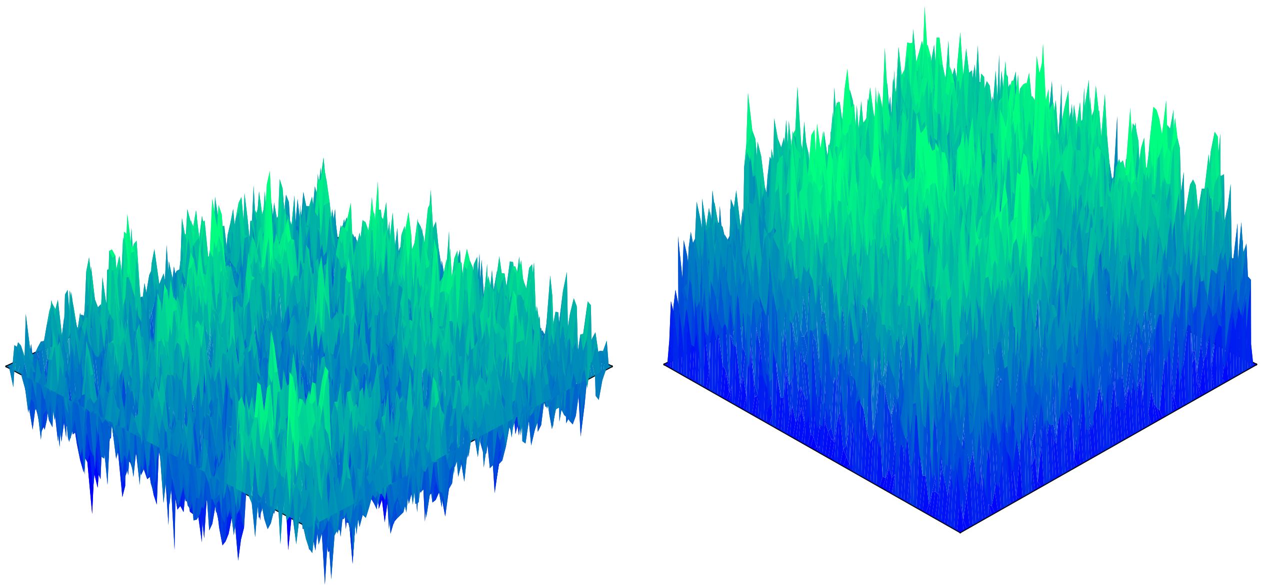

FIG. Gaussian free field and Gaussian free field conditioned to be positive.

Abstract

These lecture notes were written for a class that I taught between 2018 and 2023 for the "Probabilités et Modèles Aléatoires" Master at Sorbonne Université. The goal is to give a broad overview of various statistical mechanics models and of the question of disorder relevance, in particular in the context of pinning models.

• The first part of the notes deal with random interfaces models, the idea being to give both a general framework and develop an important example, namely the (lattice) Gaussian Free Field. This should give some gentle introduction to the subject (I hope) and provide glimpses into some of the questions that have been studied in the literature: the question of localization vs. delocalization of interfaces; the entropic repulsion phenomenon when introducing a random wall constraint; phase transitions for pinning models of interfaces.

• The second part considers a related one-dimensional model: the so-called polymer pinning model. The goal of this part is to give some introduction to the study of disordered systems, through an important example that has been extensively investigated over the past decades. The idea is to first give an overview of the disordered pinning model and of its localization phase transition, before turning to the question of disorder relevance for this model, *i.e.* determining whether some disorder of arbitrarily small intensity may change the properties of the model.

Table of contents

I Random interface models . . . 7

1 Effective interface models . . . 9

1.1 Interface $\nabla\varphi$-models without constraints . . . 10
1.1.1 Definitions and notations . . . 10
1.1.2 Two important examples . . . 12
1.1.3 The case of dimension $d = 1$ . . . 13
1.1.4 Gibbs measures and spatial Markov property . . . 14
1.2 The (lattice) Gaussian Free Field on $\mathbb{Z}^d$ . . . 17
1.2.1 The random walk representation of the covariance structure . . . 17
1.2.2 Localization and delocalization of the GFF interface . . . 21
1.2.3 Infinite volume limit of the GFF . . . 24
1.2.4 Scaling limit: the continuous-space GFF . . . 27
1.3 Hard wall constraint and entropic repulsion . . . 29
1.3.1 The case of dimension $d = 1$ . . . 29
1.3.2 The case of the GFF in dimension $d \geq 3$ . . . 35
1.3.3 The case of the GFF in dimension $d = 2$ . . . 43
1.4 A few exercises . . . 44

2 Pinning and wetting of random interfaces . . . 49

2.1 Introduction of the model and first properties . . . 49
2.1.1 Definition of the pinning and wetting of interfaces . . . 49
2.1.2 Free energy and first properties . . . 51
2.1.3 The (localization) phase transition . . . 53
2.1.4 Existence of the free energy . . . 54
2.2 Free energy and phase transition in dimension $d = 1$ . . . 62
2.2.1 Implicit formula for the free energy . . . 62
2.2.2 Critical points and critical behavior of the free energy . . . 67

2.3 Free energy and phase transition in dimension $d = 2$ 71
2.4 Free energy and phase transition in dimension $d \geq 3$ 74
2.4.1 The unconstrained case (pinning) 75
2.4.2 The constrained case (wetting) 76
2.5 A few exercises 80

II The (polymer) pinning model 83

3 Pinning on a defect line: overview of the phase diagram 85
3.1 Introduction of the (disordered) pinning model 85
3.1.1 The homogeneous pinning model 86
3.1.2 The disordered pinning model 87
3.1.3 The question of the influence of disorder 89
3.2 Solving the homogeneous model 90
3.2.1 The localization phase transition 91
3.2.2 Properties of renewal trajectories under the pinning measure . . . 95
3.3 Intermezzo: some properties of renewal processes 97
3.3.1 The positive recurrent case 97
3.3.2 The null recurrent case 98
3.3.3 The transient case 99
3.3.4 Properties of trajectories: a few exercises 99
3.4 The disordered model: free energy and phase transition 100
3.4.1 The free energy: existence and first properties 100
3.4.2 About the phase transition 106
3.4.3 Properties of trajectories: localized vs. delocalized 109
3.5 Some properties of the phase diagram 110
3.5.1 Quenched vs. annealed 110
3.5.2 Conditions for the existence of a phase transition 111
3.5.3 Disorder helps localization 114
3.5.4 Summary of the phase diagram 117
3.6 A few exercises 117

4 Relevance vs. irrelevance of disorder for the pinning model 123
4.1 Preliminaries on the question of disorder relevance 123
4.1.1 Relevance vs. irrelevance: the Harris criterion 123

4.1.2 Monotonicity of disorder relevance 125
4.2 From weak to strong disorder: the martingale approach 129
4.2.1 The martingale 129
4.2.2 Weak vs. strong disorder 130
4.2.3 Criterion for the existence of a weak disorder phase 131
4.3 Disorder irrelevance 133
4.4 Disorder relevance 136
4.4.1 About the critical behavior: the smoothing inequality 137
4.4.2 About the critical point shift 141
4.5 Back to the martingale: proof of Theorem 4.6 145
4.5.1 Weak disorder and the L^2 phase 146
4.5.2 Strong disorder and change of measure argument 147
4.5.3 The intermediate disorder regime 153
4.5.4 A few exercises 158

Bibliography **161**

A few general references

For Chapter 1:

- The survey [Vel06] by Yvan Velenik or the lecture notes [Gia01] by Giambattista Giacomin.
- For Section 1.2 on the Gaussian Free Field, the book by Sasha Friedli and Yvan Velenik *Statistical Mechanics of Lattice Systems* [FV17, Ch. 8], an excellent reference.

For Chapter 2:

- The book [Gia07] by Giambattista Giacomin *Random Polymer Models* (for the dimension $d = 1$).
- A part of the article [GL17] *Pinning and disorder relevance for the lattice Gaussian free field*, by Giambattista Giacomin and Hubert Lacoin.

For Chapters 3 and 4:

- The books [Gia07, Gia11] by Giambattista Giacomin, *Random Polymer Models* and *Disorder and critical phenomena*.

Part I

Random interface models

Chapter 1

Effective interface models

The interface models that we will study in these lecture notes were introduced as effective models for describing interfaces in statistical physics systems, but they are interesting in their own right. They describe a simplified random surface, *i.e.* a random height function $\varphi : \Lambda \to \mathbb{R}$ or $\mathbb{Z}$, where Λ is a domain of $\mathbb{Z}^d$, with $d \geq 1$. We will refer to this as an interface in dimension $d+1$. An important model in this context, on which much of this chapter will focus, is the Gaussian Free Field (GFF) on $\mathbb{Z}^d$.

Notation. We denote by $x \sim y$ if x and y are neighbors in $\mathbb{Z}^d$, that is if we have $\|x-y\|_1 = \sum_{i=1}^d |x_i - y_i| = 1$. We write $\Lambda \Subset \mathbb{Z}^d$ if Λ is a *finite* subset of $\mathbb{Z}^d$ and $\partial\Lambda := \{y \in \Lambda^c, \exists\, x \in \Lambda, x \sim y\}$ will denote the *exterior boundary* of Λ.

If $\varphi : \mathbb{Z}^d \to \mathbb{R}$ or $\mathbb{Z}$ is a given function and $\Lambda \subset \mathbb{Z}$, we will denote by φ_Λ the restriction of φ to Λ. We also denote $\varphi_x = \varphi(x)$, that we interpret as the height of the interface above the point x. If φ has values in $\mathbb{Z}$ we will talk about discrete heights models; if φ has values in $\mathbb{R}$, we will talk about continuous heights models.

Gibbs measures

Consider a physical system whose set of configurations can be described by a set Ω. If $H : \Omega \to \mathbb{R}$ is an energy function, called the Hamiltonian, which associates an energy with each possible configuration of the system, then we can consider the Gibbs measure P_β on Ω:

$$\frac{\mathrm{d}\mathrm{P}_\beta}{\mathrm{d}\mu}(\omega) = \frac{1}{Z_\beta}\,\mathrm{e}^{-\beta H(\omega)}\,. \tag{1.1}$$

Here, μ is a reference measure on Ω (not necessarily a probability measure), often given by the counting measure on Ω if Ω is discrete, or the Lebesgue measure if $\Omega \simeq \mathbb{R}^n$ for some $n \geq 1$. The parameter $\beta > 0$ is a positive parameter, related to the inverse of the temperature ($\beta = \frac{1}{k_B T}$ where k_B is Boltzmann's constant), which tunes the influence of the energy on the configurations of the system. The constant Z_β is called the *partition function*: it is the constant that renormalizes the Gibbs measure to obtain a probability on Ω. The measure P_β describes the probabilities of occurrence of possible configurations: if the energy $H(\omega)$ of a configuration is high, it will be less likely. For instance, one can show that when $\beta \to \infty$, the measure P_β concentrates on configurations with minimal energy.

Remark 1.1. Let us stress that one can modify the Hamiltonian by adding a constant, without changing the Gibbs measure P_β. Indeed, if $H'(\omega) = H(\omega) + C$ for some constant C that does not depend on $\omega \in \Omega$, then

$$\frac{\mathrm{dP}'_\beta}{\mathrm{d}\mu}(\omega) = \frac{1}{Z'_\beta}\,\mathrm{e}^{-\beta H'(\omega)} = \frac{1}{Z'_\beta\,\mathrm{e}^{\beta C}}\,\mathrm{e}^{-\beta H(\omega)}\,.$$

This very much gives the same measure $\mathrm{P}'_\beta = \mathrm{P}_\beta$ (but not the same partition function: $Z_\beta = \mathrm{e}^{\beta C} Z'_\beta$).

Remark 1.2. Gibbs measures naturally appear as measures that maximize the entropy of a system whose energy has been fixed. For example, the normal distribution is a Gibbs measure on $\Omega = \mathbb{R}$, associated with the Hamiltonian $H(x) = x^2$ (if x represents the velocity of a particle, $H(x)$ is its kinetic energy), with $\mu = \mathrm{Leb}$ as the reference measure on Ω. It can be shown that, among the probability distributions with density f on $\mathbb{R}$ with mean 0 and variance 1, the normal distribution $\mathcal{N}(0,1)$ is the one that maximizes the entropy $-\int_{\mathbb{R}} f(x)\log f(x)\mathrm{d}x$.

1.1 Interface $\nabla\varphi$-models without constraints

1.1.1 Definitions and notations

We consider a symmetric and convex function $V : \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$, called *potential*. For a finite domain $\Lambda \Subset \mathbb{Z}^d$ and a boundary condition $\xi : \mathbb{Z}^d \to \mathbb{R}$, we associate to a function $\varphi : \mathbb{Z}^d \to \mathbb{R}$ the following Hamiltonian:

$$H_\Lambda^\xi(\varphi) = H(\varphi_\Lambda \mid \xi_{\Lambda^c}) = \frac{1}{4d} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} V(\varphi_x - \varphi_y) + \frac{1}{2d} \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ x\sim y}} V(\varphi_x - \xi_y)\,. \tag{1.2}$$

The function V represents interactions in the surface and gives higher energy to larger gradients in the surface (this is referred to as a gradient model, or $\nabla\varphi$-model). Thanks to Remark 1.1, we can assume that $V(0) = 0$.

We then define an interface model on Λ by considering a Gibbs measure on the set of (continuous or discrete) surfaces

$$\Omega_\Lambda^{\mathbb{R}} = \{h : \Lambda \to \mathbb{R}\} \quad \text{ or } \quad \Omega_\Lambda^{\mathbb{Z}} = \{h : \Lambda \to \mathbb{Z}\}\,,$$

with boundary condition ξ. Taking μ_Λ to be the Lebesgue measure on $\Omega_\Lambda^{\mathbb{R}} = \mathbb{R}^\Lambda$ and μ_Λ to be the counting measure on $\Omega_\Lambda^{\mathbb{Z}} = \mathbb{Z}^\Lambda$, we consider the Gibbs measure:

$$\frac{\mathrm{dP}_{\Lambda,\beta}^\xi}{\mathrm{d}\mu_\Lambda}(\varphi) = \frac{1}{Z_{\Lambda,\beta}^\xi}\,\mathrm{e}^{-\beta H_\Lambda^\xi(\varphi)}\,. \tag{1.3}$$

The partition function $Z_{\Lambda,\beta}^\xi$ is written as follows:

$$Z_{\Lambda,\beta}^\xi = \int_{\Omega_\Lambda} \mathrm{e}^{-\beta H_\Lambda^\xi(h)} \mathrm{d}\mu_\Lambda(h)\,. \tag{1.4}$$

To simplify notation, we write $\Omega_\Lambda = \Omega_\Lambda^{\mathbb{R}/\mathbb{Z}}$; we will specify whether it is a discrete model or a continuous model if necessary. We will often denote by h an element of Ω_Λ and by φ_Λ a random variable with values in Ω_Λ: for example, for any Borel set A of Ω_Λ, we will write:

$$\mathrm{P}_{\Lambda,\beta}^\xi(\varphi_\Lambda \in A) = \frac{1}{Z_{\Lambda,\beta}^\xi} \int_A \mathrm{e}^{-\beta H_\Lambda^\xi(h)} \prod_{x\in\Lambda} \mathrm{d}\mu(h_x)\,, \quad Z_{\Lambda,\beta}^\xi = \int_{\Omega_\Lambda} \mathrm{e}^{-\beta H_\Lambda^\xi(h)} \prod_{x\in\Lambda} \mathrm{d}\mu(h_x)\,,$$

where μ denotes either the Lebesgue measure on $\mathbb{R}$ or the counting measure on $\mathbb{Z}$ depending on whether we are in the discrete or continuous case (in any case we have $\mu_\Lambda = \mu^{\otimes\Lambda}$). We also denote by $\mathrm{E}_{\Lambda,\beta}^\xi$ the expectation relative to the probability $\mathrm{P}_{\Lambda,\beta}^\xi$.

The boundary condition can be interpreted as follows: the measure $\mathrm{P}_{\Lambda,\beta}^\xi$ puts a positive mass only on the functions $\varphi : \mathbb{Z}^d \to \mathbb{R}$ that coincide with ξ on Λ^c. Another interpretation is to consider $\mathrm{P}_{\Lambda,\beta}^\xi$ as a measure on the functions $\varphi : \Lambda \to \mathbb{R}$ that are extended to $\partial\Lambda$ by setting $\varphi_{\partial\Lambda} = \xi_{\partial\Lambda}$.

Lemma 1.3. *If one has* $K_\beta := \int_{\mathbb{R}} \mathrm{e}^{-\beta V(h)} \mathrm{d}\mu(h) < +\infty$*, then, for any boundary condition* ξ*, we have* $Z^{\xi}_{\Lambda,\beta} \le (K_\beta)^{|\Lambda|} < +\infty$*. In particular, the partition function is (positive and) finite, and the Gibbs measure* $\mathrm{P}^{\xi}_{\Lambda,\beta}$ *given in* (1.3) *is well defined.*

Proof. We can easily reduce this to the case where Λ is connected: if this is not the case, we can use Exercise 1 below.

Let us first assume that the dimension is $d = 1$ and that $\Lambda = \{1, \dots, N\}$. Then

$$Z^{\xi}_{\Lambda,\beta} = \int_{\mathbb{R}^N} \prod_{i=1}^{N+1} \mathrm{e}^{-\beta V(h_i - h_{i-1})} \prod_{i=1}^{N} \mathrm{d}\mu(h_i)\,,$$

with, by convention, $h_0 = \xi_0$ and $h_{N+1} = \xi_{N+1}$ in the integral. Using the upper bound $\mathrm{e}^{-\beta V(\xi_{N+1} - h_N)} \le 1$, we easily obtain by iteration

$$Z^{\xi}_{\Lambda,\beta} \le \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{e}^{-\beta V(h_i - h_{i-1})} \prod_{i=1}^{N} \mathrm{d}\mu(h_i) \le (K_\beta)^N\,.$$

For the recurrence, we used a change of variable (and the fact that μ is invariant under translation) to obtain that for all $h' \in \mathbb{R}$, $\int_{\mathbb{R}} \mathrm{e}^{-\beta V(h-h')} \mathrm{d}\mu(h) = K_\beta$.

In the case of dimension $d \ge 2$, the argument is the same but the notation is a little more cumbersome. The idea is to consider a spanning tree of Λ, to upper bound $\mathrm{e}^{-\beta V(h_x - h_y)} \le 1$ if $x, y \in \Lambda$ are not neighbors in the tree, and to integrate step by step, each time integrating an index x that is of degree 1 in the tree. We leave it as an exercise to write a rigorous proof. □

Exercise 1. If $\Lambda, \Lambda' \Subset \mathbb{Z}^d$ are disjoints and such that $x \not\sim y$ for any $x \in \Lambda$, $y \in \Lambda'$, show that $Z^{\xi}_{\Lambda\cup\Lambda',\beta} = Z^{\xi}_{\Lambda,\beta} Z^{\xi}_{\Lambda',\beta}$.

Exercise 2. Show that, for any $a < b$, there exists a constant $c > 0$ such that, for any boundary condition ξ, we have $\mathrm{P}^{\xi}_{\Lambda,\beta}\big(\varphi_x \in [a,b]\ \forall x \in \Lambda\big) \le \mathrm{e}^{-c|\Lambda|}$.

1.1.2 Two important examples

Solid-On-Solid (SOS). When modeling an interface of the Ising model, the height function φ has values in $\mathbb{Z}$, and the energy associated with the interface is proportional to the number of disagreements between '+' and '−' spins, *i.e.*, proportional to the area of the interface. The Hamiltonian associated with an interface is therefore proportional to $\sum_{x\sim y}|\varphi_x - \varphi_y| + |\Lambda|$ (the first sum counts the area of the 'vertical' parts and the second term counts the area of the 'horizontal' parts). This therefore corresponds to the choice $V(x) = |x|$ in the Hamiltonian (1.2) (recall that the constant $|\Lambda|$ does not change the Gibbs measure, see Remark 1.1).

Gaussian Free Field (GFF). The choice of a height function φ with values in $\mathbb{R}$ and a quadratic potential $V(x) = \frac{1}{2}x^2$ is also very natural, because then $H_\lambda(\varphi)$ corresponds to the energy of a spring configuration. We will see that in this case we can show that φ_Λ is a Gaussian vector, whose expectation and covariance matrix can be characterized, which will allow us to perform many calculations later on (see Section 1.2).

1.1.3 The case of dimension $d = 1$

Let us consider the case of dimension $d = 1$, with $\Lambda = \{1, \dots, N\}$. As in Lemma 1.3, we assume that $K_\beta = \int_{\mathbb{R}} \mathrm{e}^{-\beta V(h)} \mathrm{d}\mu(h) < +\infty$. Thus, we can define a probability distribution ν_β on $\mathbb{R}$ or $\mathbb{Z}$ by setting:

$$\frac{\mathrm{d}\nu_\beta}{\mathrm{d}\mu}(x) = \frac{1}{K_\beta}\, \mathrm{e}^{-V(x)}\,. \tag{1.5}$$

Now consider $(X_i)_{i\geq 1}$ independent random variables with the same distribution ν_β (*i.e.*, with values in $\mathbb{R}$ in the continuous case and in $\mathbb{Z}$ in the discrete case). Let us consider the random walk constructed from the random variables $(X_i)_{i\geq 1}$, as follows: $S_0 = \xi_0$ and $S_k = \xi_0 + \sum_{i=1}^k X_i$ for $k \geq 1$.

We can then determine the distribution of $(S_1, \dots, S_{N+1})$, which is obtained by a simple change of variable from the product distribution $\nu_\beta^{\otimes(N+1)}$:

$$\mathrm{dP}_{(S_1,\dots,S_{N+1})}(s_1, \dots, s_{N+1}) = \frac{1}{K_\beta^{N+1}} \prod_{i=1}^{N+1} \mathrm{e}^{-V(s_i - s_{i-1})} \prod_{i=1}^{N+1} \mathrm{d}\mu(s_i)\,,$$

where we have used the convention $s_0 = \xi_0$ and the fact that μ is invariant under translation.

We then notice that the distribution of $(S_1, \dots, S_N)$ conditional on $S_{N+1} = \xi_{N+1}$ is given (both in the discrete and continuous cases) by

$$\mathrm{dP}_{(S_1,\dots,S_N)|S_{N+1}=\xi_{N+1}}(s_1, \dots, s_N) = \frac{\mathrm{dP}_{(S_1,\dots,S_{N+1})}(s_1, \dots, s_N, \xi_{N+1})}{\mathrm{dP}_{S_{N+1}}(\xi_{N+1})}$$

that is

$$\mathrm{dP}_{(S_1,\dots,S_N)|S_{N+1}=\xi_{N+1}}(s_1,\dots,s_N) = \frac{\frac{1}{K_\beta^{N+1}}\prod_{i=1}^{N+1}\mathrm{e}^{-V(s_i-s_{i-1})}\prod_{i=1}^{N}\mathrm{d}\mu(s_i)}{\left(\int_{\mathbb{R}^N}\frac{1}{K_\beta^{N+1}}\prod_{i=1}^{N+1}\mathrm{e}^{-V(s_i-s_{i-1})}\prod_{i=1}^{N}\mathrm{d}\mu(s_i)\right)},$$

where here we have set by convention $s_0 = \xi_0$ and $s_{N+1} = \xi_{N+1}$ (in both the numerator and denominator).

But this last expression is exactly the Gibbs measure defined in (1.3). Therefore, we have shown the following result.

Lemma 1.4. *If $\Lambda = \{1,\dots,N\}$, then the distribution of $\varphi_\Lambda = (\varphi_i)_{1\le i\le N}$ under $\mathrm{P}^\xi_{\Lambda,\beta}$ is that of a random walk $(S_i)_{1\le i\le N}$ with i.i.d. increments X_i with law ν_β from (1.5), started from $S_0 = \xi_0$ and conditioned to end at $S_{N+1} = \xi_{N+1}$.*

1.1.4 Gibbs measures and spatial Markov property

In this section, we will prove the following property, called *spatial Markov property*. For $\Lambda \subset \mathbb{Z}^d$, we denote by $\mathcal{F}_\Lambda = \sigma(\varphi_x, x\in\Lambda)$ the σ-algebra generated by the height function φ on Λ.

Proposition 1.5 (Spatial Markov property)**.** *Let $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^d$. Then for any non-negative measurable function $f : \Omega_{\Lambda_1} \to \mathbb{R}_+$, we have*

$$\mathrm{E}^\xi_{\Lambda_2,\beta}\big[f(\varphi_{\Lambda_1}) \mid \mathcal{F}_{\Lambda_1^c}\big] = \mathrm{E}^\phi_{\Lambda_1,\beta}\big[f(\varphi_{\Lambda_1})\big]$$

for $\mathrm{P}^\xi_{\Lambda_2,\beta}$ almost every realization ϕ.

Note that the probability $\mathrm{P}^\phi_{\Lambda_1,\beta}$ depends only on the value of ϕ on $\partial\Lambda_1$, and we therefore have

$$\mathrm{P}^\xi_{\Lambda_2,\beta}\big(\,\cdot \mid \mathcal{F}_{\Lambda_1^c}\big) = \mathrm{P}^\xi_{\Lambda_2,\beta}\big(\,\cdot \mid \mathcal{F}_{\partial\Lambda_1}\big) = \mathrm{P}^\phi_{\Lambda_1,\beta}\big(\,\cdot\,\big)\,,$$

which justifies the name spatial Markov property.

Corollary 1.6. *For any non-negative measurable function $f : \Omega_{\Lambda_1} \to \mathbb{R}_+$, we have*

$$\mathrm{E}^\xi_{\Lambda_2,\beta}\big[f(\varphi_{\Lambda_1})\big] = \mathrm{E}^\xi_{\Lambda_2,\beta}\Big[\mathrm{E}^\phi_{\Lambda_1,\beta}\big[f(\varphi_{\Lambda_1})\big]\Big]\,.$$

In other words, under $\mathrm{P}^\xi_{\Lambda_2,\beta}$, the marginal distribution of φ inside Λ_1 is the Gibbs measure $\mathrm{P}^\phi_{\Lambda_1,\beta}$ with boundary condition given by a realization of ϕ under $\mathrm{P}^\xi_{\Lambda_2,\beta}$.

Proof. First, note that for any boundary condition ϕ, we have

$$\mathrm{E}^{\phi}_{\Lambda_1,\beta}\big[f(\varphi_{\Lambda_1})\big] = \int_{h\in\mathbb{R}^{\Lambda_1}} f(h)\,\frac{1}{Z^{\phi}_{\Lambda_1,\beta}}\,\mathrm{e}^{-\beta H^{\phi}_{\Lambda_1}(h)}\mathrm{d}\mu_{\Lambda_1}(h)\,,$$

and notice that this is a measurable function of the boundary condition $(\phi_x)_{x\in\partial\Lambda_1}$ and therefore of $(\phi_x)_{x\in\Lambda_1^c}$.

Now let $g:\Omega_{\Lambda_1\setminus\Lambda_2}\to\mathbb{R}_+$ be a non-negative measurable function. Then we have

$$\begin{aligned}&\mathrm{E}^{\xi}_{\Lambda_2,\beta}\Big[g(\varphi_{\Lambda_2\setminus\Lambda_1})\mathrm{E}^{\phi}_{\Lambda_1,\beta}\big[f(\varphi_{\Lambda_1})\big]\Big]\\ &\quad=\int_{\tilde h\in\mathbb{R}^{\Lambda_2}}\int_{h\in\mathbb{R}^{\Lambda_1}} g(\tilde h_{\Lambda_2\setminus\Lambda_1})f(h)\,\frac{1}{Z^{\tilde h}_{\Lambda_1,\beta}}\,\mathrm{e}^{-\beta H^{\tilde h}_{\Lambda_1}(h)}\,\frac{1}{Z^{\xi}_{\Lambda_2,\beta}}\,\mathrm{e}^{-\beta H^{\xi}_{\Lambda_2}(\tilde h)}\mathrm{d}\mu_{\Lambda_1}(h)\mathrm{d}\mu_{\Lambda_2}(\tilde h)\,.\end{aligned}$$

Now, let us denote $\hat h$ the "concatenation" of $h\in\mathbb{R}^{\Lambda_1}$ and $\tilde h_{\Lambda_2\setminus\Lambda_1}\in\mathbb{R}^{\Lambda_2\setminus\Lambda_1}$, that is, the function $\hat h\in\mathbb{R}^{\Lambda_2}$ such that $\hat h = h$ on Λ_1 and $\hat h=\tilde h$ on $\Lambda_2\setminus\Lambda_1$. We now observe that

$$H^{\xi}_{\Lambda_2}(\tilde h) + H^{\tilde h}_{\Lambda_1}(h) = H^{\hat h}_{\Lambda_1}(\tilde h) + H^{\xi}_{\Lambda_2}(\hat h)\,.$$

Note also that $Z^{\tilde h}_{\Lambda_1,\beta}$ is a measurable function of $(\tilde h_x)_{x\in\Lambda_1^c}$, so $Z^{\tilde h}_{\Lambda_1,\beta} = Z^{\hat h}_{\Lambda_1,\beta}$. By rearranging the integrals, the expression obtained above is therefore equal to

$$\begin{aligned}&\int_{\hat h\in\mathbb{R}^{\Lambda_2}}\int_{\tilde h\in\mathbb{R}^{\Lambda_1}} g(\hat h_{\Lambda_2\setminus\Lambda_1})f(\hat h_{\Lambda_1})\,\frac{1}{Z^{\hat h}_{\Lambda_1,\beta}}\,\mathrm{e}^{-\beta H^{\hat h}_{\Lambda_1}(\tilde h)}\,\frac{1}{Z^{\xi}_{\Lambda_2,\beta}}\,\mathrm{e}^{-\beta H^{\xi}_{\Lambda_2}(\hat h)}\mathrm{d}\mu_{\Lambda_1}(\tilde h)\mathrm{d}\mu_{\Lambda_2}(\hat h)\\ &\qquad=\int_{\hat h\in\mathbb{R}^{\Lambda_2}} g(\hat h_{\Lambda_2\setminus\Lambda_1})f(\hat h_{\Lambda_1})\,\frac{1}{Z^{\xi}_{\Lambda_2,\beta}}\,\mathrm{e}^{-\beta H^{\xi}_{\Lambda_2}(\hat h)}\mathrm{d}\mu_{\Lambda_2}(\hat h) = \mathrm{E}^{\xi}_{\Lambda_2,\beta}\big[g(\varphi_{\Lambda_2\setminus\Lambda_1})f(\varphi_{\Lambda_1})\big]\,,\end{aligned}$$

where in the first equality we have integrated over $\tilde h\in\mathbb{R}^{\Lambda_1}$ and used the fact that $(Z^{\hat h}_{\Lambda_1,\beta})^{-1}\mathrm{e}^{-\beta H^{\hat h}_{\Lambda_1}(\tilde h)}$ is a probability density; the last equality follows from the definition of $\mathrm{P}^{\xi}_{\Lambda_2,\beta}$. We have therefore obtained

$$\mathrm{E}^{\xi}_{\Lambda_2,\beta}\Big[g(\varphi_{\Lambda_2\setminus\Lambda_1})\mathrm{E}^{\phi}_{\Lambda_1,\beta}\big[f(\varphi_{\Lambda_1})\big]\Big] = \mathrm{E}^{\xi}_{\Lambda_2,\beta}\big[g(\varphi_{\Lambda_2\setminus\Lambda_1})f(\varphi_{\Lambda_1})\big]\,,$$

which concludes the proof. □

Exercise 3. Using Exercise 2 and the spatial Markov property, show that for all $a<b$, there exists a constant $c>0$ such that for any boundary condition ξ and any $\Gamma\subset\Lambda$, we have $\mathrm{P}^{\xi}_{\Lambda,\beta}(\varphi_x\in[a,b]\ \forall x\in\Gamma)\le \mathrm{e}^{-c|\Gamma|}$.

We can also use the spatial Markov property to show the following corollary, as an exercise.

Exercise 4. Let $\Lambda \Subset \mathbb{Z}^d$ and $\Lambda_1, \Lambda_2 \subset \Lambda$ be "well-separated" sets, *i.e.* such that $\partial\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\partial\Lambda_2 \cap \Lambda_1 = \emptyset$ (recall that $\partial\Lambda$ denotes the exterior boundary). Show that under $\mathrm{P}^{\xi}_{\Lambda,\beta}(\,\cdot \mid \mathcal{F}_{(\Lambda_1\cup\Lambda_2)^c})$, the interfaces φ_{Λ_1}, φ_{Λ_2} are independent.

A word on infinite volume Gibbs measures.

If we want to define a Gibbs measure directly on the full space $\mathbb{Z}^d$, one option is to construct a sequence of measures $(\mathrm{P}^{\xi_n}_{\Lambda_n,\beta})_{n\geq 0}$ with $(\Lambda_n)_{n\geq 1}$ an exhaustion of $\mathbb{Z}^d$, *i.e.* and increasing sequence of finite sets of $\mathbb{Z}^d$ such that $\bigcup_n \Lambda_n = \mathbb{Z}^d$. If, for all $\Lambda \Subset \mathbb{Z}^d$, the distribution of φ_Λ under $(\mathrm{P}^{\xi_n}_{\Lambda_n,\beta})_{n\geq 0}$ converges to a distribution $\mu_{\Lambda,\beta}$ as $n \to \infty$, and if the set of distributions $(\mu_{\Lambda,\beta})_{\Lambda\Subset\mathbb{Z}^d}$ has a spatial consistency property (*i.e.*, $\mu_{\Lambda_1,\beta}$ is the marginal distribution of $\mu_{\Lambda_2,\beta}$), then we can use Kolmogorov's extension theorem to construct a measure μ_β on $\mathbb{Z}^d$. We can then show that the measure μ_β satisfies the spatial Markov property of Proposition 1.5.

However, the use of Kolmogorov's extension theorem is generally unsuitable for constructing Gibbs measures in infinite volume. Indeed, the choice of boundary conditions ξ_n becomes crucial: can one choose any sequence of boundary conditions? Is it even possible to choose a sequence of boundary conditions that works? Could it be that two sequences of boundary conditions give the same infinite-volume measure μ_β? The approach of Dobrushin, Lanford, and Ruelle (known as the *DLR* approach), consists in considering the spatial Markov property as the property that characterizes a Gibbs measure (the measure is determined by its *specifications*, *i.e.* its conditional laws). Here we give an ad hoc version for the case under consideration, but we refer to the excellent Chapter 6 of [FV17] for a more detailed discussion on infinite volume Gibbs measures.

Definition 1.7. We say that μ_β is an infinite-volume Gibbs measure associated with the Hamiltonian $H(\cdot)$ and inverse temperature β if it has the following 'coherence' property: for all $\Lambda \Subset \mathbb{Z}^d$, for any non-negative measurable function $f : \Omega_\Lambda \to \mathbb{R}_+$,

$$\mathrm{E}_{\mu_\beta}\big[f(\varphi_\Lambda) \mid \mathcal{F}_{\Lambda^c}\big] = \mathrm{E}^{\phi}_{\Lambda,\beta}\big[f(\varphi_\Lambda)\big] \quad \mu_\beta\text{-a.s.}$$

Exercise 5. Show that the set of Gibbs measures with infinite volume (associated with some Hamiltonian $H(\cdot)$ and inverse temperature β) is convex.

1.2 The (lattice) Gaussian Free Field on $\mathbb{Z}^d$

The Gaussian Free Field (or GFF) on a lattice falls within the framework of the interface models presented above: it is a height function with values in $\mathbb{R}$, with the potential $V(x) = \frac{1}{2}x^2$. In other words, the Hamiltonian is given by

$$H_\Lambda^\xi(\varphi) = \frac{1}{8d} \sum_{\substack{x,y\in\Lambda \\ x\sim y}} (\varphi_x - \varphi_y)^2 + \frac{1}{4d} \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ x\sim y}} (\varphi_x - \xi_y)^2 \,. \tag{1.6}$$

1.2.1 The random walk representation of the covariance structure

The GFF is a Gaussian field, whose mean and covariance structure can be explicitly characterized. We will begin by describing the case of a zero boundary condition $\xi \equiv 0$, before moving on to the case of an arbitrary boundary condition ξ.

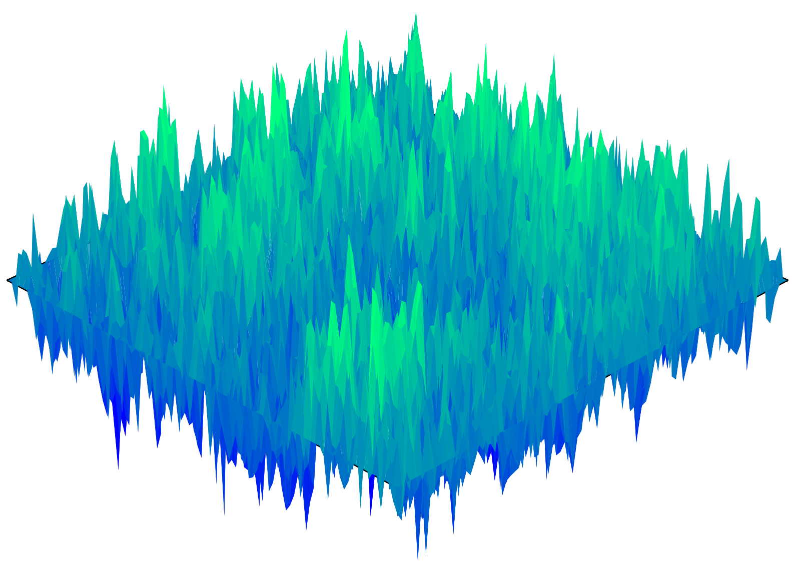

Figure 1.1: A realization of the (lattice) GFF in dimension $d = 2$, on a 100×100 grid, with zero boundary condition $\xi \equiv 0$.

The case of zero boundary condition $\xi \equiv 0$.

To state the result, let us introduce some notation. For $x \in \mathbb{Z}^d$, we denote by $\mathbf{P}_x$ the distribution of a simple random walk $(S_n)_{n\geq0}$ on $\mathbb{Z}^d$ starting from x. This is a

Markov chain started from x, with transition matrix

$$Q(x,y) = \begin{cases} \frac{1}{2d} & \text{if } x \sim y\,, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by G_Λ the Green's function of the simple random walk killed when exiting Λ: for $x, y \in \Lambda$

$$G_\Lambda(x,y) := \mathbf{E}_x\bigg[\sum_{k=0}^{T_{\Lambda^c}} \mathbf{1}_{\{S_k = y\}}\bigg] < +\infty\,, \tag{1.7}$$

where $T_A := \min\{n \geq 0, S_n \in A\}$ denotes the hitting time of the set $A \subset \mathbb{Z}^d$. The finiteness of G_Λ follows from the fact that Λ is a finite set.

Proposition 1.8 (Covariance structure of the GFF). *Under* $\mathrm{P}^0_{\Lambda,\beta} := \mathrm{P}^{\xi\equiv 0}_{\Lambda,\beta}$, *the random interface* $\varphi_\Lambda = (\varphi_x)_{x\in\Lambda}$ *is a centered Gaussian vector, with covariance matrix* $\frac{1}{\beta} G_\Lambda$.

Proof. By definition of $\mathrm{P}^0_{\Lambda,\beta}$, we have

$$\frac{\mathrm{dP}^0_{\Lambda,\beta}}{\mathrm{dLeb}_\Lambda}(\varphi) = \frac{1}{Z^0_{\Lambda,\beta}} \mathrm{e}^{-\beta H^0_\Lambda(\varphi)}\,.$$

It is therefore enough to show that $H^0_\Lambda(\varphi) = \frac{1}{2}\langle \varphi_\Lambda, G_\Lambda^{-1}\varphi_\Lambda\rangle$. By definition of the Hamiltonian, expanding the first square gives

$$\begin{aligned} 2H^0_\Lambda(\varphi) &= \frac{1}{4d}\sum_{\substack{x,y\in\Lambda\\ x\sim y}} (\varphi_x - \varphi_y)^2 + \frac{1}{2d}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}} \varphi_x^2 = \sum_{x\in\Lambda}\varphi_x^2 - \frac{1}{2d}\sum_{\substack{x\in\Lambda,y\in\Lambda\\ x\sim y}} \varphi_x\varphi_y \\ &= \sum_{x\in\Lambda}\varphi_x\Big(\varphi_x - \frac{1}{2d}\sum_{y\in\Lambda, y\sim x}\varphi_y\Big)\,. \end{aligned}$$

Now, let us consider the graph Laplacian Δ on $\mathbb{Z}^d$, which is such that $Q = I + \Delta$, that is

$$\Delta_{x,y} = \begin{cases} \frac{1}{2d} & \text{if } x \sim y\,, \\ -1 & \text{if } x = y\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let us also define the transition matrix and the Laplacian restricted to Λ: we set $Q_\Lambda = (Q(x,y))_{x,y\in\Lambda}$ and $\Delta_\Lambda = (\Delta_{x,y})_{x,y\in\Lambda}$. Then, the expression of the Hamiltonian above can be rewritten as follows:

$$H^0_\Lambda(\varphi) = \frac{1}{2}\left\langle \varphi_\Lambda, (-\Delta_\Lambda)\varphi_\Lambda \right\rangle .$$

It therefore remains to show that $-\Delta_\Lambda = I_\Lambda - Q_\Lambda = G_\Lambda^{-1}$.

But let us observe that $(I_\Lambda - Q_\Lambda)\sum_{k=0}^n Q_\Lambda^k = I_\Lambda - Q_\Lambda^{n+1}$. Now, it can be easily shown (the technical details are left as an exercise) that, for all $x, y \in \Lambda$,

$$Q_\Lambda^{n+1}(x,y) = \mathbf{P}_x\big(S_{n+1} = y, S_k \in \Lambda \;\forall\, k \le n+1\big) \le \mathbf{P}_x(T_{\Lambda^c} > n+1)\,,$$

and that additionally, there exist constants $c, C > 0$ (depending only on Λ) such that $\sup_{x\in\Lambda}\mathbf{P}_x(T_\Lambda > n) \le C\mathrm{e}^{-cn}$. We thus conclude that $\sup_{x,y\in\Lambda} Q_\Lambda^{n+1}(x,y)$ goes towards 0 as $n\to\infty$. Since we have

$$G_\Lambda(x,y) = \sum_{k=0}^{\infty}\mathbf{P}_x\big(S_k = y, S_i \in \Lambda \,\forall i \le k\big) = \sum_{k=0}^{\infty} Q_\Lambda^k(x,y) < +\infty\,,$$

we deduce that $(I_\Lambda - Q_\Lambda)G_\Lambda = (I_\Lambda - Q_\Lambda)\sum_{k=0}^\infty Q_\Lambda^k = I_\Lambda$. This concludes the proof □

The case of a general boundary condition ξ.

Let us introduce a new notation: for $\xi : \mathbb{Z}^d \to \mathbb{R}$, we define the function $u^\xi : \Lambda \to \mathbb{R}$ as the unique Q-harmonic extension of ξ_{Λ^c} inside Λ, that is, the unique solution to the Dirichlet problem on $\Lambda \Subset \mathbb{Z}^d$ with boundary condition ξ:

$$\begin{cases}(\Delta u^\xi)_x = \frac{1}{2d}\sum_{y\sim x}(u_y - u_x) = 0 & \text{for } x\in\Lambda\\ u^\xi_x = \xi_x & \text{for } x \in \Lambda^c\end{cases} \tag{1.8}$$

It is standard to show that the function defined by

$$u^\xi_x := \mathbf{E}_x\big[\xi_{S_{T_{\Lambda^c}}}\big]$$

is a solution to the Dirichlet problem; uniqueness comes from the fact that Λ is finite, so $T_{\Lambda^c} < +\infty$ $\mathbf{P}_x$-p.s. for all $x\in\Lambda$.

Proposition 1.9 (Decomposition of the GFF)**.** *For any boundary condition ξ, under $\mathrm{P}^\xi_{\Lambda,\beta}$, the random interface $\varphi_\Lambda = (\varphi_x)_{x\in\Lambda}$ is a Gaussian vector with expectation $\mathrm{E}^\xi_{\Lambda,\beta}[\varphi_x] = u^\xi_x$, with covariance matrix $\frac{1}{\beta}G_\Lambda$. Put otherwise, we can write $\varphi_\Lambda = u^\xi + \varphi^0_\Lambda$, where φ^0_Λ is a centered GFF.*

The effect of the boundary condition ξ is therefore only a translation by u^ξ of the centered GFF, that is by the harmonic extension of ξ inside Λ.

Proof. As in the centered case, it is enough to show that

$$H_\Lambda^\xi(\varphi) = \frac{1}{2}\langle(\varphi_\Lambda - u_\Lambda^\xi), G_\Lambda^{-1}(\varphi_\Lambda - u_\Lambda^\xi)\rangle + C_\Lambda^\xi\,,$$

where the constant C_Λ^ξ does not depend on φ (according to Remark 1.1, this constant does not affect the distribution $\mathrm{P}_{\Lambda,\beta}^\xi$).

Note that, on the one hand

$$\begin{aligned}2H_\Lambda^\xi(\varphi) &= \frac{1}{4d}\sum_{\substack{x,y\in\Lambda\\ x\sim y}}(\varphi_x - \varphi_y)^2 + \frac{1}{2d}\sum_{\substack{x\in\Lambda, y\in\Lambda^c\\ x\sim y}}(\varphi_x - \xi_y)^2\\ &= 2H_\Lambda^0(\varphi) - \frac{1}{2d}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}}\varphi_x\xi_y + \frac{1}{2d}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}}\xi_y^2\,.\end{aligned}$$

On the other hand, recalling that $G_\Lambda^{-1} = -\Delta_\Lambda$, we have

$$\big\langle(\varphi_\Lambda - u_\Lambda^\xi), G_\Lambda^{-1}(\varphi_\Lambda - u_\Lambda^\xi)\big\rangle = \langle\varphi_\Lambda, G_\Lambda^{-1}\varphi_\Lambda\rangle + 2\langle\varphi_\Lambda, \Delta_\Lambda\varphi_\Lambda\rangle + \langle u_\Lambda^\xi, G_\Lambda^{-1}u^\xi\rangle\,.$$

Thus, since $\langle\varphi_\Lambda, G_\Lambda^{-1}\varphi_\Lambda\rangle = 2H_\Lambda^0(\varphi)$ from the zero boundary condition case, and since u^ξ depends only on Λ and ξ (and not on φ), it remains to show that

$$\langle\varphi_\Lambda, \Delta_\Lambda\varphi_\Lambda\rangle + \frac{1}{2d}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}}\varphi_x\xi_y = \sum_{x\in\Lambda}\varphi_x(\Delta_\Lambda u)_x + \frac{1}{2d}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}}\varphi_x\xi_y$$

only depends on Λ and ξ and not on φ. But using the fact that u^ξ is harmonic on Λ, we obtain that for $x \in \Lambda$

$$0 = (\Delta u^\xi)_x = \frac{1}{2d}\sum_{y\sim x}u_y^\xi - u_x^\xi = (\Delta_\Lambda u^\xi)_x + \frac{1}{2d}\sum_{y\notin\Lambda, y\sim x}u_y^\xi\,.$$

Since $u_y^\xi = \xi_y$ for $y \notin \Lambda$, we conclude that

$$0 = \sum_{x\in\Lambda}\varphi_x(\Delta u^\xi)_x = \sum_{x\in\Lambda}\varphi_x(\Delta_\Lambda u)_x + \frac{1}{2d}\sum_{\substack{x\in\Lambda\\ y\notin\Lambda, y\sim x}}\varphi_x\xi_y\,,$$

which indeed does not depend on φ. This concludes the proof. □

Remark 1.10 (GFF associated with more general random walks)**.** We could have taken Proposition 1.9 as the definition of GFF, *i.e.* a Gaussian field with expectation u^ξ and covariance matrix $\frac{1}{\beta}G_\Lambda$. In fact, this definition can easily be generalized if one replaces the simple random walk with a random walk $(S_n)_{n\geq 0}$ on any $\mathbb{Z}^d$, with transition matrix Q. The Gaussian Free Field on $\Lambda \Subset \mathbb{Z}^d$ associated with the transition matrix Q and with boundary condition ξ is then the Gaussian vector with expectation u_Λ^ξ and covariance matrix $\frac{1}{\beta}G_\Lambda$, where:

(i) $u_x^\xi = \mathbf{E}_x[\xi_{S_{T_{\Lambda^c}}}]$ is the unique Q-harmonic extension of ξ in Λ;

(ii) $G_\Lambda(x,y) = \mathbf{E}_x\big[\sum_{k=0}^{T_{\Lambda^c}} \mathbf{1}_{\{S_k=y\}}\big] = \sum_{k=0}^{\infty} Q_\Lambda^k(x,y)$ is the Green's function of the random walk $(S_n)_{n\geq 0}$ killed when it leaves Λ.

We can then show (this is left as an exercise) that this corresponds to a Gibbs measure on Ω_Λ given by the Hamiltonian

$$H_\Lambda^\xi(\varphi) = \frac{1}{2}\sum_{x,y\in\Lambda} Q(x,y)\,(\varphi_x-\varphi_y)^2 + \sum_{x\in\Lambda, y\notin\Lambda} Q(x,y)\,(\varphi_x-\xi_y)^2\,.$$

Note that this Hamiltonian is not simply a function of the gradient of φ but may have long-range interactions (we may have terms $(\varphi_x-\varphi_y)^2$ with $x \not\sim y$). The properties of the Gaussian Free Field are determined by the covariance structure and are therefore related to the properties of the random walk $(S_n)_{n\geq 0}$ considered, through its Green's function G_Λ.

Remark 1.11 (About the dependence on β)**.** Thanks to the invariance of Gaussian variables under scaling, we know that the distribution of $\sqrt{\beta}\,\varphi_\Lambda$ under $\mathrm{P}^0_{\Lambda,\beta}$ is the same as that of φ_Λ under $\mathrm{P}^0_{\Lambda,\beta=1}$: this is a centered Gaussian vector with covariance matrix G_Λ. Thus, we can view the parameter β as a simple scaling factor (in the case of a general boundary condition, we can use Proposition 1.8 to simply scale $\varphi_\Lambda - u^\xi$).

1.2.2 Localization and delocalization of the GFF interface

The covariance structure of the GFF given in Propositions 1.8-1.9 provides information about the interface φ. One of the first questions is whether, for a GFF on $\Lambda_N := \{-N,\dots,N\}^d$ with zero boundary condition, the height of the center point φ_0 remains tight (as a random variable) as $N\to\infty$. If so, we say that the interface is *localized*; otherwise, we say that the interface is *delocalized*.

Studying the variance of φ_0 under $\mathrm{P}^0_{\Lambda_N,\beta}$ is therefore a good way to test the localization/delocalization of the interface; in the case of Gaussian variables, this even characterizes the distribution of φ_0.

Proposition 1.12 (Localization/delocalization). *For $\Lambda_N := \{-N,\dots,N\}^d$, we have*

$$\mathrm{Var}^0_{\Lambda_N,\beta}(\varphi_0) = \frac{1}{\beta} G_{\Lambda_N}(0,0) \quad \textit{with} \quad \lim_{N\to\infty} G_{\Lambda_N}(0,0) \begin{cases} = +\infty & \textit{si } d=1,2, \\ < +\infty & \textit{si } d \geq 3. \end{cases}$$

We therefore have delocalization of the interface in dimension $d=1,2$ and localization of the interface in dimension $d\geq 3$.

This proposition does not require a proof: indeed, the variance is given in Proposition 1.8. Moreover, by monotone convergence, we have $\lim_{N\to\infty} G_{\Lambda_N}(0,0) = G(0,0)$, where G is the Green's function of the simple walk on $\mathbb{Z}^d$ and is equal to $+\infty$ in dimension $d=1,2$ (because the walk is recurrent) and is finite in dimension $d\geq 3$ (because the walk is transient).

Remark 1.13. It is possible to obtain very accurate estimates of the Green's function $G_{\Lambda_N}(0,0)$: we refer to [Law13, §1.5 and 1.6]. More precisely:

1. In dimension $d=1$, we have $G_{\Lambda_N}(0,0) = N+1$.
   (*Hint: use gambler's ruin.*)
2. In dimension $d\geq 2$, we have $G_{\Lambda_N}(0,0) = \frac{2}{\pi}\log N + O(1)$ as $N\to\infty$.
3. In dimension $d\geq 3$, we have $G_{\Lambda_N}(0,0) = G(0,0) + O(N^{2-d})$.

Note also that, in dimension $d\geq 3$, we can show that for $x,y\in\mathbb{Z}^d$, we have $\mathrm{Cov}^0_{\Lambda_N,\beta}(\varphi_x,\varphi_y) = \frac{1}{\beta}G_{\Lambda_N}(x,y)$ with $\lim_{N\to\infty} G_{\Lambda_N}(x,y) = G(x,y)$ and

$$G(x,y) \sim c_d \|x-y\|^{2-d} \quad \text{when } \|x-y\| \to +\infty\,,$$

for some (explicit) constant c_d. Thus, the interface remains localized but the heights are strongly correlated.

Exercise 6. For $N\geq 1$ and $x\in\Lambda_N$, we set $\sigma_N^2(x) := G_{\Lambda_N}(x,x)$, where we recall that the Green's function is defined in (1.7). Let $\xi : \mathbb{Z}^d \to \mathbb{R}$ be any boundary condition.

1. Show that

$$\mathrm{P}^\xi_{N,\beta}\big(|\varphi_x| \geq t\big) \leq \mathrm{P}\Big(\tfrac{1}{\beta}\sigma_N^2(x)|Z| \geq t - |u^\xi_x|\Big)$$

where $Z \sim \mathcal{N}(0,1)$ and u^ξ is the harmonic extension of ξ in Λ_N.

2. Show that, for any $x \in \Lambda_N$ we have $\sigma_N^2(x) \le G_{\Lambda_{2N}}(0,0)$ and $|u_x^\xi| \le M_N^\xi := \max_{y\in\partial\Lambda_N} |\xi_y|$. Deduce that
$$\mathrm{P}^\xi_{N,\beta}\big(|\varphi_x| \ge t\big) \le \exp\bigg(-\frac{\beta (t-M_N^\xi)^2}{2G_{\Lambda_{2N}}(0,0)}\bigg) .$$
3. Assume that $M_N^\xi = o(\log N)$ in dimension $d=2$ and $M_N^\xi = o(\sqrt{\log N})$ in dimension $d\ge 3$, and show that, for any $\eta>0$
$$\lim_{N\to\infty} \mathrm{P}^\xi_{N,\beta}\Big(\max_{x\in\Lambda_N} |\varphi_x| \ge (1+\eta)\, \alpha_N \Big) = 0 \,,$$
where $\alpha_N = \sqrt{\frac{8}{\pi\beta}} \log N$ if $d=2$ and $\alpha_N = \sqrt{\frac{2d}{\beta} G(0,0)} \sqrt{\log N}$ if $d\ge 3$.

Discussion for general interfaces. Let us mention that the localization vs. delocalization phenomenon for ($\nabla\varphi$-)interfaces is in general quite difficult to study, and the phenomenology is in fact quite different depending on whether one considers a continuous or discrete height interface.

In dimension $d=1$, with the interpretation of $(\varphi_x)_{-N\le x\le N}$ as a random walk conditioned to be 0 at $-(N+1)$ and $N+1$ from Section 1.1.3, we naturally obtain the estimate $\mathrm{Var}^0_{\Lambda_N,\beta}(\varphi_0) \sim c_\beta N$, regardless of the potential V given in Hamiltonian (1.2) — provided that $\int_{\mathbb{R}} x^2 e^{-V(x)} dx < +\infty$, which ensures that $\mathrm{E}_{\mu_\beta}(X_i) = 0$ and $\mathrm{E}_{\mu_\beta}(X_i^2) < +\infty$ (with μ_β the law defined in (1.5)).

For continuous height interfaces, it turns out that the upper bounds $\mathrm{Var}^0_{\Lambda_N,\beta}(\varphi_0) \le C \log N$ in dimension $d=2$ and $\mathrm{Var}^0_{\Lambda_N,\beta}(\varphi_0) \le C$ in dimension $d\ge 3$ remain valid in a very general context (see the recent article [Dar23]). In particular, the interface in dimension $d\ge 3$ remains localized. For interfaces in dimension $d=2$, logarithmic lower bounds have been given in general settings, showing the delocalization of the interface (see the introduction to [Dar23], which reviews the literature on this subject).

In the case of discrete height interfaces, there are very few general results, and they usually focus on certain discrete height interfaces, most notably the SOS model mentioned above (and the so-called discrete Gaussian model). In these particular cases, the localization of interfaces has been shown in dimension $d\ge 3$. The case of dimension $d=2$ is quite special: it can be shown in certain examples (in particular for the SOS and discrete GFF, see [FS81, AHPS21, DCHL+22, LO24, GL25]) that there is a localization/delocalization transition (and this each time is a difficult result!), in the sense that there exists a $0 < \beta_c < \infty$ such that

$$\begin{aligned} &\limsup_{N\to\infty} \mathrm{Var}^0_{\Lambda_N,\beta}(\varphi_0) < +\infty \qquad \text{if } \beta > \beta_c\,, \\ \forall t > 0, \quad &\limsup_{N\to\infty} \mathrm{P}^0_{\Lambda_N,\beta}(|\varphi_0| > t) = 0 \qquad \text{if } \beta < \beta_c\,. \end{aligned}$$

It is conjectured that this phenomenon is general in the case of discrete interfaces: we refer to the introduction of [LO24], which gives a nice review of existing results.

1.2.3 Infinite volume limit of the GFF

The case of dimensions $d = 1, 2$

Theorem 1.14 (No infinite-volume GFF in $d = 1, 2$)**.** *In dimensions $d = 1, 2$, there does not exist an infinite-volume Gibbs measure associated with the GFF Hamiltonian* (1.6)*.*

Proof. We proceed by contradiction. Assume that μ_β is an infinite-volume Gibbs measure, *i.e.* satisfying Definition 1.7. Then, using the definition, for every $\Lambda \Subset \mathbb{Z}^d$ with $0 \in \Lambda$, and for any $a < b$, we have

$$\mu_\beta\big(\varphi_0 \in [a, b]\big) = \mathrm{E}_{\mu_\beta}\big[\mathrm{P}^\xi_{\Lambda,\beta}\big(\varphi_0 \in [a, b]\big)\big]\,.$$

We know that, under $\mathrm{P}^\xi_{\Lambda,\beta}$, φ_0 is a Gaussian variable $\mathcal{N}(m, \sigma^2)$, with $m = u^\xi_0$ and $\sigma^2 = \frac{1}{\beta} G_\Lambda(0, 0)$. Using the fact that the density of a $\mathcal{N}(m, \sigma^2)$ distribution is uniformly bounded by $\frac{1}{\sigma\sqrt{2\pi}}$, we obtain that, for any boundary condition ξ,

$$\mathrm{P}^\xi_{\Lambda,\beta}\big(\varphi_0 \in [a, b]\big) \le \frac{\sqrt{\beta/2\pi}}{\sqrt{G_\Lambda(0, 0)}}\,(b - a)\,.$$

Thus, the same upper bound holds for $\mu_\beta(\varphi_0 \in [a, b])$, and since $\Lambda \Subset \mathbb{Z}^d$ is arbitrary and $\lim_{\Lambda\uparrow\mathbb{Z}^d} G_\Lambda(0, 0) = +\infty$, we deduce that

$$\text{for all } a < b, \qquad \mu_\beta(\varphi_0 \in [a, b]) = 0\,,$$

which is a contradiction. □

Remark 1.15. The fact that no infinite-volume measure exists in dimension $d = 1, 2$ is thus essentially due to the fact that the fluctuations of the central point φ_0 are unbounded as $\Lambda \uparrow \mathbb{Z}^d$. To overcome this issue, one can define the GFF "pinned at 0", by considering $\tilde{\varphi} = \varphi - \varphi_0$ (which is a translation by the random height φ_0). One can show (exercise) that:

(i) Under $\mathrm{P}^0_{\Lambda,\beta}$, $\tilde{\Phi}_\Lambda = (\tilde{\varphi}_x)_{x\in\Lambda}$ is a centered Gaussian vector, with covariance matrix $\tilde{G}_\Lambda$ given by

$$\tilde{G}_\Lambda(x,y) = G_\Lambda(x,y) - G_\Lambda(x,0) - G_\Lambda(0,x) + G_\Lambda(0,0).$$

(ii) We have $\lim_{\Lambda\uparrow\mathbb{Z}^d} \tilde{G}_\Lambda(x,y)\tilde{G}(x,y) < +\infty$, with

$$\tilde{G}(x,y) = \mathbf{E}_x\Big[\sum_{k=0}^{T_0} \mathbf{1}_{\{S_k=y\}}\Big].$$

One can then define a Gaussian field on $\mathbb{Z}^d$ with covariance matrix $\tilde{G}$. More generally, one can define a Gibbs measure on the *gradients family* $(\varphi_x - \varphi_y)_{x,y\in\mathbb{Z}^d}$ in any dimension, rather than only on the heights; see the lecture notes [Fun05].

