THÈSE

en vue de l'obtention du grade de

Docteur de l'École Normale Supérieure de Lyon - Université de Lyon

Discipline: Physique Mathématique

Laboratoire de Physique de l'École Normale Supérieure de Lyon

École Doctorale de Physique et d'Astrophysique

Présentée et soutenue publiquement le 15 juin 2012
par **M. Quentin Berger**

Polymères en milieu aléatoire : influence d'un désordre corrélé sur le phénomène de localisation

---

Directeur de thèse: M. Fabio Toninelli

Après l'avis de: M. Giambattista Giacomin
M. Frank den Hollander

Devant la commission d'examen composée de:

M. Francis Comets	Université Paris-Diderot
M. Bernard Derrida	École Normale Supérieure
M. Krzysztof Gawedzki	École Normale Supérieure de Lyon
M. Giambattista Giacomin	Université Paris Diderot (rapporteur)
M. Frank den Hollander	Université de Leiden (rapporteur)
M. Fabio Toninelli	École Normale Supérieure de Lyon (directeur)

Polymères en milieu aléatoire : influence d'un désordre corrélé sur le phénomène de localisation

Résumé: Cette thèse porte sur l'étude de modèles de polymère en milieu aléatoire: on se concentre sur le cas d'un polymère dirigé en dimension $d+1$ qui interagit avec un défaut unidimensionnel. Les interactions sont possiblement non-homogènes, et sont représentées par des variables aléatoires. Une question importante est celle de l'influence du désordre sur le phénomène de localisation: on veut déterminer si la présence d'inhomogénéités modifie les propriétés critiques du système, et notamment les caractéristiques de la transition de phase (auquel cas le désordre est dit *pertinent*).

En particulier, nous prouvons que dans le cas où le défaut est une marche aléatoire, le désordre est pertinent en dimension $d \geqslant 3$. Ensuite, nous étudions le modèle d'accrochage sur une ligne de défauts possédant des inhomogénéités corrélées spatialement. Il existe un critère non rigoureux (dû à Weinrib et Halperin), que l'on applique à notre modèle, et qui prédit si le désordre est pertinent ou non en fonction de l'exposant critique du système homogène, noté ν^{pur}, et de l'exposant de décroissance des corrélations. Si le désordre est gaussien et les corrélations sommables, nous montrons la validité du critère de Weinrib-Halperin: nous le prouvons dans la version hiérarchique du modèle, et aussi, de manière partielle, dans le cadre (standard) non-hiérarchique. Nous avons de plus obtenu un résultat surprenant: lorsque les corrélations sont suffisamment fortes, et en particulier si elles sont non-sommables (dans le cadre gaussien), il apparaît un régime où le désordre devient toujours pertinent, l'ordre de la transition de phase étant toujours plus grand que ν^{pur}. La prédiction de Weinrib-Halperin ne s'applique alors pas à notre modèle.

Mots-clés: Polymère, Modèle d'accrochage, Modèle hiérarchique, Systèmes désordonnés, Pertinence du désordre, Critère de Harris, Corrélation, Critère de Weinrib-Halperin.

Polymers in random environment: influence of correlated disorder on the localization phenomenon

Abstract: This thesis studies models of polymers in random environment: we focus on the case of a directed polymer in dimension $d+1$ that interacts with a one-dimensional defect. The interactions are possibly inhomogeneous, and are represented by random variables. We deal with the question of the influence of disorder on the localization phenomenon: we want to determine if the presence of inhomogeneities modifies the critical properties of the system, and especially the characteristics of the phase transition (in that case disorder is said to be *pertinent*).

In particular, we prove that if the defect is a random walk, disorder is relevant in dimension $d \geqslant 3$. We then study the pinning model in random correlated environment. There is a non-rigourous criterion (due to Weinrib and Halperin), that we can apply to our model, and that predicts disorder relevance/irrelevance, according to the value of the critical exponent of the homogeneous system, denoted ν^{pur}, and of the correlation decay exponent. When disorder is Gaussian and correlations are summable, we show that the Weinrib-Halperin criterion is valid: we prove this in the hierarchical version of the model, and also, partially, in the non-hierachical (standard) framework. Moreover, we obtained a surprising result: when correlations are sufficiently strong, and in particular when they are non-summable (in the gaussian framework), a new regime in which disorder is always relevant appears, the order of the phase transition being always larger than ν^{pur}. The Weinrib-Halperin prediction therefore does not apply to our model.

Keywords: Polymer, Pinning model, Hierarchical model, Disordered system, Disorder relevance, Harris criterion, Correlation, Weinrib-Halperin criterion.

École Normale Supérieure de Lyon – Laboratoire de Physique
(CNRS U.M.R. 5672)
46 Allée d'Italie, F-69364 Lyon cedex 07, FRANCE

Remerciements

Mes plus profonds remerciements vont évidemment à mon directeur, Fabio Toninelli. Je te remercie d'avoir guidé mes premiers pas dans le monde de la recherche, en me proposant un sujet de thèse aussi intéressant. Ce furent pour moi trois années riches, où j'ai sincèrement apprécié ton incroyable disponibilité, tes conseils avisés de chercheur expérimenté, tes qualités mathématiques et humaines. J'espère que nous seront amenés à poursuivre cette collaboration dans le futur.

Je voudrais aussi remercier Giambattista Giacomin et Frank den Hollander pour avoir accepté d'être les rapporteurs de cette thèse, et d'avoir pris le temps de relire ce manuscrit. C'est un honneur que mon travail soit jugé par ces chercheurs d'exception. Je suis aussi très reconnaissant à Francis Comets, Bernard Derrida et Krysztof Gawedski d'avoir accepté d'être dans mon jury, je leur sais gré de leur intérêt pour mon travail.

Durant cette thèse (et auparavant), j'ai découvert de nombreux environnements de travail, j'ai eu de nombreuses occasions de voyager, de présenter mes travaux. Je remercie tous ceux qui m'ont permis cela. Je suis particulièrement reconnaissant à Fabio Martinelli, qui m'a si bien accueilli à l'Università Roma Tre pendant ma première moitié de thèse, tout comme l'ensemble du département de mathématique là-bas. Débarqué à Lyon, j'ai aussi profité d'un environnement très agréable à l'ENS. Merci à l'ensemble du Laboratoire de Physique, notamment à Charles, Louis-Paul, Arnaud et Thomas, qui ont fait régner dans le bureau une ambiance détendue avec des sujets de conversations malsains, et merci à Pascal et toutes ses lubies. Merci aussi à toute l'équipe de Probabilité (qui inclut évidemment Lyon 1), dont le dynamisme est vraiment motivant, je pense par exemple aux frétillants Vincent et Sébastien, à Christophe, Vincent, Cédric, à Julien et nos échanges sur nos travaux respectifs. Je remercie également tout le personnel administratif qui fait un travail remarquable, que ce soit à Paris, Rome, ou Lyon.

Si l'on remonte à ma scolarité lointaine (mais pas tant que ça), je voudrais aussi avoir une pensée pour ceux qui m'ont donné envie de prendre cette voie: Mme Portelatine au lycée, Vellu et Tosel en prépa m'ont donné le goût des mathématiques; puis, à l'ENS, où j'ai naturellement été attiré vers les probabilités par Bertoin, Le Gall et Werner (que je remercie en passant de m'avoir orienté vers Fabio).

Je voudrais maintenant faire une spéciale dédicace à Hubert Lacoin-Potter, colocataire-collaborateur-ami, dont l'intuition mathématique m'impressionne et dont la drôlerie n'est plus à démontrer. Les collaborations que nous avons eues ont beaucoup fait avancer cette thèse, et je suis heureux d'avoir rencontré une personne aussi attachante.

Je crois aussi que c'est le moment ou jamais de remercier mes proches. Si je me suis épanoui professionnellement ces dernières années, je le dois aussi en grande partie aux gens en dehors du boulot, en particulier à mes colocataires successifs, et ils furent nombreux! Plus généralement, je me sens vraiment chanceux de vous avoir rencontré les amis. Je vous remercie pour tous ces moments partagés, qui me sont très importants. Merci de m'avoir entouré de votre présence enrichissante, apaisante, revigorante et drôle...

Commençons par l'aventure parisienne, avec Sabine yogi virevoltante, Yohan pianiste rêveur, Giuseppe tanguero relax, Mat Mat gentillesse cultivée, et tous les compagnons de soirées/week-end/vacances Anna, Mathieu H, Florence, Pénélope, Sébastien (×2), Clémentine, et beaucoup d'autres avec qui j'ai partagé tant de moments spéciaux et qui m'ont rendu Paris plus qu'agréable...

J'ai ensuite découvert Rome, où je me suis vraiment senti chez moi, notamment grâce à Giovanni ami showman, Mariangela finesse incarnée, Andrea aventureux en pantoufles, Anna D sicilienne chic, Letizia artiste "guanza", Hubert (sus-mentionné) romanista du San Calisto, Giulio compagnon de cordée cinéphile et Magali compagnonne d'aperitivo humoristiquement laxiste (je l'ai entendue rire à des blagues pas décentes)...

Mon retour à Lyon m'a aussi été rendu facile, par mes retrouvailles/trouvailles, en particulier avec Chloé grimpeuse sociale de quinoa, Clément grimpeur bricoleur de soupe et Lucie sage grimpeuse de tarte; et plus généralement avec tout plein d'amis tels Olivier, Dylan, Nata, Nasta, Caro, Benjo, Pierrot, Flavia... Sans parler de la surprenante découverte, au détour d'un stage CIES, d'une jolie curieuse, tendre et gourmande, Mélanie.

Merci aussi à toute ma famille, qui m'a sans cesse encouragé, et qui m'a fait exercer ma capacité de vulgarisation scientifique. Papa, maman, soeurette, merci pour votre présence rassurante, votre soutien à la fois constant et pas encombrant. Les grands-parents, les oncles et tantes, les cousins-cousines, je vous sens aussi tous derrière moi, votre enthousiasme me porte. Merci.

Je remercie enfin tous ceux qui se sont déplacés pour assister à ma soutenance, j'apprécie votre présence, j'espère que vous avez apprécié mon exposé...

La plupart d'entre-vous s'arrêteront à ces lignes. Pour les autres, bonne lecture!

Table des Matières

Introduction

0.1. Études de polymères: motivations physiques

Les polymères sont de très longues molécules, composées d'une grande quantité de monomères reliés entre eux par des liaisons chimiques (on parle de chaîne de monomères), et interagissent fortement aussi bien avec leur environnement qu'avec eux-même. Ceci rend l'étude de leurs propriétés et de leurs interactions avec leur milieu très complexe, et fait l'objet de nombreuses études de la part de la communauté physique depuis 50 ans [De 79, DE86, Flo53], et plus récemment du côté des mathématiciens [Bol02, dH09, Gia07, Lac09].

De nombreux modèles ont ainsi été développés pour comprendre leur structure. Citons-en un en particulier, très répandu: la modélisation d'un polymère par une marche aléatoire (faiblement) auto-évitante pour représenter les contraintes d'auto-interaction et d'auto-intersection, une définition physique étant donnée dans [Edw65] (pour des résultats mathématiques, on peut se référer à l'article de survol [Le 97]). Plusieurs questions apparaissent, comme l'étude de la topologie de ces marches auto-évitantes, le but étant de comprendre les configurations spatiales des polymères et de connaître leur degré d'enchevêtrement (voir [MMO11] pour avoir un aperçu des questions posées et des outils utilisés). Nous ne considérons par la suite que des polymères qui sont étirés dans une direction de l'espace (on parle de polymère dirigé), ce qui permet de s'affranchir des contraintes topologiques de non-intersection et d'auto-interaction.

Cette thèse porte sur l'étude mathématique de modèles de polymères dirigés en milieu aléatoire. Ils sont utilisés pour décrire de nombreux phénomènes physiques, et en particulier l'influence des interactions avec l'environnement sur la configuration spatiale du polymère, et l'on dispose alors d'autant d'outils mathématiques [Bol02, dH09, Gia07, Gia11]. Nous nous concentrons alors sur le phénomène de localisation (et de délocalisation) d'un polymère sur un défaut unidimensionnel.

0.1.1. Exemples de modèles de polymères en milieu aléatoire. On donne maintenant quelques exemples de phénomènes physiques pouvant être décrits grâce aux modèles de polymères dirigés.

• Pour étudier le comportement d'une chaîne-polymère dans une solution hétérogène, on dispose du modèle dit de *Polymère dirigé en milieu aléatoire* introduit (dans un cadre différent) par Huse et Henley [HH85]. Ce modèle permet d'obtenir des résultats mathématiques qui caractérisent l'influence des impuretés sur les fluctuations transverses du polymère, voir [CSY04] pour en avoir un panorama.

• On étudie aussi le déploiement d'un polymère constitué de différents types de monomères à l'interface entre deux solvants, qui est le modèle de *Copolymère* [BdH97, OTW99, Whi02] (Figure 1).

• On s'intéresse aussi au phénomène d'accrochage sur une surface solide ou sur la membrane d'une cellule (Figure 2), ou bien à la dénaturation de l'ADN, que l'on étudie via un modèle dit de *Piégeage* ou bien d'*Accrochage* (ou des versions proches de ce modèle, cf. [Gia07]).

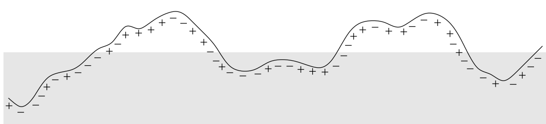

FIGURE 1. Un copolymère (à monomères + ou -) à l'interface entre deux solvants.

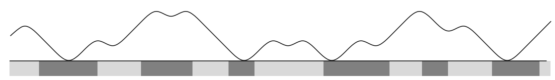

FIGURE 2. Un polymère interagissant avec une membrane de façon non homogène: les zones plus foncées représentent les zones d'attraction.

Nous nous consacrerons dans cette thèse à des modèles d'*Accrochage sur un défaut unidimensionnel*, le défaut pouvant aussi bien être une ligne fixe (accrochage sur une ligne de défauts, cf. Partie 2) qu'un autre polymère (accrochage sur une marche aléatoire, cf. Partie 1).

0.1.2. Phénomène de localisation. Le point commun entre tous les exemples décrits plus haut est que l'on observe une transition de phase, dite de *localisation*, qui est le principal objet d'étude de tous ces modèles. À haute température, l'agitation thermique est forte, et les interactions chimiques avec le milieu sont négligées devant les effets entropiques: le polymère se comporte comme s'il n'interagissait pas avec son environnement, on parle de phase *délocalisée*. À basse température, les interactions avec l'environnement deviennent très importantes: le polymère a tendance à se retrouver dans les régions énergétiquement les plus favorables de l'environnement, on parle de phase *localisée*.

Ce phénomène de localisation apparaît à une certaine température critique T_c, qui sépare les phases localisées et délocalisées, et l'une des questions centrales dans l'étude de ces systèmes désordonnés est, outre la connaissance de la valeur de T_c, de savoir comment se comporte le modèle au voisinage de la température critique.

0.1.2.1. *Dénaturation de l'ADN.* Le premier exemple que l'on peut donner de ces phénomènes est celui de la dénaturation de l'ADN: lorsque la température augmente et atteint un certain seuil (qui est environ de 90°C), certaines des liaisons chimiques qui lient les deux brins d'ADN sont cassées, et ces deux brins se décollent (Figure 3). Ce phénomène, de part la structure en double hélice de l'ADN et du caractère aléatoire de la séquence des bases, est extrêmement complexe, et est donc abordé d'une manière simplifiée.

Dans un premier temps, on oublie la structure en double hélice (Figure 4), et on considère l'ADN comme deux polymères constitués de quatre monomères possibles (les bases A, T, C, G), avec la condition que les deux séquences de "bases" soient complémentaires: une base A se trouve en face d'une base T, et une base C en face

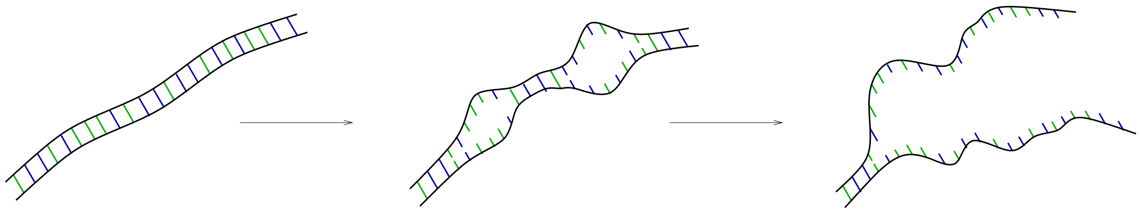

FIGURE 3. Phénomène de dénaturation de l'ADN. La question que l'on essaie de résoudre est de savoir de quelle manière s'effectue la transition entre la phase où les deux brins sont accrochés et la phase où les deux brins sont décrochés.

d'une base G. On n'observe donc que deux types de liaison entre les deux brins d'ADN (les liaisons AT et les liaisons CG) qui représentent des interactions de force différentes.

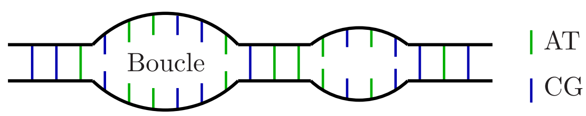

FIGURE 4. Modélisation simplifiée du phénomène de dénaturation: le polymère n'est pas enroulé et les interactions n'ont d'effet qu'aux points de contact, en dehors des boucles.

Le modèle le plus répandu pour étudier la dénaturation de l'ADN est celui introduit par Poland et Scheraga [PS66], (raffiné par Fisher [Fis66], voir [RG87] pour en avoir un panorama complet). Une fois que la séquence des bases a été fixée, de manière aléatoire le long de la chaîne, on ne prend en compte que la taille des boucles, et on suppose que les interactions sont à courte portée, *i.e.* qu'elles n'ont d'effet qu'au niveau des points de contacts entre les deux brins (Figure 4).

0.1.2.2. *Accrochage sur une ligne de défaut.* En suivant l'idée de Poland et Scheraga, on se ramène à un cadre plus général: on considère un polymère dirigé dans $\mathbb{R}^{d+1}$, qui interagit avec un défaut unidimensionnel parfaitement linéaire (cf. [Gia07]). Les interactions sont aléatoires, dans le sens où les couplages sont des variables aléatoires, qui dépendent de la position le long du défaut (Figure 5): ceci provient du fait que, aussi bien le polymère que la ligne de défauts ne sont pas homogènes (du moins peuvent ne pas l'être), et sont constitués de différents composants disposés de manière aléatoire le long des chaînes.

Comme on ne peut pas modifier l'ordre des monomères d'une chaîne, on considère que la constitution du polymère et de la ligne de défauts est fixée (de manière aléatoire), ce qui revient à considérer que la valeur des variables aléatoires représentant les interactions le long du défaut est fixée. On parle de désordre *gelé*, ou *quenched*.

Ensuite, la réduction de Poland-Scheraga consiste à estimer que la probabilité d'observer les points de contacts $\tau_1, \tau_2, \ldots$ dépend seulement de l'espacement entre ces points (*i.e.* de la taille des excursions en dehors de la ligne de défauts, ou des

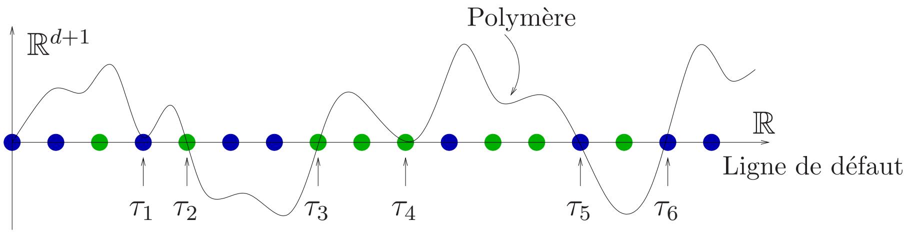

FIGURE 5. Accrochage d'un polymère dirigé sur une ligne de défauts non homogène, avec deux types d'interaction possible (représentées par deux couleurs différentes). On énumère les points de contacts $\tau_1, \tau_2, \ldots$, qui sont les seuls points où l'interaction polymère/défaut a lieu.

boucles dans le cas de l'ADN), et de l'interaction au niveau de ces points. On en déduit un modèle de méchanique statistique dont le formalisme mathémathique est détaillé dans la Section 0.1.3, et dont nous étudions les propriétés dans la Partie 2 (nous en donnons un aperçu dans le Chapitre 1).

0.1.2.3. *Accrochage sur un autre polymère.* Nous étudions aussi dans cette thèse le cas où le polymère (dirigé) s'accroche sur un défaut unidimensionnel, qui est cette fois un autre polymère (dirigé), dont la trajectoire est fixée et qui joue le rôle du désordre gelé. Nous ne considérons que le cas d'interactions homogènes, et la probabilité d'une trajectoire du polymère dépend alors uniquement du nombre de contacts entre les deux chaînes. On observe aussi une transition de phase entre un régime où le polymère est collé au défaut à basse température, et un régime où le polymère évite le défaut à haute température. Ce modèle, introduit dans [BS10], est présenté dans la Section 1.2 du Chapitre 1, et constitue l'objet d'étude de la Partie 1.

0.1.3. Modélisation mathématique. On remarque un cadre commun dans tous les exemples cités plus haut: on considère des polymères *dirigés* qui interagissent avec un défaut unidimensionnel. Les interactions n'ont lieu qu'aux points de contacts polymère/défaut, et l'intensité des interactions est représentée par des variables aléatoires.

Passons maintenant au formalisme mathématique pour décire ces phénomènes.

• **Le polymère** en dimension $d+1$ est modélisé par une marche aléatoire dirigée, *i.e.* une marche aléatoire dont une composante est déterministe et croissante. Soit $\{X_n\}_{n \geqslant 0}$ une marche aléatoire symétrique sur le réseau d-dimensionnel $\mathbb{Z}^d$, partant de 0, de loi notée $\mathbf{P}$. Alors la trajectoire d'un polymère est représentée par le graphe de la marche aléatoire dirigée $\{S_n\}_{n \geqslant 0} = (n, X_n)_{n \geqslant 0}$ (Figure 6).

• **Le défaut unidimensionnel** est alors soit la ligne $\mathbb{N} \times \{0\}$ dans le cas d'une ligne de défaut, soit lui aussi représenté par le graphe d'une marche aléatoire symétrique dirigée $(n, Y_n)_{n \geqslant 0}$ dans le cas d'un défaut étant lui-même un polymère. Les points de contact entre le polymère et le défaut ont lieu à des positions notées $\tau = \{\tau_i\}_{i \geqslant 0}$, et on notera que le polymère est en contact avec le défaut à l'instant n comme étant l'événement $\{n \in \tau\}$: c'est l'événement $\{X_n = 0\}$ dans le cas d'un défaut linéaire $\mathbb{N} \times \{0\}$, ou $\{X_n = Y_n\}$ dans le cas où le défaut est donné par une marche (n, Y_n). Les interactions sont données par deux paramètres $\beta \geqslant 0$ et $h \in \mathbb{R}$, et une séquence de variables aléatoires réelles $\omega := \{\omega_n\}_{n \geqslant 0}$ centrées unitaires, de loi notée $\mathbb{P}$. Les potentiels d'accrochages sont $-(\omega_n + h/\beta)$, et sont disposés le long du défaut, modélisant l'inhomogénéité des interactions (Figure 6).

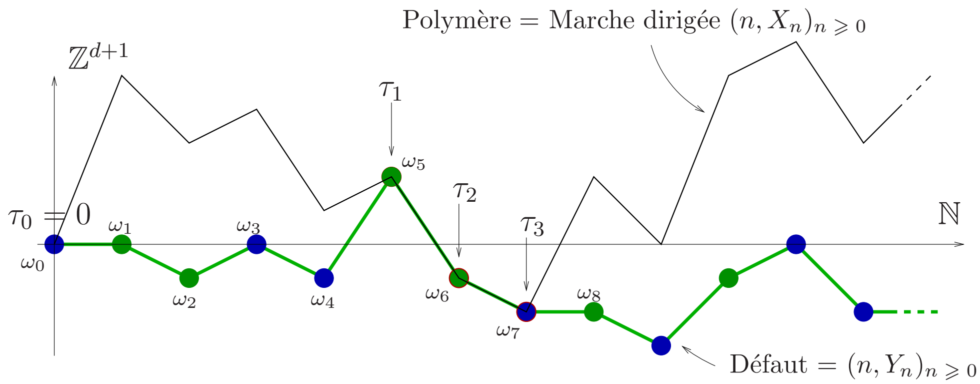

Figure 6. Les interactions entre les deux polymères sont encodées par la séquence $\{\omega_n\}_{n \geqslant 0}$ et les positions $\{\tau_i\}_{i \geqslant 0}$ des points de contact le long de la chaîne. Le hamiltonien est dans le cas ci-dessus $H = -(\omega_5 + \omega_6 + \omega_7) - 3h/\beta$. Dans la suite, nous considérerons deux cas: soit le défaut est représenté par une marche aléatoire Y et les interactions sont homogènes (*accrochage sur une marche aléatoire*, Section 1.2 et Partie 1), soit le défaut est linéaire $\mathbb{N} \times \{0\}$ mais les interactions sont non-homogènes (*accrochage sur une ligne de défaut*, Section 1.4 et Partie 2).

Il y a donc plusieurs sources d'aléatoire: la trajectoire du polymère d'une part, et l'environnement, *i.e.* la séquence ω des interactions et la forme du défaut, d'autre part. Les deux aléas ne jouent cependant pas le même rôle : l'environnement est fixé une fois pour toutes (désordre gelé), tandis que l'on étudie la configuration typique du polymère (*i.e.* des trajectoires de X) dans cet environnement.

On utilise le formalisme de Gibbs pour modéliser les interactions entre le polymère et son milieu: le défaut (Y, ω) étant gelé, on introduit le hamiltonien d'une trajectoire de longueur N

$$H_N^{Y,\omega}(X) = -\sum_{n=1}^{N} (\omega_n + h/\beta) \mathbf{1}_{\{n \in \tau\}}, \tag{0.1.1}$$

qui est la somme des potentiels d'accrochage récupérés le long de la chaîne, à chaque point de contact. La quantité $H_N^{Y,\omega}(X)$ est l'énergie associée à la trajectoire de X, et le polymère essaie alors de minimiser l'énergie qui lui est associée.

Pour un environnement donné, on définit la *mesure de polymère* $\mathbf{P}_{N,h}^{\omega,\beta}$ pour les trajectoires de longueur N comme une mesure de Boltzmann-Gibbs:

$$\mathbf{P}_N^{Y,\omega}(X_1, \ldots, X_N) = \frac{1}{Z_N^{Y,\omega}} \exp\Big(-\beta H_N^{Y,\omega}(X)\Big)\, \mathbf{P}(X_1, \ldots, X_N), \tag{0.1.2}$$

où $Z_N^{Y,\omega} = \mathbf{E}\left[\exp\left(\sum_{n=1}^N (\beta\omega_n + h)\mathbf{1}_{\{n\in\tau\}}\right)\right]$ est la *fonction de partition* du système, qui normalise $\mathbf{P}_N^{Y,\omega}$ à une loi de probabilité.

On a ainsi défini une mesure de probabilité $\mathbf{P}_N^{Y,\omega}$ qui favorise les trajectoires ayant une faible énergie (on peut voir aussi $-H_N^{Y,\omega}(X)$ comme la récompense associée à X, les trajectoires obtenant une forte récompense devenant plus probables sous $\mathbf{P}_N^{Y,\omega}$). On remarque que la mesure $\mathbf{P}_N^{Y,\omega}$ est elle-même aléatoire, car dépendant de la réalisation de l'environnement fixée. La question est alors de déterminer le comportement des trajectoires sous la mesure $\mathbf{P}_N^{Y,\omega}$ pour N très grand, et ceci pour des réalisations typiques de l'environnement.

On définit alors l'*énergie libre* du système, ou *énergie par monomère*, comme la limite (qui existe sous certaines conditions, et ne dépend que de la loi de l'environnement et pas de la réalisation de celui-ci, comme abordé dans les Sections 1.2-1.3-1.4)

$$\mathtt{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{Y,\omega} \geqslant 0. \tag{0.1.3}$$

L'énergie libre encode des propriétés cruciales du système, par exemple la densité limite de contacts polymère/défaut sous $\mathbf{P}_N^{Y,\omega}$ (cf. le Chapitre 1 pour plus de détails).

Le paramètre $\beta \geqslant 0$ peut être interprété comme l'inverse de la température (plus précisément $1/(\mathrm{k_B}T)$ au vue des poids de Boltzmann-Gibbs, avec k_B la constante de Boltzmann, le terme βH_N étant sans unité) et donne l'intensité du désordre. En revanche, h/β représente un paramètre d'accrochage homogène. Lorsque h varie, le système passe par une transition de phase au voisinage d'un certain paramètre critique $h_c(\beta)$: pour $h < h_c(\beta)$ on est dans une phase *délocalisée* où le polymère touche très peu le défaut (il est souvent suggéré qu'il a un comportement diffusif), et pour $h > h_c(\beta)$ on est dans une phase *localisée* où le polymère est accroché au défaut. Remarquons que l'on utilise la notation h/β par commodité, pour rendre les notations plus agréables (notamment pour l'écriture dde la fonction de partition $Z_N^{Y,\omega}$).

0.2. Objectifs de la thèse: rôle du désordre sur le phénomène d'accrochage

Comme annoncé précédemment, nous nous intéressons dans cette thèse à deux cas du modèle d'accrochage.

- Dans la Partie 1, le défaut est une marche aléatoire dirigée $(n, Y_n)_{n\geqslant 0}$ et où les interactions sont homogènes ($\omega_i = 0$ pour tout $i \in \mathbb{N}$): c'est le modèle d'*Accrochage sur une marche aléatoire*.
- Dans la partie 2, le défaut est linéaire ($Y_i = 0$ pour tout $i \in \mathbb{N}$) avec des interactions non-homogènes: c'est le modèle d'*Accrochage sur une ligne de défaut*. Nous considérerons en réalité toute une classe de modèles indexés par un paramètre α, qui caractérise l'écart entre les points de contacts avec la ligne de défaut (on a $\mathbf{P}(\tau_1 = k) \sim cst.k^{-(1+\alpha)}$) et qui est donc lié à la dimension d de la marche aléatoire X, en l'occurence $\alpha = 1/2$ pour $d = 1$ et $\alpha = d/2 - 1$ pour $d \geqslant 2$ (avec une correction logarithmique pour $d = 2$, cf. Section 1.1.2 pour plus de détails).

0.2.1. Quelles questions sont soulevées? Nous nous intéressons principalement aux propriétés critiques du système, c'est-à-dire au comportement des trajectoires sous $\mathbf{P}_N^{Y,\omega}$ (dans la limite $N \to \infty$), pour h proche de $h_c(\beta)$. On cherche par exemple à connaître la valeur du point critique, pour savoir pour quelles valeurs des paramètres β et h le polymère reste accroché au défaut, et comment s'effectue la transition entre la phase localisée et la phase délocalisée.

On commence par le modèle d'accrochage homogène (ligne de défaut $\mathbb{N} \times \{0\}$, $\beta = 0$) qui est non-trivial, et en réalité exactement résoluble, comme observé par Fisher [Fis84]. Nous introduisons ce modèle dans la Section 1.1.2, et nous en donnons les principales caractéristiques: on connaît à la fois le point critique $h_c(0)$ et le comportement de l'énergie libre au voisinage de ce point. On a $\mathtt{F}(0,h) \sim cst.(h - h_c)^{\nu^{\mathrm{pur}}}$, où l'exposant critique ν^{pur} dépend du paramètre α (et donc de la dimension d, voir plus haut), $\nu^{\mathrm{pur}} = \max(1, 1/\alpha)$ (avec une correction logarithmique pour $\alpha = 1$). L'exposant critique contient beaucoup d'informations sur le système et est en général le principal objet d'étude des systèmes physiques.

On souhaite alors savoir comment se comporte le système désordonné (on dit aussi modèle *quenched*), en comparaison avec les résultats sur le modèle homogène. La question est de savoir si la présence du désordre (inhomogénéité ou non-linéarité du défaut) modifie ou non les propriétés critiques du système:

- en ce qui concerne le point critique, on introduit le modèle d'accrochage *recuit* ou *annealed*, pour lequel on a "dégelé" le défaut en prenant la moyenne sur l'environnement (la fonction de partition étant $\mathbb{E}[Z_{N,h}^{\omega,\beta}]$), et qui sert d'étalon pour le modèle désordonné (cf. Section 1.4 pour une introduction plus poussée). On a facilement que le point critique *annealed* $h_c^{\mathrm{a}}(\beta)$ vérifie $h_c^{\mathrm{a}}(\beta) \leqslant h_c(\beta)$, et la question est de savoir si il y a effectivement égalité ou non.
- en ce qui concerne le comportement critique de l'énergie libre, on cherche à savoir si l'exposant critique ν^{que} du modèle *quenched* (s'il existe) est égal à ν^{pur}, et de quelle manière il est modifié si $\nu^{\mathrm{que}} \neq \nu^{\mathrm{pur}}$.

Si une quantité arbitraire de désordre change le comportement critique du système (*i.e.* points critiques *quenched* et *annealed* différents, exposant critique modifié), on dit que le désordre est *pertinent*. Nous nous attacherons dans cette thèse à vérifier la pertinence ou la non-pertinence du désordre, de manière aussi bien qualitative que quantitative.

Ces questions ont été largement abordées par les physiciens, qui ont prouvé ou conjecturé de nombreux résultats, notamment grâce à des techniques de groupes de renormalisation, qui donnent des conjectures fiables, bien que cette méthode puisse difficilement être rendue rigoureuse. Les modèles hiérarchiques (sur des réseaux en diamant) ont étés introduits dans cette optique dans de nombreux modèles (par exemple [Ble89, CEGM84, DG84]). Ils sont construits sur des réseaux possédant une structure auto-similaire, ce qui permet d'utiliser des changements d'échelle successifs entre un système de taille $2N$ et un système de taille N, de manière exacte. Nous nous penchons longuement sur le cas du modèle d'accrochage hiérachique [DHV92], que nous introduisons dans la Section 1.3, et nous donnerons les principaux résultats obtenus durant cette thèse dans le Chapitre 4.

0.2.2. Influence du désordre sur la transition de phase. Pour décider de l'influence du désordre, le physicien A. B. Harris donne une méthode générale [Har74], développée pour étudier la transition paramagnétique/ferromagnétique du modèle d'Ising, et qui permet dans le cas d'un environnement *i.i.d.* (indépendant identiquement distribué) de donner un critère pour la pertinence/non-pertinence du désordre. On souligne que le caractère *i.i.d.* du désordre est très important, et qu'il permet d'adapter la méthode pour une très grande classe de modèles.

Mentionnons que Harris considère un désordre d-dimensionnel (à ne pas confondre avec la dimension $d+1$ dans laquelle évolue notre polymère), et dans notre cas on a donc $\mathsf{d}=1$. Le critère repose sur l'exposant critique ν de la longueur de corrélation du système pur (il est montré que dans le cas du modèle d'accrochage, ν est égal à ν^{pur} l'exposant critique de l'énergie libre). Dans un cadre général (incluant le nôtre), nous pouvons reformuler le critère de Harris de la manière suivante:

• Si $\nu^{\mathrm{pur}} > 2/\mathsf{d}$, alors le désordre devrait être *non-pertinent*: pour une intensité suffisamment faible de désordre (*i.e.* β petit), les points critiques *quenched* et *annealed* sont égaux, et l'exposant critique de la longueur de corrélation *quenched* est ν^{pur}.

• Si $\nu^{\mathrm{pur}} < 2/\mathsf{d}$, alors le désordre devrait être *pertinent*: pour n'importe quelle quantité de désordre, on a $h_c^{\mathrm{a}}(\beta) < h_c(\beta)$, et le comportement critique de la longueur de corrélation *quenched* est différent de celui du modèle homogène (on prévoit que l'exposant critique *quenched* sera plus grand que $2/\mathsf{d}$).

Pour ce qui est du cas marginal $\nu^{\mathrm{pur}} = 2/\mathsf{d}$, savoir si le désordre est pertinent ou non requiert une étude plus poussée et dépend fortement du modèle.

L'argument de Harris, pour en donner une idée simplifiée, consiste à contrôler les fluctuations de l'environnement ω pour un système de taille ℓ, où $\ell = \ell(h)$ est la longueur de corrélation du système désordonné: un système de taille ℓ est censé être une bonne approximation du vrai système. Faisons pour l'instant l'hypothèse (on verra plus tard dans quel cas elle est justifiée), que pour β petit on a $\ell \sim (h - h_c(\beta))^{-\nu^{\mathrm{pur}}}$, qui signifie que le désordre est non-pertinent: les longueurs de corrélation des systèmes désordonné et homogène ont le même exposant critique. Dans notre cadre, rappelons que $\mathsf{d}=1$, et estimons l'écart type suivant (on a choisi ω centré et de variance unitaire)

$$\frac{1}{\ell}\mathbb{V}\mathrm{ar}\left(\sum_{n=1}^{\ell}\omega_n\right)^{1/2} = \ell^{-1/2}, \tag{0.2.1}$$

qui représente les fluctuations typiques de la moyenne empirique des ω_i sur un échantillon de taille ℓ. Harris justifie que le point critique "observé" du système de taille ℓ dévie du vrai point critique $h_c(\beta)$ de manière proportionelle au terme (0.2.1), et les fluctuations du désordre sont donc négligeables si et seulement si

$$\ell^{-1/2} \ll |h - h_c(\beta)|. \tag{0.2.2}$$

L'hypothèse de non-pertinence du désordre ($\ell \sim (h - h_c(\beta))^{-\nu^{\text{pur}}}$) n'est donc cohérente que si $\nu^{\text{pur}} > 2$.

On se réfère à la Section 1.4.2 pour une explication plus poussée du critère de Harris appliqué à notre étude, et pour un résumé des résultats connus dans le cas *i.i.d.*

Ces dernières années, de nombreux travaux physiques et mathématiques ont confirmé cette prédiction de manière rigoureuse [Ale08, AZ09, DGLT09, GLT10a, GT06, Ton08a] pour le modèle d'accrochage considéré, aussi bien au niveau de l'écart entre les points critiques qu'au niveau du comportement critique de l'énergie libre. Le cas marginal $\nu^{\text{pur}} = 2$ (qui est le cas $\alpha = 1/2$, *i.e.* de la dimension $d = 1$ et $d = 3$ dans notre modèle) a lui aussi été résolu dans [GLT10b], après avoir fait l'objet d'affirmations contradictoires dans la littérature physique, et il est montré que le désordre est pertinent dans ce cas, comme prédit par [DHV92].

Mentionnons que le critère de Harris a aussi été démontré dans [CCFS86] pour une classe assez générale de modèles (percolation, modèle d'Ising,...), il y est prouvé que l'exposant critique d'une certaine longueur de corrélation est plus grand que $2/\mathsf{d}$ (dans [CCFS86], la définition de la longueur de corrélation est différente de celle habituelle).

Il est donc naturel de se tourner maintenant sur le cas d'environnements présentant des corrélations spatiales. Par exemple, dans le cas de l'ADN, il a été montré dans [LK92, PBG+92] que la séquence des bases possèdait des corrélations à longue portée décroissant en loi de puissance. Il est donc question d'analyser comment ces corrélations influent sur le processus de dénaturation de l'ADN.

La méthode aboutissant au critère de Harris peut aussi être adaptée dans ce cas, et aboutit à un critère mis en place par Weinrib et Halperin [WH83] dans un cadre général, étendant la prédiction de Harris. On suppose que la fonction de corrélation (à deux points) décroît comme $r^{-\zeta}$, r étant la distance entre les points, $\zeta > 0$. Weinrib et Halperin montrent ainsi que, pour $\zeta > \mathsf{d}$, la présence de corrélations ne modifie pas le critère de pertinence/non-pertinence du désordre par rapport au cas *i.i.d.*: le désordre devrait être pertinent si $\nu^{\text{pur}} < 2/\mathsf{d}$ et non-pertinent si $\nu^{\text{pur}} > 2/\mathsf{d}$. En revanche, lorsque la fonction de corrélation décroît plus lentement, en l'occurence lorsque $\zeta < \mathsf{d}$, le critère est modifié par la présence des corrélations, et le désordre devrait alors être pertinent si $\nu^{\text{pur}} < 2/\zeta$ et non-pertinent si $\nu^{\text{pur}} > 2/\zeta$.