The case of dimensions $d \geq 3$

Theorem 1.16 (Infinite volume GFF in $d \geq 3$)**.** *Let $d \geq 3$. If ξ is a harmonic function on $\mathbb{Z}^d$, then there exists an infinite-volume Gibbs measure $\mu_\beta = \mathrm{P}^\xi_{\infty,\beta}$ on the whole lattice $\mathbb{Z}^d$, associated with the GFF Hamiltonian* (1.6)*. The measure μ_β is Gaussian, with mean ξ and covariance $(\frac{1}{\beta}G(x,y))_{x,y\in\mathbb{Z}^d}$.*

Definition 1.17 (Gaussian field)**.** A family $(W_x)_{x\in I}$ is a Gaussian field with mean $m = (m_x)_{x\in I}$ and covariance $(\Sigma_{x,y})_{x,y\in I}$ if for every finite subset $J \subset I$, $(W_y)_{y\in J}$ is a Gaussian vector with mean $m_J = (m_y)_{y\in J}$ and covariance matrix $\Sigma_J = (\Sigma_{x,y})_{x,y\in J}$.

In particular, Theorem 1.16 tells that there exist infinitely many infinite-volume Gibbs measures (at least one for each harmonic function of $\mathbb{Z}^d$). One can moreover show that if $\mathcal{G}$ denotes the (convex) set of Gibbs measures associated with the GFF (recall Exercise 5), then the measures

$$\{\mathrm{P}^\xi_{\infty,\beta},\ \xi \text{ harmonic on } \mathbb{Z}^d\}$$

are the extremal points of $\mathcal{G}$.

Proof. Note that the Gaussian law stated in the theorem is well defined. Indeed, for every $\Lambda \Subset \mathbb{Z}^d$, the law of φ_Λ under $\mathrm{P}^{\xi}_{\Lambda,\beta}$ is Gaussian with mean $u^\xi = \xi$ (since ξ is harmonic) and covariance matrix $\frac{1}{\beta}G_\Lambda$. Since in dimension $d \geq 3$ we have

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} G_\Lambda = G,$$

we can construct a Gaussian field $\varphi = (\varphi_x)_{x\in\mathbb{Z}^d}$ with mean ξ and covariance $\frac{1}{\beta}G$ (using Kolmogorov's extension theorem, see for instance [FV17, Th. 8.6]).

Let μ_β be the law of the Gaussian field $\varphi = (\varphi_x)_{x\in\mathbb{Z}^d}$ with mean ξ and covariance $\frac{1}{\beta}G$. It remains to show that the measure μ_β satisfies the Definition 1.7 of an infinite-volume Gibbs measure: given $\Lambda \Subset \mathbb{Z}^d$, we need to show that "for μ_β-almost every ϕ, the conditional law of φ_Λ given $\mathcal{F}_{\Lambda^c}$ is $\mathrm{P}^{\phi}_{\Lambda,\beta}$". Equivalently, by Proposition 1.9 and Definition 1.17, it suffices to show that for μ_β-almost every ϕ,

$$\mathrm{E}_{\mu_\beta}\big[\mathrm{e}^{i\langle t_\Lambda,\varphi_\Lambda\rangle} \mid \mathcal{F}_{\Lambda^c}\big] = \mathrm{e}^{i\langle t_\Lambda, u^\phi_\Lambda\rangle}\mathrm{e}^{-\frac{1}{2}\langle t_\Lambda, G_\Lambda t_\Lambda\rangle} \qquad \text{for all } t_\Lambda \in \mathbb{R}^\Lambda\,,$$

where we recall that $u^\phi_x = \mathbf{E}_x[\phi_{S_{T_{\Lambda^c}}}]$ is the unique harmonic extension of ϕ to Λ.

The proof essentially consists of a few lemmas, stated below, which exploit the Gaussian structure of $\mu_\beta = \mathrm{P}^{\xi}_{\infty,\beta}$. We refer to [FV17, §8.4.2] for a detailed proof.

Lemma 1.18. *For $\mathrm{P}^{\xi}_{\infty,\beta}$-almost every ϕ, we have*

$$\mathrm{E}^{\xi}_{\infty,\beta}[\varphi_x \mid \mathcal{F}_{\Lambda^c}] = u^\phi_x \qquad \textit{for all } x \in \Lambda.$$

Lemma 1.19. *Under $\mathrm{P}^{\xi}_{\infty,\beta}$, u^ϕ_Λ is a Gaussian vector with mean ξ and covariance matrix K_Λ, given by*

$$K_\Lambda(x,y) = \mathbf{E}_x\Big[\sum_{k\geq T_{\Lambda^c}} \mathbf{1}_{\{S_k=y\}}\Big] = G(x,y) - G_\Lambda(x,y).$$

Lemma 1.20. *Under $\mathrm{P}^{\xi}_{\infty,\beta}$, $\varphi_\Lambda - u^\phi_\Lambda$ is a centered Gaussian vector with covariance matrix $G_\Lambda = G - K_\Lambda$. Moreover, $\varphi_\Lambda - u^\phi_\Lambda$ is independent of $\mathcal{F}_{\Lambda^c}$ (and hence of u^ϕ_Λ).*

With these three lemmas, we obtain that for μ_β-almost every ϕ, since $(u^\phi_x)_{x\in\Lambda}$ is $\mathcal{F}_{\Lambda^c}$-measurable and $(\varphi_x - u^\phi_x)_{x\in\Lambda}$ is independent of $\mathcal{F}_{\Lambda^c}$,

$$\mathrm{E}_{\mu_\beta}\big[\mathrm{e}^{i\langle t_\Lambda,\varphi_\Lambda\rangle} \mid \mathcal{F}_{\Lambda^c}\big] = \mathrm{e}^{i\langle t_\Lambda,u^\phi_\Lambda\rangle}\mathrm{E}_{\mu_\beta}\big[\mathrm{e}^{i\langle t_\Lambda,\varphi_\Lambda - u^\phi_\Lambda\rangle}\big] = \mathrm{e}^{i\langle t_\Lambda,u^\phi_\Lambda\rangle}\mathrm{e}^{-\frac{1}{2}\langle t_\Lambda, G_\Lambda t_\Lambda\rangle}\,,$$

which is exactly what we wanted to prove. □

Exercise 7. Let $h \in \mathbb{R}$ be fixed and note that the constant function $\xi \equiv h$ is harmonic on $\mathbb{Z}^d$. We may therefore consider the infinite-volume GFF measure $\mu_\beta^h = \mu_\beta^{\xi\equiv h}$. Let $\Lambda_N = \{1, \dots, N\}^d$ and define

$$\Phi_N := \max_{x\in\Lambda_N} |\varphi_x|.$$

1. Show that
$$\limsup_{k\to\infty} \frac{\Phi_{2^k}}{\sqrt{\log 2^k}} \le \frac{\sqrt{2d}}{\sqrt{\beta}}\sqrt{G(0,0)} \qquad \mu_\beta^h\text{-a.s.}$$
2. Deduce that
$$\limsup_{N\to\infty} \frac{\Phi_N}{\sqrt{\log N}} \le \frac{\sqrt{2d}}{\sqrt{\beta}}\sqrt{G(0,0)} \qquad \mu_\beta^h\text{-a.s.}$$

   Hint. Use the monotonicity of $N \mapsto \Phi_N$.

1.2.4 Scaling limit: the continuous-space GFF

If $D \subset \mathbb{R}^d$ is a given domain in space, one may want to define a field Φ on D which would be the scaling limit of the Gaussian free field with zero boundary conditions. The idea is to consider a discretization of the domain by a grid of mesh size $\delta := \frac{1}{N} > 0$, that is, to consider a domain

$$\Lambda_N = ND \cap \mathbb{Z}^d = \left\{x \in \mathbb{Z}^d : \tfrac{x}{N} \in D\right\},$$

so that

$$D_\delta = \tfrac{1}{N}\Lambda_N$$

corresponds to a mesh of size $\delta = N^{-1}$ of the domain D. One can then define a GFF on D_δ by setting $\varphi_\delta(x) = \varphi_{Nx}$ for $x \in D_\delta$, where φ is a GFF on $\Lambda_N = ND_\delta$.

Thus, $(\varphi_\delta(x))_{x\in D_\delta}$ is a centered Gaussian field, with covariance matrix

$$G_\delta(x,y) = G_{\Lambda_N}(Nx, Ny),$$

and it turns out that the following relation holds:

$$G_\delta(x,y) \sim \delta^{d-2}\mathcal{G}_D(x,y)\,,$$

where $\mathcal{G}_D$ is the *continuous* Green function on the domain D (namely, the Green function of Brownian motion killed upon exiting D). Therefore, if one wants to obtain a non-degenerate limit, one must renormalize the φ_δ by $\delta^{1-d/2}$: one should then have that the field

$$\tilde{\varphi}_\delta = \delta^{1-d/2}\varphi_\delta$$

defined on D_δ converges to a centered Gaussian field with covariance matrix $\mathcal{G}_D$.

In dimension 1, there is essentially no difficulty: the normalization corresponds to that of the simple random walk (see Section 1.1.3), and the scaling limit of the GFF is a Brownian bridge.

In dimension $d \geq 2$, there is no normalization to perform! There is however some issue, since one can easily see that the variance

$$\mathrm{Var}^0_{D_\delta}(\varphi_\delta(0))$$

diverges logarithmically: in other words, $\mathcal{G}_D(0,0) = +\infty$. The same phenomenon occurs in dimension $d \geq 3$, since the interface is localized, but since we renormalize φ_δ by $\delta^{1-d/2}$, its variance diverges as $\delta \downarrow 0$.

It turns out that (except in dimension $d = 1$) the correct way to consider the scaling limit is to construct the continuous GFF as a distribution rather than as a genuine function (defined pointwise). More precisely, if we denote by $\mathcal{C}_c$ the space of continuous functions from D to $\mathbb{R}$ with compact support (interpreted as a space of test functions), then for any $f \in \mathcal{C}_c$ we define

$$\tilde{\varphi}_\delta(f) = \delta^d \sum_{x \in D_\delta} \tilde{\varphi}_\delta(x) f(x)\,.$$

One can then show that for every $f \in \mathcal{C}_c$, $\tilde{\varphi}_\delta(f)$ converges in distribution to $\Phi(f)$, where Φ is a random distribution, called the continuous Gaussian Free Field[1] (continuous GFF). The law of the continuous GFF Φ on D is characterized by the family $\{\Phi(f), f \in \mathcal{C}_c\}$, which is a centered Gaussian field (recall Definition 1.17) with covariance

$$\mathrm{Cov}\big(\Phi(f), \Phi(g)\big) = \int_{D\times D} f(x)g(y)\mathcal{G}_D(x,y)\mathrm{d}x\,\mathrm{d}y\,.$$

We refer to the lecture notes of Berestycki and Powell [BP21] for the definition and an overview of the properties (and areas of application) of the continuous GFF.

[1] This convergence to the continuous GFF is in fact robust and remains valid for fairly general interface models.

1.3 Hard wall constraint and entropic repulsion

The goal of this section is to study the effect of a hard-wall constraint on the properties of the interface. We restrict ourselves to the case of zero boundary conditions ($\xi \equiv 0$). For $\Lambda \Subset \mathbb{Z}^d$, let us introduce the event

$$\Omega_\Lambda^+ = \{(\varphi_x)_{x\in\Lambda} \in \Omega_\Lambda,\ \varphi_x \geq 0\ \forall\, x \in \Lambda\}\,, \tag{1.9}$$

which corresponds to a *hard-wall constraint*: the height function is required to be non-negative.

We then consider the Gibbs measure with a positivity constraint:

$$\frac{\mathrm{dP}^+_{\Lambda,\beta}}{\mathrm{d}\mu}(\varphi) = \frac{1}{Z^+_{\Lambda,\beta}}\, \mathrm{e}^{-\beta H^0_\Lambda(\varphi)}\, \mathbf{1}_{\Omega^+_\Lambda}(\varphi_\Lambda)\,, \tag{1.10}$$

where the partition function (normalizing the measure $\mathrm{P}^+_{\Lambda,\beta}$ into a probability measure) is now

$$Z^+_{\Lambda,\beta} = \int_{\Omega^+_\Lambda} \mathrm{e}^{-\beta H^0_\Lambda(\varphi)} = Z^0_{\Lambda,\beta}\, \mathrm{P}^0_{\Lambda,\beta}(\Omega^+_\Lambda)\,.$$

Let us note that, for any Borel set A of Ω_Λ,

$$\begin{aligned}\mathrm{P}^+_{\Lambda,\beta}(\varphi_\Lambda \in A) &= \int_{A\cap\Omega^+_\Lambda} \frac{1}{Z^+_{\Lambda,\beta}} \mathrm{e}^{-\beta H^0_\Lambda(h)} \mathrm{d}\mu(h) \\ &= \frac{\mathrm{P}^0_{\Lambda,\beta}(\varphi_\Lambda \in A \cap \Omega^+_\Lambda)}{\mathrm{P}^0_{\Lambda,\beta}(\Omega^+_\Lambda)} = \mathrm{P}^0_{\Lambda,\beta}(\varphi_\Lambda \in A \mid \Omega^+_\Lambda)\,,\end{aligned}$$

where we have used that, as noticed above, $Z^+_{\Lambda,\beta} = Z^0_{\Lambda,\beta}\mathrm{P}^0_{\Lambda,\beta}(\Omega^+_\Lambda)$. In other words, the measure $\mathrm{P}^+_{\Lambda,\beta}$ is simply the measure $\mathrm{P}^0_{\Lambda,\beta}$ conditioned on the event Ω^+_Λ that the interface stays above a hard wall.

In the next sections, we will show that this constraint (or conditioning) translates into a repulsion effect on the interface.

1.3.1 The case of dimension $d = 1$

In dimension $d = 1$, for simplicity, we treat only the case of a discrete interface: $\varphi_x \in \mathbb{Z}$ for all $x \in \mathbb{Z}$. For $N \geq 1$, we consider $\Lambda_N = \{1, \dots, N\}$ and we write

$$\mathrm{P}^0_N := \mathrm{P}^0_{\Lambda_N,\beta} \qquad \text{and} \qquad \mathrm{P}^+_N := \mathrm{P}^+_{\Lambda_N,\beta}\,,$$

and similarly for the associated partition functions Z_N^0 and Z_N^+.

Our goal is to compare the properties of interfaces under P_N^0 and under P_N^+. Recall that under P_N^0, the interface $(\varphi_x)_{1\leq x\leq N}$ is simply a random walk conditioned to satisfy $\varphi_0 = \varphi_{N+1} = 0$. We will therefore use the more standard notation for random walks:

- We consider $(X_i)_{i\geq 1}$ i.i.d. random variables with a *symmetric law* (with distribution ν_β given in (1.5), assuming V is symmetric), with $\sigma^2 := \mathrm{E}[X_i^2] < +\infty$;
- We define $S_n := \sum_{i=1}^n X_i$ for all $n \geq 0$, and the law P_N^0 (resp. P_N^+) corresponds to the law of $(S_k)_{0\leq k\leq N+1}$ conditioned on $S_{N+1} = 0$ (resp. $S_k \geq 0$ for all $1 \leq k \leq N$ and $S_N = 0$).

Remark 1.21. One can show that $\big(\frac{1}{\sqrt{N}}(S_k)_{0\leq k\leq \lfloor tN\rfloor}\big)_{t\in[0,1]}$ converges in distribution to a Brownian bridge under P_N^0 and to a normalized Brownian excursion under P_N^+. In what follows, we will focus on weaker properties (but potentially more robust for the analysis of other models).

Let us consider the contacts between the random walk and the line $\mathbb{Z} \times \{0\}$. For $k \leq \ell$, we define

$$\mathcal{L}_N(k,\ell) := \sum_{n=k}^{\ell} \mathbf{1}_{\{S_n=0\}}\,,$$

the number of contacts between the walk and the line between times k and ℓ, and we set $\mathcal{L}_N = \mathcal{L}_N(1,N)$ for the total number of contacts (excluding $S_0 = S_{N+1} = 0$).

If one wishes to study $\mathcal{L}_N(k,\ell)$, it is natural to start by estimating its expectation, which starts by estimating the probabilities $\mathrm{P}_N^0(S_n = 0)$ and $\mathrm{P}_N^+(S_n = 0)$. We define

$$p_n = \mathrm{P}(S_n = 0) \qquad \text{and} \qquad p_n^+ = \mathrm{P}(S_n = 0, S_k \geq 0\ \forall 1 \leq k \leq n)\,,$$

and we notice that

$$\mathrm{P}_N^0(S_n = 0) = \mathrm{P}(S_n = 0 \mid S_{N+1} = 0) = \frac{\mathrm{P}(S_n = 0, S_{N+1} = 0)}{\mathrm{P}(S_{N+1} = 0)} = \frac{p_n p_{N+1-n}}{p_{N+1}}\,, \tag{1.11}$$

and similarly,

$$\mathrm{P}_N^+(S_n = 0) = \mathrm{P}(S_n = 0 \mid S_{N+1} = 0, S_k \geq 0\ \forall 0 \leq k \leq n) = \frac{p_n^+ p_{N+1-n}^+}{p_{N+1}^+}\,. \tag{1.12}$$

Thus, an important step is to obtain estimates on p_n and p_n^+. We have the following result.

Theorem 1.22 (Asymptotic estimates for p_n, p_n^+)**.** *Assume that* $\mathrm{E}(X_i) = 0$ *and* $\mathrm{E}(X_i^2) = \sigma^2 < +\infty$. *Then, we have as* $n \to +\infty$,

$$p_n \sim \frac{c_1}{n^{1/2}}\,, \qquad p_n^+ \sim \frac{c_2}{n^{3/2}}\,,$$

where $c_1 = \frac{1}{\sqrt{2\pi\sigma^2}}$ *and* c_2 *is a constant that depends more subtly on the law of* X_1.

Proof. The asymptotic equivalence for p_n is classical and follows from the local Central Limit Theorem; see for instance [LL10, §2].

Theorem (Local CLT)**.** *Let* $(X_i)_{i\geq 0}$ *be random variables with values in* $\mathbb{Z}$, *with* $\mathrm{E}(X_i) = 0$ *and* $\sigma^2 = \mathrm{E}(X_i^2) < +\infty$. *Further, assume that the random walk* $(S_n)_{n\geq 0}$ *is aperiodic on* $\mathbb{Z}$. *Then, if* g_σ *denotes the density of the* $\mathcal{N}(0,\sigma^2)$ *distribution, we have*

$$\sup_{x\in\mathbb{Z}} \Big| \sqrt{n}\,\mathrm{P}(S_n = x) - g_\sigma(x/\sqrt{n}) \Big| \xrightarrow{n\to\infty} 0\,.$$

In particular,

$$\mathrm{P}(S_n = 0) \sim \frac{1}{\sqrt{2\pi\sigma^2 n}} \qquad \text{as } n \to \infty\,.$$

The asymptotic behavior of p_n^+ is more delicate. Most proofs rely on a combinatorial argument (the cycle lemma) and on what is known as Wiener–Hopf factorization; see for instance [AD99, Prop. 6] for a complete proof. We give below a probabilistic interpretation of the $n^{-3/2}$ factor (see Remark 1.25), but we begin by a useful lemma, valid for random walks with symmetric increments (this is a classical result of Sparre Andersen, though the proof below is taken from [DDG13, Prop. 1.3]). □

Lemma 1.23. *Let* $(X_i)_{i\geq 0}$ *be i.i.d. random variables with* symmetric *distribution and set* $S_n = \sum_{i=1}^n X_i$. *Define*

$$q_n := \mathrm{P}(S_k \geq 0\ \forall 1 \leq k \leq n) \qquad \text{and} \qquad \bar{q}_n := \mathrm{P}(S_k > 0\ \forall 1 \leq k \leq n)\,.$$

We then have the following:

(i) If the law of X_i *has no atom, then* $q_n = \bar{q}_n = \frac{1}{4^n}\binom{2n}{n}$.

(ii) In general, $\bar{q}_n \leq \frac{1}{4^n}\binom{2n}{n} \leq q_n$.

Let us make two observations which stresses how remarkable this result is: first, no assumption is made on the law (in particular, no moment condition is required); second, in the non-atomic case, the probability q_n is *universal* and does not depend on the underlying distribution.

Remark 1.24. Since $\frac{1}{4^n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}$, we obtain the asymptotic behavior of q_n in the non-atomic case. We also get that $q_n \asymp 1/\sqrt{n}$ in general. The $1/\sqrt{n}$ behavior remains universal in the non-symmetric case with $\mathrm{E}(X_i) = 0$ and $\mathrm{E}(X_i^2) < +\infty$, but is no longer universal when X_i does not have a finite second moment.

Proof of Lemma 1.23. Define

$$T = \min\{0 \le k \le n : S_k = \min_{0\le i\le n} S_i\}.$$

(It is *not* a stopping time.) Then the following equality of events holds (to see this, draw a picture):

$$\{T = k\} = \{X_k < 0, X_k + X_{k-1} < 0, \dots, X_k + \dots + X_1 < 0\} \\ \cap \{X_{k+1} \ge 0, X_{k+1} + X_{k+2} \ge 0, \dots, X_{k+1} + \dots + X_n \ge 0\}.$$

Then, using independence and the i.i.d. property, we obtain that

$$\mathrm{P}(T = k) = \mathrm{P}(S_1 < 0, \dots, S_k < 0)\,\mathrm{P}(S_1 \ge 0, \dots, S_{n-k} \ge 0) = \bar{q}_k\, q_{n-k}\,,$$

where the last equality uses the symmetry of $(S_i)_{i\ge1}$. Summing over k we obtain $\sum_{k=0}^{n} \bar{q}_k q_{n-k} = 1$. Therefore, introducing generating functions and using a Cauchy product, we get that for all $|x| < 1$,

$$\bar{Q}(x)Q(x) := \Big(\sum_{k=0}^{\infty} \bar{q}_k x^k\Big)\Big(\sum_{k=0}^{\infty} q_k x^k\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} \bar{q}_k q_{n-k}\Big)x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,.$$

We can now conclude the proof.

For (i). If the law of X_i has no atom, then clearly $\bar{q}_k = q_k$ for all $k \ge 0$, so that $\bar{Q}(x) = Q(x)$ and

$$Q(x) = \sum_{k=0}^{\infty} q_k x^k = \frac{1}{\sqrt{1-x}} \qquad \text{for all } |x| < 1\,.$$

Expanding $(1-x)^{-1/2}$ into a power series and identifying coefficients yields the desired result $q_k = \frac{1}{4^k}\binom{2k}{k}$.

For (ii). In the general case, one can approximate X_i by a distribution with no atom, in the following way. Let $(U_i)_{i\geq 1}$ be i.i.d. random variables uniformly distributed on $[-1,1]$, and for $n\in\mathbb{N}$ and $\varepsilon>0$ fixed, define

$$\tilde{X}_i := X_i + \frac{\varepsilon}{n} U_i\,,$$

which are i.i.d., symmetric, and non-atomic. Setting $\tilde{S}_k = \sum_{i=1}^k \tilde{X}_i$ and noting that $U_i\in[-1,1]$, we have

$$S_k - \varepsilon \leq \tilde{S}_k \leq S_k + \varepsilon \qquad \text{for all } 1\leq k\leq n.$$

Hence,

$$\mathrm{P}(S_k\geq \varepsilon\,,\ \forall\, 1\leq k\leq n) \leq \mathrm{P}(\tilde{S}_k \geq 0\ \forall\, 1\leq k\leq n) \leq \mathrm{P}(\tilde{S}_k\geq -\varepsilon\,,\ \forall\, 1\leq k\leq n).$$

By the part *(i)*, the middle probability is equal to $\frac{1}{4^n}\binom{2n}{n}$. Letting $\varepsilon\downarrow 0$ then yields $\bar{q}_n \leq \frac{1}{4^n}\binom{2n}{n} \leq q_n$, which concludes the proof. □

Remark 1.25 (Intuition behind $p_n^+\sim c_2 n^{-3/2}$)**.** Using Lemma 1.23, together with the local CLT, one can get some intuition for the reason one has $p_n^+ \asymp n^{-3/2}$ factor. Indeed, one may split the interval $[0,n]$ into three parts, and decompose the event $\{S_k\geq 0\ \forall 1\leq k\leq n\}$ into three pieces: $\{S_k\geq 0\ \forall k\leq n/3\}$, then $\{S_{n/3} = S_{2n/3}$ and $S_k\geq 0\}$, and finally $\{S_{n-k}\geq 0\ \forall k\leq n/3\}$. By the local CLT, the probability of the middle part is of order $1/\sqrt{n}$; by the estimates on q_n in Lemma 1.23, the first and last parts are of order $1/\sqrt{n}$. Combining these estimates yields the $n^{-3/2}$ factor.

Exercise 8. By applying the previous strategy, show rigorously that there exists $c>0$ such that $p_n^+\leq cn^{-3/2}$. *The lower bound* $p_n^+\geq c' n^{-3/2}$ *is more delicate.*

We can now use the estimates of Theorem 1.22, combined with (1.11)–(1.12), to obtain information on the number of contacts $\mathcal{L}_N$ between the interface and the line $\mathbb{N}\times\{0\}$, under the measures P_N^0 and P_N^+.

a) Trajectory properties *without* a wall constraint.

Proposition 1.26 (Returns to zero in $d=1$)**.** *For all* $0\leq a\leq b\leq 1$*, we have*

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \mathrm{E}_N^0\big[\mathcal{L}_N(\lfloor aN\rfloor, \lfloor bN\rfloor)\big] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b \frac{1}{\sqrt{u(1-u)}} \mathrm{d}u\,.$$

In particular, this shows that $\mathcal{L}_N$ is of order $\sqrt{N}$; it also shows that the contact points are "on average" spread out, in the sense that the limit above is strictly positive for any $a < b$. One can in fact show convergence in distribution, under P_N^0, of $\big(\frac{1}{\sqrt{N}}\mathcal{L}_N(0, \lfloor bN \rfloor)\big)_{b\in[0,1]}$ to the local time at 0 of a Brownian bridge.

Quick proof. Using (1.11) and Theorem 1.22, we obtain that for N large

$$\mathrm{E}_N^0\big[\mathcal{L}_N(k,\ell)\big] = \sum_{n=k}^{\ell} \frac{p_n p_{N+1-n}}{p_{N+1}} \sim c_1 \sum_{n=k}^{\ell} \frac{\sqrt{N+1}}{\sqrt{n}\sqrt{N+1-n}}$$

$$= c_1 \sqrt{N+1} \sum_{n=k}^{\ell} \frac{1}{N+1} \frac{1}{\sqrt{\frac{n}{N+1}}\sqrt{1-\frac{n}{N+1}}} \,.$$

The last term is a Riemann sum: if we have $\lim_{N\to\infty} \frac{k}{N} = a$ and $\lim_{N\to\infty} \frac{\ell}{N} = b$, then

$$\lim_{N\to\infty} \sum_{n=k}^{\ell} \frac{1}{N+1} \frac{1}{\sqrt{\frac{n}{N+1}}\sqrt{1-\frac{n}{N+1}}} = \int_a^b \frac{1}{\sqrt{u(1-u)}} \mathrm{d}u \,.$$

This concludes the proof. □

b) Trajectory properties *with* a wall constraint.

Proposition 1.27 (Entropic repulsion in $d=1$)**.** *There exists a constant $C > 0$ such that, for all $k \geq 1$,*

$$\mathrm{E}_N^+\big[\mathcal{L}_N(k, N+1-k)\big] \leq C\, k^{-1/2} \,.$$

In particular, for $k = 0$ we obtain that $\sup_N \mathrm{E}_N^+[\mathcal{L}_N] \leq C$. In other words, there is a finite (*i.e.* tight) number of contacts; for instance, by Markov's inequality we get $\mathrm{P}_N^+(\mathcal{L}_N \geq A) \leq C/A$. We also obtain that

$$\mathrm{P}_N^+\big(\exists n \in [k, N+1-k] : S_n = 0\big) = \mathrm{P}_N^+(\mathcal{L}_N(k, N+1-k) \geq 1) \leq \frac{C}{\sqrt{k}} \,.$$

This shows that the probability of having a contact with the hard wall "far from the boundary" is very small.

Quick proof. We use the same argument as above. Using (1.12) and Theorem 1.22, we obtain

$$\mathrm{E}_N^+\big[\mathcal{L}_N(k, N+1-k)\big] = \sum_{n=k}^{N+1-k} \frac{p_n^+ p_{N+1-n}^+}{p_{N+1}^+} \le C_0 \sum_{n=k}^{N+1-k} \frac{(N+1)^{3/2}}{n^{3/2}(N+1-n)^{3/2}}\,.$$

Using the symmetry of the last sum, we have

$$\mathrm{E}_N^+\big[\mathcal{L}_N(k, N+1-k)\big] \le 2C_0 \sum_{n=k}^{(N+1)/2} \frac{1}{n^{3/2}} \frac{(N+1)^{3/2}}{(N+1-n)^{3/2}} \le 2^{5/2} C_0 \sum_{n=k}^{\infty} \frac{1}{n^{3/2}}\,,$$

where we used that $N+1-n \ge (N+1)/2$ for $n \le (N+1)/2$. The conclusion follows immediately. □

Exercise 9. Show that there exists a constant $c > 0$ such that $\mathrm{P}_N^+(\mathcal{L}_N = 0) \ge c$ for all $N \ge 1$.

1.3.2 The case of the GFF in dimension $d \ge 3$

To simplify the notation in what follows, we shall assume that $\beta = 1$ and remove β from all notations (recall Remark 1.11).

We consider here the Gaussian free field on the whole lattice $\mathbb{Z}^d$ with $d \ge 3$ (*i.e.* in the infinite-volume limit), in its centered version (*i.e.* with boundary condition $\xi \equiv 0$). We recall Theorem 1.16: in that setting, the infinite-volume GFF is a centered Gaussian field with covariance matrix $(G(x,y))_{x,y\in\mathbb{Z}^d}$, where G is the Green function of the simple random walk on $\mathbb{Z}^d$. For simplicity, we denote its law by P_∞ and we also set $G_0 := G(0,0) = \mathrm{Var}_\infty(\varphi_x)$. We also recall the asymptotic behavior

$$G(x,y) \sim c_d \|x-y\|^{2-d} \quad \text{as } \|x-y\| \to +\infty\,, \tag{1.13}$$

for some constant c_d, see Remark 1.13.

Let $D \subset \mathbb{R}^d$ be a compact set of $\mathbb{R}^d$. We then consider the domain $\Lambda_N \Subset \mathbb{Z}^d$ defined by

$$\Lambda_N = (ND) \cap \mathbb{Z}^d = \big\{x \in \mathbb{Z}^d : \tfrac{x}{N} \in D\big\}\,,$$

which is a dilation (and a discretization) of D. We denote by $\Omega_N^+ = \Omega_{\Lambda_N}^+$ the event defined in (1.9) that the GFF is positive on Λ_N^+.

Our goal is then to understand the law $\mathrm{P}_\infty(\cdot \mid \Omega_N^+)$ of the Gaussian free field conditioned to be positive on Λ_N. As in dimension $d = 1$, a first step toward understanding the properties of $\mathrm{P}_\infty(\,\cdot \mid \Omega_N^+)$ is to estimate the probability $\mathrm{P}_\infty(\Omega_N^+)$. this already turns out to be a difficult task

Theorem 1.28 (Entropic repulsion of the GFF in $d=3$)**.** *In dimension $d \geq 3$, one has the following asymptotic behavior:*

$$\lim_{N\to\infty} \frac{1}{N^{d-1}\log N} \log \mathrm{P}_\infty(\Omega_N^+) = -2G_0 \mathrm{Cap}(D)\,, \tag{1.14}$$

where $\mathrm{Cap}(D)$ *is a quantity intrinsic to the domain D, called* capacity*, defined by:*

$$\mathrm{Cap}(D) := \inf\left\{ \frac{1}{2d}\int_{\mathbb{R}^d} \|\nabla f(x)\|^2 \mathrm{d}x \ ;\ f\in\mathcal{C}_c^\infty(\mathbb{R}^d), f(x)=1 \text{ for all } x\in D\right\}. \tag{1.15}$$

Moreover, there is a sequence $(a_N)_{N\geq 1}$ which verifies $a_N \sim \sqrt{4G_0\log N}$ as $N\to\infty$, such that:

- *For $\varepsilon>0$, denote by $Q_N^\varepsilon := \frac{1}{|\Lambda_N|}\sum_{x\in\Lambda_N} \mathbf{1}_{\{\frac{1}{a_N}\varphi_x \notin[1-\varepsilon,1+\varepsilon]\}}$. Then, for any $\varepsilon>0$,*

$$\lim_{N\to\infty} \mathrm{P}_\infty\big(Q_N^\varepsilon > \eta \mid \Omega_N^+\big) = 0 \qquad \text{for all } \eta>0\,.$$

- *The law of $\varphi - a_N$ under $\mathrm{P}_\infty(\,\cdot \mid \Omega_N^+)$ converges weakly to P_∞.*

This theorem is proved in [BDZ95, DG99]. We will focus here on the proof of a weaker version of (1.14), borrowed from the lecture notes [Gia01]. For a full proof of (1.14), we refer to [Gia01] which uses refinements of what we present below.

Before we start the proof, let us make a few comments.

- First, the probability decays *more slowly* than e^{-cN^d}, which would be the decay rate obtained if the $(\varphi_x)_{x\in\mathbb{Z}^d}$ were i.i.d. This means that the covariance structure of the field plays a role in the behavior of the probability.
- Second, the last two points of the theorem suggest that the interface is lifted to a height $a_N \sim \sqrt{4G_0\log N}$ and then behaves like a GFF translated by a_N.

In fact, the asymptotic behavior of $\mathrm{P}_\infty(\Omega_N^+)$ essentially comes from the "cost" of lifting the surface above Λ_N; the capacity of D appears by optimizing the way that one can lift the surface to reach a height $\approx a_N$ above Λ_N. This provides some intuition that the interface possess a certain "rigidity", which is a very different behavior from what happens in dimension $d=1$.

Remark 1.29 (Newtonian capacity of a set)**.** The capacity (1.15) of a compact set D in $\mathbb{R}^d$ is the mathematical analogue of the ability of the set D to carry an electric charge. The function f in the integral represents a potential between the set D and infinity, and $f(y) - f(x)$ is the intensity of the current between y and x. The integral $\frac{1}{2d}\int_{\mathbb{R}^d} \|\nabla f(x)\|^2 \mathrm{d}x$ is called the *Dirichlet energy* associated with the potential f.

a) Lower bound on $\mathrm{P}_\infty(\Omega_N^+)$

We first prove a lower bound on $\mathrm{P}_\infty(\Omega_N^+)$. To keep the proof slightly simpler, we aim for a weaker statement compared to the one stated in Theorem 1.28 (a factor d should be replaced by a 2). The proof we present below can be adapted to obtain the correct lower bound; we refer to the lecture notes [Gia01] for details.

Proposition 1.30. *One has* $\liminf_{N\to\infty} \dfrac{1}{N^{d-2}\log N} \log \mathrm{P}_\infty(\Omega_N^+) \geq -dG_0 \mathrm{Cap}(D)\,.$

Proof. The strategy of the proof is quite classical in probability theory (and statistical mechanics), but it is extremely powerful: it is based in a change of measure argument. The idea is as follows:

(i) Introduce another law $\hat{\mathrm{P}}$ (which depends on N) under which Ω_N^+ becomes a typical event, that is such that $\lim_{N\to\infty} \hat{\mathrm{P}}(\Omega_N^+) = 1$.

(ii) Compare the probabilities $\mathrm{P}_\infty(\Omega_N^+)$ and $\hat{\mathrm{P}}(\Omega_N^+)$. Here, we use a general inequality known as an *entropy inequality*: it involves the relative entropy $\mathcal{H}(\hat{\mathrm{P}} \mid \mathrm{P}_\infty)$ of $\hat{\mathrm{P}}$ with respect to P, see Lemma 1.31. It then remains to estimate the relative entropy $\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}_\infty)$.

Step (i). Introducing of the law $\hat{\mathrm{P}}$. Let $\alpha_N \geq 0$ and let $f : \mathbb{R}^d \to \mathbb{R}_+$ be a given function in $\mathcal{C}_c^\infty$ (*i.e.* smooth with compact support) such that $f(x) = 1$ for all $x \in D$. We then define a function $\psi = \psi_N : \mathbb{Z}^d \to \mathbb{R}_+$ with finite support, which is a scaled version of f:

$$\psi_x := \alpha_N f\big(\tfrac{x}{N}\big)\,.$$

We can then "lift" the interface by the function $\psi = \psi_N$: we denote by $\hat{\mathrm{P}}$ the law of $\varphi + \psi$, where φ is a GFF with law P_∞.

Then, since $\psi_x = \alpha_N$ for all $x \in \Lambda_N$ and $\varphi_x \sim \mathcal{N}(0, G_0)$ under P_∞, we have

$$\hat{\mathrm{P}}\big((\Omega_N^+)^c\big) = \mathrm{P}_\infty\big(\exists x \in \Lambda_n\,,\ \varphi_x + \alpha_N < 0\big) \leq |\Lambda_N| \mathrm{P}\big(\mathcal{N}(0, G_0) < -\alpha_N\big)\,,$$

by subadditivity. Then, using the inequality $\mathrm{P}(Z \le -t) = \mathrm{P}(Z \ge t) \le \frac{1}{\sqrt{2\pi}t}\mathrm{e}^{-\frac{1}{2}t^2}$ for $t \ge 0$ and $Z \sim \mathcal{N}(0,1)$, we obtain that

$$\hat{\mathrm{P}}\big((\Omega_N^+)^c\big) \le |\Lambda_N| \times \frac{1}{\sqrt{2\pi}\,\alpha_N}\mathrm{e}^{-\frac{\alpha_N^2}{2G_0}}\,.$$

To obtain a quantity that tends to 0, since $|\Lambda_N| = \mathrm{Vol}(D)N^d$, it is natural to choose

$$\alpha_N := \sqrt{2dG_0 \log N}\,, \tag{1.16}$$

so that the previous bound gives $\hat{\mathrm{P}}\big((\Omega_N^+)^c\big) \le C/\alpha_N \to 0$. In conclusion, taking α_N as in (1.16), we obtain $\lim_{N\to\infty} \hat{\mathrm{P}}(\Omega_N^+) = 1$ as desired.

Step (ii)-(a). Entropy inequality.

Lemma 1.31. *Let* $\hat{\mathrm{P}}$ *and* P *be two probability measures. We define the* relative entropy *of* $\hat{\mathrm{P}}$ *with respect to* P *by*

$$\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}) := \hat{\mathrm{E}}\Big[\log\frac{\mathrm{d}\hat{\mathrm{P}}}{\mathrm{dP}}\Big] = \mathrm{E}\Big[\frac{\mathrm{d}\hat{\mathrm{P}}}{\mathrm{dP}}\log\frac{\mathrm{d}\hat{\mathrm{P}}}{\mathrm{dP}}\Big] \tag{1.17}$$

if $\hat{\mathrm{P}}$ *is absolutely continuous with respect to* P*, and* $\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}) = +\infty$ *otherwise.*

Then, for any event A*, the following inequality holds:*

$$\mathrm{P}(A) \ge \hat{\mathrm{P}}(A)\,\exp\Big(-\frac{1}{\hat{\mathrm{P}}(A)}\big(\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}) + \mathrm{e}^{-1}\big)\Big)\,.$$

We leave as an exercise to check that one always has $\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}) \ge 0$.

Proof. Note that

$$\frac{\mathrm{P}(A)}{\hat{\mathrm{P}}(A)} = \frac{1}{\hat{\mathrm{P}}(A)}\hat{\mathrm{E}}\Big[\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\,\mathbf{1}_A\Big] = \hat{\mathrm{E}}\Big[\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\;\Big|\; A\Big]\,,$$

so that, by Jensen's inequality, we obtain

$$\log\frac{\mathrm{P}(A)}{\hat{\mathrm{P}}(A)} \ge \hat{\mathrm{E}}\Big[\log\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\;\Big|\; A\Big] = -\frac{1}{\hat{\mathrm{P}}(A)}\hat{\mathrm{E}}\Big[\Big(\log\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\Big)\mathbf{1}_A\Big]\,.$$

It remains to estimate the last term:

$$\begin{aligned}\hat{\mathrm{E}}\Big[\Big(\log\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\Big)\mathbf{1}_A\Big] = \mathrm{E}\Big[\Big(\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\log\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\Big)\mathbf{1}_A\Big] &\le \mathrm{E}\Big[\Big(\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\log\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}} + \mathrm{e}^{-1}\Big)\mathbf{1}_A\Big] \\ &\le \mathrm{E}\Big[\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}}\log\frac{\mathrm{dP}}{\mathrm{d}\hat{\mathrm{P}}} + \mathrm{e}^{-1}\Big] = \mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}) + \mathrm{e}^{-1}\,,\end{aligned}$$

where we used the fact that $x\log x + \mathrm{e}^{-1} \geq 0$ for the last inequality. This concludes the proof of the lemma. □

Combining Lemma 1.31 with *Step (i)*, this shows that

$$\mathrm{P}_\infty(\Omega_N^+) \geq (1+o(1))\exp\Big(-(1+o(1))\big[\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}_\infty) + \mathrm{e}^{-1}\big]\Big)\,. \tag{1.18}$$

Step (ii)-(b). Estimating $\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}_\infty)$. To estimate the relative entropy $\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}_\infty)$, note that $\hat{\mathrm{P}}$ is the law of a Gaussian field with mean ψ and covariance matrix G. Thus, since ψ has finite support, the density $\frac{\mathrm{d}\hat{\mathrm{P}}}{\mathrm{d}\mathrm{P}_\infty}$ vanishes outside a compact set. Using the form of the Gaussian density, we have

$$\frac{\mathrm{d}\hat{\mathrm{P}}}{\mathrm{d}\mathrm{P}_\infty} = \frac{\mathrm{e}^{-\frac12\langle\varphi-\psi, G^{-1}(\varphi-\psi)\rangle}}{\mathrm{e}^{-\frac12\langle\varphi, G^{-1}\varphi\rangle}}\,,$$

so that

$$\log\frac{\mathrm{d}\hat{\mathrm{P}}}{\mathrm{d}\mathrm{P}_\infty} = -\frac12\langle\psi, G^{-1}\psi\rangle + \langle\varphi, G^{-1}\psi\rangle = \frac12\langle\psi, G^{-1}\psi\rangle + \langle(\varphi-\psi), G^{-1}\psi\rangle\,.$$

Since $\hat{\mathrm{E}}[\varphi_x] = \psi_x$, we deduce that[2]

$$\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}_\infty) = \hat{\mathrm{E}}\Big[\log\frac{\mathrm{d}\hat{\mathrm{P}}}{\mathrm{d}\mathrm{P}_\infty}\Big] = \frac12\langle\psi, G^{-1}\psi\rangle\,. \tag{1.19}$$

In our case, we have $G^{-1} = -\Delta$, so we obtain (as in the Hamiltonian (1.6))

$$\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}_\infty) = \frac{1}{8d}\sum_{\substack{x,y\in\mathbb{Z}^d\\ x\sim y}}(\psi_x-\psi_y)^2 = \frac{\alpha_N^2}{2}\frac{1}{2d}\sum_{x\in\mathbb{Z}^d}\sum_{y\sim x}\Big(f\big(\tfrac{x}{N}\big) - f\big(\tfrac{y}{N}\big)\Big)^2$$

where we have used that ψ is of the form $\psi_x = \alpha_N f(\frac{x}{N})$ (and lost a factor 2 since pairs $\{x,y\}$ in the first sum are counted twice). The sum over $\mathbb{Z}^d$ is a Riemann sum, and we thus have (recall that f is smooth with compact support):

$$\lim_{N\to\infty}\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\sum_{y\sim x}\Big(\frac1N\big(f\big(\tfrac{x}{N}\big) - f\big(\tfrac{y}{N}\big)\big)\Big)^2 = \int_{\mathbb{R}^d}\|\nabla f(x)\|^2\mathrm{d}x\,.$$

[2] Note that this computation is general for Gaussian vectors with the same covariance matrix.

Therefore, as $N \to \infty$,

$$\mathcal{H}(\hat{\mathrm{P}}|\mathrm{P}_\infty) = (1+o(1))\frac{\alpha_N^2}{2}\, N^{d-2}\, \frac{1}{2d}\int_{\mathbb{R}^d} \|\nabla f(x)\|^2 \mathrm{d}x$$
$$= (1+o(1)) d G_0\, N^{d-2} \log N\, \frac{1}{2d}\int_{\mathbb{R}^d} \|\nabla f(x)\|^2 \mathrm{d}x\,,$$

recalling the choice (1.16) of α_N. Together with (1.18), this therefore shows that, for any function $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with $f(x) \equiv 1$ on D,

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathrm{P}_\infty(\Omega_N^+) \geq -dG_0 \times \frac{1}{2d}\int_{\mathbb{R}^d} \|\nabla f(x)\|^2 \mathrm{d}x\,.$$

Optimizing over the choice of the function f and recalling the definition (1.15) of $\mathrm{Cap}(D)$, this concludes the proof of Proposition 1.30. □

b) Upper bound on $\mathrm{P}_\infty(\Omega_N^+)$

Proposition 1.32. *One has* $\limsup_{N\to\infty} \frac{1}{N^{d-2}\log N} \log \mathrm{P}_\infty(\Omega_N^+) \leq -C_D\,.$
Here, the constant C_D is explicit (see the end of the proof), and depends on the domain D and on the dimension.

Before the proof, let us introduce the following terminology: we say that $x = (x_1,\dots,x_d) \in \mathbb{Z}^d$ is *even* if $\sum_{i=1}^d x_i$ is even; we say that $x \in \mathbb{Z}^d$ is *odd* if $\sum_{i=1}^d x_i$ is odd. We denote

$$\Lambda_N^{\mathrm{even}} = \{x \in \Lambda_N : x \text{ even}\}\,, \qquad \Lambda_N^{\mathrm{odd}} = \{x \in \Lambda_N : x \text{ odd}\}\,.$$

Proof. The idea is to condition on $\mathcal{F}_{\Lambda_N^{\mathrm{odd}}}$, *i.e.*, on the values of the GFF on odd sites. Indeed, a site $x \in \Lambda_N^{\mathrm{even}}$ is surrounded by odd sites, so by the spatial Markov property (Proposition 1.5), conditioning on $\mathcal{F}_{\Lambda_N^{\mathrm{odd}}}$ amounts to fixing the value of all neighbors of $x \in \Lambda_N^{\mathrm{even}}$. Conditionally on $\mathcal{F}_{\Lambda_N^{\mathrm{odd}}}$, *i.e.*, on the value of the GFF ϕ on Λ_N^{odd}, the variables $(\varphi_x)_{x\in\Lambda_N^{\mathrm{even}}}$ are then independent Gaussian random variables, with mean $m_x^\phi := \frac{1}{2d}\sum_{y\sim x}\phi_y$ (this is the harmonic extension of ϕ at x) and variance 1 (starting from x, the random walk exits $\Lambda_N^{\mathrm{even}}$ in one step, so the number of visits to x is exactly 1).

Therefore, conditioning with respect to $\mathcal{F}_{\Lambda_N^{\mathrm{odd}}}$ and using the spatial Markov property, we obtain

$$\mathrm{P}_\infty(\Omega_N^+) = \mathrm{E}_\infty\Big[\mathrm{P}^\phi_{\Lambda_N^{\mathrm{even}}}\big(\Omega^+_{\Lambda_N^{\mathrm{even}}}\big)\mathbf{1}_{\Omega^+_{\Lambda_N^{\mathrm{odd}}}}\Big] = \mathrm{E}_\infty\Big[\Big\{\prod_{x\in\Lambda_N^{\mathrm{even}}}\mathrm{P}(Z+m_x^\phi>0)\Big\}\mathbf{1}_{\Omega^+_{\Lambda_N^{\mathrm{odd}}}}\Big],$$

where $Z\sim\mathcal{N}(0,1)$. Writing

$$\mathrm{P}(Z+m_x^\phi>0) = \mathrm{P}(Z<m_x^\phi) = 1-\mathrm{P}(Z>m_x^\phi) \le \exp(-\mathrm{P}(Z>m_x^\phi)),$$

we deduce the following inequality:

$$\mathrm{P}_\infty(\Omega_N^+) \le \mathrm{E}_\infty\Big[\exp\Big(-\sum_{x\in\Lambda_N^{\mathrm{even}}}\mathrm{P}(Z>m_x^\phi)\Big)\mathbf{1}_{\Omega^+_{\Lambda_N^{\mathrm{odd}}}}\Big].$$

We then split this expectation into two parts, according to whether the sum in the exponential is large or not. To do this, let $\varepsilon\in(0,1)$, and introduce the following event (measurable with respect to $\mathcal{F}_{\Lambda_N^{\mathrm{odd}}}$):

$$A_N := \Big\{\big|\big\{x\in\Lambda_N^{\mathrm{even}} : m_x^\phi \ge (1-\varepsilon)\sqrt{4\log N}\big\}\big| \ge (1-\varepsilon)|\Lambda_N^{\mathrm{odd}}|\Big\}.$$

In words, A_N corresponds to the event that the vast majority of the means $m_x^\phi = \frac{1}{2d}\sum_{y\sim x}\phi_y$ are large.

Step 1. On the event A_N^c. Note that on the event A_N^c, there are at least $\varepsilon|\Lambda_N^{\mathrm{even}}|$ sites $x\in\Lambda_N^{\mathrm{even}}$ such that $m_x^\phi\ge(1-\varepsilon)\sqrt{4\log N}$: in that case we have

$$\begin{aligned}\sum_{x\in\Lambda_N^{\mathrm{even}}}\mathrm{P}(Z>m_x^\phi) &\ge \varepsilon|\Lambda_N^{\mathrm{odd}}|\,\mathrm{P}\big(Z>(1-\varepsilon)\sqrt{4\log N}\big)\\ &\ge c_\varepsilon N^d\frac{1}{\sqrt{\log N}}\mathrm{e}^{-\frac12(1-\varepsilon)^2 4\log N} \ge N^{d-2+2\varepsilon+o(1)}.\end{aligned}$$

where we have used that $\mathrm{P}(Z>t)\sim\frac{1}{t\sqrt{2\pi}}\mathrm{e}^{-\frac12 t^2}$ as $t\to\infty$. We conclude that

$$\mathrm{P}_\infty(\Omega_N^+\cap A_N^c) \le \mathrm{E}_\infty\Big[\exp\Big(-\sum_{x\in\Lambda_N^{\mathrm{even}}}\mathrm{P}(Z>m_x^\phi)\Big)\mathbf{1}_{A_N^c}\Big] \le \exp\big(-N^{d-2+2\varepsilon+o(1)}\big),$$

which is negligible compared to $\mathrm{P}_\infty(\Omega_N^+)$ thanks to Proposition 1.30.

Step 2. On the event A_N. First of all, let us start by Writing

$$\mathrm{P}_\infty(\Omega_N^+\cap A_N) \le \mathrm{P}_\infty\big(\Omega^+_{\Lambda_N^{\mathrm{odd}}}\cap A_N\big).$$

Now, note that on the event $A_N \cap \Omega^+_{\Lambda_N^{\mathrm{odd}}}$, since all m_x^ϕ are non-negative, keep only the large ones we have

$$\sum_{x\in\Lambda_N^{\mathrm{even}}} m_x^\phi \geq (1-\varepsilon)^2 |\Lambda_N^{\mathrm{odd}}| \sqrt{4\log N}\,.$$

Now, using the definition of m_x^ϕ, we get that

$$\sum_{x\in\Lambda_N^{\mathrm{even}}} m_x^\phi = \frac{1}{2d}\sum_{x\in\Lambda_N^{\mathrm{even}}}\sum_{y\sim x}\phi_y = \sum_{y\in\Lambda_N^{\mathrm{odd}}}\phi_y\,,$$

so we end up with

$$\mathrm{P}_\infty\big(\Omega^+_{\Lambda_N^{\mathrm{odd}}}\cap A_N\big) \leq \mathrm{P}_\infty\Big(\sum_{y\in\Lambda_N^{\mathrm{odd}}}\phi_y \geq (1-\varepsilon)^2|\Lambda_N^{\mathrm{odd}}|\sqrt{4\log N}\Big)\,.$$

It remains to observe that, under P_∞, the sum $\sum_{y\in\Lambda_N^{\mathrm{odd}}}\phi_y$ is a centered Gaussian random variable, with variance

$$\sigma_N^2 = \sum_{x,y\in\Lambda_N^{\mathrm{odd}}} G(x,y)\,.$$

Hence, using that $\mathrm{P}(Z>t)\leq \mathrm{e}^{-\frac12 t^2}$ for $Z\sim\mathcal{N}(0,1)$, we get

$$\begin{aligned}\mathrm{P}_\infty(\Omega_N^+\cap A_N) &\leq \exp\Big(-\frac{1}{2\sigma_N^2}\times 4(1-\varepsilon)^4|\Lambda_N^{\mathrm{odd}}|^2\log N\Big)\\ &= \exp\Big(-2(1-\varepsilon)^4\,\frac{(N/2)^{2d}}{\sigma_N^2}\mathrm{Vol}(D)^2\log N\Big)\,.\end{aligned}$$

To estimate σ_N^2, we can use the asymptotic $G(x,y)\sim c_d\|x-y\|^{2-d}$ from (1.13): setting $\mathcal{G}_N(a,b)=N^{d-2}G(aN,bN)$, by a Riemann sum we obtain

$$\lim_{N\to\infty}\frac{1}{(N/2)^{2d}}\sum_{x,y\in\Lambda_N^{\mathrm{odd}}}\mathcal{G}_N\big(\tfrac{x}{N},\tfrac{y}{N}\big) = c_d\int_{D\times D}\mathcal{G}(x,y)\mathrm{d}x\mathrm{d}y\,,$$

where $\mathcal{G}(x,y)=\|x-y\|^{2-d}$. We deduce that, as $N\to\infty$,

$$\frac{\sigma_N^2}{(N/2)^{2d}} = \frac{1}{(N/2)^{2d}}\sum_{x,y\in\Lambda_N^{\mathrm{odd}}}G(x,y)\sim c_d N^{2-d}\int_{D\times D}\mathcal{G}(x,y)\mathrm{d}x\mathrm{d}y\,.$$

This concludes the proof, with $C_D := \frac{1}{c_d}\mathrm{Vol}(D)^2\big(\int_{D\times D}\|x-y\|^{2-d}\mathrm{d}x\mathrm{d}y\big)^{-1}$. $\square$

c) Properties of trajectories under the conditional law

In the course of the proof, we have showed that $\mathrm{P}_\infty(\Omega_N^+ \cap A_N^c) \ll \mathrm{P}_\infty(\Omega_N^+)$. It easily follows that $\lim_{N\to\infty} \mathrm{P}_\infty(A_N^c \mid \Omega_N^+) = 0$. One could in fact proceed identically for an event A'_N in which the roles of $\Lambda_N^{\mathrm{even}}$ and Λ_N^{odd} are exchanged. We thus obtain the following corollary, which gives some important information on the behavior of the interface under the conditional law $\mathrm{P}_\infty(\,\cdot \mid \Omega_N^+)$.

Corollary 1.33. *For any $\varepsilon \in (0,1)$, define the event*

$$\tilde{A}_N := \Big\{ \Big| \Big\{ x \in \Lambda_N \,,\, \tfrac{1}{2d} \sum_{y\sim x} \varphi_y \geq (1-\varepsilon)\sqrt{4\log N} \Big\} \Big| \geq (1-\varepsilon)|\Lambda_N| \Big\}$$

Then we have $\lim_{N\to\infty} \mathrm{P}_\infty(\tilde{A}_N \mid \Omega_N^+) = 1$.

Note that this is some weaker version of the second part of Theorem 1.28.

Exercise 10. Prove the above result. Prove also the same result for the event $\bar{A}_N = \big\{ \frac{1}{|\Lambda_N|} \sum_{x\in\Lambda_N} \varphi_x \geq (1-\varepsilon)\sqrt{4\log N} \big\}$.

1.3.3 The case of the GFF in dimension $d = 2$

In this section, let us simply provide some statement on the entropic repulsion phenomenon in dimension $d = 2$. Note here that there is no infinite-volume Gibbs measure, so we need to introduce some slightly different setting. Consider two compact sets $D \subset V$ of $\mathbb{R}^2$ that are "well separated", that is such that $d(D, \partial V) > 0$. Define $\Lambda_N = ND \cap \mathbb{Z}^d$ and $V_N = NV \cap \mathbb{Z}^d$. The article [BDG01] then proves the following, somehow analogous to Theorem 1.28.

Theorem 1.34 (Entropic repulsion of the GFF in $d = 2$)**.** *In dimension $d = 2$, we have the following asymptotic behavior*

$$\lim_{N\to\infty} \frac{1}{(\log N)^2} \log \mathrm{P}^0_{V_N}(\Omega^+_{\Lambda_N}) = -4g\, \mathrm{Cap}_V(D)\,,$$

where $g = \frac{2}{\pi}$ and $\mathrm{Cap}_V(D)$ is the relative capacity of D inside V, defined by

$$\mathrm{Cap}_V(D) = \inf\Big\{ \frac{1}{2}\int_{\mathbb{R}^2} \|\nabla f(x)\|^2 \mathrm{d}x \,,\, f \geq 1 \text{ on } D, f = 0 \text{ on } V^c \Big\}$$

Moreover, for every $\varepsilon \in (0,1)$, we have

$$\sup_{x\in\Lambda_N} \mathrm{P}^0_{V_N}\Big(\Big| \frac{\varphi_x}{2\sqrt{g}\log N} - 1 \Big| > \varepsilon \;\Big|\; \Omega^+_{\Lambda_N} \Big) \xrightarrow[N\to\infty]{} 0\,.$$

1.4 A few exercises

In this section, for simplicity, we consider the domain $\Lambda_N = \{1, \dots, N\}^d$. We denote by $\mathrm{P}_N^0 := \mathrm{P}_{\Lambda_N}^0$ the law of a Gaussian free field on Λ_N with zero boundary condition. We also denote by $G_N(x, y) = G_{\Lambda_N}(x, y)$ the Green function inside Λ_N, *i.e.* the Green function of a random walk killed when it exits Λ_N.

a) Adapting Propositions 1.30 and 1.32 to the finite-volume GFF

The goal of the following is to adapt (as an exercise) Propositions 1.30 and 1.32 to the case of a finite-volume Gaussian Free Field, with no "buffer" between the boundary and the hard wall constraint. The three following exercises will prove the following result, as well as some consequence.

Proposition 1.35. *There exist constants $c_d, c'_d > 0$ such that, as $N \to \infty$*

$$-c_d + o(1) \leq \frac{1}{N^{d-1}\log N} \log \mathrm{P}_N^0(\Omega_N^+) \leq -c'_d + +o(1) \qquad \textit{in dimension } d \geq 3\,,$$

$$-c_2 + o(1) \leq \frac{1}{N(\log N)^2} \log \mathrm{P}_N^0(\Omega_N^+) \leq -c'_2 + o(1) \qquad \textit{in dimension } d = 2\,.$$

Exercise 11 (Lower bound in Proposition 1.35)**.** Let us define

$$\sigma_N^2 = \max_{x \in \Lambda_N} \mathrm{Var}_N^0(\varphi_x)\,.$$

1. Show that $\sigma_N^2 \leq G_0 := G(0, 0)$ in dimension $d \geq 3$ and $\sigma_N^2 \sim \frac{2}{\pi} \log N$ in dimension $d = 2$.
   (Hint: you can use Remark 1.13.)

Consider $(\alpha_N)_{N \geq 0}$ a sequence of positive real numbers such that $\alpha_N \to +\infty$. For φ with law P_N^0, let us denote $\hat{\mathrm{P}}_N$ denote the law of $\varphi + \alpha_N$.

2. (a) Show that $\hat{\mathrm{P}}_N((\Omega_N^+)^c) \leq N^d \mathrm{P}(Z < -\frac{\alpha_N}{\sigma_N})$, where $Z \sim \mathcal{N}(0, 1)$.
   (b) Take $\alpha_N = \sqrt{2d}\,\sigma_N \log N$ and show that $\lim_{N \to \infty} \hat{\mathrm{P}}_N(\Omega_N^+) = 1$.
3. Compute the relative entropy $\mathcal{H}(\hat{\mathrm{P}}_N \mid \mathrm{P}_N^0)$.
4. Conclude that the lower bounds of Proposition 1.35 hold (in $d \geq 3$ and $d = 2$), and make the constants explicit.

Exercise 12 (Consequence of the lower bound)**.** Let σ_N be defined as in Exercise 11, and take $\alpha_N = \sqrt{2d}\,\sigma_N \log N$. Then, for $\varepsilon \in (0,1)$, define the event

$$A_N = \Big\{ |\{x \in \Lambda_N, \varphi_x \geq \alpha_N\}| \geq k_N \Big\} ,$$

where k_N is a quantity to be determined (a smaller k_N gives a broader event).

1. Show that $\mathrm{P}_N^0(A_N) \leq \binom{N^d}{k_N} \mathrm{P}(\sigma_N Z \geq \alpha_N)^{k_N}$, where $Z \sim \mathcal{N}(0,1)$.
2. Deduce that if $k_N = N^{d-1}\alpha_N^2$, then for any constant C (arbitrarily large), we have $\mathrm{P}_N^0(A_N) \leq \exp(-C N^{d-1}\alpha_N^2)$ for N large enough.
3. Conclude that with that choice of k_N we have $\lim_{N\to\infty} \mathrm{P}_N^0(A_N \mid \Omega_N^+) = 0$.

Exercise 13 (Upper bound in Proposition 1.35)**.** Let us introduce the set

$$\Gamma_N = \Big\{ x \in \Lambda_N, x \text{ even}, d(x, \Lambda_N^c) = 1 \Big\} .$$

1. Show that we have, for any event $A \in \sigma\{\varphi_x, x \in \Lambda_N \setminus \Gamma_N\}$

$$\mathrm{P}_N^0(\Omega_N^+ \cap A) = \mathrm{E}_N^0\Big[\prod_{x\in\Gamma_N} \mathrm{P}(Z < m_x^\varphi) \mathbf{1}_{\tilde\Omega_N^+ \cap A} \Big] ,$$

   where $Z \sim \mathcal{N}(0,1)$ and where $\tilde\Omega_N^+ = \{\varphi_x > 0\,, \forall x \in \Lambda_N \setminus \Gamma_N\}$.
   Recall the notation $m_x^\varphi = \frac{1}{2d}\sum_{y\sim x} \varphi_y$.

Let A_N denote the event $\big\{ |\{x \in \Gamma_N, m_x^\varphi \leq c_d\sqrt{\log N}\}| \leq \varepsilon N^{d-1} \big\}$, where c_d is a constant to be determined and $\varepsilon \in (0,1)$ is fixed.

2. Show that we have

$$\mathrm{P}_N^0(\Omega_N^+ \cap A_N^c) \leq \mathrm{P}\big(Z < c_d\sqrt{\log N}\big)^{\varepsilon N^{d-1}} .$$

3. For which value of c_d do we have $\mathrm{P}(Z < c_d\sqrt{\log N})^{\varepsilon N^{d-1}} \ll \mathrm{P}_N^0(\Omega_N^+)$? Deduce that for that choice of constant c_d we have that

$$\mathrm{P}_N^0(\Omega_N^+) = \mathrm{P}_N^0(\Omega_N^+ \cap A_N) .$$

   (Recall that we have proven the lower bound in Proposition 1.35.)
4. Show that, on the other hand,

$$\mathrm{P}_N^0\big(\Omega_N^+ \cap A_N\big) \leq \mathrm{P}_N^0\Big(\sum_{x\in\Gamma_N} m_x^\varphi \geq (1-\varepsilon) c_d N^{d-1}\sqrt{\log N}\,, \tilde\Omega_N^+ \Big).$$

Let us introduce the sets

$$\tilde{\Gamma}_N^{(1)} = \{x \in \Lambda_N \text{ odd}, d(x, \Lambda_N^c) = 1\}, \quad \tilde{\Gamma}_N^{(2)} = \{x \in \Lambda_N \text{ odd}, d(x, \Lambda_N^c) = 2\},$$

and let us denote $M_B = \sum_{x\in B} \varphi_x$ for any $B \subset \Lambda_N$.

5. Show that if $\varphi_x > 0$ for all $x \in \Lambda_N \setminus \Gamma_N$ then $\sum_{x\in\Gamma_N} m_x^\varphi \le M_{\tilde{\Gamma}_N^{(1)}} + \frac{1}{2d} M_{\tilde{\Gamma}_N^{(2)}}$.
6. Deduce that

$$\begin{aligned}\mathrm{P}_N^0\big(\Omega_N^+ \cap A_N\big) &\le \mathrm{P}_N\Big(M_{\tilde{\Gamma}_N^{(1)}} + \tfrac{1}{2d} M_{\tilde{\Gamma}_N^{(2)}} \ge (1-\varepsilon) c_d N^{d-1}\sqrt{\log N}\Big) \\ &\le \mathrm{P}_N\Big(M_{\tilde{\Gamma}_N^{(1)}} \ge \tfrac{1-\varepsilon}{2} c_d N^{d-1}\sqrt{\log N}\Big) \\ &\qquad + \mathrm{P}_N\Big(M_{\tilde{\Gamma}_N^{(2)}} \ge (1-\varepsilon) d c_d N^{d-1}\sqrt{\log N}\Big).\end{aligned}$$

7. Show that $M_\Upsilon \sim \mathcal{N}(0, \sigma_\Upsilon^2)$, where $\sigma_\Upsilon^2 := \sum_{x\in\Upsilon} \mathbf{E}_x[V_\Upsilon]$ with $V_\Upsilon = \sum_{i=1}^{T_{\Lambda_N^c}} \mathbf{1}_{S_i\in\Upsilon}$ is the number of visits of a simple random walk to the set Υ before exiting Λ_N. Show that there are constants c_1, c_2 such that $\sigma^2_{\tilde{\Gamma}_N^{(1)}} \le c_1 N^{d-1}$ and $\sigma^2_{\tilde{\Gamma}_N^{(2)}} \le c_2 N^{d-1}$. *(Bonus: make the constants c_1, c_2 explicit.)*
8. Conclude that the upper bounds of Proposition 1.35 hold (in $d \ge 3$ and $d = 2$), and make the constants explicit.

b) Probability that the GFF remains "small" in a domain

We now give another useful lemma as a detailed exercise. It provides a complement to the upper bound given in Exercise 3 on the probability that the GFF remains "small" in some given domain Γ.

Lemma 1.36. *There exists a constant $c > 0$ such that, for any $N \ge 2$ and any $\Gamma \subset \Lambda_N$,*

$$\mathrm{P}_N^0\big(|\varphi_x| \le 1 \ \textit{for all } x \in \Gamma\big) \ge \mathrm{e}^{-c\,\alpha_N^2 |\Gamma|}.$$

Here, $\alpha_N = \log N$ in dimension $d = 2$ and $\alpha_N = \sqrt{\log N}$ in dimension $d \ge 3$.

Exercise 14 (Proof of Lemma 1.36)**.** Recall that the Exercise 6 shows that there is a constant $c_d > 0$ such that $\lim_{N\to\infty} \mathrm{P}_N^0(\max_{x\in\Lambda_N} |\varphi_x| \ge c_d\alpha_N) = 0$. We fix such a constant in the sequel. For a set Υ, we denote by

$$\Upsilon^{\text{even}} = \{x \in \Upsilon, x \text{ even}\}, \qquad \Upsilon^{\text{odd}} = \{x \in \Upsilon, x \text{ odd}\}.$$

1. By conditioning on $\mathcal{F}_{\Lambda_N^{\mathrm{odd}}}$, show that
$$\mathrm{P}_N^0\big(|\varphi_x| \leq 1 \ \forall\, x \in \Gamma\big) \geq \mathrm{E}_N^0\Bigg[\Big\{\prod_{x\in\Gamma^{\mathrm{even}}} \mathrm{P}\big(|Z + m_x^\varphi| \leq 1\big)\Big\}\mathbf{1}_{\{|\varphi_x|\leq 1\ \forall\, x\in\Gamma^{\mathrm{odd}}\}}\Bigg],$$
where $Z \sim \mathcal{N}(0,1)$ and $m_x^\varphi = \frac{1}{2d}\sum_{y\sim x}\varphi_x$.
2. Show that $\mathrm{P}(|Z + m_x^\varphi| \leq 1) \geq 2 \times \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(1+|m_x^\varphi|)^2}$. Deduce that
$$\mathrm{P}_N^0\big(|\varphi_x| \leq 1 \ \forall\, x \in \Gamma\big) \geq \Big(\sqrt{2/\pi}\ \mathrm{e}^{-\frac{1}{2}(1+c_d\alpha_N)^2}\Big)^{|\Gamma^{\mathrm{even}}|} \mathrm{P}_N^0\big(|\varphi_x| \leq 1\ \forall x \in \Gamma^{\mathrm{odd}}\ ;\ |\varphi_x| \leq c_d\alpha_N\ \forall x \in \Lambda_N^{\mathrm{odd}}\big)\,.$$
3. Proceeding in the same way (that is, by conditioning on $\mathcal{F}_{\Lambda_N^{\mathrm{even}}}$), show that
$$\mathrm{P}_N^0\big(|\varphi_x| \leq 1\ \forall x \in \Gamma^{\mathrm{odd}}\ ;\ |\varphi_x| \leq c_d\alpha_N\ \forall x \in \Lambda_N^{\mathrm{odd}}\big) \geq \Big(\sqrt{2/\pi}\ \mathrm{e}^{-\frac{1}{2}(1+c\alpha_N)^2}\Big)^{|\Gamma^{\mathrm{odd}}|}\mathrm{P}_N^0\big(|\varphi_x| \leq c_d\alpha_N\ \forall x \in \Lambda_N\big)\,.$$
4. Conclude the proof of the Lemma, using Exercise 6.

Remark 1.37. One can easily adapt the proof to obtain that, for any sequence M_N with $M_N = o(\log N)$ as $N \to \infty$, we have
$$\inf_{|\xi_{|\partial\Lambda_N}|\leq M_N} \mathrm{P}_N^\xi\big(|\varphi_x| \leq 1 \ \text{ for all } x \in \Gamma\big) \geq \mathrm{e}^{-c\,\alpha_N^2|\Gamma|}\,.$$

Here, the infimum is taken over boundary conditions ξ such that $\sup_{x\in\partial\Lambda_N}|\xi_x| \leq M_N$. Indeed, it suffices to note that Exercise 6 shows that $\mathrm{P}_N^\xi(\max_{x\in\Lambda_N}|\varphi_x| \geq c_d\alpha_N) \to 0$ uniformly for such boundary conditions.

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Chapter 2

Pinning and wetting of random interfaces

The goal of this section is to study an interface in dimension $d+1$ which interacts with a hyperplane $\mathbb{Z}^d \times \{0\}$. We will consider two models of effective interfaces that we have studied in the previous chapter:

- the unconstrained case, when the height of the interface can take negative values — this is known as a *pinning* model;
- the case with a hard wall constraint, when the height of the interface is conditioned to remain positive — this is known as a *wetting* model.

2.1 Introduction of the model and first properties

2.1.1 Definition of the pinning and wetting of interfaces

Let $\Lambda \Subset \mathbb{Z}^d$. For parameters $\beta \geq 0$ and $u \in \mathbb{R}$ and a boundary condition $\xi : \mathbb{Z}^d \to \mathbb{R}$, we consider the following Gibbs measures: for $a > 0$,

$$\frac{\mathrm{dP}^{\xi}_{\Lambda,\beta,u}}{\mathrm{d}\mu} = \frac{1}{W^{\xi}_{\Lambda,\beta,u}} \exp\Big(-\beta H^{\xi}_{\Lambda}(\varphi) + u \sum_{x\in\Lambda} \mathbf{1}_{\{|\varphi_x| \leq a\}}\Big)$$

and

$$\frac{\mathrm{dP}^{\xi,+}_{\Lambda,\beta,u}}{\mathrm{d}\mu} = \frac{1}{W^{\xi,+}_{\Lambda,\beta,u}} \exp\Big(-\beta H^{\xi}_{\Lambda}(\varphi) + u \sum_{x\in\Lambda} \mathbf{1}_{\{\varphi_x \in [0,a]\}}\Big) \mathbf{1}_{\Omega^+_{\Lambda}}(\varphi)\,.$$

In these notations, we have kept all the parameters of the model; notice that when $u = 0$, we recover the models of Chapter 1. The partition functions of the models are denoted by $W^{\xi}_{\Lambda,\beta,u}$ and $W^{\xi,+}_{\Lambda,\beta,u}$. are the partition functions of the model. For simplicity, we will consider in the sequel only the case $\beta = 1$ and we will omit it from the notations; we will also omit the dependence on the parameter $a > 0$, which is fixed once and for all (for instance, $a = \frac{1}{2}$).

The Gibbs measures introduced above correspond to modifying the law P_Λ^ξ of the interface by introducing an interaction with the hyperplane $\mathbb{Z}^d \times \{0\}$, when the interface is at a height smaller than a; one then speaks of *contact* between the interface and the hyperplane. The parameter u controls the intensity of the interaction: if $u < 0$, the hyperplane is repulsive (the interface is penalized if it has many contacts with the hyperplane); if $u > 0$, the hyperplane is attractive (the interface is favored if it has many contacts).

In fact, we will rather write the Gibbs measures $\mathrm{P}_{\Lambda,u}^\xi$ and $\mathrm{P}_{\Lambda,u}^{\xi,+}$ as Gibbs measures with respect to P_Λ^ξ, in the following way:

$$\frac{\mathrm{dP}_{\Lambda,u}^\xi}{\mathrm{dP}_\Lambda^\xi} = \frac{1}{Z_{\Lambda,u}^\xi} \exp\Big(u \sum_{x\in\Lambda} \vartheta_x\Big) \quad \text{and} \quad \frac{\mathrm{dP}_{\Lambda,u}^{\xi,+}}{\mathrm{dP}_\Lambda^\xi} = \frac{1}{Z_{\Lambda,u}^{\xi,+}} \exp\Big(u \sum_{x\in\Lambda} \vartheta_x\Big) \mathbf{1}_{\Omega_\Lambda^+}(\varphi)\,,$$

where we have set $\vartheta_x := \vartheta_x \mathbf{1}_{\{|\varphi_x| \le a\}}$ for simplicity. The partition functions are then given by the following expressions:

$$\begin{aligned} Z_{\Lambda,u}^\xi &= \frac{W_{\Lambda,u}^\xi}{W_{\Lambda,u=0}^\xi} = \mathrm{E}_\Lambda^\xi\Big[\exp\Big(u\sum_{x\in\Lambda}\vartheta_x\Big)\Big] \\ \text{and} \quad Z_{\Lambda,u}^{\xi,+} &= \frac{W_{\Lambda,u}^{\xi,+}}{W_{\Lambda,u=0}^\xi} = \mathrm{E}_\Lambda^\xi\Big[\exp\Big(u\sum_{x\in\Lambda}\vartheta_x\Big)\mathbf{1}_{\Omega_\Lambda^+}(\varphi)\Big]. \end{aligned} \tag{2.1}$$

In particular, for $u = 0$, we have $Z_{\Lambda,u=0}^\xi = 1$ and $Z_{\Lambda,u=0}^{\xi,+} = \mathrm{P}_\Lambda^\xi(\Omega_\Lambda^+)$. For the unconstrained measure, one speaks of the *pinning model*; for the measure with a hard wall constraint, one speaks of the *wetting model*.

The general goal of this chapter is to understand the behavior of the interface under $\mathrm{P}_{\Lambda,u}^\xi$ and $\mathrm{P}_{\Lambda,u}^{\xi,+}$, as a function of the parameter $u \in \mathbb{R}$. In fact, we will see that a phase transition occurs at some critical point u_c, and we will give some information on the properties of this phase transition.

We will focus on the case $\Lambda_N = \{1, \dots, N-1\}^d$ and in most cases on the boundary condition $\xi \equiv 0$. We will moreover denote by $\mathrm{P}_{N,u}^0$ and $\mathrm{P}_{N,u}^+$, resp. $Z_{N,u}^0$ and $Z_{N,u}^+$, the Gibbs measures, resp. the partition functions, in this case. Finally, we will focus: in dimension $d = 1$ on the discrete case; in dimension $d \ge 2$ on the GFF case.

2.1.2 Free energy and first properties

Definition 2.1. We define the free energy (without/with constraint) with zero boundary condition as the following limit:

$$\mathtt{F}^0(u) = \lim_{N\to\infty} \frac{1}{N^d} \log Z^0_{N,u}\,, \qquad \text{and} \qquad \mathtt{F}^+(u) = \lim_{N\to\infty} \frac{1}{N^d} \log Z^+_{N,u}\,. \tag{2.2}$$

The "true" definition of the free energy would be $\mathtt{F}^{0/+}(u) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z^{0/+}_{\Lambda,u}$, but we focus on the definition (2.2), which is easier to handle. Our first result give some properties of the free energy.

Proposition 2.2. *We have the following properties:*

(i) Both limits in (2.2) *exist;*

(ii) For all $u \in \mathbb{R}$*, we have* $0 \le \mathtt{F}^+(u) \le \mathtt{F}^0(u) \le \max(u, 0)$*;*
in particular $\mathtt{F}^0(u) = \mathtt{F}^+(u) = 0$ *for all* $u \le 0$*.*

(iii) The maps $u \mapsto \mathtt{F}^{0/+}(u)$ *are non-decreasing and convex.*

The proof of point (i), in particular in the case of dimension $d \ge 2$, is somewhat technical and we postpone the proof to later on, see Section 2.1.4.

Proof. Let us start by proving point (ii). We have the inequality

$$Z^0_{N,u} \ge Z^+_{N,u} \ge \mathrm{E}^0_N\Big[\exp\Big(u \sum_{x\in\Lambda_N} \vartheta_x\Big) \mathbf{1}_{\{\varphi_x > a\,\forall x\in\Lambda_N\}}\Big] = \mathrm{P}^0_N\big(\varphi_x > a\ \forall x \in \Lambda_N\big)\,,$$

where the last identity follows by the fact that all the ϑ_x are equal to zero on the event $\{\varphi_x > a\ \forall x \in \Lambda_N\}$. Now, in dimension $d \ge 2$ (for the GFF), one just needs to adapt the proof of Proposition 1.35 to obtain that there exists a constant $c > 0$ such that, for $N \ge 2$

$$Z^+_{N,u} \ge \mathrm{P}^0_N\big(\varphi_x > a\ \forall x \in \Lambda_N\big) \ge \mathrm{e}^{-c_0 N^{d-1}\log N}\,. \tag{2.3}$$

In dimension $d = 1$, for a discrete height interface and since we had fixed $a \in (0, 1)$, the lower bound that we obtain is

$$Z^+_{N,u} \ge \mathrm{P}(S_N = 0, S_k > 0\ \forall 1 \le k \le N-1) \ge cN^{-3/2}\,, \tag{2.4}$$

recalling also Theorem 1.22. In the end, taking the logarithm, dividing by N^d and taking the limit, we indeed obtain $\mathtt{F}^0(u) \ge \mathtt{F}^+(u) \ge 0$ for all $u \in \mathbb{R}$.

We also clearly have that $\mathtt{F}^+(u) \le \mathtt{F}^0(u)$ for any u, since $Z^+_{N,u} \le Z^0_{N,u}$. For the upper bound on $\mathtt{F}^+(u)$, we simply observe that $u\vartheta_x \le \max(u,0)$, since $\vartheta_x \ge 0$ and we can bound $\vartheta_x \le 1$ when $u \ge 0$. Thus, we directly obtain that $Z^{0/+}_{N,u} \le \exp(\max(u,0)|\Lambda_N|)$. Taking the logarithm, dividing by N^d and taking the limit $N \to \infty$, we obtain $\mathtt{F}^0(u) \le \max(u,0)$.

Let us now prove point (iii). Let us set, for $N \ge 1$,

$$\mathtt{F}^0_N(u) := \log Z^0_{N,u} \quad \text{and} \quad \mathtt{F}^+_N(u) := \log Z^+_{N,u}\,.$$

It suffices to show that, for all $N \ge 1$, the maps $u \mapsto \mathtt{F}^{0/+}_N(u)$ are non-decreasing and convex, since the (pointwise) limit of a sequence of non-decreasing and convex functions is itself non-decreasing and convex.

The monotonicity of $\mathtt{F}^{0/+}_N$ in u is clear since $u \mapsto Z^{0/+}_{N,u}$ is non-decreasing, as the expectation of a non-decreasing function in u (for any realization of φ, since $\vartheta_x \ge 0$). Another way to see the monotonicity of $\mathtt{F}^{0/+}_N$, which will be useful later on, is simply to differentiate $\mathtt{F}^{0/+}_N$. Let us do the calculation for $\mathtt{F}^+_N$, the analogue computations for $\mathtt{F}^0_N$ being almost identical: in view of (2.1), we have

$$\frac{\partial}{\partial u}\mathtt{F}^+_N(u) = \frac{\partial}{\partial u}\log Z^+_{N,u} = \frac{1}{Z^+_{N,u}}\mathrm{E}^0_N\Big[\Big(\sum_{x\in\Lambda_N}\vartheta_x\Big)\exp\Big(u\sum_{x\in\Lambda_N}\vartheta_x\Big)\mathbf{1}_{\Omega^+_{\Lambda_N}}(\varphi)\Big]$$

so that in particular

$$\frac{\partial}{\partial u}\mathtt{F}^+_N(u) = \mathrm{E}^+_{N,u}\Big[\sum_{x\in\Lambda_N}\vartheta_x\Big]\,. \tag{2.5}$$

Thus, it is clear that $\frac{\partial}{\partial u}\mathtt{F}^+_N(u) \ge 0$, and therefore $\mathtt{F}^+_N$ is non-decreasing.

To show the convexity of $\mathtt{F}^{0/+}_N$, we compute its second derivative. Repeating the computation (2.5) (and recalling the expression (2.1)), we have

$$\begin{aligned}\frac{\partial^2}{\partial u^2}\mathtt{F}^+_N(u) = \frac{1}{Z^+_{N,u}}&\mathrm{E}^0_N\Big[\Big(\sum_{x\in\Lambda_N}\vartheta_x\Big)^2\exp\Big(u\sum_{x\in\Lambda_N}\vartheta_x\Big)\mathbf{1}_{\Omega^+_{\Lambda_N}}(\varphi)\Big]\\ &-\frac{1}{(Z^+_{N,u})^2}\mathrm{E}^0_N\Big[\Big(\sum_{x\in\Lambda_N}\vartheta_x\Big)^2\exp\Big(u\sum_{x\in\Lambda_N}\vartheta_x\Big)\mathbf{1}_{\Omega^+_{\Lambda_N}}(\varphi)\Big]^2.\end{aligned}$$

We can rewrite the above as

$$\frac{\partial^2}{\partial u^2}\mathtt{F}_N^+(u) = \mathrm{E}_{N,u}^+\Big[\Big(\sum_{x\in\Lambda_N}\vartheta_x\Big)^2\Big] - \mathrm{E}_{N,u}^+\Big[\sum_{x\in\Lambda_N}\vartheta_x\Big]^2 = \mathrm{Var}_{N,u}^+\Big(\sum_{x\in\Lambda_N}\vartheta_x\Big) \geq 0\,,$$

which shows the convexity of $u \mapsto \mathtt{F}_N^+(u)$. □

2.1.3 The (localization) phase transition

The fact that the free energies $\mathtt{F}^0$ and $\mathtt{F}^+$ are non-decreasing and convex allows us to define the following critical points:

$$\begin{aligned} u_c^0 &:= \sup\big\{u\in\mathbb{R},\ \mathtt{F}^0(u)=0\big\} = \inf\big\{u\in\mathbb{R},\ \mathtt{F}^0(u)>0\big\}\,,\\ u_c^+ &:= \sup\big\{u\in\mathbb{R},\ \mathtt{F}^+(u)=0\big\} = \inf\big\{u\in\mathbb{R},\ \mathtt{F}^+(u)>0\big\}\,. \end{aligned} \tag{2.6}$$

These critical points $u_c^{0/+}$ mark a phase transition, which is called the *pinning transition* or *localization transition*.

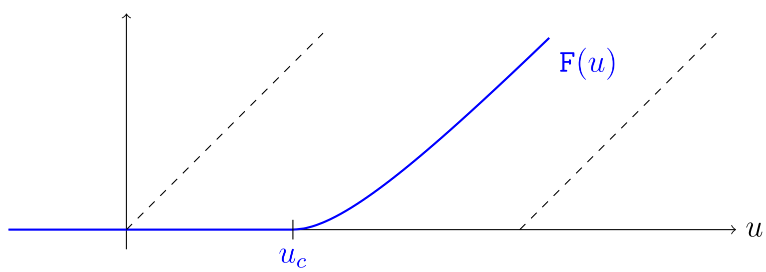


Figure 2.1: We have represented the free energy $\mathtt{F}(u)$ as a function of u (in the case $\mathtt{F}^0$ or $\mathtt{F}^+$): it is a non-negative, non-decreasing convex function. We have also represented the bounds $\mathtt{F}(u) \leq \max(u,0)$ and $\mathtt{F}(u) \geq \min(u-c_a, 0)$, cf. Exercise 15.

Let us recall the computation done in (2.5): we have showed that

$$\frac{\partial}{\partial u}\Big(\frac{1}{N^d}\log Z_{N,u}^{0/+}\Big) = \mathrm{E}_{N,u}^{0/+}\Big[\frac{1}{N^d}\sum_{x\in\Lambda_N}\vartheta_x\Big]\,,$$

which is the average density of contacts under $\mathrm{P}_{N,u}^{0/+}$. We can now use the following.

Claim 2.3. *If $(f_N)_{N\geq 1}$ is a sequence of convex functions that converges pointwise to f, then f is convex. Additionally, at every point x where f is differentiable (i.e. for all but at most a countable number of x), we have $f'(x) = \lim_{N\to\infty} f(x)$.*

With this claim we obtain that for any $u \in \mathbb{R}$ such that the derivative $\frac{\partial \mathtt{F}^{0/+}}{\partial u}$ exists[1],

$$\frac{\partial \mathtt{F}^{0/+}}{\partial u} = \lim_{N\to\infty} \mathrm{E}^{0/+}_{N,u}\left[\frac{1}{N^d}\sum_{x\in\Lambda_N}\vartheta_x\right]. \tag{2.7}$$

We deduce the following description of the phase transition:

- If $u < u_c^{0/+}$, then $\mathtt{F}^{0/+}(u) = 0$ and $\frac{\partial \mathtt{F}^{0/+}}{\partial u} = 0$. Thus, the asymptotic density of contacts under $\mathrm{P}^{0/+}_{N,u}$ is zero and this is called of a *delocalized* phase.
- If $u > u_c^{0/+}$, then $\mathtt{F}^{0/+}(u) > 0$ and by convexity $\frac{\partial}{\partial u}\mathtt{F}^{0/+} > 0$ (if the derivative exists, which is the case in practice). Thus, the asymptotic density of contacts under $\mathrm{P}^{0/+}_{N,u}$ is strictly positive and this is called a *localized* phase.

The main questions we want to answer in the rest of this chapter are the following: (i) can one compute u_c^0 or u_c^+ explicitly (or have a precise characterization)? (ii) can one give the behavior of the free energy when approaching the critical point? This would for instance allow us to describe how the contact density increases when crossing the critical point $u_c^{0/+}$.

Exercise 15. The goal of this exercise is to give lower bounds on the free energy and some consequences on the critical points.

1. Show that $u_c^+ \geq u_c^0 \geq 0$.
2. (a) Show that $Z^{0/+}_{N,u} \geq \mathrm{e}^{u|\Lambda_N|}\mathrm{P}^0_N(|\varphi_x| \leq a\ \forall x \in \Lambda_N)$.
   (b) Deduce that $\mathtt{F}^{0/+}(u) \geq \min(u - c_a, 0)$, where $\mathrm{e}^{c_a} = \frac{\int_{[-a,a]}\mathrm{e}^{-V(h)}\mathrm{d}\mu}{\int_{\mathbb{R}}\mathrm{e}^{-V(h)}\mathrm{d}\mu}$.
   Hint. Review the proof of Lemma 1.3.
   (c) Conclude that $u_c^0 \leq u_c^+ \leq c_a$.

2.1.4 Existence of the free energy

In this section, we prove the existence of the free energy, *i.e.* item *(i)* of Proposition 2.2. We start with the case of dimension $d = 1$ and we turn to the case of dimension $d \geq 2$ afterwards (it is much more technical).

a) The case of dimension $d = 1$

Recall that in dimension $d = 1$, we consider discrete height interfaces. Again, let focus on the case with a hard-wall constraint; the unconstrained case being com-

[1] The only possible point of non-differentiability will turn out to be the critical point.

pletely analogous. Thanks to the interpretation of the interface as a random walk from Section 1.1.3, we can write (for $a \in (0,1)$ so that $\vartheta_x = \mathbf{1}_{\{|\varphi_x| \le a\}} = \mathbf{1}_{\{\varphi_x = 0\}}$),

$$Z^+_{N,u} = \mathrm{E}\Big[\exp\Big(u \sum_{k=1}^{N-1} \mathbf{1}_{\{S_k=0\}} \Big) \mathbf{1}_{\Omega_N^+} \,\Big|\, S_N = 0 \Big] ,$$

where $(S_k)_{k\ge 1}$ is a random walk starting from 0, *i.e.* $S_k = \sum_{i=1}^k X_i$ for i.i.d. random variables $(X_i)_{i\ge 1}$.

Let us then introduce the modified partition function as

$$\tilde Z^+_{N,u} := \mathrm{E}\Big[\exp\Big(u \sum_{i=1}^{N} \mathbf{1}_{\{S_i=0\}} \Big) \mathbf{1}_{\Omega_N^+} \mathbf{1}_{\{S_N=0\}} \Big] = \mathrm{P}(S_N = 0) \mathrm{e}^u Z^+_{N,u} .$$

Thanks to Theorem 1.22 (the local CLT), we have that $\lim_{N\to\infty} \frac{1}{N} \log \mathrm{P}(S_N = 0) = 0$, so that the existence of the free energy follows from the existence of the limit $\lim_{N\to\infty} \frac{1}{N} \log \tilde Z^+_{N,u}$. We therefore have

$$\mathtt{F}^+(u) = \lim_{N\to\infty} \frac{1}{N} \log \tilde Z^+_{N,u} .$$

The advantage of working with the modified partition function $\tilde Z^+_{N,u}$ is that we have the following "super-multiplicativity" property. For any $N, M \ge 1$, inserting the indicator function $\mathbf{1}_{\{S_N=0\}}$ inside the expectation, we have

$$\tilde Z^+_{N+M,u} \ge \mathrm{E}\Big[\exp\Big(u \sum_{i=1}^{N+M} \mathbf{1}_{\{S_i=0\}} \Big) \mathbf{1}_{\Omega_{N+M}^+} \mathbf{1}_{\{S_N=0\}} \mathbf{1}_{\{S_{N+M}=0\}} \Big] ,$$

which, thanks to the Markov property, is equal to

$$\mathrm{E}\Big[\exp\Big(u \sum_{i=1}^{N} \mathbf{1}_{\{S_i=0\}} \Big) \mathbf{1}_{\Omega_N^+} \mathbf{1}_{\{S_N=0\}} \Big] \mathrm{E}\Big[\exp\Big(u \sum_{i=1}^{M} \mathbf{1}_{\{S_i=0\}} \Big) \mathbf{1}_{\Omega_M^+} \mathbf{1}_{\{S_M=0\}} \Big] .$$

This shows that $\tilde Z^+_{N+M,u} \ge \tilde Z^+_{N,u} \tilde Z^+_{M,u}$, so that that the sequence $(\log \tilde Z^{0/+}_{N,u})_{N\ge 1}$ is super-additive, which allows one to use Fekete's lemma (which is classical).

Lemma 2.4 (Fekete's lemma). *Let $(u_n)_{n\ge 1}$ be a super-additive sequence of real numbers, that is such that one has $u_{n+m} \ge u_n + u_m$ for all $n, m \ge 1$. Then*

$$\lim_{n\to\infty} \frac{u_n}{n} = \sup_{n\ge 1} \frac{u_n}{n} .$$

In particular, the limit always exists and is finite if the supremum is finite.

This shows that the free energy exists. □

Remark 2.5 (Finite-volume criterion)**.** In fact, let us state the following corollary of Fekete's lemma.

Corollary 2.6. *Let us set* $\tilde{Z}^{0/+}_{N,u} := e^{u} P(S_N = 0) Z^{0/+}_{N,u}$. *Then, we have*

$$\mathtt{F}^{0/+}(u) = \sup_{N \geq 1} \frac{1}{N} \log \tilde{Z}^{0/+}_{N,u} = \sup_{N\geq 1} \left\{ \frac{1}{N} \log Z^{0/+}_{N,u} + \frac{u}{N} + \frac{1}{N} \log P(S_N = 0) \right\}.$$

This corollary is quite useful to obtain what is called a *finite-volume criterion* for localization. More precisely, we have that $\mathtt{F}^{0/+}(u) > 0$ if and only if there is some $N > 0$ such that $\log Z^{0/+}_{N,u} > -u - \log P(S_N = 0)$; moreover, if one finds such N, Corollary 2.6 provides a lower bound on the free energy.

b) The case of the GFF in dimension $d \geq 2$

Recall that in dimension $d \geq 2$, we focus on the case of the GFF. Note that we may restrict to the case $u \geq 0$ since we already know that $\mathtt{F}(u) = 0$ otherwise.

The idea is to find a form of super-additivity property for $(\log Z^{0/+}_{N,u})_{N\geq 1}$, at least approximately; this is provided by (2.11) below. One difficulty is that one cannot constrain the GFF to take the value 0 on a domain since this is an event of probability zero (this problem also arises in dimension $d = 1$ for continuous surfaces).

We proceed in several steps, which we summarize in three lemmas. (Here we only work with the unconstrained case, the hard wall constraint being completely analogous.)

Lemma 2.7 (Change of boundary condition)**.** *For* $\xi : \mathbb{Z}^d \to \mathbb{R}$ *a boundary condition, we denote* $M^{\xi}_N := \sup_{x\in\partial\Lambda_N} |\xi_x|$. *Then, there exists a constant* $c_u = 4(|u| + c) > 0$ *such that, if* $M^{\xi}_N \leq \sqrt{N}$, *we have*

$$-c_u N^{d-\frac{1}{2}} M^{\xi}_N - 1 \leq \log \left(\frac{Z^{\xi}_{N,u}}{Z^{0}_{N,u}} \right) \leq N^{d-\frac{1}{2}} M^{\xi}_N + 1 .$$

Lemma 2.8 (Quasi-monotonicity along dyadic scales)**.** *There exist* $c_0 > 0$ *and* $N_0 = N_0(c_u)$ *such that for all* $N \geq N_0$ *the sequence* $\left(\frac{1}{(2^k N)^d} \log Z^0_{2^k N,u} - c_0 (2^k N)^{-1/4}\right)_{k\geq 1}$

is non-decreasing. In particular, the following limit exists:

$$f(u) = \lim_{k\to\infty} \frac{1}{2^{kd}} \log Z^0_{2^k,u} = \sup_{k\geq \log_2 N_0} \left\{ \frac{1}{2^{kd}} \log Z^0_{2^k,u} - c_0 2^{-k/4} \right\}.$$

Lemma 2.9 (Filling the gaps between dyadic scales)**.** *We have the following inequality:*

$$f(u) \leq \liminf_{N\to\infty} \frac{1}{N^d} \log Z^0_{N,u} \leq \limsup_{N\to\infty} \frac{1}{N^d} \log Z^0_{N,u} \leq f(u)\,.$$

A corollary of all these lemmas, in analogy with Corollary 2.6, we obtain the following.

Corollary 2.10. *In dimension $d \geq 2$, we have*

$$\mathtt{F}^{0/+}(u) = \lim_{N\to\infty} \frac{1}{N^d} \log Z^{0/+}_{N,u} = \sup_{N\geq N_0} \left\{ \frac{1}{N^d} \log Z^{0/+}_{N,u} - \frac{c_0}{N^{1/4}} \right\}.$$

Proof of Corollary 2.10. The existence of the limit is given by Lemmas 2.8 and 2.9. Lemma 2.8 moreover ensures that for all $N \geq N_0$,

$$\begin{aligned} f(u) = \lim_{k\to\infty} \frac{1}{2^{kd}N^d} \log Z^0_{2^k N,u} &= \lim_{k\to\infty} \left(\frac{1}{2^{kd}N^d} \log Z^0_{2^k N,u} - c_0 (2^k N)^{-1/4} \right) \\ &\geq \frac{1}{N^d} \log Z^0_{N,u} - c_0 N^{-1/4}\,, \end{aligned}$$

by monotonicity in k. This gives the bound $f(u) \geq \sup_{N\geq N_0}\{\frac{1}{N^d} \log Z^0_{N,u} - c_0 N^{-1/4}\}$, the other bound being obvious. □

The rest of the section is devoted to the proofs of Lemmas 2.7, 2.8 and 2.9; this is rather technical and can be skipped on a first reading.

Proof of Lemma 2.7. The main observation is that we know how to control the difference between the Hamiltonians with boundary condition ξ and with zero boundary condition: writing $\Lambda = \Lambda_N$, we have

$$H^0_\Lambda(\varphi) - H^\xi_\Lambda(\varphi) = \frac{1}{8d} \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ x\sim y}} \left[\varphi_x^2 - (\varphi_x^2 - \xi_y)^2 \right] = \frac{1}{4d} \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ x\sim y}} \varphi_x \xi_y - \frac{1}{8d} \sum_{\substack{x\in\Lambda, y\notin\Lambda \\ x\sim y}} \xi_y^2\,.$$

Using the definition (2.1) of $Z^\xi_{\Lambda,u}$, we can therefore write

$$Z^\xi_{\Lambda,u} = \frac{1}{W^\xi_\Lambda}\int_{\mathbb{R}^\Lambda} \mathrm{e}^{-H^\xi_\Lambda(\varphi)+H^0_\Lambda(\varphi)-H^0_\Lambda(\varphi)}\mathrm{e}^{u\sum_{x\in\Lambda}\vartheta_x}\mathrm{d}\mu = \frac{W^0_\Lambda}{W^\xi_\Lambda}\,\mathrm{E}^0_\Lambda\Big[\mathrm{e}^{-H^\xi_\Lambda(\varphi)+H^0_\Lambda(\varphi)+u\sum_{x\in\Lambda}\vartheta_x}\Big]\,,$$

with

$$W^\xi_\Lambda = \int_{\mathbb{R}^\Lambda} \mathrm{e}^{-H^\xi_\Lambda(\varphi)+H^0_\Lambda(\varphi)-H^0_\Lambda(\varphi)}\mathrm{d}\mu = W^0_\Lambda\,\mathrm{E}^0_\Lambda\Big[\mathrm{e}^{-H^\xi_\Lambda(\varphi)+H^0_\Lambda(\varphi)}\Big]\,.$$

Going back to the expression of $-H^\xi_\Lambda(\varphi)+H^0_\Lambda(\varphi)$ above and noting that the last term does not depend on φ, we deduce that

$$Z^\xi_{\Lambda,u} = \frac{\mathrm{E}^0_\Lambda\big[\mathrm{e}^{\Phi^\xi_\Lambda+u\sum_{x\in\Lambda}\vartheta_x}\big]}{\mathrm{E}^0_\Lambda\big[\mathrm{e}^{\Phi^\xi_\Lambda}\big]} \qquad \text{where} \qquad \Phi^\xi_\Lambda = \frac{1}{4d}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}}\varphi_x\xi_y\,. \tag{2.8}$$

Controlling the denominator of (2.8). Let us show that

$$1 \le \mathrm{E}^0_\Lambda\big[\mathrm{e}^{\Phi^\xi_\Lambda}\big] \le \exp\big(\tfrac{1}{2}N^{d-\frac{1}{2}}M^\xi_N\big)\,. \tag{2.9}$$

To get this, notice that Φ^ξ_Λ is a linear combination of $(\varphi_x)_{x\in\Lambda}$. Thus, under P^0_Λ, Φ^ξ_Λ is a *centered* Gaussian random variable, and

$$\mathrm{E}^0_\Lambda\big[\mathrm{e}^{\Phi^\xi_\Lambda}\big] = \mathrm{e}^{\frac{1}{2}\mathrm{Var}^0_\Lambda(\Phi^\xi_\Lambda)}\,.$$

The lower bound in (2.9) being obvious, it remains to show that $\mathrm{Var}^0_\Lambda(\Phi^\xi_\Lambda) \le (M^\xi_N)^2N^{d-1} \le M^\xi_N N^{d-\frac{1}{2}}$, the last inequality following from our assumption $M^\xi_N \le \sqrt{N}$. Now, we can estimate the variance:

$$\begin{aligned}\mathrm{Var}^0_\Lambda(\Phi^\xi_\Lambda) &= \frac{1}{(4d)^2}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}}\sum_{\substack{x'\in\Lambda, y'\notin\Lambda\\ x'\sim y'}} G_\Lambda(x,x')\xi_y\xi_{y'}\\ &\le (M^\xi_N)^2\frac{1}{(4d)^2}\sum_{\substack{x\in\Lambda, y\notin\Lambda\\ x\sim y}}\sum_{\substack{x'\in\Lambda, y'\notin\Lambda\\ x'\sim y'}} G_\Lambda(x,x') = (M^\xi_N)^2\,\mathrm{Var}^0_\Lambda(\Phi^{\xi\equiv1}_\Lambda)\,.\end{aligned}$$

One can in fact compute explicitly $\mathrm{Var}^0_\Lambda(\Phi^{\xi\equiv1}_\Lambda)$. Indeed, noticing that $W^{\xi\equiv1}_\Lambda = W^{\xi\equiv h}_\Lambda$ (simply by a change of variables $\varphi\mapsto\varphi+h$), going back to the identity above (2.8), one obtains that for any $h\in\mathbb{R}$

$$1 = \frac{W^{\xi\equiv h}_\Lambda}{W^0_\Lambda} = \mathrm{E}^0_\Lambda\Big[\mathrm{e}^{-H^{\xi\equiv h}_\Lambda(\varphi)+H^0_\Lambda(\varphi)}\Big] = \mathrm{E}^0_\Lambda\Big[\mathrm{e}^{h\Phi^{\xi\equiv1}_\Lambda-\frac{1}{8d}\sum_{x\in\Lambda,y\notin\Lambda,x\sim y}h^2}\Big]\,.$$

Since we know that $\Phi_\Lambda^{\xi\equiv 1}$ is a centered Gaussian we have $\mathrm{E}_\Lambda^0[\mathrm{e}^{h\Phi_\Lambda^{\xi\equiv 1}}] = \mathrm{e}^{\frac12 \mathrm{Var}_\Lambda^0(\Phi_\Lambda^{\xi\equiv 1})}$, so that

$$\mathrm{Var}_\Lambda^0(\Phi_\Lambda^{\xi\equiv 1}) = \frac{1}{4d} \sum_{x\in\Lambda, y\notin\Lambda, x\sim y} 1 \le N^{d-1}\,.$$

This concludes the proof of (2.9).

Upper bound in Lemma 2.7. For the numerator in (2.8), we split the expectation into two parts, depending on whether Φ_Λ^ξ is smaller or larger than $b_u\, N^{d-\frac12}\, M_N^\xi$ with $b_u^2 = 4(u+c)$ for some appropriate constant c. The first term is

$$\mathrm{E}_\Lambda^0\Big[\mathrm{e}^{\Phi_\Lambda^\xi + u\sum_{x\in\Lambda}\vartheta_x}\mathbf{1}_{\{\Phi_\Lambda^\xi \le b_u N^{d-\frac12} M_N^\xi\}}\Big] \le \mathrm{e}^{b_u N^{d-\frac12} M_N^\xi}\ Z_{\Lambda,u}^0\,.$$

The second term is (recall that we only treat the case $u\ge 0$)

$$\begin{aligned}\mathrm{E}_\Lambda^0\Big[\mathrm{e}^{\Phi_\Lambda^\xi + u\sum_{x\in\Lambda}\vartheta_x}\mathbf{1}_{\{\Phi_\Lambda^\xi > b_u N^{d-\frac12} M_N^\xi\}}\Big] &\le \mathrm{e}^{u|\Lambda|}\, \mathrm{E}_\Lambda^0\Big[\mathrm{e}^{2\Phi_\Lambda^\xi}\Big]^{1/2} \mathrm{P}_\Lambda^0\Big(\Phi_\Lambda^\xi > b_u N^{d-\frac12} M_N^\xi\Big)^{1/2}\\ &\le \mathrm{e}^{uN^d}\mathrm{e}^{N^{d-1}(M_N^\xi)^2}\mathrm{e}^{-\frac14 b_u^2 N^d}\,,\end{aligned}$$

where we have bounded $\vartheta_x\le 1$ and used Cauchy–Schwarz inequality, then used that Φ_Λ^ξ is a centered Gaussian random variable with variance $\mathrm{Var}_\Lambda^0(\Phi_\Lambda^\xi) \le N^{d-1}(M_N^\xi)^2$ as seen above. Now, since we took $b_u^2 = 4(u+c)$, using that $M_N^\xi \le \sqrt{N}$ we deduce that

$$\mathrm{E}_\Lambda^0\Big[\mathrm{e}^{\Phi_\Lambda^\xi + u\sum_{x\in\Lambda}\vartheta_x}\mathbf{1}_{\{\Phi_\Lambda^\xi > b_u N^{d-\frac12} M_\Lambda^\xi\}}\Big] \le \mathrm{e}^{-cN^d} \le \tfrac12 Z_{N,u}^0\,, \tag{2.10}$$

where we also applied the lower bound (2.3) on $Z_{N,u}^0$ (having chosen the constant c appropriately). Going back to (2.8), we deduce that

$$Z_{\Lambda,u}^\xi \le \big(\tfrac12 + \exp\big(b_u N^{d-\frac12} M_N^\xi\big)\big)\ Z_{N,u}^0\,,$$

which gives the desired upper bound.

Lower bound in Lemma 2.7. Starting from (2.8) and with the upper bound (2.9) on the denominator, we have

$$\begin{aligned} Z_{\Lambda,u}^\xi &\ge \mathrm{e}^{-\frac12 N^{d-\frac12} M_N^\xi}\, \mathrm{E}_\Lambda^0\Big[\mathrm{e}^{\Phi_\Lambda^\xi + u\sum_{x\in\Lambda}\vartheta_x}\Big]\\ &\ge \mathrm{e}^{-(\frac12 + b_u)N^{d-\frac12} M_N^\xi}\, \mathrm{E}_\Lambda^0\Big[\mathrm{e}^{u\sum_{x\in\Lambda}\vartheta_x}\mathbf{1}_{\{\Phi_\Lambda^\xi > -b_u N^{d-\frac12} M_\Lambda^\xi\}}\Big]\,.\end{aligned}$$

Now, we can write

$$\mathrm{E}_\Lambda^0\Big[\mathrm{e}^{u\sum_{x\in\Lambda}\vartheta_x}\mathbf{1}_{\{\Phi_\Lambda^\xi > -b_u N^{d-\frac12} M_\Lambda^\xi\}}\Big] = Z_{N,u}^0 - \mathrm{E}_\Lambda^0\Big[\mathrm{e}^{u\sum_{x\in\Lambda}\vartheta_x}\mathbf{1}_{\{\Phi_\Lambda^\xi < -b_u N^{d-\frac12} M_\Lambda^\xi\}}\Big] \le \tfrac12 Z_{N,u}^0\,,$$

where the last inequality is due to (2.10), using also some symmetry. We conclude that

$$Z^{\xi}_{\Lambda,u} \geq \frac{1}{2} \mathrm{e}^{-(\frac{1}{2}+b_u)N^{d-\frac{1}{2}} M_N^{\xi}} \, Z^0_{N,u} \, ,$$

which gives the desired lower bound. □

Proof of Lemma 2.8. Let us start by showing that there exists N_0 such that

$$\frac{1}{(2N)^d} \log Z^0_{2N,u} \geq \frac{1}{N^d} \log Z^0_{N,u} - N^{-\frac{1}{4}} \qquad \text{for } N \geq N_0 \, . \tag{2.11}$$

The idea is to decompose the cube $\{1, \dots, 2N-1\}^d$ into 2^d blocks $(\Lambda_N^{(v)})_{v\in\{0,1\}^d}$ of size N, where $\Lambda_N^{(v)} = \Lambda_N + Nv$. Note that the blocks $\Lambda_N^{(v)}$ are well separated by a boundary: we set $\Gamma := \Lambda_{2N} \setminus (\bigcup_{v\in\{0,1\}^d} \Lambda_N^{(v)})$. By imposing that the GFF is not too large on Γ and conditioning on $\mathcal{F}_\Gamma$, we obtain the following inequality, due to an application of the spatial Markov property:

$$\begin{aligned} Z^0_{2N,u} &\geq \mathrm{E}^0_{2N}\Big[\prod_{v\in\{0,1\}^d} Z^{\phi}_{\Lambda_v,u} \mathbf{1}_{\{|\phi_x| \leq (\log N)^3 \ \forall x\in\Gamma\}} \Big] \\ &\geq \Big(Z^0_{N,u} \mathrm{e}^{-c_u N^{d-\frac{1}{2}} (\log N)^3} \Big)^{2^d} \mathrm{P}\big(|\phi_x| \leq (\log N)^3 \ \forall x \in \Gamma\big) \, , \end{aligned}$$

where we have used Lemma 2.7 for the inequality to change the boundary condition on the sub-boxes to $\xi \equiv 0$. Now, we have

$$\mathrm{P}\big(\exists\, x \in \Gamma \, , |\phi_x| > (\log N)^3\big) \leq |\Gamma| \mathrm{e}^{-\frac{1}{2}(\log N)^3/\sigma_N^2} \, ,$$

where $\sigma_N^2 = \max_{x\in\Lambda_{2N}} \mathrm{Var}^0_{2N}(\phi_x)$ satisfies $\sigma_N^2 \leq C$ in dimension $d \geq 3$ and $\sigma_N^2 \leq C \log N$ in dimension $d = 2$. We therefore obtain

$$\mathrm{P}\big(|\phi_x| \leq (\log N)^3 \ \forall x \in \Gamma\big) \geq 1 - (2N)^d \mathrm{e}^{-c(\log N)^2} \geq \tfrac{1}{2}$$

for N large enough. We have thus shown that

$$\frac{1}{2^d} \frac{1}{N^d} \log Z^0_{2N,u} \geq \frac{1}{N^d} \log Z^0_{N,u} - c_u N^{-\frac{1}{2}} (\log N)^3 - \log 2 \geq \frac{1}{N^d} \log Z^0_{N,u} - N^{-\frac{1}{4}}$$

for N large (how large one ned to take N depends on c_u). We have thus proved inequality (2.11).

Let us now set, for $N \geq N_0$,

$$U_N := \frac{1}{N^d} \log Z^0_{N,u} - c_0 N^{1/4}\,, \quad \text{with} \;\; c_0 := (1-2^{-1/4})^{-1}\,.$$

(Note that c_0 verifies $1 + c_0 2^{-1/4} = c_0$.) Using (2.11), we have for any $N \geq N_0$,

$$U_{2N} \geq \frac{1}{N^d} \log Z^0_{N,u} - N^{-\frac14} - c_0 (2N)^{-1/4} = U_N\,.$$

Thus, the sequence $(U_{2^k N})_{k\geq 0}$ is non-decreasing. This shows that $\lim_{k\to\infty} U_{2^k N}$ exists and, because $\lim_{k\to\infty} (2^k N)^{-1/4} = 0$, it is equal to $f(u) = \lim_{k\to\infty} \frac{1}{2^{kd}} \log Z^0_{2^k,u}$. □

Proof of Lemma 2.9. Let us start with the *lower bound* in Lemma 2.9. We apply the same idea as before, writing

$$N = q_N 2^{k_N} + r_N \qquad \text{with } k_N := \tfrac12 \lceil \log_2 N \rceil\,, \quad r_N \leq 2^{k_N}\,.$$

Note that $r_N \leq \sqrt{N}$ and $q_N \sim \sqrt{N}$ when $N \to \infty$.

We decompose Λ_N into q_N boxes of size 2^{k_N}, denoted $\Lambda_N^{(v)} = \Lambda_{2^{k_N}} + 2^{k_N} v$ for $v \in \{0,\ldots,q_N-1\}^d$. Conditioning on $\mathcal{F}_\Gamma$ where $\Gamma = \Lambda_N \setminus (\bigcup_v \Lambda_N^{(v)})$ and using the spatial Markov property (and the fact that $u\vartheta_x \geq 0$ on Γ), we obtain

$$\begin{aligned} Z^0_{N,u} &\geq \mathrm{E}^0_N \bigg[\prod_{v\in\{0,\ldots,q_N-1\}^d} Z^{\phi}_{\Lambda_v,u} \mathbf{1}_{\{|\phi_x| \leq (\log N)^3 \; \forall x\in\Gamma\}} \bigg] \\ &\geq \Big(Z^0_{2^{k_N}} \mathrm{e}^{-c 2^{k_N(d-\frac12)} (\log N)^3} \Big)^{(q_N)^d} \mathrm{P}\big(|\phi_x| \leq (\log N)^3 \; \forall x \in \Gamma \big) \end{aligned}$$

using again Lemma 2.7 to change the boundary condition to $\xi \equiv 0$ in all sub-boxes. Now, exactly as in the proof of Lemma 2.8, the last probability is larger than $\frac12$. This shows that

$$\begin{aligned} \frac{1}{N^d} \log Z^0_{N,u} &\geq \frac{(q_N)^d}{N^d} \log Z^0_{2^{k_N},u} - \frac{c}{N^d} 2^{k_N(d-\frac12)} (\log N)^3 - \log 2 \\ &\geq \frac{(q_N)^d}{(q_N+1)^d} \frac{1}{2^{k_N d}} \log Z^0_{2^{k_N},u} - c' 2^{-\frac12 k_N} (\log N)^3\,, \end{aligned}$$

where for the second inequality we have used the fact that $2^{k_N} \leq N \leq (1+q_N) 2^{k_N}$. Taking the liminf as $N \to \infty$ and using Lemma 2.8 together with the fact that $k_N, q_N \to \infty$, we deduce the lower bound in Lemma 2.9.

We now turn the *upper bound* in Lemma 2.9. We introduce $\hat{k}_N = 2\lceil \log_2 N \rceil$ and we write

$$2^{\hat{k}_N} = \hat{q}_N N + \hat{r}_N \,, \qquad \hat{r}_N < N \,.$$

This time, we decompose $\Lambda_{2^{\hat{k}_N}}$ into $\hat{q}_N$ boxes of size N, denoted $\Lambda_N^{(v)} = \Lambda_N + Nv$. With the same inequalities as above (replacing N by $2^{\hat{k}_N}$ and 2^{k_N} by N), we have

$$\begin{aligned}\frac{1}{2^{\hat{k}_N d}} \log Z^0_{2^{\hat{k}_N},u} &\geq \frac{(\hat{q}_N)^d}{2^{\hat{k}_N d}} \log Z^0_{N,u} - \frac{c}{2^{\hat{k}_N d}} N^{d-\frac{1}{2}} (\log 2^{\hat{k}_N})^3 - \log 2 \\ &\geq \frac{(\hat{q}_N)^d}{(\hat{q}_N+1)^d} \frac{1}{N^d} \log Z^0_{N,u} - c' N^{-\frac{1}{2}} (\log N)^3 \,.\end{aligned}$$

Taking the limsup as $N \to \infty$ and using Lemma 2.8 together with the fact that $\hat{k}_N, \hat{q}_N \to \infty$, we deduce the upper bound in Lemma 2.9. □

2.2 Free energy and phase transition in dimension $d = 1$

In dimension $d = 1$ (recall we treat discrete height interfaces), it turns out that we are able to give an implicit formula for the free energy, both in the model with and without constraint: this is Proposition 2.12 below. We then deduce the following result, which gives the critical behavior of the free energy.

Theorem 2.11 (Critical behavior of the pinning/wetting model in $d = 1$)**.** *In dimension $d = 1$, one has $u_c^0 = 0$ and $u_c^+ > 0$. Moreover, there exist (explicit) constants $c_0, c_+ > 0$ such that*

$$\begin{aligned} \mathtt{F}^0(u) &\sim c_0 u^2 && \text{as } u \downarrow 0 \,, \\ \mathtt{F}^+(u) &\sim c_+ (u - u_c^+)^2 && \text{as } u \downarrow u_c^+ \,. \end{aligned}$$

The rest of this section is devoted to the proof of Theorem 2.11.

2.2.1 Implicit formula for the free energy

Recall the notations $p_n = \mathrm{P}(S_N = 0)$ and $p_n^+ := \mathrm{P}(S_n = 0, S_k \geq 0 \ \forall k \leq n)$, and introduce the following Laplace transforms: for $\lambda > 0$

$$U^0(\lambda) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} p_n \quad \text{and} \quad U^+(\lambda) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} p_n^+ \,. \tag{2.12}$$

Note that the functions $\lambda \mapsto U^{0/+}(\lambda)$ are decreasing and strictly convex. The main result of this subsection is the following.

Proposition 2.12. *The free energy $\mathtt{F}^0(u)$ is the solution in λ of the equation*

$$\frac{U^0(\lambda)-1}{U^0(\lambda)} = \mathrm{e}^{-u} \quad \iff \quad U^0(\lambda) = \frac{1}{1-\mathrm{e}^{-u}}$$

if a solution exists, and $\mathtt{F}^0(u) = 0$ otherwise. The same result holds for $\mathtt{F}^+(u)$ by replacing $U^0(\lambda)$ with $U^+(\lambda)$.

We prove this proposition first in the unconstrained case and then show how to adapt it to the constrained case.

a) The unconstrained case (pinning)

Step 1. Rewriting of the problem. Recall the notation $\tilde{Z}^0_{N,u} = \mathrm{P}(S_N = 0)\mathrm{e}^u Z^0_{N,u}$, that is,

$$\tilde{Z}^0_{N,u} = \mathrm{E}\Big[\exp\Big(u\sum_{i=1}^N \mathbf{1}_{\{S_i=0\}}\Big)\mathbf{1}_{\{S_N=0\}}\Big],$$

and that $\mathtt{F}^0(u) = \lim_{N\to\infty}\frac{1}{N}\log Z^0_{N,u}$.

Let us now introduce what is called the *grand canonical* partition function: for $u \in \mathbb{R}$ and $\lambda > 0$, we set

$$\tilde{\mathcal{Z}}^0_{u,\lambda} := \sum_{N=1}^{\infty} \mathrm{e}^{-\lambda N}\tilde{Z}^0_{N,u} \in [0,\infty]\,. \tag{2.13}$$

We emphasize that $\tilde{\mathcal{Z}}^0_{u,\lambda}$ may be infinite. Note moreover that by definition of the free energy $\mathtt{F}^0(u)$, we have that $\mathrm{e}^{\mathtt{F}^0(u)} = \lim_{N\to\infty}(\tilde{Z}^0_{N,u})^{1/N}$, which is the radius of convergence of the series. We therefore easily obtain that

$$\tilde{\mathcal{Z}}^0_{u,\lambda} < +\infty \ \text{ for } \lambda < \mathtt{F}^0(u)\,, \qquad \tilde{\mathcal{Z}}^0_{u,\lambda} = +\infty \ \text{ for } \lambda > \mathtt{F}^0(u)\,.$$

In other words,

$$\mathtt{F}^0(u) = \inf\big\{\lambda \in \mathbb{R}\,,\, \tilde{\mathcal{Z}}^0_{u,\lambda} < +\infty\big\}\,,$$

keeping in mind that $\tilde{\mathcal{Z}}^0_{u,\lambda} = +\infty$ for all $\lambda < 0$ (since $Z^0_{N,u} \geq cN^{-3/2}$ as shown in (2.4)). The computation of the free energy is therefore reduced to determining whether the series (2.13) converges or not.

Remark 2.13. The partition function $\tilde{Z}^0_{N,u}$ is associated to a Gibbs measure: for a system size $N \in \mathbb{N}$ and $u \in \mathbb{N}$,

$$\frac{\mathrm{d}\tilde{\mathrm{P}}^0_{N,u}}{\mathrm{dP}} = \frac{1}{\tilde{Z}^0_{N,u}} \exp\Big(u \sum_{i=1}^N \mathbf{1}_{\{S_i=0\}}\Big) \mathbf{1}_{\{S_N=0\}}\,, \tag{2.14}$$

where the reference measure dP is the law of a random walk $(S_n)_{n\geq 0}$ starting from 0. Similarly, the *grand canonical* partition function $\tilde{\mathcal{Z}}^0_u(\lambda)$ is associated to a *grand canonical* Gibbs measure, in which the system size N is itself random (but with a law tuned by the parameter λ):

$$\frac{\mathrm{d}\tilde{\mathcal{P}}^0_{u,\lambda}}{\mathrm{d}(\mathrm{P}\otimes m)} = \frac{1}{\tilde{\mathcal{Z}}^0_{u,\lambda}} \mathrm{e}^{-\lambda N} \exp\Big(u \sum_{i=1}^N \mathbf{1}_{\{S_i=0\}}\Big) \mathbf{1}_{\{S_N=0\}}$$

where m is the counting measure on $\mathbb{N}$; note that having a finite $\tilde{\mathcal{Z}}^0_{u,\lambda}$ is important here. This measure corresponds to considering the Gibbs measure $\tilde{\mathrm{P}}^0_{N,u}$ but where the system size N is a random with geometric law, $\tilde{\mathcal{P}}^0_{u,\lambda}(N=n) = (1-\mathrm{e}^{-\lambda})\mathrm{e}^{-\lambda(n-1)}$.