Nous retrouvons cette prédiction dans notre cadre (où $\mathsf{d} = 1$) en reprenant le raisonnement fait plus haut, cette fois dans le cas corrélé. Comme pour (0.2.2), les fluctuations du désordre sont négligeables seulement si

$$\frac{1}{\ell}\mathbb{V}\text{ar}\left(\sum_{n=1}^{\ell}\omega_n\right)^{1/2} \asymp \ell^{-\min(1,\zeta)/2} \ll |h - h_c(\beta)|, \tag{0.2.3}$$

où l'exposant $-\min(1,\zeta)/2$ vient d'un calcul explicite au vu de la forme $r^{-\zeta}$ des corrélations (avec des corrections logarithmiques pour $\zeta = 1$). Ainsi de la même manière que précédemment, l'hypothèse de non-pertinence du désordre ($\ell \sim (h - h_c(\beta))^{-\nu^{\text{pur}}}$) n'est cohérente que si $\nu^{\text{pur}} > 2/\min(1,\zeta)$.

0.2.3. Description des résultats obtenus. La première partie de cette thèse porte sur le modèle d'accrochage sur une marche aléatoire (avec interactions homogènes), dont nous introduisons les principaux résultats dans la Section 1.2. Jusqu'à présent, il était connu que le désordre était non-pertinent pour $d = 1, 2$, et pertinent pour $d \geqslant 4$, ceci uniquement du point de vue de l'écart entre les points critiques. Notre travail a complété ces résultats en traitant le cas de la dimension $d = 3$, et en comparant les exposants critiques des systèmes désordonnés et homogènes.

• Dans le Chapitre 2, nous montrons que le désordre est aussi pertinent en dimension $d = 3$, et en particulier que les points critiques *quenched* et *annealed* sont différents. Ceci qui a abouti à la publication de l'article [BT10], écrit en collaboration avec F. Toninelli.

• Dans le Chapitre 3, nous nous intéressons à l'exposant critique de l'énergie libre du système désordonné, et nous montrons que la transition de phase est toujours d'ordre au moins 2, ce qui souligne la pertinence du désordre en dimension $d \geqslant 4$, où la transition de phase du modèle homogène est d'ordre 1 (comme mentionné plus haut). Ce travail a donné lieu à la publication de l'article [BL11], en collaboration avec H. Lacoin.

Dans la deuxième partie, nous nous concentrons sur le modèle d'accrochage sur une ligne de défauts, dans le cas d'un environnement corrélé. Le principal objectif de cette thèse était de tester la prédiction de Weinrib et Halperin concernant la pertinence/non-pertinence du désordre. Nous avons obtenu de nombreux résultats, en particulier la confirmation de ce critère dans certains cas (voir ci-dessous), et aussi la mise en évidence d'un régime où de très fortes corrélations font apparaître un comportement inattendu du sytème, où le désordre est toujours pertinent.

• Dans le Chapitre 4, nous considérons la version hiérarchique du modèle, dans le cas d'un environnement gaussien corrélé (avec corrélations à longue portée). Le principal résultat obtenu est que le critère de Weinrib-Halperin est vérifié, dans le cas correspondant à $\zeta > 1$: le désordre est pertinent pour $\nu^{\text{pur}} \leqslant 2$ et non-pertinent pour $\nu^{\text{pur}} > 2$. Cela concerne aussi bien la différence entre les exposants critiques que l'écart entre les points critiques (à part pour le cas $\nu^{\text{pur}} = 2$), bien qu'il reste une zone où le désordre est pertinent mais où le système *annealed* possède un comportement atypique (cf. Section 4.4.3) et où nous ne sommes pas arrivés à montrer que $h_c^{\text{que}}(\beta) < h_c^{\text{a}}(\beta)$. On retrouve ainsi le même critère pour la pertinence du désordre que dans le cas d'un environnement $i.i.d.$, et on remarque que le cas marginal $\nu^{\text{pur}} = 2$ est également traité, en ce qui concerne les points critiques. Le cas correspondant à $\zeta < 1$ est lui aussi étudié dans le Chapitre 4, et il est montré qu'il apparaît alors un comportement différent: il n'y a plus de transition de phase, et on ne peut donc pas vérifier ni infirmer le critère de Weinrib-Halperin. Ce travail a permis la publication de l'article [BT11], écrit en collaboration avec F. Toninelli, et on peut se référer à la Section 1.3 pour avoir un aperçu plus précis des résultats.

• Le Chapitre 5 reprend le modèle d'accrochage sur une ligne de défauts dans le cas non hiérarchique, avec là aussi un environnement gaussien corrélé. Nous avons été en mesure de prouver seulement des résultats partiels dans le cas $\zeta > 1$: nous prouvons notamment la pertinence du désordre si $\nu^{\mathrm{pur}} < 2$. Pour $\zeta < 1$, on observe le même phénomène que dans le modèle hiérarchique, à savoir l'apparition d'un nouveau régime sans transition de phase. Nous regroupons d'autres résultats dans ce chapitre, et en particulier une analyse du système *annealed*.

Les Chapitres 6 et 7 considèrent le cas d'un environnement borné (il existe une constante C telle que $|\omega_i| \leqslant C$ $\mathbb{P}$-presque sûrement pour tout $i \in \mathbb{N}$, en contraste avec les cas gaussien), possédant de très fortes corrélations. Dans ce cas, la transition de phase ne disparaît pas comme dans le cas gaussien évoqué ci-dessus, lorsque $\zeta < 1$. Cependant on observe tout de même un régime différent, que nous appelons *fortement pertinent*: on connaît exactement le point critique, et la transition de phase est d'ordre supérieur au cas homogène, montrant ainsi la pertinence du désordre pour toute valeur de ν^{pur}. Le critère de Weinrib-Halperin est donc ici mis en défaut.

• Dans le Chapitre 6, en vertu des caractéristiques spéciales de l'environnement considéré, nous donnons un encadrement précis (à une constante près) de l'énergie libre, et nous décrivons très bien le comportement des trajectoires sous la mesure de polymère. Soulignons que le comportement critique du système est toujours très différent du modèle homogène, le désordre étant *toujours pertinent*. Ce travail a fait l'objet de l'article [BL], co-écrit avec H. Lacoin.

• Dans le Chapitre 7, nous considérons un environnement très général, et nous donnons une condition suffisante pour l'apparition d'un régime *fortement pertinent* décrit plus haut. Nous exposons de plus un cas naturel (dit des *Signes gaussiens*) où nous montrons que, pourvu que les corrélations soient fortes ($\zeta < 1$ avec les notations utilisées plus haut), la transition de phase est toujours d'ordre infini.

Notations

Nous donnons ici quelques notations, utilisées de manière répétées dans la suite de la thèse.

i.i.d.	indépendant(es) et identiquement distribué(es);
p.s.	presque sûrement, (*a.s.* pour almost surely);
$\langle x, y \rangle$	(ou $x \cdot y$) produit scalaire canonique sur $\mathbb{R}^d$, on note aussi $\\|\cdot\\|$ la norme euclidienne associée;
$\vert \cdot \vert$	désigne la cardinalité d'un ensemble fini (dans le cas discret), ou bien la mesure de Lebesgue d'un ensemble (pour un Borélien de $\mathbb{R}$);
$a \vee b$	(resp. $a \wedge b$), désigne $\max(a, b)$ (resp. $\min(a, b)$).

Pour deux fonctions $f(x)$ et $g(x)$, on introduit les notations classiques suivantes, lorsque x tend vers une valeur $a \in \mathbb{R} \cup \{-\infty, +\infty\}$

$f(x) = o(g(x))$	si $f(x)/g(x) \to 0$ lorsque $x \to a$;
$f(x) = O(g(x))$	si $f(x)/g(x)$ reste borné lorsque $x \to a$;
$f(x) \sim g(x)$	si $f(x)/g(x) \to 1$ lorsque $x \to a$;
$f(x) \asymp g(x)$	si $f(x) = O(g(x))$ et $g(x) = O(f(x))$ lorsque $x \to a$.

Nous préciserons toujours si les comportements asymptotiques dépendent des constantes du modèle que nous considérons.

Les constantes (celles dont la valeur précise n'est pas importante) seront de manière générale notées C, c, C', c'... et pour alléger les notations, leurs valeurs pourront changer d'une ligne à l'autre.

CHAPTER 1

Présentation préliminaire des modèles étudiés

1.1. Modèle d'accrochage homogène

Les modèles d'accrochages que nous allons étudier dans cette thèse sont déjà intéressants dans leur version homogène, c'est-à-dire sans désordre: les outils techniques utilisés se montrent fructueux aussi dans l'étude des sytèmes désordonnés. De plus, il est important de donner les résultats dans le cas de sytèmes purs (*i.e.* sans désordre), afin de les comparer à ceux obtenus pour des systèmes désordonnés, soulignant ainsi dans quelle mesure la présence d'inhomogénéités affecte les propriétés critiques du système.

1.1.1. Accrochage d'une marche aléatoire sur une ligne de défauts. Nous introduisons ici le modèle d'accrochage de la marche aléatoire simple d-dimensionelle sur une ligne de défaut dont nous avons parlé dans la Section 0.1.3. Soit $X = (X_n)_{n\in\mathbb{N}}$ la marche aléatoire simple symétrique, à valeurs dans $\mathbb{Z}^d$, pour $d \geqslant 1$, dont la loi est notée $\mathbf{P}$. Cette séquence X est définie par $X_0 = 0$ $\mathbf{P}$-*p.s.*, et ses incréments $(X_i - X_{i-1})_{i\in\mathbb{N}}$ sont *i.i.d.*, satisfaisant

$$\mathbf{P}(X_1 = +e_k) = \mathbf{P}(X_1 = -e_k) = \frac{1}{2d}, \qquad \text{pour tout } k \in \{1, \dots, d\}, \tag{1.1.1}$$

où e_k désigne le k^{e} vecteur de la base canonique de $\mathbb{R}^d$.

Introduisons maintenant les interactions de la marche aléatoire dirigée (n, X_n) (le polymère), avec la ligne de défauts $\mathbb{N}\times\{0\}$. Pour $h \in \mathbb{R}$, qui représente l'intensité des interactions, on modifie la mesure $\mathbf{P}$, en donnant une récompense (ou une pénalité suivant le signe de h), à chaque fois que la trajectoire de X touche 0. On obtient ainsi une *mesure de polymère* $\mathbf{P}_{N,h}$ pour les trajectoires de longueur $N \in 2\mathbb{N}$, qui est une transformation de Gibbs de la loi $\mathbf{P}$, donnée par sa dérivée de Radon-Nikodym

$$\frac{\mathrm{d}\mathbf{P}_{N,h}}{\mathrm{d}\mathbf{P}} := \frac{1}{Z_{N,h}} \exp\left(h \sum_{n=1}^{N} \mathbf{1}_{\{X_n=0\}}\right) \mathbf{1}_{\{X_N=0\}}, \tag{1.1.2}$$

$$\text{avec} \qquad Z_{N,h} := \mathbf{E}\left[\exp\left(h \sum_{n=1}^{N} \mathbf{1}_{\{X_n=0\}}\right) \mathbf{1}_{\{X_N=0\}}\right]. \tag{1.1.3}$$

La quantité $Z_{N,h}$ est la fonction de partition du système homogène. On a imposé la contrainte $\{X_N = 0\}$ pour des raisons techniques, notamment pour assurer la sur-additivité de $\log Z_{N,h}$. Cette condition impose que N soit pair dans le cas de la marche simple, mais ne modifie pas les propriétés essentielles du système.

Nous nous intéressons maintenant aux propriétés des trajectoires sous la mesure $\mathbf{P}_{N,h}$, quand la longueur du polymère tend vers l'infini (*i.e.* quand $N \to \infty$), pour les différentes valeurs de h possibles. On étudie plus particulièrement le nombre de contacts de X avec 0, afin de savoir si les trajectoires sont accrochées à la ligne de défaut, ou au contraire si elles sont repoussées par la ligne. On étudie *l'énergie libre* du système, définie par

$$\mathtt{F}(h) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h}, \tag{1.1.4}$$

dont l'existence est prouvée grâce à la sur-additivité de $\log Z_{N,h}$. La fonction $h \mapsto \mathtt{F}(h)$ est convexe et croissante, car c'est la limite de fonctions convexes et croissantes en h. De plus, il est facile de voir que $\mathtt{F}(h) \geqslant 0$ pour tout h, par exemple en dimension 1, en utilisant l'inégalité

$$Z_{N,h} \geqslant e^h \times \mathbf{P}(\text{le premier retour de } X_n \text{ en } 0 \text{ a lieu en } N) \geqslant C N^{-3/2}, \tag{1.1.5}$$

où la dernière inégalité est classique. On vérifie aussi sans problème que $\mathtt{F}(h) \leqslant 0$ (et donc $\mathtt{F}(h) = 0$) si $h \leqslant 0$.

Remarquons aussi que l'énergie libre est liée à la limite de la fraction de contacts dans le système de taille N,

$$\frac{1}{N}\mathbf{E}_{N,h}\left[\sum_{n=1}^{N} \mathbf{1}_{\{X_n=0\}}\right] = \frac{\partial}{\partial h}\frac{1}{N}\log Z_{N,h}. \tag{1.1.6}$$

Grâce à la convexité, et si on suppose que $\mathtt{F}'(h)$ existe (ce qui est le cas pour presque tout h), cette égalité passe à la limite quand $N \to \infty$, et l'on obtient donc que

$$\mathtt{F}'(h) = \lim_{N\to\infty} \mathbf{E}_{N,h}\left[\frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{\{X_n=0\}}\right]. \tag{1.1.7}$$

On peut donc séparer les comportements possibles du polymère en deux phases, voir Figure 1.1, partagées par un point critique $h_c := \inf\{h;\, \mathtt{F}(h) > 0\}$:

- Une phase *Délocalisée* pour $h < h_c$, où $\mathtt{F}(h) = 0$, et où la densité de contacts asymptotique est nulle;
- Une phase *Localisée* pour $h > h_c$, où $\mathtt{F}(h) > 0$, et où la densité de contacts asymptotique est strictement positive.

Pour $h = h_c$ le polymère peut être localisé ou delocalisé selon que $\mathtt{F}'(h_{c+})$ soit positif ou nul.

1.1.2. Généralisation du modèle. Pour l'étude de ce modèle d'accrochage, il est en fait préférable de le considérer dans un cadre plus général. La première remarque est que la mesure de polymère $\mathbf{P}_{N,h}$ modifie la loi des instants de retour en 0, mais pas la loi des excursions de X en dehors de 0 (conditionellement aux retours). On ne s'intéresse donc qu'aux instants de retour en 0, que l'on modélise par un processus de renouvellement.

Définition 1.1.1 (Processus de renouvellement)**.** Un processus $\tau = (\tau_i)_{i\,\geqslant\,0}$, de loi $\mathbf{P}$, est appelé *processus de renouvellement discret* si $\tau_0 = 0$ $\mathbf{P}$-*p.s.* (on dit qu'il est non retardé), et si les $(\tau_i - \tau_{i-1})_{i\in\mathbb{N}}$ sont des variables aléatoires *i.i.d.* à valeurs dans

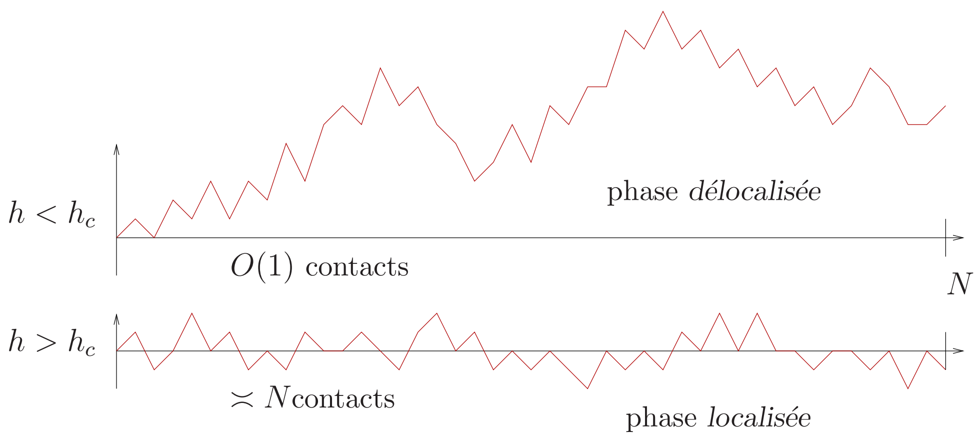

FIGURE 1.1. Cette figure donne un exemple de trajectoires typiques sous la mesure de polymère $\mathbf{P}_{N,h}$, lorsque $h < h_c$ (phase *délocalisée*, le polymère se trouve loin de la ligne de défauts) et lorsque $h > h_c$ (phase *localisée*, le polymère est accroché à la ligne de défauts).

$\mathbb{N}$. On appellera l'instant τ_n un point de renouvellement, les $(\tau_i - \tau_{i-1})$ les temps inter-arrivées (en référence à la marche aléatoire), et les intervalles $\{\tau_{i-1}, \dots, \tau_i - 1\}$ des excursions. On notera, pour $n \in \mathbb{N}$, $\mathrm{K}(n) := \mathbf{P}(\tau_1 = n)$ et on appellera $\mathrm{K}(\cdot)$ la *loi inter-arrivées*. On appelera $m_K = \mathbf{E}[\tau_1]$ la moyenne de τ_1 et on notera aussi $\mathrm{K}(\infty) = \mathbf{P}(\tau_1 = \infty) = 1 - \sum_{n \in \mathbb{N}} \mathrm{K}(n)$. On suppose aussi que $\mathrm{K}(n) > 0$ pour tout $n \in \mathbb{N}$ (dans le cas de la marche aléatoire simple en dimension d, il suffit de diviser par deux les temps de retour en 0).

On peut distinguer plusieurs cas :

- Si $\mathrm{K}(\infty) > 0$ ($\mathrm{K}(\cdot)$ est une sous-probabilité), alors τ est fini p.s.: le processus est dit *transient*;
- Si $\mathrm{K}(\infty) = 0$ ($\mathrm{K}(\cdot)$ est une probabilité), alors τ est infini *p.s.*: le processus est dit *récurrent*, ou *persistant*. Si $m_K = +\infty$, le processus est récurrent nul, et si $m_K < +\infty$, le processus est récurrent positif.

Remarquons qu'un processus de renouvellement est transient si et seulement si le nombre de renouvellement est fini, *i.e.* si $\mathbf{E}[|\tau|] < +\infty$. On a donc un critère :

$$\tau \text{ est transient } \iff \mathbf{E}[|\tau|] = \sum_{n \geqslant 1} \mathbf{P}(n \in \tau) < +\infty.$$

Ainsi, pour une marche aléatoire symétrique $X = \{X_n\}_{n \geqslant 0}$ issue de 0 et à valeurs dans $\mathbb{Z}^d$, on note $\tau_0 := 0$, et ensuite on définit par récurrence, pour $i \in \mathbb{N}$

$$\tau_i := \inf\{n > \tau_{i-1},\, X_n = 0\}. \tag{1.1.8}$$

Le processus $\tau = (\tau_i)_{i \geqslant 0}$ des instants de retour en 0 de la marche aléatoire X est bien un processus de renouvellement (recurrent pour $d = 1, 2$ et transient pour $d \geqslant 3$), et la variable aléatoire τ_1 est le premier temps de retour de X en 0, cf. Figure 1.2.

De plus, il est connu [DK11] que l'on a dans ce cas

$$\mathrm{K}(n) \sim \frac{C_d}{n^{1+\alpha}}, \qquad \text{pour } n \to \infty, \tag{1.1.9}$$

avec $\alpha = 1/2$ pour $d = 1$, $\alpha = 0$ pour $d = 2$ (avec une correction logarithmique), et $\alpha = d/2 - 1$ pour $d \geqslant 3$.

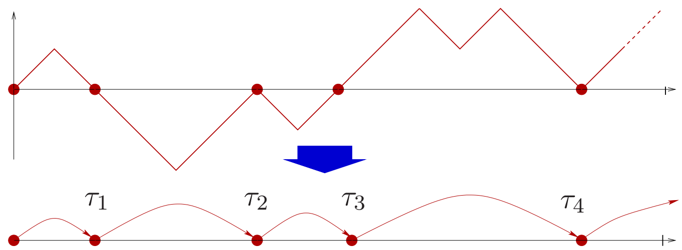

FIGURE 1.2. Processus de renouvellement des temps de retour de la marche aléatoire en 0. Pour la marche simple, les temps de retour sont toujours pairs. On les divise par 2 pour obtenir un renouvellement satisfaisant l'hypothèse $\mathrm{K}(n) > 0$ pour tout $n \in \mathbb{N}$.

Grâce à cette observation, nous supposons dans la suite que la loi inter-arrivées possède une queue à puissance:

Hypothèse 1.1.2 (Hypothèse sur la forme du renouvellement)**.** On suppose que

$$\mathrm{K}(n) = \frac{L(n)}{n^{1+\alpha}}, \qquad \text{pour } n \in \mathbb{N} \tag{1.1.10}$$

où $\alpha \geqslant 0$, et où $L(\cdot)$ est une fonction à variations lentes (voir [BGT87]). Nous nous placerons dans la suite dans le cas où $\alpha > 0$ et où $\lim_{n\to\infty} L(n) =: c_K > 0$, afin d'éviter certains détails techniques (ce qui exclut le cas de la marche simple en dimension $d = 2$).

Avec un léger abus de notations, on note aussi τ l'ensemble $\tau := \{\tau_0, \tau_1, \ldots\}$ des points de renouvellement (ou ensemble des points de contact), et l'on notera $\{n \in \tau\}$ l'événement que n est un point de renouvellement (ou un point de contact). Pour le renouvellement lié à une marche aléatoire X défini ci-dessus, on a donc $\{n \in \tau\} = \{X_n = 0\}$.

La définition de la mesure de polymère se généralise très bien au cadre des renouvellements. Pour $h \in \mathbb{R}$, on définit $\mathbf{P}_{N,h}$ comme une tranformation de Gibbs de la loi $\mathbf{P}$, en donnant une récompense h à chaque point de renouvellement:

$$\frac{\mathrm{d}\mathbf{P}_{N,h}}{\mathrm{d}\mathbf{P}} = \frac{1}{Z_{N,h}} \exp\left(h \sum_{n=1}^{N} \delta_n\right) \delta_N, \tag{1.1.11}$$

où on a noté $\delta_n = \mathbf{1}_{\{n\in\tau\}}$, et où $Z_{N,h} = \mathbf{E}\left[\exp\left(h\sum_{n=1}^N \delta_n\right)\delta_N\right]$ est la fonction de partition du système pur (nous noterons parfois dans la suite $Z_{N,h}^{\mathrm{pur}}$ pour éviter les confusions avec le système désordonné).

Remarque 1.1.3. Pour simplifier les calculs, on peut supposer que le renouvellement τ est récurrent, sans perdre aucune généralité. En effet, dans le cas contraire, il suffit de poser $\widetilde{\mathrm{K}}(n) := (1-\mathrm{K}(\infty))^{-1}\mathrm{K}(n)$, qui définit un processus de renouvellement récurrent de loi $\widetilde{\mathbf{P}}$, et d'étudier la fonction de partition

$$\widetilde{Z}_{N,h} = \widetilde{\mathbf{E}}\left[e^{h\sum_{n=1}^N \delta_n}\delta_N\right] = \mathbf{E}\left[e^{h\sum_{n=1}^N \delta_n}(1-\mathrm{K}(\infty))^{-\sum_{n=1}^N \delta_n}\delta_N\right] = Z_{N,h-\log(1-K(\infty))}. \tag{1.1.12}$$

On peut donc se ramener à un processus de renouvellement récurrent par une simple translation du paramètre h, et nous ne traiterons donc que ce cas par la suite.

L'énergie libre $\mathtt{F}(h) := \lim_{N\to\infty} N^{-1}\log Z_{N,h}$ est bien définie par sur-additivité de $\log Z_{N,h}$ (en utilisant la structure de renouvellement, *i.e.* l'indépendance des excursions). Les propriétés générales de $\mathtt{F}(h)$ (convexité, non-décroissance et non-négativité) ont déjà été données dans la Section 1.1.1, dans le cas particulier du modèle d'accrochage de la marche simple. En particulier, l'inégalité $\mathtt{F}(h) \geqslant 0$ suit simplement du fait que $\lim_{N\to\infty} N^{-1}\log K(N) = 0$.

Ce modèle général est en fait exactement résoluble (l'article fondateur étant [Fis84]), dans le sens où l'on a une formule explicite pour l'énergie libre, qui permet de calculer le point critique h_c.

Proposition 1.1.4 ([Gia07], Prop.1.1). *L'énergie libre* $\mathtt{F}(h)$ *est la solution de l'équation en* b

$$\widehat{\mathrm{K}}(b) := \sum_{n\in\mathbb{N}} \mathrm{K}(n)e^{-bn} = e^{-h}, \tag{1.1.13}$$

si une telle solution existe (i.e. si $\sum_{n\in\mathbb{N}}\mathrm{K}(n) \geqslant e^{-h}$*), et* $\mathtt{F}(h) = 0$ *sinon. En particulier, le point critique est* $h_c = -\log(1-K(\infty))$.

On a aussi, de manière classique, une description de la mesure de polymère [Gia07, Ch.2]. Notons $\widetilde{\mathrm{K}}^h(n) := e^h e^{-bn}\mathrm{K}(n)$ (où b est la solution de (1.1.13), $b=0$ s'il n'y a pas de solution), et $\widetilde{\mathbf{P}}^h$ la loi d'un processus de renouvellement $\widetilde{\tau}^h$ associé à cette loi inter-arrivée (remarquons que dans tous les cas $\sum_{n\in\mathbb{N}}\widetilde{\mathrm{K}}^h(n) \leqslant 1$).

Lemme 1.1.5 ([Gia07], Th.2.3). *On possède la formule suivante pour la fonction de partition:*

$$Z_{N,h} = e^{N\mathtt{F}(h)}\widetilde{\mathbf{P}}^h(N\in\tau). \tag{1.1.14}$$

De plus, pour tout événement A *qui est* $\sigma(\tau\cap[0,N])$*-mesurable, on a*

$$\mathbf{P}_{N,h}(A) = \widetilde{\mathbf{P}}^h\left(A \middle| N\in\widetilde{\tau}^h\right). \tag{1.1.15}$$

La Proposition 1.1.4, et notamment (1.1.13), permet en outre d'avoir le comportement critique de $\mathtt{F}(h)$ près du point critique.

Théorème 1.1.6 ([Gia07], Th.2.1)**.** *Si le processus de renouvellement τ est récurrent, alors $h_c = 0$. De plus, sous l'Hypothèse 1.1.2, on a*

$$\mathtt{F}(h) \overset{h\searrow 0}{\sim} \begin{cases} \left(\frac{\alpha}{\Gamma(1-\alpha)c_K}\right)^{1/\alpha} h^{1/\alpha} & si\ \alpha < 1, \\ \frac{1}{c_K}|\ln h|^{-1} h & si\ \alpha = 1, \\ \frac{1}{m_K} h & si\ \alpha > 1. \end{cases} \tag{1.1.16}$$

La preuve consiste à observer que le comportement asymptotique de $\widehat{\mathrm{K}}(b)$ quand $b \searrow 0$ peut être obtenu, grâce à un Théorème Abélien (voir [BGT87, Th.1.7.1]), à partir du comportement de $\mathrm{K}(n)$ quand $n \to \infty$. Ceci permet d'inverser (1.1.13) quand $h \searrow 0$ et donc d'obtenir (1.1.16). On peut retrouver ce résultat grâce à l'équation alternative (1.1.21), en utilisant le comportement asymptotique de $\mathbf{P}(n \in \tau)$.

L'équation (1.1.13) vérifiée par $\mathtt{F}(h)$ permet, en outre du comportement critique de $\mathtt{F}(\cdot)$, d'obtenir plus d'informations sur l'énergie libre. Par exemple, comme $\widehat{\mathrm{K}}(\cdot)$ est analytique, le Théorème des fonctions implicites (cf. [KK83, Th.8.6]) assure que $\mathtt{F}(\cdot)$ est analytique en tout point $h \neq h_c$.

L'exposant $\nu^{\mathrm{pur}} = 1 \vee 1/\alpha$ caractérise le comportement critique de $\mathtt{F}$, et donc la transition de phase, et est appelé *exposant critique* (ou *ordre de la transition de phase*) du système pur. On ne prendra pas en compte les corrections logarithmiques lorsque l'on parlera de l'exposant critique (par exemple, $\nu^{\mathrm{pur}} = 1$ pour $\alpha = 1$).

Souvent, dans la suite, nous exclurons de la discussion le cas $\alpha = 1$ pour éviter d'alourdir l'énoncé des résultats et leur preuve par certains détails techniques, causés par les corrections logarithmiques dans (1.1.16) et dans (1.1.18) ci-dessous. Cependant, les méthodes présentées ici fonctionnent aussi dans ce cas, qui ne cache rien de profondément différent.

1.1.2.1. *Processus de renouvellement et fonction de renouvellement.* Un outil très important dans l'étude des processus de renouvellement est la *fonction de renouvellement*, $\mathbf{P}(n \in \tau)$. Voici un théorème très important de la théorie du renouvellement (pour une preuve, voir [Asm03, Ch.I, Th.2.2]) :

Théorème 1.1.7 (Théorème de Renouvellement)**.** *Si* $\mathrm{K}(.)$ *est apériodique (i.e. si la loi de τ_1 n'est pas concentrée sur un sous réseau de $\mathbb{N}$), alors*

$$\mathbf{P}(n \in \tau) \xrightarrow{n\to\infty} \frac{1}{m_K} \in [0,1].$$

Ce Théorème permet de comprendre à quoi fait référence le caractère récurrent *positif* ou récurrent *nul* : si $m_K < \infty$, il y a asymptotiquement une probabilité strictement positive pour qu'un site n soit visité par le renouvellement, alors que si $m_K = \infty$ cette probabilité tend vers 0 quand $n \to \infty$. On remarque que, par la loi des grands nombres, on a presque sûrement

$$\lim_{N\to\infty} \frac{1}{N} |\tau \cap \{1, \ldots, N\}| = \frac{1}{m_K},$$

où $|\cdot|$ désigne la cardinalité d'un ensemble. Ceci signifie que si $m_K = \infty$, il y a asymptotiquement une densité nulle de renouvellements.

Remarquons aussi que si $m_K = \infty$, on a très peu d'informations sur le comportement asymptotique de $\mathbf{P}(n \in \tau)$. Sous l'Hypothèse 1.1.2, le comportement asymptotique de la fonction de renouvellement est aussi connu dans le cas $m_K = \infty$, *i.e.* si $K(\infty) > 0$ ou si $\alpha \leqslant 1$.

Proposition 1.1.8. *Sous l'Hypothèse 1.1.2, on a*
• *Si* $\sum_{n\in\mathbb{N}} \mathrm{K}(n) < 1$, *(cas transient),*

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \frac{K(n)}{K(\infty)^2}; \tag{1.1.17}$$

• *Si* $\sum_{n\in\mathbb{N}} \mathrm{K}(n) = 1$ *(cas récurrent), alors si* $\alpha > 1$ *on a* $\mathbf{P}(n \in \tau) \overset{n\to\infty}{\to} 1/m_K$. *De plus, on a*

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \begin{cases} \frac{\alpha \sin(\pi\alpha)}{c_K \pi} n^{\alpha-1} & si\ \alpha \in (0,1) \\ \frac{1}{c_K} (\ln n)^{-1} & si\ \alpha = 1 \end{cases} \tag{1.1.18}$$

Les résultats (1.1.17) et (1.1.18)-($\alpha = 1$) sont détaillés dans [BGT87] (on peut aussi trouver (1.1.17) dans [Gia07, App.A.6]). Le résultat (1.1.18)-($\alpha < 1$) est plus délicat, et la preuve est assez récente, dans [Don97, Th.B]. Cette Proposition permet aussi de relier le comportement asymptotique de $\mathrm{K}(n)$ au nombre moyen de contacts $|\tau \cap [0, N]|$ sous la mesure $\mathbf{P}$.

Le Lemme 1.1.5 permet de connaître précisément la mesure $\mathbf{P}_{N,h}$, qui est exactement $\widetilde{\mathbf{P}}^h(\cdot | N \in \widetilde{\tau}^h)$. La Proposition 1.1.8 permet alors d'accéder au comportement du nombre de contacts (ou de la fraction de contact, cf. (1.1.7)) sous $\mathbf{P}_{N,h}$, pour les différentes valeurs de h.

Proposition 1.1.9 ([Gia07], Th.2.4)**.** *Si* τ *est récurrent (i.e. si* $\sum_{n\in\mathbb{N}} \mathrm{K}(n) = 1$*), on a*

• *Quand* $h < 0$, *pour tout* $k \in \mathbb{N}$ *on a*

$$\lim_{N\to\infty} \mathbf{P}_{N,h} \left(|\tau \cap [0, N]| = k + 1 \right) = (1 - e^h) e^{kh}. \tag{1.1.19}$$

• *Quand* $h = 0$, *et* $\alpha \in (0, 1)$, $N^{-\alpha} |\tau \cap [0, N]|$ *converge en loi vers l'inverse d'une loi* α*-stable sous* $\mathbf{P}(\cdot | N \in \tau) = \mathbf{P}_{N,h}$.

• *Quand* $h > 0$, *(ou* $h = 0$ *et* $\alpha > 1$*)*

$$\frac{1}{N} |\tau \cap [0, N]| \overset{N\to\infty}{\to} \frac{1}{m_{\widetilde{\mathrm{K}}^h}}, \quad \mathbf{P}_{N,h} - p.s., \quad avec\ m_{\widetilde{\mathrm{K}}^h} := e^h \sum_{n\in\mathbb{N}} n e^{-\mathtt{F}(h)n} \mathrm{K}(n) < \infty, \tag{1.1.20}$$

et on remarque qu'avec le calcul fait en (1.1.7), on a $\frac{1}{m_{\widetilde{\mathrm{K}}^h}} = \mathtt{F}'(h)$.

1.1.2.2. *Résolution alternative du modèle homogène.*

Proposition 1.1.10. *L'énergie libre* $\mathtt{F}(h)$ *peut être définie alternativement à* (1.1.13) *comme la solution de l'équation en* b

$$\widehat{\mathbf{P}}(b) := \sum_{n\in\mathbb{N}} e^{-bn}\mathbf{P}(n\in\tau) := \frac{1}{e^h-1} \tag{1.1.21}$$

si une telle solution existe, et $\mathtt{F}(h)=0$ *sinon. On retrouve grâce à* (1.1.21) *que le point critique est* $h_c=-\log(1-\mathrm{K}(\infty))$*, et on peut en déduire le comportement critique de l'énergie libre, au voisinage de* h_c*, i.e. Théorème 1.1.6.*

Démonstration Nous prouvons ici la définition (1.1.21) de b de manière directe, car les idées utilisées seront reprises plus tard, notamment dans la Section 5.3.3.

Posons b la solution de l'équation (1.1.21) si une telle solution existe, et $b=0$ sinon, et montrons que $b=\mathtt{F}(h)$. L'idée consiste à utiliser le développement binomial de $(1+e^h-1)^{\sum\delta_i}$, comme cela est fait dans [BGdH10] ou [BS10]:

$$e^{h\sum_{n=1}^N\delta_n}\delta_N = (1+e^h-1)^{\sum_{n=1}^{N-1}\delta_n}e^h\delta_N = e^h\sum_{m=0}^{N-1}(e^h-1)^m \sum_{0<i_1<\ldots<i_m\leqslant N-1}\delta_{i_1}\ldots\delta_{i_m}\delta_N. \tag{1.1.22}$$

En prenant l'espérance et en utilisant la propriété de renouvellement, on obtient

$$Z_{N,h} = \frac{e^h}{e^h-1}\sum_{m=1}^{N}(e^h-1)^m \sum_{0=i_0<i_1<\ldots<i_m=N}\prod_{k=1}^m \mathbf{P}(i_k-i_{k-1}\in\tau)$$
$$= \frac{e^h}{e^h-1}e^{bN}\widehat{\mathbf{P}}^h(n\in\tau), \tag{1.1.23}$$

où on a défini $\widehat{\mathrm{K}}^h(n\in\tau) := (e^h-1)e^{-bn}\mathbf{P}(n\in\tau)$, qui est la loi inter-arrivée d'un nouveau processus de renouvellement de loi $\widehat{\mathbf{P}}^h$ (qui est récurrent positif si $b>0$). On en déduit de manière classique que $\mathtt{F}(h)=b$, grâce au Théorème de Renouvellement 1.1.7.

Remarquons aussi que (1.1.21) n'a pas de solution si $\sum_{n\in\mathbb{N}}\mathbf{P}(n\in\tau)<(e^h-1)^{-1}$. Cela donne que le point critique est $h_c=0$ si le renouvellement est récurrent, et l'on retrouve aussi que $h_c=-\log(1-\mathrm{K}(\infty))$, car $\sum_{n\in\mathbb{N}}\mathbf{P}(n\in\tau)=1/\mathrm{K}(\infty)$ (le nombre de points de contacts pour un renouvellement transient est une variable géométrique de paramètre $1-\mathrm{K}(\infty)$). □

Nous avons inclus dans cette thèse la démonstration ci-dessus car ce raisonnement sera repris plus tard, mais nous aurions pu montrer plus directement que (1.1.13) est équivalente à (1.1.21), en utilisant l'équation de renouvellement

$$\mathbf{P}(n\in\tau) = \mathbf{1}_{\{n=0\}} + \sum_{n=1}^{N} s1.1.3et\mathrm{K}(n)\mathbf{P}(N-n), \tag{1.1.24}$$

qui donne $\widehat{\mathbf{P}}(\lambda)=1+\widehat{\mathbf{P}}(\lambda)\widehat{\mathrm{K}}(\lambda)$. Pour $b>0$, on obtient ainsi $\widehat{\mathbf{P}}(b)=(\widehat{\mathrm{K}}(b)-1)^{-1}$, avec $\widehat{\mathrm{K}}(b)=e^h$, cf. (1.1.13).

À partir de l'équation (1.1.21), on peut obtenir de nouveau le comportement critique (1.1.16) de l'énergie libre (au voisinage de 0 si $\mathrm{K}(\infty)=0$). En effet, pour

$h > 0$, l'égalité (1.1.21) donne

$$\widehat{\mathbf{P}}\left(\mathtt{F}(h)\right) = (e^h - 1)^{-1} \overset{h\downarrow 0}{\sim} h^{-1}, \tag{1.1.25}$$

et on peut extraire $\mathtt{F}$ à partir de la fonction inverse de la tranformée de Laplace $\widehat{\mathbf{P}}$

$$\mathtt{F}(h) = \widehat{\mathbf{P}}^{-1}\left((e^h - 1)^{-1}\right) \overset{h\searrow 0}{\sim} \widehat{\mathbf{P}}^{-1}\left(h^{-1}\right). \tag{1.1.26}$$

Le comportement de $\mathtt{F}(h)$ au voisinage du point critique depend donc uniquement du comportement asymptotique de $\sum_{n=1}^{N} \mathbf{P}(n \in \tau)$ (qui détermine par le Théoreme Abélien [BGT87, Th.1.7.1] celui de $\widehat{\mathbf{P}}(b)$ quand $b \searrow 0$), c'est-à-dire du nombre moyen de points de contacts au point critique $h = 0$. Souvent, même pour des modèles d'accrochages non homogènes, on peut relier le comportement critique de $\mathtt{F}$ et le nombre moyen de contacts au point critique (voir par exemple le Corollaire 1.2.6). Nous verrons par contre, dans le Chapitre 6, un exemple un peu surprenant où le comportement critique de l'énergie libre n'est nullement prédit par le nombre de contacts au point critique.

Avant de terminer cette section, nous mentionnons une petite application de la Proposition 1.1.10. Si on considère l'intersection de deux processus de renouvellement récurrents indépendants τ et τ' d'exposant α, α' respectivement, alors on obtient de nouveau un processus de renouvellement, que l'on note $\check{\tau}$. Il est difficile de connaître le comportement de $\check{\mathrm{K}}(n) = \check{\mathbf{P}}(\check{\tau}_1 = n)$, mais on sait que $\check{\mathbf{P}}(n \in \check{\tau}) = \mathbf{P}(n \in \tau)\mathbf{P}'(n \in \tau')$, et le comportement de $\mathbf{P}(n \in \tau)$ et $\mathbf{P}'(n \in \tau')$ sont connus précisément grâce à la Proposition 1.1.8. Ainsi on sait que $\check{\tau}$ est récurrent si et seulement si $\alpha \wedge 1 + \alpha' \wedge 1 \geqslant 1$ (auquel cas $\sum_{n\in\mathbb{N}} \check{\mathbf{P}}(n \in \check{\tau}) = +\infty$).