Step 2. Computation of $\tilde{\mathcal{Z}}^0_{N,\lambda}$. It turns out that one can compute explicitly the grand canonical partition function $\tilde{\mathcal{Z}}^0_{N,\lambda}$ from (2.13). Let us introduce the sequence of successive returns to 0 of the random walk: $\tau_0 = 0$ and, iteratively, for $k \geq 1$

$$\tau_k := \min\{n > \tau_{k-1}\,, S_n = 0\}\,. \tag{2.15}$$

We can then decompose the partition function $\tilde{Z}^0_{N,u}$ according to the number of returns to 0, noting also that $\{S_N = 0\} = \bigcup_{k=1}^N \{\tau_k = N\}$ where the union is disjoint. We have

$$\tilde{Z}^0_{N,u} = \sum_{k=1}^N \mathrm{E}\Big[\exp\Big(u\sum_{i=1}^N \mathbf{1}_{\{S_i=0\}}\Big)\mathbf{1}_{\{\tau_k=N\}}\Big] = \sum_{k=1}^N \mathrm{e}^{uk}\mathrm{P}(\tau_k = N)\,,$$

where we have used that $\tau_k = N$ if and only if $S_N = 0$ and $\sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} = k$. Then, forming the grand canonical partition function, we obtain

$$\begin{aligned}\tilde{\mathcal{Z}}^0_{u,\lambda} := \sum_{N=1}^\infty \mathrm{e}^{-\lambda N}\tilde{Z}^0_{N,u} &= \sum_{N=1}^\infty \sum_{k=1}^N \mathrm{e}^{-\lambda N}\mathrm{e}^{uk}\mathrm{P}(\tau_k = N)\\ &= \sum_{k=1}^\infty \mathrm{e}^{uk}\sum_{N=k}^\infty \mathrm{e}^{-\lambda N}\mathrm{P}(\tau_k=N) = \sum_{k=1}^\infty \mathrm{e}^{uk}\mathrm{E}[\mathrm{e}^{-\lambda\tau_k}]\,.\end{aligned}$$

Since the increments $(\tau_k - \tau_{k-1})_{k\geq 1}$ are i.i.d. (by the strong Markov property), we deduce that

$$\tilde{\mathcal{Z}}^0_{u,\lambda} = \sum_{k=1}^{\infty} \Big(\mathrm{e}^u \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]\Big)^k = \begin{cases} \frac{\mathrm{e}^u \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]}{1-\mathrm{e}^u \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]} < +\infty & \text{if } \mathrm{e}^u \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}] < 1\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, since the function $\lambda \in \mathbb{R}_+ \mapsto \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]$ is non-increasing and continuous (with value $\mathrm{P}(\tau_1 < +\infty)$ at $\lambda = 0$), we have the following characterization for the free energy:

$$\mathtt{F}^0(u) \text{ is the solution in } \lambda \text{ of } \; \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}] = \mathrm{e}^{-u} \tag{2.16}$$

if a solution exists, and $\mathtt{F}^0(u) = 0$ otherwise.

Step 3. Conclusion of the proof. It only remains to relate the Laplace transform $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]$ of τ_1 to $U^0(\lambda) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} p_n$. The following lemma, combined with (2.16), directly allows one to conclude the proof of Proposition 2.12. □

Lemma 2.14. *Let $(S_n)_{n\geq 0}$ be a Markov chain on a countable state space E starting from 0, and let $\tau_1 = \min\{n \geq 1, S_n = 0\}$. For all $\lambda \geq 0$, setting $U(\lambda) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} \mathrm{P}(S_n = 0)$, one has*

$$\mathrm{E}\big[\mathrm{e}^{-\lambda\tau_1}\big] = \frac{U(\lambda) - 1}{U(\lambda)}\,.$$

Proof. We start from the following decomposition of the probability $p_n = \mathrm{P}(S_n = 0)$: for $n \geq 1$, we have $p_n = \sum_{k=1}^{n} \mathrm{P}(\tau_1 = k) p_{n-k}$. Then, using that $p_n = 1$ for $n = 0$, we obtain

$$U(\lambda) = 1 + \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda n} \sum_{k=1}^{n} \mathrm{P}(\tau_1 = k) p_{n-k} = 1 + \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda k} \mathrm{P}(\tau_1 = k) \sum_{n=k}^{\infty} \mathrm{e}^{-\lambda(n-k)} p_{n-k}\,,$$

which gives $U(\lambda) = 1 + \mathrm{E}\big[\mathrm{e}^{-\lambda\tau_1}\big] U(\lambda)$, which is the desired identity. □

Remark 2.15. Note that one can also take $\lambda = 0$ in Lemma 2.14 (or take $\lambda \downarrow 0$), which shows that

$$\mathrm{P}(\tau_1 < +\infty) = \frac{\sum_{n=1}^{\infty} \mathrm{P}(S_n = 0)}{1 + \sum_{n=1}^{\infty} \mathrm{P}(S_n = 0)}\,.$$

This recovers in particular the fact that $\mathrm{P}(\tau_1 < +\infty) = 1$, *i.e.* the Markov chain is recurrent, if and only if $\sum_{n=1}^{\infty} \mathrm{P}(S_n = 0) = +\infty$, *i.e.* $G(0,0) = +\infty$. In the present case, since $\mathrm{P}(S_n = 0) \sim c_1 n^{-1/2}$ (recall Theorem 1.22), we have $\sum_{n=1}^{\infty} \mathrm{P}(S_n = 0) = +\infty$, hence $\mathrm{P}(\tau_1 < +\infty)$.

b) The constrained case (wetting)

The constrained case is completely analogous, but let us briefly go over the proof. Considering the (constrained) grand canonical partition function[2]

$$\tilde{\mathcal{Z}}^+_{u,\lambda} = \sum_{N=1}^{\infty} \mathrm{e}^{-\lambda N}\, \tilde{Z}^+_{N,u}\,,$$

we again obtain that $\mathtt{F}^+(u) = \inf\{\lambda \in \mathbb{R}, \tilde{\mathcal{Z}}^+_{u,\lambda} < +\infty\}$.

To compute $\tilde{\mathcal{Z}}^+_{u,\lambda}$ explicitly, we introduce the stopping times $(\tau_k^+)_{k\geq 0}$ as follows: let $\tau_0^+ = 0$ and then define iteratively

$$\tau_k^+ = \min\{n > \tau_{k-1},\ S_n = 0 \text{ and } S_j > 0 \ \text{ for all } \tau_{k-1} < j < n\}\,, \tag{2.17}$$

with the convention that $\min\emptyset = +\infty$ (in particular $\tau_k^+ = +\infty$ if $\tau_{k-1}^+ = +\infty$). With this definition of τ_k^+, we have

$$\mathrm{P}(\tau_1^+ = j) = \mathrm{P}(S_j = 0,\ S_i > 0\ \forall 1 \leq i \leq j-1)$$

and we again notice that we have $\tau_k^+ = N$ if and only if $S_N = 0$, $S_i \geq 0\ \forall i \leq N$ and $\sum_{i=1}^N \mathbf{1}_{\{S_i = 0\}} = k$. We can therefore decompose $\Omega_N^+ \cap \{S_N = 0\} = \bigcup_{k=1}^N \{\tau_k^+ = N\}$, so, analogously to what we did for $\tilde{Z}^0_{N,u}$, we obtain

$$\tilde{Z}^+_{N,u} = \sum_{k=1}^{N} \mathrm{e}^{uk} \mathrm{P}(\tau_k^+ = N)\,.$$

Repeating the same computation as above for $\tilde{\mathcal{Z}}^+_{u,\lambda}$, we therefore obtain

$$\tilde{\mathcal{Z}}^+_{u,\lambda} = \sum_{k=1}^{\infty} \Big(\mathrm{e}^u \mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}]\Big)^k = \begin{cases} \frac{\mathrm{e}^u \mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}]}{1-\mathrm{e}^u\mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]} < +\infty & \text{if } \mathrm{e}^u\mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}] < 1\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, we have the following characterization for the free energy with the hard wall constraint:

$$\mathtt{F}^+(u) \text{ is the solution in } \lambda \text{ of } \ \mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}] = \mathrm{e}^{-u} \tag{2.18}$$

if a solution exists, and $\mathtt{F}^+(u) = 0$ otherwise.

[2] Let us note that Remark 2.13 remains valid in the constrained case.

Again, with the same proof as for Lemma 2.14, we obtain the following relation between the Laplace transform $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}]$ and $U^+(\lambda) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} p_n^+$, where we recall that $p_n^+ = \mathrm{P}(S_n = 0, S_i \geq 0\ \forall i \leq n)$: for all $\lambda \geq 0$, we have

$$\mathrm{E}\big[\mathrm{e}^{-\lambda\tau_1^+}\big] = \frac{U^+(\lambda) - 1}{U^+(\lambda)}\,.$$

This concludes the proof of Proposition 2.12. □

Remark 2.16. Let us emphasize here that, in the same way as in Remark 2.15, by taking $\lambda = 0$ we obtain

$$\mathrm{P}(\tau_1^+ < +\infty) = \frac{\sum_{n=1}^{\infty} p_n^+}{1 + \sum_{n=1}^{\infty} p_n^+}\,.$$

In particular, since we have $p_n^+ \sim c_2 n^{-3/2}$ thanks to Theorem 1.22, we obtain that $\Sigma_+ := \sum_{n=1}^{\infty} p_n^+ < +\infty$, hence $\mathrm{P}(\tau_1^+ < +\infty) = \frac{\Sigma_+}{1+\Sigma_+} < 1$.

Exercise 16. Consider $(S_n)_{n\geq 0}$ the random walk with i.i.d. increments $(X_i)_{i\geq 1}$ with law given by $\mathrm{P}(X_i = \pm 1) = \frac{1}{4}$ and $\mathrm{P}(X_i = 0) = \frac{1}{2}$; in other words, $S_n = SRW_{2n}$, where $(SRW_k)_{k\geq 0}$ is the (nearest-neighbor symmetric) simple random walk. Let us introduce

$$T_- := \min\{k \geq 0, S_k = -1\} \quad \text{and} \quad G(x) = \mathrm{E}[x^{T_-}] \ \text{ for } x \in [0,1]\,.$$

1. Show that $G(x) = \frac{1}{4}x(1 + 2G(x) + G(x)^2)$ and deduce a formula for $G(x)$.
2. Show that $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1}] = \frac{1}{2}\mathrm{e}^{-\lambda}(1 + G(\mathrm{e}^{-\lambda}))$. Compute $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]$ and then $\mathtt{F}^0(u)$.
3. Show that $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}] = \frac{1}{2}\mathrm{e}^{-\lambda}(1 + \frac{1}{2}G(\mathrm{e}^{-\lambda}))$. Compute $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}]$ and then $\mathtt{F}^+(u)$.

2.2.2 Critical points and critical behavior of the free energy

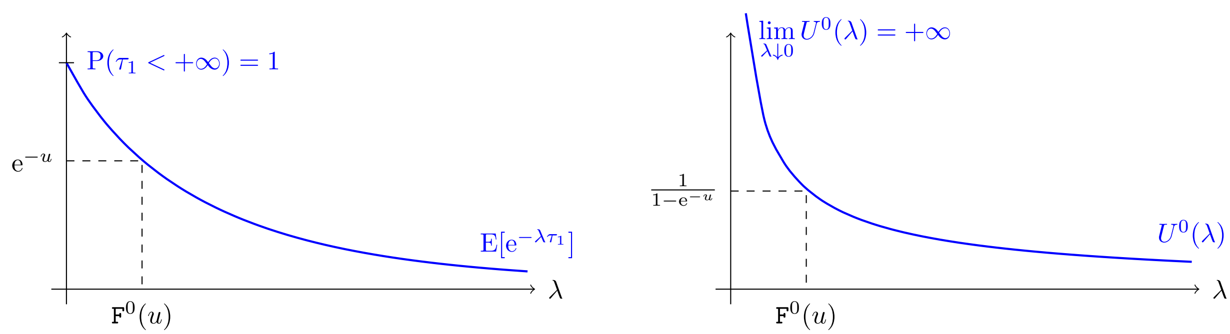
Using Proposition 2.12 or the characterizations (2.16) and (2.18), we now compute the critical points and determine the critical behavior of the free energy. Let us first collect some properties of the Laplace transforms that we have seen in the previous subsection.

Remark 2.17. The functions $\lambda \mapsto \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}], \mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}]$, resp. $\lambda \mapsto U^0(\lambda), U^+(\lambda)$ are strictly decreasing and convex on $\mathbb{R}_+$, with limits 0, resp. 1, as $\lambda \to \infty$. Moreover, we have

$$\lim_{\lambda\downarrow 0} \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}] = \mathrm{P}(\tau_1 < +\infty) = 1\,, \qquad \lim_{\lambda\downarrow 0} U^0(\lambda) = +\infty\,,$$
$$\lim_{\lambda\downarrow 0} \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}] = \mathrm{P}(\tau_1^+ < +\infty) < 1\,, \qquad \lim_{\lambda\downarrow 0} U^+(\lambda) = 1 + \Sigma_+ < +\infty\,.$$

a) Computing the critical points u_c^0, u_c^+

The unconstrained case (pinning). Let us represent graphically the characterization (2.16) of $\mathtt{F}^0$ (on the left) and the characterization of Proposition 2.12 (on the right), using the properties of the functions $\lambda \mapsto \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}], U^0(\lambda)$ recalled in Remark 2.17 above:

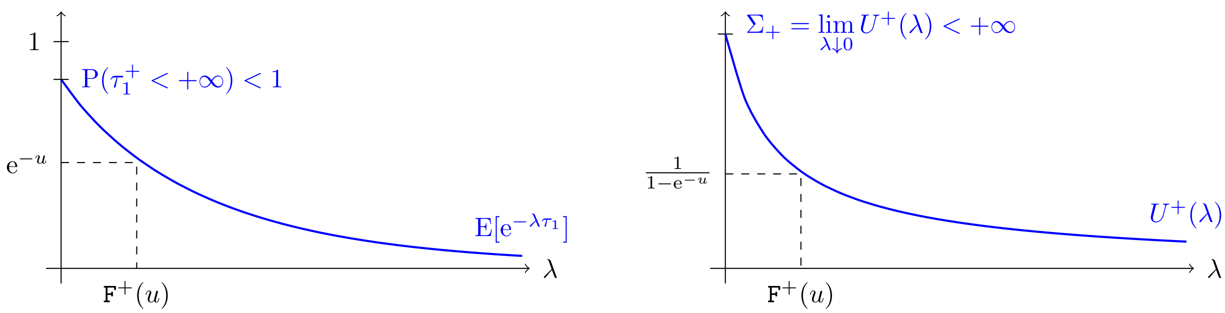


Note that, by the properties of the function $\lambda \mapsto \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]$ recalled in Remark 2.17, the equation $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1}] = \mathrm{e}^{-u}$ admits a unique solution in $\mathbb{R}_+^*$ for every u such that $\mathrm{e}^{-u} \in (0,1)$, *i.e.* for every $u > 0$. Using the characterization (2.16), this shows that $\mathtt{F}^0(u) > 0$ if and only if $u > 0$: in other words, this proves that

$$u_c^0 = 0\,.$$

One could also have used Proposition 2.12, observing that, thanks to the properties of the function $\lambda \mapsto U^0(\lambda)$ given in Remark 2.17, the equation $U^0(\lambda) = \frac{1}{1-\mathrm{e}^{-u}}$ admits a unique solution in $\mathbb{R}_+^*$ for every u such that $\frac{1}{1-\mathrm{e}^{-u}} \in (1,\infty)$, *i.e.* for every $u > 0$.

The constrained case (wetting). Let us represent graphically the characterization (2.18) of $\mathtt{F}^+$ (on the left) and the characterization of Proposition 2.12 (on the right), using the properties of the functions $\lambda \mapsto \mathrm{E}[\mathrm{e}^{-\lambda\tau_1^+}], U^+(\lambda)$ recalled above:



By the properties of the function $\lambda \mapsto E[e^{-\lambda\tau_1^+}]$ given in Remark 2.17, the equation $E[e^{-\lambda\tau_1^+}] = e^{-u}$ admits a unique solution in $\mathbb{R}_+^*$ for every u such that $e^{-u} \in (0, P(\tau_1^+ < +\infty))$, *i.e.* for every $u > -\log P(\tau_1^+ < +\infty)$. Using the characterization (2.18), this shows that $\mathtt{F}^+(u) > 0$ if and only if $u > -\log P(\tau_1^+ < +\infty)$: in other words, this proves that

$$u_c^+ = -\log P(\tau_1^+ < +\infty) > 0\,.$$

One could also have used Proposition 2.12, observing that the equation $U^+(\lambda) = \frac{1}{1-e^{-u}}$ admits a unique solution in $\mathbb{R}_+^*$ for every u such that $\frac{1}{1-e^{-u}} \in (1, 1+\Sigma_+)$, *i.e.* for every $u > \log(\frac{1+\Sigma_+}{\Sigma_+})$. We thus have $u_c^+ = \log(\frac{1+\Sigma_+}{\Sigma_+}) > 0$ (see Remark 2.16 to recover the value given above).

b) Critical behavior of the free energy

Now that we have the explicit expression of the critical points, we can describe the behavior of the free energy in the neighborhood of u_c^0, u_c^+, using again Proposition 2.12. The idea is to start from the observation that for every $u > u_c^0$, resp. $u > u_c^+$, we have

$$U^0(\mathtt{F}^0(u)) = \frac{1}{1-e^{-u}} \quad \text{and} \quad U^+(\mathtt{F}^+(u)) = \frac{1}{1-e^{-u}}\,. \tag{2.19}$$

Note that, since the functions $u \mapsto \mathtt{F}^{0/+}(u)$ are continuous, we have $\lim_{u\downarrow u_c^0} \mathtt{F}^0(u) = 0$ and $\lim_{u\downarrow u_c^+} \mathtt{F}^+(u) = 0$. We therefore need to study the behavior of $U^{0/+}(\lambda)$ as $\lambda \downarrow 0$. This is given by the following lemma.

Lemma 2.18. *As $\lambda \downarrow 0$, we have the following behaviors:*

$$U^0(\lambda) \sim c_1\sqrt{\pi}\,\lambda^{-1/2} \quad \textit{and} \quad U^+(0) - U^+(\lambda) \sim 2c_2\sqrt{\pi}\,\lambda^{1/2}\,.$$

where c_1, c_2 are the constants appearing in Theorem 1.22.

Remark 2.19. As a side remark, note that the formulas (2.19) show that the free energies $\mathtt{F}^0, \mathtt{F}^+$ are analytic on $(0,+\infty$ and $(u_c^+,+\infty)$ respectively. Indeed, the functions $\lambda \mapsto U^0(\lambda), U^+(\lambda)$ are analytic and bijective from $(0,+\infty)$ to $(0,+\infty)$, $(0, \frac{1}{1-\mathrm{e}^{-u_c^+}})$ respectively. The implicit function theorem then shows that the inverse functions of U^0, U^+ are analytic, which proves the claim property (by composing with the analytic function $u \mapsto \frac{1}{1-\mathrm{e}^{-u}}$).

Let us now use Lemma 2.18 together with (2.19) to prove Theorem 2.11.

- In the unconstrained case, as $u \downarrow u_c^0 = 0$ we have

$$c_1\sqrt{\pi}\,\mathtt{F}^0(u)^{-1/2} \sim U^0(\mathtt{F}^0(u)) = \frac{1}{1-\mathrm{e}^{-u}} \sim \frac{1}{u}\,.$$

This gives Theorem 2.11, with the constant $c_0 = (\pi c_1^2)^{-1} = 2\sigma^2$ (recall Theorem 1.22).

- In the constrained case, since $\frac{1}{1-\mathrm{e}^{-u_c^+}} = U^+(0) = 1+\Sigma_+$, as $u \downarrow u_c^+$ we have

$$2c_2\sqrt{\pi}\,\mathtt{F}^+(u)^{1/2} \sim U^+(0)-U^+(\mathtt{F}^+(u)) = \frac{1}{1-\mathrm{e}^{-u_c^+}}-\frac{1}{1-\mathrm{e}^{-u}} \sim \frac{\mathrm{e}^{-u_c^+}}{(1-\mathrm{e}^{-u_c^+})^2}(u-u_c^+)\,.$$

This gives Theorem 2.11, with the constant $c_+ = \frac{1}{4\pi c_2^2}\frac{\mathrm{e}^{-2u_c^+}}{(1-\mathrm{e}^{-u_c^+})^4} = \frac{\Sigma_+^2(1+\Sigma_+)^2}{4\pi c_2^2}$.

Proof of Lemma 2.18. Let us start with the unconstrained case. By Theorem 1.22, we have $p_n \sim c_1 n^{-1/2}$ with $c_1 = \frac{1}{\sqrt{2\pi\sigma^2}}$. It is easy to show that the contribution of small n is negligible as $\lambda \downarrow 0$, which allows us to obtain

$$U^0(\lambda) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n} p_n \sim c_1 \sum_{n=1}^{\infty} n^{-1/2} e^{-\lambda n}\,.$$

We now write a Riemann sum approximation:

$$\sum_{n=1}^{\infty} n^{-1/2}\mathrm{e}^{-\lambda n} = \lambda^{-1/2}\sum_{n=1}^{\infty} \lambda(\lambda n)^{-1/2}\mathrm{e}^{-\lambda n} \sim \lambda^{-1/2}\int_0^\infty x^{-1/2}\mathrm{e}^{-x}\,\mathrm{d}x \quad \text{as } \lambda \downarrow 0\,.$$

Since $\int_0^\infty x^{-1/2}\mathrm{e}^{-x}\mathrm{d}x = \Gamma(1/2) = \sqrt{\pi}$, we conclude that $U^0(\lambda) \sim c_1\sqrt{\pi}\,\lambda^{-1/2}$.

For the constrained case, we write, analogously to the above calculation (the contribution of small n is also negligible here): since $U^+(0) = \sum_{n=0}^\infty p_n^+$, we have

$$U^+(0) - U^+(\lambda) = \sum_{n=0}^{\infty}(1-\mathrm{e}^{-\lambda n})p_n^+ \sim c_2\sum_{n=1}^{\infty}\frac{1-\mathrm{e}^{-\lambda n}}{n^{3/2}}\,.$$

We then write again a Riemann sum approximation (notice the integrability at 0 and at $+\infty$):

$$\sum_{n=1}^{\infty} \frac{1-\mathrm{e}^{-\lambda n}}{n^{3/2}} = \lambda^{1/2} \sum_{n=1}^{\infty} \lambda \frac{1-\mathrm{e}^{-\lambda n}}{(\lambda n)^{3/2}} \sim \lambda^{1/2} \int_0^\infty \frac{1-\mathrm{e}^{-x}}{x^{3/2}} \mathrm{d}x \quad \text{as } \lambda \downarrow 0\,,$$

using again a Riemann sum . Since $\int_0^\infty \frac{1-\mathrm{e}^{-x}}{x^{3/2}} \mathrm{d}x = 2\sqrt{\pi}$, we deduce that $U^+(0) - U^+(\lambda) \sim 2c_2\sqrt{\pi}\,\lambda^{1/2}$. □

2.3 Free energy and phase transition in dimension $d = 2$

Contrary to the case of dimension $d = 1$ for discrete interfaces, there is no explicit formula for the free energy in the case of the GFF. One can nevertheless obtain information on the phase transition by more flexible methods, at least in the unconstrained case.

Theorem 2.20 (Critical behavior for the pinning of the GFF in $d = 2$)**.** *In dimension $d = 2$, we have $u_c^0 = 0$. Additionally, there exists a constant c_0 such that*

$$\mathtt{F}^0(u) \sim c_0 \frac{u}{\sqrt{\log 1/u}} \qquad \text{as } u \downarrow 0\,.$$

Let us emphasize that the precise critical behavior of the free energy $\mathtt{F}^0$ is announced in [CM13, Fact 2.4], citing a "variant of the proof" of [BV01, Thm. 2.4] (but without giving the constant). In fact, I do not believe that a written proof exists somewhere... We shall prove here, using a method employed in [GL17], a slightly weaker result, which provides bounds: as $u \downarrow 0$

$$(1+o(1))\,\frac{a}{2}\,\frac{u}{\sqrt{\log 1/u}} \le \mathtt{F}^0(u) \le (1+o(1))\,a\sqrt{2}\,\frac{u}{\sqrt{\log 1/u}}\,. \tag{2.20}$$

The lower bound in particular shows that $\mathtt{F}^0(u) > 0$ for every $u > 0$ (arbitrarily small), which therefore shows that $u_c^0 = 0$.

The constrained case is significantly more difficult.

Theorem 2.21 (Wetting of the GFF in $d = 2$)**.** *In dimension $d = 2$, we have that $u_c^+ > 0$. However, the critical behavior of $\mathtt{F}^+$ in the neighborhood of u_c^+ is unknown.*

The fact that $u_c^+ > 0$ is proved in [CV00] and this is a nontrivial result — we do not provide a proof here, but we refer to the lecture notes [Gia01, Vel06] where an idea of the proof is given. The critical behavior of $\mathtt{F}^+$ in the neighborhood of u_c^+ is completely open; to my knowledge there is no result in this direction, even partial. If I had to bet, I would guess that $\lim_{u\downarrow u_c^+} \frac{\log \mathtt{F}^+(u)}{\log(u-u_c^+)} = +\infty$; in other words, the free energy decays faster than any polynomial.

Proof of the bounds in (2.20).

Lower bound in (2.20). We can use the convexity of $u \mapsto \log Z^0_{N,u}$ to obtain the following inequality:

$$\log Z^0_{N,u} \geq u \times \frac{\partial}{\partial_u} \log Z^0_{N,u}\Big|_{u=0} = u\, \mathrm{E}^0_N\Big[\sum_{x\in\Lambda_N} \vartheta_x\Big]\,,$$

reusing the computation (2.5). Using Corollary 2.10, called the *finite-volume criterion*, we obtain that for all $N \geq N_0$ (note that N_0 is bounded from below by a constant, uniformly in $u \in [0, \frac{1}{2}]$), we have

$$\mathtt{F}^0(u) \geq \frac{1}{N^d} \log Z^0_{N,u} - N^{-1/4} \geq \frac{u}{N^d} \mathrm{E}^0_N\Big[\sum_{x\in\Lambda_N} \vartheta_x\Big] - c_0 N^{-\frac{1}{4}}\,. \tag{2.21}$$

We will then apply this inequality with a suitably chosen $N = N(u)$.

Before doing so, let us estimate the lower bound (in the limit $N \to \infty$, since we will choose $N(u)$ such that $\lim_{u\downarrow 0} N(u) = +\infty$). We have $\mathrm{E}^0_N[\vartheta_x] = \mathrm{P}^0_N(\varphi_x \in [-a, a])$, and the variable φ_x has under P^0_N a centered Gaussian distribution with variance $\sigma_N^2(x) := G_{\Lambda_N}(x,x)$. One can show that $\max_{x\in\Lambda_N} \sigma_N(x) = \sigma_N(0)$, so that for any $x \in \Lambda_N$ we have

$$\mathrm{P}^0_N\big(\varphi_x \in [-a,a]\big) \geq \mathrm{P}\big(\sigma_N(0) Z \in [-a,a]\big)\,.$$

Now, recall from Remark 1.13 that $\sigma_N(0) \sim \frac{2}{\pi} \log N$, so that

$$\mathrm{P}\big(\sigma_N(0) Z \in [-a,a]\big) \sim \frac{2a}{\sqrt{2\pi}\sigma_N(0)} \sim \frac{a}{\sqrt{\log N}}\,.$$

All together, we obtain that

$$\frac{1}{N^d} \mathrm{E}^0_N\Big[\sum_{x\in\Lambda_N} \vartheta_x\Big] \geq (1+o(1)) \frac{a}{\sqrt{\log N}} \qquad \text{as } N \to \infty\,.$$

Now, let $\varepsilon \in (0,1)$ and choose $N(u) = (1/u)^{4+\varepsilon}$ in (2.21): this gives that as $u \downarrow 0$,

$$\mathtt{F}^0(u) \geq (1+o(1))\frac{au}{\sqrt{\log N(u)}} - c_0 N(u)^{-1/4} \geq (1+o(1))\frac{1}{\sqrt{4+\varepsilon}}\frac{u}{\sqrt{\log 1/u}}\,.$$

Since ε is arbitrary, we deduce the lower bound in (2.20).

Remark 2.22. One could improve this lower bound to obtain a factor $a/\sqrt{2}$ instead of $a/2$: it suffices to note that Corollary 2.10 remains valid when replacing $c_0 N^{-1/4}$ by $c_\eta N^{-1/2+\eta}$, for arbitrary $\eta > 0$. This in particular allows one to choose $N_u = (1/u)^{2+\varepsilon}$ and to gain a factor $\sqrt{2}$ in the lower bound of (2.20). Exercise 18 below improves this factor even further.

Upper bound in (2.20). Let us begin by proving the following fact: for all $N \geq 1$

$$\mathtt{F}^0(u) \leq \frac{1}{N^d}\log \hat{Z}_{N,u} \qquad \text{where} \qquad \hat{Z}_{N,u} := \sup_{\xi} Z^{\xi}_{N,u}\,, \tag{2.22}$$

the supremum being taken over all boundary conditions $\xi : \mathbb{Z}^d \to \mathbb{R}$. Indeed, by decomposing Λ_{2N} into sub-boxes $\Lambda_N^{(v)} = \Lambda_N + Nv$ for $v \in \{0,1\}^d$ and applying the spatial Markov property (conditioning on $\Gamma = \Lambda_{2N} \setminus (\bigcup_{v\in\{0,1\}^d} \Lambda_N^{(v)})$), we obtain the following: for any boundary condition ξ,

$$Z^{\xi}_{2N,u} = \mathrm{E}^0_N\Big[\prod_{v\in\{0,1\}^d} Z^{\phi}_{\Lambda_N^{(v)},u}\Big] \leq \big(\hat{Z}_{N,u}\big)^{2^d}\,.$$

We conclude that $\frac{1}{(2N)^d}\log \hat{Z}_{2N,u} \leq \frac{1}{N^d}\log \hat{Z}_{N,u}$ for all $N \geq 1$, hence

$$\mathtt{F}^0(u) = \lim_{k\to\infty}\frac{1}{(2^k N)^d}\log Z^0_{2^k N,u} \leq \lim_{k\to\infty}\frac{1}{(2^k N)^d}\log \hat{Z}_{2^k N,u} \leq \frac{1}{N^d}\log \hat{Z}_{N,u}\,,$$

by monotonicity of the sequence $\big(\frac{1}{(2^k N)^d}\log \hat{Z}_{2^k N,u}\big)_{k\geq 1}$.

Let us now bound $\hat{Z}_{N,u}$. For any boundary condition ξ, we have

$$\frac{\partial}{\partial u}\log Z^{\xi}_{N,u} = \frac{1}{Z^{\xi}_{N,u}}\mathrm{E}^{\xi}_N\Big[\Big(\sum_{x\in\Lambda_N}\vartheta_x\Big)\,\mathrm{e}^{u\sum_{x\in\Lambda_N}\vartheta_x}\Big] \leq \mathrm{e}^{uN^d}\mathrm{E}^{\xi}_N\Big[\sum_{x\in\Lambda_N}\vartheta_x\Big]\,,$$

where we have used that $u \geq 0$ to bound $Z^{\xi}_{N,u} \geq 1$ and we have bounded $\vartheta_x \leq 1$ in the exponential. Now note that under P^{ξ}_N the variable φ_x is Gaussian with mean m^{ξ}_x and variance $\sigma_N^2(x) = G_{\Lambda_N}(x,x)$. Thus, for any boundary condition ξ, we have

$$\mathrm{E}^{\xi}_N[\vartheta_x] = \mathrm{P}\big(\sigma_N(x)Z + m^{\xi}_x \in [-a,a]\big) \leq \mathrm{P}\big(\sigma_N(x)Z \in [-a,a]\big) = \mathrm{E}^0_N[\vartheta_x]\,,$$

where we used the fact that the density of the $\mathcal{N}(0,1)$ distribution is increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$ to see that the function $t \mapsto \mathrm{P}(Z+t \in [-b,b])$ is maximized at $t=0$.

We deduce that

$$\frac{\partial}{\partial u} \log Z^{\xi}_{N,u} \leq \mathrm{e}^{uN^d} \mathrm{E}^0_N\Big[\sum_{x\in\Lambda_N} \vartheta_x \Big] \quad \Longrightarrow \quad \log Z^{\xi}_{N,u} \leq \frac{\mathrm{e}^{uN^d}-1}{N^d}\, \mathrm{E}^0_N\Big[\sum_{x\in\Lambda_N} \vartheta_x \Big] , \quad (2.23)$$

where we simply integrated the first inequality between 0 and u to obtain the second one. Since this inequality holds for any boundary condition ξ, we conclude that, for all $N \geq 1$,

$$\mathtt{F}^0(u) \leq \frac{1}{N^d} \log \hat{Z}_{N,u} \leq \frac{\mathrm{e}^{uN^d}-1}{N^d} \times \frac{1}{N^d} \mathrm{E}^0_N\Big[\sum_{x\in\Lambda_N} \vartheta_x \Big] . \qquad (2.24)$$

In the same way as for the lower bound[3], we obtain

$$\frac{1}{N^d} \mathrm{E}^0_N\Big[\sum_{x\in\Lambda_N} \vartheta_x \Big] \leq (1+o(1))\, \frac{a}{\sqrt{\log N}} \qquad \text{as } N \to \infty .$$

Thus, choosing $N(u) = (\varepsilon/u)^{1/2}$ for some fixed $\varepsilon > 0$ and plugging this choice in (2.24), we obtain

$$\mathtt{F}^0(u) \leq (1+o(1))\, \frac{\mathrm{e}^{\varepsilon}-1}{\varepsilon}\, \frac{au}{\sqrt{\log N(u)}} = (1+o(1))\, \frac{\mathrm{e}^{\varepsilon}-1}{\varepsilon}\, \frac{au\sqrt{2}}{\sqrt{\log(1/u)}} .$$

Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, this gives the upper bound in (2.20). □

2.4 Free energy and phase transition in dimension $d \geq 3$

Let us state the two main results of this section, for the critical behavior of the pinning and the wetting models of the GFF in dimension $d \geq 3$. We separate the statements into two parts since the results are quite different and of quite a different difficulty.

Theorem 2.23 (Critical behavior for the pinning of the GFF in $d \geq 3$). *In dimension $d \geq 3$, we have $u_c^0 = 0$. Moreover, we have the following asymptotic behavior:*

[3] One needs to verify that $\sigma_N(x) \sim \frac{2}{\pi} \log N$ in the "bulk" of Λ_N, *i.e.* when $d(x, \Lambda_N^c) \geq N^{1-o(1)}$.

there exists a constants c_0 such that

$$\mathtt{F}^0(u) \sim c_0 u \qquad \text{as} \;\; u \downarrow 0\,.$$

Theorem 2.24 (Critical behavior for the wetting of the GFF in $d \geq 3$)**.** *In dimension $d \geq 3$, we have $u_c^+ = 0$. Moreover, we have the following asymptotic behaviors: there exists a constant c_+ such that*

$$\mathtt{F}^+(u) = \exp\Big(- (1+o(1)) c_+ \log\big(1/u\big)^2 \Big) \qquad \text{as} \;\; u \downarrow 0\,.$$

The fact that $u_c^+ = 0$ had already been observed in [BDZ00], but the precise critical behavior of Theorem 2.24 was proved much more recently, in [GL18]. We follow here the ideas of the proof in [GL18].

2.4.1 The unconstrained case (pinning)

We begin by proving Theorem 2.23, namely that $\mathtt{F}^0(u) \sim c_0 u$ as $u \downarrow 0$. The proof shows that the constant is equal to

$$c_0 = \mathrm{P}\big(|Z| \leq a/\sqrt{G_0}\big)\,,$$

where $Z \sim \mathcal{N}(0,1)$ and $G_0 = G(0,0)$ is the Green function at 0 of the simple random walk on $\mathbb{Z}^d$. The proof strategy is very similar to the case of dimension $d = 2$.

Lower bound. We again use the convexity of $u \mapsto \log Z_{N,u}^0$ to obtain that

$$\frac{1}{N^d} \log Z_{N,u}^0 \geq u \times \frac{1}{N^d} \frac{\partial}{\partial u} \log Z_{N,u}^0 \Big|_{u=0} = u \times \frac{1}{N^d} \mathrm{E}_{N,u}^0 \Big[\sum_{x \in \Lambda_N} \vartheta_x \Big]\,.$$

Now, we have $\mathrm{E}_N^0[\vartheta_x] = \mathrm{P}(\sigma_N(x) Z \in [-a,a])$, where $\sigma_N^2(x) = G_{\Lambda_N}(x,x)$ is the variance of φ_x under P_N^0. Now, we have $\sigma_N(x) \leq G(0,0) =: G_0$ for all $x \in \Lambda_N$: this yields that

$$\mathtt{F}^0(u) = \lim_{N\to\infty} \frac{1}{N^d} \log Z_{N,u}^0 \geq u \times \mathrm{P}\big(\sqrt{G_0} Z \in [-a,a]\big)\,.$$

Upper bound. We use the inequality (2.22) together with (2.23) (note that they are valid in any dimension $d \geq 2$). In the same way as in (2.24), they yield the following: for all $N \geq 1$

$$\mathtt{F}^0(u) \leq \frac{e^{uN^d} - 1}{N^d} \frac{1}{N^d} \mathrm{E}_N^0 \Big[\sum_{x \in \Lambda_N} \vartheta_x \Big]\,.$$

This time, note that $E_N^0[\vartheta_x] = P(\sigma_N(x) Z \in [-a,a])$ with $\sigma_N^2(x) \geq (1+o(1)) G_0$, uniformly for $x \in \Lambda_N$ such that $N - |x| \to \infty$. This gives that

$$\mathtt{F}^0(u) \leq (1+o(1)) \frac{e^{uN^d}-1}{N^d} P(\sqrt{G_0} Z \in [-a,a]) \,.$$

Now, applying this to some $N = N(u)$ such that $\lim_{u\downarrow 0} uN(u)^d = 0$ (in particular $\lim_{u\downarrow 0} N(u) = +\infty$), this gives that

$$\mathtt{F}^0(u) \leq (1+o(1)) u \, P\big(\sqrt{G_0} Z \in [-a,a]\big)$$

and concludes the proof. □

2.4.2 The constrained case (wetting)

We now prove Theorem 2.24. We give a complete proof of the lower bound, allowing us in particular to show that $u_c^+ = 0$. The upper bound is somewhat more technical: we will give the essential steps of the proof. Let us mention that one can actually obtain the precise value of the constant: $c_+ = \frac{G_0}{2a^2}$.

Proof of the lower bound in Theorem 2.24.

Step 1. Softening the hard wall. The first step consists in considering a slightly different model, by making the hard wall constraint "softer". For $K \geq 0$, let us set

$$Z_{N,u,K}^{\xi} := E_N^{\xi}\Big[\exp\Big(u \sum_{x\in\Lambda_N} \mathbf{1}_{\{\varphi_x \in [0,a]\}} - K \sum_{x\in\Lambda_N} \mathbf{1}_{\{\varphi_x < 0\}} \Big) \Big] \,,$$

and we denote by $P_{N,u,K}^{\xi}$ the Gibbs measure naturally associated with this partition function. In this case, the surface is simply *penalized* when it takes negative values (instead of being forbidden), the parameter K controlling the strength of the penalty. Note that in the case $K = +\infty$, we recover $Z_{N,u}^{\xi,+} = Z_{N,u,\infty}^{\xi}$. We prove the following lemma that compares the free energies when the hard-wall constraint is replaced by a "softer" wall.

Lemma 2.25. *For any $K \geq 0$, denote $\mathtt{F}_K(u) := \lim_{N\to\infty} \frac{1}{N^d} \log Z_{N,u,K}^0$. Then we have the bounds:*

$$\mathtt{F}_K(u) \geq \mathtt{F}_\infty(u) = \mathtt{F}^+(u) \geq \mathtt{F}_K(u) - e^{-K} \,.$$

Proof. The upper bound is obvious by monotonicity in K, so we focus on the lower bound. For any $A \subset \Lambda_N$, let us introduce the event

$$E_A := \big\{ \varphi \in \mathbb{R}^{\Lambda_N}, \{x \in \Lambda_N, \varphi < 0\} = A \big\} ,$$

and note that $E_\emptyset = \Omega_N^+$. Thus, decomposing according to the set A of points where $\varphi_x < 0$, we obtain

$$Z^0_{N,u,K} = \sum_{A \subset \Lambda_N} \mathrm{e}^{-K|A|} \mathrm{E}^0_N \Big[\exp\Big(u \sum_{x \in \Lambda_N} \mathbf{1}_{\{\varphi_x \in [0,a]\}} \Big) \mathbf{1}_{E_A} \Big] .$$

We show just below that, for any $A \subset \Lambda_N$, we have

$$\mathrm{E}^0_N \Big[\exp\Big(u \sum_{x \in \Lambda_N} \mathbf{1}_{\{\varphi_x \in [0,a]\}} \Big) \mathbf{1}_{E_A} \Big] \leq Z^+_{N,u} . \tag{2.25}$$

Indeed, with this inequality and the previous decomposition, we obtain

$$Z^0_{N,u,K} \leq Z^+_{N,u} \sum_{A \subset \Lambda_N} \mathrm{e}^{-K|A|} = Z^+_{N,u} \big(1 + \mathrm{e}^{-K}\big)^{N^d} \leq Z^+_{N,u} \exp(\mathrm{e}^{-K} N^d) ,$$

where we have used the binomial formula, and then the inequality $1 + x \leq \mathrm{e}^x$. We conclude that $\frac{1}{N^d} \log Z^+_{N,u,K} \leq \frac{1}{N^d} \log Z^+_{N,u} + \mathrm{e}^{-K}$, which proves the desired lower bound on $\mathtt{F}^+(u)$ when taking the limit $N \to \infty$.

It remains to prove (2.25). We simply write what the expectation corresponds to: recalling the definition of the Gibbs measure P^0_N, we obtain

$$\mathrm{E}^0_N \Big[\exp\Big(u \sum_{x \in \Lambda_N} \mathbf{1}_{\{\varphi_x \in [0,a]\}} \Big) \mathbf{1}_{E_A} \Big] = \frac{1}{W^0_N} \int_{E_A} \mathrm{e}^{u \sum_{x \in \Lambda_N} \mathbf{1}_{\{h_x \in [0,a]\}}} \mathrm{e}^{-H^0_N(h)} \mathrm{d}h .$$

We now perform a change of variables $h \to \tilde{h}$ by setting $\tilde{h}_x = -h_x$ for $x \in A$ and $\tilde{h}_x = h_x$ for $x \notin A$, so that if $h \in E_A$ then $\tilde{h} \in E_\emptyset$: the Jacobian of the change of variables is equal to 1 and we have $\mathbf{1}_{\{\tilde{h}_x \in [0,a]\}} \geq \mathbf{1}_{\{h_x \in [0,a]\}}$ and also $H^0_N(\tilde{h}) \leq H^0_N(h)$ (we have reduced some differences $h_x - h_y$). Thus, we obtain

$$\mathrm{E}^0_N \Big[\exp\Big(u \sum_{x \in \Lambda_N} \mathbf{1}_{\{\varphi_x \in [0,a]\}} \Big) \mathbf{1}_{E_A} \Big] \leq \frac{1}{W^0_N} \int_{E_\emptyset} \mathrm{e}^{u \sum_{x \in \Lambda_N} \mathbf{1}_{\{\tilde{h}_x \in [0,a]\}}} \mathrm{e}^{-H^0_N(\tilde{h})} \mathrm{d}\tilde{h} = Z^+_{N,u} ,$$

which is the desired inequality. □

Step 2. Raising the boundary condition. The following lemma allows us to change (*i.e.* raise) the boundary conditions. Its proof is left as an exercise.

Lemma 2.26. *For any $h \in \mathbb{R}$, we have*

$$\mathtt{F}_K(u) = \lim_{N\to\infty} \frac{1}{N^d} \mathrm{E}^h_\infty\big[\log Z^\phi_{N,u,K}\big] ,$$

where P^h_∞ is the infinite-volume GFF with mean (constant equal to) h.

Exercise 17. Prove Lemma 2.26. One may use Lemma 2.7 on the change of boundary conditions, as well as Exercise 7.

Step 3. Exploiting convexity. Thanks to Jensen's inequality, for any boundary condition ϕ, we have

$$\begin{aligned}\log Z^\phi_{N,u,K} &= \log \mathrm{E}^\phi_N\Big[\exp\Big(u \sum_{x\in\Lambda_N} \mathbf{1}_{\{\varphi_x\in[0,a]\}} - K \sum_{x\in\Lambda_N} \mathbf{1}_{\{\varphi_x<0\}}\Big)\Big] \\ &\geq \mathrm{E}^\phi_N\Big[u \sum_{x\in\Lambda_N} \mathbf{1}_{\{\varphi_x\in[0,a]\}} - K \sum_{x\in\Lambda_N} \mathbf{1}_{\{\varphi_x<0\}}\Big] .\end{aligned}$$

Thanks to the spatial Markov property for the Gibbs measure P^h_∞, we have the identity $\mathrm{E}^h_\infty\big[\mathrm{E}^\phi_N[F(\varphi_{\Lambda_N})]\big] = \mathrm{E}^h_\infty[F(\varphi_{\Lambda_N})]$. Thus, taking the expectation under P^h_∞ in the previous inequality, we obtain

$$\begin{aligned}\frac{1}{N^d}\mathrm{E}^h_\infty\big[\log Z^\phi_{N,u,K}\big] &\geq \frac{1}{N^d}\sum_{x\in\Lambda_N}\big(u\mathrm{P}^h_\infty(\varphi_x\in[0,a]) - K\mathrm{P}^h_\infty(\varphi_x<0)\big) \\ &= u\mathrm{P}\big(\sqrt{G_0}Z + h \in [0,a]\big) - K\mathrm{P}\big(\sqrt{G_0}Z + h < 0\big)\end{aligned}$$

for $Z \sim \mathcal{N}(0,1)$. We have also used here that under P^h_∞ we have $\varphi_x \sim \mathcal{N}(h, G_0)$. Thanks to Lemmas 2.25 and 2.26, we conclude that, for any $h\in\mathbb{R}$ and $K\geq 0$:

$$\mathtt{F}^+(u) \geq \mathtt{F}_K(u) - \mathrm{e}^{-K} \geq u\mathrm{P}\big(Z+h\in[0,b]\big) - K\mathrm{P}\big(Z+h<0\big) - \mathrm{e}^{-K} ,$$

where we have set $b = a/\sqrt{G_0}$ (and replaced $h/\sqrt{G_0}$ by h).

Conclusion. It remains to optimize over the choices of K and h as functions of u, b and to perform Gaussian calculations. Let us take

$$K = \log\big(1/u\big)^3 , \qquad h = \tfrac{1}{b}\log\big(3K/u\big) = (1+o(1))\tfrac{1}{b}\log\big(1/u\big) ,$$

which in particular both go to $+\infty$ as $u \downarrow 0$. Using that $\mathrm{P}(Z > t) \sim \frac{1}{t\sqrt{2\pi}} \mathrm{e}^{-t^2/2}$ as $t \to \infty$, we obtain

$$\mathrm{P}\big(Z + h \in [0,b]\big) \sim \frac{1}{h\sqrt{2\pi}} \mathrm{e}^{-(h-b)^2/2}, \quad \mathrm{P}\big(Z + h < 0\big) \sim \frac{1}{h\sqrt{2\pi}} \mathrm{e}^{-h^2/2},$$

so when u is sufficiently small we obtain

$$u\mathrm{P}\big(Z + h \in [0,b]\big) - K\mathrm{P}\big(Z + h < 0\big) \geq \frac{1}{h\sqrt{2\pi}} \mathrm{e}^{-h^2/2}\big(u\mathrm{e}^{bh} - 2K\big) \geq \frac{K}{h\sqrt{2\pi}} \mathrm{e}^{-h^2/2},$$

where we used the definition of h for the last equality.

With the above choices of h and K, we thus obtain

$$\mathtt{F}^+(u) \geq \frac{K}{h\sqrt{2\pi}} \mathrm{e}^{-h^2/2} - \mathrm{e}^{-K} = \mathrm{e}^{-(1+o(1))\frac{1}{2b^2}\log(1/u)^2} - \mathrm{e}^{-\log(1/h)^3},$$

which gives the desired lower bound, the second term being negligible compared to the first one. □

Sketch of proof of the upper bound in Theorem 2.24. The upper bound in Theorem 2.24 is more technical and is treated in detail in Section 3 of [GL18]: after another reduction of the problem, it also relies on Gaussian calculations similar to the conclusion above. We summarize here the main steps of the proof, and refer to [GL18] for details. The goal is simply to bound $\mathtt{F}_K(u)$ for $K > 0$.

Step 1. Reduction to a sub-grid. We consider a large (but fixed) size L, then we use Hölder's inequality to reduce to partitions function where $\sum_{x\in\Lambda_N} \vartheta_x$ is replaced by a sum where the points lie on a sub-grid spaced by L. More precisely,

$$Z_{N,u}^+ \leq \prod_{v\in\{1,\dots,L\}^d} \mathrm{E}_N^0\Big[\exp\Big(L^d \sum_{x\in\Lambda_{N,L}^{(v)}} \big(\mathbf{1}_{\{\varphi_x\in[0,a]\}} - K\mathbf{1}_{\{\varphi_x<0\}}\big)\Big)\Big]^{1/L^d},$$

where $\Lambda_{N,L}^{(v)} = \Lambda_N \cap \{v + L\mathbb{Z}^d\}$ is a sub-grid of Λ_N and verify that $\Lambda_N = \bigcup_v \Lambda_{N,L}^{(v)}$.

Step 2. Reducing to a single cell of width L. We then use the spatial Markov property to decouple points at distance L. We consider a set Γ that separates the points of $\Lambda_{N,L}^{(v)} = \Lambda_N \cap \{v + L\mathbb{Z}^d\}$ into "cells", so that conditionally on this set Γ the values of φ_x for $x \in \Lambda_{N,L}^{(v)}$ are independent. We are then reduced to estimating expectations of the type

$$\mathrm{E}^{\phi}_{\Lambda_L^{(v)}}\Big[\exp\Big(L^d\big(u\mathbf{1}_{\{\varphi_{x_0}\in[0,a]\}}-K\mathbf{1}_{\{\varphi_{x_0}<0\}}\big)\Big)\Big],$$

where x_0 is the center of the box $\Lambda_L^{(v)}$ and ϕ is the boundary condition on each "cell" of All together, we obtain something of the form

$$Z^+_{N,u,K}\lesssim\bigg(\sup_{\phi}\mathrm{E}^{\phi}_{\Lambda_L}\Big[\exp\Big(L^d\big(u\mathbf{1}_{\{\varphi_{x_0}\in[0,a]\}}-K\mathbf{1}_{\{\varphi_{x_0}<0\}}\big)\Big)\Big]\bigg)^{N^d/L^d},$$

where the supremum is over all possible boundary conditions ϕ.

Step 3. Estimating the worst boundary condition With boundary condition ϕ, we have that φ_{x_0} has mean $m^{\phi}_{x_0}$ and variance σ_L^2, with $\sigma_L^2\to G_0$ as $L\to\infty$. Taking the worst boundary condition therefore simply corresponds to considering

$$\begin{aligned}&\sup_{h\in\mathbb{R}}\mathrm{E}\Big[\exp\Big(L^d\big(u\mathbf{1}_{\{\sigma_L Z+h\in[0,a]\}}-K\mathbf{1}_{\{\sigma_L Z+h<0\}}\big)\Big)\Big]\\&=\sup_{h\in\mathbb{R}}\mathrm{E}\Big[\exp\Big(L^d\big(u\mathbf{1}_{\{Z+h\in[0,b_L]\}}-K\mathbf{1}_{\{Z+h<0\}}\big)\Big)\Big],\end{aligned}$$

where we have set $b_L:=a/\sigma_L$, with $b_L\to a/\sqrt{G_0}$ as $L\to\infty$. Then, using explicit Gaussian calculations (that we skip here but are detailed in [GL18, p. 587]: the supremum is attained for h close to $\frac{b_L}{2}+\frac{1}{b_L}\log(\frac{1}{uL^d})$), we get that the above is bounded by $1+\exp(-(1+o(1))\frac{b_L^2}{2}\log(1/u)^2)$.

Conclusion All together, we get an upper bound of the type

$$Z^+_{N,u,K}\le\Big(1+\exp\Big(-(1+o(1))\frac{b_L^2}{2}\log(1/u)^2\Big)\Big)^{N^d/L^d}\le \mathrm{e}^{\frac{N^d}{L^d}\exp(-(1+o(1))\frac{b_L^2}{2}\log(1/u)^2)},$$

which gives that $\mathtt{F}^+_K(u)\le\frac{1}{L^d}\exp(-(1+o(1))\frac{b_L^2}{2}\log(1/u)^2$. This concludes the (sketch of the) proof. □

2.5 A few exercises

Exercise 18 (Pinning of the GFF in dimension $d=2$)**.** In this exercise we improve the lower bound (2.20), to obtain

$$\mathtt{F}^0(u)\ge(1+o(1))\frac{au}{\sqrt{\log(1/u)}}\qquad\text{as } u\downarrow 0\,. \tag{2.26}$$

Let us introduce

$$\check{Z}_{N,u} := \inf_{\xi:\mathbb{Z}^d\to[-1,1]} Z^{\xi}_{N,u}\,,$$

where the infimum is taken over boundary conditions with values in $[-1,1]$.

1. Show that, for any $u \geq 0$, for all $N \geq 1$

$$\check{Z}_{2N,u} \geq (\check{Z}_{N,u})^{2^d} \inf_{\xi:\mathbb{Z}^d\to[-1,1]} \mathrm{P}^{\xi}_N\big(\varphi_x \in [-1,1]\,,\ \forall\, x \in \Gamma\big)$$

   where $\Gamma = \Lambda_{2N} \setminus (\bigcup_{v\in\{0,1\}^d} \Lambda^{(v)}_N)$ (as seen in the proof of Lemma 2.8).

2. Using Lemma 1.36 (and also Remark 1.37), show that there exists a constant $c > 0$ such that

$$\frac{1}{(2N)^d} \log \check{Z}_{2N,u} \geq \frac{1}{N^d} \log \check{Z}_{N,u} - cN^{-1}(\log N)^2\,.$$

   Deduce that, if N is sufficiently large, the sequence

$$\Big(\frac{1}{(2^k N)^d} \log \check{Z}_{2^k N,u} - 3c(2^k N)^{-1}(\log 2^k N)^2\Big)_{k\geq 0}$$

   is non-decreasing.

3. Conclude that, for any N sufficiently large, we have

$$\mathtt{F}^0(u) \geq \frac{1}{N^d} \log \check{Z}_{N,u} - 3cN^{-1}(\log N)^2\,.$$

4. Show that if ξ takes values in $[-1,1]$, then

$$\mathrm{P}^{\xi}_N\big(\varphi_x \in [-a,a]\big) \geq \mathrm{P}\big(\sigma_N(x) Z \in [1-a, 1+a]\big)$$

   where $Z \sim \mathcal{N}(0,1)$. Using the fact that $\max_{x\in\Lambda_N} \sigma^2_N(x) = (1+o(1))\frac{2}{\pi}\log N$, show that for any fixed $\eta > 0$, for N sufficiently large

$$\inf_{\xi:\mathbb{Z}^d\to[-1,1]} \mathrm{E}^{\xi}_N\Big[\sum_{x\in\Lambda_N} \vartheta_x\Big] \geq (1-\eta) N^d \frac{a}{\sqrt{\log N}}\,.$$

5. Reusing the proof of the lower bound of (2.20), show that for all N sufficiently large we have

$$\mathtt{F}^0(u) \geq (1-\eta)\frac{au}{\sqrt{\log N}} - 3cN^{-1}(\log N)^2\,,$$

6. Applying this inequality for some appropriate $N = N(u)$, deduce the lower bound (2.26).

Exercise 19 (The co-membrane model)**.** Let us consider the following measure, in dimension $d \geq 2$:

$$\frac{\mathrm{dP}^{\mathrm{co}}_{N,u}}{\mathrm{dP}^0_{\Lambda_N}} = \frac{1}{Z^{\mathrm{co}}_{N,u}} \exp\Big(u \sum_{x\in\Lambda_N} \Delta_x \Big) , \quad \text{where } \Delta_x = \mathbf{1}_{\{\varphi_x > 0\}} ,$$

and $Z^{\mathrm{co}}_{N,u}$ is the partition function of the model. For A an event, we set

$$Z^{\mathrm{co}}_{N,u}(A) = \mathbf{E}^0_{\Lambda_N}\Big[\exp\Big(u \sum_{x\in\Lambda_N} \Delta_x \Big) \mathbf{1}_A \Big] ,$$

so that $\mathrm{P}^{\mathrm{co}}_{N,u}(A) = Z^{\mathrm{co}}_{N,u}(A)/Z^{\mathrm{co}}_{N,u}$. We denote again $\Omega_N^+ := \{\varphi_x > 0 \,\forall\, x \in \Lambda_N\}$.

1. Show that for all $u \geq 0$, we have $\mathrm{e}^{uN^d} \mathrm{P}^0_{\Lambda_N}(\Omega_N^+) \leq Z^{\mathrm{co}}_{N,u} \leq \mathrm{e}^{uN^d}$.
2. Show that for all $u \leq 0$, we have $\mathrm{P}^0_{\Lambda_N}(\Omega_N^+) \leq Z^{\mathrm{co}}_{N,u} \leq 1$ for all $u \leq 0$.
3. Conclude that
$$\mathtt{F}^{\mathrm{co}}(u) := \lim_{N\to\infty} \frac{1}{N^d} \log Z^{\mathrm{co}}_{N,u} = \max(0,u) .$$
4. Let $u \neq 0$ and set $\alpha_N := -\frac{2}{|u|} \log \mathrm{P}^0_{\Lambda_N}(\Omega_N^+)$. Recall the behavior of α_N.
   (a) If $u < 0$, then setting $A_N := \big\{ \sum_{x\in\Lambda_N} \Delta_x \geq \alpha_N \big\}$, show that
   $$Z^{\mathrm{co}}_{N,u}(A_N) \leq \mathrm{e}^{u\alpha_N} \leq \mathrm{e}^{-|u|\alpha_N} Z^{\mathrm{co}}_{N,u} .$$
   Conclude that $\lim_{N\to\infty} \mathbf{P}^{\mathrm{co}}_{N,h}(A_N) = 0$.
   (b) If $u > 0$, then setting $B_N := \big\{ \sum_{x\in\Lambda_N} (1-\Delta_x) \geq \alpha_N \big\}$, show that
   $$Z^{\mathrm{co}}_{N,h}(B_N) \leq \mathrm{e}^{uN^d - u\alpha_N} \leq \mathrm{e}^{-|u|\alpha_N} Z^{\mathrm{co}}_{N,u} .$$
   Conclude that $\lim_{N\to\infty} \mathbf{P}^{\mathrm{co}}_{N,h}(B_N) = 0$.

Part II

The (polymer) pinning model

Chapter 3

Pinning on a defect line: overview of the phase diagram

The pinning model that we consider in the following chapters can be used to represent a polymer which interacts with a one-dimensional line of defects. It has a long history and in fact different origins: it was introduced by Poland and Scheraga [PS70] to describe the DNA denaturation transition; it was also studied by Fisher [Fis84] as a model of pinning of a random walk on a hard wall (and is related to the wetting model introduced in the previous chapter).

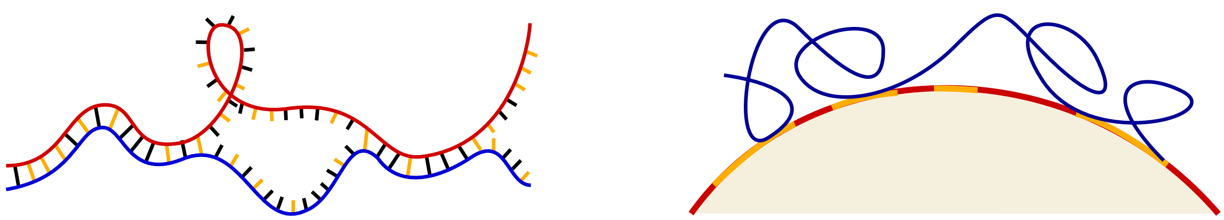


Figure 3.1: Two examples of physical/bio-physical situations described by the pinning model: DNA denaturation (left) and pinning of a protein on a cell (right).

The model has been intensively studied in the past decades from a mathematical point of view, in particular regarding the question of *disorder relevance*. Some excellent references are the books [Gia07, Gia11] by Giacomin, which give a nice overview of the research activity around these models.

3.1 Introduction of the (disordered) pinning model

We directly introduce the model in a general framework, where the interactions between the polymer and the defect line can be inhomogenous, *i.e.* we see the pinning model as an instance of a *disordered system*. The notation will be slightly different compared to the previous chapters (in particular concerning the parameters of the model), in order to agree with the vast majority of the literature.

Let $(S_i)_{i\geq 0}$ be a Markov chain on $\mathbb{Z}^d$ (for some $d \geq 1$), starting from $S_0 = 0$; we denote by P its law. For $n \in \mathbb{N}$, we consider the trajectory $(i, S_i)_{1\leq i\leq n}$, which may be interpreted as a directed polymer (see Figure 3.2 for an illustration). Then, the polymer interacts with a defect line $\mathbb{N} \times \{0\}$ when it touches it, that is when $S_i = 0$. Since interactions only occur when $S_i = 0$, we can directly consider the set of return times $\tau = \{i, S_i = 0\}$, which is what is called a *renewal process*: $\tau_0 = 0$, and $(\tau_k - \tau_{k-1})_{k\geq 1}$ are i.i.d. $\mathbb{N}$-valued random variables. The renewal process $\tau = (\tau_k)_{k\geq 0}$ will be our reference model, and its law is denoted by P.

With a slight abuse of notation, we may also view $\tau = \{\tau_k\}_{k\geq 0}$ as a set of points in $\mathbb{N}$, called *renewal points* (and representing the contact points of the polymer with the defect line). In particular, we denote by $\{n \in \tau\}$ the event that n is a renewal point, *i.e.* that there exists $k \geq 0$ such that $\tau_k = n$. Let us now introduce the notation:

$$\text{for } n \geq 1, \quad K(n) := \mathrm{P}(\tau_1 = n), \qquad K(\infty) = \mathrm{P}(\tau_1 = +\infty),$$

which we refer to as the *inter-arrival distribution*. We say that the renewal process τ is

- *persistent* (or recurrent) if $K(\infty) = 0$ — in particular $|\tau| = +\infty$ almost surely;
- *finite* (or transient) if $K(\infty) > 0$ — in particular $|\tau| < +\infty$ follows a geometric law with parameter $K(\infty)$.

3.1.1 The homogeneous pinning model

The homogeneous pinning model consists in modifying the law of the renewal process τ by considering the following Gibbs measure, analogously to (2.14): for a length $N \geq 1$ and for $h \in \mathbb{R}$ (the pinning parameter), we define

$$\frac{\mathrm{dP}_{N,h}}{\mathrm{dP}}(\tau) := \frac{1}{Z_{N,h}} \exp\Big(h \sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big) \mathbf{1}_{\{N\in\tau\}} . \tag{3.1}$$

Recall that we have introduced a pinning model of one-dimensional interfaces in Section 2.2. The definition (3.1) in fact includes both the unconstrained case and the constrained case considered in Section 2.2: it is enough to consider as the underlying renewal process the process τ defined in (2.15) or the process τ^+ defined in (2.17).

In all the following, we make the following assumption about the renewal process, which is the one usually made in the literature.

Assumption. *There exist $\alpha \geq 0$ and a slowly varying function $L(\cdot)$ (see Remark 3.1 below) such that, for $n \geq 1$:*

$$K(n) := \mathbf{P}(\tau_1 = n) = L(n) n^{-(1+\alpha)} . \tag{$*$}$$

The assumption ($*$) is satisfied for instance if $\tau = \{n, S_{2n} = 0\}$ where $(S_n)_{n\geq 0}$ is the simple random walk on $\mathbb{Z}^d$: one has

- if $d = 1$, then $\alpha = \frac{1}{2}$ and $\lim_{n\to\infty} L(n) = \frac{1}{2\sqrt{\pi}}$ (see for instance [Fel66, Ch. III]);
- if $d = 2$, then $\alpha = 0$ and $L(n) \sim \frac{\pi}{(\log n)^2}$ (cf. [JP72, Thm. 4]);
- if $d \geq 3$, then $\alpha = \frac{d}{2} - 1$ and $\lim_{n\to\infty} L(n) = c_d$ (cf. [DK11, Thm. 4]).

We will often consider the slightly simpler framework where the slowly varying function converges to a constant, *i.e.* there is a constant c_1 such that

$$K(n) := \mathbf{P}(\tau_1 = n) \sim c_1\, n^{-(1+\alpha)} \qquad \text{as } n \to \infty . \tag{$\hat{*}$}$$

Remark 3.1 (Slowly-varying functions)**.** A slowly varying function $L(\cdot)$ is a function that satisfies $\lim_{n\to\infty} \frac{L(tn)}{L(n)} = 1$ for all $t > 0$; a standard example is a poly-logarithmic function, $L(n) \sim c\,(\log n)^\kappa$ as $n \to \infty$. A complete reference for slowly varying functions is [BGT89]. A summary of useful properties can be found in [Gia07, App. A.4], but let us give a few ones:

- for all $0 < a < b < \infty$, the convergence $\lim_{n\to\infty} \frac{L(tn)}{L(n)} = 1$ is uniform in $t \in [a, b]$;
- $L(n) = n^{o(1)}$ as $n \to \infty$;
- $\sum_{n=1}^{x} n^{\gamma-1} L(n) \overset{x\to\infty}{\sim} \frac{1}{\gamma} n^\gamma L(x)$ if $\gamma > 0$ and $\sum_{n=x}^{+\infty} n^{\gamma-1} L(n) \overset{x\to\infty}{\sim} \frac{-1}{\gamma} n^\gamma L(x)$ if $\gamma < 0$.

3.1.2 The disordered pinning model

We now defined a disordered version of the pinning model. Consider a sequence $\omega = (\omega_i)_{i\geq 0}$ of i.i.d. random variables, with law denoted by $\mathbb{P}$. The variables ω_i represent the inhomogeneities along the defect line (or the polymer) and will be thought as perturbations of the interactions between the polymer and the defect line, see Figure 3.2. We assume in all the following that $\mathbb{E}[\omega_i] = 0$, $\mathbb{E}[\omega_i^2] = 1$.

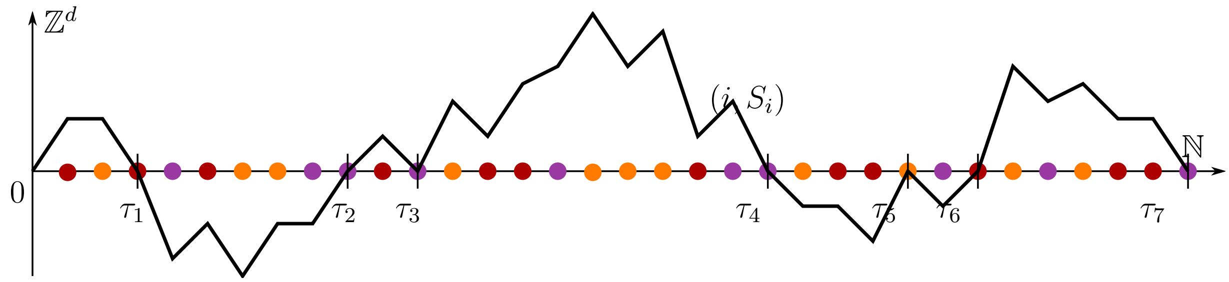


Figure 3.2: The polymer trajectory is represented by the sequence (i, S_i), which represents the position of the i-th monomer. Interactions take place along the defect line, at sites where S_i returns to 0, that is at times $\tau_1, \tau_2, \dots$ The defect line (or the polymer, or both) is inhomogeneous, represented by random variables $(\omega_i)_{i\geq 0}$ attached to the different sites.

For a given realization of ω (this is called *quenched* or *frozen* disorder) and for $\beta \geq 0$ (the inverse temperature or intensity of disorder), $h \in \mathbb{R}$ (the homogeneous pinning parameter), we define for $N \geq 1$ the *disordered pinning measure* $\mathrm{P}_{N,h}^{\beta,\omega}$ as a Gibbs measure with respect to the reference law P of τ:

$$\frac{\mathrm{dP}_{N,h}^{\beta,\omega}}{\mathrm{dP}}(\tau) := \frac{1}{Z_{N,h}^{\beta,\omega}} \exp\Big(\sum_{i=1}^{n}(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}\Big)\mathbf{1}_{\{N\in\tau\}}\,. \tag{3.2}$$

Here, the partition function of the model is given by

$$Z_{N,h}^{\beta,\omega} := \mathrm{E}\Big[\exp\Big(\sum_{i=1}^{N}(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}\Big)\mathbf{1}_{\{N\in\tau\}}\Big]\,. \tag{3.3}$$

The measure $\mathrm{P}_{N,\beta,h}^{\omega}$ corresponds to giving a reward $h+\beta\omega_i$ (or a penalty depending on the sign) if the polymer touches the defect line at site i. One can thus view the disordered model as a *random perturbation* of the homogeneous pinning model; note that if $\beta = 0$, one indeed recovers the homogeneous model. Note that $\mathrm{P}_{N,\beta,h}^{\omega}$ and $Z_{N,h}^{\beta,\omega}$ depend on the realization of the disorder ω, in fact, $\mathrm{P}_{N,\beta,h}^{\omega}$ is a random measure and $Z_{N,h}^{\beta,\omega}$ is a random variable.

Notice that we have added in (3.2)-(3.3) the indicator function $N \in \tau$, forcing the endpoint of the polymer to be pinned; this corresponds to having a zero boundary condition. One can also remove this constraint and consider the *free* model:

$$\frac{\mathrm{d}\tilde{\mathrm{P}}_{N,h}^{\beta,\omega}}{\mathrm{dP}}(\tau) := \frac{1}{Z_{N,h}^{\beta,\omega}} \exp\Big(\sum_{i=1}^{n}(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}\Big)\,, \tag{3.4}$$

with partition function

$$\tilde{Z}_{N,h}^{\beta,\omega} := \mathrm{E}\Big[\exp\Big(\sum_{i=1}^{N}(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}\Big)\Big]. \tag{3.5}$$

We will see later that this free model has properties close to the original model (3.2)-(3.3), see in particular Remark 3.13 below.

Remark 3.2. An important observation is that, by a translation of the parameter h, one can always reduce to the case where $K(\infty)=0$, that is to the case where τ is *persistent*. Indeed, if we set $\hat{h} = h - \log \mathrm{P}(\tau_1 < +\infty)$, one can write the partition function by decomposing it according to the number and positions of the renewals points:

$$\begin{aligned} Z_{N,\hat{h}} &= \sum_{k=1}^{N} \sum_{t_0=0<t_1<\cdots<t_k=N} \prod_{j=1}^{k} \mathrm{e}^{\hat{h}+\beta\omega_{t_j}} K(t_j - t_{j-1}) \\ &= \sum_{k=1}^{N} \sum_{t_0=0<t_1<\cdots<t_k=N} \prod_{j=1}^{k} \mathrm{e}^{h+\beta\omega_{t_j}} \hat{K}(t_j - t_{j-1}) = \hat{\mathrm{E}}\Big[\mathrm{e}^{\sum_{i=1}^{N}(h+\beta\omega_i)\mathbf{1}_{\{i\in\hat{\tau}\}}}\mathbf{1}_{\{N\in\hat{\tau}\}}\Big], \end{aligned}$$

where we have set $\hat{K}(n) := K(n)/\mathrm{P}(\tau_1 < +\infty) = \mathrm{P}(\tau_1 = n \mid \tau_1 < +\infty)$, which is the inter-arrival law of a renewal process $\hat{\tau}$. Note that the condition $(*)$ remains satisfied (with $\hat{L}(n) = L(n)/\mathrm{P}(\tau_1 < +\infty)$) and that $\hat{K}$ satisfies $\sum_{n=1}^{\infty}\hat{K}(n) = 1$, *i.e.* $\hat{K}(\infty) = 0$, so that the renewal is *persistent*.

3.1.3 The question of the influence of disorder

The aim of this chapter and the next one is to study the question of disorder relevance, which amounts to comparing the behaviors of the homogeneous and disordered pinning models. More precisely, we will try to answer the following questions (in that order):

- Can one describe precisely the phase transition of the homogeneous model?
- Does the disordered model also exhibit a phase transition?
- What can be said about the phase transition of the disordered model? In particular:
  - Can we estimate the critical point?
  - Can we estimate the behavior in the neighborhood of the critical point?

– Does the phase transition of the disordered model have different features from those of the homogeneous one?

These questions are at the core of the notion of *disorder relevance*. Disorder is said to be *relevant* if an arbitrarily weak intensity of disorder (represented in our case by the parameter β) modifies the critical behavior of the model. Disorder is said to be *irrelevant* if, provided that the intensity of disorder is small enough, the phase transition of the disordered model has the same features as the one of the homogeneous model.

3.2 Solving the homogeneous model

As already seen in Chapter 2, the free energy is an important physical quantity that encodes key properties of the model.

Proposition 3.3 (Existence of the free energy)**.** *The free energy*

$$\mathtt{F}(h) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h} = \lim_{N\to\infty} \frac{1}{N} \log \tilde{Z}_{N,h}$$

exists, is positive, and satisfies $\mathtt{F}(h) = 0$ *if* $h \le 0$*. Moreover,* $h \mapsto \mathtt{F}(h)$ *is increasing and convex and at every point where* $\mathtt{F}$ *is differentiable, one has*

$$\frac{\partial}{\partial h}\mathtt{F}(h) = \lim_{N\to\infty} \mathrm{E}_{N,h}\Big[\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big] = \lim_{N\to\infty} \tilde{\mathrm{E}}_{N,h}\Big[\frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big] .$$

This proposition is the exact analogue of Proposition 2.2 and the proof is identical. In particular, the existence of the free energy is due to the super-additivity of $\log Z_{N,h}$ (and to Fekete's Lemma).

The only point that deserves a comment is the fact that the limit is the same if one replaces the partition function $Z_{N,h}$ by its *free* version $\tilde{Z}_{N,h}$ (for which one does not have super-additivity). We leave this as an exercise.

Exercise 20. We assume that the inter-arrival law of the renewal process τ satisfies ($*$), that is $K(n) = \mathrm{P}(\tau_1 = n) = L(n)n^{-(1+\alpha)}$.

1. Show that $\tilde{Z}_{N,h} = Z_{N,h} + \sum_{k=1}^{N} Z_{N-k,h}\mathrm{P}(\tau_1 \ge k)$.
2. Show that there exists a constant C such that $\mathrm{P}(\tau_1 \ge k)/\mathrm{P}(\tau_1 = k) \le Ck^{2+\alpha}$ for all $k \ge 1$ (one may use the second item of Remark 3.1), show that

$$Z_{N,h} \le \tilde{Z}_{N,h} \le (1 + C\mathrm{e}^{-h}N^{2+\alpha})Z_{N,h} .$$

3. Conclude that $\mathtt{F}(h) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h} = \lim_{N\to\infty} \frac{1}{N} \log \tilde{Z}_{N,h}$.

3.2.1 The localization phase transition

Proposition 3.3 shows that one can define the critical point

$$h_c := \sup\left\{h \in \mathbb{R},\ \mathtt{F}(h) = 0\right\} = \inf\left\{h \in \mathbb{R},\ \mathtt{F}(h) > 0\right\}.$$

As remarked in Section 2.1.3 (see (2.7)), the critical point h_c marks a transition between a *delocalized* phase for $h < h_c$ (zero asymptotic contact density) and a *localized* phase for $h > h_c$ (strictly positive asymptotic contact density).

Exercise 21. Show that $0 \le h_c \le -\log(\sup_{n\ge 1} K(n))$.

In analogy with (2.16) (see also Proposition 2.12 and Lemma 2.14), one obtains the following characterization for the free energy.

Proposition 3.4 (Characterization of the free energy)**.** *The free energy* $\mathtt{F}(h)$ *is the solution in* λ *of*

$$\mathrm{E}\left[\mathrm{e}^{-\lambda\tau_1}\right] = \mathrm{e}^{-h}$$

if a solution exists, and $\mathtt{F}(h) = 0$ *otherwise.*

One deduces the following generalization of Theorem 2.11 (the generalization is relative to assumption ($*$) on the renewal process).

Theorem 3.5 (Critical point and critical behavior)**.** *One has*

$$h_c = -\log \mathrm{P}(\tau_1 < +\infty).$$

Additionally, under assumption ($*$) *and if* $\alpha > 0$, *there exists a slowly varying function* $\hat{L}(\cdot)$ *such that, as* $u \downarrow 0$,

$$\mathtt{F}(h_c + u) \sim \hat{L}(1/u)\, u^{\nu} \qquad \text{with} \qquad \nu := \max\left(\tfrac{1}{\alpha}, 1\right).$$

We can make the slowly varying function $\hat{L}(\cdot)$ *explicit in the following cases:*

- *If* $m_\tau := \mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 < +\infty\}}] < +\infty$, *then* $\mathtt{F}(h_c + u) \sim \dfrac{\mathrm{P}(\tau_1 < +\infty)}{m_\tau} u$ *as* $u \downarrow 0$.
- *If* ($\hat{*}$) *holds with* $\alpha \in (0,1)$, *then* $\mathtt{F}(h_c + u) \sim \left(\dfrac{\alpha \mathrm{P}(\tau_1 < +\infty)}{c_1 \Gamma(1-\alpha)}\right)^{1/\alpha} u^{1/\alpha}$ *as* $u \downarrow 0$.

If $\alpha = 0$, *then* $\mathtt{F}(h_c + u) = o(u^p)$ *as* $u \downarrow 0$, *for all* $p > 0$.

Let us emphasize that thanks to Remark 3.2, one can always reduce to the case $h_c = 0$ simply by changing the inter-arrival law $K(n)$ to $\hat{K}(n) = K(n)/\mathrm{P}(\tau_1 < +\infty)$, which amounts to a translation of the parameter h.

Proof of Theorem 3.5. The fact that $h_c = -\log \mathrm{P}(\tau_1 < +\infty)$ is simply related to the fact that $\lim_{\lambda\downarrow 0} \mathrm{E}[\mathrm{e}^{-\lambda\tau_1}] = \mathrm{P}(\tau_1 < +\infty)$, so that according to Proposition 3.4 one has $\mathtt{F}(h) > 0$ if and only if $\mathrm{e}^{-h} < \mathrm{P}(\tau_1 < +\infty)$.

For the critical behavior, the proof is similar to that of Theorem 2.11. The idea is to obtain the behavior of $\mathrm{E}[\mathrm{e}^{-\lambda\tau_1}]$ as $\lambda \downarrow 0$ and to use Proposition 3.4 to conclude, taking into account the fact that $\mathtt{F}(h) \downarrow 0$ as $h \downarrow h_c$. We use the following lemma.

Lemma 3.6 (Aboue the Laplace transform of τ_1)**.** *Under assumption* ($*$)*, there exists a slowly varying function* $\tilde{L}(\cdot)$ *such that*

$$\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda\tau_1}\big] \sim \tilde{L}(1/\lambda)\lambda^{\min(\alpha,1)} \qquad \text{as } \lambda \downarrow 0\,.$$

- *When* $m_\tau := \mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1<+\infty\}}] < +\infty$*, one can take* $\tilde{L}(x) \equiv m_\tau$*.*
- *If* $\alpha \in (0,1)$ *in* ($*$)*, then one can take* $\tilde{L}(x) = \frac{\Gamma(1-\alpha)}{\alpha} L(x)$*.*

With this lemma and Proposition 3.4, also using that $\mathrm{e}^{-h_c} = \mathrm{P}(\tau_1 < +\infty)$, since we have that $\mathtt{F}(h) \downarrow 0$ when $h \downarrow h_c$ (by continuity of $\mathtt{F}$), one obtains

$$\mathrm{e}^{-h_c}(h-h_c) \sim \mathrm{e}^{-h_c} - \mathrm{e}^{-h} = \mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\mathtt{F}(h)\tau_1}\big] \sim \tilde{L}(1/\mathtt{F}(h))\mathtt{F}(h)^{\min(\alpha,1)}$$

as $h \downarrow h_c$. By inverting this relation, one obtains Theorem 3.5 (the slowly varying function $\hat{L}(\cdot)$ is related to the function $L(\cdot)$ and to the exponent $\min(\alpha, 1)$, see [BGT89, Thm. 1.5.13]). In particular:

- If $m_\tau < +\infty$, one has $\mathrm{e}^{-h_c}(h-h_c) \sim m_\tau \mathtt{F}(h)$, which gives the desired result;
- If ($\hat{*}$) holds with $\alpha \in (0,1)$, then one has $\mathrm{e}^{-h_c}(h-h_c) \sim c_1 \frac{\Gamma(1-\alpha)}{\alpha} \mathtt{F}(h)^{1/\alpha}$, which gives the desired result. □

Proof of Lemma 3.6. We start by writing $\mathrm{P}(\tau_1 = +\infty) = \sum_{n=1}^{\infty} \mathrm{P}(\tau_1 = n)$, so that

$$\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda\tau_1}\big] = \sum_{n=1}^{\infty} (1 - \mathrm{e}^{-\lambda_n})\mathrm{P}(\tau_1 = n)\,. \tag{3.6}$$

We focus on the case where $m_\tau < +\infty$ and on the case $\alpha \in (0,1)$; we leave the other cases ($\alpha = 1$ with $m_\tau = +\infty$ and $\alpha = 0$) as exercises (see after the proof).

Case $m_\tau < +\infty$. Note that for all n, the function $\lambda \mapsto \frac{1-\mathrm{e}^{-\lambda n}}{\lambda}$ is decreasing (and bounded from above by n) and that $\lim_{\lambda\downarrow 0} \frac{1-\mathrm{e}^{-\lambda n}}{\lambda} = n$. Therefore, by monotone (or dominated) convergence, one gets that

$$\lim_{\lambda\downarrow 0} \frac{1}{\lambda}\Big(\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda \tau_1}\big]\Big) = \sum_{n=1}^{\infty} n \mathrm{P}(\tau_1 = n) = m_\tau \,,$$

which gives the desired result.

Case $\alpha \in (0,1)$. In this case, let us write

$$\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda\tau_1}\big] = \sum_{n=1}^{\infty} \frac{1-\mathrm{e}^{-\lambda_n}}{n^{1+\alpha}} L(n) = \lambda^\alpha L(1/\lambda) \sum_{n=1}^{\infty} \lambda \frac{1-\mathrm{e}^{-\lambda_n}}{(\lambda n)^{1+\alpha}} \frac{L(n)}{L(1/\lambda)} \,.$$

In the case where $\lim_{x\to\infty} L(x) = c_1$ one has $\frac{L(n)}{L(1/\lambda)} \to 1$ if $n, 1/\lambda$ are large. It then suffices to use the convergence of the Riemann sum (then an integration by parts)

$$\lim_{\lambda\downarrow 0} \sum_{n=1}^{\infty} \lambda \frac{1-\mathrm{e}^{-\lambda_n}}{(\lambda n)^{1+\alpha}} = \int_0^\infty \frac{1-\mathrm{e}^{-x}}{x^{1+\alpha}} \mathrm{d}x = \frac{1}{\alpha} \int_0^\infty x^{-\alpha} \mathrm{e}^{-x} \mathrm{d}x = \frac{\Gamma(1-\alpha)}{\alpha} \,.$$

The case of a general slowly varying function $L(\cdot)$ is a bit more technical (see [BGT89, Cor. 8.1.7]): the idea is that the main contribution to the sum is for n of order $1/\lambda$ (one can control the sum for $n \le \varepsilon/\lambda$ and the sum for $n \ge 1/\varepsilon\lambda$). Then, for such $n \asymp 1/\lambda$ one gets $L(n)/L(1/\lambda) \to 1$ by definition of a slowly varying function, so one can use the same Riemann approximation argument. □

Exercise 22 (Case $\alpha = 1$, first version)**.** Assume that $\mathrm{P}(\tau_1 = n) \sim c_1 n^{-2}$ as $n \to \infty$, for some constant $c_1 > 0$. Starting from (3.6), we admit that, as $\lambda \downarrow 0$ one has

$$\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda\tau_1}\big] \sim c_1 \sum_{n=1}^{\infty} \frac{1-\mathrm{e}^{-\lambda n}}{n^2} \,.$$

1. Show that (we omit integer parts to simplify the notation)

$$\sum_{n=1}^{\infty} \frac{1-\mathrm{e}^{-\lambda n}}{n^2} \ge \sum_{n=1}^{1/\lambda} \Big(\frac{\lambda}{n} - \frac{1}{2}\lambda^2 \Big) \ge \lambda \Big(\log \frac{1}{\lambda} - \frac{1}{2} \Big) \,.$$

2. Show that
$$\sum_{n=1}^{\infty} \frac{1-\mathrm{e}^{-\lambda n}}{n^2} \le \sum_{n=1}^{1/\lambda} \frac{\lambda}{n} + \sum_{n=1/\lambda+1}^{\infty} \frac{1}{n^2} \le \lambda \log \frac{1}{\lambda} + 1 + \lambda\,.$$
3. Deduce that, as $\lambda \downarrow 0$,
$$\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda \tau_1}\big] \sim c_1 \lambda \log \frac{1}{\lambda}.$$
4. Deduce the behavior of $\mathtt{F}(h)$ as $h \downarrow h_c$ in this case.

Exercise 23 (Case $\alpha = 1$, general version)**.** Assume that $\mathrm{P}(\tau_1 = n) = L(n) n^{-2}$ with $m_\tau := \sum_{n=1}^{\infty} \frac{L(n)}{n} = +\infty$. We set
$$L_*(n) := \sum_{k=1}^{n} \frac{L(k)}{k},$$
which therefore satisfies $\lim_{n\to\infty} L_*(n) = +\infty$.

1. Show that for all $t > 0$ we have $\lim_{n\to\infty} \frac{L_*(n) - L_*(tn)}{L(n)} = \log \frac{1}{t}$.
2. Deduce that $\lim_{n\to\infty} \frac{L_*(n)}{L(n)} = +\infty$ and then that $L_*(\cdot)$ is slowly varying.
3. Show that, as $\lambda \downarrow 0$, one has $\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda \tau_1}\big] \sim \lambda L^*(1/\lambda)$.
   (One may use the third property of Remark 3.1.)

This therefore proves Lemma 3.6 in the case $\alpha = 1$, with $\tilde{L} = L_*$.

Exercise 24 (Case $\alpha = 0$)**.** Assume that $\mathrm{P}(\tau_1 = n) = L(n) n^{-1}$, and note that $\sum_{k=1}^{\infty} L(n) n^{-1} = \mathrm{P}(\tau_1 < +\infty) < +\infty$. We set
$$L^*(n) = \sum_{k=n}^{\infty} \frac{L(k)}{k}\,,$$
which therefore satisfies $\lim_{n\to\infty} L^*(n) = 0$.

1. As in the previous exercise, show that $\lim_{n\to\infty} \frac{L^*(n)}{L(n)} = +\infty$ and that $L^*(\cdot)$ is slowly varying.
2. Show that, as $\lambda \downarrow 0$, one has $\mathrm{P}(\tau_1 < +\infty) - \mathrm{E}\big[\mathrm{e}^{-\lambda \tau_1}\big] \sim L^*(1/\lambda)$.
   (One may use the third property of Remark 3.1.)

This therefore proves Lemma 3.6 in the case $\alpha = 0$, with $\tilde{L} = L^*$.

Exercise 25 (Pinning of the $d = 2$ random walk)**.** We recall that in the case of the simple random walk in dimension $d = 2$, we have that $\mathrm{P}(\tau_1 = n) \sim \frac{\pi}{n(\log n)^2}$. Since $\mathrm{P}(\tau_1 < \infty) = 1$ we have $h_c = 0$. Using the previous exercise, show that

$$\mathtt{F}(u) = \exp\Big(- \frac{\pi + o(1)}{u} \Big) \qquad \text{as} \quad u \downarrow 0 .$$

3.2.2 Properties of renewal trajectories under the pinning measure

Let us introduce, for $h \in \mathbb{R}$ the following inter-arrival distribution:

$$\forall\, n \in \mathbb{N} \qquad K_h(n) := \mathrm{e}^h K(n) \mathrm{e}^{-\mathtt{F}(h) n} . \tag{3.7}$$

We then consider a renewal process $\tau^{(h)} = (\tau_k^{(h)})_{k \geq 1}$ with law denoted $\mathrm{P}^{(h)}$, by setting its inter-arrival distribution as $\mathrm{P}^{(h)}(\tau_1^{(h)} = n) = K_h(n)$. Notice that, depending on h, $\tau^{(h)}$ may be persistent or finite:

$$\begin{aligned} &\text{if } h \geq h_c\,, \qquad \sum_{n=1}^{\infty} K_h(n) = \mathrm{e}^h \mathrm{E}[\mathrm{e}^{-\mathtt{F}(h)\tau_1}] = 1 \\ &\text{if } h < h_c\,, \qquad \sum_{n=1}^{\infty} K_h(n) = \mathrm{e}^h \mathrm{P}(\tau_1 < +\infty) < 1 \,. \end{aligned}$$

In particular, if $h \geq h_c$ then the renewal $\tau^{(h)}$ is persistent (recurrent), whereas if $h < h_c$ the renewal $\tau^{(h)}$ is finite (transient). We can then interpret the pinning measure $\mathrm{P}_{N,h}$ as follows.

Proposition 3.7. *We have* $Z_{N,h} = \mathrm{e}^{\mathtt{F}(h)N} \mathrm{P}^{(h)}(N \in \tau^{(h)})$ *for all* $N \in \mathbb{N}$*. Additionally, the pinning measure* $\mathrm{P}_{N,h}$ *can be described as follows:*

$$\mathrm{P}_{N,h}(\,\cdot\,) = \mathrm{P}^{(h)}(\,\cdot \mid N \in \tau^{(h)}) \,.$$

Proof. Let us start by proving the first point. By decomposing the partition function according to the number and positions of the renewal points, we have

$$\begin{aligned} Z_{N,h} &= \sum_{k=1}^{N} \sum_{t_0 = 0 < t_1 < \cdots < t_k = N} \prod_{j=1}^{k} \mathrm{e}^h K(t_j - t_{j-1}) \\ &= \mathrm{e}^{\mathtt{F}(h)N} \sum_{k=1}^{N} \sum_{t_0 = 0 < t_1 < \cdots < t_k = N} \prod_{j=1}^{k} K_h(t_j - t_{j-1}) \,, \end{aligned}$$

where we used the definition (3.7) of K_h (and noted that $\prod_{j=1}^k \mathrm{e}^{-\mathtt{F}(h)(t_j-t_{j-1})} = \mathrm{e}^{\mathtt{F}(h)N}$). We therefore get that $Z_{N,h} = \mathrm{e}^{\mathtt{F}(h)N}\mathrm{P}^{(h)}(N \in \tau^{(h)})$ by using the same decomposition of the event $\{N \in \tau^{(h)}\}$ according to the number and positions of the renewal points.

Let us now show the second point. For all $t_0 = 0 < t_1 < \cdots < t_k = N$, we have

$$\begin{aligned}\mathrm{P}_{N,h}\big(\tau_1 = t_1, \dots, \tau_k = t_k\big) &= \frac{1}{Z_{N,h}}\,\mathrm{e}^{kh}\prod_{j=1}^k K(t_j - t_{j-1}) \\ &= \frac{1}{\mathrm{P}^{(h)}(N \in \tau^{(h)})}\prod_{j=1}^k \mathrm{e}^h \mathrm{e}^{-\mathtt{F}(h)(t_j-t_{j-1})}K(t_j - t_{j-1})\,.\end{aligned}$$

where we have used that $Z_{N,h} = \mathrm{e}^{\mathtt{F}(h)N}\mathrm{P}^{(h)}(N \in \tau^{(h)})$ as seen above. Together with the definition (3.7) of $K_h(\cdot)$, this shows that

$$\mathrm{P}_{N,h}\big(\tau_1 = t_1, \dots, \tau_k = t_k\big) = \mathrm{P}^{(h)}\big(\tau_1 = t_1, \dots, \tau_k = t_k \mid N \in \tau^{(h)}\big)$$

and thus $\mathrm{P}_{N,h}(\tau = A) = \mathrm{P}^{(h)}(\tau^{(h)} = A \mid N \in \tau^{(h)})$ for every $A \subset \{0, \dots, N\}$. $\square$

We can then prove the following result.

Theorem 3.8 (Infinite volume limit). *We have the following infinite-volume limit for $\mathrm{P}_{N,h}$: for any $h \in \mathbb{R}$, as $N \to \infty$*

$$\mathrm{P}_{N,h}(\,\cdot\,) = \mathrm{P}^{(h)}(\,\cdot \mid N \in \tau^{(h)}) \Longrightarrow \mathrm{P}^{(h)}\,.$$

Let us stress that this statement hides three different regimes:

- If $h > h_c$, then $\tau^{(h)}$ is positive recurrent, $\mathrm{E}^{(h)}[\tau_1^{(h)}] < +\infty$. n Note also that $K_h(n)$ decays exponentially fast in that case. Exercise 27 below shows that the largest gap between renewal points under $P_{N,h}$ is asymptotic to $\frac{1}{\mathtt{F}(h)}\log N$.
- If $h = h_c$, then $\tau^{(h)}$ is recurrent. It is either positive or null recurrent, depending on whether $\sum_{n=1}^\infty nK(n)$ is finite or infinite. Note that $K_h(n)$ decays polynomially, see ($*$).
- If $h < h_c$, then $\tau^{(h)}$ is transient. Note that $K_h(n)$ decays polynomially, see ($*$).

Proof. Let $0 < t_1 < \ldots < t_k$ be fixed. For $N \geq t_k$, thanks to Proposition 3.7,

$$\begin{aligned} \mathrm{P}_{N,h}\big(\tau_1 = t_1, \ldots, \tau_k = t_k\big) &= \frac{1}{\mathrm{P}^{(h)}(N \in \tau^{(h)})} \mathrm{P}^{(h)}\big(\tau_1 = t_1, \ldots, \tau_k = t_k, N \in \tau^{(h)}\big) \\ &= \frac{\mathrm{P}^{(h)}(N - t_k \in \tau^{(h)})}{\mathrm{P}^{(h)}(N \in \tau^{(h)})} \mathrm{P}^{(h)}\big(\tau_1 = t_1, \ldots, \tau_k = t_k\big) . \end{aligned}$$

To conclude the proof, it is enough to show that for every $t \in \mathbb{N}$ we have

$$\lim_{N \to \infty} \frac{\mathrm{P}^{(h)}(N - t \in \tau^{(h)})}{\mathrm{P}^{(h)}(N \in \tau^{(h)})} = 1 , \tag{3.8}$$

which is the object of the next section which deals with properties of renewal processes. □

3.3 Intermezzo: some properties of renewal processes

We now give some useful estimates for renewal processes. We consider a renewal process $\tau = (\tau_k)_{k \geq 1}$, which we assume to be aperiodic, *i.e.* $\gcd\{n, K(n) > 0\} = 1$. An important quantity to study is the *renewal mass function*

$$u(n) := \mathrm{P}(n \in \tau) .$$

3.3.1 The positive recurrent case

The following theorem is useful in the case of a positive recurrent renewal process, *i.e.* when $\mathrm{E}[\tau_1] < +\infty$; it also gives a partial result in the null recurrent or transient cases, where $\mathrm{E}[\tau_1] = +\infty$. In particular, it proves (3.8) when $\tau_1^{(h)}$ is positive recurrent, that is when $h > h_c$, or $h = h_c$ with $\sum_{n \geq 1} nK(n) < +\infty$.

Theorem 3.9 (Standard renewal theorem)**.** *Let $\tau = (\tau_k)_{k \geq 0}$ be an aperiodic renewal process. Then*

$$\lim_{n \to \infty} \mathrm{P}(n \in \tau) = \frac{1}{\mathrm{E}[\tau_1]} .$$

Proof. Let us give a quick proof in the recurrent case; it is easy to see that in the transient case where $K(\infty) > 0$, one has $\lim_{n \to \infty} \mathrm{P}(n \in \tau) = 0$.

To every persistent renewal one can associate a Markov chain $(S_i)_{i \geq 0}$ on $\mathbb{Z}$ starting from 0 such that τ has the same law as the return times to 0 of $(S_i)_{i \geq 0}$, that is $\tau = \{i, S_i = 0\}$. For this, simply consider the Markov chain with transition matrix

$Q(0,x) = K(x-1)$ and $Q(x,x-1) = 1$ for $x \geq 1$ (draw a picture to understand). The chain is then recurrent if and only if $\mathrm{P}(\tau_1 < +\infty) = 1$ and is positive recurrent if and only if $\mathrm{E}[\tau_1] < +\infty$.

In the recurrent case, one easily computes the invariant measures of the Markov chain (unique up to a multiplicative factor): they are given $\mu(x) = \mu(0)\mathrm{P}(\tau_1 \geq x)$. Notice that μ have a finite mass if and only if $\sum_{x\geq 0} \mathrm{P}(\tau_1 \geq x) = \mathrm{E}[\tau_1] < +\infty$. In the positive recurrent case, since the chain is aperiodic $\mathrm{P}(n \in \tau) = \mathrm{P}(S_n = 0)$ converges to $\pi(0) = \frac{1}{\mathrm{E}[\tau_1]}$, where $\pi(x) = \frac{\mathrm{P}(\tau_1 \geq x)}{\mathrm{E}[\tau_1]}$ is the unique invariant probability. In the null recurrent case, $\mathrm{P}(n \in \tau) = \mathrm{P}(S_n = 0)$ converges to $0 = \frac{1}{\mathrm{E}[\tau_1]}$. □

3.3.2 The null recurrent case

Let us now assume that $\mathrm{P}(\tau_1 < +\infty) = 1$ but that $\mathrm{E}[\tau_1] = +\infty$. To estimate the renewal mass function $u(n) = \mathrm{P}(n \in \tau)$, we assume that the inter-arrival law satisfies ($*$) with $\alpha \in [0,1]$. The following result proves (3.8) when $\tau^{(h)}$ is null recurrent, that is when $h = h_c$ and $\sum_{n\geq 1} nK(n) = +\infty$ (notice that K_h satisfies ($*$) with $\alpha \in [0,1]$).

This is in fact a difficult result: the case $\alpha = 1$ is due to [Eri70] (notice the analogy with the renewal theorem above), the case $\alpha \in (0,1)$ to [Don97], the case $\alpha = 0$ to [Nag12, AB16].

Theorem 3.10 (Renewal theorem with infinite mean)**.** *Assume that τ is persistent, i.e.* $\mathrm{P}(\tau_1 < +\infty) = 1$, *and that* $\mathrm{P}(\tau_1 = n)$ *satisfies* ($*$) *with* $\alpha \in [0,1]$. *Then, as* $n \to \infty$

$$\mathrm{P}(n \in \tau) \sim \begin{cases} \dfrac{1}{\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq n\}}]} & \text{if } \alpha = 1\,; \\ \dfrac{\alpha \sin(\pi\alpha)}{\pi} L(n)^{-1} n^{-(1-\alpha)} & \text{if } \alpha \in (0,1); \\ \dfrac{1}{\mathrm{P}(\tau_1 \geq n)^2}\, \mathrm{P}(\tau_1 = n) & \text{if } \alpha = 0\,. \end{cases} \tag{3.9}$$

Let us stress that, in the case $\alpha = 1$, $\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq n\}}]$ is equal to $L_*(n) := \sum_{k=1}^n \frac{L(k)}{k}$, which is a slowly varying function that verifies $\lim_{n\to\infty} L_*(n)/L(n) = +\infty$, see Exercise 23, Q.2. Similarly, in the case $\alpha = 0$, $\mathrm{P}(\tau_1 \geq n)$ is equal to $L^*(n) := \sum_{k=n}^\infty \frac{L(k)}{k}$, which is a slowly varying function that verifies $\lim_{n\to\infty} L^*(n)/L(n) = +\infty$, see Exercise 24, Q.1.

3.3.3 The transient case

Let us finally treat the transient case, and assume that $K(\infty) = \mathrm{P}(\tau_1 = +\infty) > 0$. The following result allows proves (3.8) when $\tau^{(h)}$ is transient, that is when $h < h_c$ (notice that K_h satisfies $(*)$).

Theorem 3.11 (Transient renewal theorem)**.** *Assume that $K(\infty) > 0$ and that $(*)$ holds. Then, as $n \to \infty$,*

$$\mathrm{P}(n \in \tau) \sim \frac{1}{\mathrm{P}(\tau_1 = \infty)^2} \mathrm{P}(\tau_1 = n) \,. \tag{3.10}$$

This result is not that easy: we refer to [Gia07, Thm. A.4] for a proof. Note the analogy with the recurrent case with $\alpha = 0$ (also, this theorem admits a converse, see [AB16, Thm. 1.4]).

3.3.4 Properties of trajectories: a few exercises

Exercise 26. Let us denote by $H_N(\tau) := \sum_{i=1}^{N-1} \mathbf{1}_{\{i \in \tau\}}$ the total number of renewal points before N.

1. Assume that $h > h_c$. Show that for any $\varepsilon > 0$ we have that

$$\lim_{N\to\infty} \mathrm{P}_{N,h}\big(|N^{-1} H_N(\tau) - \mathtt{F}'(h)| > \varepsilon\big) = 0\,.$$

   Bonus: Show that $\mathrm{P}_{N,h}\big(|H_N(\tau) - \mathtt{F}'(h)N| > \varepsilon N\big) \le \mathrm{e}^{-c_\varepsilon N}$ *for some* $c_\varepsilon > 0$.

2. Assume that $h < h_c$. Show that

$$\forall\, k \ge 0 \qquad \lim_{N\to\infty} \mathrm{P}_{N,h}(H_N(\tau) = k) = (k+1) q_h^2 (1-q_h)^k\,,$$

   where q_h is a parameter to be determined.
   (The right-hand side is the law of the sum of two independent $\mathrm{Geom}_0(q_h)$ *r.v.)*

Exercise 27. Let us denote by $M_N := \max\{\tau_k - \tau_{k-1}, \tau_k \le N\}$ the largest "gap" in the renewal process before N, that is the size of the largest interval without a renewal point.

1. Assume that $h > h_c$.
   (a) Show that for any $\varepsilon > 0$, $\lim_{N\to\infty} \mathrm{P}_{N,h}(M_N > \frac{1+\varepsilon}{\mathtt{F}(h)} \log N) = 0$.
   (b) Show that for any $\varepsilon > 0$, $\lim_{N\to\infty} \mathrm{P}_{N,h}(M_N < \frac{1-\varepsilon}{\mathtt{F}(h)} \log N) = 0$.

2. Assume that $h < h_c$. Show that, if $a_N \to \infty$, then
$$\lim_{N\to\infty} \mathrm{P}_{N,h}(M_N < N - a_N) = 0\,.$$

Exercise 28. Denote again $H_N(\tau) := \sum_{i=1}^{N-1} \mathbf{1}_{\{i\in\tau\}}$ and assume that $h = h_c$.

1. Suppose that $m_c := \sum_{n=1}^{\infty} nK_h(n) < +\infty$. Show that, for any $\varepsilon > 0$, we have that $\lim_{N\to\infty} \mathrm{P}_{N,h}\big(|N^{-1}H_N(\tau) - m_c| > \varepsilon\big) = 0$.
2. Suppose that $\alpha \in (0,1)$ in (∗).
   (a) Show that, if $a_N \to \infty$, then $\lim_{N\to\infty} \mathrm{P}_{N,h}\big(H_N(\tau) \geq a_N N^\alpha / L(N)\big) = 0$.
   (b) Show that, if $\varepsilon_N \downarrow 0$, then $\lim_{N\to\infty} \mathrm{P}_{N,h}\big(H_N(\tau) \leq \varepsilon_N N^\alpha / L(N)\big) = 0$.
   (This is more difficult, see e.g. (4.7).)

3.4 The disordered model: free energy and phase transition

From now on and in view of Remark 3.2, we will assume, in order to simplify some statements, that the renewal τ is persistent, that is $\mathrm{P}(\tau_1 < +\infty) = 1$. This implies in particular that the critical point of the homogeneous model is $h_c = 0$.

3.4.1 The free energy: existence and first properties

Let us start by defining and proving the existence of the free energy, which has a so-called self-averaging property.

Theorem 3.12 (Existence of the quenched free energy)**.** *The free energy is defined as the limit*
$$\mathtt{F}(\beta,h) = \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h}^{\beta,\omega}\,. \tag{3.11}$$
This limit exists $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$ and is constant $\mathbb{P}$-a.s., and we have
$$\mathtt{F}(\beta,h) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big] = \sup_{N\geq 1} \frac{1}{N}\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big]\,.$$

Remark 3.13. One can also define the free energy as in (3.11), by replacing the partition function with its *free* version $\tilde Z_{N,h}^{\beta,\omega}$, see (3.5). Indeed, it is enough to see that, by adapting Exercise 20, one has
$$Z_{N,h}^{\beta,\omega} \leq \tilde Z_{N,h}^{\beta,\omega} \leq (1 + Ce^{-(h+\beta\omega_N)} N^{2+\alpha}) Z_{N,h}^{\beta,\omega}\,. \tag{3.12}$$

This then shows that $\mathtt{F}(\beta,h) = \lim_{N\to\infty} \frac{1}{N} \log \tilde Z_{N,h}^{\beta,\omega} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log \tilde Z_{N,h}^{\beta,\omega}$. However, the last identity of Theorem 3.12 is *not* satisfied for the free partition function $\tilde Z_{N,h}^{\beta,\omega}$. Using (3.12), one has instead that there is a constant $c > 0$ (which depends on h, β) such that

$$\forall\, N \in \mathbb{N} \qquad \mathtt{F}(\beta,h) \geq \frac{1}{N} \mathbb{E}\big[\log \tilde Z_{N,h}^{\beta,\omega} \big] - c\, \frac{\log N}{N} \,. \tag{3.13}$$

Some properties

Before proving Theorem 3.12, let us give some important properties.

Proposition 3.14 (Properties of the free energy)**.** *The free energy satisfies the following:*
(i) for all $\beta \geq 0$, $h \in \mathbb{R}$, one has $\mathtt{F}(\beta,h) \in [0,+\infty)$;
(ii) the functions $h \mapsto \mathtt{F}(\beta,h)$ and $\beta \mapsto \mathtt{F}(\beta,h)$ are convex and increasing.
(iii) the function $(\beta,h) \mapsto \mathtt{F}(\beta,h)$ is convex;

Proof. The computations are quite similar to those of Proposition 2.2, but give useful estimates.

(i). For the lower bound, by introducing the indicator $\mathbf{1}_{\{\tau_1 = N\}}$, one gets

$$Z_{N,h}^{\beta,\omega} \geq \mathrm{e}^{h+\beta\omega_N} \mathrm{P}(\tau_1 = N) \,.$$

Using property $(*)$ for $\mathrm{P}(\tau_1 = N)$, taking the log, dividing by N and letting $N \to \infty$ gives $\mathtt{F}(\beta,h) \geq 0$.

For the upper bound, by bounding $(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}$ by $(h+\beta\omega_i)^+$, one gets

$$Z_{N,h}^{\beta,\omega} \leq \mathrm{e}^{\sum_{i=1}^N (h+\beta\omega_i)^+} \mathrm{P}(N \in \tau) \,.$$

Using the law of large numbers and the fact that $\lim_{N\to\infty} \frac{1}{N} \log \mathrm{P}(N\in\tau) = 0$, see Section 3.3, one gets $\mathtt{F}(\beta,h) \leq \mathbb{E}[(h+\beta\omega_1)^+] < +\infty$.

(ii). Set, for $N \geq 1$,

$$\mathtt{F}_N^\omega(\beta,h) := \log Z_{N,h}^{\beta,\omega} = \log \mathrm{E}\Big[\exp\Big(\sum_{i=1}^N (h+\beta\omega_i) \mathbf{1}_{\{i\in\tau\}} \Big) \mathbf{1}_{\{N\in\tau\}} \Big] \,.$$

(Note that $\mathtt{F}_N^\omega(\beta,h)$ depends on ω!) Let us first show that, for all $N \geq 1$ and every realization of ω, the functions $h \mapsto \mathtt{F}_N^\omega(\beta,h)$ and $\beta \mapsto \mathtt{F}_N^\omega(\beta,h)$ are convex.

By differentiating, one gets

$$\frac{\partial}{\partial h}\mathtt{F}_N^\omega(\beta,h) = \mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^N \mathbf{1}_{\{i\in\tau\}}\Big]\,,\quad \frac{\partial}{\partial \beta}\mathtt{F}_N^\omega(\beta,h) = \mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^N \omega_i\mathbf{1}_{\{i\in\tau\}}\Big]\,, \tag{3.14}$$

and

$$\frac{\partial^2}{\partial^2 h}\mathtt{F}_N^\omega(\beta,h) = \mathrm{Var}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^N \mathbf{1}_{\{i\in\tau\}}\Big]\,,\quad \frac{\partial^2}{\partial^2 \beta}\mathtt{F}_N^\omega(\beta,h) = \mathrm{Var}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^N \omega_i\mathbf{1}_{\{i\in\tau\}}\Big]\,,$$

which shows the convexity of the two functions.

The monotonicity of $h\mapsto \mathtt{F}_N^\omega(\beta,h)$ (for all $N\geq 1$, for all ω) follows directly from (3.14). However, it is not true that $\beta\mapsto \mathtt{F}_N^\omega(\beta,h)$ is monotone (for example if the first N variables ω are negative), but it is enough to show that $\beta\mapsto \mathbb{E}[\mathtt{F}_N^\omega(\beta,h)]$ is non-decreasing. This function being convex, it is therefore enough to show that its derivative at $\beta=0$ is non-negative. But since $\mathrm{P}_{N,h}^{\beta=0,\omega}=\mathrm{P}_{N,h}$ does not depend on ω, one has

$$\frac{\partial}{\partial \beta}\mathbb{E}\big[\mathtt{F}_N^\omega(\beta,h)\big]\big|_{\beta=0} = \frac{1}{N}\mathbb{E}\mathrm{E}_{N,h}\Big[\sum_{i=1}^N \omega_i\mathbf{1}_{\{i\in\tau\}}\Big] = 0\,,$$

thanks to the assumption that $\mathbb{E}[\omega_i]=0$.

(iii). For the convexity of $(\beta,h)\mapsto \mathtt{F}(\beta,h)$, it is enough to show that for all $N\geq 1$ and every realization of ω, the function $(\beta,h)\mapsto \mathtt{F}_N^\omega(\beta,h)$ is convex. The same computations as above gives

$$\mathrm{Hess}(\mathtt{F}_N^\omega(\beta,h)) = \Big(\mathrm{Cov}_{N,h}^{\beta,\omega}\big(\Sigma_a,\Sigma_b\big)\Big)_{1\leq a,b\leq 2}\,,$$

where $\Sigma_1=\sum_{i=1}^N \mathbf{1}_{\{i\in\tau\}}$ and $\Sigma_2=\sum_{i=1}^N \omega_i\mathbf{1}_{\{i\in\tau\}}$. This is a positive definite matrix, which shows the desired convexity. □

Remark 3.15. The observation made in Section 2.1.3-(2.7) remains valid here: for every h such that the derivative $\frac{\partial}{\partial h}\mathtt{F}(\beta,h)$ exists, this derivative is the limit of $\frac{\partial}{\partial h}\frac{1}{N}\mathtt{F}_N^\omega(\beta,h)$ (and of $\frac{\partial}{\partial h}\frac{1}{N}\mathbb{E}[\mathtt{F}_N^\omega(\beta,h)]$), that is

$$\frac{\partial}{\partial h}\mathtt{F}(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^N \mathbf{1}_{\{i\in\tau\}}\Big] = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^N \mathbf{1}_{\{i\in\tau\}}\Big]\,.$$

Proof of Theorem 3.12

A quick proof consists in using Kingman's subadditive ergodic theorem [Kin73] as a black box (this theorem can be seen as a random version of Fekete's lemma). This theorem is extremely useful in many situations; we give its statement here.

We denote by $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ the shift operator: if $\omega = (\omega_i)_{i\geq 1}$, then for $k \geq 0$

$$\theta^k \omega = (\omega_{i+k})_{i\geq 0}\,,$$

that is $(\theta^k\omega)_i = \omega_{i+k}$ for all $i \geq 0$.

Theorem (Kingman's subadditive ergodic)**.** *Let $(X_n)_{n\geq 0}$ be a sequence of measurable functions from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$. Assume that for all $n, m \geq 1$, for all $\omega \in \mathbb{R}^{\mathbb{N}}$,*

$$X_{n+m}(\omega) \leq X_n(\omega) + X_m(\theta^n \omega)\,.$$

Then, if $\omega = (\omega_i)_{i\geq 0}$ are i.i.d. random variables[1] *with law $\mathbb{P}$, one has the $\mathbb{P}$-almost sure convergence* $\lim_{n\to\infty} \frac{1}{n} X_n(\omega) = \inf_{n\geq 1} \frac{1}{n}\mathbb{E}[X_n]$.

One can apply this theorem to the sequence $-\log Z_{N,h}^{\beta,\omega}$, which satisfies the assumptions of ergodic subadditivity. Indeed by inserting the indicator $\mathbf{1}_{\{N\in\tau\}}$ inside the expectation, one gets

$$Z_{N+M,h}^{\beta,\omega} \geq \mathrm{E}\bigg[\exp\bigg(\Big(\sum_{i=1}^{N} + \sum_{i=N+1}^{N+M}\Big)(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}\bigg)\mathbf{1}_{\{N\in\tau\}}\mathbf{1}_{\{N+M\in\tau\}}\bigg] = Z_{N,h}^{\beta,\omega} Z_{M,h}^{\beta,\theta^N\omega}\,,$$

where we have used Markov's property and wrote that $\sum_{i=N+1}^{N+M}(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}} = \sum_{i=1}^{M}(h+\beta\theta^N\omega_i)\mathbf{1}_{\{N+i\in\tau\}}$. This shows the (ergodic) superadditivity of $\log Z_{N,h}^{\beta,\omega}$: for all $N, M \geq 1$,

$$\log Z_{N+M,h}^{\beta,\omega} \geq \log Z_{N,h}^{\beta,\omega} + \log Z_{M,h}^{\beta,\theta^N\omega}\,, \tag{3.15}$$

so that Kingman's ergodic theorem yields the existence of the free energy.

Pedestrian proof of Theorem 3.12. We prove here the existence of the free energy by an ad-hoc procedure, which uses the superadditivity of $\log Z_{N,h}^{\beta,\omega}$ and a quasi-subadditivity of $\log Z_{N,h}^{\beta,\omega}$, applying the law of large numbers a number of times.

[1] The result remains valid under the weaker assumption of ergodicity, *i.e.* that every θ-invariant event has $\mathbb{P}$-probability 0 or 1.

Step 1. By the superadditivity (3.15) of $\log Z_{N,h}^{\beta,\omega}$, one directly gets that the deterministic sequence $(\mathbb{E}[\log Z_{N,h}^{\beta,\omega}])_{N\geq 1}$ is superadditive. By Fekete's lemma, one therefore gets the existence (and characterization) of the limit:

$$\mathtt{F} := \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big] = \sup_{N\to\infty} \frac{1}{N}\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big]\,. \tag{3.16}$$

Step 2. Lower bound. We now show that $\mathbb{P}$-a.s., for $\mathtt{F}$ defined in (3.16),

$$\liminf_{N\to\infty} \frac{1}{N}\log Z_{N,h}^{\beta,\omega} \geq \mathtt{F}\,. \tag{3.17}$$

Fix $\varepsilon>0$ and consider $N_0 = N_0(\varepsilon)$ such that $\frac{1}{N_0}\mathbb{E}[\log Z_{N_0,h}^{\beta,\omega}] \geq \mathtt{F}-\varepsilon$. Then, for $N\geq 1$, write the Euclidean division $N = m_N N_0 + q_N$, where $0\leq q_N < N_0$. By decomposing into blocks of size m_N and using the superadditivity property (3.15), we get that

$$\log Z_{N,h}^{\beta,\omega} \geq \sum_{j=0}^{m_N-1} \log Z_{m_N,h}^{\beta,\theta^{jN_0}\omega} + \log Z_{q_N,h}^{\beta,\theta^{m_N N_0}\omega}\,.$$

We now note that the random variables $(\log Z_{m_N,h}^{\beta,\theta^{jN_0}\omega})_{j\geq 0}$ are i.i.d. (they depend on disjoint blocks of the variables ω_i): by the law of large numbers, we obtain that $\mathbb{P}$-a.s.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{m_N-1} \log Z_{m_N,h}^{\beta,\theta^{jN_0}\omega} = \lim_{N\to\infty}\frac{m_N}{N}\frac{1}{m_N}\sum_{j=0}^{m_N-1} \log Z_{m_N,h}^{\beta,\theta^{jN_0}\omega} = \frac{1}{N_0}\mathbb{E}\big[\log Z_{N_0,h}^{\beta,\omega}\big]\,,$$

which is $\geq \mathtt{F}-\varepsilon$ by definition of N_0. One can also show that the remaining term $\log Z_{q_N,h}^{\beta,\theta^{m_N N_0}\omega}$ is negligible:

$$\lim_{m\to\infty}\frac{1}{m}\max_{q\in\{0,\dots,N_0-1\}}\big|\log Z_{q_N,h}^{\beta,\theta^{mN_0}\omega}\big| = 0 \qquad \mathbb{P}\text{-a.s.}$$

(We leave this as an exercise.) We therefore have$\liminf_{N\to\infty}\frac{1}{N}\log Z_{N,h}^{\beta,\omega} \geq \mathtt{F}-\varepsilon$, which proves (3.17) since $\varepsilon>0$ is arbitrary.

Step 3. Upper bound. We now show that $\mathbb{P}$-a.s., for $\mathtt{F}$ defined in (3.16),

$$\limsup_{N\to\infty}\frac{1}{N}\log Z_{N,h}^{\beta,\omega} \leq \mathtt{F}\,. \tag{3.18}$$

For this, we show that a quasi-subadditivity property holds of $\log Z_{N,h}^{\beta,\omega}$, analogous to (3.15): there exists a constant $C > 0$ such that, for all $N, M \geq 1$,

$$\log Z_{N+M,h}^{\beta,\omega} \leq \log Z_{N,h}^{\beta,\omega} + \log Z_{M,h}^{\beta,\theta^N\omega} + \log\big(1 + Ce^{-(h+\beta\omega_N)} \min(N,M)^{2+\alpha}\big) . \quad (3.19)$$

We prove (3.19) below, but let us first show how it implies (3.18). Let $N_0 \geq 0$ be large, and for $N \geq N_0$, write the Euclidean division $N = m_N N_0 + q_N$, where $0 \leq q_N < N_0$. Decomposing into blocks of size m_N as above and applying (3.19) repeatedly, we get that

$$\log Z_{N,h}^{\beta,\omega} \leq \sum_{j=0}^{m_N-1} \log Z_{m_N,h}^{\beta,\theta^{jN_0}\omega} + \log Z_{q_N,h}^{\beta,\theta^{m_N N_0}\omega} + \beta \sum_{j=0}^{m_N} |\omega_{jN_0}| + c\, m_N \log N_0 ,$$

where we have also used that $\log(1 + Ce^{-(h+\beta\omega_i)} N_0^{2+\alpha}) \leq \beta|\omega_i| + c \log N_0$ (for N_0 sufficiently large, depending only on h).

As before, applying the law of large numbers, neglecting the term $\frac{1}{N} \log Z_{q_N,h}^{\beta,\theta^{m_N N_0}\omega}$ (and noting that $\lim_{N\to\infty} \frac{m_N}{N} = \frac{1}{N_0}$), one gets for every (large) N_0

$$\limsup_{N\to\infty} \frac{1}{N} \log Z_{N,h}^{\beta,\omega} \leq \frac{1}{N_0} \mathbb{E}\big[\log Z_{N_0,h}^{\beta,\omega}\big] + \frac{\beta}{N_0} \mathbb{E}[|\omega_1|] + \frac{c}{N_0} \log N_0 .$$

The bound (3.18) then follows by letting $N_0 \to \infty$, recalling the definition (3.16) of $\mathtt{F}$.

It remains to show the quasi-subadditivity property (3.19). By decomposing the partition function according to the last renewal point before N (at position a) and the first renewal point after N (at position b), one has the identity

$$Z_{N+M,h}^{\beta,\omega} = Z_{N,h}^{\beta,\omega} Z_{M,h}^{\beta,\theta^N\omega} + \sum_{a=0}^{N-1} \sum_{b=N+1}^{N+M} Z_{a,h}^{\beta,\omega} K(b-a) \mathrm{e}^{h+\beta\omega_b} Z_{N+M-b,h}^{\beta,\theta^b\omega} ,$$

where the first term is due to the event $\{N \in \tau\}$ (*i.e.* when $a = b = N$). Similarly, one has the following last-point and first-point decompositions:

$$Z_{N,h}^{\beta,\omega} = \sum_{a=0}^{N-1} Z_{a,h}^{\beta,\omega} K(N-a) \mathrm{e}^{h+\beta\omega_N} , \quad Z_{M,h}^{\beta,\theta^N\omega} = \sum_{b=N+1}^{N+M} K(b-N) \mathrm{e}^{h+\beta\omega_b} Z_{N+M-b,h}^{\beta,\theta^b\omega} .$$

Combining these two identities, we get that

$$\sum_{a=0}^{N-1}\sum_{b=N+1}^{N+M} Z_{a,h}^{\beta,\omega} K(b-a)\mathrm{e}^{h+\beta\omega_b} Z_{N+M-b,h}^{\beta,\theta^b\omega} \\ \le Z_{N,h}^{\beta,\omega} Z_{M,h}^{\beta,\theta^N\omega} \times \mathrm{e}^{-(h+\beta\omega_N)} \max_{0\le a<N<b\le N+M} \frac{K(b-a)}{K(N-a)K(b-N)} .$$

This yields the inequality (3.19) once we realize that

$$\max_{0\le a<N<b\le N+M} \frac{K(b-a)}{K(N-a)K(b-N)} \le C\min(N,M)^{2+\alpha} .$$

Indeed, this follows from the fact that $K(n) = L(n)n^{-(1+\alpha)}$ (recall $(*)$), from which we deduce that $\frac{K(b-a)}{K(N-a)} \le c$ and $K(b-N) \ge c' M^{-(2+\alpha)}$ uniformly in $0\le a<N$ and $N<b\le N+M$, recalling Remark 3.1; symmetrically we have $\frac{K(b-a)}{K(b-N)} \le c$ and $K(N-a) \ge c' N^{-(2+\alpha)}$. □

3.4.2 About the phase transition

The fact that $h\mapsto \mathtt{F}(\beta,h)$ is non-negative, non-decreasing and convex shows that one can define the critical point: for any fixed $\beta\ge 0$, define

$$h_c(\beta) := \inf\{h, \mathtt{F}(\beta,h)>0\} = \sup\{h, \mathtt{F}(\beta,h)=0\} ,$$

with the convention $\sup\emptyset = -\infty$.