En étudiant la fonction de partition $\check{Z}_{N,h} := \mathbf{E}\left[\exp\left(h \sum_{n=1}^{N} \mathbf{1}_{\{n\in\check{\tau}\}}\right) \mathbf{1}_{\{N\in\tau\}}\right]$, et en adoptant des notations naturelles, on a $\check{h}_c = 0$, et on déduit le comportement critique de l'énergie libre grâce à de celui de la transformée de Laplace de $\check{\mathbf{P}}(n \in \tau)$. En particulier, on obtient que l'exposant critique est $\check{\nu}^{\mathrm{pur}} = 1 \vee (\alpha \wedge 1 + \alpha' \wedge 1 - 1)^{-1}$.

1.1.3. Accrochage d'une marche aléatoire à temps continu. Une généralisation possible du modèle introduit dans la Section 1.1.2, est de prendre une marche aléatoire non pas indexée par $n \in \mathbb{N}$, mais par un réel $t \in \mathbb{R}_+$. Nous considérons cette généralisation, qui n'apporte pas grand chose si on n'étudie que le modèle homogène, parce qu'elle nous sera très utile dans l'étude du modèle d'accrochage sur une marche aléatoire, voir la Section 1.2. On considère donc $(X_t)_{t\in\mathbb{R}_+}$ une marche aléatoire à temps continu de loi $\mathbb{P}^X$, à valeurs dans $\mathbb{Z}^d$, $d \geqslant 1$, partant de 0, et de taux de saut 1. On suppose que la marche est symétrique et irréductible, et que les sauts ont un deuxième moment fini.

De la même manière que dans la Section 1.1.2, pour $\beta \in \mathbb{R}$ et $t \in \mathbb{R}_+$, on définit une mesure de Gibbs sur les trajectoires de longueur t, qui est une transformation de la loi $\mathbb{P}^X$: la mesure de polymère $\mu_{t,\beta}$. Elle est absolument continue par rapport à $\mathbb{P}^X$, et sa dérivée de Radon-Nikodym est

$$\frac{\mathrm{d}\mu_{t,\beta}}{\mathrm{d}\mathbb{P}^X}(X) = \frac{1}{Z_{t,\beta}} e^{\beta L_t(X,0)}\, \mathbf{1}_{\{X_t=0\}}, \tag{1.1.27}$$

où $L_t(X,0) := \int_0^t \mathbf{1}_{\{X_s=0\}}\,\mathrm{d}s$ est le temps local de X en 0, et où

$$Z_{t,\beta} := \mathbb{E}^X\left[e^{\beta L_t(X,0)}\,\mathbf{1}_{\{X_t=0\}}\right] \tag{1.1.28}$$

est la fonction de partition du système. Sous la mesure $\mu_{t,\beta}$ la marche X reçoit une récompense β pour rester en contact avec 0. La contrainte que la marche X soit égale à 0 à son extrémité X_t est présente pour des raisons techniques que nous verrons plus tard, mais cela n'affecte les propriétés de ce système que de manière marginale. Ce modèle est étudié en détails dans la Section 3.A, et nous en donnons ici les principales caractéristiques.

On définit l'énergie libre du sytème homogène

$$\mathtt{F}(\beta) := \lim_{t\to\infty}\frac{1}{t}\log Z_{t,\beta}, \tag{1.1.29}$$

qui existe grâce à la sur-additivité de $\log Z_{t,\beta}$, de la même manière que précédemment (un peu de précautions sont nécessaires pour généraliser au temps continu, on trouvera des détails dans [BS10, Sec.2]). L'énergie libre possède les mêmes propriétés que dans le cas discret, elle est positive ou nulle, croissante et convexe, et l'on obtient ainsi l'existence d'un point critique $\beta_c := \inf\{\beta,\, \mathtt{F}(\beta)>0\}$, qui marque la transition entre une phase délocalisée pour $\beta<\beta_c$ (où $\mathtt{F}(\beta)=0$), et une phase localisée pour $\beta>\beta_c$ (où $\mathtt{F}(\beta)>0$).

Ce modèle est lui aussi exactement résoluble, car on réussit à avoir une formule explicite pour l'énergie libre, cf. Proposition 3.A.3. On notera dans la suite $p_t(\cdot) := \mathbb{P}^X(X_t=\cdot)$ le noyau de transition de X au temps t, et on pose $G := \int_0^\infty p_t(0)\,\mathrm{d}t$ ($G<\infty$ quand $d\geqslant 3$).

Proposition 1.1.11. *Pour $d\geqslant 1$, le point critique est $\beta_c = G^{-1}$ (avec la convention que $G^{-1}=0$ si $G=\infty$, pour $d=1,2$). On a aussi le comportement critique de $\mathtt{F}$ au voisinage de β_c:*

- *pour $d=1,3$,*
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c}{\sim} c_0(\beta-\beta_c)^2. \tag{1.1.30}$$
- *pour $d=2$,*
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c}{=} \exp\left(-c_0\,\frac{1+o(1)}{\beta}\right). \tag{1.1.31}$$
- *pour $d=4$,*
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c}{\sim} c_0(\beta-\beta_c^{\mathrm{a}})/\log(\beta-\beta_c). \tag{1.1.32}$$
- *pour $d\geqslant 5$*
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c}{\sim} c_0(\beta-\beta_c). \tag{1.1.33}$$

(c_0 est une constante explicite, qui ne dépend que de G, de la dimension et du deuxième moment du noyau de transition).

Ce résultat est établi en partie dans [BS10] (en ce qui concerne le point critique), et en partie dans la Section 3.A, et donne ainsi les exposants critiques du système pur, à mettre en parallèle avec le Théorème 1.1.6.

1.2. Modèle d'accrochage sur une marche aléatoire

1.2.1. Introduction du modèle. Dans la premiere partie de la thèse, nous étudions un modèle où la marche aléatoire dirigée $(n, X_n)_{n \geqslant 0}$ (ou $(t, X_t)_{t \geqslant 0}$ dans le cas à temps continu) s'accroche non pas sur une ligne déterministe $\mathbb{N} \times \{0\}$ (ou $\mathbb{R} \times \{0\}$), mais sur une autre marche aléatoire dont la trajectoire est fixée (désordre gelé, ou quenched).

1.2.1.1. *Modèle à temps continu.* Soient $X = (X_s)_{s \geqslant 0}$ et $Y = (Y_s)_{s \geqslant 0}$ deux marches aléatoires à temps continu sur $\mathbb{Z}^d$, $d \geqslant 1$, partant de 0, avec des taux de sauts respectifs 1 et $\rho \geqslant 0$. On suppose de plus que leurs incréments (*i.e.* leurs sauts) ont la même loi, qui est symétrique et possède un deuxième moment fini. La loi des incréments est aussi non dégénérée (leur matrice de covariance est non-singulière) et telle que les marches X et Y sont irréductibles. On appelle $\mathbb{P}^X$, $\mathbb{P}^{Y,\rho}$ les lois respectives de X et de Y.

Pour $\beta \in \mathbb{R}$ (quand β est positif, on peut l'interpréter comme l'inverse de la température), pour $t \in \mathbb{R}_+$, et pour une réalisation fixée de Y (désordre gelé), on définit la mesure de polymère sur le trajectoires de longueur t, $\mu^Y_{t,\beta}$, comme une transformation de Gibbs de la loi $\mathbb{P}^X$, qui dépend de la réalisation de Y. On donne à une trajectoire de X une probabilité proportionelle à $\exp(\beta L_t(X,Y))$, où $L_t(X,Y) := \int_0^t \mathbf{1}_{\{X_s = Y_s\}}\, \mathrm{d}s$ est le temps local d'intersection entre X et Y:

$$\frac{\mathrm{d}\mu^Y_{t,\beta}}{\mathrm{d}\mathbb{P}^X}(X) = \frac{1}{Z^Y_{t,\beta}} e^{\beta L_t(X,Y)}\, \mathbf{1}_{\{X_t = Y_t\}}, \tag{1.2.1}$$

avec la fonction de partition $Z^Y_{t,\beta} := \mathbb{E}^X \left[e^{\beta L_t(X,Y)}\, \mathbf{1}_{\{X_t = Y_t\}} \right]$.

Le paramètre ρ, en exposant dans la loi $\mathbb{P}^{Y,\rho}$ de Y, que l'on peut faire varier, représente l'intensité du désordre: plus ρ est grand, plus la marche Y possède des variations importantes, alors que si $\rho = 0$, Y est réduit à la ligne déterministe $\mathbb{R} \times \{0\}$.

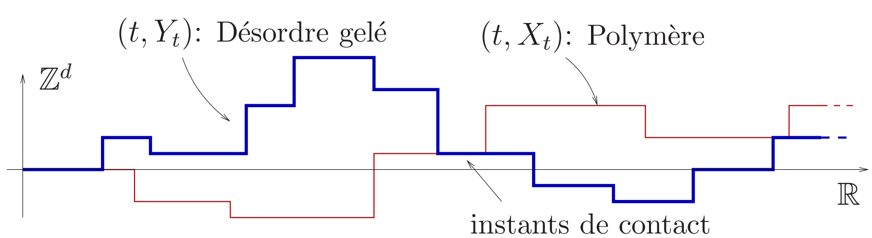

FIGURE 1.3. Accrochage de la marche dirigée (t, X_t) à temps continu sur le défaut aléatoire, ou désordre gelé, qui est la marche dirigée (t, Y_t). Les marches X et Y sont de taux de saut respectifs 1 et ρ.

Pour une trajectoire de $Y = (Y_s)_{s \geqslant 0}$ fixée, on définit aussi la fonction de partition le long d'un intervalle de temps $[t_1, t_2]$, par

$$Z^Y_{[t_1,t_2],\beta} := Z^{\theta_{t_1} Y}_{t_2 - t_1, \beta}, \tag{1.2.2}$$

où $\theta_t Y := (Y_{s+t} - Y_t)_{s \geqslant 0}$ (θ_t est l'opérateur de translation le long du temps, qui préserve la loi de Y).

Remarque 1.2.1 (Sur-additivité)**.** Une des propriétés fondamentales de la fonction de partition *accrochée* ou *pinned*, (et qui est la raison pour laquelle on a imposé la contrainte $\mathbf{1}_{\{X_t=Y_t\}}$), est la sur-additivité stochastique de $\log Z_{t,\beta}^Y$. En effet, pour $0 \leqslant s \leqslant t$ et $\beta \in \mathbb{R}$,

$$Z_{t,\beta}^Y \geqslant \mathbb{E}^X \left[\mathbf{1}_{\{X_s=Y_s\}} e^{\beta L_t(X,Y)} \mathbf{1}_{\{X_t=Y_t\}} \right] = Z_{s,\beta}^Y Z_{[s,t],\beta}^Y. \tag{1.2.3}$$

Cette remarque s'applique pour la fonction de partition sur un intervalle

$$Z_{[u,w],\beta}^Y \geqslant Z_{[u,v],\beta}^Y Z_{[v,w],\beta}^Y, \qquad \text{pour tout } u \leqslant v \leqslant w. \tag{1.2.4}$$

Cette propriété est cruciale, et permet avec un peu de travail (voir [BS10] pour plus de détails) de démontrer la Proposition 1.2.2 suivante, *i.e.* l'existence de l'énergie libre (appelée aussi dans ce contexte exposant de Lyapunov).

Proposition 1.2.2 (voir [BS10] Théorème 1.1 et Corollaire 1.3)**.** *La limite*

$$\mathtt{F}(\beta,\rho) := \lim_{t\to\infty} \frac{1}{t} \log Z_{t,\beta}^Y \geqslant 0 \tag{1.2.5}$$

existe et est constante $\mathbb{P}^{Y,\rho}$*-p.s., elle est appelée l'énergie libre avec* désordre gelé *ou énergie libre* quenched*. La fonction* $\beta \mapsto \mathtt{F}(\beta,\rho)$ *est convexe croissante, et il existe un point critique* quenched $\beta_c^{\mathrm{que}}(\rho) := \inf\{\beta,\, \mathtt{F}(\beta,\rho) > 0\}$ *tel que*

$$\mathtt{F}(\beta,\rho) > 0 \Leftrightarrow \beta > \beta_c^{\mathrm{que}}(\rho).$$

L'énergie libre du système moyenné (ou système recuit*, ou encore* annealed*) est appelée énergie libre* annealed *et est définie par*

$$\mathtt{F}^{\mathrm{a}}(\beta,\rho) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{Y,\rho} \left[Z_{t,\beta}^Y \right]. \tag{1.2.6}$$

On a aussi l'existence d'un point critique annealed $\beta_c^{\mathrm{a}}(\rho) := \inf\{\beta,\, \mathtt{F}^{\mathrm{a}}(\beta,\rho) > 0\}$. *Grâce à l'inégalité de Jensen, on obtient* $\mathtt{F}(\beta,\rho) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,\rho)$*, de sorte que l'on ait* $\beta_c^{\mathrm{que}}(\rho) \geqslant \beta_c^{\mathrm{a}}(\rho)$.

S'il n'y a pas d'ambiguité, on notera β_c^{que} (resp. β_c^{a}) à la place de $\beta_c^{\mathrm{que}}(\rho)$ (resp. $\beta_c^{\mathrm{a}}(\rho)$).

Comme dans le cas homogène, le point critique $\beta_c^{\mathrm{que}}(\rho)$ marque bien la transition entre le régime délocalisé et le régime localisé. La quantité $\partial_\beta \mathtt{F}(\beta,\rho)$ est aussi dans ce cas la fraction de contact asymptotique entre X et Y sous $\mu_{\beta,t}^Y$, qui est nulle dans la phase délocalisée, pour $\beta < \beta_c(\rho)$, et qui est strictement positive dans la phase localisée, pour $\beta > \beta_c$ (cf. Figure 1.1 dans le cas homogène).

Remarque 1.2.3. Comme il est remarqué dans [BS10], le modèle *recuit* (ou *annealed*) est simplement le modèle homogène défini dans la Section 1.1.3, avec la fonction de partition

$$\mathbb{E}^Y[Z_{t,\beta}^Y] = \mathbb{E}^{X-Y} \left[\exp\left(\beta L_t(X-Y,0)\right) \mathbf{1}_{\{(X-Y)_t=0\}} \right], \tag{1.2.7}$$

qui décrit la marche $X-Y$, de taux de saut $1+\rho$, recevant une récompense β pour rester en contact avec 0. Par une renormalisation en t afin que la marche aléatoire $X-Y$ ait un taux de saut de 1, on obtient que

$$\mathtt{F}^{\mathrm{a}}(\beta,\rho)=(1+\rho)\mathtt{F}\left(\beta/(1+\rho),0\right), \tag{1.2.8}$$

et l'on notera $\mathtt{F}(\beta)$ au lieu de $\mathtt{F}(\beta,0)$, de manière cohérente avec la Section 1.1.3.

Grâce à la Proposition 1.1.11, on connaît le point critique et comportement critique du modèle annealed. En particulier, si $d\geqslant 3$ on a

$$\beta_c^{\mathrm{a}}=\frac{1+\rho}{G}>0.$$

1.2.1.2. *Modèle à temps discret.* On peut définir l'analogue du modèle d'accrochage sur une marche aléatoire dans le cas discret, en prenant $X=\{X_n\}_{n\geqslant 0}$ et $Y=\{Y_n\}_{n\geqslant 0}$ deux marches aléatoires à temps discret sur $\mathbb{Z}^d$, $d\geqslant 1$, partant de 0. Notons $\mathbb{P}^X$ et $\mathbb{P}^Y$ leurs lois respectives. On suppose que la marche aléatoire X est apériodique, et que ses incréments $(X_i-X_{i-1})_{i\geqslant 1}$ sont *i.i.d.*, symétriques et ont un deuxième moment fini ($\mathbb{E}^X\left[\|X_1\|^2\right]<\infty$, où $\|\cdot\|$ représente la norme euclidienne sur $\mathbb{Z}^d$). De plus, la matrice de covariance de X_1, que l'on appelle Σ_X, est non-singulière. On suppose que les mêmes hypothèses sont aussi vérifiées par les incréments de Y (et on appelle Σ_Y la matrice de covariance de Y_1).

Pour $\beta\in\mathbb{R}$, $N\in\mathbb{N}$ et pour une réalisation fixée de Y, on définit de la même manière que précédemment une transformation de Gibbs de $\mathbb{P}^X$: la mesure de polymère $\mathbb{P}^Y_{N,\beta}$, qui donne à une trajectoire de X une probabilité proportionelle à $\exp(\beta L_N(X,Y))$, où $L_N(X,Y):=\sum_{n=1}^N \mathbf{1}_{\{X_n=Y_n\}}$ est le temps local d'intersection entre X et Y. On définit

$$\frac{\mathrm{d}\mathbb{P}^Y_{N,\beta}}{\mathrm{d}\mathbb{P}^X}:=\frac{1}{Z^Y_{N,\beta}}e^{\beta L_N(X,Y)}\mathbf{1}_{\{X_N=Y_N\}}, \tag{1.2.9}$$

avec la fonction de partition $Z^Y_{N,\beta}=\mathbb{E}^X[e^{\beta L_N(X,Y)}\ \mathbf{1}_{\{X_N=Y_N\}}$. On a l'existence de l'énergie libre *quenched*

$$\mathtt{F}(\beta,\mathbb{P}^Y):=\lim_{N\to\infty}\frac{1}{N}\log Z^Y_{N,\beta}$$

(la preuve est donnée dans [BS10]), et l'énergie libre *annealed* est

$$\mathtt{F}^{\mathrm{a}}(\beta,\mathbb{P}^Y):=\lim_{N\to\infty}\frac{1}{N}\log \mathbb{E}^Y[Z^Y_{N,\beta}]. \tag{1.2.10}$$

Comme pour le modèle à temps continu, on définit les point critiques *quenched* et *annealed*, β_c^{que} et β_c^{a}, comme les valeurs où les énergies libres respectives deviennent non-nulles, et grâce à l'inégalité de Jensen on a $\mathtt{F}(\beta,\mathbb{P}^Y)\leqslant \mathtt{F}^{\mathrm{a}}(\beta,\mathbb{P}^Y)$, ce qui donne $\beta_c^{\mathrm{a}}\leqslant\beta_c^{\mathrm{que}}$.

Les propriétés de $\mathtt{F}^{\mathrm{a}}(\cdot,\mathbb{P}^Y)$ sont bien connues. En effet, la Remarque 1.2.3 s'applique aussi dans le cadre discret, et l'on obtient donc que le système annealed est en fait le modèle homogène d'accrochage de la marche $X-Y$ sur $\mathbb{N}\times\{0\}$, introduit dans la Section 1.1.1. On connaît donc le point critique annealed, et le comportement critique annealed, qui est similaire à celui de la Proposition 1.1.11.

Dans la Partie 1, nous étudierons les propriétés de la transition de phase du système désordonné lorsque β varie. Nous estimerons notamment le point critique, et aussi le comportement de l'énergie libre du système quand β est proche du point critique, ou bien quand β est grand (*i.e.* à basse température).

Il est à noter que le modèle d'accrochage sur une marche aléatoire est lié par exemple à une version du modèle parabolique d'Anderson avec un unique catalyseur mouvant [BS10], au modèle de copolymère ou au modèle d'accrochage non homogène (cf. Section 1.4). On trouvera de plus amples précisions, ainsi que d'autres motivations physiques dans l'introduction de [BS10].

1.2.2. Propriétés critiques. Il est maintenant naturel de se demander quelle influence possède le désordre sur le modèle d'accrochage sur une marche aléatoire. La question que l'on se pose est donc de savoir si le fait de s'accrocher sur une marche aléatoire Y, plutôt que sur la ligne $\mathbb{R} \times \{0\}$ comme dans la Section 1.1, modifie le comportement du système. On dit que le désordre est *pertinent* si les propriétés critiques du système désordonné sont modifiées par rapport au modèle homogène: si l'exposant critique *quenched* de l'énergie libre (s'il existe) est différent de celui du système pur, ou si l'inégalité $\beta_c^{\mathrm{a}} \leqslant \beta_c^{\mathrm{que}}$ est stricte.

Nous avons discuté dans la Section 0.2.2 du *critère de Harris* dans le cas d'un environnement *i.i.d.*, qui permet de décider du caractère pertinent ou non du désordre, de manière heuristique. Ici, l'environnement n'est pas *i.i.d.*, car la position de la marche Y à l'instant n dépend crucialement (et uniquement) de sa position à l'instant $n-1$: on dit que le désordre est Markovien.

Un méthode consiste à calculer la variance de $Z_{N,\beta}^Y$ (ou $Z_{t,\beta}^Y$) au point critique annealed $\beta = \beta_c$: l'idée de Harris [Har74] rappelée dans l'Introduction est légèrement différente, mais consiste de la même manière à savoir si $Z_{N,\beta}^Y$ est concentré autour de sa moyenne. Si cette variance reste bornée, alors c'est un bon indicateur que l'énergie libre quenched possède le même point critique que l'énergie libre annealed (et aussi le même comportement critique), et le désordre serait non-pertinent. Si la variance de $Z_{N,\beta=\beta_c}^Y$ diverge, on peut penser que le désordre est pertinent.

Dans le cas des dimensions $d=1$ et $d=2$, le point critique annealed est $\beta_c^{\mathrm{a}} = 0$. Il est donc immédiat que $Z_{N,\beta=0}^Y = 1$ (donc la variance est nulle), et le désordre devrait donc être non-pertinent. Il est beaucoup plus difficile de calculer le comportement de $\mathbb{V}\mathrm{ar}^Y(Z_{N,\beta=\beta_c^{\mathrm{a}}}^Y)$ au point critique annealed dans le cas des dimensions $d \geqslant 3$, en particulier à cause du caractère non *i.i.d.* du désordre. Nous détaillerons dans la Section 2.1.3 un calcul heuristique permettant de voir que, en dimension $d \geqslant 3$, la variance au point critique annealed $\mathbb{V}\mathrm{ar}^Y(Z_{N,\beta=\beta_c^{\mathrm{a}}}^Y)$ diverge. Il est donc présagé que dans le cas des dimensions $d \geqslant 3$, on ait $\beta_c^{\mathrm{que}}(\beta) > \beta_c^{\mathrm{a}}$, et qu'ainsi le désordre soit pertinent.

Mentionnons ici que dans la Section 1.4.2, nous utilisons une méthode (détaillée) qui est légèrement différente, basée elle aussi sur les mêmes idées du critère de Harris, afin d'obtenir un critère pour la pertinence/non-pertinence du désordre, cette fois dans le cas *i.i.d.*

Nous donnons ici un résumé des différents résultats obtenus pour ce modèle, qui confirment ces prédictions aussi bien pour le cas discret que pour le cas continu.

Considérons dans un premier temps l'écart entre les points critiques quenched et annealed:

Théorème 1.2.4 (Écart entre les points critiques)**.** *En dimension $d=1$ et $d=2$, les points critiques quenched et annealed sont tous les deux égaux à 0, $\beta_c^{\mathrm{que}} = \beta_c^{\mathrm{a}} = 0$, aussi bien dans le cas discret que dans le cas continu, pour tout $\rho > 0$.*

En dimension $d \geqslant 3$, on a $\beta_c^{\mathrm{que}} > \beta_c^{\mathrm{a}} > 0$ dans le cas discret et dans le cas continu, pour tout $\rho > 0$. De plus, dans le cas continu, on sait contrôler l'écart entre les points critiques:

- *Si $d \geqslant 5$, il existe $a > 0$, telle que $\beta_c^{\mathrm{que}} - \beta_c^{\mathrm{a}} \geqslant a\,\rho$, pour tout $\rho \in [0,1]$.*
- *Si $d = 4$, pour tout $\delta > 0$, il existe $a_\delta > 0$, telle que $\beta_c^{\mathrm{que}} - \beta_c^{\mathrm{a}} \geqslant a_\delta\,\rho^{1+\delta}$, pour tout $\rho \in [0,1]$.*
- *Si $d = 3$, pour tout $\varsigma > 2$, il existe $a_\varsigma > 0$ telle que $\beta_c^{\mathrm{que}} - \beta_c^{\mathrm{a}} \geqslant \exp(-a_\varsigma \rho^{-\varsigma})$, pour tout $\rho \in (0,1]$.*

On pense que le cas de la dimension $d=3$ est bien un cas marginal, et que l'écart entre les points critiques est d'ordre $\exp(-c\rho^{-1})$.

Les différentes parties de ce Théorème sont prouvées dans différents articles: le cas des dimensions $d=1,2$ et $d \geqslant 4$ est traité par Birkner et Sun dans [BS10] (cas discret et cas continu), tandis que le cas de la dimension $d=3$ a eté résolu par [BT10] (cas discret), et de manière indépendente par [BS11] (cas continu). Il est ainsi montré, qu'en ce qui concerne l'écart entre les points critiques, le désordre est pertinent si et seulement si $d \geqslant 3$.

La méthode utilisée pour montrer $\beta_c^{\mathrm{que}} > \beta_c^{\mathrm{a}}$ pour $d \geqslant 3$ est similaire dans les trois articles cités ci-dessus. Nous en donnons ici une idée rapide: elle consiste à estimer le moment fractionnaire $\mathbb{E}^Y\left[(Z_{N,\beta}^Y)^\gamma\right]$ pour un certain $\gamma < 1$. Si pour un certain $\beta_0 > \beta_c^{\mathrm{a}}$, la quantité $\frac{1}{N}\log \mathbb{E}^Y\left[(Z_{N,\beta_0}^Y)^\gamma\right]$ converge vers 0, alors une application facile de l'inégalité de Jensen montre que $\mathtt{F}(\beta_0, \mathbb{P}^Y) \leqslant 0$ (et est donc égal à 0 à cause de (1.2.5)), ce qui donne $\beta_c^{\mathrm{que}} \geqslant \beta_0 > \beta_c^{\mathrm{a}}$. Pour estimer le moment fractionnaire, on introduit une autre mesure $\widetilde{\mathbb{P}}^Y$ pour le désordre, absolument continue par rapport à $\mathbb{P}^Y$, et l'inégalité de Hölder donne alors

$$\mathbb{E}^Y\left[(Z_{N,\beta_0}^Y)^\gamma\right] \leqslant \widetilde{\mathbb{E}}^Y[Z_{N,\beta_0}^Y]^\gamma \mathbb{E}^Y\left[\left(\frac{\mathrm{d}\widetilde{\mathbb{P}}^Y}{\mathrm{d}\mathbb{P}^Y}\right)^{-\frac{\gamma}{1-\gamma}}\right]^{1-\gamma}. \tag{1.2.11}$$

Le choix de la mesure $\widetilde{\mathbb{P}}^Y$ est crucial: le désordre sous $\widetilde{\mathbb{P}}^Y$ doit être moins favorable à la localisation que sous $\mathbb{P}^Y$, afin que le terme $\widetilde{\mathbb{E}}^Y[Z_{N,\beta_0}^Y]$ soit petit ($\ll \mathbb{E}^Y[Z_{N,\beta_0}^Y]$), mais suffisamment proche de la mesure $\mathbb{P}^Y$ pour que le terme de (1.2.11) qui dépend de la densité relative de $\widetilde{\mathbb{P}}^Y$ par rapport à $\mathbb{P}^Y$ ne soit pas trop grand.

Le dernier ingrédient est alors une procédure dite de *coarse-graining*, qui permet de ramener l'étude du système lorsque $N \to \infty$ à des estimées sur des systèmes de taille finie. Cette technique consiste ainsi à découper le système en blocs de taille $N = N(\beta_0 - \beta_c^{\mathrm{a}})$, et d'effectuer des estimées sur chacun des blocs séparément. Il suffit alors de montrer que $\mathbb{E}^Y\left[(Z_{N,\beta_0}^Y)^\gamma\right]$ est très petit pour N de l'ordre de la longueur de corrélation du système annealed (qui s'avère être de l'ordre de $\mathtt{F}(\beta_0 - \beta_c^{\mathrm{a}})^{-1}$, voir

[Gia08]), la procédure de *coarse-graining* permettant de “recoller” les estimées sur les différents blocs.

Cette methode a été developpée dans [DGLT09, GLT10b, GLT11] pour le modèle d'accrochage sur une ligne de défauts, et l'application au cas du modèle d'accrochage sur une marche aléatoire demande une extension non triviale. Nous verrons les détails de cette méthode dans le Chapitre 2, qui est basé sur l'article [BT10] écrit en collaboration avec F. Toninelli, et qui montre que $\beta_c^{\mathrm{que}} > \beta_c^{\mathrm{a}}$ en dimension $d = 3$, dans le cadre discret.

Comme nous l'avons mentionné dans l'Introduction, une deuxième façon de voir l'influence du désordre est d'étudier le changement de comportement pour l'énergie libre, lorsque β est proche du point critique β_c^{que}, par rapport au comportement critique du système pur. C'est ce à quoi est consacré le Chapitre 3, basé sur l'article [BL11] écrit en collaboration avec H. Lacoin, et qui se concentre sur le cadre continu. On peut effectivement, dans ce cadre, jouer sur le paramètre ρ, et faire ainsi varier de façon continue l'intensité du désordre. Nous avons notamment montré que le désordre provoque un phénomème de lissage de la transition de phase en dimension $d \geqslant 3$, c'est à dire que la transition de phase est toujours d'ordre au moins 2 en présence de désordre.

Théorème 1.2.5. *Pour tout $d \geqslant 3$, $\rho > 0$, $\beta > 0$, on a*

$$\mathtt{F}(\beta,\rho) \leqslant d\frac{G^2}{4\rho}\left(\beta - \beta_c^{\mathrm{que}}(\rho)\right)_+^2 . \tag{1.2.12}$$

La preuve de ce résultat, basée sur des idées de grandes déviations pour le désordre, est similaire à celle utilisée dans [AW90, GT06], pour montrer que la transition de phase du modèle d'accrochage (hiérarchique ou non) sur une ligne de défauts en environnement *i.i.d.* est au moins du deuxième ordre. Cependant, la nature du désordre est ici différente, ce qui oblige à développer de nouvelles idées.

Ce résultat confirme que le désordre est pertinent en dimension $d \geqslant 4$, car le système pur possède alors une transition de phase d'ordre 1 (voir Proposition 1.1.11). Il ne permet cependant pas de conclure dans le cas $d = 3$ qui est le cas marginal (d'après le critère de Harris, cf. Section 0.2.2, car on a $\nu^{\mathrm{pur}} = 2$), et où le désordre est pertinent, du moins en ce qui concerne les points critiques. Pour $d = 1, 2$, où les points critiques quenched et annealed coïncident, nous montrons que les exposants critiques quenched et annealed coïncident aussi. Dans le cas de la dimension $d = 2$, on sait déjà que $\beta_c^{\mathrm{que}} = 0$, et l'inégalité de Jensen donne $\mathtt{F}(\beta,\rho) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,\rho) = (1+\rho)\mathtt{F}(\beta/(1+\rho))$, et $\mathtt{F}$ décroît donc exponentiellement, comme c'est le cas pour le système pur: le désordre est alors non-pertinent. Dans cas $d = 1$, nous verrons dans la Proposition 3.1.3 et la Remarque 3.1.8 que le modèle quenched a le même exposant critique $\nu^{\mathrm{que}} = 2$ que le modèle annealed.

On déduit aussi du Théorème 1.2.5 des considérations sur le nombre de contacts au point critique, de la même manière que dans [Lac10]

Corollaire 1.2.6. *Pour $d \geqslant 3$, $\rho > 0$ et $\varepsilon > 0$ fixé, on a, sous $\mathbb{P}^{Y,\rho}$,*

$$\lim_{t\to\infty} \mu^Y_{t,\beta_c^{\mathrm{que}}(\rho)}\left(L_t(X,Y) \geqslant t^{1/2+\varepsilon}\right) = 0, \tag{1.2.13}$$

en probabilité.

Ce Corollaire contraste avec ce qui se passe dans le modèle pur au point critique β_c, où on a typiquement $L_t(X,0) \asymp t$ si $d \geqslant 5$ (voir Proposition 1.1.9 et Corollaire 3.A.5), et où pour $d=4$, on a que $L_t(X,0)/\log t$ converge en loi vers une variable exponentielle, cf. [ET60].

1.3. Modèle d'accrochage hiérarchique

Comme nous l'avons évoqué dans la Section 0.2, les versions hiérarchiques des systèmes désordonnés permettent de conjecturer de manière fiable le comportement des modèles non hiérarchiques associés. C'est pour cela que je me suis longuement penché sur le modèle d'accrochage (sur un réseau) hiérarchique durant ma thèse, dont les principales contributions ont donné lieu à un article [BT11] (avec F. Toninelli), et sont reprises en détail dans le Chapitre 4.

1.3.1. Introduction du réseau hiérarchique. En mécanique statistique, les modèles hiérarchiques sur des réseaux en diamant, qu'ils soient homogènes ou désordonnés, sont des outils importants pour l'étude des propriétés critiques, cf. [Ble89, CEGM84, DG84]. Les réseaux en diamants sont en effets construits de manière à être auto-similaires, ce qui permet un étude itérative par renormalisations successives, les renormalisations étant exactes grâce aux symétries du système. Cela signifie que l'on peut écrire de façon simple et sans aucune approximation la fonction de partition d'un système de taille N en termes de celle d'un système de taille $N/2$.

1.3.1.1. *La mesure de polymère.* La version hiérarchique du modèle d'accrochage a été introduite par B. Derrida, V. Hakim et J. Vannimenus [DHV92], et reprise dans [GLT10a], et correspond au modèle d'accrochage sur une famille croissante de réseaux, construits de la manière suivante. Ayant fixé un entier $B \geqslant 2$, on considère D_0 le graphe constitué d'une arête simple entre deux points A_0 et A_1, puis l'on construit la suite de graphes $(D_n)_{n\geqslant 0}$ de manière itérative, pour tout $n \geqslant 0$: D_{n+1} est obtenu à partir de D_n, en remplaçant chaque arête par B copies en parallèle de deux arêtes en série (cf. Figure 1.4).

Sur le réseau D_n, tous les chemins de A_0 à A_1 sont équivalents. En l'absence d'interactions, le polymère est modélisé par un chemin dirigé allant de A_0 à A_1 dans D_n (chemin qui est donc de longueur 2^n), et l'on munit l'ensemble des chemins dirigés de D_n de la mesure uniforme, que l'on note $\mathbf{P}_n$. On note ensuite Γ_0 le chemin le plus à droite, qui jouera le rôle de la ligne de défauts. On introduit ensuite un modèle d'accrochage non-homogène, dans le sens où les interactions entre le polymère et la ligne de défauts Γ_0 n'est pas constante égale à h le long de celle-ci, mais dépend de manière aléatoire de la position sur la ligne.

On considère $\omega := (\omega_i)_{i\in\mathbb{N}}$ une suite de variable aléatoire centrées, de variance unitaire, de loi notée $\mathbb{P}$ (nous donnerons par la suite plus d'hypothèses concernant la loi $\mathbb{P}$), que l'on interprète comme étant le désordre sur les arêtes de Γ_0.

On définit $\omega_{I_{k,p}} := (\omega_{(k-1)2^p+1}, \dots, \omega_{k2^p})$ (avec $I_{k,p} := \{(k-1)2^p+1, \dots, k2^p\}$) et on fait l'hypothèse que pour tout $p \geqslant 0$, les variables $(\omega_{I_{k,p}})_{k\in\mathbb{N}}$ ont la même loi. Cette hypothèse est naturelle car les $\omega_{I_{k,p}}$ sont les restrictions de ω à des blocs du

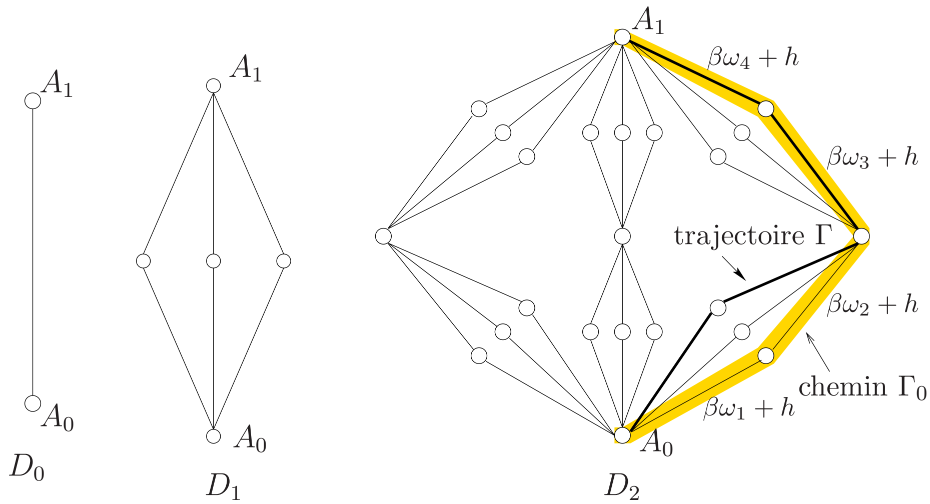

FIGURE 1.4. Représentation des trois premières itérations de la construction des graphes hiérarchiques, pour $B = 3$. Sur le troisième graphe, on a représenté le chemin particulier Γ_0 en jaune, qui est la ligne de défauts, et une autre trajectoire Γ par une ligne épaisse. Le hamiltonien de la trajectoire Γ représentée ci-dessus est (au signe près) $H = \beta(\omega_3 + \omega_4) + 2h$

réseau hiérarchique, et cela permet d'avoir une invariance par changement d'échelle, lorsque l'on passe de D_n à D_{n+1}. Nous ferons par la suite des hypothèses plus précises sur l'environnement ω, et nous considérerons en particulier le cas où ω est une séquence gaussienne, dont les corrélations respectent la structure hiérarchique. Pour $\beta \geqslant 0$ (l'inverse de la température), $h \in \mathbb{R}$ (le paramètre d'accrochage) et e une arête de Γ_0 (on notera $e \in \Gamma_0$, que l'on peut considérer comme étant un entier: sa position dans le chemin Γ_0), la récompense associée à $e \in \Gamma_0$ est $\beta\omega_e + h$. Le polymère récupère cette récompense si l'arête $e \in \Gamma_0$ est contenue dans la trajectoire de Γ. La récompense totale d'une trajectoire Γ du polymère dans D_n est alors la somme des récompenses collectées sur Γ_0 le long de la trajectoire, voir la Figure 1.4. Autrement dit, le hamiltonien de la trajectoire Γ_n est $H(\Gamma) = \sum_{e\in\Gamma}(\beta\omega_e + h)\mathbf{1}_{\{e\in\Gamma_0\}}$ (il faudrait ajouter un signe "$-$" devant le hamiltonien pour être cohérent avec les notations physiques, mais on s'en affranchit ici).

On peut alors définir la mesure de polymère sur les chemins dirigés de D_n, comme une transformation de Gibbs de la loi uniforme $\mathbf{P}_n$:

$$\frac{\mathrm{d}\mathbf{P}^{\omega}_{n,h}}{\mathrm{d}\mathbf{P}_n}(\Gamma) := \frac{1}{Z^{\omega}_{n,h}} \exp\left(\sum_{k=1}^{2^n} (\beta\omega_k + h)\mathbf{1}_{\{\gamma_k\in\Gamma_0\}}\right), \tag{1.3.1}$$

où on a noté γ_k la k^{e} arête de Γ, et où

$$Z^{\omega}_{n,h} := \mathbf{E}_n\left[\exp\left(\sum_{k=1}^{2^n} (\beta\omega_k + h)\mathbf{1}_{\{\gamma_k\in\Gamma_0\}}\right)\right] \tag{1.3.2}$$

est la fonction de partition du système. On note que la mesure de polymère est elle-même une variable aléatoire, qui dépend de la réalisation de ω, de $\beta \geqslant 0$ et de $h \in \mathbb{R}$.

Notons, pour $i \geqslant 1$,

$$Z_n^{(i)} = Z_{n,h}^{\omega,(i)} := \mathbf{E}_n \left[\exp\left(\sum_{k=1}^{2^n} (\beta \omega_{2^n(i-1)+k} + h)\delta_k \right) \right], \tag{1.3.3}$$

de telle façon que $Z_{n,h}^{\omega} := Z_n^{(1)}$. Grâce à la structure hiérarchique du réseau D_n, on vérifie aisément la relation de récurrence suivante:

$$Z_{n+1}^{(i)} = \frac{Z_n^{(2i-1)} Z_n^{(2i)} + B - 1}{B}, \tag{1.3.4}$$

pour $n \in \mathbb{N} \cup \{0\}$ et $i \in \mathbb{N}$, avec $Z_0^{(i)} = e^{\beta\omega_i + h}$.

La relation de récurrence (1.3.4) peut en fait être définie pour n'importe quel $B \neq 0$. Pour $B > 1$, $Z_n^{(i)}$ est positive (ce qui est necessaire pour pouvoir l'interpréter comme une fonction de partition), et nous utiliserons cette relation de récurrence comme définition pour la fonction de partition $Z_{n,h}^{\omega} := Z_n^{(1)}$. La Remarque 2.1 dans [GLT10a] assure que l'on peut toujours se restreindre au cas $B \in (1,2)$: si $B > 2$ on considère $Z_{n,h}^{\omega}/(B-1)$, qui satisfait la même récursion, avec B remplacé par $B/(B-1) \in (1,2)$. Nous ne traiterons donc dans la suite que le cas $B \in (1,2)$.

1.3.1.2. *Interprétation en terme d'arbres de Galton-Watson.* On peut aussi considérer la construction du graphe D_n de manière différente, en observant que D_n est constitué de $2B$ copies de D_{n-1}, chacune étant elle-même constituée de $2B$ copies de D_{n-2}... (cf. Figure 1.4). Le réseau D_n possède ainsi, de part sa construction itérative, une structure de branchement, ce qui permet d'introduire le parallèle suivant avec les arbres de Galton-Watson.

On prend $1 < B < 2$, et on définit $\mathbf{P}_n$ comme étant la loi d'un arbre de Galton-Watson $\mathcal{T}_n$ de profondeur $n+1$, où la loi du nombre d'enfants est concentrée en 0 avec probabilité $\frac{B-1}{B}$, et en 2 avec probabilité $\frac{1}{B}$. Le nombre moyen d'enfants est donc $2/B > 1$, et le processus de Galton-Watson est sur-critique. Ce processus de Galton-Watson nous donne donc un arbre aléatoire, avec un ensemble aléatoire de descendants, et l'on définit $\mathcal{R}_n \subset \{1, \ldots, 2^n\}$ l'ensemble des individus présents à la n^{e} génération (qui sont les feuilles de $\mathcal{T}_n$).

On peut définir pour $p \in \mathbb{N} \cup \{0\}$ et $k \in \mathbb{N}$,

$$I_{k,p} := \{(k-1)2^p + 1, \ldots, k2^p\} \tag{1.3.5}$$

le k^{e} bloc de taille 2^p. On définit la distance hiérarchique $d(\cdot,\cdot)$ sur $\mathbb{N}$ en posant $d(i,j) = p$ si i,j sont contenus dans le même bloc de taille 2^p mais pas dans le même bloc de taille 2^{p-1}. En d'autres termes, $d(i,j)$ est simplement la moitié de la distance d'arbre entre i et j, si $\mathbb{N}$ est considéré comme l'ensemble des feuilles d'un arbre binaire infini.

On mentionne ici la Proposition 4.2.1, qui permet de contrôler la mesure $\mathbf{P}_n$, et de donner une formule simple pour $\mathbf{E}_n \left[\prod_{i \in I} \delta_i \right]$, où $\delta_i = 1$ si l'individu i est présent à la génération n (*i.e.* si $i \in \mathcal{R}_n$), et $\delta_i = 0$ sinon. En particulier, on a $\mathbf{E}_n[\delta_i] = B^{-n}$ pour tout $i \in \{1, \ldots, 2^n\}$.

En utilisant la structure récursive de l'arbre de Galton-Watson $\mathcal{T}_n$, on peut vérifier facilement que la définition (1.3.4) de $Z_n^{(i)}$ est bien équivalente à la formule (1.3.3) (rappelons que (1.3.3) avait été écrite pour B entier, avec $\mathbf{P}_n$ la mesure uniforme sur les chemins dans le réseau en diamants). Il est utile d'indiquer explicitement la dépendance en h et en ω, la dépendance en β restant implicite pour simplifier les notations.

Il est aussi pratique de définir

$$H_{n,h}^{\omega,(i)} = \sum_{k \in I_{i,n}} (\beta\omega_k + h)\delta_k, \tag{1.3.6}$$

le hamiltonien du i^{e} bloc de taille 2^n (on écrit $H_{n,h}^{\omega}$ à la place de $H_{n,h}^{\omega,(1)}$ s'il n'y a pas d'ambiguité). Cela permet de redéfinir la mesure de polymère de la même manière que précédement: $\frac{\mathrm{d}\mathbf{P}_{n,h}^{\omega}}{\mathrm{d}\mathbf{P}_n} := \frac{1}{Z_{n,h}^{\omega}} \exp\left(H_{n,h}^{\omega}\right)$.

1.3.1.3. *Environnement gaussien et énergie libre.* À partir de maintenant, pour plus de clarté dans l'énoncé des résultats, nous considérons le cas d'un environnement gaussien. Soit donc $\omega := \{\omega_i\}_{i\in\mathbb{N}}$ une suite de variables gaussiennes centrées et normalisées (*i.e.* $\mathbb{E}[\omega_i] = 0$ et $\mathbb{E}[\omega_i^2] = 1$), de loi notée $\mathbb{P}$. Le fait de se restreindre à une séquence gaussienne simplifie grandement l'analyse du problème, car les corrélations sont dans ce cas entièrement codées par les corrélations à deux points, *i.e.* par la matrice de Covariance (ou de Corrélation) de la loi $\mathbb{P}$, que l'on note $K = (\kappa_{ij})_{i,j\geqslant 0}$ avec $\kappa_{ij} := \mathbb{E}[\omega_i\omega_j]$. Pour respecter la structure hiérarchique du système, on suppose que κ_{ij} ne dépend que de la distance (hiérarchique) $d(i,j)$, et pour i,j tels que $d(i,j) = p$ on écrit $\kappa_{ij} =: \kappa_p$, pour tout entier p. Cela implique en particulier que pour tout $j \in \mathbb{N}$, les variables $\omega_{I_{k,p}}$ pour $k \in \mathbb{N}$ (rappelons que $\omega_{I_{k,p}} := (\omega_{(k-1)2^p+1}, \ldots, \omega_{k2^p})$) sont de même loi. On fait, pour simplifier les calculs, le choix explicite

$$\kappa_p = \kappa^p, \qquad \text{pour un certain } \kappa \in [0,1). \tag{1.3.7}$$

Remarquons que le cas $\kappa = 0$ correspond au cas d'un environnement *i.i.d.*

Dans la suite, nous conserverons cette définition pour la séquence ω, la matrice de covariance K (et donc la valeur de κ) étant fixée une fois pour toute.

On peut définir l'énergie libre *quenched* du modèle, dans le cas de la séquence gaussienne corrélée ω définie ci-dessus:

$$\mathtt{F}(\beta,h) := \lim_{n\to\infty} \frac{1}{2^n} \log Z_{n,h}^{\omega} \overset{\mathbb{P}-a.s}{=} \lim_{n\to\infty} \frac{1}{2^n} \mathbb{E}[\log Z_{n,h}^{\omega}] \geqslant 0. \tag{1.3.8}$$

On définit aussi le point critique par $h_c(\beta) = h_c^{\text{que}}(\beta) := \inf\{h,\, \mathtt{F}(\beta,h) > 0\}$. L'inégalité $\mathtt{F}(\beta,h) \geqslant 0$ suit simplement du fait que $Z_{n,h}^{\omega} \geqslant (B-1)/B$. Le fait que $\mathtt{F}(\beta,h) < \infty$ est une conséquence facile du fait que $Z_{n,h}^{\omega} \leqslant \exp(\sum_{i=1}^{2^n}(\beta|\omega_i| + h))$, et qu'on a supposé que les ω_i avaient un moment d'ordre 2 fini.

La preuve de l'existence de la limite (1.3.8) est presque identique à celle de [GLT10a, Th.1.1] ou le désordre est *i.i.d.* Il est classique de voir que la séquence $(2^{-n}\mathbb{E}[\log Z_{n,h}^{\omega,\beta}])_{n\in\mathbb{N}}$ converge, et on utilise une propriété de concentration de $\log Z_{n,h}^{\omega,\beta}$

autour de sa moyenne (qui est la condition générale pour avoir l'existence de l'énergie libre, voir la preuve du Théorème 4.2.2) qui permet d'obtenir la convergence presque sûre de (1.3.8).

1.3.2. Résultats pour le système homogène. Il est pratique de définir $S_n^{(i)} = \sum_{k\in I_{i,n}} \delta_k$ le nombre de points de contacts (ou de descendants) sur le bloc $I_{i,n}$, et on écrit $S_n = S_n^{(1)}$ s'il n'y a pas d'ambiguité. On a la relation $S_n^{(i)} = S_{n-1}^{(2i-1)} + S_{n-1}^{(2i)}$.

Le modèle pur est le modèle où $\beta = 0$: la fonction de partition est $Z_{n,h} = \mathbf{E}_n\left[\exp\left(hS_n\right)\right]$ et vérifie

$$Z_{n+1,h} = \frac{1}{B}(Z_{n,h})^2 + \frac{B-1}{B}, \tag{1.3.9}$$

avec $Z_{0,h} = e^h$. On appelle $\mathtt{F}(h) = \lim_{n\to\infty} 2^{-n}\log Z_{n,h}$ l'énergie libre associée. Nous ajouterons parfois dans la suite un exposant "pur" aux différentes quantités liées au système homogène, afin d'éviter de possibles confusions.

L'équation de récurrence (1.3.9) possède deux points fixes, $x_0 = 1$ et $x_1 = B-1$. Le point $x_1 = B-1$ est stable et attractif, alors que le point $x_0 = 1$ est instable, et correspond à la condition initiale $Z_{0,h} = e^h = 1$, pour $h = 0$. Par conséquent le modèle homogène possède une transition de phase au point critique $h_c = h_c(\beta = 0) = 0$.

Théorème 1.3.1. *[GLT10a, Théorème 1.2] Pour tout $B \in (1,2)$, on a $\mathtt{F}(h) = 0$ pour tout $h \leqslant 0$. De plus, il existe une constante $c := c(B) > 0$ telle que pour tout $0 \leqslant h \leqslant 1$, on ait*

$$ch^{\nu^{\mathrm{pur}}} \leqslant \mathtt{F}(h) \leqslant c^{-1}h^{\nu^{\mathrm{pur}}} \tag{1.3.10}$$

avec

$$\nu^{\mathrm{pur}} = \frac{\log 2}{\log(2/B)} > 1. \tag{1.3.11}$$

L'exposant ν^{pur} est l'exposant critique du système pur. Nous avons gardé les mêmes notations que dans le modèle homogène introduit dans la Section 1.1, car il ne risque pas d'y avoir de confusion. On remarque que ν^{pur} est une fonction croissante de B, et que l'on a $\nu^{\mathrm{pur}} = 2$ pour $B = B_c := \sqrt{2}$.

On possède aussi quelques estimées concernant le modèle pur, notamment la fonction de partition d'un système de taille 2^n. La proposition suivante permet de dire que le régime critique, où la fonction de partition reste proche de la fonction de partition au point critique $Z_{n,h_c} = Z_{n,0} = 1$, correspond à $h = O((B/2)^n)$.

Proposition 1.3.2. Estimées sur le modèle homogène

(1) *Il existe des constantes $a_0 > 0$ et $c_0 > 0$ telles que pour tout $n \geqslant 0$, si on a $h \leqslant a_0\,(B/2)^n$, alors*

$$\mathbf{E}_n\left[\exp\left(hS_n\right)\right] \leqslant \exp\left(c_0 h(2/B)^n\right). \tag{1.3.12}$$

(2) *Il existe une constante $c > 0$ telle que pour tout $n \geqslant 0$ et $u \geqslant 0$, on ait*

$$\mathbf{E}_n\left[\exp\left(hS_n\right)\right] \leqslant c\exp\left(ch^{\nu^{\mathrm{pur}}}2^n\right), \tag{1.3.13}$$

où ν^{pur} est l'exposant critique du système pur, donné par (1.3.11).

Nous verrons dans la Section 4.A (cf. Corollaire 4.A.2) que cette proposition permet notamment de montrer que la mesure de polymère $\mathbf{P}_{n,h}$ reste proche (dans un sens à définir) de la mesure $\mathbf{P}_{n,h_c} = \mathbf{P}_{n,0}$ dans le régime critique $h = O((B/2)^n)$. Notons que l'on possède des bornes inférieures similaires, quitte à changer les constantes: $\log \mathbf{E}_n [\exp(hS_n)]$ est d'ordre $h(2/B)^n$ pour $h = O((2/B)^n)$, et d'ordre $h^{\nu^{\mathrm{pur}}} 2^n$ pour $h \geqslant (2/B)^n$.

1.3.3. Environnement *i.i.d.*: résultats connus. Jusqu'à présent, le cas où la séquence $\omega = (\omega_i)_{i\in\mathbb{N}}$ est *i.i.d.* avait principalement été étudié. On rappelle ici les principaux résultats déjà connus dans ce cas, sur le rôle du désordre dans la transition de phase. On supposera que $\omega = (\omega_i)_{i\in\mathbb{N}}$ est une suite *i.i.d.* de variables gaussiennes standard ($\kappa = 0$).

Dans le cas*i.i.d.*, la fonction de partition annealed $Z^{\mathrm{a}}_{n,h} := \mathbb{E} Z^{\omega}_{n,h}$ est égale à la fonction de partition du modèle homogène, avec le paramètre $h + \beta^2/2$. En effet, en prenant l'espérance dans la relation de récurrence (1.3.4), on obtient

$$Z^{\mathrm{a}}_{n+1,h} = \frac{(Z^{\mathrm{a}}_{n,h})^2 + B - 1}{B}, \tag{1.3.14}$$

avec $Z^{\mathrm{a}}_{0,h} = e^h \mathbb{E}[e^{\beta\omega}] = e^{h+\beta^2/2}$. On sait donc que le point critique annealed est $h^{\mathrm{a}}_c(\beta) = -\beta^2/2$, et que l'exposant critique annealed est $\nu^{\mathrm{a}} = \nu^{\mathrm{pur}} = \log 2/\log(2/B)$.

Comme dans le cas de la Section 1.2, on peut comparer les énergies libres quenched et annealed, via l'inégalité de Jensen $\mathtt{F}(\beta,h) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,h)$, qui donne directement $h^{\mathrm{que}}_c(\beta) \geqslant h^{\mathrm{a}}_c(\beta)$, cf. Figure 1.5.

Nous avons vu dans la Section 0.2 que le critère de Harris prédit si le désordre est pertinent ou non selon que ν^{pur} est plus petit ou plus grand que 2. En se référant au Théorème 1.3.1, cela correspond donc à $B < \sqrt{2}$ ou $B > \sqrt{2}$, respectivement. Dans le cas $B = \sqrt{2}$, si on étudie le comportement de la variance de $Z^{\omega,\beta}_{N,h}$ au point critique annealed $h^{\mathrm{a}}_c(\beta)$ en fonction de N, on peut deviner le critère de pertinence/non-pertinence, comme discuté dans la Section 1.2.2. En effet, nous trouvons dans ce cas que la variance diverge (voir [DHV92, GLT10a]), ce qui suggère que le désordre est pertinent.

Dans le cas d'un environnement *i.i.d.*, la question de la pertinence du désordre est maintenant mathématiquement bien comprise. Le Théorème suivant rassemble les principaux résultats connus:

Théorème 1.3.3. Pertinence du désordre dans le cas *i.i.d.*

Pour tout $B \in (1,2)$, la transition de phase est continue, pour une quantité arbitraire de désordre: il existe $c(B) > 0$ tel que pour tout $\beta > 0$, on ait

$$\mathtt{F}(\beta,h) \leqslant \frac{c(B)}{\beta^2} \left(h - h_c(\beta)\right)_+^2. \tag{1.3.15}$$

- *Le désordre est* pertinent *pour $B \geqslant \sqrt{2}$.*

Lorsque $\nu^{\mathrm{pur}} < 2$, i.e. quand $B > B_c := \sqrt{2}$, (1.3.15) implique que la transition de phase est plus lisse dans le système désordonné que dans le système pur. De plus,

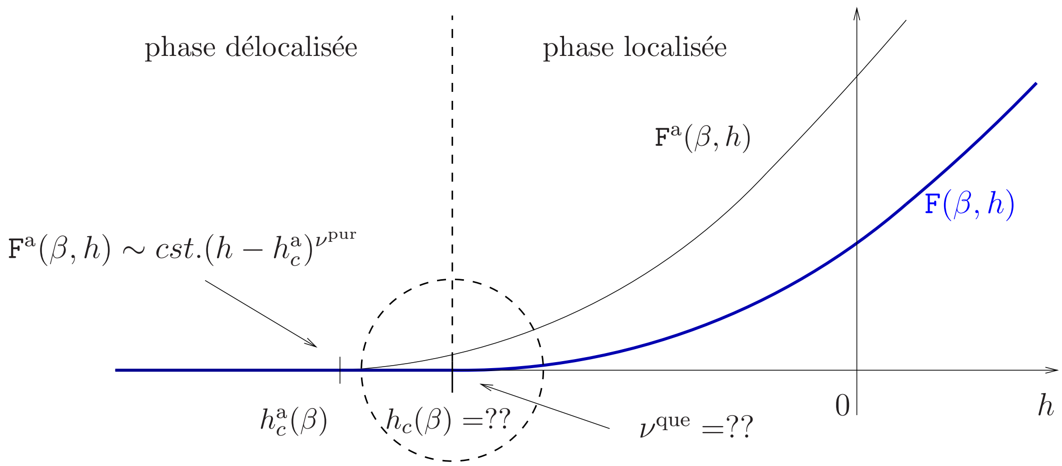

FIGURE 1.5. Comparaison des courbes de l'énergie libre quenched et annealed, à $\beta > 0$ fixé. Le principal objectif de l'étude du système désordonné est de donner les caractéristiques de la courbe de l'énergie libre quenched $\mathtt{F}(\beta,h)$. Un outil important est la comparaison avec le système annealed, qui donne déjà $\mathtt{F}(\beta,h) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,h)$, et $h_c(\beta) \geqslant h_c^{\mathrm{a}}(\beta)$. Une question est donc de savoir si $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$ ou non, et si $\mathtt{F}(\beta,h)$ possède un exposant critique ν^{que} égal à celui annealed ν^{a} (qui dans le cas *i.i.d.* est $\nu^{\mathrm{a}} = \nu^{\mathrm{pur}}$). Le cas où le désordre possède des corrélations devient ainsi plus complexe, car l'analyse du système annealed est beaucoup plus difficile.

pour $B \geqslant \sqrt{2}$, on a $h_c^{\mathrm{que}}(\beta) > h_c^{\mathrm{a}}(\beta)$, et plus précisément pour tout $\varepsilon > 0$ et $\beta \leqslant \beta_0$ petit, on a

$$\begin{gathered} c\beta^{\frac{2}{2-\nu^{\mathrm{pur}}}} \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant c'\beta^{\frac{2}{2-\nu^{\mathrm{pur}}}}, \\ \exp\left(-\frac{c(\varepsilon)}{\beta^{2+\varepsilon}}\right) \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant \exp\left(-\frac{c'}{\beta^2}\right). \end{gathered} \tag{1.3.16}$$

• *Le désordre est* non-pertinent *pour $B < \sqrt{2}$.*
Si $B < \sqrt{2}$, on a $h_c^{\mathrm{que}}(\beta) = h_c^{\mathrm{a}}(\beta)$ pour β petit, et de plus, l'énergie libre quenched possède le même comportement critique que dans le cas homogène:

$$\mathtt{F}^{\mathrm{a}}(\beta,h) \geqslant \mathtt{F}^{\mathrm{que}}(\beta,h) \geqslant (1-\eta)\mathtt{F}^{\mathrm{a}}(\beta,h)$$

pour tout $h \in (0,1)$, où η peut être rendu arbitrairement petit en prenant β petit.

Ces résultats ont été obtenus par une série d'articles dus à G. Giacomin, H. Lacoin et F. Toninelli: l'article [LT09] démontre (1.3.15), et [GLT10a, GLT10b] concernent les points critiques. Ce Théorème est à mettre en relation avec le Théorème 1.3.8 et la Proposition 4.3.5 de la Section suivante, obtenus durant le cours de ma thèse, et qui en sont l'équivalent pour le modèle hiérarchique corrélé.

1.3.4. Modèle hiérarchique avec environnement corrélé. Nous nous plaçons maintenant dans le cadre de l'étude du modèle d'accrochage hiérarchique en environnement corrélé, effectuée dans le Chapitre 4, basé sur l'article [BT11]. Nous considérons, comme mentionné auparavant, une suite $\omega := \{\omega_i\}_{i\in\mathbb{N}}$ de variables gaussiennes centrées et normalisées, de loi $\mathbb{P}$. La matrice de Covariance de cette loi est $K = (\kappa_{ij})_{i,j\geqslant 0}$ avec $\kappa_{ij} := \mathbb{E}[\omega_i\omega_j] = \kappa_{d(i,j)}$ (qui ne dépend que de la distance hiérarchique entre i et j). Rappelons notre hypothèse

$$\kappa_p = \kappa^p, \qquad \text{pour un certain } 0 < \kappa < 1. \tag{1.3.17}$$

L'hypothèse (1.3.17), notamment le fait que $\kappa < 1$, permet d'obtenir l'inégalité de concentration nécessaire à l'existence de l'énergie libre quenched (se reporter à la Section 4.2 pour plus de détails).

On verra dans le Théorème 1.3.9 que l'on exclut le cas $\kappa \geqslant 1/2$ dans notre étude. La raison en est simple: le modèle ne possède plus de transition de phase dans le modèle quenched, et le modèle annealed n'est plus correctement défini. Pour $\kappa = 0$, on retrouve le modèle désordonné avec un environnement *i.i.d.*

Il est possible de construire explicitement une loi gaussienne vérifiant (1.3.17), de la manière suivante. Soit $\mathcal{I} = \{I_{k,p}, p \geqslant 0, k \in \mathbb{N}\}$ où $I_{k,p}$ est défini dans (1.3.5), et soit $\{\widehat{\omega}_I\}_{I\in\mathcal{I}}$ une famille de variables gaussiennes standard *i.i.d.* $\mathcal{N}(0,1)$, de loi notée $\widehat{\mathbb{P}}$. On a alors l'égalité en loi suivante:

$$\omega_i := \sum_{I\in\mathcal{I}; i\in I} \widehat{\kappa}_I \widehat{\omega}_I, \tag{1.3.18}$$

avec $\widehat{\kappa}_{I_{k,p}} := \widehat{\kappa}_p := \sqrt{\kappa^p - \kappa^{p+1}}$. En effet, il suffit de vérifier que la famille gaussienne ainsi construite possède la bonne structure de corrélations, la somme dans le membre droite de (1.3.18) étant bien définie car $\sum_p \widehat{\kappa}_p^2 = 1 < \infty$.

Remarque 1.3.4. Ce choix pour la structure des corrélations correspond dans le cas non hiérarchique à une décroissance en loi de puissance. En effet, on peut comparer la distance hiérarchique $d(\cdot,\cdot)$, à la distance usuelle (sur l'axe $\mathbb{N}$), en remarquant par exemple que $d(1,k) = \lfloor \log k/\log 2\rfloor$ pour tout $k \in \mathbb{N}$. Le coefficient de corrélation entre i et $j = i+k$ vérifie ainsi, lorsque i est fixé et k est grand, $\kappa_{ij} = \kappa^{d(i,j)} \sim k^{-\zeta}$, où $\zeta = \log(1/\kappa)/\log 2 > 0$. Remarquons que $\zeta > 0$ pour tout $\kappa < 1$, et que $\zeta > 1$ si $\kappa < 1/2$ (corrélations sommables), et que $\zeta < 1$ si $\kappa > 1/2$ (corrélations non-sommables).

1.3.4.1. *Le modèle annealed.* On définit comme dans le cas *i.i.d.*, la fonction de partition *annealed* $Z^{\mathrm{a}}_{n,h} := \mathbb{E}[Z^{\omega}_{n,h}]$, que l'on est capable de rendre explicite, grâce à la nature gaussienne du désordre. Un calcul donne

$$Z^{\mathrm{a}}_{n,h} = \mathbf{E}_n\left[\exp\left(\left(\frac{\beta^2}{2}+h\right)\sum_{k=1}^{2^n}\delta_k + \beta^2/2\sum_{p=1}^{n}\kappa_p \sum_{\substack{1\leqslant i,j\leqslant 2^n\\ d(i,j)=p}} \delta_i\delta_j\right)\right]. \tag{1.3.19}$$

Les termes $\delta_i\delta_j$, dus aux corrélations du désordre, compliquent grandement l'analyse du problème par rapport au cas du modèle annealed avec désordre *i.i.d.*, cf. Section 1.3.2. Il sera utile pour la suite d'observer que ce terme d'interaction ("à deux corps") est positif ou nul, car on a choisi des corrélations positives.

Nous soulignons que Dyson [Dys69] définit aussi une version hiérarchique du modèle d'Ising ferromagnétique (qui ressemble, au moins au niveau formel, à notre modèle annealed, cf. (1.3.19)). En combinant les résultats qu'il obtient sur ce modèle hiérarchique avec les inégalités de corrélations de Griffiths, il trouve un critère pour l'existence d'une transition de phase ferromagnétique pour le modèle d'Ising unidimensionnel non hiérarchique, où les couplages décroissent comme $J_{i-j} \sim |i-j|^{-\zeta}$. Remarquons que dans notre cas, il n'existe aucun type d'inégalité permettant de donner directement des résultats sur le modèle d'accrochage non hiérarchique à partir de ceux obtenus dans le cadre hiérarchique.

On définit aussi l'énergie libre *annealed*: $\mathtt{F}^{\mathrm{a}}(\beta,h) := \lim_{n\to\infty} \frac{1}{2^n} \log \mathbb{E}[Z^{\omega}_{n,h}] \geqslant 0$ (l'existence de la limite est montrée dans la Proposition 4.2.3), et on appelle comme d'habitude son point critique $h_c^{\mathrm{a}}(\beta)$.

Nous verrons dans la Proposition 4.2.4 que l'on peut montrer que $h_c^{\mathrm{a}}(\beta)$ est d'ordre $-\beta^2$ pour β petit. Cependant, il paraît très difficile de calculer explicitement sa valeur, contrairement au cas d'un environnement *i.i.d.* où il est immédiat d'obtenir $h_c^{\mathrm{a}}(\beta) = -\beta^2/2$. Cela rend l'étude du système corrélé beaucoup plus complexe.

Remarque 1.3.5. Le caractère fini de l'énergie libre annealed dépend de la sommabilité des corrélations. En effet, si on se restreint à l'événement où tous les δ_n sont égaux à 1, on obtient

$$e^{2^n\left((h+\beta^2/2)+\beta^2/2\sum_{p=1}^n \kappa_p 2^{p-1}\right)} \geqslant Z^{\mathrm{a}}_{n,h} \geqslant \left(\frac{1}{B}\right)^{2^n} e^{2^n\left((h+\beta^2/2)+\beta^2/2\sum_{p=1}^n \kappa_p 2^{p-1}\right)}. \quad (1.3.20)$$

Ceci montre que $\mathtt{F}^{\mathrm{a}}(\beta,h) = \infty$ (et, on peut dire, $h_c^{\mathrm{a}}(\beta) = -\infty$), sauf si

$$K_\infty := \sum_{p=0}^{\infty} \kappa_p 2^p < +\infty \,; \quad i.e. \;\; \kappa < 1/2. \quad (1.3.21)$$

En ce qui concerne l'énergie libre quenched, le cas $\kappa > 1/2$ est, de même, est moins intéressant comme le montre le Théorème 1.3.9.

Dans le cas d'un désordre *i.i.d.*, nous avons vu que, trivialement, le modèle pur et annealed ont le même comportement critique. Le modèle annealed est ici beaucoup plus complexe que le modèle homogène standard, notamment à cause du terme d'interaction "à deux corps" $\delta_i\delta_j$ dans (1.3.19), provenant des corrélations. L'un des résultats principaux de ma thèse est que, néanmoins, si le paramètre κ safisfait $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ et β est petit, le modèle annealed, au voisinage du point critique, possède le même comportement que le modèle pur.

Théorème 1.3.6. *Si $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$, alors il existe des constantes $\beta_0 > 0$ et $c_1 > 0$ telles que, pour tout $\beta \leqslant \beta_0$, et pour tout ensemble non vide $I \subset \{1,\ldots,2^n\}$ on ait*

$$\left(e^{-c_1\beta^2}\right)^{|I|} \mathbf{E}_n\left[\delta_I\right] \leqslant \mathbf{E}_n\left[\delta_I e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\right] \leqslant \left(e^{c_1\beta^2}\right)^{|I|} \mathbf{E}_n\left[\delta_I\right], \quad (1.3.22)$$

où $\delta_I := \prod_{i\in I}\delta_i$.De plus, la fonction de partition au point critique converge vers 1 exponentiellement vite:

$$e^{-c_2\beta^2(4\kappa/B^2)^n} \leqslant Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \leqslant 1. \quad (1.3.23)$$

Cela permet d'obtenir que, pour $\beta \leqslant \beta_0$, et pour tout $u \in [0,1]$, on a

$$\mathtt{F}\left(e^{-c_1\beta^2}u\right) \leqslant \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}+u) \leqslant \mathtt{F}\left(e^{c_1\beta^2}u\right). \tag{1.3.24}$$

Pour une formulation plus complète de ce résultat, se reporter au Chapitre 4 (Théorème 4.3.1 et Proposition 4.3.2).

On a donc entre autres que, si $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ et β est petit, la mesure de polymère annealed au point critique,

$$\frac{\mathrm{d}\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}{\mathrm{d}\mathbf{P}_n} := \frac{1}{Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}} \exp\left(H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\right),$$

est "proche" de celle du système pur au point $h=0$, au vu de (1.3.22). Ces résultats ont été obtenus dans [BT11] (voir Chapitre 4), et sont les outils de base avec lesquels nous avons attaqué le problème de la pertinence du désordre dans le cas corrélé, voir le Théorème 1.3.8 ci-dessous. Il est important de remarquer le Théorème 1.3.6 ne présuppose pas de connaître la valeur exacte de $h_c^{\mathrm{a}}(\beta)$ (il y a effectivement peu d'espoir de connaître sa valeur de manière générale).

Le cas $B^2/4 \leqslant \kappa < 1/2$ est plus subtil car le terme "à deux corps" $\delta_i\delta_j$ qui apparaît dans (1.3.19), possède un effet non négligeable sur le comportement critique, et est ainsi plus difficile à traiter. Dans le même article [BT11] (voir Chapitre 4), des résultats partiels dans ce cas sont donnés, montrant que certaines propriétés critiques au niveau annealed sont modifiées par rapport au cas $\kappa < 1/2 \wedge B^2/4$.

Théorème 1.3.7. *Soit $B^2/4 < \kappa < 1/2$ et $\beta > 0$. Contrairement à (1.3.23), la fonction de partition annealed, au point critique, ne converge pas vers 1. On a en réalité*

$$\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} \leqslant \frac{1}{\beta\sqrt{\kappa}}\left(\frac{B}{2\sqrt{\kappa}}\right)^n. \tag{1.3.25}$$

De plus, le nombre moyen de descendants à la génération n, sous la mesure de polymère annealed au point critique, vérifie

$$\mathbf{E}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}[S_n] = \mathbf{E}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\left[\sum_{i=1}^{2^n}\delta_i\right] \leqslant \frac{c(B)}{\beta}\frac{1}{\kappa^{(n+1)/2}} \ll \mathbf{E}_n[S_n] = \left(\frac{2}{B}\right)^n. \tag{1.3.26}$$

Ce résultat montre que, si la fonction de partition $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ converge vers une constante Z_∞, alors cette constante vérifie $Z_\infty \leqslant B/2\sqrt{\kappa} < 1$. Ceci contraste avec le résultat (1.3.23), ce qui suggère un comportement critique atypique. Remarquons que le nombre moyen de points de contacts au point critique est ausi beaucoup plus faible que dans le cas homogène, où il y en a, en moyenne, $(2/B)^n$ dans un système de taille 2^n. Ce Théorème, bien qu'il montre que certaines propriétés critiques du système annealed sont modifiées par la présence de corrélations, n'implique rien sur le comportement de l'énergie libre annealed, et ne donne par exemple aucune borne sur l'exposant critique.

Par contre, la borne (1.3.26) sur la fraction de contact au point critique annealed suggère, comme mentionné dans la Section 1.1.2 (où l'on relie le nombre de contacts au point critique au comportement de l'énergie libre), que l'exposant critique de l'énergie libre annealed (s'il existe) est plus grand que l'exposant critique $\nu^{\mathrm{pur}} = \log 2/\log(2/B)$, la transition de phase du modèle annealed étant donc plus lisse. En l'occurence, nous pensons que l'exposant critique *annealed* devrait être $\nu^{\mathrm{a}} = \log 2/\log(1/\sqrt{\kappa})$ (cf. Section 4.4.3).

1.3.4.2. *Influence du désordre dans le cas corrélé.* Le critère de Harris concernant la pertinence/non-pertinence du désordre dans le cas d'un environnement *i.i.d.* peut être généralisé au cas d'un environnement corrélé, comme l'ont fait Weinrib et Halperin [WH83] dans un cadre différent (mais néanmoins général), voir la Section 0.2.2. Suivant les idées de ces auteurs, le critère de Harris pourrait être modifié lorsque les corrélations sont trop fortes. Il y a alors deux cas possibles. Si $\kappa < 1/2$, les corrélations sont sommables, et selon le critère de Weinrib-Halperin, on peut prévoir que le critère de Harris de pertinence/non-pertinence du désordre ne soit pas modifié: il devrait y avoir pertinence du désordre si $\nu^{\mathrm{pur}} < 2$ (*i.e.* $B > \sqrt{2}$), et non-pertinence si $\nu^{\mathrm{pur}} > 2$ (*i.e.* $B < \sqrt{2}$). Si $1/2 < \kappa < 1$, les corrélations ne sont pas sommables, et une modification de ce critère devrait être observée. Le désordre serait pertinent si $\nu^{\mathrm{pur}} < 2/\zeta$ (où $\zeta = \log(1/\kappa)/\log 2$ code la décroissance des corrélations, cf. Remarque 1.3.4), ce qui correspond à $B > 2\sqrt{\kappa}$, et il serait non-pertinent si $\nu^{\mathrm{pur}} > 2/\zeta$, *i.e.* si $B < 2\sqrt{\kappa}$. Nous verrons par la suite, que le cas $\kappa > 1/2$ est atypique, et que le critère de Weinrib-Halperin n'est alors plus opportun.

Avec l'aide cruciale du Théorème 1.3.6, il devient possible de prouver que, au moins pour $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$, le critère de Harris décidant de la pertinence du désordre n'est pas modifié par la présence de corrélations dans l'environnement, ce qui est en accord avec la prédiction de Weinrib-Halperin (dans le cas $\kappa < 1/2$).

Théorème 1.3.8. *Soit* $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$.

- *Si* $1 < B \leqslant B_c = \sqrt{2}$, *le désordre est pertinent: les points critiques quenched et annealed sont différents pour tout* $\beta > 0$, *et on a*
  - *si* $B < B_c$, *il existe une constante* $c_3 > 0$ *telle que pour tout* $0 \leqslant \beta \leqslant 1$

$$(c_3)^{-1}\beta^{\frac{2}{2-\nu^{\mathrm{pur}}}} \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant c_3\beta^{\frac{2}{2-\nu^{\mathrm{pur}}}} \ ; \tag{1.3.27}$$

  - *si* $B = B_c$, *il existe une constante* $c_4 > 0$ *et un certain* $\beta_0 > 0$ *tels que* $0 \leqslant \beta \leqslant \beta_0$

$$\exp\left(-\frac{c_4}{\beta^4}\right) \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant \exp\left(-\frac{c_4^{-1}}{\beta^{2/3}}\right). \tag{1.3.28}$$

- *Si* $B_c < B < 2$, *le désordre est non-pertinent: il existe un certain* $\beta_0 > 0$ *tel que* $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$ *pour tout* $0 < \beta \leqslant \beta_0$. *Plus précisément, pour tout* $\eta > 0$, *en prenant* $u > 0$ *suffisamment petit,* $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta) + u) \geqslant (1-\eta)\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta) + u)$.

Nous avons démontré ce résultat dans [BT11] (repris dans le Chapitre 4). En poussant plus loin les calculs, il est sûrement possible d'améliorer la borne supérieure

(1.3.28) à $\exp(-c_4^{-1}/\beta^2)$ et la borne inférieure à $\exp(-c_4(\epsilon)/\beta^{2+\epsilon})$ pour tout $\epsilon > 0$, comme c'est le cas dans le modèle sans corrélations $\kappa = 0$, cf. Théorème 1.3.3. Nous ne donnerons pas plus de détails dans cette direction.

Dans le cas $B^2/4 \leqslant \kappa < 1/2$, nous avons vu dans le Théorème 1.3.7 que les corrélations ont un effet important sur le comportement du système annealed. On s'attend par conséquent à ce que le Théorème 1.3.6 ne soit plus valable: une conjecture naturelle serait que dans (1.3.27), $\nu^{\text{pur}} = \log 2/\log(2/B)$ soit remplacé par l'exposant du modèle annealed ν^{a}. Comme discuté précédemment, on devrait avoir $\nu^{\text{a}} > \nu^{\text{pur}}$, et pour $B < B_c$, l'écart entre les points critiques quenched et annealed serait ainsi réduit lorsque $B^2/4 \leqslant \kappa < 1/2$.

Remarquons en passant que l'inégalité (1.3.15) peut être montrée pour le modèle hiérarchique corrélé pour tout $B \in (1,2)$ et $\beta > 0$, si $\kappa < 1/2$. Nous ne donnerons pas la preuve dans cette thèse car, grâce à la sommabilité des corrélations (garantie par la condition $\kappa < 1/2$), la démonstration est similaire au cas du modèle hiérarchique *i.i.d.*, cf. [LT09]. On peut aussi se reporter à la Section 5.4.1, où un résultat équivalent est prouvé dans le modèle corrélé non hiérarchique. Cela montre que si $\kappa < 1/2$, l'exposant critique *quenched* est toujours plus grand ou égal à 2, et donc plus grand que ν^{pur} dans le cas où $\nu^{\text{pur}} < 2$. Le désordre est donc pertinent si $\nu^{\text{pur}} < 2$, ce qui est en accord avec le critère de Weinrib-Halperin.

Nous considérons enfin le cas $\kappa > 1/2$. Nous avons déjà vu que le système annealed n'est pas bien défini, son énergie libre étant infinie. Le résultat suivant souligne que l'énergie libre quenched (qui est finie) est strictement positive pour tout valeur de $h \in \mathbb{R}$, et qu'ainsi le système ne possède pas de transition de phase entre les régimes localisé et délocalisé.

Théorème 1.3.9. *Si $\kappa > 1/2$, alors $\mathtt{F}(\beta, h) > 0$ pour tout $\beta > 0, h \in \mathbb{R}$, et ainsi $h_c(\beta) = -\infty$. Il existe une constante $c_5 > 0$ telle que pour tout $h \leqslant -1$ et $\beta > 0$*

$$\mathtt{F}(\beta, h) \geqslant \exp\left(-c_5|h|(|h|/\beta^2)^{\log 2/\log(2\kappa)}\right). \tag{1.3.29}$$

La preuve de ce Théorème est faite dans [BT11, Sec.3.1], et nous ne la présenterons pas cette thèse. En effet, la preuve est très similaire à celle du Théorème 1.4.6 (voir Section 1.4.3), dans le cas du modèle d'acrochage non hiérarchique.

Les corrélations modifient donc complètement les propriétés critiques du système, et la transition de phase n'apparaît plus. Il est donc impossible de vérifier (ou d'infirmer) la prédiction de Weinrib-Halperin dans ce cas, car le système ne possède pas de comportement critique à proprement parler, le point critique étant $-\infty$.

1.4. Modèle d'accrochage non-homogène

On s'intéresse de nouveau au modèle d'accrochage sur une ligne de défauts de la Section 1.1, mais comme dans la Section précédente, on introduit des inhomogénéités dans le système, modélisant le caractère aléatoire des interactions entre le polymère et la ligne de défauts. Les notations sont similaires à celle de la Section 1.3, mais les deux modèles étant traités dans des Chapitres différents, aucune confusion n'est possible.

1.4.1. Introduction. On considère une séquence ergodique de variables aléatoires $\omega := \{\omega_i\}_{i\in\mathbb{N}}$, centrées et de variance unitaire. On note sa loi $\mathbb{P}$, et on fera par la suite différentes hypothèses supplémentaires sur la loi $\mathbb{P}$: la Section 1.4.2 traite par exemple le cas où la suite ω est *i.i.d.*, les Sections 1.4.3 et 1.4.4 plusieurs cas où ω présente une structure de corrélation particulière.