By monotonicity of $\beta\mapsto \mathtt{F}(\beta,h)$, we have $\mathtt{F}(\beta,h)\ge \mathtt{F}(0,h)$ and hence $h_c(\beta)\le h_c(0)=0$, see Theorem 3.5 (recall we assumed that $\mathrm{P}(\tau_1<+\infty)=1$). We also have the upper bound $\mathtt{F}(\beta,h)\le \mathbb{E}[(h+\beta\omega_1)^+]$ (recall the proof of Proposition 3.14), but this does not allow us to deduce anything about the critical point (since $\mathbb{E}[(h+\beta\omega_1)^+]>0$ for all $h\in\mathbb{R}$, unless ω_1 has bounded support); note however that $\lim_{h\to-\infty}\mathbb{E}[(h+\beta\omega_1)^+]=0$. We refer to Figure 3.3 for an illustration.

In view of Remark 3.15, the point $h_c(\beta)$ marks a phase transition between a *localized* phase $\mathtt{F}(\beta,h)>0$, with an asymptotically positive density of contacts, and a *delocalized* phase where $\mathtt{F}(\beta,h)=0$, with an asymptotically null density of contacts. We denote by $\mathcal{L}$, $\mathcal{D}$ the corresponding regions of the parameter space $(\beta,h)\in\mathbb{R}_+\times\mathbb{R}$,

$$\mathcal{L} := \big\{(\beta,h)\,;\, \mathtt{F}(\beta,h)>0\big\} ,\quad \mathcal{D} := \big\{(\beta,h)\,;\, \mathtt{F}(\beta,h)=0\big\} .$$

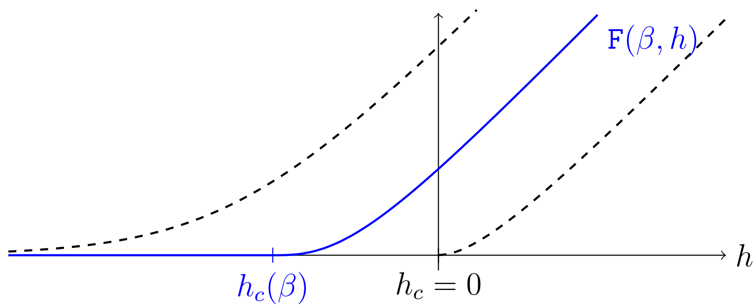


Figure 3.3: We have represented the free energy $h \mapsto \mathtt{F}(\beta, h)$ as a function of h (for fixed $\beta > 0$): it is a convex, increasing function, nonnegative. We have also represented as dashed lines the bounds $\mathtt{F}(\beta, h) \le \mathbb{E}[(h + \beta\omega_1)^+]$ and $\mathtt{F}(\beta, h) \ge \mathtt{F}(0, h)$.

Note that since $(\beta, h) \mapsto \mathtt{F}(\beta, h)$ is convex, the delocalized region $\mathcal{D}$ is convex, that is, the function $\beta \mapsto h_c(\beta)$ is concave. We refer to Figure 3.5 on page 118 for an overview of the phase diagram.

Condition for the existence of a phase transition.

The following proposition shows that an assumption is necessary in order to really have a phase transition, *i.e.* in order to have $h_c(\beta) > -\infty$.

Proposition 3.16 (Absence of phase transition)**.** *If* $\mathbb{E}[e^{\beta\omega_1}] = +\infty$ *for all* $\beta > 0$, *then* $\mathtt{F}(\beta, h) > 0$ *for all* $h \in \mathbb{R}$ *and all* $\beta > 0$. *In particular* $h_c(\beta) = -\infty$ *for all* $\beta > 0$.

Its proof is given in the form of an exercise.

Exercise 29 (Proof of Proposition 3.16)**.** The goal is to obtain a lower bound on the free energy using a specific renewal trajectory that only visits large values of ω; this is referred to as a *rare sites strategy*. We define $T_0 = 0$, and then by iteration $T_k := \min\big\{i > T_{k-1},\, \omega_i > 2\frac{|h|}{\beta}\big\}$.

1. Show that for all $n \ge 1$ we have $Z^{\beta,\omega}_{T_n,h} \ge \prod_{i=1}^{n} K(T_i - T_{i-1}) e^{|h|}$.
2. Show that, $\mathbb{P}$-a.s.

$$\mathtt{F}(\beta, h) \ge \liminf_{n\to\infty} \frac{1}{T_n} \log Z^{\beta,\omega}_{T_n,h} = \liminf_{n\to\infty} \frac{n}{T_n} \frac{1}{n} \log Z^{\beta,\omega}_{T_n,h}\,.$$

and then that

$$\mathtt{F}(\beta,h) \geq \frac{1}{\mathbb{E}(T_1)}\big(|h| + \mathbb{E}[\log K(T_1)]\big)\,.$$

3. Using the fact that there exists a constant $c>0$ such that $K(n) \geq cn^{-(2+\alpha)}$ for all $n \geq 1$, deduce that

$$\begin{aligned}\mathtt{F}(\beta,h) &\geq \frac{1}{\mathbb{E}(T_1)}\big(|h| - (2+\alpha)\log \mathbb{E}(T_1) + \log c\big)\\ &= \mathbb{P}\Big(\omega_1 > 2\frac{|h|}{\beta}\Big)\,\Big(|h| + \log c + (2+\alpha)\log \mathbb{P}\big(\omega_1 > 2|h|/\beta\big)\Big)\,.\end{aligned}$$

4. Show that if $\mathbb{E}(\mathrm{e}^{\beta\omega_1}) = +\infty$ for all $\beta>0$, then $\liminf_{t\to\infty} -\frac{1}{t}\log \mathbb{P}(\omega_1 > t) = 0$. Deduce that for any $A<0$ there exists an $h<A$ such that

$$\mathtt{F}(\beta,h) \geq \frac{1}{2}|h|\,\mathbb{P}\Big(\omega_1 > 2\frac{|h|}{\beta}\Big) > 0\,,$$

and conclude. □

In order to have a phase transition, we therefore assume the following:

$$\exists\, \bar\beta \in (0,+\infty] \;\text{ such that }\qquad \lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega_i}] < +\infty \quad \forall\, \beta \in (0,\bar\beta)\,. \qquad (**)$$

Lemma 3.17 (Annealed bound)**.** *Assume that* (**) *holds. Then for any $\beta \geq 0$ such that $\lambda(\beta) < +\infty$, we have*

$$\mathtt{F}(\beta,h) \leq \mathtt{F}(0,h+\lambda(\beta))\,. \tag{3.20}$$

Thus we have $h_c(\beta) \geq -\lambda(\beta)$, and in particular, $h_c(\beta) > -\infty$ for all $\beta \in [0,\bar\beta)$.

Proof. Assume that $\lambda(\beta) < +\infty$, otherwise there is nothing to prove. Using Jensen's inequality, we have

$$\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big] \leq \log \mathbb{E}\big[Z_{N,h}^{\beta,\omega}\big] = \log \mathrm{E}\Big[\exp\Big(\sum_{i=1}^{N}(h+\lambda(\beta))\mathbf{1}_{\{i\in\tau\}}\Big)\mathbf{1}_{\{N\in\tau\}}\Big]\,,$$

where we have used the fact that the ω_i are i.i.d., together with the definition of $\lambda(\beta)$. We recognize the partition function of the homogeneous model, with parameter $h+\lambda(\beta)$: we therefore deduce that

$$\mathtt{F}(\beta,h) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big] \leq \lim_{N\to\infty}\frac{1}{N}\log Z_{N,h+\lambda(\beta)} = \mathtt{F}(0,h+\lambda(\beta))\,.$$

Since $\mathtt{F}(0,h+\lambda(\beta)) = 0$ for $h+\lambda(\beta) \leq h_c = 0$, this shows that $h_c(\beta) \geq -\lambda(\beta)$. □

3.4.3 Properties of trajectories: localized vs. delocalized

We now give some properties of the renewal τ under the pinning measure $\mathrm{P}_{N,h}^{\beta,\omega}$, depending on whether $h < h_c(\beta)$ or $h > h_c(\beta)$.

a) Localized phase $h > h_c(\beta)$. We have the following result, due to [GT06a] (see also [GZ24, Thm. 1.2] for an improved result).

Theorem 3.18 (Differentiability)**.** *The free energy $h \mapsto \mathtt{F}(\beta, h)$ is infinitely differentiable in the localized phase $\mathcal{L} = \{(\beta, h), \mathtt{F}(\beta, h) > 0\}$.*

In particular, recalling Remark 3.15, for any $h > h_c(\beta)$, under $\mathrm{P}_{N,h}^{\beta,\omega}$ we asymptotically have a positive density of contacts:

$$\frac{\partial}{\partial h}\mathtt{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N}\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big] = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big] > 0.$$

where the limit exists $\mathbb{P}$-a.s. since $\mathtt{F}$ is differentiable. We refer to Exercise 31 below for a stronger result on trajectories in the localized phase.

Let us stress that [GT06a] also shows that the correlations $\mathrm{Cov}_{N,h}^{\beta,h}(\mathbf{1}_{\{i\in\tau\}}, \mathbf{1}_{\{j\in\tau\}})$ decay exponentially fast. It is then deduced [GT06a, Thm. 2.5] that, if M_N is the size of the largest excursion of the renewal process, there exists a constant $C_{\beta,h}$ such that for all $\varepsilon > 0$ one has $\lim_{N\to\infty} \mathrm{P}_{N,h}^{\beta,\omega}\big(|\frac{M_N}{\log N} - C_{\beta,h}| > \varepsilon\big) = 0$, in $\mathbb{P}$-probability. This result is therefore comparable to Exercise 27 in the homogeneous case. We also refer to [GZ24] for a number of properties (and a recent account of references) in the localized phase.

b) Delocalized phase $h < h_c(\beta)$. We now prove the following result, which shows that there are very few contacts in the delocalized phase.

Proposition 3.19 (Delocalized phase)**.** *If $h < h_c(\beta)$, then there exists a constant $C_h > 0$ such that*

$$\mathbb{E}\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big] \leq \frac{C_h}{h_c(\beta) - h}\ \log N\,.$$

Proof. By convexity of $h \mapsto \log Z_{N,h}^{\beta,\omega}$, setting $u = h_c(\beta) - h$, we have the inequality

$$\log Z_{N,h_c(\beta)}^{\beta,\omega} = \log Z_{N,h+u}^{\beta,\omega} \geq \log Z_{N,h}^{\beta,\omega} + u\,\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big]\,,$$

where we have used the computation (3.14) of the derivative of $\log Z_{N,h}^{\beta,\omega}$. Taking expectations and using the last identity of Theorem 3.12, we get that for any $N \geq 1$,

$$\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big] + u\mathbb{E}\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big] \leq \mathbb{E}\big[\log Z_{N,h_c(\beta)}^{\beta,\omega}\big] \leq \mathtt{F}(\beta, h_c(\beta)) = 0\,.$$

Then, using the lower bound $Z_{N,h}^{\beta,\omega} \geq \mathrm{e}^{\beta\omega_N + h}\mathrm{P}(\tau_1 = N)$, we obtain the inequality

$$\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big] \geq \beta\mathbb{E}[\omega_N] + h + \log\mathrm{P}(\tau_1 = N) = h + \log\mathrm{P}(\tau_1 = N)\,,$$

so that we end up with

$$\mathbb{E}\mathrm{E}_{N,h}^{\beta,\omega}\Big[\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\}}\Big] \leq \frac{-h - \log\mathrm{P}(\tau_1 = N)}{u}\,,$$

with $u = h_c(\beta) - h$. Now, according to assumption $(*)$ on $\mathrm{P}(\tau_1 = N)$, we have that $\log\mathrm{P}(\tau_1 = N) \sim -(1+\alpha)\log N$ as $N \to \infty$, which concludes the proof. (Note that we can choose C_h arbitrarily close to $1+\alpha$ if we choose N large enough.) □

3.5 Some properties of the phase diagram

In this section we prove some (lower and upper) bounds on the critical curve $h_c(\beta)$. Throughout, we assume that condition $(**)$ holds: this in particular implies that $\mathbb{E}[\omega_1^2] < +\infty$. Recall that we assumed for simplicity that $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$.

3.5.1 Quenched vs. annealed

Recall that under assumption $(**)$, we have the upper bound (3.20), due to Jensen's inequality:

$$\mathtt{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\big[\log Z_{N,h}^{\beta,\omega}\big] \leq \lim_{N\to\infty} \frac{1}{N}\log\mathbb{E}[Z_{N,h}^{\beta,\omega}] = \mathtt{F}(0, h + \lambda(\beta))\,.$$

The averaged partition function $\mathbb{E}[Z_{N,h}^{\beta,h}]$ (here equal to $Z_{N,h+\lambda(\beta)}$) is called the *annealed* partition function: it corresponds to taking expectation of the Gibbs weight both over the renewal process and over the disorder.

Remark 3.20. One may consider the Gibbs measure associated with the annealed partition function: it is a measure both on the renewal τ and on the disorder ω, with reference measure $\mathrm{P} \otimes \mathbb{P}$, namely

$$\frac{\mathrm{dP}_{N,h}^{\beta,\mathrm{a}}}{\mathrm{dP} \otimes \mathbb{P}}(\tau,\omega) = \frac{1}{Z_{N,h}^{\beta,\mathrm{a}}} \exp\Big(\sum_{i=1}^{N} (h+\beta\omega_i) \mathbf{1}_{\{i\in\tau\}} \Big) \mathbf{1}_{\{N\in\tau\}} ,$$

with $Z_{N,h}^{\beta,\mathrm{a}} := \mathbb{E}[Z_{N,h}^{\beta,h}]$.

Thus, we call *annealed* free energy the limit

$$\mathtt{F}^{\mathrm{a}}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z_{N,h}^{\beta,\omega}] = \mathtt{F}(0, h+\lambda(\beta)) ,$$

and we denote by $h_c^{\mathrm{a}}(\beta) = -\lambda(\beta)$ the corresponding annealed critical point. Note that we have the lower bound $h_c(\beta) \geq h_c^{\mathrm{a}}(\beta) = -\lambda(\beta)$ and that one easily obtains (recall $\mathbb{E}[\omega_1^2] = 1$)

$$h_c^{\mathrm{a}}(\beta) = -\lambda(\beta) \sim -\frac{1}{2}\beta^2 \qquad \text{as} \;\; \beta \downarrow 0 .$$

Recalling that we assume (∗∗), let us denote

$$\bar\beta := \sup\big\{ \beta \geq 0 \,, \lambda(\beta) < +\infty \big\} \in (0,+\infty] . \tag{3.21}$$

Then it is clear that we have the following dichotomy for the existence of a phase transition for the annealed model:

(i) If $\beta < \bar\beta$, then $h_c^{\mathrm{a}}(\beta) > -\infty$;

(ii) If $\beta > \bar\beta$, then $h_c^{\mathrm{a}}(\beta) = -\infty$ (in fact, $\mathtt{F}^{\mathrm{a}}(\beta,h) = +\infty$ for all $h \in \mathbb{R}$).

3.5.2 Conditions for the existence of a phase transition

As far as the disordered (or quenched) model is defined, we have the following criterion for the existence of a phase transition. (We refer to Figure 3.5 for an illustration.)

Theorem 3.21 (Criterion for a phase transition)**.** *Let $\bar\beta$ be defined in* (3.21)*. Then we have the following dichotomy:*

1. *If $\beta < (1+\alpha)\bar\beta$, then $h_c(\beta) > -\infty$;*
2. *If $\beta > (1+\alpha)\bar\beta$, then $h_c(\beta) = -\infty$.*

The question of whether $h_c(\beta) < -\infty$ or $h_c(\beta) = -\infty$ at $\beta = (1+\alpha)\bar\beta$ is open and should depend in a more refined way on the laws of τ and ω. There are a number of cases that can be treated: for instance, if one has $\lambda(\bar\beta) = +\infty$, then $h_c((1+\alpha)\bar\beta) = -\infty$; but the converse is false. We refer to Proposition 3.25 below.

Exercise 30. Show that the critical curve is left-continuous at $\beta = (1+\alpha)\bar{\beta}$, namely

$$\lim_{\beta \uparrow (1+\alpha)\bar{\beta}} h_c(\beta) = h_c((1+\alpha)\bar{\beta}) .$$

Proof of Theorem 3.21. We prove the two items separately.

Proof of item 1. Let us fix $\beta < (1+\alpha)\bar{\beta}$. The goal is to show that $\mathtt{F}(\beta, h) = 0$ for h sufficiently negative. Our proof is inspired by [Ton08a, Thm. 2.1] Let us fix some $\gamma \in (0,1)$ such that $\frac{1}{1+\alpha} < \gamma < (1+\alpha)\frac{\bar{\beta}}{\beta}$, so in particular $\lambda(\gamma\beta) < +\infty$.

Using that $\log Z_{N,h}^{\beta,\omega} = \frac{1}{\gamma} \log (Z_{N,h}^{\beta,\omega})^{\gamma}$, we obtain by Jensen's inequality that

$$\mathtt{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{\gamma N} \mathbb{E}[\log (Z_{N,h}^{\beta,\omega})^{\gamma}] \lim_{N\to\infty} \frac{1}{\gamma N} \log \mathbb{E}[(Z_{N,h}^{\beta,\omega})^{\gamma}] . \tag{3.22}$$

Then, we decompose the partition function $Z_{N,h}^{\beta,\omega}$ according to the number and positions of renewal points, *i.e.*

$$Z_{N,h}^{\beta,\omega} = \sum_{k=1}^{N} \sum_{t_0=0<t_1<\cdots<t_k=N} \prod_{i=1}^{k} K(t_i - t_{i-1}) \mathrm{e}^{h+\beta\omega_{t_i}} .$$

Therefore, using that for $\gamma \in (0,1]$ we have $(\sum a_i)^{\gamma} \le \sum (a_i)^{\gamma}$ for any collection of nonnegative real numbers $a_i \ge 0$, we obtain that

$$\begin{aligned}(Z_{N,h}^{\beta,\omega})^{\gamma} &\le \sum_{k=1}^{N} \sum_{t_0=0<t_1<\cdots<t_k=N} \prod_{i=1}^{k} K(t_i - t_{i-1})^{\gamma} \mathrm{e}^{h+\beta\omega_{t_i}} \\ &= \sum_{k=1}^{N} \sum_{t_0=0<t_1<\cdots<t_k=N} \prod_{i=1}^{k} \hat{K}(t_i - t_{i-1}) \mathrm{e}^{\gamma h + \lambda(\gamma\beta) + \log C_\gamma} ,\end{aligned}$$

where we have set $\hat{K}(n) := \frac{1}{C_\gamma} K(n)^{\gamma}$, with $C_\gamma = \sum_{n=1}^{\infty} K(n)^{\gamma}$, which is finite since $\gamma < \frac{1}{1+\alpha}$. We have thus proven that $(Z_{N,h}^{\beta,\omega})^{\gamma} \le \hat{Z}_{N,\hat{h}}$, where $\hat{Z}_{N,\hat{h}}$ is the partition function of a homogeneous pinning model of a renewal $\hat{\tau}$ with inter-arrival distribution $\hat{K}(\cdot)$, and with pinning parameter $\hat{h} := \gamma h + \lambda(\gamma\beta) - \log C_\gamma$. Going back to (3.22), we therefore get that

$$\mathtt{F}(\beta, h) \le \gamma^{-1} \hat{\mathtt{F}}(\gamma h + \gamma\lambda(\beta) - \log C_\gamma) ,$$

where $\hat{\mathtt{F}}$ is the free energy associated to . Since $\hat{K}(\cdot)$ also satisfies ($*$) and verifies $\sum_{n=1}^{\infty} \hat{K}(n) = 1$, we conclude from Theorem 3.5 that $\hat{h}_c = 0$, *i.e.* that $\mathtt{F}(\beta, h) = 0$

for any $h \le -\frac{1}{\gamma}(\lambda(\gamma\beta) + \log C_\gamma)$. This concludes that, for any $\gamma \in (\frac{1}{1+\alpha}, \frac{\beta/\beta^*}{(1+\alpha)})$, we have

$$h_c(\beta) \ge -\frac{\lambda(\gamma\beta) + \log C_\gamma}{\gamma} > -\infty\,. \tag{3.23}$$

Proof of item 2. Let us fix $\beta > (1+\alpha)\bar{\beta}$. The goal is to show that $\mathtt{F}(\beta,h) > 0$ for all $h \in \mathbb{R}$. We use a *rare sites strategy* similar to what is done in Exercise 29.

Let us define $T_0 = 0$ and by iteration $T_k := \min\{i > T_{k-1}, \omega_i \ge t\}$, where $t = t(\beta,h)$ with be fixed below. Then, considering the trajectory $\tau_1 = T_1$, $\tau_2 = T_2$, etc., which only visits the ω's that are larger than t, we get that, for any $n \in \mathbb{N}$,

$$Z^{\beta,\omega}_{T_n,h} \ge \prod_{i=1}^n K(T_i - T_{i-1}) e^{\beta t + h}\,.$$

Then, taking the logarithm and dividing by n, we get that

$$\frac{1}{n} \log Z^{\beta,\omega}_{T_n,h} \ge \beta t + h + \frac{1}{n} \sum_{i=1}^n \log K(T_i - T_{i-1})\,,$$

so that, since the $(T_i - T_{i-1})_{i\ge 0}$ are i.i.d., by applying the law of large numbers (twice) we get

$$\mathtt{F}(\beta,h) = \lim_{n\to\infty} \frac{1}{T_n} \log Z^{\beta,\omega}_{T_n,h} = \lim_{n\to\infty} \frac{n}{T_n}\frac{1}{n} \log Z^{\beta,\omega}_{T_n,h} \ge \frac{1}{\mathbb{E}[T_1]}\big(\beta t + h + \mathbb{E}[\log K(T_1)]\big)\,.$$

Notice that T_1 is a geometric random variable with parameter $\mathbb{P}(\omega \ge t)$, so in particular, $\mathbb{E}[T_1] = \mathbb{P}(\omega \ge t)^{-1}$. Now, for any $\varepsilon \in (0,1)$, there is some constant $c_\varepsilon > 0$ such that $K(n) \ge c_\varepsilon n^{-(1+\alpha+\varepsilon)}$ for all $n \ge 1$ (recall Remark 3.1), so that

$$\mathbb{E}[\log K(T_1)] \ge \log c_\varepsilon - (1+\alpha-\varepsilon)\mathbb{E}[\log T_1] \ge \log c_\varepsilon - (1+\alpha-\varepsilon) \log \mathbb{E}[T_1]\,,$$

where we have used Jensen's for the second inequality. All together, we have showed that, for any $\varepsilon \in (0,1)$,

$$\mathtt{F}(\beta,h) \ge \mathbb{P}(\omega \ge t)\big(\beta t + h + \log c_\varepsilon + (1+\alpha+\varepsilon) \log \mathbb{P}(\omega \ge t)\big)\,.$$

We now use the following claim, that we leave as an exercise.

Claim 3.22. *Let $\bar{\beta} := \sup\{\beta \ge 0\,, \mathbb{E}[e^{\beta\omega}] < +\infty\}$. Then, we have that*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\omega \ge t) = -\bar{\beta}\,.$$

tll together, is we have chosen t large enough (how large depends on ε), we have that

$$\mathtt{F}(\beta,h) \geq \mathbb{P}(\omega \geq t)\big((\beta-(1+\alpha+\varepsilon)(\bar\beta+\varepsilon))t + h + \log c_\varepsilon\big)\,.$$

We can now conclude the proof. Fix $h \in \mathbb{R}$ and $\varepsilon \in (0,1)$ small enough so that $\beta-(1+\alpha+\varepsilon)(\bar\beta+\varepsilon) = \eta > 0$. Then, we can fix t large enough so that ηt is strictly larger $-(h+\log c_\varepsilon)$. The above inequality then yields that $\mathtt{F}(\beta,h)$ is bounded from below by $\mathbb{P}(\omega \geq t)(\eta t + h + \log c_\varepsilon)$, which is strictly positive, as wanted. □

3.5.3 Disorder helps localization

In this section, we show that $h_c(\beta) < 0$ for all $\beta > 0$, and, in fact, we prove a quantitative upper bound on $h_c(\beta)$. This is interpreted as the fact that disorder makes it easier for the polymer to localize, in the sense that it lowers the critical point: for $h \in (h_c(\beta), 0)$, even if $h + \beta\omega_i$ is on average strictly negative (hence repulsive), localization still occurs!

Theorem 3.23 (Disorder helps localization)**.** *Assume that* $\mathbb{E}[\omega_1] = 0$ *and also that* $K(1), K(2) > 0$*. Then for all* $\beta > 0$*, we have* $h_c(\beta) < 0$*. More precisely,*

$$\forall\, \beta \geq 0 \qquad h_c(\beta) \leq -\frac{p'}{1+p'}\,\mathbb{E}\big[\log(p\mathrm{e}^{\beta\omega_1} + 1 - p)\big]\,,$$

with $p := \mathrm{P}(1 \in \tau \mid 2 \in \tau) = \frac{K(1)^2}{K(1)^2+K(2)} \in (0,1)$ *and* $p' = \mathrm{P}(2\in\tau) = K(1)^2 + K(2)$.

Let us emphasize that one has $\log(p\mathrm{e}^{\beta\omega_1} + 1 - p) > p\beta\omega_1$ whenever $\omega_1 \neq 0$, by Jensen's inequality; the inequality is strict since $p \in (0,1)$ and $\mathrm{e}^{\beta\omega_1} \neq 1$. This shows that the upper bound is indeed strictly negative for all $\beta > 0$.

Let us mention that one can additionally obtain estimates on the upper bound as $\beta \downarrow 0$ or $\beta \uparrow +\infty$ (we leave this as an exercise, recall that $\mathbb{E}[\omega_1^2] = 1$):

$$\mathbb{E}\big[\log(p\mathrm{e}^{\beta\omega_1} + 1 - p)\big] \sim \begin{cases} \frac{p}{2}\,\beta^2 & \text{as } \beta \downarrow 0\,,\\ \beta\mathbb{E}[\omega_1^+] & \text{as } \beta \uparrow +\infty\,. \end{cases}$$

Proof of Theorem 3.23. The proof we give here follows the one in [Gia07, §5.2] (let us mention that the proof of $h_c(\beta) < 0$ was first given in [AS06]). The main idea is to consider "pairs of renewal points" and to observe that between these renewal points, the *effective disorder* has strictly positive mean, by using a so-called *partial annealing* procedure.

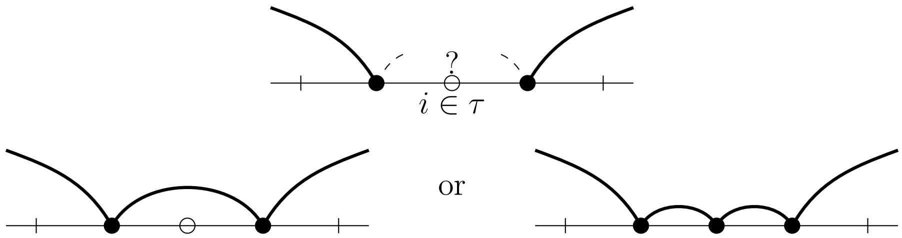


Figure 3.4: Conditionally on having $i \in \mathcal{I}_N$, the renewal may either: (i) visit the point i (that is, $i \in \tau$, with probability p) and collect $\mathrm{e}^{h+\beta\omega_i}$; (ii) not visit the point i (that is $i \notin \tau$ with probability $1-p$) and collect 1.

Let us consider N even and, for a given realization of τ, we define

$$\mathcal{I}_N = \mathcal{I}_N(\tau) := \Big\{ n \in \{1, \ldots, N-1\} \text{ odd }, \, n-1 \in \tau \text{ and } n+1 \in \tau \Big\},$$

corresponding to the "pairs of points" mentioned above. We denote $\hat{\vartheta}_i := \mathbf{1}_{\{i \in \mathcal{I}_N\}}$.

For fixed β, h, we define the following *effective disorder*

$$\hat{\omega}_i = \hat{\omega}_i(\beta, h) := \log \mathrm{E}\Big[\mathrm{e}^{(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}} \,\Big|\, \hat{\vartheta}_i = 1 \Big] = \log\big(p\mathrm{e}^{h+\beta\omega_i} + 1 - p \big),$$

where $p := \mathrm{P}(1 \in \tau \mid 2 \in \tau) = \frac{K(1)^2}{K(1)^2 + K(2)}$, see Figure 3.4. Let us stress that the $(\hat{\omega}_i)_{i\geq 1}$ are i.i.d. and that $\mathbb{E}[\hat{\omega}_1] > ph$ thanks to Jensen's inequality (the inequality is strict since $h + \beta\omega_i$ is not a constant).

Lemma 3.24. *For all even N, one has*

$$Z_{N,h}^{\beta,\omega} = \mathrm{E}\Big[\exp\Big(\sum_{i=1}^{N} (h + \beta\omega_i)(1 - \hat{\vartheta}_i)\mathbf{1}_{\{i\in\tau\}} + \sum_{i=1}^{N} \hat{\omega}_i \hat{\vartheta}_i \mathbf{1}_{\{i\in\tau\}} \Big) \mathbf{1}_{\{N\in\tau\}} \Big].$$

Proof. Let us also introduce the set $\mathcal{J}_N := \{ i \in \{0, \ldots, N\}, i \in \tau, \hat{\vartheta}_i = 0 \}$. Then, conditioning on $\mathcal{J}_N, \mathcal{I}_N$, we obtain

$$\mathrm{E}\Big[\mathrm{e}^{\sum_{i=1}^{N}(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}} \mathbf{1}_{\{N\in\tau\}} \,\Big|\, \mathcal{I}_N, \mathcal{J}_N \Big] = \mathrm{e}^{\sum_{i\in\mathcal{J}_N}(h+\beta\omega_i)} \mathbf{1}_{\{N\in\tau\}} \mathrm{E}\Big[\mathrm{e}^{\sum_{i\in\mathcal{I}_N}(h+\beta\omega_i)} \,\Big|\, \mathcal{I}_N, \mathcal{J}_N \Big].$$

Next, observe that, conditionally on $\mathcal{I}_N$ (and $\mathcal{J}_N$), the $(\mathbf{1}_{\{i\in\tau\}})_{i\in\mathcal{I}_N}$ are independent Bernoulli variables with parameter p (see Figure 3.4 for an illustration). Thus, we obtain

$$\mathrm{E}\Big[\mathrm{e}^{\sum_{i\in\mathcal{I}_N}(h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}} \,\Big|\, \mathcal{I}_N, \mathcal{J}_N \Big] = \prod_{i\in\mathcal{I}_N} \Big(p\mathrm{e}^{h+\beta\omega_i} + 1 - p \Big) = \mathrm{e}^{\sum_{i\in\mathcal{I}_N} \hat{\omega}_i}.$$

The conditional expectation is therefore equal to

$$\mathrm{E}\Big[\mathrm{e}^{\sum_{i=1}^N (h+\beta\omega_i)\mathbf{1}_{\{i\in\tau\}}}\mathbf{1}_{\{N\in\tau\}}\,\Big|\,\mathcal{I}_N,\mathcal{J}_N\Big] = \mathrm{e}^{\sum_{i\in\mathcal{J}_N}(h+\beta\omega_i)+\sum_{i\in\mathcal{I}_N}\hat\omega_i}\mathbf{1}_{\{N\in\tau\}}$$
$$= \exp\Big(\sum_{i=1}^N (h+\beta\omega_i)(1-\hat\vartheta_i)\mathbf{1}_{\{i\in\tau\}} + \sum_{i=1}^N \hat\omega_i\hat\vartheta_i\Big)\mathbf{1}_{\{N\in\tau\}}\,.$$

Taking the expectation again yields the conclusion of the lemma. □

Let us now conclude the proof Theorem 3.23. Since $\mathtt{F}(\beta,h) = \sup_{N\ge 1}\frac{1}{N}\mathbb{E}[\log Z_{N,h}^{\beta,\omega}]$ by Theorem 3.12, we have the finite-volume criterion:

$$\mathtt{F}(\beta,h)>0 \quad\Longleftrightarrow\quad \text{there exists an } N \text{ such that } \mathbb{E}[\log Z_{N,h}^{\beta,\omega}]>0\,. \tag{3.24}$$

To obtain a lower bound on $Z_{N,h}^{\beta,h}$, we apply Jensen's inequality to the conditional probability $\mathrm{P}(\,\cdot\mid N\in\tau)$: with Lemma 3.24, we obtain that

$$Z_{N,h}^{\beta,\omega} \ge \mathrm{P}(N\in\tau)\exp\Big(\mathrm{E}\Big[\sum_{i=1}^N (h+\beta\omega_i)(1-\hat\vartheta_i)\mathbf{1}_{\{i\in\tau\}} + \sum_{i=1}^N \hat\omega_i\hat\vartheta_i\mathbf{1}_{\{i\in\tau\}}\,\Big|\,N\in\tau\Big]\Big)\,.$$

Taking the logarithm and then the expected value, we get that $\mathbb{E}[\log Z_{N,h}^{\beta,\omega}]$ is bounded from below by

$$\log\mathrm{P}(N\in\tau) + h\mathrm{E}\Big[\sum_{i=1}^N (1-\hat\vartheta_i)\mathbf{1}_{\{i\in\tau\}}\,\Big|\,N\in\tau\Big] + \mathbb{E}[\hat\omega_i]\mathrm{E}\Big[\sum_{i=1}^N \hat\vartheta_i\,\Big|\,N\in\tau\Big]\,.$$

It therefore remains to estimate the various terms, but before doing so, let us make a few observations. First, we can focus on the case $h\le 0$, since otherwise we have the lower bound $\mathtt{F}(\beta,h)\ge\mathtt{F}(0,h)$. Thus, in the second term we may bound $1-\hat\vartheta_i\le 1$. Similarly, by definition of $\hat\omega_i$, we have $\hat\omega_i = h+\log(p\mathrm{e}^{\beta\omega_1}+(1-p)e^{-h}) \ge h+\log(p\mathrm{e}^{\beta\omega_1}+1-p)$. We therefore conclude that

$$\mathbb{E}[\log Z_{N,h}^{\beta,\omega}] \ge \log\mathrm{P}(N\in\tau) + h(U_N+\hat U_N) + \mathbb{E}\big[\log\big(p\mathrm{e}^{\beta\omega_1}+1-p\big)\big]\hat U_N\,,$$

$$\text{where } U_N = \mathrm{E}\Big[\sum_{i=1}^N \mathbf{1}_{\{i\in\tau\}}\,\Big|\,N\in\tau\Big] \quad\text{and}\quad \hat U_N = \mathrm{E}\Big[\sum_{i=1}^N \hat\vartheta_i\,\Big|\,N\in\tau\Big]\,.$$

Using the results of Section 3.3, see in particular Theorem 3.10, we obtain the following estimates, that we leave as an exercise to the reader (let us ignore the case

$\alpha = 0$ for simplicity):

$$U_N = \sum_{i=1}^{N} \frac{\mathrm{P}(i \in \tau)\mathrm{P}(N-i \in \tau)}{\mathrm{P}(N \in \tau)} \sim \begin{cases} \dfrac{N}{\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}]} & \text{if } \alpha \ge 1\,, \\ c_\alpha L(N)^{-1} N^{\alpha} & \text{if } \alpha \in (0,1), \end{cases}$$

and

$$\hat{U}_N = \sum_{i=1}^{N} \frac{\mathrm{P}(i-1 \in \tau)\mathrm{P}(2 \in \tau)\mathrm{P}(N-i-1 \in \tau)}{\mathrm{P}(N \in \tau)} \sim \mathrm{P}(2 \in \tau) U_N\,.$$

We conclude that

$$\mathbb{E}[\log Z_{N,h}^{\beta,\omega}] \ge \log \mathrm{P}(N \in \tau) + (1 + o(1))\Big((1+p')h + p'\mathbb{E}\big[\log\big(p\mathrm{e}^{\beta\omega_1} + 1 - p\big)\big]\Big) U_N\,,$$

with $p' = \mathrm{P}(2 \in \tau)$. Since $\log \mathrm{P}(N \in \tau) = o(U_N)$, this shows that, provided that we have $h > -\frac{p'}{1+p'}\mathbb{E}\big[\log\big(p\mathrm{e}^{\beta\omega_1} + 1 - p\big)\big]$, we get $\mathbb{E}[\log Z_{N,h}^{\beta,\omega}] > 0$ for N large enough, which implies that $\mathtt{F}(\beta, h) > 0$. This concludes the proof. □

3.5.4 Summary of the phase diagram

Let us give a summary of the bounds that we obtained in this section in Figure 3.4 below, which illustrates the phase diagram.

3.6 A few exercises

Exercise 31 (Localized phase)**.** Assume that $h > h_c(\beta)$ and recall that $\mathtt{F}$ is differentiable on $(h_c(\beta), +\infty)$, see Theorem 3.18. Let us also define $H_N(\tau) := \sum_{i=1}^{N-1} \mathbf{1}_{\{i \in \tau\}}$ the number of renewal points in $\{1, \ldots, N-1\}$.

1. Show that $\mathrm{E}_{N,h}^{\beta,\omega}\big[\mathrm{e}^{uH_N(\tau)}\big] = \frac{Z_{N,h+u}^{\beta,\omega}}{Z_{N,h}^{\beta,\omega}}$ and deduce that for any $u \in \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{N} \log \mathrm{E}_{N,h}^{\beta,\omega}\big[\mathrm{e}^{uH_N(\tau)}\big] = \mathtt{F}(\beta, h+u) - \mathtt{F}(\beta, h)\,,$$

   where the limit is $\mathbb{P}$-a.s. and in $L^1(\mathbb{P})$.

2. Show that, for any $\varepsilon > 0$, there is a constant $c_\varepsilon(\beta, h)$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathrm{P}_{N,h}^{\beta,\omega}\Big(N^{-1}H_N(\tau) \notin \big[\tfrac{\partial}{\partial h}\mathtt{F}(\beta,h) - \varepsilon, \tfrac{\partial}{\partial h}\mathtt{F}(\beta,h) + \varepsilon\big]\Big) \le -c_\varepsilon(\beta, h)\,.$$

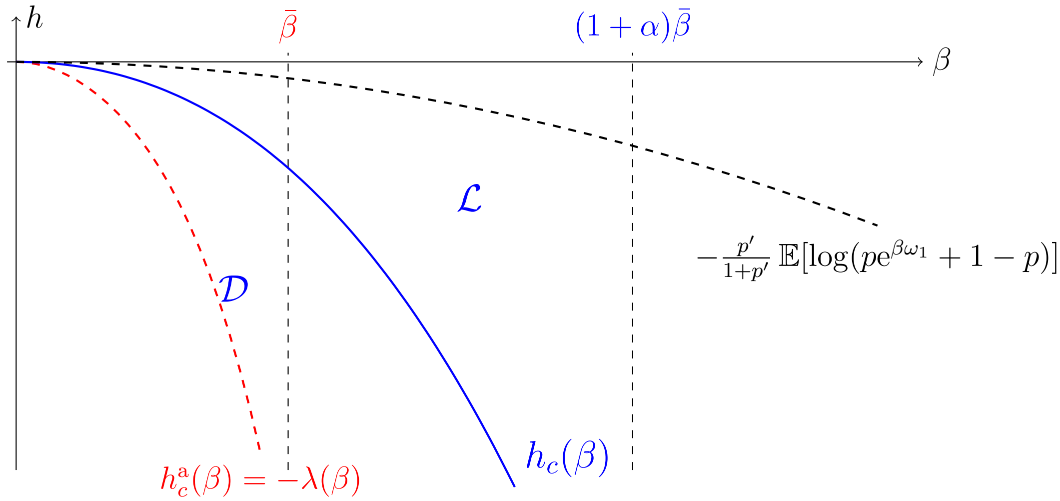


Figure 3.5: Phase diagram: the critical curve $\beta \mapsto h_c(\beta)$ is concave and separates the localized phase $\mathcal{L}$ (where $\mathtt{F}(\beta,h) > 0$, *i.e.* $h > h_c(\beta)$) from the delocalized phase $\mathcal{D}$ (where $\mathtt{F}(\beta,h) = 0$, *i.e.* $h \le h_c(\beta)$). We have also summarized the bounds obtained: one has $h_c(\beta) \ge h_c^{\mathrm{a}}(\beta) = -\lambda(\beta)$ and $h_c(\beta) \le -\frac{p'}{1+p'}\mathbb{E}[\log(pe^{\beta\omega}+1-p)]$. Note that both the lower and upper bounds are of order β^2 as $\beta \downarrow 0$. Also, $h_c^{\mathrm{a}}(\beta) = -\lambda(\beta)$ "explodes" at $\bar\beta$ whereas $h_c(\beta)$ "explodes" at $(1+\alpha)\bar\beta$, see Theorem 3.21.

Exercise 32 (Case of a heavy-tailed disorder)**.** We consider Exercise 29 in the particular case where $\mathbb{P}(\omega_1 > t) \sim t^{-\gamma}$ for some $\gamma > 1$. The goal is to give the asymptotic behavior of $\mathtt{F}(\beta,h)$ as $h \to -\infty$ (with $\beta > 0$ fixed).

Let $h < 0$ and define $T_0 = 0$ and then by iteration $T_k := \min\{i > T_{k-1},\, \omega_i > t\}$ with $t := \frac{|h|}{\beta}$.

1. Repeating the proof of Theorem 3.21-*(ii)*, show that
$$\mathtt{F}(\beta,h) \ge \frac{1}{\mathbb{E}(T_1)}\Big(\mathbb{E}\big(\beta\omega_1 + h \,\big|\, \omega_1 > t\big) + \mathbb{E}(\log K(T_1))\Big)\,.$$
2. Deduce that
$$\mathtt{F}(\beta,h) \ge \beta\mathbb{E}\big((\omega_1 - t)\mathbf{1}_{\{\omega_1 > t\}}\big) + \mathbb{P}(\omega_1 > t)\mathbb{E}(\log K(T_1))\,.$$
3. Show that $\mathbb{E}\big((\omega_1 - t)\mathbf{1}_{\{\omega_1>t\}}\big) \sim \frac{1}{\gamma-1}t^{1-\gamma}$ and $\mathbb{E}(\log K(T_1)) = O(\log t)$ as $t \to \infty$.
4. Deduce that we have
$$\mathtt{F}(\beta,h) \ge (1-o(1))\frac{\beta}{\gamma-1}\,t^{1-\gamma}\,, \qquad \text{as} \quad t \to +\infty\,.$$

5. On the other hand, show that $\mathtt{F}(\beta,h) \le \mathbb{E}((\beta\omega_1 + h)^+) = \beta\mathbb{E}((\omega_1 - t)\mathbf{1}_{\{\omega_1 > t\}})$.
6. Conclude that we have

$$\mathtt{F}(\beta,h) \sim \frac{\beta^\gamma}{\gamma-1}\,|h|^{1-\gamma}\,, \quad \text{as} \quad h \to -\infty\,.$$

About the case $\beta = (1+\alpha)\bar\beta$ in Theorem 3.21

In the next two exercises, we consider the case $\beta = \beta_0 := (1+\alpha)\bar\beta$ in Theorem 3.21, where we recall that $\bar\beta : \sup\{\beta \ge 0, \lambda(\beta) < +\infty\}$. We show the following, which is extracted from [Ler15].

Proposition 3.25. *Assume that τ verifies assumption $(*)$ and let $\beta_0 := (1+\alpha)\bar\beta$.*
(i) If $\lambda(\bar\beta) < +\infty$ and $\sum_{n\ge0} n^{-1}L(n)^{\frac{1}{1+\alpha}} < +\infty$, then $h_c(\beta_0) > -\infty$.
(ii) If $\lambda(\bar\beta) = +\infty$, then $h_c(\beta_0) = -\infty$.
(iii) There are cases where $\lambda(\bar\beta) < +\infty$ but $h_c(\beta_0) = -\infty$.

We can make the following conjecture, supported by a similar result in the context of the Derrida–Retaux model, which is a hierarchical version of (a simplified version of) the model; we refer to [CDH+19] for an overview of the questions for this model.

Conjecture 3.26. *Assume $(\hat{*})$, i.e. that $K(n) \sim c_1 n^{-(1+\alpha)}$ as $n \to \infty$, and let $\beta_0 := (1+\alpha)\bar\beta$. Then, we have that*

$$h_c(\beta_0) > -\infty \qquad \text{if and only if} \qquad \mathbb{E}[\omega_1 e^{\bar\beta\omega_1}] < +\infty\,.$$

Exercise 33 (Proof of Proposition 3.25-*(i)*)**.**
1. Verify that the proof of Theorem 3.21-*(i)* still holds with $\gamma = \frac{1}{1+\alpha} \in (0,1)$.
2. Deduce that

$$h_c(\beta_0) \ge -(1+\alpha)\Big(\lambda(\bar\beta) + \log\sum_{n=1}^{\infty} K(n)^{\frac{1}{1+\alpha}}\Big) > -\infty\,.$$

Exercise 34 (Proof of Proposition 3.25-*(iii)*)**.** We start from the following lower bound proven in the proof of Theorem 3.21-*(ii)*:

$$\mathtt{F}(\beta_0,h) \ge \mathbb{P}(\omega \ge t)\big(\beta_0 t + h + \mathbb{E}[\log K(T_1)]\big)\,,$$

where T_1 is a geometric random variable with parameter $\mathbb{P}(\omega \ge t)$. We also make some assumptions on $K(\cdot)$ and the distribution of ω: we assume that

$$K(n) \sim (\log n)^a n^{-(1+\alpha)} \quad \text{as } n\to\infty \quad \text{and} \quad \mathbb{P}(\omega \ge t) \sim t^\kappa e^{-\bar\beta t} \quad \text{as } t \to \infty\,,$$

where $a \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ are parameters we can play with.

1. Show that $\mathbb{E}[e^{\bar\beta\omega}] = +\infty$ for any $\kappa \geq -1$.
2. Show that, with the above assumption, we have that

$$\mathbb{E}[\log K(T_1)] = -(1+\alpha)\bar\beta t + (1+o(1))(a+\kappa(1+\alpha))\log t \qquad \text{as } t \to \infty\,.$$

3. Deduce that provided that we have $a > -\kappa(1+\alpha)$, then $\mathtt{F}(\beta_0, h) > 0$ for all $h \in \mathbb{R}$. Conclude.

Exercise 35 (Proof of Proposition 3.25-*(ii)*)**.** Let us assume here that $\lambda(\bar\beta) = +\infty$. The idea is to improve the *rare-site strategy* used in the proof of Theorem 3.21-*(ii)* by introducing some "m-rare sites strategy".

Part I. Preliminary estimates.

1. Show that $u \mapsto \lambda'(u)$ is a bijection from $[0, \bar\beta)$ to $[0, +\infty)$. Show in particular that $\lambda'(u) \to +\infty$ as $u \uparrow \bar\beta$.
2. For any $t > 0$, define $I(t) := \sup_{u>0}\{ut - \lambda(u)\}$. Let also θ_t be such that $\lambda'(\theta_t) = t$, in such a way that $I(t) = t\theta_t - \lambda(\theta_t)$
   (a) Show that $\theta_t \uparrow \bar\beta$ as $t \uparrow \infty$.
   (b) Deduce that $\lim_{t\to\infty} \bar\beta t - I(t) = +\infty$.

For $m \geq 1$, let us define $p_m(t) := \mathbb{P}\big(\frac{1}{m}\sum_{i=1}^m \omega_i > t\big)$, for any $t > 0$.

3. Show that for any fixed $t > 0$, we have that $\lim_{m\to\infty} \frac{1}{m} \log p_m(t) = -I(t) < 0$. *(You can simply use Cramer's theorem.)*
4. Using properties of slowly varying functions (see Remark 3.1), deduce that for any fixed $t > 0$, we have

$$\lim_{m\to\infty} \frac{1}{m} \log K(p_m(t)^{-1}) = -(1+\alpha) I(t)\,.$$

Part II. The rare-stretch strategy. Let $h \in \mathbb{R}$.

5. Show that, for any (fixed) constant C_0, we may choose t large enough so such that $\beta_0 t + h - (1+\alpha)I(t) \geq C_0$.
6. With t chosen as in the previous question, show that we can fix $m \geq 2$ (that may depend on t) so that $\frac{1}{m}\log K(p_m(t)^{-1}) \geq -(1+\alpha)I(t) - 1$.

With $t > 0$ and $m \geq 2$ chosen as above, we are now ready to define the rare "m-stretch" strategy, by identifying blocks of length m where the empirical mean of the ω's is larger than t. More precisely, define $T_0 = 0$ and then by iteration

$$T_k := \inf\Big\{ j > T_{k-1} \, , \frac{1}{m} \sum_{i=jm}^{(j+1)m-1} \omega_i \geq t \Big\} \, .$$

7. Adapting the rare-site strategy, show that for any $n \geq 1$

$$Z_{mT_n}^{\beta_0,\omega} \geq K(T_1 m) K(1)^{m-1} \mathrm{e}^{m(\beta_0 t + h)} \prod_{k=2}^{n} K(1+(T_k - T_{k-1} - 1)m) K(1)^{m-1} \mathrm{e}^{m(\beta_0 t + h)} \, .$$

8. Using the law of large numbers (as in the proof of Theorem 3.21-*(ii)*), show that

$$\mathtt{F}(\beta_0, h) \geq \frac{1}{m\mathbb{E}[T_1]} \Big(m(\beta_0 t + h) + (m-1) \log K(1) + \mathbb{E}[\log K(1 + m(T_1 - 1))] \Big) \, .$$

Now, let us admit that there is a constant $C > 0$ such that the following holds: if G is a geometric random variable with parameter $p \in (0, \frac{1}{2}]$, then for any $m \geq 2$

$$\mathbb{E}[\log K(1 + m(G-1))] \geq \log K(p^{-1}) - C \log m \, .$$

9. Deduce that

$$\begin{aligned} \mathtt{F}(\beta_0, h) &\geq p_m(t) \Big(\beta_0 t + h + \log K\big(p_m(t)^{-1}\big) + \log K(1) - C \, \frac{\log m}{m} \Big) \\ &\geq p_m(t) \big(C_0 + \log K(1) - C \, \mathrm{e}^{-1} \big) \, . \end{aligned}$$

10. Conclude the proof of Proposition 3.25.

Chapter 4

Relevance vs. irrelevance of disorder for the pinning model

The question of the influence of disorder is a general problem for physical systems: it consists in determining whether the characteristics of the system (in particular the properties of the phase transition) are affected by the presence of inhomogeneities. In other words, the question is whether the system is "stable" when disorder is introduced.

This chapter is concerned with this question in the framework of the pinning model: this has been studied extensively over the past decades, and a (nearly) complete answer has been provided in recent years.

4.1 Preliminaries on the question of disorder relevance

The main question we want to answer is the following: do the properties of the (phase transition of the) disordered pinning model differ from those of the homogeneous or annealed model?

4.1.1 Relevance vs. irrelevance: the Harris criterion

Let us frame the question of disorder relevance in a general context. Consider a homogeneous physical system in dimension $d \geq 1$ (say on $\mathbb{Z}^d$), endowed with a parameter h that may vary (such as the temperature, an external field, etc.). Assume further that the system undergoes a phase transition as the parameter h crosses some critical threshold h_c. One then asks about the influence of disorder on this phase transition, in the following terms. Let $(\omega_x)_{x\in\mathbb{Z}^d}$ be i.i.d. centered, unit-variance random variables, and let $\beta > 0$ be a parameter tuning the intensity of the disorder. We then consider the previous system perturbed by the *quenched* disorder (ω_x) by locally modifying the parameter h: at the site x, the parameter is now $h + \beta\omega_x$.

The first question to ask is whether the disordered model still exhibits a phase transition (this is not necessarily the case, as we have seen in the previous chapter). If a phase transition does occur at some critical point $h_c(\beta)$, then one may try to compare the characteristics of the phase transition of the disordered (*i.e.* quenched) model with those of the annealed one[1]. This leads to the question of relevance vs. irrelevance of disorder:

- If, for a sufficiently small disorder intensity β, the quenched model has the same critical properties as the annealed model, then disorder is said to be *irrelevant*;
- If, for any arbitrary disorder intensity $\beta > 0$, the quenched model has different critical properties from the annealed model, then disorder is said to be *relevant*.

One may address the question of disorder relevance either in terms of critical points or in terms of the critical behavior of the free energy. Let us thus formulate the relevance vs. irrelevance of disorder in this spirit (in the framework of the pinning model, for simplicity):

- If there is some $\varepsilon_0 > 0$ such that for all $\beta \in (0, \varepsilon_0)$ one has

$$h_c(\beta) = h_c^{\mathrm{a}}(\beta) \quad \text{and} \quad \mathtt{F}(\beta, h_c(\beta) + u) \approx \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta) + u) \quad \text{as} \;\; u \downarrow 0\,,$$

  then disorder is said to be *irrelevant*.
- If for any $\beta > 0$ one has

$$h_c(\beta) \neq h_c^{\mathrm{a}}(\beta) \quad \text{or} \quad \mathtt{F}(\beta, h_c(\beta) + u) \not\approx \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta) + u) \quad \text{as} \;\; u \downarrow 0\,,$$

  then disorder is said to be *relevant*.

The Harris criterion in general

The physicist A.B. Harris gave in his paper [Har74] very general predictions to determine whether disorder is relevant or not. The prediction is based on the critical exponent of the *correlation length*[2] $\xi = \xi(h)$ of the homogeneous model. If this

[1] The annealed model is in fact the natural model to compare with the quenched one.

[2] I will not give a precise definition here, but informally $\xi(h)$ is the scale at which the decay of correlations is observed. Roughly speaking, this means that $|\operatorname{Cov}_h(\vartheta_x, \vartheta_{x+y})| \approx \exp(-|y|/\xi(h))$, or put differently $\xi(h)^{-1} = \lim_{|y|\to\infty} -\frac{1}{|y|}\log|\operatorname{Cov}_h(\vartheta_x, \vartheta_{x+y})|$.

correlation length has a critical exponent ν, *i.e.* if $\xi(h_c+u) \sim u^{-\nu}$ as $u \downarrow 0$, then Harris predicts that one should have

$$\text{disorder irrelevance if } \nu > 2/d,$$
$$\text{disorder relevance if } \nu < 2/d.$$

This is what is now known as the *Harris criterion*. Note that the case $\nu = 2/d$, called *marginal*, is not included in the prediction; in practice, it depends on the details of the model under consideration.

The Harris criterion for the pinning model

As far as the pinning model is concerned, we have a one-dimensional system, *i.e.* $d = 1$. Moreover, it was shown in [Gia08] that homogeneous model correlation length is related to the free energy as follows: $\xi(h) = 1/\mathtt{F}(h)$. Thus, in view of Theorem 3.5, the critical exponent of the correlation length is $\nu := \max(\frac{1}{\alpha}, 1)$. The Harris criterion therefore gives the following prediction for the pinning model:

$$\begin{aligned} &\text{disorder irrelevance if } \alpha < 1/2, \\ &\text{disorder relevance if } \alpha > 1/2. \end{aligned} \tag{4.1}$$

In the past decades, the pinning model has been extensively studied, in particular as a testing ground for the Harris criterion. The pinning model indeed possess several advantages: (i) It is a one-dimensional model, whose homogeneous version is very well understood. (ii) It is in fact a family of models, parametrized by the inter-arrival distribution $K(\cdot)$ and a continuous parameter $\alpha > 0$ (see ($*$)): this allows in particular to cover the whole range of the Harris criterion, including the marginal case $\alpha = \frac{1}{2}$ (which contains the pinning model of the simple random walk on $\mathbb{Z}$).

The Harris criterion (4.1) has now been completely proved for the disorder pinning model; and the marginal case $\alpha = 1/2$ has also been fully treated. The purpose of this chapter is essentially to present the results that prove the criterion (4.1) and some of the ideas behind their proofs.

4.1.2 Monotonicity of disorder relevance

Let us stress that we have shown the inequality $h_c(\beta) \geq h_c^{\mathrm{a}}(\beta)$, see Lemma 3.17. Hence, we have disorder relevance (in terms of critical point shift) if and only if

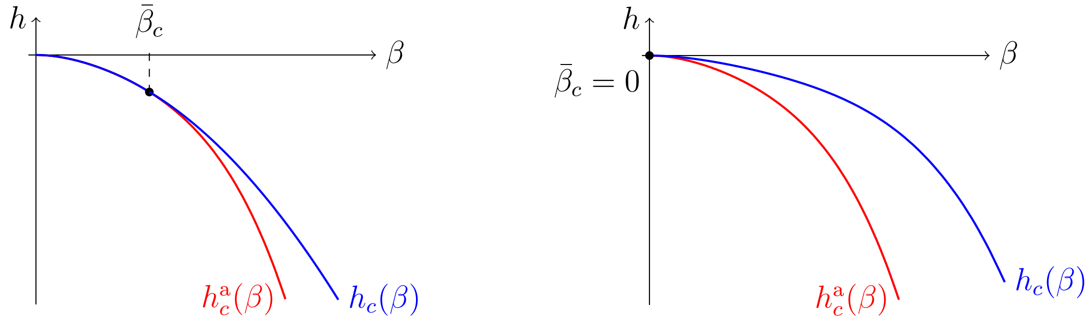


Figure 4.1: The monotonicity of $\beta \mapsto h_c(\beta) - h_c^{\mathrm{a}}(\beta)$ shows that the quenched and annealed critical points are equal up to a certain $\bar\beta_c$. If $\bar\beta_c > 0$ (left), disorder is *irrelevant*; if $\bar\beta_c = 0$ (right), disorder is *relevant*.

$h_c(\beta) > h_c^{\mathrm{a}}(\beta)$ for any $\beta > 0$. The following proposition shows that the critical point shift, *i.e.* $h_c(\beta) - h_c^{\mathrm{a}}(\beta)$, is monotone in β.

Proposition 4.1 (Monotonicity of the critical point shift)**.** *The function $\beta \mapsto h_c(\beta) - h_c^{\mathrm{a}}(\beta)$ is non-decreasing.*

In particular, there is a critical value

$$\bar\beta_c = \inf\left\{\beta, h_c(\beta) > h_c^{\mathrm{a}}(\beta)\right\} \in [0,\infty)\,, \tag{4.2}$$

such that the quenched and annealed critical points are equal for all $\beta \le \beta_c$, and differ for all $\beta > \bar\beta_c$. One can view this as another phase transition, from a *weak disorder* phase ($\beta < \bar\beta_c$) to a *(very) strong disorder* ($\beta > \bar\beta_c$). Let us stress that one can indeed show that $\bar\beta_c < +\infty$ (see [Ton08a] in the case where $\beta^* := \sup\{\beta, \lambda(\beta) < +\infty\} = +\infty$, or Theorem 3.21 in the case $\beta^* < +\infty$.)

Let us also observe that disorder relevance can be reformulated by saying that this critical value is $\beta_c = 0$: in other words, disorder is relevant if for any $\beta > 0$ the system is in the (very) strong disorder phase. We refer to Figure 4.1 for an illustration.

Exercise 36. Recall that $\beta^* = \sup\{\beta,\, \lambda(\beta) < +\infty\}$.

1. Show that $\beta \mapsto h_c(\beta) - h_c^{\mathrm{a}}(\beta)$ is continuous on $(0,\beta^*)$ if $\lambda(\beta^*) = +\infty$ and on $(0,\beta^*]$ if $\lambda(\beta^*) < +\infty$.
2. Deduce that at $\beta = \bar\beta_c$ we have $h_c(\bar\beta_c) = h_c^{\mathrm{a}}(\bar\beta_c)$.

Proof of Proposition 4.1. The proof is extracted from [GLT11, Prop. 6.1]. To prove the result, it is enough to show that, for all $u \in \mathbb{R}$, the function $\beta \mapsto \mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta) + u)$

is non-increasing. Indeed, by definition of the critical point $h_c(\beta)$ we have

$$h_c(\beta) - h_c^{\mathrm{a}}(\beta) = \inf\big\{u;\ \mathtt{F}(\beta, h_c^{\mathrm{a}} + u) > 0\big\}\,,$$

which will then be non-decreasing.

Recall that $h_c^{\mathrm{a}}(\beta) = -\lambda(\beta)$ with $\lambda(\beta) = \log \mathbb{E}[\mathrm{e}^{\beta\omega_1}]$. Let us consider

$$Z^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)+u} = Z^{\beta,\omega}_{N,u-\lambda(\beta)} = \mathrm{E}\Big[\mathrm{e}^{\sum_{i=1}^N (u+\beta\omega_i - \lambda(\beta))\vartheta_i}\vartheta_N\Big]\,,$$

where we have introduced the notation $\vartheta_i := \mathbf{1}_{\{i\in\tau\}}$. We only need to show that, for all $N\in\mathbb{N}$ and all $u\in\mathbb{R}$, the function

$$f_N : \beta \mapsto \mathbb{E}\big[\log Z^{\beta,\omega}_{N,u-\lambda(\beta)}\big]$$

is non-increasing. Differentiating with respect to β, we have

$$\begin{aligned}\frac{\partial f_N}{\partial\beta} &= \mathbb{E}\Bigg[\frac{1}{Z^{\beta,\omega}_{N,u-\lambda(\beta)}}\mathrm{E}\Bigg[\Big(\sum_{i=1}^N(\omega_i - \lambda'(\beta))\vartheta_i\Big)\mathrm{e}^{\sum_{i=1}^N (u+\beta\omega_i-\lambda(\beta))\vartheta_i}\,\vartheta_N\Bigg]\Bigg]\\ &= \mathrm{E}\Bigg[\mathrm{e}^{u\sum_{i=1}^N\vartheta_i}\,\vartheta_N\,\mathbb{E}\Bigg[\frac{1}{Z^{\beta,\omega}_{N,u-\lambda(\beta)}}\Big(\sum_{i=1}^N(\omega_i-\lambda'(\beta))\vartheta_i\Big)\mathrm{e}^{\sum_{i=1}^N(\beta\omega_i-\lambda(\beta))\vartheta_i}\Bigg]\Bigg]\,.\end{aligned}$$

The core of the argument is to show that, for any realization of τ, that is for any fixed $(\vartheta_i)_{1\le i\le N}\in\{0,1\}^N$, we have

$$\mathbb{E}\Bigg[\frac{1}{Z^{\beta,\omega}_{N,u-\lambda(\beta)}}\Big(\sum_{i=1}^N(\omega_i-\lambda'(\beta))\vartheta_i\Big)\mathrm{e}^{\sum_{i=1}^N(\beta\omega_i-\lambda(\beta))\vartheta_i}\Bigg] \le 0\,, \tag{4.3}$$

which yields that $\frac{\partial f_N}{\partial\beta}\le 0$ and thus that $\beta\mapsto f_N(\beta)$ is non-decreasing.