On considère $\mathbf{P}$ la loi d'un processus de renouvellement τ, qui vérifie l'Hypothèse 1.1.2. Comme pour le modèle homogène, on introduit une modification de Gibbs de la loi $\mathbf{P}$ qui prend en compte les interactions (inhomogènes, encodées par la séquence ω) d'une trajectoire de τ avec la ligne de défauts. Pour $\beta \geqslant 0$ et $h \in \mathbb{R}$, on définit comme dans les sections précédentes le hamiltonien $H^{\omega,\beta}_{N,h} := \sum_{n=1}^{N}(\beta\omega+h)\delta_n$, où on a utilisé la notation $\delta_n := \mathbf{1}_{\{n\in\tau\}}$. La mesure de polymère $\mathbf{P}^{\omega,\beta}_{N,h}$ pour une trajectoire de longueur N est définie par sa dérivée de Radon-Nykodim par rapport à la mesure $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}^{\omega,\beta}_{N,h}}{\mathrm{d}\mathbf{P}} := \frac{1}{Z^{\omega,\beta}_{N,h}} \exp(H^{\omega,\beta}_{N,h})\delta_N, \tag{1.4.1}$$

où $Z^{\omega,\beta}_{N,h} := \mathbf{E}\left[\exp(H^{\omega,\beta}_{N,h})\delta_N\right]$ est la fonction de partition du système.

La fonction de partition possède la propriété suivante, déjà soulignée dans la Remarque 1.2.1: $Z^{\omega,\beta}_{N+M,h} \geqslant Z^{\omega,\beta}_{N,h} Z^{\theta^N\omega,\beta}_{M,h}$ pour tout $N, M \in \mathbb{N}$, où θ est l'opérateur de translation de l'environnement, c'est-à-dire $\theta((\omega_i)_{i\in\mathbb{N}}) = (\omega_{i+1})_{i\in\mathbb{N}}$. Cette propriété montre que la suite $(\log Z^{\omega,\beta}_{N,h})_{N\in\mathbb{N}}$ est sur-additive au sens ergodique, ce qui permet de définir, grâce au Théorème ergodique sur-additif de Kingman [Kin73], l'énergie libre *quenched* du système, de la même manière que dans la Section 1.2: c'est la la limite

$$\mathtt{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z^{\omega,\beta}_{N,h} = \sup_{n\in\mathbb{N}} \frac{1}{N} \mathbb{E} \log Z^{\omega,\beta}_{N,h}. \tag{1.4.2}$$

Cette limite existe et est constante $\mathbb{P}$-*p.s.*, et la fonction $h \mapsto \mathtt{F}(\beta,h)$ est positive ou nulle, convexe et croissante. Il existe un point critique *quenched* $h_c(\beta) = h_c^{\mathrm{que}}(\beta) := \inf\{h\,;\, \mathtt{F}(\beta,h) > 0\}$, tel que $\mathtt{F}(\beta,h) > 0$ si et seulement si $h > h_c(\beta)$.

Comme pour le système homogène, on obtient avec un calcul rapide que

$$\partial_h \mathtt{F}(\beta,h) = \lim_{N\to\infty} N^{-1} \mathbf{E}^{\omega,\beta}_{N,h}[|\tau\cap[0,N]|] \quad \mathbb{P}-p.s.,$$

en tout point h où $\mathtt{F}(\beta,h)$ est différentiable. Le point critique $h_c(\beta)$ marque donc, comme nous l'avons déjà vu maintes fois, la transition entre une phase *délocalisée* pour $h < h_c(\beta)$, où il y a une densité nulle de contacts, et une phase *localisée* pour $h > h_c(\beta)$, où il y a une densité strictement positive de contacts.

On souhaite maintenant comparer ce modèle à celui sans désordre de la Section 1.1, afin de savoir comment le désordre modifie les propriétés critiques du système. On s'intéresse par exemple au calcul de la valeur explicite du point critique $h_c(\beta)$, ou à l'ordre de la transition de phase de l'énergie libre quenched, ou encore au comportement des trajectoires au point critique $h_c(\beta)$.

L'un des principaux outils utilisés dans l'optique de l'analyse de la pertinence du désordre, est l'étude de la fonction de partition annealed $Z^{\mathrm{a}}_{N,h} := \mathbb{E}[Z^{\omega,\beta}_{N,h}]$ (et de manière plus générale, du système annealed associé). On peut définir l'énergie libre annealed par $\mathtt{F}^{\mathrm{a}}(\beta,h) := \lim_{N\to\infty} \frac{1}{N}\log \mathbb{E}[Z^{\omega}_{N,h}]$, et on a aussi un point critique annealed $h_c^{\mathrm{a}}(\beta)$, tel que $\mathtt{F}^{\mathrm{a}}(\beta,h) > 0$ si et seulement si $h > h_c^{\mathrm{a}}(\beta)$. L'étude du système annealed apporte de nombreux outils pour l'étude du système désordonné, et par exemple une simple application de l'inégalité de Jensen montre que $\mathtt{F}(\beta,h) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,h)$, ce qui donne $h_c^{\mathrm{a}}(\beta) \leqslant h_c(\beta)$.

On obtient, grâce à une autre utilisation de l'inégalité de Jensen, et en utilisant le fait que les ω_i sont centrés, que $\mathtt{F}(\beta,h) \geqslant \mathtt{F}(0,h)$ (on note aussi $\mathtt{F}(h) = \mathtt{F}(0,h)$, en accord avec la Section 1.1), ce qui donne $h_c(\beta) \leqslant h_c(0) = 0$. Mentionnons que dans [AS06], dans le cas d'un environnement *i.i.d.* (cf. Section 1.4.2) et sous des conditions assez générales, il est prouvé que l'inégalité $h_c(\beta) \leqslant 0$ est en réalité stricte, montrant ainsi que le désordre possède un effet “localisateur”.

On détermine la pertinence du désordre aussi bien en terme d'écart entre les points critiques (l'inégalité $h_c^{\mathrm{a}}(\beta) \leqslant h_c(\beta)$ est-elle stricte?), qu'en terme de différence des exposants critiques de l'énergie libre (qui est l'objet d'étude principal dans la majorité de la littérature).

1.4.2. Critère de Harris, influence du désordre dans le cas *i.i.d.* Nous donnons maintenant des résultats dans le cas, largement étudié ces dernières années [Ale08, AZ09, DGLT09, GLT10b, GLT11, GT06, Lac10, Ton07, Ton08a, Ton08b], où la séquence $\omega = (\omega_i)_{i\in\mathbb{N}}$ est *i.i.d.*, et où l'on suppose que ω_1 possède des moments exponentiels

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_1}] < \infty, \quad \forall \beta > 0. \tag{1.4.3}$$

Dans ce cas, le modèle annealed est en réalité exactement égal au modèle homogène de la Section 1.1.2, avec le paramètre $h + \lambda(\beta)$,

$$\mathbb{E}[Z^{\omega,\beta}_{N,h}] = \mathbf{E}\left[\exp\left((h+\lambda(\beta))\sum_{n=1}^{N}\delta_n\right)\delta_N\right].$$

Il est possible de calculer explicitement $\lambda(\beta)$ dans des cas particuliers: on donne l'exemple où ω_1 est de loi gaussienne $\mathcal{N}(0,1)$, auquel cas on obtient $\lambda(\beta) = \beta^2/2$.

On peut dans le cas *i.i.d.* expliciter la méthode du critère de Harris, qui indique si le désordre est pertinent ou non, en fonction de l'exposant critique homogène $\nu^{\mathrm{pur}} = 1 \vee 1/\alpha$, cf. Théorème 1.1.6. Si $\alpha < 1/2$, le désordre devrait être non-pertinent pour β petit, et si $\alpha > 1/2$, le désordre devrait être pertinent pour tout $\beta > 0$. Les physiciens se sont particulièrement intéressés au cas $\alpha = 1/2$, dit *marginal*, où des prédictions contradictoires ont été données par [DHV92] (pertinence) et [FLNO86] (non-pertinence).

Nous donnons maintenant une explication rapide de la méthode donnant le critère de Harris, dans le cadre que nous considérons ici. Pour β petit, on développe l'énegie libre pour h très proche de $h_c^{\mathrm{a}}(\beta)$, autour de l'énergie libre annealed

$$\mathbb{E}\log Z^{\beta,\omega}_{N,h} = \mathbb{E}\log\left[\mathbb{E}Z^{\beta,\omega}_{N,h} + (Z^{\beta,\omega}_{N,h} - \mathbb{E}Z^{\beta,\omega}_{N,h})\right] \backsimeq \log \mathbb{E}Z^{\beta,\omega}_{N,h} - \frac{1}{2}\frac{\mathbb{V}\mathrm{ar}(Z^{\beta,\omega}_{N,h})}{(\mathbb{E}Z^{\beta,\omega}_{N,h})^2} + \cdots, \tag{1.4.4}$$

où on a simplement développé le logarithme au deuxième ordre.

On note $h = h_c^{\mathrm{a}}(\beta) + \Delta$, et on prend Δ petit, dépendant de β. Pour savoir si $\mathtt{F}(\beta, h)$ se comporte comme $\mathtt{F}^{\mathrm{a}}(\beta, h)$, il faut savoir si la fonction de partition quenched $Z_{N,h}^{\omega,\beta}$ reste concentrée autour de sa moyenne, la fonction de partition annealed $Z_{N,h}^{\mathrm{a}}$. Pour cela, on estime $\mathbb{V}\mathrm{ar}(Z_{N,h}^{\omega,\beta})/\mathbb{E}[Z_{N,h}^{\omega,\beta}]^2$, comme suggéré par (1.4.4). Cette quantité diverge toujours quand N croît vers $+\infty$. Cependant on montre qu'en réalité, pour décider si les comportements des énergies libres *quenched* et *annealed* sont différents, il suffit de comparer les termes de (1.4.4), pour N de l'ordre de la longueur de corrélation du modèle annealed. Nous ne détaillons pas ici la définition de la longueur de corrélation du système (voir [Gia08]), mais mentionnons simplement qu'elle est de l'ordre de l'inverse de l'énergie libre (voir [Gia08] pour le modèle homogène, [Ton07] pour des cas particulier du système d'accrochage désordonné). On doit donc estimer $\mathbb{V}\mathrm{ar}(Z_{N,h}^{\omega,\beta})/\mathbb{E}[Z_{N,h}^{\omega,\beta}]^2$ pour $N = 1/\mathtt{F}(0,\Delta) = 1/\mathtt{F}(\Delta)$, et voir si cette quantité peut être rendue beaucoup plus petite que $\log \mathbb{E} Z_{N,h}^{\beta,\omega}$ (qui est d'ordre 1 lorsque $N = 1/\mathtt{F}(\Delta)$), en prenant β et $\Delta = \Delta(\beta)$ petit. Ainsi, au vu de (1.4.4), si le terme $\mathbb{V}\mathrm{ar}(Z_{N,h}^{\omega,\beta})/\mathbb{E}[Z_{N,h}^{\omega,\beta}]^2$ est proche de 0 pour $N = 1/\mathtt{F}(\Delta)$, on devrait avoir $\mathtt{F}(\beta, h) \sim \mathtt{F}^{\mathrm{a}}(\beta, h)$, et cela suggère que le désordre est non-pertinent; dans le cas contraire, cela suggère que le désordre est pertinent.

Pour faire le calcul et simplifier les notations, on se place dans le cas d'un environnement gaussien normalisé, où on a donc $\lambda(\beta) = \beta^2/2$. On obtient après un rapide calcul, en utilisant $h = h_c^{\mathrm{a}}(\beta) + \Delta = -\beta^2/2 + \Delta$,

$$\mathbb{E}\left[(Z_{N,h}^{\omega,\beta})^2\right] = \mathbf{E}^{\otimes 2}\left[\exp\left(\frac{\beta^2}{2}\sum_{n=1}^{N}\delta_n\delta_n' + \Delta\sum_{n=1}^{N}(\delta_n + \delta_n')\right)\delta_N\delta_N'\right], \tag{1.4.5}$$

où $\mathbf{P}^{\otimes 2}$ est la loi jointe de deux processus de renouvellement indépendants τ et τ', et où on a gardé la notation δ_n (resp. δ_n') pour l'indicatrice que n soit un point de renouvellement de τ (resp. de τ').

Comme on prend $N = 1/\mathtt{F}(\Delta)$, on peut montrer que la fonction de partition $Z_{N,\Delta} = Z_{N,h}^{\mathrm{a}}$ est d'ordre $\mathbf{P}(N \in \tau)$, ce qui permet d'obtenir

$$\begin{aligned}\frac{\mathbb{V}\mathrm{ar}(Z_{N,h}^{\omega,\beta})}{\mathbb{E}[Z_{N,h}^{\omega,\beta}]^2} &\geqslant cst. \times \left(\mathbf{E}^{\otimes 2}\left[e^{\frac{\beta^2}{2}\sum_{n=1}^{N}\delta_n\delta_n'}\,\middle|\, N \in \tau\cap\tau'\right] - 1\right)\\ &\geqslant cst. \times \left(e^{c'\beta^2\sum_{n=1}^{N}\mathbf{P}^{\otimes 2}(n\in\tau\cap\tau')} - 1\right) = cst. \times \left(e^{c'\beta^2\sum_{n=1}^{1/\mathtt{F}(\Delta)}\mathbf{P}(n\in\tau)^2} - 1\right),\end{aligned} \tag{1.4.6}$$

où on a utilisé dans un premier temps l'inégalité de Jensen et le fait qu'il existe une constante $c > 0$ telle que $\mathbf{P}^{\otimes 2}(n \in \tau\cap\tau' | N \in \tau\cap\tau') \geqslant c\mathbf{P}^{\otimes 2}(n \in \tau\cap\tau')$, et dans un deuxième temps l'indépendance de τ et τ'.

Si $\sum_{n=1}^{\infty}\mathbf{P}(n\in\tau)^2 = +\infty$, ce qui est le cas pour $\alpha \geqslant 1/2$ (grâce à la Proposition 1.1.8), alors pour n'importe quel β (arbitrairement petit), on prend aussi $\Delta = \Delta(\beta)$ très petit, de sorte que $\beta^2\sum_{n=1}^{1/\mathtt{F}(\Delta)}\mathbf{P}(n\in\tau)^2 \gg 1$. Alors le terme $\mathbb{V}\mathrm{ar}(Z_{N,h}^{\omega,\beta})/\mathbb{E}[Z_{N,h}^{\omega,\beta}]^2$ diverge (et est ainsi beaucoup plus grand que $\log\mathbb{E}Z_{N,h}^{\omega,\beta}$ qui est d'ordre 1) quand $\beta \searrow 0$.

De manière plus précise, avec l'aide de la Proposition 1.1.8, qui donne $\mathbf{P}(n \in \tau) \asymp n^{\alpha\wedge 1-1}$ pour $\alpha \neq 1$, on obtient pour $\alpha \in (1/2, 1)$ (les cas $\alpha > 1$ et $\alpha = 1/2$ étant traités de manière similaire)

$$\sum_{n=1}^{1/\mathtt{F}(\Delta)} \mathbf{P}(n \in \tau)^2 \overset{\Delta\to 0}{\sim} c\mathtt{F}(\Delta)^{1-2\alpha} \sim c'\Delta^{(1-2\alpha)/\alpha},$$

où on a aussi utilisé le Théorème 1.1.6. Si $\alpha \in (1/2, 1)$, alors pour que le terme $\mathbb{V}\mathrm{ar}(Z_{N,h}^{\omega,\beta})/\mathbb{E}[Z_{N,h}^{\omega,\beta}]^2$ diverge quand $\beta \searrow 0$, il faut donc que $\beta^2\Delta^{(1-2\alpha)/\alpha} \gg 1$, *i.e.* $\Delta(\beta) \ll \beta^{2\alpha/(2\alpha-1)}$ (on obtient $\Delta(\beta) \ll \beta^2$ pour $\alpha > 1$, et $\Delta(\beta \ll e^{-\beta^2})$ pour $\alpha = 1/2$). En reprenant (1.4.4), on aurait $h_c(\beta) \geqslant h_c^{\mathrm{a}}(\beta) + \Delta(\beta) > h_c^{\mathrm{a}}(\beta)$, pour tout $\beta > 0$, avec le choix de $\Delta(\beta)$ ci-dessus. Le désordre devrait donc être pertinent dès que $\sum_{n=1}^{\infty} \mathbf{P}(n \in \tau)^2 = +\infty$, c'est-à-dire dès que $\alpha \geqslant 1/2$, et on aurait aussi une borne inférieure sur l'écart entre les points critiques, donné par le choix de $\Delta(\beta)$.

D'autre part, on étudie le cas où $\sum_{n\in\mathbb{N}} \mathbf{P}(n \in \tau)^2 < +\infty$. On obtient à partir de (1.4.5) que

$$\frac{\mathbb{V}\mathrm{ar}(Z_{N,h}^{\omega,\beta})}{\mathbb{E}[Z_{N,h}^{\omega,\beta}]^2} = \mathbf{E}_{N,\Delta}^{\otimes 2}\left[e^{\frac{\beta^2}{2}\sum_{n=1}^{N}\delta_n\delta'_n}\right] - 1, \tag{1.4.7}$$

où $\mathbf{P}_{N,\Delta}^{\otimes 2}$ est la loi produit de deux mesures de polymère dans le cas homogène, avec paramètre Δ ($\mathbf{P}_{N,\Delta}$ est introduite dans la Section 1.1.2).

Avec le choix $N = 1/\mathtt{F}(\Delta)$, on sait que $Z_{N,\Delta}$ est proche de 1, et on se convainc facilement que la mesure de polymère $\mathbf{P}_{N,\Delta}$ est proche de celle originelle $\mathbf{P}$ (dans un certain sens, voir par exemple le Théorème 1.3.6 dans le cas hiérarchique). On conclut que si $\sum_{n\in\mathbb{N}} \mathbf{P}(n \in \tau)^2 < +\infty$, *i.e.* si le renouvellement $\tau \cap \tau'$ est transient, alors pour β tendant vers 0, la quantité $\mathbf{E}_{N,\Delta}^{\otimes 2}\left[e^{\frac{\beta^2}{2}\sum_{n=1}^{N}\delta_n\delta'_n}\right]$ converge vers 1 uniformément en N (par un calcul facile, $|\tau \cap \tau'|$ étant une variable géométrique). Cela suggère que le désordre est non pertinent dès que $\sum_{n=1}^{\infty} \mathbf{P}(n \in \tau)^2 < +\infty$. Ce raisonnement peut en réalité être rendu rigoureux, comme cela est le cas dans [Ale08], ou dans la Section 4.5, dans le cas du modèle hiérarchique.

On possède donc un critère simple, sur la pertinence du désordre:

$$\text{Le désordre est pertinent} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \mathbf{P}(n \in \tau)^2 = +\infty. \tag{1.4.8}$$

Ce critère a été longuement étudié (de manière implicite du côté des physiciens), et il a été montré récemment de manière rigoureuse que si $\sum_{n=1}^{\infty} \mathbf{P}(n \in \tau)^2 < +\infty$ alors le désordre est non-pertinent, et que dans le cas où la fonction à variation lente de l'Hypothèse 1.1.2 converge vers une constante positive, alors le désordre est pertinent pour tout $\alpha \geqslant 1/2$. Ce critère est donc proche d'être complètement démontré, et nous rassemblons maintenant les différents résultats soulignant la pertinence ou la non-pertinence du désordre.

Le premier résultat concerne la non-pertinence du désordre pour $\alpha < 1/2$, et a été obtenu par K. Alexander [Ale08], (une preuve alternative est donnée dans [Ton08b]).

En ce qui concerne la pertinence du désordre, les travaux se sont principalement concentrés sur l'écart entre les points critiques quenched et annealed, en utilisant la méthode des moments fractionnaires citée précédemment (Section 1.2.2), qui permet de se ramener à des estimées de taille finie. Ce travail a été effectué dans une série d'articles, par K. Alexander, N. Zygouras, B. Derrida, G. Giacomin, H. Lacoin et F. Toninelli [Ale08, AZ09, DGLT09] pour le cas $\alpha > 1/2$, puis par G. Giacomin, H. Lacoin et F. Toninelli [GLT10b, GLT11] pour $\alpha = 1/2$.

Mentionnons aussi que D. Cheliotis et F. den Hollander [CdHar] ont obtenu une caractérisation variationnelle de l'énergie libre, en utilisant un principe de grandes déviations pour un processus très général [BGdH10], qui leur permet de retrouver certains résultats déjà évoqués, que l'on a regroupé dans le Théorème suivant.

Théorème 1.4.1. Écart entre les points critiques dans le cas *i.i.d.*

- *Si $\alpha < 1/2$, alors le désordre est non-pertinent.*

Il existe un β_0 (qui dépend de la loi du renouvellement $\mathbf{P}$ et de la loi de l'environnement $\mathbb{P}$) tel que, pour tout $\beta \in (0, \beta_0)$, on ait, pour tout $u \in (0,1)$

$$\begin{aligned} h_c(\beta) = h_c^{\mathrm{a}}(\beta) = -\lambda(\beta) \\ \mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta) + u) \geqslant (1 - \eta(\beta))\mathtt{F}(u), \end{aligned} \tag{1.4.9}$$

où $\eta(\beta)$ tend vers 0 quand β tend vers 0. Rappellons que l'inégalité de Jensen donne aussi $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta) + u) \leqslant \mathtt{F}(u)$.

- *Si $\alpha \geqslant 1/2$, alors le désordre est pertinent.*

Pour tout $\beta > 0$ on a $h_c(\beta) > h_c^{\mathrm{a}}(\beta)$, et pour tout $\varepsilon > 0$, il existe une constante $c > 0$ telle que, pour tout $\beta \in (0,1)$

$$\begin{aligned} &\text{si } \alpha > 1/2 \qquad c\beta^{2 \vee \frac{2\alpha}{2\alpha - 1}} \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant c^{-1}\beta^{2 \vee \frac{2\alpha}{2\alpha-1}}, \\ &\text{si } \alpha = 1/2 \qquad \exp\left(-c(\varepsilon)/\beta^{2+\varepsilon}\right) \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant \exp\left(-c/\beta^2\right). \end{aligned} \tag{1.4.10}$$

Dans le cas $\alpha < 1/2$, et pour β suffisamment petit (*i.e.* à haute température), il n'y a donc aucune modification de l'exposant critique, ni de déplacement du point critique. Dans [Ton08a], F. Toninelli montre que lorsque le désordre est constitué de variables non bornées, il y a un déplacement du point critique quand β est grand (*i.e.* à basse température), pour toute valeur de $\alpha > 0$. Le cas $\alpha = 0$ (avec une fonction à variation lente qui décroît suffisament vite, voir l'Hypothèse 1.1.2) est traité dans [AZ10], où il est montré que les points critiques quenched et annealed sont égaux pour tout $\beta > 0$.

G. Giacomin et F. Toninelli [GT06] ont montré un résultat complémentaire, concernant le comportement critique de l'énergie libre quenched. Sous certaines conditions assez générales sur la loi de l'environnement (se reporter à [Gia07, Th.5.6], un environnement gaussien convient), la présence du désordre rend la transition de phase d'ordre au moins 2: dans le cas d'un environnement gaussien (*i.i.d.*, centré et normalisé), pour tout $\beta > 0$, $\alpha \in [0, \infty)$ et $h \geqslant h_c(\beta)$ on a

$$\mathtt{F}(\beta, h) \leqslant (1+\alpha)(2\beta^2)^{-1}(h - h_c(\beta))^2. \quad (1.4.11)$$

Mentionnons que la constante $(2\beta^2)^{-1}$ dépend de l'environnement considéré, voir [Gia07, Th.5.6].

Ce résultat souligne la pertinence du désordre quand $\alpha > 1/2$, car dans ce cas l'exposant critique *annealed* (ou pur) est $\nu^{\text{pur}} < 2$. Ainsi, l'exposant critique ν^{que} de l'énergie libre *quenched*, s'il existe, est nécessairement différent de celui de l'énergie libre *annealed*, en l'occurence $\nu^{\text{que}} > \nu^{\text{a}} = \nu^{\text{pur}}$. On parle d'un phénomène de *lissage* de la transition de phase par le désordre.

1.4.3. Modèle d'accrochage en environnement corrélé. La direction naturelle à prendre dans l'étude du modèle d'accrochage désordonné est de considérer désormais un environnement non plus *i.i.d.*, mais possédant des corrélations spatiales, de la même manière que dans la Section 1.3.4. Dans les Chapitres 5, 6 et 7, nous nous intéressons donc à une séquence $\omega := (\omega_i)_{i\in\mathbb{N}}$ ergodique, et pour rendre possible l'analyse du problème, nous nous concentrerons sur des cas où ω possède une structure de corrélation particulière.

1.4.3.1. *Critère de Weinrib-Halperin: influence du désordre dans le cas corrélé.* Rappelons ici les prédictions que l'on peut faire concernant la pertinence du désordre, au vu du critère de Weinrib et Halperin [WH83] (cf. Section 0.2.2). On suppose que la fonction de corrélation (à deux points) décroît comme $r^{-\zeta}$, r étant la distance entre les points, $\zeta > 0$. On devrait avoir, dans le cas d'un système unidimensionnel, que le désordre est pertinent si $\nu^{\text{pur}} < 2/(\zeta \wedge 1)$ et non pertinent si $\nu^{\text{pur}} > 2/(\zeta \wedge 1)$. On devrait ainsi observer une modification du critère de Harris à partir du moment où $\zeta > 1$.

Jusqu'à présent, seul le cas d'un environnement gaussien possédant des corrélations à portée finie a été considéré, par J. Poisat dans [Poi11, Poi12]. Il est montré que dans ce cas, le critère de Harris reste valable, à savoir que le désordre est pertinent si $\alpha > 1/2$ et non-pertinent si $\alpha < 1/2$. Nous nous attacherons dans le Chapitre 5 à donner des résultats dans le cas où l'environnement est gaussien avec des corrélations à longue portée (dont l'intensité décroît en puissance), et qui confirment en partie la prédiction de Weinrib-Halperin. Un résultat surprenant est notamment que lorsque les corrélations deviennent trop fortes (*i.e.* quand $\zeta < 1$ pour reprendre les notations précédentes), alors le système désordonné ne possède plus de transition de phase à proprement parler, et reste dans la phase localisée pour tous les paramètres $h \in \mathbb{R}$. Ce phénomène provient du caractère non borné des variables aléatoires ω_i. Nous considérons donc dans les Chapitres 6 et 7 des environnements bornés, afin de conserver la transition de phase, et de pouvoir étudier l'influence de très fortes corrélations sur celle-ci.

1.4.3.2. *Environnement gaussien corrélé.* Il est naturel de considérer dans un premier temps un environnement gaussien, pour rendre certains calculs explicites (comme c'est le cas dans le modèle hiérarchique, cf. Section 1.3.4). Nous étudierons ce cas en détails dans le Chapitre 5, où les résultats obtenus sont regroupés.

On considère une séquence $\omega := \{\omega_n\}_{n \geqslant 0}$ de variables gaussiennes centrées, de variance unitaire. La loi de ω est notée $\mathbb{P}$, caractérisée par la matrice de covariance $\Upsilon := (\Upsilon_{ij})_{i,j \in \mathbb{N}}$, avec $\Upsilon_{ij} = \mathbb{E}[\omega_i \omega_j]$, qui est symétrique définie positive. On fait l'hypothèse naturelle que les covariances Υ_{ij} ne dépendent que de la distance entre i et j, c'est-à-dire $\Upsilon_{ij} = \rho_{|i-j|}$, avec $\rho_0 = 1$. En d'autres termes, on considère $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ un processus gaussien stationnaire centré, caractérisé par sa fonction de corrélation $(\rho_k)_{k \geqslant 0}$, avec $\rho_0 = 1$. On suppose que $\lim_{k\to\infty} |\rho_k| = 0$, de sorte que la suite ω soit ergodique (voir [CFS82, Ch.14 §2, Th.2], [Itô44] pour le théorème original), ainsi l'énergie libre quenched $\mathtt{F}(\beta, h)$ existe, tout comme le point critique quenched $h_c^{\mathrm{que}}(\beta)$, comme annoncé dans la Section 1.4.1.

Hypothèse 1.4.2. Il est en réalité naturel de considérer, comme le suggèrent Weinrib-Halperin, des corrélations qui décroissent en loi de puissance. On suppose que $\rho_k \geqslant 0$ pour tout $k \geqslant 0$, et on fait l'hypothèse qu'il existe un certain $\zeta > 0$ et une constante $c_0 > 0$ telle que

$$\rho_k \overset{k\to\infty}{\sim} c_0 k^{-\zeta}. \tag{1.4.12}$$

Mentionnons que dans la plupart des résultats obtenus dans le cas $\zeta > 1$, cette hypothèse peut être affaiblie, en ne supposant ni la non-négativité des ρ_k, ni la décroissance en loi de puissance, mais simplement la sommabilité des corrélations. On se réfère au Chapitre 5 pour plus de précisions. Dans la suite, on se réfère au cas de corrélations "sommables" si $\zeta > 1$ et "non-sommables" si $\zeta < 1$.

Remarque 1.4.3. Dans le cas d'une suite convexe $(\rho_k)_{k\in\mathbb{N}}$ (qui est en particulier positive, puisque $\rho_k \to 0$), ce qui est par exemple le cas lorsque $\rho_k := (1+k)^{-\zeta}$, il est standard qu'une telle loi gaussienne existe, car c'est une fonction de corrélation de Pólya [Pól49] (cf. aussi [Luk83, Th.1.2.2.]). Nous donnons maintenant une construction explicite d'un processus gaussien avec une telle fonction de corrélation. Soit $\mathcal{I} = \{[a,b] \cap \mathbb{N};\, a, b \in \mathbb{N},\, a \leqslant b\}$, et soit $\{\widehat{\omega}_I\}_{I\in\mathcal{I}}$ une famille de variables gaussiennes *i.i.d.* $\mathcal{N}(0,1)$, de loi notée $\widehat{\mathbb{P}}$. On a alors l'égalité suivante en loi:

$$\omega_i := \sum_{I\in\mathcal{I};\, i\in I} \widehat{\rho}_I \widehat{\omega}_I, \tag{1.4.13}$$

où $\widehat{\rho}_I := \widehat{\rho}_{|I|}$ ne dépend que de la taille de I, et où on a pris $\widehat{\rho}_k := \sqrt{\rho_{k-1} - 2\rho_k + \rho_{k+1}}$ pour tout $k \in \mathbb{N}$ (la racine carrée étant bien définie, grâce à la convexité de la suite $(\rho_k)_{k \geqslant 0}$). On calcule directement (grâce à une sommation par partie) que la famille gaussienne ainsi construite possède la structure de corrélation voulue, la somme dans le membre de droite de (1.4.13) étant bien définie, car $\sum_{k\in\mathbb{N}} k\widehat{\rho}_k^2 = 1 < \infty$.

Il est possible de donner d'autres exemples de séquences gaussiennes dont la fonction de corrélation n'est pas forcément convexe, mais tout de même positive et à décroissance en loi de puissance. Dans [BDZ95], les auteurs considèrent le cas d'un processus gaussien dont la fonction de corrélation est égale à la fonction de Green (donc positive) d'une marche aléatoire X transiente sur $\mathbb{Z}^d$. Ils donnent aussi une construction de marches aléatoires transientes dont les noyaux de transition sont dans le domaine de loi α-stables ($\alpha \in (0, 2\wedge d)$), et montrent qu'alors la fonction de Green est asymptotiquement équivalente à $c_{\alpha,d}|x|^{-d+\alpha}$, où la constante $c_{\alpha,d}$ est explicite (cf. aussi [BD94]).

Le Chapitre 5 est dédié à l'étude de l'influence des corrélations sur le modèle d'accrochage désordonné, et nous rassemblons ici les principaux résultats (bien que partiels) sur ce modèle. Rappelons que le Chapitre 4 correspond à l'étude du modèle hiérarchique dans le cas de corrélations sommables, et permet ainsi de présager des résultats dans le modèle non hiérarchique.

On commence par le cas sommable $\zeta > 1$. On note $Z^{\mathrm{a}}_{N,h} := \mathbb{E}[Z^{\omega,\beta}_{N,h}]$ la fonction de partition annealed, et on montre que sous l'Hypothèse 1.4.2 et si $\zeta > 1$, l'énergie libre annealed définie par $\mathtt{F}^a(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z^{\mathrm{a}}_{N,h}$ existe et est finie pour tout $\beta \geqslant 0, h \in \mathbb{R}$ (voir la Proposition 5.2.1). On a ainsi l'existence d'un point critique annealed $h_c^{\mathrm{a}}(\beta) = \inf\{h, \mathtt{F}^{\mathrm{a}}(\beta,h) > 0\}$, qui sépare (comme d'habitude) une phase délocalisée d'une phase localisée.

De la même manière que dans le modèle hiérarchique, on peut donner une expression explicite de la fonction de partition annealed grâce à la nature gaussienne du désordre, de la même forme que (1.3.19)

$$Z^{\mathrm{a}}_{N,h} = \mathbf{E}\left[\exp\left((\beta^2/2+h)\sum_{n=1}^{N}\delta_n + \beta^2\sum_{n=1}^{N}\delta_n\sum_{k=1}^{N-n}\rho_k\delta_{n+k}\right)\delta_N\right]. \tag{1.4.14}$$

Contrairement au cas *i.d.d.* (cf. Section 1.4.2), le modèle annealed n'est donc pas équivalent au modèle homogène standard de la Section 1.1.2, car des termes $\delta_n\delta_{n+k}$ dus aux corrélations apparaissent. Le modèle annealed est intéressant en soi, en tant que modèle homogène possédant des interactions à longue portée, et le premier résultat concernant l'énergie libre *annealed* est le suivant.

Théorème 1.4.4. *On fixe $\beta > 0$ et on suppose que $\sum_{k\in\mathbb{N}} k|\rho_k| < \infty$. Alors il existe une constante $c > 0$ telle que*

$$\mathtt{F}(c^{-1}u) \leqslant \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta)+u) \leqslant \mathtt{F}(cu) \tag{1.4.15}$$

tant que $\mathtt{F}^{\mathrm{a}}(h_c^{\mathrm{a}}(\beta)+u) \leqslant 1$. On a ainsi $\nu^{\mathrm{a}} = \nu^{\mathrm{pur}} = 1 \vee 1/\alpha$.

Ce Théorème, à rapprocher du Théorème 1.3.6 dans le cas hiérarchique, montre donc que, si $\sum_{k\in\mathbb{N}} k|\rho_k| < \infty$ (*i.e.* $\zeta > 2$), le système annealed possède le même comportement que le système homogène. La difficulté, comme dans le modèle hiérarchique, consiste à contrôler l'effet des interactions à longue portée sur le système annealed. Ce résultat ne donne cependant aucune indication sur le critère de pertinence/non-pertinence du désordre, même si l'étude du système annealed en constitue une première étape.

Soulignons aussi que les résultats obtenus dans le cadre hiérarchique (Section 1.3), bien que n'impliquant rien dans le cas présent, donnent en vertu de la Remarque 1.3.4 une idée précise de ce que l'on peut attendre ici, en utilisant la (supposée) correspondance $\zeta = \log(1/\kappa)/\log 2$. La condition $\sum_{k\in\mathbb{N}} k|\rho_k| < \infty$ correspond au choix $\kappa < 1/4$, et au vu du Théorème 1.3.6 on peut penser que le Théorème 1.4.4 reste valable sous la condition plus faible $\zeta > 2(1 \wedge \alpha) \vee 1$. On peut comparer la Figure 4.1 du Chapitre 4 et la Figure 5.1 du Chapitre 5, pour avoir une idée de la correspondance entre les modèles hiérarchique et non hiérarchique.

Même si l'étude du système annealed reste encore en chantier, nous possédons des résultats sur le système désordonné, notamment un résultat similaire à la Proposition 4.3.5 (qui était dans le cas hiérarchique).

Proposition 1.4.5. *Sous l'Hypothèse 1.4.2 et si $\zeta > 1$, pour tout $\alpha > 0$, tout $\beta > 0$ et $h \in \mathbb{R}$, on a*

$$\mathtt{F}(\beta, h) \leqslant (1+\alpha)(2\beta^2 \Upsilon_\infty)^{-1} \left(h - h_c(\beta)\right)_+^2, \tag{1.4.16}$$

où on a noté $\Upsilon_\infty := \left(1 + 2\sum_{k\in\mathbb{N}} \rho_k\right) \in (0, +\infty)$.

Le système désordonné possède ainsi une transition de phase d'ordre au moins 2, et a donc un comportement critique différent du système homogène dans le cas où $\nu^{\mathrm{pur}} < 2$. La pertinence du désordre est donc montrée pour $\alpha > 1/2$. Par contre, la non-pertinence du désordre pour $\alpha < 1/2$, prédite par le critère de Weinrib-Halperin, est plus difficile à traiter que dans le cas *i.i.d.*, car l'outil principal est la comparaison du système désordonné avec le système annealed, qu'il est très complexe de manipuler dans le cas présent (le cadre hiérarchique permettait une étude plus simple du modèle annealed, cf. Section 1.3.4).

Dans le cas non-sommable, comme pour la Remarque 1.3.5, nous montrons que sous l'Hypothèse 1.4.2, $\zeta < 1$, l'énergie libre annealed est $\mathtt{F}^{\mathrm{a}}(\beta, h) = +\infty$. Nous avons aussi un résultat similaire au Théorème 1.3.9, qui indique que le système désordonné devient lui aussi moins intéressant, car ne possédant aucune transition de phase.

Théorème 1.4.6. *Sous l'Hypothèse 1.4.2 et si $\zeta < 1$, on a $\mathtt{F}(\beta, h) > 0$ pour tout $\beta > 0, h \in \mathbb{R}$, de sorte que $h_c(\beta) = -\infty$. Il existe une constante $c > 0$ telle que pour tout $\beta > 0$ et $h \leqslant -1$*

$$\mathtt{F}(\beta, h) \geqslant \exp\left(-c|h|(|h|/\beta^2)^{1/(1-\zeta)}\right). \tag{1.4.17}$$

Ainsi, lorsque les corrélations sont trop fortes (*i.e.* si $\zeta < 1$), la transition de phase disparaît, et il n'y a donc aucun espoir de vérifier ou d'infirmer le critère de Weinrib-Halperin, qui devrait donner une modification du critère de Harris (cf. Section 1.4.2).

1.4.4. Effet de très fortes corrélations sur la transition de phase. La démonstration du Théorème 1.4.6 (cf. Section 5.2.3), suggère que, lorsque les corrélations sont très fortes, le point critique est $h_c(\beta) = -\beta \operatorname{ess\,sup}(\omega_1)$ (avec la définition $\operatorname{ess\,sup}(\omega_1) = \inf\{a \in \mathbb{R}, \mathbb{P}(\omega_1 > a) = 0\}$). En effet, on a trivialement que $h_c(\beta) \geqslant -\beta \operatorname{ess\,sup}(\omega_1)$, car tous les ω_i sont plus petit que $\operatorname{ess\,sup}(\omega_1)$ (avec probabilité 1), et si de très grandes régions où ω est très proche de $\operatorname{ess\,sup}(\omega_1)$ apparaissent (ce qui arrive lorsque le corrélations sont très fortes), alors la stratégie consistant à ne viser que ces zones permet aux trajectoires d'être localisées dès que $h > -\beta \operatorname{ess\,sup}(\omega_1) + \varepsilon$, pour $\varepsilon > 0$ arbitraire.

Sous l'Hypothèse 1.4.2, on devrait donc observer le comportement suivant: pour $\zeta > 1$, le critère de Harris est vérifié, et lorsque ζ décroît vers 1 (*i.e.* les corrélations augmentent), le point critique $h_c(\beta)$ est poussé vers $-\beta \operatorname{ess\,sup}(\omega_1)$, valeur qu'il atteint lorsque $\zeta < 1$.

Nous nous concentrons, dans les Chapitres 6 et 7 au cas où la séquence $\omega := \{\omega_n\}_{n\in\mathbb{N}}$ (de loi toujours notée $\mathbb{P}$) est constituée de variables bornées ($\operatorname{ess\,sup}(\omega_1) < \infty$), afin de pouvoir observer une transition de phase même lorsque les corrélations sont très fortes (nous préciserons plus tard ce que nous voulons dire par là). Pour des raisons techniques et de notations, on suppose que ω est à valeurs dans $\{-1,0\}^{\mathbb{N}}$. Dans ce cas, les ω_i ne sont ni des variables centrées, ni de variance 1, mais un ajustement des paramètre β et h dans la fonction de partition $Z_{N,h}^{\omega,\beta}$ ($h \mapsto h + \beta\mathbb{E}[\omega_1]$, $\beta \mapsto \mathbb{V}\mathrm{ar}(\omega_1)^{1/2}\,\beta$), permet de se ramener à ce cas.