To show (4.3), note that $\mathrm{e}^{\sum_{i=1}^N(\beta\omega_i-\lambda(\beta))\vartheta_i}$ is positive and has expectation 1 (with respect to $\mathbb{E}$), by definition of $\lambda(\beta)$: it can therefore be interpreted as a probability density. Thus, for a fixed realization of τ, we define the probability measure $\mathbb{P}^{(\tau)}_{N,\beta}$ as follows:

$$\frac{\mathrm{d}\mathbb{P}^{(\tau)}_{N,\beta}}{\mathrm{d}\mathbb{P}}(\omega) = \prod_{i=1}^N \mathrm{e}^{(\beta\omega_i-\lambda(\beta))\vartheta_i}\,. \tag{4.4}$$

Notice that this is a product measure: under $\mathbb{P}^{(\tau)}_{N,\beta}$ the $(\omega_i)_{1\le i\le N}$ are independent, with law $\mathbb{P}$ if $\vartheta_i=0$ (*i.e.* $i\notin\tau$) and law $\mathbb{P}_\beta$ if $\vartheta_i = 1$ (*i.e.* $i\in\tau$), where $\mathbb{P}_\beta$ is the

β-*tilted* law of ω, that is

$$\frac{\mathrm{d}\mathbb{P}_\beta}{\mathrm{d}\mathbb{P}}(\omega_i) = \mathrm{e}^{\beta\omega_i - \lambda(\beta)} = \frac{\mathrm{e}^{\beta\omega_i}}{\mathbb{E}[\mathrm{e}^{\beta\omega_i}]}\,. \tag{4.5}$$

With the definition (4.4), we can thus rewrite the left-hand side of (4.3) as

$$\mathbb{E}^{(\tau)}_{N,\beta}\bigg[\frac{1}{Z^{\beta,\omega}_{N,u-\lambda(\beta)}}\Big(\sum_{i=1}^{N}(\omega_i - \lambda'(\beta))\vartheta_i\Big)\bigg]\,.$$

We then use the Harris inequality, also called the FKG inequality (for Fortuin–Kasteleyn–Ginibre, in a more general setting). Its proof is left as an exercise (see Exercises 37 below).

Theorem (Harris–FKG inequality)**.** *Let $(\omega_i)_{1\le i\le n}$ be independent real random variables (not necessarily identically distributed) and let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two non-decreasing functions (i.e., non-decreasing coordinate-wise). Then we have*

$$\mathbb{E}\big[f(\omega_1,\dots,\omega_n)g(\omega_1,\dots,\omega_n)\big] \ge \mathbb{E}\big[f(\omega_1,\dots,\omega_n)\big]\mathbb{E}\big[g(\omega_1,\dots,\omega_n)\big]\,.$$

To apply the Harris–FKG inequality, observe that for any $\beta \ge 0$, the functions

$$(\omega_1,\dots,\omega_N) \mapsto Z^{\beta,\omega}_{N,u-\lambda(\beta)} = \mathrm{E}\Big[\mathrm{e}^{\sum_{i=1}^{N}(u+\beta\omega_i-\lambda(\beta))\vartheta_i}\vartheta_N\Big]$$

$$(\omega_1,\dots,\omega_N) \mapsto \sum_{i=1}^{N}(\omega_i - \lambda'(\beta))\vartheta_i$$

are non-decreasing. Thus, $(\omega_1,\dots,\omega_N) \mapsto 1/Z^{\beta,\omega}_{N,u-\lambda(\beta)}$ is non-increasing and thanks to the fact that $\mathbb{P}^{(\tau)}_{N,\beta}$ is a product measure, we can use the Harris–FKG inequality (the inequality is reversed if one of the functions is non-increasing), which shows that

$$\mathbb{E}^{(\tau)}_{N,\beta}\bigg[\frac{1}{Z^{\beta,\omega}_{N,u-\lambda(\beta)}}\Big(\sum_{i=1}^{N}(\omega_i - \lambda'(\beta))\vartheta_i\Big)\bigg] \le \mathbb{E}^{(\tau)}_{N,\beta}\bigg[\frac{1}{Z^{\beta,\omega}_{N,u-\lambda(\beta)}}\bigg]\sum_{i=1}^{N}\mathbb{E}^{(\tau)}_{N,\beta}\big[\omega_i - \lambda'(\beta)\big]\vartheta_i\,.$$

It remains to notice that in the case $\vartheta_i = 1$ we have $\mathbb{E}^{(\tau)}_{N,\beta}\big[\omega_i - \lambda'(\beta)\big] = 0$, since by the definition (4.5) of $\mathbb{P}_\beta$ we have

$$\mathbb{E}_\beta[\omega_i] = \frac{1}{\mathbb{E}[\mathrm{e}^{\beta\omega_i}]}\mathbb{E}[\omega_i\mathrm{e}^{\beta\omega_i}] = \frac{\partial}{\partial\beta}\log\mathbb{E}[\mathrm{e}^{\beta\omega_i}] = \lambda'(\beta)\,.$$

This concludes the proof of (4.3) and thus of Proposition 4.1. □

Exercise 37 (Proof of the Harris–FKG inequality)**.** The proof is done by induction.

1. Case $n = 1$. Let ω_1, ω_1' be two independent copies of ω_1 and let f, g be two non-decreasing functions. Show that $\mathbb{E}[(f(\omega_1) - g(\omega_1))(f(\omega_1') - g(\omega_1'))] \geq 0$, then conclude.
2. Induction step, case $n \geq 2$. Define $\tilde{f}(\omega_1) = \mathbb{E}[f(\omega_1, \dots, \omega_n) \mid \omega_1]$ and $\tilde{g}(\omega_1) = \mathbb{E}[g(\omega_1, \dots, \omega_n) \mid \omega_1]$. Applying the inequality for $n = 1$ to $\tilde{f}, \tilde{g}$, prove the induction step. Conclude.

4.2 From weak to strong disorder: the martingale approach

4.2.1 The martingale

To get some insight as to whether the quenched and annealed models behave similarly, one possibility is to look at the quenched system at the annealed critical point $h_c^{\mathrm{a}}(\beta) = -\lambda(\beta)$. Let us define (recall $\vartheta_i = \mathbf{1}_{\{i \in \tau\}}$)

$$W_N^{\beta,\omega} := \tilde{Z}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega} = \mathrm{E}\Big[\exp\Big(\sum_{i=1}^{N} (\beta\omega_i - \lambda(\beta))\vartheta_i\Big)\Big], \tag{4.6}$$

where we have considered here the partition function of the free model (*i.e.* without the constraint $N \in \tau$), see (3.5).

If $W_N^{\beta,\omega}$ stays close to $\mathbb{E}[W_N^{\beta,\omega}] = 1$, this suggests that the quenched and annealed models behave similarly (at least at $h_c^{\mathrm{a}}(\beta)$). If, on the other hand, $W_N^{\beta,\omega}$ goes to 0 as $N \to \infty$, this means that the expectation $\mathbb{E}[W_N^{\beta,\omega}] = 1$ is carried by atypical realizations of the environment ω; this suggests that the quenched model will behave differently from the annealed model (at least at $h_c^{\mathrm{a}}(\beta)$).

Lemma 4.2 (Martingale property)**.** *For any $\beta \geq 0$, the sequence $(W_N^{\beta,\omega})_{N \geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_N := \sigma\{\omega_i, i \leq N\}$. It is a non-negative martingale, so the following limit exists $\mathbb{P}$-almost surely:*

$$W_\infty^{\beta,\omega} := \lim_{N \to \infty} W_N^{\beta,\omega}.$$

Exercise 38. Prove Lemma 4.2.

The question now is whether this limit is degenerate or not, *i.e.* $W_\infty^{\beta,\omega} = 0$ or not, as this should be associated to strong or weak disorder. An important observation is the following (see Exercise 39 for its proof):

$$\text{the event } \{W_\infty^{\beta,\omega} = 0\} \text{ belongs to the tail } \sigma\text{-algebra } \mathcal{Q} = \bigcap_{k\geq 0} \mathcal{Q}_k\,,$$

where $\mathcal{Q}_k = \sigma\{\omega_i, i > k\}$. By Kolmogorov's 0-1 law, we have $\mathrm{P}(W_\infty^{\beta,\omega} = 0) = 0$ or 1, which can be summarized by the following dichotomy.

Lemma 4.3 (Weak and strong disorder)**.** *We have*

$$\lim_{N\to\infty} W_N^{\beta,\omega} = W_\infty^{\beta,\omega}\,, \qquad \text{with} \quad \begin{cases} \text{either } W_\infty^{\beta,\omega} > 0 & \mathbb{P}\text{-a.s.} \quad \text{(weak disorder)},\\ \text{or } W_\infty^{\beta,\omega} = 0 & \mathbb{P}\text{-a.s.} \quad \text{(strong disorder)}. \end{cases}$$

We again use the denomination *weak* and *strong* disorder, but here it has a slightly different meaning from that introduced in Section 4.1.2. We however use the same terminology because it is conjectured that the two notions coincide.

Exercise 39. The goal is to show that, for any $k \geq 1$, we have $\{W_\infty^{\beta,\omega} = 0\} \in \mathcal{Q}_k$.

1. Show that, for any $N \geq k$, we have

$$W_N^{\beta,\omega} \geq \mathrm{P}(k \in \tau)\, e^{-\beta \sum_{i=1}^k |\omega_k| - k\lambda(\beta)}\; W_{N-k,\beta}^{\theta^k \omega}$$

   and also

$$W_N^{\beta,\omega} \leq e^{\beta \sum_{i=1}^k |\omega_k| - k\lambda(\beta)} \sum_{j=k}^{N} \mathrm{P}(\tau_1 \geq j-k) W_{N-j,\beta}^{\theta^j \omega}\,.$$

2. Deduce that, for any $k \geq 1$, $\{W_\infty^{\beta,\omega} = 0\} = \bigcap_{j\geq k} \{\lim_{n\to\infty} W_{n,\beta}^{\theta^j \omega} = 0\}$.
3. Conclude.

4.2.2 Weak vs. strong disorder

The following result, based on a monotonicity property of the event $\{W_\infty^{\beta,\omega} = 0\}$, shows that there is a phase transition between a weak disorder phase and a strong disorder phase.

Proposition 4.4 (Phase transition for the martingale)**.** *There exists some* $\tilde\beta_c \in [0,\infty)$ *such that:*

$$\begin{aligned} W_\infty^{\beta,\omega} > 0 \ \ \mathbb{P}\text{-a.s.} &\quad \text{for } \beta < \tilde\beta_c\,,\\ W_\infty^{\beta,\omega} = 0 \ \ \mathbb{P}\text{-a.s.} &\quad \text{for } \beta > \tilde\beta_c\,. \end{aligned}$$

We will show below, see Theorem 4.8, that if $W_\infty^{\beta,\omega} > 0$ $\mathbb{P}$-a.s. then we have $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$ (and the quenched and annealed critical behaviors are similar). In particular, this shows that $\tilde{\beta}_c \geq \bar{\beta}_c$. On the other hand, we will see in Theorem 4.17 (combined with Theorem 4.6) below that $\bar{\beta}_c = 0$ as soon as $\tilde{\beta}_c = 0$, and in particular $\bar{\beta}_c = \tilde{\beta}_c = 0$ in that case. It is therefore natural to wonder whether we in fact always have $\bar{\beta}_c = \tilde{\beta}_c$, that is whether the two notions of weak disorder ($W_\infty^{\beta,\omega} > 0$ or $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$) and strong/very strong disorder ($W_\infty^{\beta,\omega} = 0$ or $h_c(\beta) > h_c^{\mathrm{a}}(\beta)$) coincide.

This is still a conjecture, but the analogous result has been proven very recently [JL24, JL25] in the context of directed polymers (a closely related model), solving a long-standing problem.

Conjecture 4.5 (Strong and very strong disorder coincide)**.** *If $\alpha \neq 0$, then strong disorder, i.e. $W_\infty^{\beta,\omega} = 0$, implies very strong disorder, i.e. $h_c(\beta) > h_c^{\mathrm{a}}(\beta)$. In particular, we have $\bar{\beta}_c = \tilde{\beta}_c$.*

Let us stress that this would also show that *weak disorder holds* at $\beta = \tilde{\beta}_c$, since we have $h_c(\bar{\beta}_c) = h_c^{\mathrm{a}}(\bar{\beta}_c)$, see Exercise 36. The case $\alpha = 0$ is more delicate, and in particular one may indeed have $\bar{\beta}_c > \tilde{\beta}_c$ (see [Viv23] in the context of directed polymers, see also Remark 4.12 below).

The core of the proof of Proposition 4.4 relies on the same tools as those of Proposition 4.1, here we give only the main steps, in the form of an exercise.

Exercise 40 (Proof of Proposition 4.4)**.** Let us fix $\gamma \in (0,1)$.

1. Show that $\lim_{N\to\infty} \mathbb{E}\big[(W_N^{\beta,\omega})^\gamma\big] = \mathbb{E}\big[(W_\infty^{\beta,\omega})^\gamma\big]$.
2. Deduce that $W_\infty^{\beta,\omega} = 0$ $\mathbb{P}$-a.s. if and only if $\lim_{N\to\infty} \mathbb{E}\big[(W_N^{\beta,\omega})^\gamma\big] = 0$.
3. Show that for all $N \in \mathbb{N}$, the function$\beta \mapsto \mathbb{E}\big[(W_N^{\beta,\omega})^\gamma\big]$ is non-increasing.
   (Hint: follow the ideas in the proof of Proposition 4.1.)
4. Conclude.

4.2.3 Criterion for the existence of a weak disorder phase

Let us now give the main result of this section, which allows to determine whether the martingale really goes through a phase transition, *i.e.* whether $\tilde{\beta}_c = 0$ or $\tilde{\beta}_c > 0$. Notice that determining whether $\tilde{\beta}_c = 0$ is related to a (weak) notion of disorder relevance; notice also that showing that $\tilde{\beta}_c > 0$ will show that $\bar{\beta}_c > 0$ (once we know that $\bar{\beta}_c \geq \tilde{\beta}_c$, see Theorem 4.8).

In the following, we assume that the underlying renewal process is recurrent and satisfies condition $(*)$.

Theorem 4.6 (Criterion for having $\tilde{\beta}_c = 0$)**.** *We have*

$$\tilde{\beta}_c = 0 \quad \textit{if and only if} \quad \sum_{n=0}^{\infty} \mathrm{P}(n \in \tau)^2 = +\infty \,.$$

In particular, recalling Theorem 3.10, we have $\tilde{\beta}_c > 0$ if $\alpha < \frac{1}{2}$ and $\tilde{\beta}_c = 0$ if $\alpha > \frac{1}{2}$.

For the second part of the theorem, one needs to refer to Section 3.3. In particular, we deduce that if $\mathrm{E}[\tau_1] < +\infty$ (for instance if $\alpha > 1$), then $\mathrm{P}(n \in \tau) \to \mathrm{E}[\tau_1]^{-1}$, which shows that $\sum_{n=0}^{\infty} \mathrm{P}(n \in \tau)^2 = +\infty$. The case $\alpha = 1$ falls in the same framework, since then $\mathrm{P}(n \in \tau) \sim \mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq n\}}]$ is a slowly varying function.

If $\alpha \in (0,1)$, then by Theorem 3.10 we have that $\mathrm{P}(n \in \tau) \sim c_\alpha L(n)^{-1} n^{-(1-\alpha)}$ and thus

$$\sum_{n=0}^{\infty} \mathrm{P}(n \in \tau)^2 = +\infty \quad \Longleftrightarrow \quad \sum_{n=0}^{\infty} \frac{1}{L(n)^2 n^{1+2\alpha-1}} = +\infty \,.$$

We recover the criterion given in Theorem 4.6: the sum is finite (so $\tilde{\beta} + c > 0$) if $\alpha < 1/2$ and infinite (so $\tilde{\beta}_c = 0$) if $\alpha > 1/2$.

In the case $\alpha = 1/2$, we obtain that $\tilde{\beta}_c = 0$ if and only if $\sum_{n \geq 1} \frac{1}{nL(n)^2} = +\infty$. Note that in the case of the simple random walk, the slowly varying function $L(n)$ converges to a constant: this means that the sum is infinite, so that $\tilde{\beta}_c = 0$; this suggests that the disorder is relevant in this case.

Remark 4.7 (Relation with the intersection of two renewals)**.** Notice that if τ, τ' are two independent copies of a renewal process, then the intersection $\tau \cap \tau'$ is again a renewal process. Moreover, the renewal $\tau \cap \tau'$ is finite (transient) if and only if $\mathrm{E}[|\tau \cap \tau'|] < +\infty$. Indeed, thanks to the renewal property, to total number of renewal points $|\tau \cap \tau'|$ is a geometric variable with parameter $\mathrm{P}((\tau \cap \tau')_1 = +\infty) = \mathrm{E}[|\tau \cap \tau'|]^{-1}$. In summary, $\tau \cap \tau'$ is transient if and only if

$$\mathrm{E}[|\tau \cap \tau'|] = \sum_{n=0}^{\infty} \mathrm{P}(n \in \tau \cap \tau') = \sum_{n=0}^{\infty} \mathrm{P}(n \in \tau)^2 < +\infty \,.$$

We postpone the proof of Theorem 4.6 to Section 4.5 below. The next two sections deal with the question of disorder relevance/irrelevance: Section 4.3 shows that having $W_\infty^\beta > 0$ a.s. implies that $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$, which shows in particular that $\bar\beta_c := \inf\{\beta > 0, h_c(\beta) > h_c^{\mathrm{a}}(\beta)\} \geq \tilde\beta_c$; Section 4.4 shows that the criterion of Theorem 4.6 also hold for $\bar\beta_c$, and in particular we have $\bar\beta_c = 0$ if and only if $\tilde\beta_c = 0$, see (4.11) below.

4.3 Disorder irrelevance

In this section, we show the following result: it shows that, in the weak disorder regime, the quenched and annealed critical points are equal and the critical behaviors are similar. The proof is adapted from [Lac10].

Theorem 4.8 (A condition for disorder irrelevance). *Let us assume that $\beta > 0$ is such that $\lim_{N\to\infty} W_N^{\beta,\omega} = W_\infty^{\beta,\omega} > 0$ a.s. Then we have*

$$h_c(\beta) = h_c^{\mathrm{a}}(\beta)\,.$$

In particular, one has that $\bar\beta_c := \inf\{\beta,\, h_c(\beta) > h_c^{\mathrm{a}}(\beta)\} \geq \tilde\beta_c$. Additionally, we have that

$$\mathtt{F}(\beta, h_c(\beta) + u) = u^{\nu + o(1)} \qquad \text{as } u \downarrow 0\,,$$

where $\nu := \max(\frac{1}{\alpha}, 1)$ is the critical exponent of the annealed model.

Combined with Theorem 4.6 (and Proposition 4.4), an immediate corollary is the following.

Corollary 4.9 (Disorder irrelevance). *If $\sum_{n=1}^\infty \mathrm{P}(n \in \tau)^2 < +\infty$, then disorder is* irrelevant.

Proof. Theorem 4.6 shows that $\tilde\beta_c > 0$. Then, Theorem 4.8 shows that for all $\beta < \tilde\beta_c$ we have $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$; in other words $\bar\beta_c \geq \tilde\beta_c > 0$. Additionally, for all $\beta < \tilde\beta_c$ we have that $\mathtt{F}(\beta, h_c(\beta) + u) = \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta) + u)^{1+o(1)}$ as $u \downarrow 0$. □

Let us stress that Theorem 4.8 is in fact interesting only in the case where $\sum_{n=1}^\infty \mathrm{P}(n \in \tau)^2 < +\infty$ since otherwise we have $W_\infty^{\beta,\omega} = 0$ a.s. for any $\beta > 0$, by Theorem 4.6. We therefore only need to prove Theorem 4.8 in that case; in particular, me may assume that $\alpha \in (0, \frac{1}{2}]$, leaving aside the case $\alpha = 0$ for simplicity (see Remark 4.12 for a discussion).

Remark 4.10. The critical behavior $\mathtt{F}(\beta, h_c(\beta)+u) = u^{\nu+o(1)}$ can in fact be improved if one strengthen the assumption of the theorem. More precisely (if $\alpha \neq 0$), [Ale08, Ton08b] show that in the irrelevant disorder regime $\sum_{n=1}^{\infty} \mathrm{P}(n \in \tau)^2 < +\infty$, if β is fixed small enough, we have that $\mathtt{F}(\beta, h_c(\beta)+u) \asymp \mathtt{F}(0,u)$. In fact, [GT09] shows that, if β is small enough, then $\mathtt{F}(\beta, h_c(\beta)+u) \sim \mathtt{F}(0,u)$ as $u \downarrow 0$.

Step 1. Properties of $\mathrm{P}_{N,h}^{\beta,\omega}$ at $h = h_c^{\mathrm{a}}(\beta)$. The following lemma shows that if we have $W_\infty^{\beta,\omega} > 0$ a.s., then the *free* pinning measure $\tilde{\mathrm{P}}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}$, *i.e.* without the constraint $\{N \in \tau\}$, see (3.4), is “close to” the reference measure P. Here, “close to” means that events that are typical under P are also typical under $\tilde{\mathrm{P}}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}$.

Lemma 4.11. *Let $(A_N)_{N\geq 0}$ be a sequence of $\mathcal{F}_N$-measurable events, where $\mathcal{F}_N := \sigma\{\tau \cap [0,N]\}$. Then if $\lim_{N\to\infty} W_N^{\beta,\omega} = W_\infty^{\beta,\omega} > 0$ a.s., we have that*

$$\lim_{N\to\infty} \mathrm{P}(A_N) = 1 \quad \Longrightarrow \quad \lim_{N\to\infty} \mathrm{P}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}(A_N) = 1 \quad \text{in } L^1(\mathbb{P})\,.$$

Proof. We work with the complement of A_N. Let $\varepsilon > 0$ be arbitrary and write

$$\mathbb{E}\big[\mathrm{P}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}(A_N^c)\big] \leq \mathbb{P}\big(W_N^{\beta,\omega} < \varepsilon\big) + \mathbb{E}\big[\mathrm{P}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}(A_N^c)\mathbf{1}_{\{W_N^{\beta,\omega}\geq\varepsilon\}}\big]\,.$$

For the second term, recalling that $W_N^{\beta,\omega} = \tilde{Z}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}$ is the *free* partition function, we have

$$\begin{aligned}\mathrm{P}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}(A_N^c)\mathbf{1}_{\{W_N^{\beta,\omega}\geq\varepsilon\}} &= \frac{1}{W_N^{\beta,\omega}}\mathrm{E}\Big[\mathrm{e}^{\sum_{i=1}^N(\beta\omega_i-\lambda(\beta))\vartheta_i}\mathbf{1}_{A_N^c}\Big]\mathbf{1}_{\{W_N^{\beta,\omega}\geq\varepsilon\}}\\ &\leq \frac{1}{\varepsilon}\mathrm{E}\Big[\mathrm{e}^{\sum_{i=1}^N(\beta\omega_i-\lambda(\beta))\vartheta_i}\mathbf{1}_{A_N^c}\Big]\,.\end{aligned}$$

Taking the expectation, recalling that $\mathbb{E}[\mathrm{e}^{\beta\omega_i-\lambda(\beta)}] = 1$, we deduce that

$$\mathbb{E}\big[\mathrm{P}_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega}(A_N^c)\big] \leq \mathbb{P}\big(W_N^{\beta,\omega} < \varepsilon\big) + \varepsilon^{-1}\mathrm{P}(A_N^c)\,.$$

Taking the limit as $N \to \infty$, the upper bound converges to $\mathbb{P}(W_\infty^{\beta,\omega} < \varepsilon)$, which can be made arbitrarily small by taking $\varepsilon \downarrow 0$ since we assumed that $W_\infty^{\beta,\omega} > 0$ a.s. □

Step 2. Finite-size and convexity argument. Let $u > 0$. In order to show that $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$, we need to show that $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta)+u) > 0$. Recall that by

Theorem 3.12 and Remark 3.13 (see (3.13)), there exists a constant c ($= c_{h,\beta}$) such that the free energy satisfies

$$\forall\, N \in \mathbb{N} \qquad \mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta) + u) \geq \frac{1}{N}\mathbb{E}\big[\log \tilde{Z}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)+u}\big] - c\frac{\log N}{N}\,.$$

Now, by convexity of $h \mapsto \log \tilde{Z}^{\beta,\omega}_{N,h}$, we have

$$\log \tilde{Z}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)+u} \geq \log \tilde{Z}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)} + u\tilde{\mathrm{E}}^{\beta,\omega}_{N,h_c^{\mathrm{a}}}\Big[\sum_{i=1}^{N} \vartheta_i\Big]\,.$$

Therefore, using the bound $\log \tilde{Z}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)} \geq \log \mathrm{P}(\tau_1 > N) \geq -c' \log N$, we get that

$$\forall\, N \in \mathbb{N} \qquad \mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta) + u) \geq \frac{u}{N}\mathbb{E}\bigg[\mathrm{E}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)}\Big[\sum_{i=1}^{N} \vartheta_i\Big]\bigg] - c''\,\frac{\log N}{N}\,.$$

Define $U_N := \sum_{i=1}^N \mathrm{P}(i \in \tau)$ and notice that thanks to Theorem 3.10 (recall $\alpha \in (0,1)$) we have $U_N \sim c_\alpha N^\alpha L(N)^{-1}$. Let us now consider the sequence of events $A_N := \big\{ \sum_{i=1}^N \vartheta_i \geq k_N \big\}$, where $k_N := \varepsilon_N U_N$ is an integer, with $\varepsilon_N \downarrow 0$. We now show that

$$\mathrm{P}(A_N^c) = \mathrm{P}(\tau_{k_N} > N) \leq C k_N \mathrm{P}(\tau_1 > N) \leq C' \varepsilon_N U_N N^{-\alpha} L(N) \xrightarrow{N\to\infty} 0\,, \qquad (4.7)$$

where we have used that $U_N \sim c_\alpha N^\alpha L(N)^{-1}$ and $\varepsilon_N \downarrow 0$.

In order to show (4.7), we use a truncation argument: we write

$$\begin{aligned}\mathrm{P}\big(\tau_{k_N} > N\big) &\leq \mathrm{P}\big(\exists\, i \leq k_N\,, \tau_i - \tau_{i-1} > N\big) + \mathrm{P}\Big(\sum_{i=1}^{k_N} (\tau_i - \tau_{i-1}) \mathbf{1}_{\{\tau_i - \tau_{i-1} \leq N\}} > N\Big) \\ &\leq k_N \mathrm{P}(\tau_1 > N) + \frac{k_N}{N}\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq N\}}]\,,\end{aligned}$$

where for the second inequality we have used subadditivity and Markov's inequality. We now notice that $\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq N\}}] \sim (1-\alpha)^{-1} L(N) N^{1-\alpha}$ (by properties of regularly varying functions, see Remark 3.1) so that $N^{-1}\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq N\}}] \leq C\mathrm{P}(\tau_1 > N)$. This shows (4.7).

Now, in view of (4.7) and using Lemma 4.11, we obtain that $\mathbb{E}[\mathrm{P}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)}(A_N)] \geq \frac{1}{2}$ for N large enough, so that

$$\mathbb{E}\bigg[\mathrm{E}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)}\Big[\sum_{i=1}^{N} \vartheta_i\Big]\bigg] \geq k_N \mathbb{E}\big[\mathrm{P}^{\beta,\omega}_{N,h_c^{\mathrm{a}}(\beta)}(A_N)\big] \geq \frac{1}{2} k_N\,.$$

We conclude that for all sufficiently large N

$$\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta)+u) \geq \frac{1}{2N}\big(u\, k_N U_N - c\log N\big)\,. \tag{4.8}$$

Let us now fix some arbitrary $\eta \in (0,\alpha)$ and choose $k_N = N^{\alpha-\eta}$, *i.e.* take $\varepsilon_N := N^{-\eta} U_N N^{\alpha} \sim c_\alpha N^{-\eta} L(N)^{-1} \to 0$. We then take $N = N_u := u^{-\frac{1}{\alpha}}$ in (4.8) to obtain that for u small enough (*i.e.* for N_u large enough)

$$\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta)+u) \geq \frac{u^{1/\alpha}}{2}\big(u^{-\eta/\alpha} - c'\log(1/u)\big) \geq c\, u^{(1-\eta)/\alpha}\,,$$

where we have used that $\log(1/u) = o(u^{-\eta/\alpha})$ as $u \downarrow 0$.

Since η is arbitrary, this shows that $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta)+u) \geq u^{\frac{1}{\alpha}+o(1)}$ as $u \downarrow 0$. In particular, this shows that $h_c(\beta) \geq h_c^{\mathrm{a}}(\beta)$, so the critical points are equal since the reverse inequality always holds. This shows in particular that for any $\beta < \tilde\beta_c$, then $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$, *i.e.* we also have $\beta \leq \bar\beta_c$, so that $\bar\beta_c \geq \tilde\beta_c$.

As far as the critical behavior is concerned, recall that we already know that $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta)+u) \leq \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta)+u) = \mathtt{F}(0,u)$ by Lemma 3.17 and that $\mathtt{F}(0,u) = u^{\nu+o(1)}$, see Theorem 3.5. This gives that $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta)+u) \leq u^{\nu+o(1)}$ and concludes the proof of Theorem 4.8, recalling that we only treat the case where $\alpha \in (0,\frac12]$, for which $\nu = \frac{1}{\alpha}$. □

Remark 4.12 (The case $\alpha = 0$)**.** In the above proof, we have left aside the case $\alpha = 0$, which is a bit special. The above proof cannot be adapted, but the conclusion of Theorem 4.8 remains valid. In fact, it is shown in [AZ10] that if $\lambda(\beta) < +\infty$ for all $\beta > 0$, then $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$ for all $\beta > 0$! Once we know that $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$, the annealed bound $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta)+u) \leq \mathtt{F}(0,u)$ (see Lemma 3.17), combined with Theorem 3.5, shows that $\mathtt{F}(\beta, h_c(\beta)+u)$ decays faster than any power as $u \downarrow 0$.

4.4 Disorder relevance

As mentioned in Section 4.1.1, disorder relevance could be observed both in terms of a difference in the (quenched vs. annealed) critical behavior or in terms of a different (quenched vs. annealed) critical point. The next two sections address these two questions.

4.4.1 About the critical behavior: the smoothing inequality

In this section, we show the following result, which tells that the phase transition of the disordered model is at least of order 2. This was first proven in [GT06b], but we also refer to see [GZ25, Thm. 1.5] for the most general statement given below, and [CdH13] for sharp estimates on the constants.

Theorem 4.13 (Smoothing of the phase transition)**.** *For any $\beta > 0$ such that $h_c(\beta) > -\infty$, there exists a constant $C_\beta > 0$ such that*

$$\mathtt{F}(\beta, h_c(\beta) + u) \leq C_\beta \, u^2 \qquad \text{for } u \in (0,1).$$

Since we have $\mathtt{F}^{\mathrm{a}}(\beta, h_c(\beta) + u) = \mathtt{F}(0, u) = u^{\nu + o(1)}$ as $u \downarrow 0$ with $\nu = \max(\frac{1}{\alpha}, 1)$ (see Theorem 3.5), an immediate corollary is the following.

Corollary 4.14. *If $\alpha > 1/2$, then disorder is* relevant*: for any $\beta > 0$, the phase transition of the quenched free energy is* smoother *than that of the annealed one.*

Proof. Indeed, for any $\beta > 0$, we have

$$\mathtt{F}(\beta, h_c(\beta) + u) = O(u^2) \ll u^{\nu + o(1)} = \mathtt{F}^{\mathrm{a}}(\beta, h_c(\beta) + u) \qquad \text{as } u \downarrow 0. \qquad \square$$

Notice that Theorem 4.13 is very general and for instance its proof will *not* rely on the fact that disorder is relevant or that the quenched and annealed critical points are different. In fact, Derrida and Retaux [DR14] introduced some simplified (recursive) version of the model: in the disorder relevant regime, sharper estimates on the free energy have been obtained, proving that the phase transition is of infinite order, see [CDD+21]. This leads to the following conjecture put forward in [DR14].

Conjecture 4.15 (Derrida–Retaux conjecture)**.** *For any $\beta \in (0, (1+\alpha)\beta^*)$ (recall Theorem 3.21), then if $h_c(\beta) > h_c^{\mathrm{a}}(\beta)$, there is a constant K_β such that*

$$\mathtt{F}(\beta, h_c(\beta) + u) = \exp\Big(-(1+o(1)) \frac{K_\beta}{\sqrt{u}} \Big) \qquad \text{as } u \downarrow 0\,.$$

Of course, this is (much) better than the smoothing inequality of Theorem 4.13, and even showing that the phase transition gets smoothen more than quadratically is an important challenge. Let us note that this conjecture is not fully proven in the Derrida–Retaux recursive model, but [CDD+21] identifies (under some integrability condition) the correct behavior $\exp(-u^{-\frac{1}{2}+o(1)})$, which is already an impressive achievement.

Proof of Theorem 4.13. We focus on the case $\omega_i \sim \mathcal{N}(0,1)$, which simplifies some arguments.

The idea of the proof is to obtain a *lower bound* on the free energy: more precisely we get a lower bound on $\mathtt{F}(\beta,h)$ in terms of $\mathtt{F}(\beta,h+u)$. We will show the following lower bound: for any β,h, there exists C_β such that for all $u\in(0,1)$,

$$\mathtt{F}(\beta,h) \geq K_{\beta,h,u}\big(\mathtt{F}(\beta,h+u) - C_\beta\, u^2\big)\,, \tag{4.9}$$

for some constant $K_{\beta,h,u}>0$ that depends on all parameters. Taking $h=h_c(\beta)$, the left-hand side is zero, which thus shows $\mathtt{F}(\beta,h+u)\leq C_\beta\, u^2$, as desired.

We now prove (4.9) for a fixed u such that $h+u>h_c(\beta)$ (otherwise there is nothing to prove). We will use a rare-stretch strategy, in the spirit of Exercises 35 and 29.

Let us fix $\varepsilon>0$ small enough so that $(1-\varepsilon)\mathtt{F}(\beta,h+u)>\mathtt{F}(\beta,h)$, and for ℓ fixed but large (how large depends on ε,β,h as we will see below), we introduce the event

$$\mathcal{A}_\ell = \Big\{\omega\ ;\ \frac{1}{\ell}\log Z^{\beta,\omega}_{\ell,h} \geq (1-\varepsilon)\mathtt{F}(\beta,h+u)\Big\}\,.$$

Note that this event depends only on $(\omega_1,\ldots,\omega_\ell)$, which we refer to as a *block* of length ℓ.

Let us now observe that, $\mathbb{P}$-a.s., $\lim_{\ell\to\infty}\frac{1}{\ell}\log Z^{\beta,\omega}_{\ell,h} = \mathtt{F}(\beta,h) < \mathtt{F}(\beta,h+u)$, so $\mathcal{A}_\ell$ is atypical, in the sense that

$$\lim_{\ell\to\infty}\mathbb{P}(\mathcal{A}_\ell)=0\,.$$

An environment $\omega\in\mathcal{A}_\ell$ will be called *favorable*: the free energy $\mathtt{F}^\omega_\ell := \frac{1}{\ell}\log Z^{\beta,\omega}_{\ell,h}$ is strictly larger than its typical value $\mathtt{F}(\beta,h)$. Equivalently, we will say that the *block* $(\omega_1,\ldots,\omega_\ell)$ is favorable. Let us now introduce the set of *favorable* blocks in ω: we denote by $\mathcal{I}$ the indices of these blocks (excluding the first for simplicity),

$$\mathcal{I} := \big\{ i\geq 1\,, \theta^{i\ell}\omega\in\mathcal{A}_\ell\big\}\,.$$

Note that since $\mathcal{A}_\ell$ is atypical, favorable blocks are rare. We then obtain a lower bound on the partition function by using a *rare stretch strategy*, which consists in visiting only and all favorable blocks.

Let us write $\mathcal{I}=\{i_k\}_{k\geq1}$ with $i_1<i_2<\cdots$ the indices of favorable blocks. Then, keeping only the contribution of trajectories of τ that visit the first m rare favorable

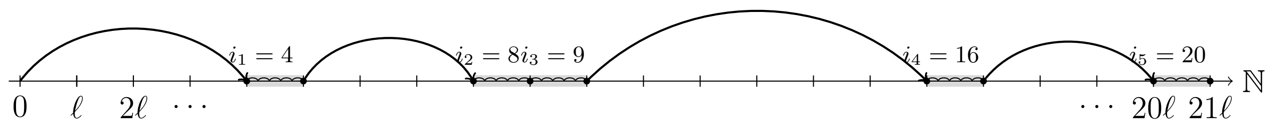


Figure 4.2: Illustration of the rare stretch strategy. Here, we consider trajectories that visit *exactly* the first $m=5$ favorable blocks; these are trajectories of length $(i_m+1)\ell$. Inside favorable regions, we have $\frac{1}{\ell}\log Z_{\ell,h}^{\beta,\theta^i\omega} \geq (1-\varepsilon)\mathtt{F}(\beta,h+u)$ by definition of $\mathcal{A}_\ell$.

regions, we obtain the lower bound (see Figure 4.2 for an illustration):

$$Z_{(i_m+1)\ell,h}^{\beta,\omega} \geq \prod_{k=1}^{m} \mathrm{P}\big(\tau_1 = (i_k - i_{k-1} - 1)\ell\big) Z_{\ell,h}^{\beta,\theta^{i_k\ell}\omega},$$

with the convention that $\mathrm{P}(\tau_1 = 0) = 1$ if $i_k = i_{k-1}+1$.

Notice that, by definition of $\mathcal{A}_\ell$, inside a favorable block $i \in \mathcal{I}$, we have

$$\frac{1}{\ell}\log Z_{\ell,h}^{\beta,\theta^i\omega} \geq (1-\varepsilon)\mathtt{F}(\beta,h+u).$$

Thus, taking the logarithm and dividing by $(i_m+1)\ell$, we get

$$\frac{1}{(i_m+1)\ell}\log Z_{(i_m+1)\ell,h}^{\beta,\omega} \\ \geq \frac{m}{i_m+1}\Big((1-\varepsilon)\mathtt{F}(\beta,h+u) + \frac{1}{m\ell}\sum_{k=1}^{m}\log \mathrm{P}\big(\tau_1 = (i_k-i_{k-1}-1)\ell\big)\Big).$$

Letting $m\to\infty$, we have $i_m\to\infty$, so the left-hand side converges $\mathbb{P}$-a.s. to $\mathtt{F}(\beta,h)$. On the other hand, we have that $(i_k - i_{k-1})_{k\geq 1}$ are i.i.d. with a geometric law of parameter $p_\ell := \mathbb{P}(\mathcal{A}_\ell)$. Therefore, by the law of large numbers, we get that $\lim_{m\to\infty}\frac{1}{m}(i_m+1) = \mathbb{E}[i_1] = (p_\ell)^{-1}$ $\mathbb{P}$-a.s. All together, using also the law of large numbers for $(\log \mathrm{P}(\tau_1 = (i_k-i_{k-1}-1)\ell))_{k\geq 1}$, we end up with

$$\mathtt{F}(\beta,h) \geq p_\ell\Big((1-\varepsilon)\mathtt{F}(\beta,h+u) + \frac{1}{\ell}\mathbb{E}\big[\log \mathrm{P}(\tau_1 = (i_1-1)\ell)\big]\Big).$$

It remains to estimate the last term. Recalling our assumption $(*)$, for ℓ large enough we have that $\mathrm{P}(\tau_1 = i\ell) \geq (i\ell)^{-(1+\alpha+\varepsilon)}$ for all $i\geq 1$, so that

$$\log \mathrm{P}(\tau_1 = (i_1-1)\ell) \geq -(1+\alpha+\varepsilon)\log((i_1-1)\ell)\mathbf{1}_{\{i_1\geq 2\}}.$$

Therefore, we have that

$$\mathbb{E}\Big[\log \mathrm{P}(\tau_1 = (i_1-1)\ell)\Big] \geq -(1+\alpha+\varepsilon)\mathbb{P}(i_1\geq 2)\mathbb{E}\Big[\log((i_1-1)\ell)\,\Big|\, i_1\geq 2\Big]\,.$$

Now, notice that conditionally on $i_1 \geq 2$, $i_1 - 1$ has geometric law of parameter p_ℓ, *i.e.* it has the same law as i_1. Using also Jensen's inequality, we therefore get that

$$\mathbb{E}\Big[\log((i_1-1)\ell)\,\Big|\, i_1\geq 2\Big] = \mathbb{E}[\log(i_1\ell)] \leq \log \mathbb{E}[i_1] + \log \ell = -\log p_\ell + \log \ell\,.$$

Overall, we end up with

$$\mathtt{F}(\beta,h) \geq p_\ell\Big((1-\varepsilon)\mathtt{F}(\beta,h+u) + (1+\alpha+\varepsilon)\frac{1}{\ell}\log p_\ell - c\frac{\log \ell}{\ell}\Big)\,.$$

This part of the proof makes no use of the law of ω, but it remains to estimate $p_\ell = \mathbb{P}(\mathcal{A}_\ell)$. We have the following lemma in the case where $\omega_i \sim \mathcal{N}(0,1)$.

Lemma 4.16. *If* $(\omega_i)_{i\geq 1}$ *are i.i.d.* $\mathcal{N}(0,1)$, *then we have*

$$\liminf_{\ell\to\infty} \frac{1}{\ell}\log \mathbb{P}(\mathcal{A}_\ell) \geq -\frac{1}{2\beta^2}u^2\,.$$

With this lemma at hand, for ℓ large enough (how large depends on β, h, ε), we have

$$\mathtt{F}(\beta,h) \geq p_\ell\Big((1-\varepsilon)\mathtt{F}(\beta,h+u) - \frac{1+\alpha+2\varepsilon}{2\beta^2}\,u^2 - c\frac{\log \ell}{\ell}\Big)\,.$$

Taking ℓ even larger so that the last term is negligible, we get

$$\mathtt{F}(\beta,h) \geq p_\ell\Big((1-\varepsilon)\mathtt{F}(\beta,h+u) - \frac{1+\alpha+3\varepsilon}{2\beta^2}\,u^2\Big)\,,$$

which gives (4.9) with $C_\beta = \frac{1+\alpha+3\varepsilon}{(1-\varepsilon)2\beta^2}$ and $K_{\beta,h,u} = p_\ell$. □

Let us note that in the case of Gaussian disorder $\omega_i \sim \mathcal{N}(0,1)$, we have identified the constant C_β in Theorem 4.13: since $\varepsilon > 0$ is arbitrary, we get that for all $u \in (0,1)$,

$$\mathtt{F}(\beta,h_c(\beta)+u) \leq C_\beta\, u^2 \qquad \text{with} \quad C_\beta := \frac{1+\alpha}{2\beta^2}\,.$$

This constant is in fact optimal: in the general case, [CdH13] shows that one has $\mathtt{F}(\beta,h_c(\beta)+u) \leq (1+o(1))C_\beta\, u^2$ as $u\downarrow 0$, with $C_\beta \sim \frac{1+\alpha}{2\beta^2}$ as $\beta \downarrow 0$.

Proof of Lemma 4.16. The proof relies on the following entropy (or change of measure) inequality, that we already encountered in Lemma 1.31: for any law $\tilde{\mathbb{P}}$, we have

$$p_\ell = \mathbb{P}(\mathcal{A}_\ell) \geq \tilde{\mathbb{P}}(\mathcal{A}_\ell) \exp\Big(- \frac{1}{\tilde{\mathbb{P}}(\mathcal{A}_\ell)} \big(\mathcal{H}(\tilde{\mathbb{P}} \mid \mathbb{P}) + \mathrm{e}^{-1} \big) \Big) . \qquad (4.10)$$

Let us take here $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}_\ell$ the law of $(\omega_1 + u/\beta, \ldots, \omega_\ell + u/\beta)$, so that

$$\tilde{\mathbb{P}}(\mathcal{A}_\ell) = \tilde{\mathbb{P}}\Big(\frac{1}{\ell} \log Z_{\ell,h}^{\beta,\omega} \geq (1-\varepsilon)\mathtt{F}(\beta, h+u) \Big) = \mathbb{P}\Big(\frac{1}{\ell} \log Z_{\ell,h+u}^{\beta,\omega} \geq (1-\varepsilon)\mathtt{F}(\beta, h+u) \Big) ,$$

since $Z_{\ell,h}^{\beta,\omega+u/\beta} = Z_{\ell,h+u}^{\beta,\omega}$. Thus we get that $\lim_{\ell\to\infty} \tilde{\mathbb{P}}(\mathcal{A}_\ell) = 1$, because $\frac{1}{\ell} \log Z_{\ell,h+u}^{\beta,\omega}$ converges $\mathbb{P}$-a.s. to $\mathtt{F}(\beta, h+u) > 0$.

We therefore get from (4.10) that

$$\liminf_{\ell\to\infty} \frac{1}{\ell} \log p_\ell \geq - \liminf_{\ell\to\infty} \frac{1}{\ell} \mathcal{H}(\tilde{\mathbb{P}}_\ell \mid \mathbb{P}) .$$

Now, since the density of $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}_\ell$ w.r.t. $\mathbb{P}$ is a product, we have that $\mathcal{H}(\tilde{\mathbb{P}}_\ell \mid \mathbb{P}) = \ell \mathcal{H}(\tilde{\mathbb{P}}_1 \mid \mathbb{P})$, so we only need to compute the relative entropy in the case $\ell = 1$.

Let $\mathbb{P}_\lambda$ denote the distribution of $\omega + \lambda$. For Gaussian variables $\omega \sim \mathcal{N}(0,1)$, we have $\frac{\mathrm{d}\mathbb{P}_\lambda}{\mathrm{d}\mathbb{P}}(\omega) = \mathrm{e}^{\lambda\omega - \frac{1}{2}\lambda^2}$, so that the relative entropy is

$$\mathcal{H}(\mathbb{P}_\lambda \mid \mathbb{P}) = \mathbb{E}_\lambda \Big[\log \frac{\mathrm{d}\mathbb{P}_\lambda}{\mathrm{d}\mathbb{P}} \Big] = \lambda \mathbb{E}_\lambda[\omega_1] - \frac{1}{2}\lambda^2 = \frac{1}{2}\lambda^2 .$$

Applying this with $\lambda = u/\beta$, this shows that $\mathcal{H}(\tilde{\mathbb{P}}_1 \mid \mathbb{P}) = \frac{1}{2}(u/\beta)^2$, which concludes the proof. □

4.4.2 About the critical point shift

We now turn to the question of the critical point shift. Recall the definition $\bar{\beta}_c := \inf\{\beta,\, h_c(\beta) > h_c^{\mathrm{a}}(\beta)\}$, so that disorder relevance in terms of critical points amounts to having $\bar{\beta}_c = 0$.

The main result here is that the criterion of Theorem 4.6 still holds for $\bar{\beta}_c$. Since we already have $\bar{\beta}_c \geq \tilde{\beta}_c$, one only needs to prove that $\bar{\beta}_c = 0$ in the case where $\tilde{\beta}_c = 0$, *i.e.* $\sum_{n=1}^\infty \mathrm{P}(n \in \tau)^2 = +\infty$.

Theorem 4.17 (Criterion for the critical point shift)**.** *If* $\sum_{n=1}^\infty \mathrm{P}(n \in \tau)^2 = +\infty$, *then* $h_c(\beta) > h_c^{\mathrm{a}}(\beta)$ *for all* $\beta > 0$.

In other words, combined with Theorem 4.8, we have that

$$\bar\beta_c = 0 \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \mathrm{P}(n\in\tau)^2 = +\infty\,. \tag{4.11}$$

This theorem has been proven in [BL18], but builds upon a series of papers; let us cite [DGLT09] then [AZ09] which first treated the case $\alpha > \frac12$ and then [GLT10, GLT11] which (almost) treated the marginal case $\alpha = \frac12$.

Estimates on the critical point shift.

The proof(s) of Theorem 4.17 in [AZ09, DGLT09, GLT10, GLT11, BL18] in fact all provided not only the existence of a critical point shift, but lower bounds on $h_c(\beta) - h_c^{\mathrm{a}}(\beta)$ in the $\beta\downarrow 0$ regime. In particular, in the case $\alpha > \frac12$, [AZ09, DGLT09] proved that $h_c(\beta) - h_c^{\mathrm{a}}(\beta) \geq \beta^{\frac{2}{2-\nu}+o(1)}$, where $\nu = \max(\frac{1}{\alpha},1)$ is the critical exponent of the homogeneous free energy; note that a matching upper bound had been provided in [Ale08, Ton08a]. The marginal case $\alpha = \frac12$ is more delicate, and the critical point shift is smaller than any power of β.

Further results have been obtained in the following years, sharpening these estimates: let us now present the best estimates obtained so far. The case $\alpha > 1$ is proven in [BCP$^+$14] (we exclude the case $\alpha = 1$ for simplicity); the case $\alpha \in (\frac12, 1)$ is proven in [CTT17]; the case $\alpha = \frac12$ is proven in [BL18].

Theorem 4.18 (Critical point shift)**.** *Assume that the underlying renewal process τ is persistent and verifies $(*)$, i.e. $\mathrm{P}(\tau_1 = n) = L(n) n^{-(1+\alpha)}$. Recall that we assumed that $\mathbb{E}[\omega]=0$, $\mathbb{E}[\omega^2]=1$.*

(i) If $\mathrm{E}[\tau_1] < +\infty$ (in particular if $\alpha > 1$), then we have that

$$h_c(\beta) - h_c^{\mathrm{a}}(\beta) \sim \frac{1}{\mathrm{E}[\tau_1]}\, \frac{\alpha}{2(1+\alpha)} \beta^2 \qquad \text{as } \beta\downarrow 0\,.$$

(ii) If $\alpha \in (\frac12, 1)$, then there exists some slowly varying function $\psi(\cdot)$ (which depends only $L(\cdot)$ and α) such that

$$h_c(\beta) - h_c^{\mathrm{a}}(\beta) \sim \mathbf{c}_\alpha\, \psi(1/\beta)\, \beta^{\frac{2\alpha}{2\alpha-1}} \qquad \text{as } \beta\downarrow 0\,.$$

In the above, the constant $\mathbf{c}_\alpha$ is universal*: it depends only on α but not on the distribution of ω*

(iii) If $\alpha = \frac{1}{2}$, denoting R^{-1} an asymptotic inverse of $N \mapsto R_N := \sum_{n=1}^{N} \mathrm{P}(n \in \tau)^2$, i.e. such that $R^{-1}(R_N) \sim N$ as $N \to \infty$, we have that

$$h_c(\beta) - h_c^{\mathrm{a}}(\beta) = R^{-1}\big((1+o(1))\beta^{-2}\big)^{-\frac{1}{2}+o(1)} \qquad \text{as } \beta \downarrow 0\,.$$

In particular, if $\lim_{n\to\infty} L(n) = c_1$ as in ($\hat{*}$), then we have $R_N \sim \frac{1}{(2\pi c_1)^2} \log N$ (recall Theorem 3.10), so that

$$h_c(\beta) - h_c^{\mathrm{a}}(\beta) = \exp\Big(- (1+o(1)) \frac{2(\pi c_1)^2}{\beta^2} \Big)\,.$$

About the proof of Theorems 4.17 and 4.18

First of all, let us stress that in Section 4.5.2 we prove that a divergent series $\sum_{n=1}^{\infty} \mathrm{P}(n \in \tau)^2 = +\infty$ implies that $\lim_{N\to\infty} W_N^{\beta,\omega} = 0$ for all $\beta > 0$, *i.e.* that $\tilde{\beta}_c = 0$. Because of Theorem 4.8, this is a strictly weaker result than Theorem 4.17 since one may still have that $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$.

The full proof of Theorems 4.17 and 4.18 is quite involved (and relies on some of the ideas developed in Section 4.5.2 below), but let us give an outline of the general strategy.

Step 1. Fractional moment. The first idea, developed in [DGLT09], is to reduce to the study of a fractional moment of teh partition function. The main goal is to show that $\mathtt{F}(\beta,h) = 0$ for $h = h_c(\beta) + u$ with u small enough; if we show this say for $u \le \Delta(\beta)$, then we would get that $h_c(\beta) - h_c^{\mathrm{a}}(\beta) \ge \Delta(\beta)$.

One therefore needs an upper bound on the free energy. We use the following: for any $\gamma \in (0,1)$,

$$\mathtt{F}(\beta,h) = \lim_{N\to\infty} \frac{1}{\gamma N} \mathbb{E}\big[\log (Z_{N,h}^{\beta,\omega})^{\gamma}\big] \le \lim_{N\to\infty} \frac{1}{\gamma N} \log \mathbb{E}\big[(Z_{N,h}^{\beta,\omega})^{\gamma}\big]$$

where we have used Jensen's inequality. It therefore remains to show that the fractional moment $\mathbb{E}[(Z_{N,h}^{\beta,\omega})^{\gamma}]$ does not grow to fast (in fact stays bounded) when $h = h_c(\beta) + u$ with u small enough. Notice that when $u = 0$, we have $Z_{N,h_c^{\mathrm{a}}(\beta)}^{\beta,\omega} = W_N^{\beta,\omega}$, which always have a bounded fractional moment.

Step 2. Coarse-graining procedure. The second step consists in dividing the system into large blocks of length $L = 1/\mathtt{F}(u)$, where $\mathtt{F}(u)$ is the free energy of the homogeneous pinning model. The idea is then to perform some "coarse-graining procedure" to reduce the estimate of the fractional moment $\mathbb{E}[(Z_{N,h}^{\beta,\omega})^\gamma]$ to estimates on blocks of length L. One proceed as follows.

We decompose the partition function $Z_{N,h}^{\beta,\omega}$ according to the blocks visited by the renewal trajectory, and write, for a system of size $N = mL$

$$Z_{mL,h}^{\beta,\omega} = \sum_{\ell=1}^{m} \sum_{1\le i_1<\cdots<i_\ell=m} Z_{i_1,\dots,i_\ell}\,,$$

where $Z_{i_1,\dots,i_\ell}$ is the partition function restricted to trajectories of τ visiting blocks with indices $i_1,\dots,i_\ell$. Then, one can use the bound $(\sum_i a_i)^\gamma \le \sum_i (a_i)^\gamma$, valid for any family of non-negative a_i's and any $\gamma\in(0,1)$, to bound the fractional moment

$$\mathbb{E}\big[(Z_{mL,h}^{\beta,\omega})^\gamma\big] \le \sum_{\ell=1}^{m} \sum_{1\le i_1<\cdots<i_\ell=m} \mathbb{E}\big[(Z_{i_1,\dots,i_\ell})^\gamma\big]\,.$$

The main goal is then to bound the fractional moment as follows:

$$\mathbb{E}[(Z_{i_1,\dots,i_\ell})^\gamma] \le C\,\varepsilon^\ell \prod_{j=1}^{\ell} (i_j - i_{j-1})^{-\gamma'(1+\alpha)}\,, \tag{4.12}$$

where $\prod_{j=1}^{\ell}(i_j-i_{j-1})^{-\gamma'(1+\alpha)}$ is some "coarse-grained probability" of visiting exactly $i_1,\dots,i_k$ (and $\gamma'\in(0,1)$ is close to γ) and ε can be made arbitrarily small by choosing h close enough to $h_c^{\mathrm{a}}(\beta)$. Then, with (4.12) at hand, we get that

$$\mathbb{E}\big[(Z_{mL,h}^{\beta,\omega})^\gamma\big] \le C\sum_{\ell=1}^{m} \sum_{1\le i_1<\cdots<i_\ell=m} \varepsilon^\ell \prod_{j=1}^{\ell} (i_j - i_{j-1})^{-\gamma'(1+\alpha)}\,,$$

which is the partition function of a homogeneous pinning model; we leave as an exercise to prove that if $\varepsilon\sum_{i\ge1} i^{-\gamma'(1+\alpha)} < 1$ then $\mathbb{E}[(Z_{mL,h}^{\beta,\omega})^\gamma]\le 1$.

Step 3. The change of measure argument. One is then reduced to showing (4.12), and in fact the product structure of $Z_{i_1,\dots,i_k}$ allows one to treat all the blocks separately. The idea is to use a change of measure argument, based on some (rare) event $\mathcal{A}\in\sigma\{\omega_1,\dots,\omega_L\}$. Define the "change of measure" function

$$g_{i_1,\dots,i_\ell}(\omega) := \prod_{j=1}^{\ell} g(\theta^{i_j L}\omega)\,, \qquad \text{with } g(\omega) = \mathbf{1}_{\mathcal{A}^c} + \mathbb{P}(\mathcal{A})^{\frac{\gamma}{1-\gamma}}\mathbf{1}_{\mathcal{A}}\,, \tag{4.13}$$

which will therefore only affects the blocks with indices $i_1, \dots, i_k$. Then, by Hölder's inequality, we have that

$$\mathbb{E}[(Z_{i_1,\dots,i_\ell})^\gamma] = \mathbb{E}[(g_{i_1,\dots,i_\ell} Z_{i_1,\dots,i_\ell})^\gamma (g_{i_1,\dots,i_\ell})^{-\gamma}] \le \mathbb{E}[g_{i_1,\dots,i_\ell} Z_{i_1,\dots,i_\ell}]^\gamma \mathbb{E}[(g_{i_1,\dots,i_\ell})^{-\frac{\gamma}{1-\gamma}}]^{1-\gamma},$$

and note that $\mathbb{E}[g^{-\frac{\gamma}{1-\gamma}}] = \mathbb{P}(\mathcal{A}^c) + 1 \le 2$ so that $\mathbb{E}[(g_{i_1,\dots,i_\ell} Z_{i_1,\dots,i_\ell})^{-\frac{1-\gamma}{\gamma}}]^{1-\gamma} \le 2^{(1-\gamma)\ell}$.

Therefore, because of the product structure of $Z_{i_1,\dots,i_k}$, we are reduced to showing that $\mathbb{E}[g(\omega) Z^{\beta,\omega}_{[a,b],h}] \le \varepsilon' \mathrm{P}(b-a \in \tau)$, with $Z^{\beta,\omega}_{[a,b],h}$ the partition function with starting point at a and ending point at b; the delicate point is that one needs to show this somehow uniformly over $0 \le a \le b \le L$ with $b-a$ large. In practice, the fact that we have a system of length $L = 1/\mathtt{F}(u)$ allows us to reduce the estimate at $h = h_c^{\mathrm{a}}(\beta) + u$ to an estimate at $h = h_c^{\mathrm{a}}(\beta)$; for instance, one has that $\mathbb{E}[Z^{\beta,\omega}_{[a,b],h}] \le cst.\, \mathrm{P}(b-a \in \tau)$. In view of the form (4.13) of $g(\omega)$, it all boils down to finding some event $\mathcal{A} \in \sigma\{\omega_1, \dots, \omega_L\}$ such that, when L is large enough (*i.e.* u small enough), we have

$$\mathbb{P}(\mathcal{A}^c) \le (\varepsilon'/2)^{(1-\gamma)/\gamma} \quad \text{and} \quad \mathbb{E}\big[\mathbf{1}_{\mathcal{A}} Z^{\beta,\omega}_{[a,b],h_c^{\mathrm{a}}(\beta)}\big] \le \varepsilon'/2 \mathrm{P}(b-a \in \tau), \tag{4.14}$$

uniformly over $0 \le a \le b \le L$ with $b-a$ large.

The choice of the event $\mathcal{A}$ is actually the hardest part of the proof, and the fact that the estimate (4.14) needs to hold uniformly over $a < b$ with $b-a$ large makes it even more difficult. We give in Section 4.5.2 a proof of the fact that $\lim_{N\to\infty} W_N^{\beta,\omega} = 0$ based on a similar (but more direct) change of measure argument (see Lemma 4.20): we only deal with the case where $a = 0$, which makes the choice of the event $\mathcal{A}$ simpler.

4.5 Back to the martingale: proof of Theorem 4.6

The main goal of this section is to prove Theorem 4.6.

The first part of the criterion is to show that $\tilde{\beta}_c > 0$ when $\sum_{n=1}^\infty \mathrm{P}(n \in \tau)^2 < +\infty$; thanks to Theorem 4.8, this also shows that $\bar{\beta}_c > 0$ under this condition. The proof relies on a second moment computation, and is performed in Section 4.5.1.

The second part of the criterion is to show that $\tilde{\beta}_c = 0$ when $\sum_{n=1}^\infty \mathrm{P}(n \in \tau)^2 = +\infty$; as mentioned above, this is strictly easier than showing $\bar{\beta}_c = 0$ in that case. The proof relies on a change of measure argument (see in particular Lemma 4.20), and is performed in Section 4.5.2.

We also discuss in Section 4.5.3 the question of the so-called intermediate disorder regime (in the case where $\tilde\beta_c = 0$), and we present some recent advances.

4.5.1 Weak disorder and the L^2 phase

In this section, we show that $\tilde\beta_c > 0$ when $\sum_{n=0}^\infty \mathrm{P}(n\in\tau)^2 < +\infty$. We actually give a lower bound on $\tilde\beta_c$ by controlling the second moment of $W_N^{\beta,\omega}$.

We use so-called replicas to express the second moment of $W_N^{\beta,\omega}$: we write

$$\begin{aligned}(W_N^{\beta,\omega})^2 &= \mathrm{E}\Big[\exp\Big(\sum_{i=1}^N (\beta\omega_i - \lambda(\beta))\vartheta_i\Big)\Big]\mathrm{E}\Big[\exp\Big(\sum_{i=1}^N (\beta\omega_i - \lambda(\beta))\vartheta_i'\Big)\Big]\\ &= \mathrm{E}^{\otimes 2}\Big[\exp\Big(\sum_{i=1}^N (\beta\omega_i - \lambda(\beta))(\vartheta_i+\vartheta_i')\Big)\Big],\end{aligned}$$

where τ,τ' are two independent copies of the renewal process and we used again the notation $\vartheta_i = \mathbf{1}_{\{i\in\tau\}}$, $\vartheta_i' = \mathbf{1}_{\{i\in\tau'\}}$. Observe that

$$\mathbb{E}\Big[\mathrm{e}^{(\beta\omega_i-\lambda(\beta))(\vartheta_i+\vartheta_i')}\Big] = \mathrm{e}^{\lambda_2(\beta)\vartheta_i\vartheta_i'} \quad \text{with } \lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta)\,,$$

since the only time when the expectation dos not give 1 is when $\vartheta_i + \vartheta_i' = 2$, so that taking the expectation of $(W_N^{\beta,\omega})^2$ and using Fubini–Tonelli, we get

$$\mathbb{E}\big[(W_N^{\beta,\omega})^2\big] = \mathrm{E}^{\otimes 2}\Big[\exp\Big(\lambda_2(\beta)\sum_{i=1}^N \mathbf{1}_{\{i\in\tau\cap\tau'\}}\Big)\Big]. \tag{4.15}$$

Then, by monotone convergence, we have

$$\sup_{N\geq 1}\mathbb{E}\big[(W_N^{\beta,\omega})^2\big] = \mathrm{E}^{\otimes 2}\Big[\exp\Big(\lambda_2(\beta)\sum_{i=1}^\infty \mathbf{1}_{\{i\in\tau\cap\tau'\}}\Big)\Big].$$

Since $\tau\cap\tau'$ is transient, $1+\sum_{i=1}^\infty \mathbf{1}_{\{i\in\tau\cap\tau'\}}$ is a geometric random variable with parameter $p := \mathrm{E}[|\tau\cap\tau'|]^{-1} > 0$ (see Remark 4.7) with $\mathrm{E}[|\tau\cap\tau'|] = \sum_{n=0}^\infty \mathrm{P}(n\in\tau)^2$.

All together, we conclude that

$$\sup_{N\geq 1}\mathbb{E}\big[(W_N^{\beta,\omega})^2\big] < +\infty \quad \Leftrightarrow \quad \beta < \beta_2\,,$$

where β_2 is defined by (recall that $\lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta)$)

$$\beta_2 := \sup\big\{\beta,\ \lambda_2(\beta) < \log\big(\tfrac{1}{1-p}\big)\big\}. \tag{4.16}$$

Indeed, note that a geometric random variable $G \sim \mathrm{Geom}(p)$ has a finite generating function $\mathbb{E}[s^G] < +\infty$ if and only if $(1-p)|s| < 1$.

Notice that we have $\beta_2 > 0$ since $\lambda(2\beta) - 2\lambda(\beta) \sim \beta^2$ as $\beta \downarrow 0$, so that in particular $\lambda(2\beta) - 2\lambda(\beta)$ can be made arbitrarily small.

In conclusion, for all $\beta < \beta_2$, the martingale $(W_N^{\beta,\omega})_{N\geq 0}$ is bounded in $L^2(\mathbb{P})$. It therefore converges in $L^2(\mathbb{P})$, hence in $L^1(\mathbb{P})$, and in particular

$$\mathbb{E}[W_\infty^{\beta,\omega}] = \lim_{N\to\infty} \mathbb{E}[W_N^{\beta,\omega}] = 1\,,$$

which excludes having $W_\infty^{\beta,\omega} = 0$ $\mathbb{P}$-a.s. We have thus shown that $W_\infty^{\beta,\omega} > 0$ $\mathbb{P}$-a.s. for all $\beta < \beta_2$, and thus $\tilde{\beta}_c \geq \beta_2 > 0$, which concludes the proof. □

Remark 4.19. A consequence of the results of [AB18] is that $\tilde{\beta}_c > \beta_2$ for all $\alpha < 2/5$. It is conjectured that this is actually the case for all $\alpha < 1/2$. Notice that having $\tilde{\beta}_c > \beta_2$ means there exists a regime $\beta \in (\beta_2, \tilde{\beta}_c)$ where the martingale converges a.s. but not in L^2.

4.5.2 Strong disorder and change of measure argument

Let us now show that $\tilde{\beta}_c = 0$ when $\sum_{n=0}^{\infty} \mathrm{P}(n \in \tau)^2 = +\infty$. To do this, we fix $\beta > 0$ and we show that $W_\infty^{\beta,\omega} = 0$ a.s. In fact, it is actually enough to show that $W_N^{\beta,\omega} \to 0$ in probability. We will use the following easy (and classical) observation, that characterize the convergence in probability for non-negative random variables.

Lemma 4.20. *Let $(Z_N)_{N\geq 0}$ be a sequence of non-negative random variables. The three following statements are then equivalent:*

(i) $Z_N \xrightarrow{\mathbb{P}} 0$ *as* $N \to \infty$*;*

(ii) $\lim_{N\to\infty} \mathbb{E}[Z_N \wedge 1] = 0$*;*

(iii) there exists a sequence of events $(A_N)_{N\geq 1}$ such that:

$$\lim_{N\to\infty} \mathbb{P}(A_N) = 1 \quad \text{and} \quad \lim_{N\to\infty} \mathbb{E}[Z_N \mathbf{1}_{A_N^c}] = 0\,.$$

The item (iii) above can be interpreted as a change of measure argument. Indeed, if one defines the measure μ_N by $\mu_N(A) = \mathbb{E}[Z_N \mathbf{1}_A]$, *i.e.* interpreting Z_N as a Radon–Nikodym derivative, then we have that $Z_N \xrightarrow{\mathbb{P}} 0$ if and only if there is some event A_N such that $\mathbb{P}(A_N) \to 1$ and $\mu_N(A_N^c) \to 0$, in other words if and only if μ_N

becomes singular with respect to $\mathbb{P}$ as $N \to \infty$. (We refer to after the proof for a discussion in the context of the pinning model.)

Proof of Lemma 4.20. First of all, notice that: for any event A_N, bounding $Z_N \wedge 1$ by Z_N on A_N or by 1 on A_N^c, we have

$$\mathbb{E}[Z_N \wedge 1] \le \mathbb{P}(A_N) + \mathbb{E}[Z_N \mathbf{1}_{A_N^c}] .$$

Hence, item (iii) implies item (ii), which in turns implies item (i) by Markov's inequality: for any $\varepsilon > 0$ we have $\mathbb{P}(Z_N > \varepsilon) \le \varepsilon^{-1}\mathbb{E}[Z_N \wedge 1]$.

Finally, assuming (i), consider the event $A_N := \{Z_N > \varepsilon_N\}$, where ε_N goes to 0 slowly enough so that $\lim_{\mathbb{N}\to\infty} \mathbb{P}(A_N) = 0$. We also clearly have that $\mathbb{E}[Z_N \mathbf{1}_{\{A_N^c\}}] \le \varepsilon_N \to 0$ as $N \to \infty$, which shows item (iii) and concludes the proof. ☐

Note that $W_N^{\beta,\omega}$ is non-negative and verifies $\mathbb{E}[W_N^{\beta,\omega}] = 1$. Therefore, $W_N^{\beta,\omega}$ is a probability density and we may define the *size-biased measure* $\hat{\mathbb{P}}_{N,\beta}$ as follows:

$$\frac{\mathrm{d}\hat{\mathbb{P}}_{N,\beta}}{\mathrm{d}\mathbb{P}}(\omega) = W_N^{\beta,\omega} .$$

There is an interpretation of the size-biased distribution $\hat{\mathbb{P}}_{N,\beta}$: for any event A,

$$\hat{\mathbb{P}}_{N,\beta}(A) = \mathbb{E}\big[W_N^{\beta,\omega}\mathbf{1}_A\big] = \mathrm{E}\Big[\mathbb{E}\Big[\prod_{i=1}^{N} \mathrm{e}^{(\beta\omega_i - \lambda(\beta))\mathbf{1}_{\{i\in\tau\}}}\Big]\Big] = \mathrm{E}\big[\mathbb{P}_{N,\beta}^{(\tau)}(A)\big] , \qquad (4.17)$$

where $\mathbb{P}_{N,\beta}^{(\tau)}$ is defined in (4.4). Recall that under $\mathbb{P}_{N,\beta}^{(\tau)}$ the $(\omega_i)_{1\le i\le N}$ are independent, distributed according either to $\mathbb{P}$ or $\mathbb{P}_\beta$ depending on whether $i \in \tau$, see (4.4)-(4.5). In other words, the distribution of $(\omega_i)_{1\le i\le N}$ under $\hat{\mathbb{P}}_{N,\beta}$ is obtained as follows:

- First sample a renewal process τ under P;
- Then, for every $i \in \tau$, modify the distribution of the ω_i to is β-*tilted* version $\mathbb{P}_\beta$ (recall (4.5)).

Let us also stress that under $\hat{\mathbb{P}}_{N,\beta}$ the ω_i are no longer independent.

To conclude, to prove that $W_N^{\beta,\omega} \xrightarrow{\mathbb{P}} 0$, Lemma 4.20 tells that we simply need to find a sequence of events $(A_N)_{N\ge1}$ that are typical under $\mathbb{P}$, *i.e.* $\lim_{N\to\infty} \mathbb{P}(A_N) = 1$, but become atypical under $\hat{\mathbb{P}}_{N,\beta}$, *i.e.* $\lim_{N\to\infty} \hat{\mathbb{P}}_{N,\beta}(A_N) = 0$. This is in fact a statistical testing problem of finding a good statistics to discriminate between $\hat{\mathbb{P}}_{N,\beta}$ and $\mathbb{P}$.

It remains to choose the events A_N appropriately and this depends on the problem at hand. We now treat separately the cases $\alpha > \frac{1}{2}$ and $\alpha = \frac{1}{2}$.

a) The case $\alpha > \frac{1}{2}$

Let us define

$$U_N := \sum_{i=1}^{N} \mathrm{P}(i \in \tau) \sim \begin{cases} \dfrac{N}{\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}]} & \text{if } \alpha \ge 1\,, \\ c_\alpha N^\alpha L(N)^{-1} & \text{if } \alpha \in (0,1)\,, \end{cases} \tag{4.18}$$

where the asymptotic equivalent comes from Theorems 3.9 and 3.10. Notice that we have $U_N \gg \sqrt{N}$ if $\alpha > \frac{1}{2}$. Let us consider the event

$$A_N = \big\{ X_N \ge \varepsilon_N \lambda'(\beta) U_N \big\}\,, \quad \text{where} \quad X_N := \sum_{i=1}^{N} \omega_i\,, \tag{4.19}$$

for a sequence ε_N which goes to 0 slowly enough so that we still have $\varepsilon_N U_N \gg \sqrt{N}$; this is where the condition $\alpha > \frac{1}{2}$ is needed. Recall that $\lambda(\beta) = \log \mathbb{E}[\mathrm{e}^{\beta\omega_1}]$ and that $\mathbb{E}_\beta[\omega_1] = \lambda'(\beta)$, with $\mathbb{P}_\beta$ the β-tilted law of ω, see (4.5). In particular, we get from (4.17) that

$$\hat{\mathbb{E}}_{N,\beta}[X_N] = \mathrm{E}\big[\mathbb{E}^{(\tau)}_{N,\beta}[X_N]\big] \quad \text{with} \quad \mathbb{E}^{(\tau)}_{N,\beta}[X_N] = \lambda'(\beta) \sum_{i=1}^{N} \vartheta_i\,, \tag{4.20}$$

where we have used that $\mathbb{E}^{(\tau)}_{N,\beta}[\omega_i] = \mathbb{E}[\omega_i] = 0$ if $i \notin \tau$ (*i.e.* if $\vartheta_i = 0$) and $\mathbb{E}^{(\tau)}_{N,\beta}[\omega_i] = \mathbb{E}_\beta[\omega_i] = \lambda'(\beta)$ if $i \in \tau$ (*i.e.* if $\vartheta_i = 1$). This shows that $\hat{\mathbb{E}}_{N,\beta}[X_N] = \lambda'(\beta) U_N$.

We now need to show that $\lim_{N\to\infty} \mathbb{P}(A_N) = 0$ and that $\lim_{N\to\infty} \hat{\mathbb{P}}_{N,\beta}(A_N^c) = 1$.

(i) Proof of $\lim_{N\to\infty} \mathbb{P}(A_N) = 0$. This estimate is easy: since $\mathbb{E}[X_N] = 0$ and $\mathbb{V}\mathrm{ar}(X_N) = N$, Chebyshev's inequality directly gives that

$$\mathbb{P}(A_N) \le \frac{1}{\lambda'(\beta)^2} \frac{N}{\varepsilon_N^2 U_N^2} \xrightarrow{N\to\infty} 0\,, \tag{4.21}$$

since $\varepsilon_N U_N \gg \sqrt{N}$.

(ii) Proof of $\lim_{N\to\infty} \hat{\mathbb{P}}_{N,\beta}(A_N^c) = 0$. Recalling the interpretation (4.17) of $\hat{\mathbb{P}}_{N,\beta}$, we have that

$$\hat{\mathbb{P}}_{N,\beta}(A_N^c) = \mathrm{E}\Big[\mathbb{P}^{(\tau)}_{N,\beta}\big(X_N \le \varepsilon_N \lambda'(\beta) U_N\big)\Big]\,.$$

Let us now estimate $\mathbb{P}^{(\tau)}_{N,\beta}(X_N \le \varepsilon_N \lambda'(\beta) U_N)$ for any given realization of τ. If one has $\hat{\mathbb{E}}^{(\tau)}_{N,\beta}[X_N] \ge 2\varepsilon_N \lambda'(\beta) U_N$, we use Chebyshev's inequality to obtain

$$\mathbb{P}^{(\tau)}_{N,\beta}\big(X_N \le \varepsilon_N \lambda'(\beta) U_N\big) = \mathbb{P}^{(\tau)}_{N,\beta}\big(X_N - \hat{\mathbb{E}}^{(\tau)}_{N,\beta}[X_N] \le -\varepsilon_N \lambda'(\beta) U_N\big) \le \frac{\mathbb{V}\mathrm{ar}^{(\tau)}_{N,\beta}(X_N)}{\varepsilon_N^2 \lambda'(\beta)^2 U_N^2} .$$

Then, recalling that the $(\omega_i)_{1\le i\le N}$ are independent under $\mathbb{P}^{(\tau)}_{N,\beta}$, with law $\mathbb{P}$ if $i \in \tau$ and $\mathbb{P}_\beta$ if $i \notin \tau$, we get that

$$\mathbb{V}\mathrm{ar}^{(\tau)}_{N,\beta}(X_N) = \sum_{i=1}^N \mathbb{V}\mathrm{ar}^{(\tau)}_{N,\beta}(\omega_i) = \sum_{i=1}^N \big(\mathbf{1}_{\{i\notin\tau\}} + \sigma_\beta^2 \mathbf{1}_{\{i\in\tau\}}\big) \le \max(1,\sigma_\beta^2) N ,$$

where we recall that $\mathbb{V}\mathrm{ar}(\omega_i) = 1$ and we denoted $\sigma_\beta^2 := \mathbb{V}\mathrm{ar}_\beta(\omega_i)$; note that it verifies $\sigma_\beta^2 = \lambda''(\beta)$ and goes to 1 as $\beta \downarrow 0$.

Bounding $\mathbb{P}^{(\tau)}_{N,\beta}(X_N \le \varepsilon_N \lambda'(\beta) U_N)$ by 1 on the event $\hat{\mathbb{E}}^{(\tau)}_{N,\beta}[X_N] < 2\varepsilon_N \lambda'(\beta) U_N$, we end up with

$$\hat{\mathbb{P}}_{N,\beta}(A_N^c) \le \frac{\max(1,\sigma_\beta^2)}{\lambda'(\beta)^2} \frac{N}{\varepsilon_N^2 U_N^2} + \mathrm{P}\big(\hat{\mathbb{E}}^{(\tau)}_{N,\beta}[X_N] < 2\varepsilon_N \lambda'(\beta) U_N\big) . \tag{4.22}$$

Again, the first term goes to 0 as $N \to \infty$ since we have $\varepsilon_N U_N \gg \sqrt{N}$. For the second term, using that $\mathbb{E}^{(\tau)}_{N,\beta}[X_N] = \lambda'(\beta) \sum_{i=1}^N \vartheta_i$, see (4.20), we get that it is equal to

$$\mathrm{P}\Big(\sum_{i=1}^N \vartheta_i < 2\varepsilon_N U_N\Big) \le \mathrm{P}\big(\tau_{k_N} > N\big) \qquad \text{where} \qquad k_N := \lceil 2\varepsilon_N U_N \rceil .$$

It remains to see that, as $\varepsilon_N \downarrow 0$, this probability goes to 0. To do so, we use a truncation argument similar to (4.7): we have that

$$\mathrm{P}\big(\tau_{k_N} > N\big) \le k_N \mathrm{P}(\tau_1 > N) + \frac{k_N}{N} \mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}] ,$$

where for the second inequality we have used subadditivity and Markov's inequality. We now treat the different cases separately.

(a) Case $\alpha \in (0,1)$. Then we have that $\mathrm{P}(\tau_1 > N) \sim \alpha^{-1} L(N) N^{-\alpha}$ and also that $\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1\le N\}}] \sim (1-\alpha)^{-1} L(N) N^{1-\alpha}$; we have used properties of regularly varying functions, see Remark 3.1. We therefore get that both terms above are of the

same order, namely

$$\mathrm{P}\big(\tau_{k_N} > N\big) \le C k_N N^{-\alpha} L(N) \le C \varepsilon_N U_N N^{-\alpha} L(N) \xrightarrow{N\to\infty} 0\,,$$

recalling that $U_N \sim c_\alpha N^\alpha L(N)^{-1}$ (see (4.18)) and that $\varepsilon_N \downarrow 0$.

(b) Case $\alpha \ge 1$. Then, we still have that $\mathrm{P}(\tau_1 > N) \sim \alpha^{-1} L(N) N^{-\alpha}$ but $\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}] \sim L_*(N)$ for some slowly varying function $L_*(\cdot)$; note that $L_*(N) \to \mathrm{E}[\tau_1]$ in the finite mean case, and that $L_*(N) \gg L(N)$ in the case $\alpha = 1$, see e.g. Exercise 23. In particular, we always have that $\mathrm{P}(\tau_1 > N) \ll N^{-1}\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}]$, so that we get

$$\mathrm{P}\big(\tau_{k_N} > N\big) \le C k_N N^{-1} \mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}] \le C \varepsilon_N U_N N^{-1} \mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}] \xrightarrow{N\to\infty} 0\,,$$

recalling that $U_N \sim N / \mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \le N\}}]$ (see (4.18)) and that $\varepsilon_N \downarrow 0$.

b) The marginal case $\alpha = \frac{1}{2}$

For $\alpha = \frac12$, the choice of event A_N above works as long as $U_N \gg \sqrt{N}$, *i.e.* if we have $L(N) \to \infty$, but this does not cover the whole regime where $\sum_{n=1}^\infty \mathrm{P}(n \in \tau) = +\infty$.

Let us introduce the notation

$$u(n) := \mathrm{P}(n \in \tau) \qquad \text{and} \qquad R_N := \sum_{i=1}^N u(i)^2\,,$$

so that the assumption of Theorem 4.6 is that $\lim_{N\to\infty} R_N = +\infty$. We then define the event

$$A_N = \Big\{ X_N \le \tfrac14 \lambda'(\beta) R_N \Big\} \quad \text{where} \quad X_N := \sum_{i=1}^N u(i)\omega_i\,. \tag{4.23}$$

Let us stress here again that, analogously to (4.20), we have that

$$\hat{\mathbb{E}}_{N,\beta}[X_N] = \mathrm{E}\big[\mathbb{E}^{(\tau)}_{N,\beta}[X_N]\big] \quad \text{with} \quad \mathbb{E}^{(\tau)}_{N,\beta}[X_N] = \lambda'(\beta) \sum_{i=1}^N u(i)\vartheta_i\,, \tag{4.24}$$

so in particular $\hat{\mathbb{E}}_{N,\beta}[X_N] = \lambda'(\beta)\sum_{i=1}^N u(i)\mathrm{P}(i\in\tau) = \lambda'(\beta) R_N$.

As above, we need to show that $\lim_{N\to\infty} \mathbb{P}(A_N) = 0$ and $\lim_{N\to\infty} \hat{\mathbb{P}}_{N,\beta}(A_N^c) = 1$.

(i) Proof of $\lim_{N\to\infty} \mathbb{P}(A_N) = 0$. Again, this is easy: we have that $\mathbb{E}[X_N] = 0$ and $\mathbb{V}\mathrm{ar}(X_N) = \sum_{i=1}^N u(i)^2 = R_N$, so Chebyshev's inequality directly yields

$$\mathbb{P}(A_N) \le \frac{16\, R_N}{\lambda'(\beta)^2 R_N^2} \xrightarrow{N\to\infty} 0\,, \tag{4.25}$$

using that $R_N \to \infty$.

(i) Proof of $\lim_{N\to\infty} \hat{\mathbb{P}}_{N,\beta}(A_N^c) = 0$. First, let us write as above that $\hat{\mathbb{P}}_{N,\beta}(A_N^c) = \mathrm{E}[\mathbb{P}_{N,\beta}^{(\tau)}(X_N \le \frac14 \lambda'(\beta) R_N)]$. Then, if $\mathbb{E}_{N,\beta}^{(\tau)}[X_N] \ge \frac12 \lambda'(\beta) R_N$ we use Chebyshev's inequality to get that

$$\mathbb{P}_{N,\beta}^{(\tau)}\big(X_N \le \tfrac14 \lambda'(\beta) R_N\big) \le \frac{16\, \mathbb{V}\mathrm{ar}_{N,\beta}^{(\tau)}(X_N)}{\lambda'(\beta)^2 R_N^2} \le \frac{16 \max(1,\sigma_\beta^2) R_N}{\lambda'(\beta)^2 R_N^2}\,,$$

where we have used that $\mathbb{V}\mathrm{ar}_{N,\beta}^{(\tau)}(X_N) \le \max(1,\sigma_\beta^2) R_N$ for any realization of τ, by a simple calculation (similar to the case $\alpha > \frac12$). Bounding $\mathbb{P}_{N,\beta}^{(\tau)}(X_N \le \frac14 \lambda'(\beta) R_N)$ by 1 if $\mathbb{E}_{N,\beta}^{(\tau)}[X_N] < \frac12 \lambda'(\beta) R_N$, we therefore obtain, similarly to (4.22)

$$\hat{\mathbb{P}}_{N,\beta}(A_N) \le \frac{16 \max(1,\sigma_\beta^2)}{\lambda'(\beta)^2 R_N} + \mathrm{P}\Big(\mathbb{E}_{N,\beta}^{(\tau)}[X_N] < \tfrac12 \lambda'(\beta) R_N\Big)\,. \tag{4.26}$$

The first term goes to 0 since $R_N \to \infty$. For the second term, recalling (4.24) and denoting $Y_N = Y_N(\tau) := \sum_{i=1}^N u(i)\vartheta_i$, we get that it is equal to

$$\mathrm{P}\big(Y_N < \tfrac12 R_N\big) = \mathrm{P}\big(Y_N - \mathrm{E}[Y_N] \le -\tfrac12 R_N\big) \le \frac{4\, \mathrm{Var}(Y_N)}{R_N^2}$$

where we have used that $\mathrm{E}[Y_N] = R_N$ and then Chebyshev's inequality. It therefore only remains to show that $\mathrm{Var}(Y_N) = o(R_N^2)$ as $N \to \infty$.

Let us compute $\mathrm{Var}(Y_N)$:

$$\begin{aligned}\mathrm{Var}(Y_N) &= \sum_{i=1}^N u(i)^2\, \mathrm{Var}(\vartheta_i) + 2 \sum_{i=1}^N \sum_{j=i+1}^N u(i)u(j)\big(\mathrm{E}[\vartheta_i\vartheta_j] - \mathrm{E}[\vartheta_i]\mathrm{E}[\vartheta_j]\big)\\ &\le R_N + 2 \sum_{i=1}^N u(i)^2 \sum_{j=i+1}^N u(j)\big(u(j-i) - u(j)\big).\end{aligned}$$

Note that by Theorem 3.10 we have $u(n) \sim cL(n)^{-1}n^{-1/2}$. Thus, if $j \to +\infty$ with $(j-i)/j \to 1$, we have that $u(j-i)/u(j) \to 1$; in other words, for any $\varepsilon > 0$ there exists C_ε such that $u(j-i) - u(j) \le \varepsilon u(j)$ for all $j \ge C_\varepsilon\, i$. We deduce that

$$\mathrm{Var}(Y_N) \le R_N + 2\varepsilon \sum_{i=1}^{N} u(i)^2 \sum_{j=C_\varepsilon i}^{N} u(j)^2 + 2\sum_{i=1}^{N} u(i)^2 \sum_{j=i+1}^{C_\varepsilon i} u(j)u(j-i)\,.$$

Then, using that $u(n) \sim cL(n)^{-1}n^{-1/2}$ and properties of slowly varying functions (see Remark 3.1), this is bounded by

$$R_N + 2\varepsilon R_N^2 + C'_\varepsilon \sum_{i=1}^{N} \frac{1}{L(i)^3 i^{3/2}} \sum_{k=1}^{(C_\varepsilon-1)i} \frac{1}{L(k)k^{1/2}} \le 3\varepsilon R_N^2 + C''_\varepsilon \sum_{i=1}^{N} \frac{1}{L(i)^4 i}\,.$$

Now, we can use that $\frac{1}{L(N)^2} = o(R_N)$ as $N \to \infty$, see [BGT89, Prop. 1.5.9a] (this is similar to Question 2 in Exercise 23), to get that the last term is $o(R_N^2)$. Since $\varepsilon > 0$ is arbitrary, this shows that $\mathrm{Var}(Y_N) = o(R_N^2)$, as claimed. □

4.5.3 The intermediate disorder regime

Let us stress that when $\tilde{\beta}_c = 0$, *i.e.* when $\sum_{n=1}^{\infty} \mathrm{P}(n \in \tau)^2 = +\infty$ by Theorem 4.6, then we have that $W_N^{\beta,\omega} \to 0$ for any $\beta > 0$. On the other hand, we have that $W_N^{\beta=0,\omega} = 1$. The question is then whether there exists some regime where $\beta_N \downarrow 0$ but for which one has that $W_N^{\beta_N,\omega}$ converges neither to 0 or 1, but to some non-trivial random variable $\mathcal{W} \in (0,+\infty)$. This regime is called the *intermediate disorder regime*, since it corresponds to the regime where disorder *kicks in*: disorder is still present in the limit, but it is not strong enough to make $W_N^{\beta_N,\omega}$ vanish in the limit. In fact, this regime is where the transition between weak and strong disorder really occurs.

Notice that, in the case where $\sum_{n=1}^{\infty} \mathrm{P}(n \in \tau) = +\infty$, the proof of Theorem 4.6 can be improved to the following, which identifies the intermediate disorder regime.

Theorem 4.21 (Intermediate disorder scaling: from weak to strong disorder)**.** *Assume that* $\sum_{n=1}^{\infty} \mathrm{P}(n \in \tau)^2 = +\infty$ *and define*

$$R_N := \sum_{n=1}^{N} \mathrm{P}(n \in \tau)^2\,.$$

Then, if $(\beta_N)_{N\geq 1}$ be a non-negative vanishing sequence, we have that
(i) If $\beta_N^2 R_N \to 0$, then $W_N^{\beta_N,\omega} \xrightarrow{\mathbb{P}} 1$.
(ii) If $\beta_N^2 R_N \to +\infty$, then $W_N^{\beta_N,\omega} \xrightarrow{\mathbb{P}} 0$.

Before we start the proof, let us notice that in the case $\alpha > \frac{1}{2}$, then thanks to Theorems 3.9 and 3.10 (and the properties of slowly varying functions, see Remark 3.1) we have that, as $N \to \infty$

$$R_N = \sum_{n=1}^{N} \mathrm{P}(n \in \tau)^2 \sim \begin{cases} \dfrac{N}{\mathrm{E}[\tau_1 \mathbf{1}_{\{\tau_1 \leq N\}}]^2} & \text{if } \alpha \geq 1\,, \\ c'_\alpha N^{2\alpha-1} L(N)^{-2} & \text{if } \alpha \in (\frac{1}{2}, 1)\,. \end{cases}$$

Recalling the definition $U_N := \sum_{n=1}^{N} \mathrm{P}(n \in \tau)$ and in view of (4.18), we therefore get that

$$R_N \sim c''_\alpha N^{-1} U_N^2 \qquad \text{as } N \to \infty\,. \tag{4.27}$$

Thus, in the case $\alpha > \frac{1}{2}$, we can therefore replace the condition $\beta_N^2 R_N \to 0$ (or $+\infty$) by $\beta_N^2 N^{-1} U_N^2 \to 0$ (or $+\infty$).

Proof of Theorem 4.21. For item (i), one simply need to control the second moment of $W_N^{\beta_N,\omega}$. With the same calculation as in Section 4.5.1, see (4.15), recalling that $\lambda_2(\beta) := \lambda(2\beta) - \lambda(\beta)$, we need to show that

$$\mathbb{E}\big[(W_N^{\beta_N,\omega})^2\big] = \mathrm{E}^{\otimes 2}\Big[\exp\Big(\lambda_2(\beta_N) \sum_{i=1}^{N} \mathbf{1}_{\{i \in \tau\cap\tau'\}}\Big)\Big] \to 1 \quad \text{if } \beta_N^2 R_N \to 0\,. \tag{4.28}$$

Indeed, together with the fact that $\mathbb{E}[W_N^{\beta_N,\omega}] = 1$, this shows that $\mathbb{V}\mathrm{ar}(W_N^{\beta_N,\omega}) \to 0$, and as a consequence $W_N^{\beta_N,\omega} \to 1$ in probability (and in $L^2(\mathbb{P})$). But this is a general fact for renewal processes.

Lemma 4.22. *Let $\rho = (\rho_i)_{i\geq 0}$ be a persistent (i.e. recurrent) renewal process, and let $U_N := \sum_{n=1}^{N} \mathrm{P}(n \in \rho)$. Then, for any non-negative vanishing sequence $\varepsilon_N \downarrow 0$, we have*

$$\mathrm{E}\Big[\exp\Big(\frac{\varepsilon_N}{U_N} \sum_{i=1}^{N} \mathbf{1}_{\{i\in\rho\}}\Big)\Big] \xrightarrow[N\to\infty]{} 1\,.$$

We leave the proof as a exercise (see Exercise 41 below), but applying it to $\rho = \tau \cap \tau'$ directly gives (4.28), since $\lambda_2(\beta) \sim \beta^2$ as $\beta \downarrow 0$ and $R_N := \sum_{n=1}^{N} \mathrm{P}(n \in \rho)$.

As far item (ii) is concerned, it follows exactly as in Section 4.5.2. Let us start with the case $\alpha > \frac{1}{2}$. Then, in view of (4.21) and (4.22), we get that $\mathbb{P}(A_N) \to 0$ and $\hat{\mathbb{P}}_{N,\beta}(A_N^c) \to 0$ as soon as $\lambda'(\beta_N)^2 U_N^2 \gg N$ and $a_N \downarrow 0$ slowly enough (e.g. $a_N = \lambda'(\beta)^{-1/2} U_N^{-1/2} N^{1/4}$). Since $\lambda'(\beta) \sim \beta$ as $\beta \downarrow 0$, we therefore get that $W_N^{\beta_N,\omega} \stackrel{\mathbb{P}}{\to} 0$ provided that $\beta_N U_N N^{-1/2} \to +\infty$. In the marginal case $\alpha = \frac{1}{2}$, one also gets from (4.25) and (4.26) that $\mathbb{P}(A_N) \to 0$ and $\hat{\mathbb{P}}_{N,\beta}(A_N^c) \to 0$ as soon as $\lambda'(\beta_N)^2 R_N \to +\infty$. This concludes the proof that $W_N^{\beta_N,\omega} \stackrel{\mathbb{P}}{\to} 0$ if $\beta_N^2 R_N \to +\infty$, recalling also (4.27) in the case $\alpha > \frac{1}{2}$. □

Intermediate disorder and scaling limits

Let us now review some of the results in the intermediate disorder regime, *i.e.* when $\lim_{N\to\infty} \beta_N R_N^{1/2} \to \hat{\beta} \in (0,\infty)$. We first present the case $\alpha \in (\frac{1}{2}, 1)$ (we exclude the case $\alpha \geq 1$ by simplicity) and then turn to the marginal case $\alpha = \frac{1}{2}$.

The case $\alpha \in (\frac{1}{2}, 1)$. The case $\alpha \in (\frac{1}{2}, 1)$ is called *subcritical*, and is treated in [CSZ17a] with quite a broad generality. The result is the following; recall that when $\alpha \in (\frac{1}{2}, 1)$, we have $R_N \sim c'_\alpha N^{2\alpha-1} L(N)^{-2}$.

Theorem (Intermediate disorder regime, subcritical case)**.** *Assume that* $(*)$ *holds with* $\alpha \in (\frac{1}{2}, 1)$ *and that* $(\beta_N)_{N\geq 1}$ *is a vanishing sequence such that*

$$\lim_{N\to\infty} \beta_N R_N^{-1/2} = \hat{\beta} \in (0, +\infty)\,.$$

Then, we have the following convergence in distribution for the partition function:

$$W_N^{\beta_N,\omega} \xrightarrow[N\to\infty]{(d)} \mathcal{W}_{\hat{\beta}}\,,$$

where $\mathcal{W}_{\hat{\beta}} \in (0, +\infty)$ $\mathbb{P}$*-a.s. is a non-trivial random variable.*

Let us stress that we have treated above only the case where $h = h_c^{\mathrm{a}}(\beta)$, but the analogous result holds in some window around the annealed critical point, namely assuming that $h = h_N$ verifies $h_N - h_c^{\mathrm{a}}(\beta) \sim \hat{h}\, L(N) N^{-\alpha}$ for some $\hat{h} \in \mathbb{R}$. Additionally, in the same intermediate disorder regime, [CSZ16] shows that the convergence also holds at the level of Gibbs measures: in other words, $\mathrm{P}_{N,h_N}^{\beta_N,\omega}(\frac{\tau}{N} \in \cdot)$ converges in distribution[3] to some *random* measure $\mathbf{P}_{\hat{h}}^{\hat{\beta}}$, called the *continuum disordered pinning model.*

[3] In the space of probability measure on closed subsets of $(0,1)$, for the vague topology.

The marginal case $\alpha = \frac{1}{2}$. The case $\alpha = \frac{1}{2}$, called *marginal* or *critical*, is much more difficult, and there are few results for the intermediate disorder limit in the pinning model. Let us give the main result of [CSZ17b] in this context.

Theorem 4.23 (Intermediate disorder regime, critical case)**.** *Let $\alpha = \frac{1}{2}$ and assume that $R_N = \sum_{n=1}^{N} \mathrm{P}(n \in \tau)^2 \to \infty$ and that $\lim_{N\to\infty} \beta_N R_N^{1/2} = \hat{\beta} \in (0, +\infty)$. Then, there is a phase transition at $\hat{\beta} = 1$: more precisely*

(i) If $\hat{\beta} < 1$, then $\log W_N^{\beta_N,\omega} \xrightarrow[N\to\infty]{(d)} \sigma_{\hat{\beta}} \mathcal{N}(0,1) - \frac{1}{2}\sigma_{\hat{\beta}}^2$, with $\sigma_{\hat{\beta}}^2 := \log\big(\frac{1}{1-\hat{\beta}^2}\big)$.

(ii) If $\hat{\beta} \geq 1$, then $W_N^{\beta_N,\omega} \xrightarrow[N\to\infty]{(d)} 0$.

Remark 4.24 ((i) $\Rightarrow$ (ii) in Theorem 4.23)**.** In Exercise 42 below, we show how, thanks to some monotonicity in β, item (ii) can be derived from item (i) (which is indeed the difficult result). However, the proof gives no information whatsoever on how fast $W_N^{\beta_N,\omega}$ goes to 0. It would therefore be interesting to have another proof of item (ii) which provides such an estimate.

The phase transition in Theorem 4.23 can actually be seen already at the level of the second moment of $W_N^{\beta_N,\omega}$. In fact, we have the following lemma, whose proof is given in the form of an exercise (see Exercise 43).

Lemma 4.25 (Phase transition for the second moment)**.** *Assume that $R_N := \sum_{n=1}^{\infty} \mathrm{P}(n \in \tau)^2 \to +\infty$ as a slowly varying function. Then if $\lim_{N\to\infty} \beta_N R_N^{1/2} = \hat{\beta}$ we have (recall (4.15))*

$$\mathbb{E}\big[(W_N^{\beta_N,\omega})^2\big] = \mathrm{E}^{\otimes 2}\Big[\mathrm{e}^{\lambda_2(\beta_N)\sum_{i=1}^{N} \mathbf{1}_{\{i\in\tau\cap\tau'\}}}\Big] \xrightarrow[N\to\infty]{} \begin{cases} \dfrac{1}{1-\hat{\beta}^2} & \text{if } \hat{\beta} < 1\,, \\ +\infty & \text{if } \hat{\beta} \geq 1\,. \end{cases}$$

Notice that, in the case $\hat{\beta} < 1$, the limit of the second moment matches the one we expect from Theorem 4.23, that is $\mathbb{E}\big[(\mathrm{e}^{\sigma_{\hat{\beta}}\mathcal{N}(0,1) - \frac{1}{2}\sigma_{\hat{\beta}}^2})^2\big]$ with $\sigma_{\hat{\beta}}^2 = \log\big(\frac{1}{1-\hat{\beta}^2}\big)$.

The question of what happens at the critical value $\hat{\beta} = 1$ is in fact extremely rich, and has been the object of a very intense activity in the context of the directed polymer model; we refer to [CSZ25] for a very recent overview of all the progress that has been made over the last five years.

Let us explain here the general philosophy in the context of the pinning model. The idea is that, in order to obtain a non-trivial scaling limit of the model, one must consider the partition functions $W_{a,N}^{\beta,\omega}$ (*i.e.* with starting point a) and interpret them as a *random measure* on $[0,1]$, by setting $W_N^{\beta,\omega}(\mathrm{d}x) := W_{\lfloor xN \rfloor,N}^{\beta,\omega}\mathrm{d}x$. In other words, for any function $\varphi : (0,1) \to \mathbb{R}$, one may consider

$$W_N^{\beta,\omega}(\varphi) := \int_{[0,1]} \varphi(x) W_N^{\beta,\omega}(\mathrm{d}x) = \sum_{a=0}^{N-1} \varphi_N(a) W_{a,N}^{\beta,\omega} \quad \text{with } \varphi_N(a) = \int_{\frac{a}{N}}^{\frac{a+1}{N}} \varphi(x)\mathrm{d}x \,.$$

Put differently, it amounts to considering partition functions with a starting point drawn under the measure φ_N.

In that case, if $\beta_N R_N^{1/2} = 1$, the sequence of random measures $(W_N^{\beta_N,\omega}(\mathrm{d}x))_{N\geq 1}$ on $[0,1]$ should converge in distribution to some limiting non-trivial (*i.e.* random) measure $\mathcal{U}(\mathrm{d}x)$. This is in fact the main result of [CSZ23] in the context of the directed polymer model. The convergence actually holds in a critical window around $\beta_N = R_N^{-1/2}$ we should therefore obtain the following result[4]. Here, the space of measures on $[0,1]$ is equipped with the vague topology.

Conjecture 4.26. *Assume that $R_N \to \infty$ as $N \to \infty$ as a slowly varying function. Then, if $(\beta_N)_{N\geq 0}$ goes to 0 in such a way that*[5]

$$\lim_{N\to\infty} \big(\mathrm{e}^{\lambda_2(\beta_N)} - 1\big) R_{\lfloor \mathrm{e}^{\vartheta} N \rfloor} = 1 \quad \textit{for some } \vartheta \in \mathbb{R} \,, \tag{4.29}$$

then there exist some random measure $\mathcal{U}^{\vartheta}(\mathrm{d}x)$ on $[0,1]$ such that

$$W_N^{\beta_N,\omega}(\mathrm{d}x) \xrightarrow[N\to\infty]{(d)} \mathcal{U}^{\vartheta}(\mathrm{d}x) \,.$$

The limit $\mathcal{U}^{\vartheta}$ is called the critical disordered pinning measure*: it is universal, in the sense that it does not depend on the specific distribution of the disorder ω or of the renewal process τ.*

Let us mention that this conjecture has been proven in [WY25] in the case where the underlying renewal τ is the set of return times to 0 of the simple random walk on $\mathbb{Z}$: in that case one has $R_N = \frac{1}{\pi}\log N + c + o(1)$, so the critical window (4.29) then reads:

$$\beta_N = \frac{\sqrt{\pi}}{\sqrt{\log N}} \Big(1 + (1+o(1)) \frac{\vartheta + c}{\log N} \Big) \,.$$

[4]In analogy with [CSZ23], the convergence should actually hold for a *flow* of measures, that is for the two-parameter family of measures $W_{sN,tN}^{\beta,\omega}(\mathrm{d}x,\mathrm{d}y)$ on the time-interval $[sN,tN]$ with $0 \leq s < t < \infty$.

[5]Recall that $\lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta) \sim \beta^2$ as $\beta \downarrow 0$.

The generalization stated in Conjecture 4.26 would then show that the critical disordered pinning measure is truly universal.

4.5.4 A few exercises

Exercise 41 (Proof of Lemma 4.22)**.** Let ρ be a persistent renewal and consider

$$\text{for } h \geq 0 \qquad Z_{N,h} := \mathrm{E}\Big[\mathrm{e}^{h\sum_{i=1}^N \mathbf{1}_{\{i\in\rho\}}}\Big].$$

1. Show that $\mathrm{e}^{h\mathbf{1}_{\{i\in\rho\}}} = 1 + (\mathrm{e}^h - 1)\mathbf{1}_{\{i\in\rho\}}$ for any $1 \leq i \leq N$.
2. Expanding $\prod_{i=1}^N (1 + (\mathrm{e}^h - 1)\mathbf{1}_{\{i\in\rho\}})$, show that
$$Z_{N,h} = 1 + \sum_{k=1}^N (\mathrm{e}^h - 1)^k \sum_{1\leq i_1<\ldots<i_k\leq N} \mathrm{P}(i_1 \in \rho) \cdots \mathrm{P}(i_k - i_{k-1} \in \rho)\,. \tag{4.30}$$
3. Writing $U_N := \sum_{n=1}^N \mathrm{P}(n \in \rho)$, deduce that $Z_{N,h} \leq 1 + \sum_{k=1}^N (\mathrm{e}^h - 1)^k (U_N)^k$.
4. Conclude the proof of Lemma 4.22.

Exercise 42 (Critical case, $\hat\beta \geq 1$)**.** Assume that $(\beta_N)_{N\geq 1}$ verifies $\lim_{N\to\infty} \beta_N R_N^{1/2} = \hat\beta \geq 1$. Our goal is to use item (i) in Theorem 4.23 to shows that $\lim_{N\to\infty} W_N^{\beta_N} = 0$.

1. Let $(\beta'_N)_{N\geq 1}$ be such that $\beta'_N \leq \beta_N$ for all N and $\lim_{N\to\infty} \beta'_N R_N^{1/2} = \hat\beta' < 1$. Using Exercise 40, show that
$$\mathbb{E}\big[(W_N^{\beta_N,\omega})^{1/2}\big] \leq \mathbb{E}\big[(W_N^{\beta'_N,\omega})^{1/2}\big]\,.$$
2. Show that, for any random non-negative random variable X with $\mathbb{E}[X] = 1$, we have $\mathbb{E}[X^{1/2}] \leq \sqrt{2}\mathbb{E}[X \wedge 1]^{1/2}$. *(Hint: use that $X = (X \vee 1)(X \wedge 1)$.)*
3. Using Theorem 4.23-(i), deduce that
$$\limsup_{N\to\infty} \mathbb{E}\big[(W_N^{\beta_N,\omega})^{1/2}\big] \leq \sqrt{2}\,\mathbb{E}\Big[\mathrm{e}^{\sigma_{\hat\beta'}\mathcal{N}(0,1) - \frac{1}{2}\sigma_{\hat\beta'}^2} \wedge 1\Big]^{1/2}.$$
4. Show that, for any random non-negative random variable X with $\mathbb{E}[X] = 1$, we have $\mathbb{E}[X \wedge 1] \leq \mathbb{E}[X^{1/2}]$. Deduce that
$$\limsup_{N\to\infty} \mathbb{E}\big[(W_N^{\beta_N,\omega})^{1/2}\big] \leq \sqrt{2}\,\mathrm{e}^{-\frac{1}{16}\sigma_{\hat\beta'}^2} = \sqrt{2}\,(1-\hat\beta')^{\frac{1}{16}}\,.$$
5. Conclude that $W_N^{\beta_N,\omega} \xrightarrow{\mathbb{P}} 0$.

Exercise 43 (Second moment and Erdős–Taylor theorem)**.** Let ρ be a renewal process such that $U_N := \sum_{n=1}^N \mathrm{P}(n \in \rho)$ goes to $+\infty$ as a slowly varying function. Let $Z_{N,h} := \mathrm{E}\big[e^{h\sum_{i=1}^N \mathbf{1}_{\{i\in\rho\}}}\big]$ for $h \geq 0$.

1. Using Exercise 41, show that $Z_{N,h} \leq \sum_{k=0}^{\infty}(h\,U_N)^k$. Deduce that
$$\text{if } \lim_{N\to\infty} h_N U_N = \hat{h} < 1 \qquad \text{then} \quad \limsup_{N\to\infty} Z_{N,h_N} \leq \frac{1}{1-\hat{h}}.$$
2. (a) Show that there exists some $\varepsilon_N \downarrow 0$ such that $U_{\varepsilon_N N} \sim U_N$ as $N \to \infty$.
   (b) Using (4.30), show that $Z_{N,h} \geq \sum_{k=0}^{1/\varepsilon_N}(h\,U_{\varepsilon_N N})^k$.
   (c) Deduce that
$$\text{if } \lim_{N\to\infty} h_N U_N = \hat{h} < 1 \qquad \text{then} \quad \liminf_{N\to\infty} Z_{N,h_N} \geq \frac{1}{1-\hat{h}}$$
   and that $\lim_{N\to\infty} Z_{N,h_N} = +\infty$ if $\lim_{N\to\infty} h_N U_N = \hat{h} \geq 1$.
3. Conclude that
$$\text{if } \lim_{N\to\infty} h_N U_N = \hat{h} \qquad \text{then} \quad \lim_{N\to\infty} Z_{N,h_N} = \begin{cases} \dfrac{1}{1-\hat{h}} & \text{if } \hat{h} < 1\,, \\ +\infty & \text{if } \hat{h} \geq 1\,, \end{cases}$$
   and show Lemma 4.25.
4. Show the following theorem as a corollary[6], noticing that Z_{n,h_N} is a Laplace transform.

Theorem (Erdős–Taylor theorem)**.** *Assume that ρ is a persistent renewal process, with $U_N = \sum_{n=1}^N \mathrm{P}(n \in \rho)$ diverging as a slowly varying function. Then, we have the following convergence in distribution:*
$$\frac{1}{U_N}\sum_{n=1}^N \mathbf{1}_{\{n\in\rho\}} \xrightarrow[N\to\infty]{(d)} \mathrm{Exp}(1)\,.$$

[6] This dates back to Erdős and Taylor [ET60], who proved this result for the returns to zero of the simple random walk on $\mathbb{Z}^2$.

Bibliography

[AB16] K. S. Alexander and Q. Berger. Local limit theorems and renewal theory with no moments. *Electronic Journal of Probability*, 21:18 pp., 2016.

[AB18] K. S. Alexander and Q. Berger. Pinning of a renewal on a quenched renewal. *Electronic Journal of Probability*, 23:1–48, 2018.

[AD99] L. Alili and R. Doney. Wiener–hopf factorization revisited and some applications. *Stochastics and Stochastic Reports*, 66(1-2):87–102, 1999.

[AHPS21] M. Aizenman, M. Harel, R. Peled, and J. Shapiro. Depinning in integer-restricted gaussian fields and BKT phases of two-component spin models. *arXiv:2110.09498 [math.PR]*, 2021.

[Ale08] K. S. Alexander. The effect of disorder on polymer depinning transitions. *Commun. Math. Phys.*, 279(1):117–146, 2008.

[AS06] K. S. Alexander and V. Sidoravicius. Pinning of polymers and interfaces by random potentials. *Ann. Appl. Probab.*, 16(2):636–669, 2006.

[AZ09] K. S. Alexander and N. Zygouras. Quenched and annealed critical points in polymer pinning models. *Communications in Mathematical Physics*, 291(3):659–689, 2009.

[AZ10] K. S. Alexander and N. Zygouras. Equality of critical points for polymer depinning transitions with loop exponent one. *The Annals of Applied Probability*, 20(1):356 – 366, 2010.

[BCP$^+$14] Q. Berger, F. Caravenna, J. Poisat, R. Sun, and N. Zygouras. The critical curves of the random pinning and copolymer models at weak coupling. *Comm. Math. Phys.*, 326(2):507–530, 2014.

[BDG01] E. Bolthausen, J.-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic. *The Annals of Probability*, 29(4):1670–1692, 2001.

[BDZ95] E. Bolthausen, J.-D. Deuschel, and O. Zeitouni. Entropic repulsion of the lattice free field. *Communications in mathematical physics*, 170:417–443, 1995.

[BDZ00] E. Bolthausen, J. D. Deuschel, and O. Zeitouni. Absence of a wetting transition for a pinned harmonic crystal in dimensions three and larger. *Journal of Mathematical Physics*, 41(3):1211–1223, 2000.

[BGT89] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27. Cambridge university press, 1989.

[BL18] Q. Berger and H. Lacoin. Pinning on a defect line: characterization of marginal relevance and sharp asymptotics for the critical point shift. *Journal of the Institute of Mathematics of Jussieu*, 17(2):305–346, 2018.

[BP21] N. Berestycki and E. Powell. Gaussian free field, liouville quantum gravity and gaussian multiplicative chaos. *Lecture notes*, 2021.

[BV01] E. Bolthausen and Y. Velenik. Critical behavior of massless free field at depinning transition. *Communications in Mathematical Physics*, 223:161–203, 2001.

[CDD+21] X. Chen, V. Dagard, B. Derrida, Y. Hu, M. Lifshits, and Z. Shi. The derrida–retaux conjecture on recursive models. *Ann. Probab.*, 49(2):637–670, 2021.

[CdH13] F. Caravenna and F. den Hollander. A general smoothing inequality for disordered polymers. *Electron. Commun. Probab.*, 18:1–15, 2013.

[CDH+19] X. Chen, B. Derrida, Y. Hu, M. Lifshits, and Z. Shi. A max-type recursive model: some properties and open questions. In *Sojourns in Probability Theory and Statistical Physics-III: Interacting Particle Systems and Random Walks, A Festschrift for Charles M. Newman*, pages 166–186. Springer, 2019.

[CM13] L. Coquille and P. Miłoś. A note on the discrete gaussian free field with disordered pinning on $\mathbb{Z}^d$, $d \geq 2$. *Stochastic Processes and their Applications*, 123(9):3542–3559, 2013.

[CSZ16] F. Caravenna, R. Sun, and N. Zygouras. The continuum disordered pinning model. *Probability theory and related fields*, 164(1):17–59, 2016.

[CSZ17a] F. Caravenna, R. Sun, and N. Zygouras. Polynomial chaos and scaling limits of disordered systems. *Journal of the European Mathematical Society*, 19:1–65, 2017.

[CSZ17b] F. Caravenna, R. Sun, and N. Zygouras. Universality in marginally relevant disordered systems. *The Annals of Applied Probability*, 27(5):3050–3112, 2017.

[CSZ23] F. Caravenna, R. Sun, and N. Zygouras. The critical 2d stochastic heat flow. *Inventiones mathematicae*, 233(1):325–460, 2023.

[CSZ25] F. Caravenna, R. Sun, and N. Zygouras. From disordered systems to the Critical 2D Stochastic Heat Flow. *arXiv:2511.08479*, 2025.

[CTT17] F. Caravenna, F. L. Toninelli, and N. Torri. Universality for the pinning model in the weak coupling regime. *Ann. Probab.*, 45(4):2154–2209, 2017.

[CV00] P. Caputo and Y. Velenik. A note on wetting transition for gradient fields. *Stochastic processes and their applications*, 87(1):107–113, 2000.

[Dar23] P. Dario. Upper bounds on the fluctuations of a class of degenerate convex $\nabla\phi$-interface models. *arXiv:2302.00547*, 2023.

[DCHL+22] H. Duminil-Copin, M. Harel, B. Laslier, A. Raoufi, and G. Ray. Logarithmic variance for the height function of square-ice. *Communications in Mathematical Physics*, 396(2):867–902, 2022.

[DDG13] A. Dembo, J. Ding, and F. Gao. Persistence of iterated partial sums. *Annales de l'IHP Probabilités et statistiques*, 49(3):873–884, 2013.

[DG99] J.-D. Deuschel and G. Giacomin. Entropic repulsion for the free field: Pathwise characterization in $d \geq 3$. *Communications in mathematical physics*, 206(2):447–462, 1999.

[DGLT09] B. Derrida, G. Giacomin, H. Lacoin, and F. L. Toninelli. Fractional moment bounds and disorder relevance for pinning models. *Comm. Math. Phys.*, 287(3):867–887, 2009.

[DK11] R. A. Doney and D. A. Korshunov. Local asymptotics for the time of first return to the origin of transient random walk. *Statistics & Probability Letters*, 81(9):1419–1424, 2011.

[Don97] R. A. Doney. One-sided local large deviation and renewal theorems in the case of infinite mean. *Probability Theory and Related Fields*, 107(4):451–465, 1997.

[DR14] B. Derrida and M. Retaux. The depinning transition in presence of disorder: a toy model. *Journal of Statistical Physics*, 156(2):268–290, 2014.

[Eri70] K. B. Erickson. Strong renewal theorems with infinite mean. *Transactions of the American Mathematical Society*, 151(1):263–291, 1970.

[ET60] P. Erdos and S. J. Taylor. Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar*, 11:137–162, 1960.

[Fel66] W. Feller. *An Introduction to Probability Theory and its Applications, Vol. I.* John Wiley & Sons. Inc., New York-London-Sydney, 1966.

[Fis84] M. E. Fisher. Walks, walls, wetting, and melting. *Journal of Statistical Physics*, 34(5-6):667–729, 1984.

[FS81] J. Fröhlich and T. Spencer. The kosterlitz-thouless transition in two-dimensional abelian spin systems and the coulomb gas. *Communications in Mathematical Physics*, 81(4):527–602, 1981.

[Fun05] T. Funaki. Stochastic interface models. *Lectures on probability theory and statistics*, 1869:103–274, 2005.

[FV17] S. Friedli and Y. Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction.* Cambridge University Press, 2017.

[Gia01] G. Giacomin. Aspects of statistical mechanics of random surfaces. *Notes of lectures given at IHP, fall*, 2001.

[Gia07] G. Giacomin. *Random polymer models.* Imperial College Press, 2007.

[Gia08] G. Giacomin. Renewal convergence rates and correlation decay for homogeneous pinning models. *Elec. Jour. Probab.*, 13:513–529, 2008.

[Gia11] G. Giacomin. *Disorder and Critical Phenomena Through Basic Probability Models: École d'Été de Probabilités de Saint-Flour XL–2010.* Springer, 2011.

[GL17] G. Giacomin and H. Lacoin. Pinning and disorder relevance for the lattice gaussian free field. *Journal of the European Mathematical Society*, 20(1):199–257, 2017.

[GL18] G. Giacomin and H. Lacoin. Disorder and wetting transition: the pinned harmonic crystal in dimension three or larger. *The Annals of Applied Probability*, 28(1):577–606, 2018.

[GL25] A. Glazman and P. Lammers. Delocalisation and continuity in 2D: Loop $o(2)$, six-vertex, and random-cluster models. *Communications in Mathematical Physics*, 406(5):1–59, 2025.

[GLT10] G. Giacomin, H. Lacoin, and F. L. Toninelli. Marginal relevance of disorder for pinning models. *Commun. Pure Appl. Math.*, 63:233–265, 2010.

[GLT11] G. Giacomin, H. Lacoin, and F. L. Toninelli. Disorder relevance at marginality and critical point shift. *Annales de l'IHP Probabilités et Statistiques*, 47(1):148–175, 2011.

[GT06a] G. Giacomin and F. L. Toninelli. The localized phase of disordered copolymers with adsorption. *CALEA Lat. Am. J. Probab. Math. Stat.*, 1:149–180, 2006.

[GT06b] G. Giacomin and F. L. Toninelli. Smoothing effect of quenched disorder on polymer depinning transitions. *Communications in mathematical physics*, 266:1–16, 2006.

[GT09] G. Giacomin and F. L. Toninelli. On the irrelevant disorder regime of pinning models. *The Annals of Probability*, 37(5):1841–1875, 2009.

[GZ24] G. Giacomin and M. Zamparo. Concentration and fluctuation phenomena in the localized phase of the pinning model. *arXiv:2405.16991*, 2024.

[GZ25] G. Giacomin and M. Zamparo. Pinning models with contact number constraint: the effect of disorder. *arXiv:2507.10707*, 2025.

[Har74] A. B. Harris. Effect of random defects on the critical behaviour of Ising models. *Journal of Physics C: Solid State Physics*, 7(9):1671, 1974.

[JL24] S. Junk and H. Lacoin. Strong disorder and very strong disorder are equivalent for directed polymers. *arXiv:2402.02562*, 2024.

[JL25] S. Junk and H. Lacoin. Coincidence of critical points for directed polymers for general environments and random walks. *arXiv:2502.04113*, 2025.

[JP72] N. C. Jain and W. E. Pruitt. The range of random walk. *Proc. Sixth Berkeley Symp. on Math. Statist. and Probab.*, pages 31–50, 1972.

[Kin73] J. F. C. Kingman. Subadditive ergodic theory. *The Annals of Probability*, 1(6):883–899, 1973.

[Lac10] H. Lacoin. The martingale approach to disorder irrelevance for pinning models. *Electron. Commun. Probab.*, 15:418–427, 2010.

[Law13] G. F. Lawler. *Intersections of random walks*. Springer Science & Business Media, 2013.

[Ler15] V. Lerouvillois. Polymer pinning models and conditions on the existence of a phase transition. Master's thesis, ENS Lyon, 2015.

[LL10] G. F. Lawler and V. Limic. *Random walk: a modern introduction*, volume 123. Cambridge University Press, 2010.

[LO24] P. Lammers and S. Ott. Delocalisation and absolute-value-FKG in the solid-on-solid model. *Probability Theory and Related Fields*, 188(1):63–87, 2024.

[Nag12] S. V. Nagaev. The renewal theorem in the absence of power moments. *Theory of Probability & Its Applications*, 56(1):166–175, 2012.

[PS70] D. Poland and H. A. Scheraga. *Theory of helix-coil transitions in biopolymers: statistical mechanical theory of order-disorder transitions in biological macromolecules*. Academic Press, 1970.

[Ton08a] F. L. Toninelli. Disordered pinning models and copolymers: beyond annealed bounds. *The Annals of Applied Probability*, pages 1569–1587, 2008.

[Ton08b] F. L. Toninelli. A replica-coupling approach to disordered pinning models. *Communications in Mathematical Physics*, 280(2):389–401, 2008.

[Vel06] Y. Velenik. Localization and delocalization of random interfaces. *Probability Surveys*, 3:112 – 169, 2006.

[Viv23] R. Viveros. Directed polymer for very heavy tailed random walks. *The Annals of Applied Probability*, 33(1):490–506, 2023.

[WY25] R. Wei and J. Yu. The critical disordered pinning measure. *The Annals of Applied Probability*, 35(6):3844–3905, 2025.