Le fait que $\omega_i \in \{-1,0\}$ donne un environnement qui n'est constitué que d'éléments neutres ou répulsifs, de sorte que le point critique vérifie $h_c(\beta) \geqslant 0$. Lorsque les corrélations augmentent, le point critique $h_c(\beta)$ est en quelque sorte poussé vers $-\beta \operatorname{ess\,sup}(\omega_1) = 0$ et devient égal à 0 lorsque les corrélations dépassent un certain seuil, comme suggéré précédemment. Plus précisément, s'il y a dans un système de taille N des régions de taille $\gg \log N$ où ω est constant et égal à 0 avec probabilité $\geqslant \varepsilon$ (uniformément en N), le point critique est $h_c(\beta) = 0$. Le fait de connaître la valeur exacte de $h_c(\beta)$ permet une analyse plus poussée du système désordonné, et notamment de ses propriétés critiques.

Cette prédiction est vérifiée dans les Chapitres 6 et 7, et nous sommes en effet capables de donner des bornes (précises dans le Chapitre 6, significatives dans le Chapitre 7) sur l'énergie libre quenched, si les corrélations sont suffisamment fortes (en un sens à préciser). Le résultat le plus surprenant est que, lorsque le point critique est égal à 0, nous arrivons de manière très générale à avoir des bornes sur l'énergie libre. On montre entre autres que, sous certaines conditions (notamment dans le cas d'un environnement naturel basé un processus gaussien, introduit dans un moment), le désordre est toujours pertinent: l'exposant critique quenched est plus grand que celui du système pur (et peut être infini), pour toute les valeurs du paramètre $\alpha > 0$ du processus de renouvellement.

1.4.4.1. *Un premier exemple: un environnement corrélé par blocs.* Le Chapitre 6, basé sur l'article [BL] écrit en collaboration avec H. Lacoin, se concentre sur un environnement particulier, à valeurs dans $\{-1,0\}$, et construit par blocs. La construction est la suivante: on prend $\widehat{\tau} = (\widehat{\tau}_n)_{n\geqslant 0}$, $\widehat{\tau}_0 = 0$ un processus de renouvellement récurrent (de loi notée $\widehat{\mathbf{P}}$), dont la loi inter-arrivée $\widehat{\mathrm{K}}(\cdot)$ vérifie, de manière analogue à l'Hypothèse 1.1.2

$$\widehat{\mathrm{K}}(n) := \widehat{\mathbf{P}}(\widehat{\tau}_1 = n) \overset{n\to\infty}{=} (1+o(1))\frac{\widehat{c}_K}{n^{1+\widetilde{\alpha}}}, \tag{1.4.18}$$

pour un certain $\widetilde{\alpha} > 1$. Une réalisation de $\widehat{\tau}$ donne alors un découpage du système en blocs $[\widehat{\tau}_{i-1}, \widehat{\tau}_i)$, sur lesquels on donne à ω une valeur constante (on notera $\omega \equiv 0$ ou $\omega \equiv -1$). On tire à pile ou face sur chaque bloc la valeur que prend ω:

$$\omega_n = X_i, \quad \forall n \in (\tau_{i-1}, \tau_i], \tag{1.4.19}$$

où $X_i = -1$ avec probabilité $1/2$, et $X_i = 0$ avec probabilité $1/2$ (les $\{X_i\}_{i\in\mathbb{N}}$ formant une famille indépendante). Notons que, sous ces conditions, on a $\widehat{\mathbf{E}}[\widehat{\tau}_1] < \infty$, ce qui est très important pour assurer l'ergodicité de ω, et ainsi le caractère auto-moyennant de l'énergie libre (cf. Chapitre 6).

L'environnement possède donc de très grands blocs où $\omega \equiv 0$, et d'autres où $\omega \equiv -1$, la taille des différents blocs étant $i.i.d$, de loi à queue lourde. Nous arrivons dans ce cadre à montrer que le point critique est $h_c(\beta) = 0$ pour tout $\beta > 0$, et nous contrôlons de manière très précise le comportement critique du système, aussi bien lorsque $h \searrow 0_+$ (notamment concernant l'énergie libre, cf. Théorème 1.4.7), que lorsque $h = 0$ (concernant le nombre de contacts sous la mesure $\mathbf{P}^{\omega,\beta}_{N,h=0}$, cf. Théorème 1.4.8).

Le comportement de l'énergie libre est ainsi connu à une constante près.

Théorème 1.4.7. *Il existe deux constantes $C_1 > 0$ et $C_2 > 0$ (qui dépendent de β) telles que, pour tout $h \in (0,1)$, on ait*

$$C_1 h^{\frac{\widetilde{\alpha}}{(1\wedge\alpha)}} |\log h|^{1-\widetilde{\alpha}} \leqslant \mathtt{F}(\beta,h) \leqslant C_2 h^{\frac{\widetilde{\alpha}}{(1\wedge\alpha)}} |\log h|^{1-\widetilde{\alpha}}. \tag{1.4.20}$$

Nous avons aussi des informations sur la mesure de polymère $\mathbf{P}^{\omega,\beta}_{N,h=0}$, au point critique $h_c(\beta) = 0$.

Théorème 1.4.8. *Pour $\mathbb{P}$-presque tout ω, pour tout $\varepsilon > 0$, il existe un certain $a_0 = a_0(\omega,\beta,\varepsilon) \in \mathbb{R}$ tel que, pour tout $a \geqslant a_0$ et $a \leqslant N^{\frac{(1/\alpha)\wedge\alpha}{\widetilde{\alpha}}-\varepsilon}$, on ait*

$$a^{-\varepsilon-\frac{\widetilde{\alpha}(\alpha+1)-1}{1\wedge\alpha}} \leqslant \mathbf{P}^{\omega,\beta}_{N,h=0}\left(|\tau\cap[0,N]| = a\right) \leqslant a^{\varepsilon-\widetilde{\alpha}(1\vee\alpha)}. \tag{1.4.21}$$

Un conséquence est que la suite de lois $(\nu_N)_{N\geqslant 0}$ sur $\mathbb{N}$ definie par

$$\nu_N(A) := \mathbf{P}^{\omega,\beta}_{N,h=0}(|\tau\cap[0,N]| \in A) \tag{1.4.22}$$

(i.e. les lois du nombre de contacts sous $\mathbf{P}^{\omega,\beta}_{N,h=0}$), est tendue $\mathbb{P}$-p.s.

Ce Théorème indique que sous la mesure $\mathbf{P}^{\omega,\beta}_{N,h=0}$, (*i.e.* la mesure de polymère au point critique), il y a un nombre fini de contact.

Le comportement critique du système désordonné est donc vraiment différent de celui du système pur (se rappeler de la Section 1.1.2), aussi bien en ce qui concerne l'exposant critique, qu'en ce qui concerne le comportement des trajectoires au point critique, et ceci quelle que soit la valeur du paramètre α du processus de renouvellement. Le désordre est ainsi *toujours pertinent*.

Ce phénomène est dû à la présence de très grandes régions où $\omega \equiv 0$: dans le cas présent, pour un système de taille N, la plus grande zone constituée de 0 est de taille d'ordre $N^{1/\widetilde{\alpha}}$, ce qui permet aux trajectoires du polymère d'être localisées. Nous montrons aussi dans le Chapitre 6 que $\mathrm{Cov}(\omega_i,\omega_k) \sim cst.k^{1-\widetilde{\alpha}}$, ce qui souligne que l'on peut observer le régime anormal décrit par le Théorème 1.4.7 même si la fonction de corrélation décroît (relativement) rapidement. Ceci contredit le critère de Weinrib-Halperin, qui prédit que le désordre devraît être pertinent seulement si $\nu^{\mathrm{pur}} < 2/(\zeta\wedge 1)$, dans le cas où la fonction de corrélation décroît comme $r^{-\zeta}$, $\zeta > 0$. On observe ainsi l'apparition d'un comportement nouveau, que l'on appelle *fortement pertinent*, et qui est dû à la présence de très grandes zones favorables (où $\omega \equiv 0$) de taille $\gg \log N$ dans un système de taille N

Nous mentionnons aussi que, lors de la démonstration de ces Théorèmes (cf. Chapitre 6), il ressort une très bonne compréhension du comportement des trajectoires sous la mesure $\mathbf{P}_{N,h}^{\omega,\beta}$, aussi bien pour $h>0$ que pour $h=0$. Pour $h>0$, la stratégie de localisation du polymère consiste à viser les grandes régions où $\omega \equiv 0$ (en l'occurence les régions plus grandes que $|\log h| \mathtt{F}(h)^{-1}$), et à éviter les autres régions, qui ne sont pas assez profitables d'un point de vue énergétique. Pour $h=0$, afin d'avoir a contacts sous $\mathbf{P}_{N,h=0}^{\omega,\beta}$, la meilleure stratégie est de viser une zone assez grande où $\omega \equiv 0$, pour pouvoir y placer les a contacts sans pénalité énergétique. Nous donnons une idée plus précise du comportement des trajectoires dans la Section 6.2.2.

Il s'avère aussi que, malgré la construction particulière de l'environnement, l'étude de ce système désordonné fasse ressortir certaines propriétés importantes de la séquence ω, dues aux corrélations, et qui influent sur le comportement critique du système de manière cruciale. Ce travail est donc une première étape dans la compréhension de systèmes où l'environnement $\omega \in \{-1,0\}^{\mathbb{N}}$ possède une forme plus générale que celle (1.4.19) présentée ci-dessus.

1.4.4.2. *Le cas général d'un environnement corrélé.* Le Chapitre 7 se place donc dans un cadre général: on considère une séquence ω ergodique, à valeurs dans $\{-1,0\}$ et non triviale (*i.e.* $\mathbb{P}(\omega_1=0)>0$, $\mathbb{P}(\omega_1=-1)>0$). Nous donnons alors une condition suffisante sur la séquence ω pour que le point critique soit égal à sa valeur minimale possible $h_c(\beta)=0$ (comme dans le cas où ω serait constant égal à 0).

On définit les blocs où $\omega \equiv 0$ et où $\omega \equiv -1$ de la configuration ω. On considère une séquence $\omega = \{\omega_i\}_{i \geqslant -1}$ (pour simplifier les notations qui suivent), et on conditionne la séquence à avoir $\omega_{-1}=-1, \omega_0=0$, qui est un événement de probabilité positive, et n'affecte donc pas l'énergie libre. On définit les suites $(T_n)_{n\geqslant 0}$, et $(\xi_n)_{n \geqslant 1}$, en posant $T_0 := 0$, et pour tout $n \geqslant 1$

$$\begin{aligned} T_n &:= \inf\{i > T_{n-1}\ ;\ \omega_{i+1} \neq \omega_i\}, \\ \xi_n &:= T_n - T_{n-1} \end{aligned} \tag{1.4.23}$$

Le système est donc divisé en blocs de taille ξ_n, qui sont alternativement constitués de "0" et de "-1". Le conditionnement $\omega_0=-1, \omega_1=0$ permet d'identifier les blocs d'indice impair $(T_{2n}, T_{2n+1}]$ (donc de taille ξ_{2n+1}), comme étant les blocs où $\omega \equiv 0$.

On définit maintenant la variable aléatoire $\mathcal{T}_1(A)$ comme la position de la première zone de taille supérieure à A, où $\omega \equiv 0$,

$$\mathcal{T}_1(A) := \min\{T_{2i+1} \geqslant 0\,;\ i \geqslant 0,\, \xi_{2i+1} \geqslant A\}. \tag{1.4.24}$$

L'idée du Théorème 1.4.7 est de comparer le coût entropique et la récompense énergétique de la stratégie consistant à viser directement une région où $\omega \equiv 0$ de taille $\geqslant A$. En reprenant cette idée, on obtient une condition suffisante pour avoir $h_c(\beta)=0$.

Théorème 1.4.9. *Si* $\liminf_{A\to\infty} \frac{1}{A}\mathbb{E}[\log \mathcal{T}_1(A)] = 0$, *alors pour tout* $\beta > 0$ *on a* $h_c(\beta) = 0$.

Ce Théorème est en réalité beaucoup plus complet dans le Chapitre 7, sous la forme du Théorème 7.3.2, que nous n'avons pas énoncé dans son intégralité pour éviter l'accumulation de détails techniques ou de notations dans cette Introduction. Le Théorème 7.3.2 donne en effet une borne inférieure sur l'énergie libre (que nous présumons être la bonne, au moins dans le cas $\alpha > 1$), ainsi qu'une borne supérieure. Il donne aussi une condition (qui ne rejoint pas la condition du Théorème 1.4.9) sur la séquence ω pour que le point critique soit $h_c(\beta) > 0$. Le Théorème 7.3.2 permet ainsi d'obtenir déjà des résultats assez précis, concernant l'énergie libre ou la valeur du point critique, pour de nombreuses séquences ω à valeur dans $\{-1, 0\}^{\mathbb{N}}$.

Comme dans le cas du Théorème 1.4.7, obtenir une bonne borne supérieure pour l'énergie libre est beaucoup plus difficile. Le raisonnement de la Section 7.4.3 donne un méthode qui pourrait donner une borne très précise pour l'énergie libre, et qui permet aussi de faire la conjecture suivante (voir la Conjecture 7.3.3 pour une version plus complète), que la condition du Théorème 1.4.9 est en réalité un critère décidant si le point critique est égal à 0 ou non.

Conjecture 1.4.10. *Si* $\omega \in \{-1, 0\}^{\mathbb{N}}$ *est ergodique et non triviale, on a l'équivalence entre les deux conditions suivantes*

$$\begin{aligned} &(i) \quad \liminf_{A\to\infty} \frac{1}{A}\mathbb{E}[\log \mathcal{T}_1(A)] > 0, \\ &(ii) \quad h_c(\beta) > 0 \qquad \textit{pour tout } \beta > 0. \end{aligned} \tag{1.4.25}$$

Nous considérons dans le Chapitre 7 quelques applications possibles du Théorème 1.4.9 (ou mieux, de sa version complète, cf. Théorème 7.3.2), notamment dans le cas où la suite $\omega \in \{-1, 0\}^{\mathbb{N}}$ est basée sur le signe d'une séquence gaussienne corrélée. Soit $\mathsf{W} := \{\mathsf{W}_n\}_{n\in\mathbb{N}}$ un processus gaussien stationnaire centré, de loi notée $\mathbb{P}$, de matrice de covariance Υ caractérisée par sa fonction de corrélation $\rho_k := \mathbb{E}[\mathsf{W}_0\mathsf{W}_k]$, avec $\rho_0 = 1$. Nous reprenons les même notations que précédemment, car nous considérons le même objet. On fait de même l'Hypothèse 1.4.2 pour W: $\rho_k \geqslant 0$ pour tout $k \geqslant 0$, et il existe un certain $\zeta > 0$ et une constante $c_0 > 0$ telle que

$$\rho_k \overset{k\to\infty}{\sim} c_0 k^{-\zeta} \tag{1.4.26}$$

La séquence ω est alors simplement basée sur le signe de W_n. En l'occurence, on définit $\omega_i := -\mathbf{1}_{\{\mathsf{W}_i \leqslant 0\}}$, qui donne une séquence ω ergodique (car W est ergodique), de manière très naturelle. Nous nous réfererons à ce choix d'environnement comme étant le cas des *Signes gaussiens*.

Avec un peu de travail, principalement sur des estimées gaussiennes (cf. Appendice A), on obtient à partir du Théorème 7.3.2 le résultat suivant, qui recense deux comportements différents pour le modèle d'accrochage inhomogène, selon que les corrélations soient sommables ($\zeta > 1$) ou non-sommables ($\zeta < 1$).

Théorème 1.4.11. *Dans le cadre d'un environnement de Signes gaussiens, on a*

• *Si* $\zeta < 1$*, alors* $h_c(\beta) = 0$ *pour tout* $\beta > 0$*. Il existe une constante* $c > 0$ *telle que pour tout* $\beta > 0$

$$\mathtt{F}(\beta, h) \geqslant \exp\left(-c|\log h|^{1/(1-\zeta)}\mathtt{F}(h)^{-\zeta/(1-\zeta)}\right). \tag{1.4.27}$$

De plus, pour tout $\beta \in (0,1)$*, il existe un certain* $h_0 > 0$ *et une constante* $c' > 0$ *tels que, pour tout* $h \in (0, h_0)$ *on ait*

$$\mathtt{F}(\beta, h) \leqslant \exp\left(-c' h^{-\zeta}\right). \tag{1.4.28}$$

• *Si* $\zeta > 1$*, alors il existe* $\eta > 0$ *tel que pour tout* $\beta > 0$ *et* $h \in \mathbb{R}$ *on ait* $\mathtt{F}(\beta, h) \leqslant \mathtt{F}^{\mathrm{a}}(\beta, h) \leqslant \mathtt{F}(h - \eta\beta)$*, de sorte que* $h_c(\beta) \geqslant h_c^{\mathrm{a}}(\beta) \geqslant \eta\beta$*.*

Le résultat marquant est que lorsque $\zeta < 1$, la transition de phase est d'ordre infini, quelle que soit la valeur du paramètre α du processus de renouvellement. Nous sommes donc en présence d'un cas où le désordre est *toujours pertinent*. Dans le cadre du modèle d'accrochage sur une ligne de défauts, c'est le premier exemple dont on soit au courant où l'on peut montrer que la transition est d'ordre infini pour le système désordonné (mis à part le cas très spécial où $\alpha = 0$ où le système pur possède déjà une transition C_∞, et où les points critiques quenched et annealed sont toujours égaux [AZ10]).

1.4.5. Y a-t-il violation du critère de Weinrib-Halperin? Dans le cas d'un environnement de *Signes gaussiens* considéré plus haut, *i.e.* où $\omega_i = -\mathbf{1}_{\{\mathsf{W}_i \leqslant 0\}}$, un rapide calcul permet d'obtenir que $\mathbb{C}\mathrm{ov}(\omega_i, \omega_{i+r}) \sim c r^{-\zeta}$ lorsque r est grand (cf. (7.3.15)), avec ζ le même exposant de décroissance que la fonction de corrélation du processus gaussien W sous-jacent (cf. (1.4.26)). En faisant varier ζ le paramètre de décroissance des corrélations, le Théorème 1.4.11 montre que l'on observe une transition lorsque ζ devient plus petit que 1. Si $\zeta > 1$, le point critique est d'ordre β, et le critère de Weinrib-Halperin suggère que l'on devrait avoir un régime similaire au cas *i.i.d.* en ce qui concerne le critère pertinence/non-pertinence du désordre (en analogie avec l'étude faite pour le modèle hiérarchique dans le cas de corrélations sommables, cf. Chapitre 4). D'un autre côté, pour $\zeta < 1$, on a un régime où le point critique est égal à 0, et où l'énergie libre possède une transition d'ordre infini: le désordre est toujours pertinent et la prédiction de Weinrib-Halperin n'est plus valide.

Nous résumons dans le tableau suivant les différents comportements du système en fonction de ζ, passant d'un régime que nous appelons *classique* pour $\zeta > 1$, à un régime que nous appelons *fortement pertinent* pour $\zeta < 1$.

$\zeta > 1$	$\dashrightarrow$	$\zeta < 1$
$h_c(\beta) \asymp c(\zeta)\beta$? Critère de Weinrib-Halperin ?	$h_c(\beta) \to 0$	$h_c(\beta) = 0$ Transition de phase d'ordre ∞ Désordre toujours pertinent

Comme déjà évoqué précédemment, l'apparition d'un nouveau régime provient du fait que certains événements rares, à savoir la présence de très grandes zones où $\omega \equiv 0$, deviennent prépondérants dans le comportement du système (quand $\zeta < 1$ dans le cas des Signes gaussiens). En réalité, la taille de ces zones où $\omega \equiv 0$ ne dépend, en général, pas uniquement de la décroissance de la fonction de corrélation à deux points (ce qui est par contre le cas pour un environnement basé sur une séquence gaussienne), mais de la structure de corrélation globale.

Dans le cas d'un environnement construit par blocs cité plus haut (1.4.19), et étudié dans le Chapitre 6, nous avons par exemple remarqué que la fonction de corrélation décroît comme $r^{-(\widetilde{\alpha}-1)}$ (on aurait donc $\zeta = \widetilde{\alpha} - 1$ pour reprendre les notations de Weinrib-Halperin). Cependant, de très grands blocs où $\omega \equiv 0$ apparaissent (de taille $N^{1/\widetilde{\alpha}}$ dans un système de taille N). Cela montre que le critère de Weinrib-Halperin n'est pas opportun dans ce cas particulier, et en l'occurence nous prouvons que le désordre est toujours pertinent, quelle que soit la valeur de $\zeta = \widetilde{\alpha}-1$. Au vu des Théorèmes 1.4.7 et 1.4.9 et de la Conjecture 1.4.10, ceci permet d'affirmer que la condition pour observer un comportement *fortement pertinent* n'est pas le caractère non-sommable des corrélations, mais l'apparition de régions anormalement grandes où $\omega \equiv 0$ (ou plus généralement de régions où $\omega \equiv \text{ess sup}(\omega_1)$).

Le critère de Weinrib-Halperin est basé principalement sur des raisonnements de type gaussien, où connaître la décroissance de la fonction de corrélation suffit à caractériser le processus (cf. [WH83, Sec. III], où la nature gaussienne du désordre permet le calcul de moments de la fonction de partition $\mathbb{E}[(Z_N)^n]$). Le critère de Weinrib-Halperin n'est ainsi à propos que dans ce cadre, bien qu'il soit souvent cité dans la littérature dans un cadre général. Nous présumons cependant que ce critère soit valable en dehors du régime *fortement pertinent* décrit plus haut.

On conclut que l'effet des corrélations dépend très fortement de la loi du désordre, ce qui rend l'argument de Weinrib et Halperin moins robuste que le critère de Harris, où la nature *i.i.d.* du désordre permet d'adopter un point de vue très général.

Élargissement

Ce travail de thèse a permis de mieux comprendre l'influence du désordre sur le phénomène de localisation, et en particulier lorsque celui-ci possède de très fortes corrélations spatiales. Nous avons par exemple mis en avant le comportement dit *fortement pertinent* du système lorsque des régions favorables très grandes apparaissent, ce qui est le cas lorsque les corrélations ne sont pas sommables dans le cas gaussien, la condition étant différente (cf. Théorème 1.4.9) dans le cas général. Le choix fait dans le Chapitre 7 de considérer un environnement qui est basé sur la séquence des signes d'un processus gaussien est arbitraire, et l'on peut avoir des résultats similaires si l'on considère d'autres séquences à valeurs discrètes.

La question naturelle qui apparaît est celle du seuil à dépasser dans les corrélations afin d'observer ce comportement atypique. La conjecture 1.4.10 propose ainsi un critère qui permet de savoir si la séquence ω possède suffisament de zones favorables à la localisation pour que celle-ci soit effective dès que $h > 0$ (dans le cas d'un environnement $\omega \in \{-1, 0\}^{\mathbb{N}}$).

Ce travail de thèse a donc ouvert de nombreuses perspectives. Par exemple, en ce qui concerne le modèle d'accrochage sur une marche aléatoire, on possède une borne inférieure sur l'écart entre les points critiques en dimension $d \geqslant 3$ (voir Théorème 1.2.4). Nous pensons que cette borne est exactement le bon ordre de grandeur pour $\beta_c^{\mathrm{que}}(\rho) - \beta_c^{\mathrm{a}}(\rho)$, ce qui permettrait d'affirmer que le cas de la dimension $d = 3$ est bien marginal, l'écart entre les points critiques étant exponentiellement petit. Il reste cependant à donner une borne supérieure pour cet écart, la difficulté venant du fait que l'on ne maîtrise pas la quantité $\mathbb{V}\mathrm{ar}(Z_{t,\beta}^{Y})$, dont la connaissance précise donnerait l'écart entre les points critiques, en reprenant les méthodes déjà utilisées pour le modèle d'accrochage sur une ligne de défauts (voir [Ale08, Ton08b], et la Section 4.5). Pour le modèle d'accrochage non hiérarchique sur une ligne de défauts en environnement corrélé, il reste aussi à comprendre complètement le modèle annealed, et à montrer que le critère de Weinrib-Halperin est valable en dehors du régime *fortement pertinent*, notamment en termes d'écart entre les points critiques quenched et annealed.

Nous mentionnons ici que M. Birkner, A. Greven et F. den Hollander [BGdH10] ont trouvé un principe de grandes déviations *quenched* pour un processus de découpage de mots dans une séquence *i.i.d.* de lettres, fortement lié aux modèles que nous considérons ici. D. Cheliotis et F. den Hollander ont utilisé ce principe dans le cadre du modèle d'accrochage désordonné [CdHar], et E. Bolthausen, F. den Hollander et A. A. Opoku l'ont aussi adapté dans le cadre du modèle de copolymère [BdHO11]. Dans les deux cas, les auteurs ont retrouvé certains résultats connus sur la pertinence du désordre dans le cas *i.i.d.*, et leur méthode permet aussi d'obtenir une formule variationnelle pour l'énergie libre et pour les point critiques quenched et annealed, grâce au principe de grandes déviations [BGdH10]. En ce qui concerne le modèle de copolymère, les auteurs de [BdHO11], gâce à cette caractrérisation variationnelle, ont pu améliorer qualitativement les bornes connues sur la pente de la courbe critique.

On peut espérer que ce procédé donne de nombreux autres résultats, car les propriétés essentielles du systèmes sont codées dans le principe variationnel: on peut par exemple penser pouvoir donner un équivalent précis du point critique $h_c(\beta)$ lorsque $\beta \searrow 0$. Cette technique semble très robuste et pourrait aussi s'appliquer dans le cas corrélé, en adaptant le principe de grandes déviations déjà cité.

On peut ouvrir la réflexion de cette thèse à d'autres cadres, par exemple celui du copolymère, jusqu'ici toujours étudié dans un cadre *i.i.d.* Une étude de la dynamique du phénomène d'accrochage d'un polymère sur une ligne de défauts a été entreprise récemment [CLM$^+$11, CMT08], où une dynamique de Glauber est considérée, dans le cas d'un environnement homogène. L'étude du système à l'équilibre donnant de nombreuses informations sur la dynamique hors équilibre, le travail présenté ici peut ainsi donner des outils pour comprendre la dynamique d'un polymère dans le cadre d'un environnement aléatoire, *i.i.d.* ou corrélé.

Part 1

Modèle d'accrochage sur une marche aléatoire

CHAPTER 2

On the critical points in dimension $d = 3$

2.1. Introduction

We consider the Random Walk Pinning Model (we write for short RWPM): the starting point is a zero-drift random walk X on $\mathbb{Z}^d$ ($d \geqslant 1$), whose law is modified by the presence of a second random walk, Y. The trajectory of Y is fixed (quenched disorder) and can be seen as the random medium. The modification of the law of X due to the presence of Y takes the Boltzmann-Gibbs form of the exponential of a certain interaction parameter, β, times the collision local time of X and Y up to time N, $L_N(X, Y) := \sum_{1 \leqslant n \leqslant N} \mathbf{1}_{\{X_n = Y_n\}}$. If β exceeds a certain threshold value β_c^{que}, then for almost every realization of Y the walk X sticks together with Y, in the thermodynamic limit $N \to \infty$. If on the other hand $\beta < \beta_c^{\mathrm{que}}$, then $L_N(X, Y)$ is $o(N)$ for typical trajectories.

Averaging with respect to Y the partition function, one obtains the partition function of the so-called annealed model, whose critical point β_c^{a} is easily computed; a natural question is whether $\beta_c^{\mathrm{que}} \neq \beta_c^{\mathrm{a}}$ or not. In the renormalization group language, this is related to the question whether disorder is *relevant* or not. In an early version of the paper [BGdH10], Birkner *et al.* proved that $\beta_c^{\mathrm{que}} \neq \beta_c^{\mathrm{a}}$ in dimension $d \geqslant 5$. Around the same time, Birkner and Sun [BS10] extended this result to $d = 4$, and also proved that the two critical points *do coincide* in dimensions $d = 1$ and $d = 2$.

The dimension $d = 3$ is the *marginal dimension* in the renormalization group sense, where not even heuristic arguments like the "Harris criterion" (at least its most naive version) can predict whether one has disorder relevance or irrelevance. Our main result here is that quenched and annealed critical points differ also in $d = 3$.

For a discussion of the connection of the RWPM with the "parabolic Anderson model with a single catalyst", and of the implications of $\beta_c^{\mathrm{que}} \neq \beta_c^{\mathrm{a}}$ about the location of the weak-to-strong transition for the directed polymer in random environment, we refer to [BS10, Sec. 1.2 and 1.4].

Our proof is based on the idea of bounding the fractional moments of the partition function, together with a suitable change of measure argument. This technique, originally introduced in [DGLT09, GLT10b, GLT11] for the proof of disorder relevance for the random pinning model with tail exponent $\alpha \geqslant 1/2$, has also proven to be quite powerful in other cases: in the proof of non-coincidence of critical points for the RWPM in dimension $d \geqslant 4$ [BS10], in the proof that "disorder is always strong" for the directed polymer in random environment in dimension $(1+2)$ [Lac09] and finally in the proof that quenched and annealed large deviation functionals for random walks in random environments in two and three dimensions differ [YZ10]. Let us mention that for the random pinning model there is another method, developed by Alexander and Zygouras [AZ09], to prove disorder relevance: however, their method fails in the marginal situation $\alpha = 1/2$ (which corresponds to $d = 3$ for the RWPM).

To guide the reader through this Chapter, let us point out immediately what are the novelties and the similarities of our proof with respect to the previous applications of the fractional moment/change of measure method:

- the change of measure chosen by Birkner and Sun in [BS10] consists essentially in correlating positively each increment of the random walk Y with the next one. Therefore, under the modified measure, Y is more diffusive. The change of measure we use in dimension three has also the effect of correlating positively the increments of Y, but in our case the correlations have long range (the correlation between the i^{th} and the j^{th} increment decays like $|i-j|^{-1/2}$). Another ingredient which was absent in [BS10] and which is essential in $d = 3$ is a coarse-graining step, of the type of that employed in [Ton09, GLT11];
- while the scheme of the proof of our Theorem 2.1.5 has many points in common with that of [GLT11, Th.1.7], here we need new renewal-type estimates (e.g. Lemma 2.3.7) and a careful application of the Local Limit Theorem to prove that the average of the partition function under the modified measure is small (Lemmas 2.3.2 and 2.3.3).

2.1.1. Reminder of the model and of known results. Let $X = \{X_n\}_{n\geqslant 0}$ and $Y = \{Y_n\}_{n\geqslant 0}$ be two independent discrete-time random walks on $\mathbb{Z}^d$, $d \geqslant 1$, starting from 0, and let $\mathbb{P}^X$ and $\mathbb{P}^Y$ denote their respective laws. We make the following assumption:

Assumption 2.1.1. The random walk X is aperiodic. The increments $(X_i - X_{i-1})_{i\geqslant 1}$ are *i.i.d.*, symmetric and have a finite third moment ($\mathbb{E}^X\left[\|X_1\|^3\right] < \infty$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{Z}^d$). Moreover, the covariance matrix of X_1, call it Σ_X, is non-singular.

The same assumptions hold for the increments of Y (in that case, we call Σ_Y the covariance matrix of Y_1).

For $\beta \in \mathbb{R}, N \in \mathbb{N}$ and for a fixed realization of Y, we define a Gibbs transformation of the path measure $\mathbb{P}^X$: this is the polymer path measure $\mathbb{P}^Y_{N,\beta}$, absolutely continuous with respect to $\mathbb{P}^X$, given by

$$\frac{\mathrm{d}\mathbb{P}^Y_{N,\beta}}{\mathrm{d}\mathbb{P}^X}(X) = \frac{1}{Z^Y_{N,\beta}} e^{\beta L_N(X,Y)}\, \mathbf{1}_{\{X_N = Y_N\}}, \tag{2.1.1}$$

where $L_N(X,Y) = \sum_{n=1}^{N} \mathbf{1}_{\{X_n = Y_n\}}$, and where $Z^Y_{N,\beta} = \mathbb{E}^X[e^{\beta L_N(X,Y)}\, \mathbf{1}_{\{X_N = Y_N\}}]$ is the partition function that normalizes $\mathbb{P}^Y_{N,\beta}$ to a probability.

We note that the quantity $\mathbb{V}\mathrm{ar}^Y(Y_1)$ measures in a way the intensity of the disorder, if $\mathbb{V}\mathrm{ar}^Y(Y_1)$ is high then the random walk Y oscillates a lot, whereas if

$\mathbb{V}\mathrm{ar}^Y(Y_1) = 0$ then Y is constant equal to 0, and the RWPM is only the homogeneous pinning model described in Section 1.1.2

The *quenched* free energy of the model is defined by

$$\mathtt{F}(\beta, \mathbb{P}^Y) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\beta}^Y = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}^Y[\log Z_{N,\beta}^Y] \tag{2.1.2}$$

(the existence of the limit and the fact that it is $\mathbb{P}^Y$-almost surely constant and non-negative is proven in [BS10]). We define also the *annealed* partition function $\mathbb{E}^Y[Z_{N,\beta}^Y]$, and the *annealed* free energy: $\mathtt{F}^{\mathrm{a}}(\beta, \mathbb{P}^Y) := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}^Y[Z_{N,\beta}^Y]$.

The properties of $\mathtt{F}^{\mathrm{a}}(\cdot)$ are well known (see Remark 2.1.3), and we have the existence of critical points, for both *quenched* and *annealed* models, see [BS10, Cor. 1.1]:

Definition 2.1.2 (Critical points). *There exist $0 \leqslant \beta_c^{\mathrm{a}} \leqslant \beta_c^{\mathrm{que}}$ depending on the laws of X and Y such that: $\mathtt{F}^{\mathrm{a}}(\beta, \mathbb{P}^Y) = 0$ if $\beta \leqslant \beta_c^{\mathrm{a}}$ and $\mathtt{F}^{\mathrm{a}}(\beta, \mathbb{P}^Y) > 0$ if $\beta > \beta_c^{\mathrm{a}}$; $\mathtt{F}(\beta, \mathbb{P}^Y) = 0$ if $\beta \leqslant \beta_c^{\mathrm{que}}$ and $\mathtt{F}(\beta, \mathbb{P}^Y) > 0$ if $\beta > \beta_c^{\mathrm{que}}$.*

The inequality $\beta_c^{\mathrm{a}} \leqslant \beta_c^{\mathrm{que}}$ comes from a simple application of the Jensen inequality that gives $\mathtt{F}(\beta, \mathbb{P}^Y) \leqslant \mathtt{F}^{\mathrm{a}}(\beta, \mathbb{P}^Y)$. We also recall that the critical points β_c^{que} and β_c^{a} mark the transition from a delocalized to a localized regime, both for disordered and annealed systems (see the physical motivations in Introduction and in Section 1.2.1).

Remark 2.1.3. As was noticed in [BS10] and in Remark 1.2.3, the *annealed* model is just the homogeneous pinning model of Section 1.1.2 (or [Gia07, Chapter 2]), with partition function $\mathbb{E}^Y[Z_{N,\beta}^Y] = \mathbb{E}^{X-Y}\left[\exp\left(\beta \sum_{n=1}^N \mathbf{1}_{\{(X-Y)_n=0\}}\right) \mathbf{1}_{\{(X-Y)_N=0\}}\right]$, which describes the random walk $X - Y$ which receives the reward β each time it hits 0.

Before this work, the following was known about the question of the coincidence of quenched and annealed critical points:

Theorem 2.1.4 ([BS10]). *Assume that X and Y are discrete time symmetric simple random walks on $\mathbb{Z}^d$.*

If $d = 1$ or $d = 2$, the quenched and annealed critical points coincide: $\beta_c = \beta_c^{\mathrm{a}} = 0$.

If $d \geqslant 4$, the quenched and annealed critical points differ: $\beta_c > \beta_c^{\mathrm{a}} > 0$.

In dimension $d \geqslant 5$, the result was also proven (via a very different method, and for more general random walks which include those of Assumption 2.1.1) in an early version of the paper [BGdH10]. The method and result of [BS10] in dimensions $d = 1, 2$ can be easily extended beyond the simple random walk case (keeping zero mean and finite variance). On the other hand, in the case $d \geqslant 4$ new ideas are needed to make the change-of-measure argument of [BS10] work for more general random walks.

Our main result completes this picture, resolving the open case of the critical dimension $d = 3$ (for simplicity, we deal only with the discrete-time model).

Theorem 2.1.5. *Under the Assumption 2.1.1, for $d = 3$, we have $\beta_c^{\mathrm{que}} > \beta_c^{\mathrm{a}}$.*

We point out that the result holds also in the case where X (or Y) is a simple random walk, a case which a priori is excluded by the aperiodicity condition of Assumption 2.1.1; see the Remark 2.1.8. Also, it is possible to modify our change-of-measure argument to prove the non-coincidence of quenched and annealed critical points in dimension $d = 4$ for the general walks of Assumption 2.1.1, thereby extending the result of [BS10]; see Section 2.3.4 for a hint at the necessary steps.

Note. Independently of this work, M. Birkner and R. Sun [BS11] proved Theorem 2.1.5 for the continuous-time model, so that the picture is also complete fore the continuous RWPM, see Theorem 3.1.3.

2.1.2. An essential tool: a renewal-type representation for $Z^Y_{N,\beta}$. From now on, we will assume that $d \geqslant 3$.

As discussed in [BS10, Sec.4], there is a way to represent the partition function $Z^Y_{N,\beta}$ in terms of a renewal process τ; this rewriting makes the model look formally similar to the random pinning model [Gia07].

In order to introduce the representation of [BS10], we need a few definitions.

Definition 2.1.6. *We let*

(1) $p^X_n(x) = \mathbb{P}^X(X_n = x)$ *and* $p^{X-Y}_n(x) = \mathbb{P}^{X-Y}\big((X-Y)_n = x\big)$*;*
(2) $\mathbf{P}$ *be the law of a recurrent renewal* $\tau = \{\tau_0, \tau_1, \ldots\}$ *with* $\tau_0 = 0$*, i.i.d. increments and inter-arrival law given by*

$$\mathrm{K}(n) := \mathbf{P}(\tau_1 = n) = \frac{p^{X-Y}_n(0)}{G^{X-Y}} \quad \text{where} \quad G^{X-Y} := \sum_{n=1}^{\infty} p^{X-Y}_n(0) \tag{2.1.3}$$

(note that $G^{X-Y} < \infty$ *in dimension* $d \geqslant 3$*);*
(3) $z' = (e^\beta - 1)$ *and* $z = z' \, G^{X-Y}$*;*
(4) *for* $n \in \mathbb{N}$ *and* $x \in \mathbb{Z}^d$*,*

$$w(z, n, x) = z \frac{p^X_n(x)}{p^{X-Y}_n(0)}; \tag{2.1.4}$$

(5) $\check{Z}^Y_{N,z} := \frac{z'}{1+z'} Z^Y_{N,\beta}$.

Then, via the binomial expansion of $e^{\beta L_N(X,Y)} = (1+z')^{L_N(X,Y)}$ one gets

$$\begin{aligned}
\check{Z}^Y_{N,z} &= \mathbb{E}^X \left[1 + \sum_{m=1}^{N} \sum_{\tau_0 = 0 < \tau_1 < \ldots < \tau_m = N} (z')^m \prod_{i=1}^{m} \mathbf{1}_{\{X_{\tau_i} = Y_{\tau_i}\}} \right] \\
&= \sum_{m=1}^{N} \sum_{\tau_0 = 0 < \tau_1 < \ldots < \tau_m = N} \prod_{i=1}^{m} \mathrm{K}(\tau_i - \tau_{i-1}) w(z, \tau_i - \tau_{i-1}, Y_{\tau_i} - Y_{\tau_{i-1}}) \\
&= \mathbf{E}\left[W(z, \tau \cap \{0, \ldots, N\}, Y) \mathbf{1}_{N \in \tau} \right],
\end{aligned} \tag{2.1.5}$$

where we defined for any finite increasing sequence $s = \{s_0, s_1, \dots, s_l\}$

$$W(z,s,Y) = \frac{\mathbb{E}^X\left[\prod_{n=1}^l z\mathbf{1}_{\{X_{s_n}=Y_{s_n}\}}\middle| X_{s_0}=Y_{s_0}\right]}{\mathbb{E}^{X-Y}\left[\prod_{n=1}^l \mathbf{1}_{\{X_{s_n}=Y_{s_n}\}}\middle| X_{s_0}=Y_{s_0}\right]} = \prod_{n=1}^{l} w(z, s_n - s_{n-1}, Y_{s_n} - Y_{s_{n-1}}). \tag{2.1.6}$$

We remark that, taking the $\mathbb{E}^Y$-expectation of the weights, we get

$$\mathbb{E}^Y\left[w(z,\tau_n-\tau_{n-1},Y_{\tau_n}-Y_{\tau_{n-1}})\right] = z.$$

Again, we see that the *annealed* partition function is the partition function of a homogeneous pinning model:

$$\check{Z}^{Y,\mathrm{a}}_{N,z} = \mathbb{E}^Y[\check{Z}^Y_{N,z}] = \mathbf{E}\left[z^{R_N}\mathbf{1}_{\{N\in\tau\}}\right], \tag{2.1.7}$$

where we defined $R_N := |\tau \cap \{1,\dots,N\}|$. Since the renewal τ is recurrent, the *annealed* critical point is $z_c^{\mathrm{a}} = 1$.

In the sequel, we will often use the Local Limit Theorem for random walks, that one can find for instance in [DM95, Th.3] (recall that we assumed that the increments of both X and Y have finite second moments and non-singular covariance matrix):

Proposition 2.1.7 (Local Limit Theorem)**.** *Under the Assumption 2.1.1, we get*

$$\mathbb{P}^X(X_n = x) \overset{n\to\infty}{=} \frac{1}{(2\pi n)^{d/2}(\det\Sigma_X)^{1/2}}\exp\left(-\frac{1}{2n}x\cdot\left(\Sigma_X^{-1}x\right)\right) + o(n^{-d/2}), \tag{2.1.8}$$

where $o(n^{-d/2})$ is uniform for $x\in\mathbb{Z}^d$ (we use the notation $x\cdot y$ for the canonical scalar product in $\mathbb{R}^d$).
Moreover, there exists a constant $c>0$ such that for all $x\in\mathbb{Z}^d$ and $n\in\mathbb{N}$

$$\mathbb{P}^X(X_n = x) \leqslant cn^{-d/2}. \tag{2.1.9}$$

Similar statements hold for the walk Y.

In particular, from Proposition 2.1.7 and the definition of $\mathrm{K}(\cdot)$ in (2.1.3), we get $\mathrm{K}(n) \sim c_K n^{-d/2}$ as $n\to\infty$, for some positive c_K. As a consequence, for $d=3$ we get from [Don97, Th.B] that

$$\mathbf{P}(n\in\tau) \overset{n\to\infty}{\sim} \frac{1}{2\pi c_K\sqrt{n}}. \tag{2.1.10}$$

Remark 2.1.8. In Proposition 2.1.7, we supposed that the walk X is aperiodic, which is not the case for the simple random walk. If X is the symmetric simple random walk on $\mathbb{Z}^d$, then [Law96, Prop.1.2.5]

$$\mathbb{P}^X(X_n = x) \overset{n\to\infty}{=} \mathbf{1}_{\{n\leftrightarrow x\}}\frac{2}{(2\pi n)^{d/2}(\det\Sigma_X)^{1/2}}\exp\left(-\frac{1}{2n}x\cdot\left(\Sigma_X^{-1}x\right)\right) + o(n^{-d/2}), \tag{2.1.11}$$

where $o(n^{-d/2})$ is uniform for $x\in\mathbb{Z}^d$, and where $n\leftrightarrow x$ means that n and x have the same parity (so that x is a possible value for X_n). Of course, in this case Σ_X is just $1/d$ times the identity matrix. The statement (2.1.9) also holds.

Via this remark, one can adapt all the computations of the following sections, which are based on Proposition 2.1.7, to the case where X (or Y) is a simple random walk. For simplicity of exposition, we give the proof of Theorem 2.1.5 only in the aperiodic case.

2.1.3. The Harris criterion: heuristic arguments for disorder relevance. The physicist A. B. Harris provides a method [Har74] to decide in a heuristic way whether disorder is relevant or not, for (quite general) systems with *i.i.d.* environment. His argument is robust and applies to a wide class of models, but in our case the environment is not *i.i.d.*, since the position of the random walk Y at step n depends on its position at step $n-1$. However we are able to adapt the idea of Harris to derive a heuristic criterion in our case.

The key point to decide whether one has $\beta_c^{\mathrm{que}} = \beta_c^{\mathrm{a}}$ or not is to know if the *quenched* partition function $Z_{N,\beta}^Y$ stays close to its mean value, for β close or equal to β_c^{a}. We therefore stand at $\beta = \beta_c^{\mathrm{a}}$, and estimate for a small intensity of disorder (*i.e.* $\mathbb{V}\mathrm{ar}^Y(Y_1)$ small) the quantity $\mathbb{V}\mathrm{ar}^Y(Z_{N,\beta_c^{\mathrm{a}}}^Y)$: if it stays bounded, then one should have that $\beta_c^{\mathrm{que}} = \beta_c^{\mathrm{a}}$, and if it diverges one should have that $\beta_c^{\mathrm{que}} > \beta_c^{\mathrm{a}}$.

One already notes that in dimension $d = 1, 2$ one has $\beta_c^{\mathrm{a}} = 0$, and therefore one gets $Z_{N,\beta_c^{\mathrm{a}}}^Y = 1$, and $\mathbb{V}\mathrm{ar}^Y(Z_{N,\beta_c^{\mathrm{a}}}^Y) = 0$, so that disorder should be irrelevant.

We therefore focus on the case of the dimension $d \geqslant 3$, and to simplify our computations, we assume that X and Y are lazy symmetric simple random walks: $p_1^X(0) = 1/2$, $p_1^X(e_k) = p_1^X(-e_k) = 1/(4d)$ and we have some $\rho \in (0,1)$ such that $p_1^Y(0) = 1-\rho$, $p_1^Y(e_k) = p_1^Y(-e_k) = \rho/(2d)$ (where e_k is the k^{th} vector of the canonical base of $\mathbb{R}^d$). We made this choice so that both X and Y are aperiodic, and also to have $\mathbb{V}\mathrm{ar}^Y(Y_1) = \rho$, so that one is able to choose an arbitrary small intensity of disorder by taking ρ small.

We use the renewal-type representation for the partition function defined in Section 2.1.2-(2.1.5), standing at the critical point $z_c^{\mathrm{a}} = 1$, and we therefore have

$$(\check{Z}_{N,z_c^{\mathrm{a}}}^Y)^2 = \mathbf{E}^{\otimes 2}\left[W(1,\tau\cap(0,N],Y)W(1,\tau'\cap(0,N],Y)\mathbf{1}_{\{N\in\tau\cap\tau'\}}\right], \tag{2.1.12}$$

where τ and τ' are two independent renewal processes, with identical laws $\mathbf{P}$ characterized by (2.1.3) ($\mathbf{P}^{\otimes 2}$ denotes the joint law of τ and τ').

We note $\bar{\tau} = \tau\cap\tau'$, which is a recurrent renewal process since one has $\mathbf{E}^{\otimes 2}[|\bar{\tau}\cap(0,N]|] = \sum_{n=1}^N \mathbf{P}(n\in\tau)^2 \overset{N\to\infty}{\to} \infty$: recall the definition (2.1.3) that gives from Theorem 1.1.7 and Proposition 1.1.8, $\mathbf{P}(n\in\tau) \overset{n\to\infty}{\sim} cn^{-1/2}$ in dimension $d=3$, $\mathbf{P}(n\in\tau) \overset{n\to\infty}{\sim} c/\log n$ in dimension $d=4$, and $\mathbf{P}(n\in\tau) \overset{n\to\infty}{\to} c$ in dimension $d \geqslant 5$.

Recalling the definition (2.1.6), we decompose $W(1,\tau\cap(0,N],Y)$ and $W(1,\tau'\cap(0,N],Y)$ according to the points in $\bar{\tau}$, and we obtain a product of weights of excursions of $\bar{\tau}$. We then use the independence of these weights to get that $\mathbb{E}\left[(\check{Z}_{N,z_c^{\mathrm{a}}}^Y)^2\right]$

is equal to

$$\mathbf{E}^{\otimes 2}\left[\prod_{n=1}^{|\bar\tau\cap(0,N]|}\mathbb{E}^Y\left[\prod_{\tau_k\in(\bar\tau_{n-1},\bar\tau_n]} w(1,\Delta\tau_k,\mathrm{D}^Y_{(\tau_k)})\prod_{\tau'_l\in(\bar\tau_{n-1},\bar\tau_n]} w(1,\Delta\tau'_l,\mathrm{D}^Y_{(\tau'_l)})\right]\mathbf{1}_{\{N\in\bar\tau\}}\right]$$
$$=: \mathbf{E}^{\otimes 2}\left[\prod_{n=1}^{|\bar\tau\cap(0,N]|}\mathbb{W}_n\right], \quad (2.1.13)$$

where we defined $\Delta\tau_k := \tau_k-\tau_{k-1}$ and $\mathrm{D}^Y_{(\tau_k)} := Y_{\tau_k}-Y_{\tau_{k-1}}$.

We are left with estimating the weight $\mathbb{W}_n$ of each excursion of $\bar\tau$. First of all, we note that if $|\tau\cap(\bar\tau_{n-1},\bar\tau_n]| = |\tau'\cap(\bar\tau_{n-1},\bar\tau_n]| = 1$, the weight $\mathbb{W}_n$ of the excursion is $\mathbb{E}^Y[w(1,\Delta\bar\tau_n,\mathrm{D}^Y_{(\bar\tau_n)})^2] \geqslant \mathbb{E}^Y[w(1,\Delta\bar\tau_n,\mathrm{D}^Y_{(\bar\tau_n)})]^2 = 1$, and we actually believe that in all cases the weight of an excursion is $\mathbb{W}_n \geqslant 1$, as the calculations below suggest, see (2.1.18).

We also notice that if $\bar\tau_n-\bar\tau_{n-1}=1$, then the weight is equal to (after computation)

$$\mathbb{E}^Y[w(1,1,Y_1)^2] = \frac{\mathbb{E}^Y[p_1^X(Y_1)^2]}{p_1^{X-Y}(0)^2} \overset{\rho\to 0}{=} 1+\frac{\rho}{4d^2}\left(1+4d(d-1)+o(1)\right). \quad (2.1.14)$$

Then, accepting that all the weight $\mathbb{W}_n$ are larger than 1 (we partly justify this in a moment), one would have, for small ρ

$$\mathbb{E}^Y\left[(\check Z^Y_{N,z_c^{\mathrm{a}}})^2\right] \geqslant \mathbf{E}^{\otimes 2}\left[\exp\left(\log(1+cst.\rho)\sum_{n=1}^{|\bar\tau\cap(0,N]|}\mathbf{1}_{\{n-1\in\bar\tau,n\in\bar\tau\}}\right)\mathbf{1}_{\{N\in\bar\tau\}}\right]$$
$$\geqslant \exp\left(cst'.\rho K(1)^2\sum_{n=1}^{N}\mathbf{P}(n\in\bar\tau)\right), \quad (2.1.15)$$

where we used Jensen inequality. Since the renewal process $\bar\tau$ is recurrent, one has that $\mathbb{V}\mathrm{ar}^Y((\check Z^Y_{N,z_c^{\mathrm{a}}})^2) = \mathbb{E}\left[(\check Z^Y_{N,z_c^{\mathrm{a}}})^2\right]-1$ diverges when $N\to\infty$, and disorder should be relevant.

We now justify the fact that $\mathbb{W}_n \geqslant 1$ in any case. We estimate the weight of an excursion of τ, cf. (2.1.13), in the case where $\Delta\tau_k$ is large, and show that it is larger than 1. The case where $\Delta\tau_k$ is not large should be more in the spirit of (2.1.14), and $\mathbb{W}_n$ should also be larger than 1. We use the Local Limit Theorem approximation (see Proposition 2.1.7), using that $\Delta\tau_k$ is large and that $\mathrm{D}^Y_{(\tau_k)}$ is not much larger than $\rho\Delta\tau_k^{1/2}$ (its typical behavior), to get after simplifications, recalling our choice for X and Y

$$w(1,\Delta\tau_k,\mathrm{D}^Y_{(\tau_k)}) := \frac{p^X_{\Delta\tau_n}(\mathrm{D}^Y_{(\tau_k)})}{p^{X-Y}_{\Delta\tau_k}(0)} \simeq (1+2\rho)^{d/2}e^{-\frac{d}{\Delta\tau_k}\|\mathrm{D}^Y_{(\tau_k)}\|^2}. \quad (2.1.16)$$

Then, as we take ρ to be small, an expansion in ρ at the second order gives

$$\prod_{k=1}^{a} w(1, \Delta\tau_k, \mathrm{D}^Y_{(\tau_k)}) \simeq 1 + d\sum_{k=1}^{a}\left(\rho - \frac{\|\mathrm{D}^Y_{(\tau_k)}\|^2}{\Delta\tau_k}\right) + \mathrm{U} \tag{2.1.17}$$

with U of order ρ^2, and such that $\mathbb{E}^Y\mathrm{U} = 0$ (since $\mathbb{E}^Y[w(1, \Delta\tau_k, \mathrm{D}^Y_{(\tau_k)})] = 1$). Hence one has the weight of the first excursion of $\bar{\tau}$ (which is enough by translation invariance), setting $a, a' \in \mathbb{N}$ such that $\tau_a = \tau'_{a'} = \bar{\tau}_1$:

$$\mathbb{W}_1 = \mathbb{E}^Y\left[\prod_{k=1}^{a} w(1, \Delta\tau_k, \mathrm{D}^Y_{(\tau_k)})\prod_{l=1}^{a'} w(1, \Delta\tau'_l, \mathrm{D}^Y_{(\tau'_l)})\right]$$

$$\simeq 1 + d^2\sum_{k=1}^{a}\sum_{l=1}^{a'}\left(\mathbb{E}^y\left[\frac{\|\mathrm{D}^Y_{(\tau_k)}\|^2}{\Delta\tau_k}\frac{\|\mathrm{D}^Y_{(\tau'_l)}\|^2}{\Delta\tau'_l}\right] - \rho^2\right), \tag{2.1.18}$$

and we estimate the last term, using the notation $\Delta_i = Y_i - Y_{i-1}$:

$$\mathbb{E}^Y\left[\|\mathrm{D}^Y_{(\tau_k)}\|^2\|\mathrm{D}^Y_{(\tau'_l)}\|^2\right] = \sum_{i,j=\tau_{k-1}+1}^{\tau_k}\sum_{i',j'=\tau'_{l-1}+1}^{\tau'_l}\mathbb{E}^Y[(\Delta_i\cdot\Delta_j)(\Delta_{i'}\cdot\Delta_{j'})]$$

$$\geqslant \sum_{i=\tau_{k-1}+1}^{\tau_k}\sum_{i'=\tau'_{l-1}+1}^{\tau'_l}\mathbb{E}^Y[\|\Delta_i\|^2\|\Delta_{i'}\|^2] = \rho^2\Delta\tau_k\Delta\tau'_l + \sum_{i\in(\tau_{k-1},\tau_k]\cap(\tau_{l-1},\tau_l]}(\rho-\rho^2), \tag{2.1.19}$$

where we kept in the sum only the terms $i = j, i' = j'$, the only remaining non null ones being for $i = i', j = j'$ or $i = j', j = i'$ and therefore positive. As $\rho - \rho^2 \geqslant 0$, one finally gets from (2.1.18) that the weight of an excursion of $\bar{\tau}$ is larger than 1, that allows us to derive (2.1.15).

We mention that controlling $\mathbb{V}\mathrm{ar}^Y(Z^Y_{N,\beta})$ for β close to β^{a}_c would enable us to develop techniques similar to Section 4.5 (inspired from [Ale08]), and control the difference between the critical points. The above computation should therefore be pushed further, in order to get upper bounds on the critical point shift that match the ones of Theorem 3.1.3.

We note that the case of the dimension $d = 3$ is marginal, in the sense that $\mathbf{E}^{\otimes 2}[|\bar{\tau}\cap(0,N]|]$ grows only logarithmically, so that in view of (2.1.15), the quantity $\mathbb{E}^Y\left[(\check{Z}^Y_{N,z^{\mathrm{a}}_c})^2\right]$ starts to grow for N of order $e^{1/\rho}$, whereas in dimension $d \geqslant 5$ it starts to grow for N of order $1/\rho$ (for N of order $\log(1/\rho)/\rho$ in dimension $d = 4$). This is the sign that in dimension $d = 3$ the difference between the critical points is exponentially small in $1/\rho$, and stresses the marginality of this dimension in the RWPM.

2.2. Main result: the dimension $d = 3$

With the definition $\check{\mathtt{F}}(z) := \lim_{N\to\infty}\frac{1}{N}\log\check{Z}^Y_{N,z}$, to prove Theorem 2.1.5 it is sufficient to show that $\check{\mathtt{F}}(z) = 0$ for some $z > 1$.

2.2.1. The coarse-graining procedure and the fractional moment method. We consider without loss of generality a system of size proportional to $L = \frac{1}{z-1}$ (the coarse-graining length), that is $N = mL$, with $m \in \mathbb{N}$. Then, for $\mathcal{I} \subset \{1, \dots, m\}$, we define

$$Z^{\mathcal{I}}_{z,Y} := \mathbf{E}\big[W(z, \tau \cap \{0, \dots, N\}, Y)\mathbf{1}_{N\in\tau}\mathbf{1}_{E_{\mathcal{I}}}(\tau)\big], \tag{2.2.1}$$

where $E_{\mathcal{I}}$ is the event that the renewal τ intersects the blocks $(B_i)_{i\in\mathcal{I}}$ and only these blocks over $\{1, \dots, N\}$, B_i being the i^{th} block of size L:

$$B_i := \{(i-1)L + 1, \dots, iL\}. \tag{2.2.2}$$

Since the events $E_{\mathcal{I}}$ are disjoint, we can write

$$\check{Z}^Y_{N,z} := \sum_{\mathcal{I}\subset\{1,\dots,m\}} Z^{\mathcal{I}}_{z,Y}. \tag{2.2.3}$$

Note that $Z^{\mathcal{I}}_{z,Y} = 0$ if $m \notin \mathcal{I}$. We can therefore assume $m \in \mathcal{I}$. If we denote $\mathcal{I} = \{i_1, i_2, \dots, i_l\}$ ($l = |\mathcal{I}|$), $i_1 < \dots < i_l$, $i_l = m$, we can express $Z^{\mathcal{I}}_{z,Y}$ in the following way:

$$Z^{\mathcal{I}}_{z,Y} := \sum_{\substack{a_1,b_1\in B_{i_1}\\ a_1\leqslant b_1}} \sum_{\substack{a_2,b_2\in B_{i_2}\\ a_2\leqslant b_2}} \cdots \sum_{a_l\in B_{i_l}} \mathrm{K}(a_1)w(z, a_1, Y_{a_1})Z^z_{a_1,b_1} \cdots \mathrm{K}(a_l - b_{l-1})w(z, a_l - b_{l-1}, Y_{a_l} - Y_{b_{l-1}})Z^z_{a_l,N}, \tag{2.2.4}$$

where

$$Z^z_{j,k} := \mathbf{E}\big[W(z, \tau \cap \{j, \dots, k\}, Y)\mathbf{1}_{k\in\tau}\,|j \in \tau\,\big] \tag{2.2.5}$$

is the partition function between j and k.

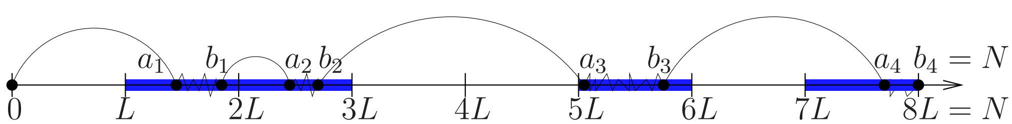

FIGURE 2.1. The coarse-graining procedure. Here $N = 8L$ (the system is cut into 8 blocks), and $\mathcal{I} = \{2, 3, 6, 8\}$ (the gray zones) are the blocks where the contacts occur, and where the change of measure procedure of the Section 2.2.2 acts.

Moreover, thanks to the Local Limit Theorem (Proposition 2.1.7), one can note that there exists a constant $c > 0$ independent of the realization of Y such that, if one takes $z \leqslant 2$ (we will take z close to 1 anyway), one has

$$w(z, \tau_i - \tau_{i-1}, Y_{\tau_i} - Y_{\tau_{i-1}}) = z\frac{p^X_{\tau_i-\tau_{i-1}}(Y_{\tau_i} - Y_{\tau_{i-1}})}{p^{X-Y}_{\tau_i-\tau_{i-1}}(0)} \leqslant c.$$

So, the decomposition (2.2.4) gives

$$Z^{\mathcal{I}}_{z,Y} \leqslant c^{|\mathcal{I}|} \sum_{\substack{a_1,b_1\in B_{i_1}\\ a_1\leqslant b_1}} \sum_{\substack{a_2,b_2\in B_{i_2}\\ a_2\leqslant b_2}} \cdots \sum_{a_l\in B_{i_l}} \mathrm{K}(a_1)Z^z_{a_1,b_1}\mathrm{K}(a_2 - b_1)Z^z_{a_2,b_2} \dots \mathrm{K}(a_l - b_{l-1})Z^z_{a_l,N}. \tag{2.2.6}$$

We now eliminate the dependence on z in the inequality (2.2.6). This is possible thanks to the choice $L = \frac{1}{z-1}$. As each $Z^z_{a_i,b_i}$ is the partition function of a system of size smaller than L, we get $W(z, \tau \cap \{a_i, \dots, b_i\}, Y) \leqslant z^L W(z = 1, \tau \cap \{a_i, \dots, b_i\}, Y)$ (recall the definition (2.1.6)). But with the choice $L = \frac{1}{z-1}$, the factor z^L is bounded by a constant c, and thanks to the equation (2.2.5), we finally get

$$Z^z_{a_i,b_i} \leqslant c Z^{z=1}_{a_i,b_i}. \tag{2.2.7}$$

Notational warning: in what follows, c, c', etc. will denote positive constants, whose value may change from line to line.

We note $Z_{a_i,b_i} := Z^{z=1}_{a_i,b_i}$ and $W(\tau, Y) := W(z = 1, \tau, Y)$. Plugging this in the inequality (2.2.6), we finally get

$$Z^{\mathcal{I}}_{z,Y} \leqslant c'^{|\mathcal{I}|} \sum_{\substack{a_1,b_1 \in B_{i_1} \\ a_1 \leqslant b_1}} \sum_{\substack{a_2,b_2 \in B_{i_2} \\ a_2 \leqslant b_2}} \dots \sum_{a_l \in B_{i_l}} \mathrm{K}(a_1) Z_{a_1,b_1} \mathrm{K}(a_2 - b_1) Z_{a_2,b_2} \dots \mathrm{K}(a_l - b_{l-1}) Z_{a_l,N}, \tag{2.2.8}$$

where there is no dependence on z anymore.

The fractional moment method starts from the observation that for any $\gamma \neq 0$

$$\check{\mathtt{F}}(z) = \lim_{N\to\infty} \frac{1}{\gamma N} \mathbb{E}^Y \left[\log \left(\check{Z}^Y_{N,z}\right)^\gamma\right] \leqslant \liminf_{N\to\infty} \frac{1}{N\gamma} \log \mathbb{E}^Y \left[\left(\check{Z}^Y_{N,z}\right)^\gamma\right]. \tag{2.2.9}$$

Let us fix a value of $\gamma \in (0, 1)$ (as in [GLT11], we will choose $\gamma = 6/7$, but we will keep writing it as γ to simplify the reading). Using the inequality $(\sum a_n)^\gamma \leqslant \sum a_n^\gamma$ (which is valid for $a_i \geqslant 0$), and combining with the decomposition (2.2.3), we get

$$\mathbb{E}^Y \left[\left(\check{Z}^Y_{N,z}\right)^\gamma\right] \leqslant \sum_{\mathcal{I} \subset \{1,\dots,m\}} \mathbb{E}^Y \left[\left(Z^{\mathcal{I}}_{z,Y}\right)^\gamma\right]. \tag{2.2.10}$$

Thanks to (2.2.9) we only have to prove that one has $\limsup_{N\to\infty} \mathbb{E}^Y \left[\left(\check{Z}^Y_{N,z}\right)^\gamma\right] < \infty$, for some $z > 1$,.

We deal with the term $\mathbb{E}^Y \left[(Z^{\mathcal{I}}_{z,Y})^\gamma\right]$ via a change of measure procedure.

2.2.2. The change of measure procedure. The idea is to change the measure $\mathbb{P}^Y$ on each block whose index belongs to $\mathcal{I}$, keeping each block independent of the others. We replace, for fixed $\mathcal{I}$, the measure $\mathbb{P}^Y(\mathrm{d}Y)$ with $g_{\mathcal{I}}(Y)\mathbb{P}^Y(\mathrm{d}Y)$, where the function $g_{\mathcal{I}}(Y)$ will have the effect of creating long range positive correlations between the increments of Y, inside each block separately. Then, thanks to the Hölder inequality, we can write

$$\mathbb{E}^Y \left[\left(Z^{\mathcal{I}}_{z,Y}\right)^\gamma\right] = \mathbb{E}^Y \left[\frac{g_{\mathcal{I}}(Y)^\gamma}{g_{\mathcal{I}}(Y)^\gamma} \left(Z^{\mathcal{I}}_{z,Y}\right)^\gamma\right] \leqslant \mathbb{E}^Y \left[g_{\mathcal{I}}(Y)^{-\frac{\gamma}{1-\gamma}}\right]^{1-\gamma} \mathbb{E}^Y \left[g_{\mathcal{I}}(Y) Z^{\mathcal{I}}_{z,Y}\right]^\gamma. \tag{2.2.11}$$

In the sequel, we will denote $\Delta_i = Y_i - Y_{i-1}$ the i^{th} increment of Y. Let us introduce, for $K > 0$ and ε_K to be chosen, the following "change of measure":

$$g_{\mathcal{I}}(Y) = \prod_{k \in \mathcal{I}} (\mathbf{1}_{F_k(Y) \leqslant K} + \varepsilon_K \mathbf{1}_{F_k(Y) > K}) \equiv \prod_{k \in \mathcal{I}} g_k(Y), \tag{2.2.12}$$

where

$$F_k(Y) = -\sum_{i,j\in B_k} M_{ij}\Delta_i \cdot \Delta_j, \tag{2.2.13}$$

and

$$\begin{cases} M_{ij} = \frac{1}{\sqrt{L\log L}}\frac{1}{\sqrt{|j-i|}} & \text{if } \ i \neq j \\ M_{ii} = 0. \end{cases} \tag{2.2.14}$$

Let us note that from the form of M, we get that $\|M\|^2 := \sum_{i,j\in B_1} M_{ij}^2 \leqslant C$, where the constant $C < \infty$ does not depend on L. We also note that F_k only depends on the increments of Y in the block labeled k.

Let us deal with the first factor of (2.2.11)**:**

$$\begin{aligned}\mathbb{E}^Y\left[g_{\mathcal{I}}(Y)^{-\frac{\gamma}{1-\gamma}}\right] &= \prod_{k\in\mathcal{I}}\mathbb{E}^Y\left[g_k(Y)^{-\frac{\gamma}{1-\gamma}}\right] \\ &= \Big(\mathbb{P}^Y(F_1(Y)\leqslant K) + \varepsilon_K^{-\frac{\gamma}{1-\gamma}}\mathbb{P}^Y(F_1(Y) > K)\Big)^{|\mathcal{I}|}. \end{aligned}\tag{2.2.15}$$

We now choose

$$\varepsilon_K := \mathbb{P}^Y(F_1(Y) > K)^{\frac{1-\gamma}{\gamma}} \tag{2.2.16}$$

such that the first factor in (2.2.11) is bounded by $2^{(1-\gamma)|\mathcal{I}|} \leqslant 2^{|\mathcal{I}|}$. The inequality (2.2.11) finally gives

$$\mathbb{E}^Y\left[\left(Z_{z,Y}^{\mathcal{I}}\right)^\gamma\right] \leqslant 2^{|\mathcal{I}|}\mathbb{E}^Y\left[g_{\mathcal{I}}(Y)Z_{z,Y}^{\mathcal{I}}\right]^\gamma. \tag{2.2.17}$$

The idea is that when $F_1(Y)$ is large, the weight $g_1(Y)$ in the change of measure is small. That is why the following lemma is useful:

Lemma 2.2.1. *We have the following limit:*

$$\lim_{K\to\infty}\varepsilon_K = \lim_{K\to\infty}\mathbb{P}^Y(F_1(Y) > K) = 0 \tag{2.2.18}$$

Proof . We already now that $\mathbb{E}^Y[F_1(Y)] = 0$, so thanks to the standard Tchebyshev inequality, we only have to prove that $\mathbb{E}^Y[F_1(Y)^2]$ is bounded. We get

$$\begin{aligned}\mathbb{E}^Y[F_1(Y)^2] &= \sum_{\substack{i,j\in B_1\\k,l\in B_1}} M_{ij}M_{kl}\mathbb{E}^Y\left[(\Delta_i\cdot\Delta_j)(\Delta_k\cdot\Delta_l)\right] \\ &= \sum_{\{i,j\}=\{k,l\}} M_{ij}^2\mathbb{E}^Y\left[(\Delta_i\cdot\Delta_j)^2\right]\end{aligned}$$

where we used that $\mathbb{E}^Y\left[(\Delta_i\cdot\Delta_j)(\Delta_k\cdot\Delta_l)\right] = 0$ if $\{i,j\} \neq \{k,l\}$. Then, we can use the Cauchy-Schwarz inequality to get

$$\mathbb{E}^Y[F_1(Y)^2] \leqslant \sum_{\{i,j\}=\{k,l\}} M_{ij}^2\mathbb{E}^Y\left[\|\Delta_i\|^2\|\Delta_j\|^2\right] \leqslant \|M\|^2\sigma_Y^4. \tag{2.2.19}$$

□

We are left with the estimation of $\mathbb{E}^Y\left[g_{\mathcal{I}}(Y)Z^{\mathcal{I}}_{z,Y}\right]$. We set $P_{\mathcal{I}} := \mathbf{P}\left(E_{\mathcal{I}}, N\in\tau\right)$, that is the probability for τ to visit the blocks $(B_i)_{i\in\mathcal{I}}$ and only these ones, and to visit also N. We now use the following two statements.

Proposition 2.2.2. *For any $\eta>0$, there exists $z>1$ sufficiently close to 1 (or L sufficiently big, since $L=(z-1)^{-1}$) such that for every $\mathcal{I}\subset\{1,\dots,m\}$ with $m\in\mathcal{I}$, we have*

$$\mathbb{E}^Y\left[g_{\mathcal{I}}(Y)Z^{\mathcal{I}}_{z,Y}\right] \leqslant \eta^{|\mathcal{I}|}P_{\mathcal{I}}. \tag{2.2.20}$$

Proposition 2.2.2 is the core of this Chapter and is proven in the next section.

Lemma 2.2.3. *[GLT11, Lemma 2.4] There exist three constants $C_1=C_1(L)$, C_2 and L_0 such that (with $i_0:=0$)*

$$P_{\mathcal{I}} \leqslant C_1 C_2^{|\mathcal{I}|}\prod_{j=1}^{|\mathcal{I}|}\frac{1}{(i_j-i_{j-1})^{7/5}} \tag{2.2.21}$$

for $L\geqslant L_0$ and for every $\mathcal{I}\in\{1,\dots,m\}$.

Thanks to these two statements and combining with the inequalities (2.2.10) and (2.2.17), we get

$$\mathbb{E}^Y\left[\left(\check{Z}^Y_{N,z}\right)^\gamma\right] \leqslant \sum_{\mathcal{I}\subset\{1,\dots,m\}}\mathbb{E}^Y\left[\left(Z^{\mathcal{I}}_{z,Y}\right)^\gamma\right] \leqslant C_1^\gamma \sum_{\mathcal{I}\subset\{1,\dots,m\}}\prod_{j=1}^{|\mathcal{I}|}\frac{(3C_2\eta)^\gamma}{(i_j-i_{j-1})^{7\gamma/5}}. \tag{2.2.22}$$

Since $7\gamma/5=6/5>1$, we can set

$$\widetilde{K}(n)=\frac{1}{\widetilde{c}n^{6/5}},\qquad\text{where }\ \widetilde{c}=\sum_{i=1}^{+\infty}i^{-6/5}<+\infty, \tag{2.2.23}$$

and $\widetilde{K}(\cdot)$ is the inter-arrival probability of some recurrent renewal $\widetilde{\tau}$. We can therefore interpret the right-hand side of (2.2.22) as a partition function of a homogeneous pinning model of size m (see Figure 2.2), with the underlying renewal $\widetilde{\tau}$, and with pinning parameter $\log[\widetilde{c}(3C_2\eta)^\gamma]$:

$$\mathbb{E}^Y\left[\left(\check{Z}^Y_{N,z}\right)^\gamma\right] \leqslant C_1^\gamma\mathbf{E}_{\widetilde{\tau}}\left[\left(\widetilde{c}(3C_2\eta)^\gamma\right)^{|\widetilde{\tau}\cap\{1,\dots,m\}|}\right]. \tag{2.2.24}$$

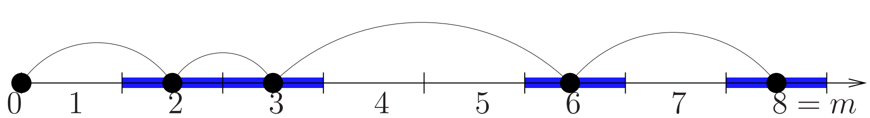

FIGURE 2.2. The underlying renewal $\widetilde{\tau}$ is a subset of the set of blocks $(B_i)_{1\leqslant i\leqslant m}$ (*i.e.* the blocks are reinterpreted as points) and the inter-arrival distribution is $\widetilde{K}(n)=1/\left(\widetilde{c}n^{6/5}\right)$.

Thanks to Proposition 2.2.2, we can take η arbitrary small. Let us fix $\eta := 1/((4C_2)\widetilde{c}^{1/\gamma})$. Then,

$$\mathbb{E}^Y\left[\left(\check{Z}^Y_{N,z}\right)^\gamma\right] \leqslant C_1^\gamma \tag{2.2.25}$$

for every N. This implies, thanks to (2.2.9), that $\check{\mathtt{F}}(z) = 0$, and we are done. $\square$

Remark 2.2.4. The coarse-graining procedure reduced the proof of delocalization to the proof of Proposition 2.2.2. Thanks to the inequality (2.2.8), one has to estimate the expectation, with respect to the $g_{\mathcal{I}}(Y)-$modified measure, of the partition functions Z_{a_i,b_i} in each visited block. We will show (this is Lemma 2.3.1) that the expectation with respect to this modified measure of $Z_{a_i,b_i}/\mathbf{P}(b_i - a_i \in \tau)$ can be arbitrarily small if L is large, and if $b_i - a_i$ is of the order of L. If $b_i - a_i$ is much smaller, we can deal with this term via elementary bounds.

2.3. Proof of the main tool: Proposition 2.2.2

As pointed out in Remark 2.2.4, Proposition 2.2.2 relies on the following key lemma:

Lemma 2.3.1. *For every ε and $\delta > 0$, there exists $L > 0$ such that*

$$\mathbb{E}^Y \left[g_1(Y) Z_{a,b} \right] \leqslant \delta \mathbf{P}(b - a \in \tau) \tag{2.3.1}$$

for every $a \leqslant b$ in B_1 such that $b - a \geqslant \varepsilon L$.

Given this lemma, the proof of Proposition 2.2.2 is very similar to the proof of [GLT11, Prop.2.3], so we will sketch only a few steps. The inequality (2.2.8) gives us

$$\begin{aligned}
&\mathbb{E}^Y \left[g_{\mathcal{I}}(Y) Z_{z,Y}^{\mathcal{I}} \right] \\
&\leqslant c^{|\mathcal{I}|} \sum_{\substack{a_1,b_1 \in B_{i_1} \\ a_1 \leqslant b_1}} \sum_{\substack{a_2,b_2 \in B_{i_2} \\ a_2 \leqslant b_2}} \cdots \sum_{a_l \in B_{i_l}} \mathrm{K}(a_1) \mathbb{E}^Y \left[g_{i_1}(Y) Z_{a_1,b_1} \right] \mathrm{K}(a_2 - b_1) \mathbb{E}^Y \left[g_{i_2}(Y) Z_{a_2,b_2} \right] \cdots \\
&\qquad \cdots \mathrm{K}(a_l - b_{l-1}) \mathbb{E}^Y \left[g_{i_l}(Y) Z_{a_l,N} \right] \\
&= c^{|\mathcal{I}|} \sum_{\substack{a_1,b_1 \in B_{i_1} \\ a_1 \leqslant b_1}} \sum_{\substack{a_2,b_2 \in B_{i_2} \\ a_2 \leqslant b_2}} \cdots \sum_{a_l \in B_{i_l}} \mathrm{K}(a_1) \mathbb{E}^Y \left[g_1(Y) Z_{a_1 - L(i_1-1), b_1 - L(i_1-1)} \right] \mathrm{K}(a_2 - b_1) \ldots \\
&\qquad \cdots \mathrm{K}(a_l - b_{l-1}) \mathbb{E}^Y \left[g_1(Y) Z_{a_l - L(m-1), N - L(m-1)} \right].
\end{aligned} \tag{2.3.2}$$

The terms with $b_i - a_i \geqslant \varepsilon L$ are dealt with via Lemma 2.3.1, while for the remaining ones we just observe that $\mathbb{E}^Y[g_1(Y) Z_{a,b}] \leqslant \mathbf{P}(b - a \in \tau)$ since $g_1(Y) \leqslant 1$. One has then

$$\begin{aligned}
\mathbb{E}^Y \left[g_{\mathcal{I}}(Y) Z_{z,Y}^{\mathcal{I}} \right] \leqslant c^{|\mathcal{I}|} &\sum_{\substack{a_1,b_1 \in B_{i_1} \\ a_1 \leqslant b_1}} \sum_{\substack{a_2,b_2 \in B_{i_2} \\ a_2 \leqslant b_2}} \ldots \sum_{a_l \in B_{i_l}} \mathrm{K}(a_1) \left(\delta + \mathbf{1}_{\{b_1 - a_1 \leqslant \varepsilon L\}} \right) \mathbf{P}(b_1 - a_1 \in \tau) \\
&\cdots \mathrm{K}(a_l - b_{l-1}) \left(\delta + \mathbf{1}_{\{N - a_l \leqslant \varepsilon L\}} \right) \mathbf{P}(N - a_l \in \tau).
\end{aligned} \tag{2.3.3}$$

From this point on, the proof of Theorem 2.2.2 is identical to the proof of Proposition 2.3 in [GLT11] (one needs of course to choose $\varepsilon = \varepsilon(\eta)$ and $\delta = \delta(\eta)$ sufficiently small). $\square$

2.3.1. Proof of Lemma 2.3.1. Let us fix a, b in B_1, such that $b - a \geqslant \varepsilon L$. The small constants δ and ε are also fixed. We recall that for a fixed configuration of τ such that $a, b \in \tau$, we have $\mathbb{E}^Y\big[W(\tau \cap \{a, \dots, b\}, Y)\big] = 1$ because $z = 1$. We can therefore introduce the probability measure (always for fixed τ)

$$\mathrm{d}\mathbb{P}_\tau(Y) = W(\tau \cap \{a, \dots, b\}, Y)\, \mathrm{d}\mathbb{P}^Y(Y) \tag{2.3.4}$$

where we do not indicate the dependence on a and b. Let us note for later convenience that, in the particular case $a = 0$, the definition (2.1.6) of W implies that for any function $f(Y)$

$$\mathbb{E}_\tau[f(Y)] = \mathbb{E}^X \mathbb{E}^Y\big[f(Y) | X_i = Y_i \ \forall i \in \tau \cap \{1, \dots, b\}\big]. \tag{2.3.5}$$

With the definition (2.2.5) of $Z_{a,b} := Z_{a,b}^{z=1}$, we get

$$\begin{aligned} \mathbb{E}^Y\left[g_1(Y) Z_{a,b}\right] &= \mathbb{E}^Y \mathbf{E}\big[g_1(Y) W(\tau \cap \{a, \dots, b\}, Y) \mathbf{1}_{b \in \tau} \,|a \in \tau\,\big] \\ &= \widehat{\mathbf{E}} \mathbb{E}_\tau[g_1(Y)] \mathbf{P}(b - a \in \tau), \end{aligned} \tag{2.3.6}$$

where $\widehat{\mathbf{P}}(\cdot) := \mathbf{P}(\cdot | a, b \in \tau)$, and therefore we have to show that $\widehat{\mathbf{E}} \mathbb{E}_\tau[g_1(Y)] \leqslant \delta$.

With the definition (2.2.12) of $g_1(Y)$, we get that for any K

$$\widehat{\mathbf{E}} \mathbb{E}_\tau[g_1(Y)] \leqslant \varepsilon_K + \widehat{\mathbf{E}} \mathbb{P}_\tau\left(F_1 < K\right). \tag{2.3.7}$$

If we choose K big enough, ε_K is smaller than $\delta/3$ thanks to the Lemma 2.2.1. We now use two lemmas to deal with the second term. The idea is to first prove that $\mathbb{E}_\tau[F_1]$ is big with a $\widehat{\mathbf{P}}-$probability close to 1, and then that its variance is not too large.

Lemma 2.3.2. *For every $\zeta > 0$ and $\varepsilon > 0$, one can find two constants $u = u(\varepsilon, \zeta) > 0$ and $L_0 = L_0(\varepsilon, \zeta) > 0$, such that for every $a, b \in B_1$ such that $b - a \geqslant \varepsilon L$,*

$$\widehat{\mathbf{P}}\left(\mathbb{E}_\tau[F_1] \leqslant u\sqrt{\log L}\right) \leqslant \zeta, \tag{2.3.8}$$

for every $L \geqslant L_0$.

Choose $\zeta = \delta/3$ and fix $u > 0$ such that (2.3.8) holds for every L sufficiently large. If $2K = u\sqrt{\log L}$ (and therefore we can make ε_K small enough by choosing L large), we get that

$$\begin{aligned} \widehat{\mathbf{E}} \mathbb{P}_\tau\left(F_1 < K\right) &\leqslant \widehat{\mathbf{E}} \mathbb{P}_\tau\big[F_1 - \mathbb{E}_\tau[F_1] \leqslant -K\big] + \widehat{\mathbf{P}}\left(\mathbb{E}_\tau[F_1] \leqslant 2K\right) \\ &\leqslant \frac{1}{K^2} \widehat{\mathbf{E}} \mathbb{E}_\tau\left[(F_1 - \mathbb{E}_\tau[F_1])^2\right] + \delta/3. \end{aligned} \tag{2.3.9}$$

Putting this together with (2.3.7) and with our choice of K, we have

$$\widehat{\mathbf{E}} \mathbb{E}_\tau[g_1(Y)] \leqslant 2\delta/3 + \frac{4}{u^2 \log L} \widehat{\mathbf{E}} \mathbb{E}_\tau\left[(F_1 - \mathbb{E}_\tau[F_1])^2\right] \tag{2.3.10}$$

for $L \geqslant L_0$. Then we just have to prove that $\widehat{\mathbf{E}} \mathbb{E}_\tau\left[(F_1 - \mathbb{E}_\tau[F_1])^2\right] = o(\log L)$. Indeed,

Lemma 2.3.3. *For every $\varepsilon > 0$ there exists some constant $c = c(\varepsilon) > 0$ such that*

$$\widehat{\mathbf{E}}\mathbb{E}_\tau\left[(F_1 - \mathbb{E}_\tau[F_1])^2\right] \leqslant c\,(\log L)^{3/4} \tag{2.3.11}$$

for every $L > 1$ and $a, b \in B_1$ such that $b - a \geqslant \varepsilon L$.

We finally get that

$$\widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)] \leqslant 2\delta/3 + c(\log L)^{-1/4}, \tag{2.3.12}$$

and there exists a constant $L_1 > 0$ such that for $L > L_1$

$$\widehat{\mathbf{E}}\mathbb{E}_\tau[g_1(Y)] \leqslant \delta. \tag{2.3.13}$$

□

2.3.2. Proof of Lemma 2.3.2. Up to now, the proof of Theorem 2.1.5 is quite similar to the proof of the main result in [GLT11]. Starting from the present section, instead, new ideas and technical results are needed.

Let us fix a realization of τ such that $a, b \in \tau$ (so that it has a non-zero probability under $\widehat{\mathbf{P}}$) and let us note $\tau \cap \{a, \dots b\} = \{\tau_{R_a} = a, \tau_{R_a+1}, \dots, \tau_{R_b} = b\}$ (recall that $R_n = |\tau \cap \{1, \dots, n\}|$). We observe (just go back to the definition of $\mathbb{P}_\tau$) that, if f is a function of the increments of Y in $\{\tau_{n-1}+1, \dots, \tau_n\}$, g of the increments in $\{\tau_{m-1}+1, \dots, \tau_m\}$ with $R_a < n \neq m \leqslant R_b$, and if h is a function of the increments of Y not in $\{a+1, \dots, b\}$ then

$$\begin{aligned}&\mathbb{E}_\tau\big[f\big(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\big)g\big(\{\Delta_i\}_{i\in\{\tau_{m-1}+1,\dots,\tau_m\}}\big)h\big(\{\Delta_i\}_{i\notin\{a+1,\dots,b\}}\big)\big]\\ &\quad= \mathbb{E}_\tau\big[f\big(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\big)\big]\mathbb{E}_\tau\big[g\big(\{\Delta_i\}_{i\in\{\tau_{m-1}+1,\dots,\tau_m\}}\big)\big]\mathbb{E}^Y\big[h\big(\{\Delta_i\}_{i\notin\{a+1,\dots,b\}}\big)\big],\end{aligned} \tag{2.3.14}$$

and that

$$\begin{aligned}&\mathbb{E}_\tau\big[f\big(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\big)\big]\\ &\qquad= \mathbb{E}^X\mathbb{E}^Y\big[f\big(\{\Delta_i\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\big)|X_{\tau_{n-1}} = Y_{\tau_{n-1}}, X_{\tau_n} = Y_{\tau_n}\big]\\ &\qquad= \mathbb{E}^X\mathbb{E}^Y\big[f\big(\{\Delta_{i-\tau_{n-1}}\}_{i\in\{\tau_{n-1}+1,\dots,\tau_n\}}\big)|X_{\tau_n-\tau_{n-1}} = Y_{\tau_n-\tau_{n-1}}\big].\end{aligned} \tag{2.3.15}$$

We want to estimate $\mathbb{E}_\tau[F_1]$: since the increments Δ_i for $i \in B_1 \setminus \{a+1, \dots, b\}$ are *i.i.d.* and centered (like under $\mathbb{P}^Y$), we have

$$\mathbb{E}_\tau[F_1] := \sum_{i,j=a+1}^{b} M_{ij}\mathbb{E}_\tau[-\Delta_i \cdot \Delta_j]. \tag{2.3.16}$$

Via a time translation, one can always assume that $a = 0$ and we do so from now on.

The key point is the following

Lemma 2.3.4. (1) *If there exists $1 \leqslant n \leqslant R_b$ such that $i, j \in \{\tau_{n-1}+1, \dots, \tau_n\}$, then*

$$\mathbb{E}_\tau[-\Delta_i \cdot \Delta_j] = A(r) \overset{r\to\infty}{\sim} \frac{C_{X,Y}}{r} \tag{2.3.17}$$

where $r = \tau_n - \tau_{n-1}$ (in particular, note that the expectation depends only on r) and $C_{X,Y}$ is a positive constant which depends on $\mathbb{P}^X, \mathbb{P}^Y$;

(2) *otherwise,* $\mathbb{E}_\tau[-\Delta_i \cdot \Delta_j] = 0$.

Proof of Lemma 2.3.4 **Case (2).** Assume that $\tau_{n-1} < i \leqslant \tau_n$ and $\tau_{m-1} < j \leqslant \tau_m$ with $n \neq m$. Thanks to (2.3.14)-(2.3.15) we have that

$$\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}^X \mathbb{E}^Y[\Delta_i | X_{\tau_{n-1}} = Y_{\tau_{n-1}}, X_{\tau_n} = Y_{\tau_n}] \times \mathbb{E}^X \mathbb{E}^Y[\Delta_j | X_{\tau_{m-1}} = Y_{\tau_{m-1}}, X_{\tau_m} = Y_{\tau_m}] \quad (2.3.18)$$

and both factors are immediately seen to be zero, since the laws of X and Y are assumed to be symmetric.

Case (1). Without loss of generality, assume that $n = 1$, so we only have to compute

$$\mathbb{E}^Y \mathbb{E}^X \left[\Delta_i \cdot \Delta_j \,|X_r = Y_r\right]. \quad (2.3.19)$$

where $r = \tau_1$. Let us fix $x \in \mathbb{Z}^3$, and denote $\mathbb{E}^Y_{r,x}[\cdot] = \mathbb{E}^Y[\cdot\,|Y_r = x]$.

$$\begin{aligned}\mathbb{E}^Y[\Delta_i \cdot \Delta_j \,|Y_r = x] &= \mathbb{E}^Y_{r,x}\left[\Delta_i \cdot \mathbb{E}^Y_{r,x}\left[\Delta_j \,|\Delta_i\right]\right] = \mathbb{E}^Y_{r,x}\left[\Delta_i \cdot \frac{x - \Delta_i}{r-1}\right] \\ &= \frac{x}{r-1} \cdot \mathbb{E}^Y_{r,x}\left[\Delta_i\right] - \frac{1}{r-1}\mathbb{E}^Y_{r,x}\left[\|\Delta_i\|^2\right] \\ &= \frac{1}{r-1}\left(\frac{\|x\|^2}{r} - \mathbb{E}^Y_{r,x}\left[\|\Delta_1\|^2\right]\right), \quad (2.3.20)\end{aligned}$$

where we used the fact that under $\mathbb{P}^Y_{r,x}$ the law of the increments $\{\Delta_i\}_{i \leqslant r}$ is exchangeable. Then, we get

$$\begin{aligned}\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] &= \mathbb{E}^X \mathbb{E}^Y\left[\Delta_i \cdot \Delta_j \mathbf{1}_{\{Y_r = X_r\}}\right] \mathbb{P}^{X-Y}(Y_r = X_r)^{-1} \\ &= \mathbb{E}^X\Big[\mathbb{E}^Y\left[\Delta_i \cdot \Delta_j \,|Y_r = X_r\right] \mathbb{P}^Y(Y_r = X_r)\Big] \mathbb{P}^{X-Y}(Y_r = X_r)^{-1} \\ &= \frac{1}{r-1}\Bigg(\mathbb{E}^X\left[\frac{\|X_r\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right] \mathbb{P}^{X-Y}(Y_r = X_r)^{-1} \\ &\quad -\mathbb{E}^X \mathbb{E}^Y\left[\|\Delta_1\|^2 \mathbf{1}_{\{Y_r = X_r\}}\right] \mathbb{P}^{X-Y}(Y_r = X_r)^{-1}\Bigg) \\ &= \frac{1}{r-1}\left(\mathbb{E}^X\left[\frac{\|X_r\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right] \mathbb{P}^{X-Y}(Y_r = X_r)^{-1} - \mathbb{E}^X \mathbb{E}^Y\left[\|\Delta_1\|^2 \,|Y_r = X_r\right]\right).\end{aligned}$$

Next, we study the asymptotic behavior of $A(r)$ and we prove (2.3.17) with $C_{X,Y} = tr(\Sigma_Y) - tr\left((\Sigma_X^{-1} + \Sigma_Y^{-1})^{-1}\right)$. Note that $tr(\Sigma_Y) = \mathbb{E}^Y(||Y_1||^2) := \sigma_Y^2$. The fact that $C_{X,Y} > 0$ is just a consequence of the fact that, if A and B are two positive-definite matrices, one has that $A - B$ is positive definite if and only if $B^{-1} - A^{-1}$ is [HJ85, Cor. 7.7.4(a)].

To prove (2.3.17), it is enough to show that

$$\mathbb{E}^X \mathbb{E}^Y\left[\|\Delta_1\|^2 \,|Y_r = X_r\right] \overset{r\to\infty}{\to} \mathbb{E}^X \mathbb{E}^Y\left[\|\Delta_1\|^2\right] = \sigma_Y^2, \quad (2.3.21)$$

and that

$$B(r) := \frac{\mathbb{E}^X\left[\frac{\|X_r\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right]}{\mathbb{P}^{X-Y}(X_r = Y_r)} \overset{r\to\infty}{\to} tr\left((\Sigma_X^{-1} + \Sigma_Y^{-1})^{-1}\right). \quad (2.3.22)$$

To prove (2.3.21), write

$$\begin{aligned}\mathbb{E}^X\mathbb{E}^Y\left[\|\Delta_1\|^2 \,|Y_r = X_r\right] &= \mathbb{E}^Y\left[\|\Delta_1\|^2\,\mathbb{P}^X(X_r = Y_r)\right]\mathbb{P}^{X-Y}(X_r = Y_r)^{-1}\\ &= \sum_{y,z\in\mathbb{Z}^d}\|y\|^2\mathbb{P}^Y(Y_1 = y)\frac{\mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y+z)}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)}. \quad (2.3.23)\end{aligned}$$

Thanks to the Local Limit Theorem (Proposition 2.1.7), the term $\frac{\mathbb{P}^X(X_r=y+z)}{\mathbb{P}^{X-Y}(X_r-Y_r=0)}$ is uniformly bounded from above, and so there exist a constant $c > 0$ such that for all $y \in \mathbb{Z}^d$

$$\sum_{z\in\mathbb{Z}^d}\frac{\mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y+z)}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)} \leqslant c. \quad (2.3.24)$$

If we can show that for every y fixed $\mathbb{Z}^3$, this term goes to 1 as r goes to infinity, then from (2.3.23), a dominated convergence argument would give that

$$\mathbb{E}^X\mathbb{E}^Y\left[\|\Delta_1\|^2\,|Y_r = X_r\right] \overset{r\to\infty}{\longrightarrow} \sum_{y\in\mathbb{Z}^d}\|y\|^2\mathbb{P}^Y(Y_1 = y) = \sigma_Y^2. \quad (2.3.25)$$

We are now left with proving that the right term of (2.3.24) goes to 1 as r goes to infinity for any fixed $y \in \mathbb{Z}^d$.

We use the Local Limit Theorem to get

$$\begin{aligned}&\sum_{z\in\mathbb{Z}^d}\mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y+z)\\ &\quad= \sum_{z\in\mathbb{Z}^d}\frac{c_Xc_Y}{r^d}e^{-\frac{1}{2(r-1)}z\cdot\left(\Sigma_Y^{-1}z\right)}e^{-\frac{1}{2r}(y+z)\cdot\left(\Sigma_X^{-1}(y+z)\right)} + o(r^{-d/2})\\ &\quad= (1+o(1))\sum_{z\in\mathbb{Z}^d}\frac{c_Xc_Y}{r^d}e^{-\frac{1}{2}z\cdot\left(\Sigma_Y^{-1}z\right)}e^{-\frac{1}{2r}z\cdot\left(\Sigma_X^{-1}z\right)} + o(r^{-d/2}), \quad (2.3.26)\end{aligned}$$

where $c_X = (2\pi)^{-d/2}(\det\Sigma_X)^{-1/2}$ and similarly for c_Y (the constants are different in the case of simple random walks: see Remark 2.1.8), and where we used that y is fixed to neglect $y/\sqrt{r}$.

Using the same reasoning, we also have (with the same constants c_X and c_Y)

$$\begin{aligned}\mathbb{P}^{X-Y}(X_r = Y_r) &= \sum_{z\in\mathbb{Z}^3}\mathbb{P}^Y(Y_r = z)\mathbb{P}^X(X_r = z)\\ &= \sum_{z\in\mathbb{Z}^d}\frac{c_Xc_Y}{r^d}e^{-\frac{1}{2r}z\cdot\left(\Sigma_Y^{-1}z\right)}e^{-\frac{1}{2r}z\cdot\left(\Sigma_X^{-1}z\right)} + o(r^{-d/2}). \quad (2.3.27)\end{aligned}$$

Putting this together with (2.3.26) (and recalling that $\mathbb{P}^{X-Y}(X_r = Y_r) \sim c_{X,Y} r^{-d/2}$), we have, for every $y \in \mathbb{Z}^d$

$$\sum_{z\in\mathbb{Z}^d} \frac{\mathbb{P}^Y(Y_{r-1} = z)\mathbb{P}^X(X_r = y+z)}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)} \overset{r\to\infty}{\longrightarrow} 1. \tag{2.3.28}$$

To deal with the term $B(r)$ in (2.3.22), we apply the Local Limit Theorem as in (2.3.27) to get

$$\mathbb{E}^X\left[\frac{\|X_r\|^2}{r}\mathbb{P}^Y(Y_r = X_r)\right] = \frac{c_Y c_X}{r^d}\sum_{z\in\mathbb{Z}^d}\frac{\|z\|^2}{r} e^{-\frac{1}{2r}z\cdot\left(\Sigma_Y^{-1}z\right)} e^{-\frac{1}{2r}z\cdot\left(\Sigma_X^{-1}z\right)} + o(r^{-d/2}). \tag{2.3.29}$$

Together with (2.3.27), we finally get

$$B(r) = \frac{\sum_{z\in\mathbb{Z}^d}\frac{\|z\|^2}{r}e^{-\frac{1}{2r}z\cdot\left((\Sigma_Y^{-1}+\Sigma_X^{-1})z\right)} + o(r^{-d/2})}{\sum_{z\in\mathbb{Z}^d} e^{-\frac{1}{2r}z\cdot\left((\Sigma_Y^{-1}+\Sigma_X^{-1})z\right)} + o(r^{-d/2})} = (1+o(1))E\left[\|\mathcal{N}\|^2\right], \tag{2.3.30}$$

where $\mathcal{N} \sim \mathcal{N}\left(0, (\Sigma_Y^{-1} + \Sigma_X^{-1})^{-1}\right)$ is a centered Gaussian vector of covariance matrix $(\Sigma_Y^{-1} + \Sigma_X^{-1})^{-1}$. Therefore, $E\left[\|\mathcal{N}\|^2\right] = tr\left((\Sigma_Y^{-1} + \Sigma_X^{-1})^{-1}\right)$ and (2.3.22) is proven. □

Remark 2.3.5. For later purposes, we remark that with the same method one can prove that, for any polynomials U and V such that $\mathbb{E}^Y[\|U(\{\|\Delta_k\|\}_{k\leqslant k_0})\|] < \infty$, we have

$$\mathbb{E}^X\mathbb{E}^Y\left[U(\{\|\Delta_k\|\}_{k\leqslant k_0})\, V\left(\frac{\|X_r\|}{\sqrt{r}}\right)\middle| Y_r = X_r\right] \overset{r\to\infty}{\to} \mathbb{E}^Y\left[U(\{\|\Delta_k\|\}_{k\leqslant k_0})\right] E\left[V(\|\mathcal{N}\|)\right], \tag{2.3.31}$$

where $\mathcal{N}$ is as in (2.3.30).

Let us now quickly sketch the proof: as in (2.3.23), we can write

$$\mathbb{E}^X\mathbb{E}^Y\left[U(\{\|\Delta_k\|\}_{k\leqslant k_0})\, V\left(\frac{\|X_r\|}{\sqrt{r}}\right)\middle| Y_r = X_r\right] = \sum_{y_1,\ldots,y_{k_0}\in\mathbb{Z}^d} U(\{\|y_k\|\}_{k\leqslant k_0}) \sum_{z\in\mathbb{Z}^d} V\left(\frac{\|z\|}{\sqrt{r}}\right)\mathbb{P}^X(X_r = z)\frac{\mathbb{P}^Y(Y_{r-k_0} = z - y_1 - \ldots - y_{k_0})}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)}. \tag{2.3.32}$$

Then using the Local Limit Theorem the same way as in (2.3.27), one can show that for any $y_1, \ldots, y_{k_0}$, we get similarly as (2.3.30)

$$\sum_{z\in\mathbb{Z}^d} V\left(\frac{\|z\|}{\sqrt{r}}\right)\mathbb{P}^X(X_r = z)\frac{\mathbb{P}^Y(Y_{r-k_0} = z - y_1 - \ldots - y_{k_0})}{\mathbb{P}^{X-Y}(X_r - Y_r = 0)} \overset{r\to\infty}{\to} E\left[V(\|\mathcal{N}\|)\right]. \tag{2.3.33}$$

Using a uniform bound for $\mathbb{P}^Y(Y_{r-k_0} = z - y_1 - \ldots - y_{k_0})$ and $\mathbb{P}^{X-Y}(X_r - Y_r = 0)$, we also see that this term is uniformly bounded for $y_1, \ldots, y_{k_0} \in \mathbb{Z}^d$ by some constant times $\mathbb{E}^X\left[V\left(\frac{\|X_r\|}{\sqrt{r}}\right)\right]$. Then, as for (2.3.21), we get the result thanks to a dominated convergence argument.

Given Lemma 2.3.4, we can resume the proof of Lemma 2.3.2, and lower bound the average $\mathbb{E}_\tau[F_1]$. Recalling (2.3.16) and the fact that we reduced to the case $a=0$, we get

$$\mathbb{E}_\tau[F_1] = \sum_{n=1}^{R_b}\left(\sum_{\tau_{n-1}<i,j\leqslant\tau_n} M_{ij}\right)A(\Delta\tau_n), \tag{2.3.34}$$

where $\Delta\tau_n := \tau_n-\tau_{n-1}$. Using the definition (2.2.14) of M, we see that there exists a constant $c>0$ such that for $1<m\leqslant L$

$$\sum_{i,j=1}^m M_{ij} \geqslant \frac{c}{\sqrt{L\log L}} m^{3/2}. \tag{2.3.35}$$

On the other hand, thanks to Lemma 2.3.4, there exists some $r_0>0$ and two constants c and c' such that $A(r)\geqslant\frac{c}{r}$ for $r\geqslant r_0$, and $A(r)\geqslant -c'$ for every r. Plugging this into (2.3.34), one gets

$$\begin{aligned}\sqrt{L\log L}\,\mathbb{E}_\tau[F_1] &\geqslant c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n}\mathbf{1}_{\{\Delta\tau_n\geqslant r_0\}} - c'\sum_{n=1}^{R_b}(\Delta\tau_n)^{3/2}\mathbf{1}_{\{\Delta\tau_n\leqslant r_0\}}\\ &\geqslant c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n} - c' R_b.\end{aligned} \tag{2.3.36}$$

Therefore, we get for any positive $B>0$ (independent of L)

$$\begin{aligned}\widehat{\mathbf{P}}\left(\mathbb{E}_\tau[F_1]\leqslant g\sqrt{\log L}\right) &\leqslant \widehat{\mathbf{P}}\left(\frac{1}{\sqrt{L\log L}}\left(c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n}-c'\,R_b\right)\leqslant u\sqrt{\log L}\right)\\ &\leqslant \widehat{\mathbf{P}}\left(\frac{1}{\sqrt{L\log L}}\left(c\sum_{n=1}^{R_b}\sqrt{\Delta\tau_n}-c'\sqrt{L}B\right)\leqslant u\sqrt{\log L}\right)+\widehat{\mathbf{P}}\left(R_b>B\sqrt{L}\right)\\ &\leqslant \widehat{\mathbf{P}}\left(\sum_{n=1}^{R_{b/2}}\sqrt{\Delta\tau_n}\leqslant(1+o(1))\frac{u}{c}\sqrt{L}\log L\right)+\widehat{\mathbf{P}}(R_b>B\sqrt{L}).\end{aligned} \tag{2.3.37}$$

Now we show that for B large enough, and $L\geqslant L_0(B)$,

$$\widehat{\mathbf{P}}(R_b>B\sqrt{L})\leqslant\zeta/2, \tag{2.3.38}$$

where ζ is the constant which appears in the statement of Lemma 2.3.2. We start with getting rid of the conditioning in $\widehat{\mathbf{P}}$ (recall $\widehat{\mathbf{P}}(\cdot)=\mathbf{P}(\cdot|b\in\tau)$ since we reduced to the case $a=0$). If $R_b>B\sqrt{L}$, then either $|\tau\cap\{1,\dots,b/2\}|$ or $|\tau\cap\{b/2+1,\dots,b\}|$ exceeds $\frac{B}{2}\sqrt{L}$. Since both random variables have the same law under $\widehat{\mathbf{P}}$, we have

$$\widehat{\mathbf{P}}(R_b>B\sqrt{L})\leqslant 2\widehat{\mathbf{P}}\left(R_{b/2}>\frac{B}{2}\sqrt{L}\right)\leqslant 2c\mathbf{P}\left(R_{b/2}>\frac{B}{2}\sqrt{L}\right), \tag{2.3.39}$$

where in the second inequality we applied Lemma 2.A.1. Now, we can use the Lemma 2.A.3 in the Appendix, to get that (recall $b \leqslant L$)

$$\mathbf{P}\left(R_{b/2} > \frac{B}{2}\sqrt{L}\right) \leqslant \mathbf{P}\left(R_{L/2} > \frac{B}{2}\sqrt{L}\right) \stackrel{L\to\infty}{\to} \mathbf{P}\left(\frac{|\mathcal{Z}|}{\sqrt{2\pi}} \geqslant B\frac{c_K}{\sqrt{2}}\right), \qquad (2.3.40)$$

with $\mathcal{Z}$ a standard Gaussian random variable and c_K the constant such that $\mathrm{K}(n) \sim c_K n^{-3/2}$. The inequality (2.3.38) then follows for B sufficiently large, and $L \geqslant L_0(B)$.

We are left to prove that for L large enough and u small enough

$$\widehat{\mathbf{P}}\left(\sum_{n=1}^{R_{b/2}} \sqrt{\Delta\tau_n} \leqslant \frac{u}{c}\sqrt{L}\log L\right) \leqslant \zeta/2. \qquad (2.3.41)$$

The conditioning in $\widehat{\mathbf{P}}$ can be eliminated again via Lemma 2.A.1. Next, one notes that for any given $A > 0$ (independent of L)

$$\mathbf{P}\left(\sum_{n=1}^{R_{b/2}} \sqrt{\Delta\tau_n} \leqslant \frac{u}{c}\sqrt{L}\log L\right) \leqslant \mathbf{P}\left(\sum_{n=1}^{A\sqrt{L}} \sqrt{\Delta\tau_n} \leqslant \frac{u}{c}\sqrt{L}\log L\right) + \mathbf{P}\left(R_{b/2} < A\sqrt{L}\right). \qquad (2.3.42)$$

Thanks to the Lemma 2.A.3 in Appendix and to $b \geqslant \varepsilon L$, we have

$$\limsup_{L\to\infty} \mathbf{P}\left(\frac{R_{b/2}}{\sqrt{L}} < A\right) \leqslant \mathbf{P}\left(\frac{|\mathcal{Z}|}{\sqrt{2\pi}} < A c_K \sqrt{\frac{2}{\varepsilon}}\right),$$

which can be arbitrarily small if $A = A(\varepsilon)$ is small enough, for L large. We now deal with the other term in (2.3.42), using the exponential Bienaymé-Chebyshev inequality (and the fact that the $\Delta\tau_n$ are *i.i.d.*):

$$\mathbf{P}\left(\frac{1}{\sqrt{L\log L}} \sum_{n=1}^{A\sqrt{L}} \sqrt{\Delta\tau_n} < \frac{u}{c}\sqrt{\log L}\right) \leqslant e^{(u/c)\sqrt{\log L}}\mathbf{E}\left[\exp\left(-\sqrt{\frac{\tau_1}{L\log L}}\right)\right]^{A\sqrt{L}}. \qquad (2.3.43)$$

To estimate this expression, we remark that, for L large enough,

$$\begin{aligned}\mathbf{E}\left[1-\exp\left(-\sqrt{\frac{\tau_1}{L\log L}}\right)\right] &= \sum_{n=1}^{\infty} \mathrm{K}(n)\left(1-e^{-\sqrt{\frac{n}{L\log L}}}\right) \\ &\geqslant c' \sum_{n=1}^{\infty} \frac{1-e^{-\sqrt{\frac{n}{L\log L}}}}{n^{3/2}} \geqslant c''\sqrt{\frac{\log L}{L}}, \end{aligned} \qquad (2.3.44)$$

where the last inequality follows from keeping only the terms with $n \leqslant L$ in the sum, and noting that in this range $1 - e^{-\sqrt{\frac{n}{L\log L}}} \geqslant c\sqrt{n/(L\log L)}$. Therefore,

$$\mathbf{E}\left[\exp\left(-\sqrt{\frac{\tau_1}{L\log L}}\right)\right]^{A\sqrt{L}} \leqslant \left(1 - c''\sqrt{\frac{\log L}{L}}\right)^{A\sqrt{L}} \leqslant e^{-c'' A\sqrt{\log L}}, \qquad (2.3.45)$$

and, plugging this bound in the inequality (2.3.43), we get

$$\mathbf{P}\left(\frac{1}{\sqrt{L\log L}}\sum_{n=1}^{A\sqrt{L}}\sqrt{\Delta\tau_n}\leqslant\frac{u}{c}\sqrt{\log L}\right)\leqslant e^{[(u/c)-c''A]\sqrt{\log L}}, \tag{2.3.46}$$

that goes to 0 if $L\to\infty$, provided that u is small enough. This concludes the proof of Lemma 2.3.2. □

2.3.3. Proof of Lemma 2.3.3. We can write

$$-F_1+\mathbb{E}_\tau[F_1]=S_1+S_2:=\sum_{i\neq j=a+1}^{b}M_{ij}D_{ij}+\sum_{i\neq j}{}^{'}M_{ij}D_{ij} \tag{2.3.47}$$

where we denoted

$$D_{ij}=\Delta_i\cdot\Delta_j-\mathbb{E}_\tau[\Delta_i\cdot\Delta_j] \tag{2.3.48}$$

and $\sum'$ stands for the sum over all $1\leqslant i\neq j\leqslant L$ such that either i or j (or both) do not fall into $\{a+1,\dots,b\}$. This way, we have to estimate

$$\begin{aligned}\mathbb{E}_\tau[(F_1-\mathbb{E}_\tau[F_1])^2]&\leqslant 2\mathbb{E}_\tau[S_1^2]+2\mathbb{E}_\tau[S_2^2]\\&=2\sum_{i\neq j=a+1}^{b}\sum_{k\neq l=a+1}^{b}M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]+2\sum_{i\neq j}{}^{'}\sum_{k\neq l}{}^{'}M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}].\end{aligned} \tag{2.3.49}$$

Remark 2.3.6. We easily deal with the part of the sum where $\{i,j\}=\{k,l\}$. In fact, we trivially bound $\mathbb{E}_\tau\left[(\Delta_i\cdot\Delta_j)^2\right]\leqslant\mathbb{E}_\tau\left[\|\Delta_i\|^2\|\Delta_j\|^2\right]$. Suppose for instance that $\tau_{n-1}<i\leqslant\tau_n$ for some $R_a<n\leqslant R_b$: in this case, the Remark 2.3.5 tells that $\mathbb{E}_\tau\left[\|\Delta_i\|^2\|\Delta_j\|^2\right]$ converges to $\mathbb{E}^Y[\|\Delta_1\|^2\|\Delta_2\|^2]=\sigma_Y^4$ as $\tau_n-\tau_{n-1}\to\infty$. If, on the other hand, $i\notin\{a+1,\dots,b\}$, we know that $\mathbb{E}_\tau\left[\|\Delta_i\|^2\|\Delta_j\|^2\right]$ equals exactly $\mathbb{E}^Y\left[\|\Delta_1\|^2\right]\mathbb{E}_\tau\left[\|\Delta_j\|^2\right]$ which is also bounded. As a consequence, we have the following inequality, valid for every $1\leqslant i,j\leqslant L$:

$$\mathbb{E}_\tau\left[(\Delta_i\cdot\Delta_j)^2\right]\leqslant c \tag{2.3.50}$$

and then

$$\sum_{i\neq j=1}^{L}\sum_{\{k,l\}=\{i,j\}}M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]\leqslant c\sum_{i\neq j=1}^{L}M_{ij}^2\leqslant c' \tag{2.3.51}$$

since the Hilbert-Schmidt norm of M was chosen to be finite.

Upper bound on $\mathbb{E}_\tau[S_2^2]$. This is the easy part, and this term will be shown to be bounded even without taking the average over $\widehat{\mathbf{P}}$.

We have to compute $\sum'_{i\neq j}\sum'_{k\neq l}M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]$. Again, thanks to (2.3.14)-(2.3.15), we have $\mathbb{E}_\tau[D_{ij}D_{kl}]\neq 0$ only in the following case (recall that thanks to Remark 2.3.6 we can disregard the case $\{i,j\}=\{k,l\}$):

$$i=k\notin\{a+1,\dots,b\}\text{ and }\tau_{n-1}<j\neq l\leqslant\tau_n\text{ for some }R_a<n\leqslant R_b. \tag{2.3.52}$$

One should also consider the cases where i is interchanged with j and/or k with l. Since we are not following constants, we do not keep track of the associated combinatorial factors. Under the assumption (2.3.52), $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}_\tau[\Delta_i \cdot \Delta_l] = 0$ (cf. (2.3.14)) and we will show that

$$\mathbb{E}_\tau[D_{ij} D_{il}] = \mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)] \leqslant \frac{c}{r} \tag{2.3.53}$$

where $r = \tau_n - \tau_{n-1} =: \Delta\tau_n$. Indeed, using (2.3.14)-(2.3.15), we get

$$\begin{aligned}\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)] &= \sum_{\nu,\mu=1}^{3} \mathbb{E}^Y[\Delta_i^{(\nu)} \Delta_i^{(\mu)}] \mathbb{E}^X \mathbb{E}^Y [\Delta_{j-\tau_{n-1}}^{(\nu)} \Delta_{l-\tau_{n-1}}^{(\mu)} | X_{\tau_n - \tau_{n-1}} = Y_{\tau_n - \tau_{n-1}}] \\ &= \sum_{\nu,\mu=1}^{3} \Sigma_Y^{\nu\mu}\, \mathbb{E}^X \mathbb{E}^Y \left[\Delta_{j-\tau_{n-1}}^{(\nu)} \Delta_{l-\tau_{n-1}}^{(\mu)} | X_r = Y_r\right]. \end{aligned} \tag{2.3.54}$$

In the remaining expectation, we assume without loss of generality that $\tau_{n-1} = 0$, $\tau_n = r$. Like for instance in the proof of (2.3.17), one writes

$$\mathbb{E}^X \mathbb{E}^Y \left[\Delta_j^{(\nu)} \Delta_l^{(\mu)} | X_r = Y_r\right] = \frac{\mathbb{E}^X \left[\mathbb{E}^Y \left[\Delta_j^{(\nu)} \Delta_l^{(\mu)} | Y_r = X_r\right] \mathbb{P}^Y(Y_r = X_r)\right]}{\mathbb{P}^{X-Y}(X_r = Y_r)} \tag{2.3.55}$$

and

$$\mathbb{E}^Y \left[\Delta_j^{(\nu)} \Delta_l^{(\mu)} \middle| Y_r = X_r\right] = \frac{1}{r(r-1)} X_r^{(\nu)} X_r^{(\mu)} - \frac{1}{r-1} \mathbb{E}^Y[\Delta_j^{(\nu)} \Delta_j^{(\mu)} | Y_r = X_r]. \tag{2.3.56}$$

An application of the Local Limit Theorem like in (2.3.21), (2.3.22) then leads to (2.3.53).

We are now able to bound

$$\begin{aligned}\mathbb{E}_\tau\left[S_2^2\right] &= c \sum_{i \notin \{a+1,\dots,b\}} \sum_{n=R_a+1}^{R_b} \sum_{\tau_{n-1} < j \neq l \leqslant \tau_n} M_{ij} M_{il} \mathbb{E}_\tau[D_{ij} D_{il}] \\ &\leqslant \frac{c}{L \log L} \sum_{i \notin \{a+1,\dots,b\}} \sum_{n=R_a+1}^{R_b} \sum_{\tau_{n-1} < j,l \leqslant \tau_n} \frac{1}{\sqrt{|i-j|}} \frac{1}{\sqrt{|i-l|}} \frac{1}{\Delta\tau_n}. \end{aligned} \tag{2.3.57}$$

Assume for instance that $i > b$ (the case $i \leqslant a$ can be treated similarly):

$$\begin{aligned}&\frac{c}{L \log L} \sum_{i>b} \sum_{n=R_a+1}^{R_b} \sum_{\tau_{n-1} < j,l \leqslant \tau_n} \frac{1}{\sqrt{i-j}} \frac{1}{\sqrt{i-l}} \frac{1}{\Delta\tau_n} \\ &\leqslant \frac{c}{L \log L} \sum_{i>b} \sum_{n=R_a+1}^{R_b} \sum_{\tau_{n-1} < j,l \leqslant \tau_n} \frac{1}{(i-\tau_n)\Delta\tau_n} \leqslant \frac{c}{L \log L} (b-a) \sum_{i=1}^{L} \frac{1}{i} \leqslant c'. \end{aligned} \tag{2.3.58}$$

Upper bound on $\mathbb{E}_\tau[S_1^2]$. Thanks to time translation invariance, one can reduce to the case $a = 0$. We have to distinguish various cases (recall Remark 2.3.6: we assume that $\{i, j\} \neq \{k, l\}$).

(1) Assume that $\tau_{n-1} < i, j \leqslant \tau_n$, $\tau_{m-1} < k, l \leqslant \tau_m$, with $1 \leqslant n \neq m \leqslant R_b$. Then, thanks to (2.3.14), we get $\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}_\tau[D_{ij}]\mathbb{E}_\tau[D_{kl}] = 0$, because $\mathbb{E}_\tau[D_{ij}] = 0$. For similar reasons, one has that $\mathbb{E}_\tau[D_{ij}D_{kl}] = 0$ if one of the indexes, say i, belongs to one of the intervals $\{\tau_{n-1}+1, \ldots, \tau_n\}$, and the other three do not.

(2) Assume that $\tau_{n-1} < i, j, k, l \leqslant \tau_n$ for some $n \leqslant R_b$. Using (2.3.15), we have

$$\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}^Y\mathbb{E}^X\left[D_{ij}D_{kl}\,\middle|\,X_{\tau_{n-1}} = Y_{\tau_{n-1}}, X_{\tau_n} = Y_{\tau_n}\right],$$

and with a time translation we can reduce to the case $n = 1$ (we call $\tau_1 = r$). Thanks to the computation of $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j]$ in Section 2.3.2, we see that $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}_\tau[\Delta_k \cdot \Delta_l] = -A(r)$ so that

$$\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)] - A(r)^2 \leqslant \mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)]. \tag{2.3.59}$$

(a) If $i = k$, $j \neq l$ (and $\tau_{n-1} < i, j, l \leqslant \tau_n$ for some $n \leqslant R_b$), then

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)] \leqslant \frac{c}{\Delta\tau_n}. \tag{2.3.60}$$

The computations are similar to those we did in Section 2.3.2 for the computation of $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j]$. See Appendix 2.A.1 for details.

(b) If $\{i, j\} \cap \{k, l\} = \emptyset$ (and $\tau_{n-1} < i, j, k, l \leqslant \tau_n$ for some $n \leqslant R_b$), one gets

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)] \leqslant \frac{c}{(\Delta\tau_n)^2}. \tag{2.3.61}$$

See Appendix 2.A.2 for a (sketch of) the proof, which is analogous to that of (2.3.60).

(3) The only remaining case is that where $i \in \{\tau_{n-1}+1, \ldots, \tau_n\}$, $j \in \{\tau_{m-1}+1, \ldots, \tau_m\}$ with $m \neq n \leqslant R_b$, and each of these two intervals contains two indexes in i, j, k, l. Let us suppose for definiteness $n < m$ and $k \in \{\tau_{n-1}+1, \ldots, \tau_n\}$. Then $\mathbb{E}_\tau[\Delta_i \cdot \Delta_j] = \mathbb{E}_\tau[\Delta_k \cdot \Delta_l] = 0$ (cf. Lemma 2.3.4), and $\mathbb{E}_\tau[D_{ij}D_{kl}] = \mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)]$. We will prove in Appendix 2.A.3 that

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_k \cdot \Delta_l)] \leqslant \frac{c}{\Delta\tau_n\Delta\tau_m} \tag{2.3.62}$$

and that

$$\mathbb{E}_\tau[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)] \leqslant \frac{c}{\Delta\tau_m}. \tag{2.3.63}$$

We are now able to compute $\mathbb{E}_\tau[S_1^2]$. We consider first the contribution of the terms whose indexes i, j, k, l are all in the same interval $\{\tau_{n-1}+1, \ldots, \tau_n\}$, *i.e.*

case (2) above. Recall that we drop the terms $\{i, j\} = \{k, l\}$ (see Remark 2.3.6):

$$\sum_{\substack{\tau_{n-1}<i,j,k,l\leqslant\tau_n\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}] \leqslant \frac{c}{\Delta\tau_n}\sum_{\substack{l\in\{i,j\}\ \text{or}\ k\in\{i,j\}\\ \tau_{n-1}<i,j,k,l\leqslant\tau_n}} M_{ij}M_{kl} + \frac{c}{\Delta\tau_n^2}\sum_{\substack{\{i,j\}\cap\{k,l\}=\emptyset\\ \tau_{n-1}<i,j,k,l\leqslant\tau_n}} M_{ij}M_{kl}$$

$$\leqslant \frac{c'}{L\log L}\left[\frac{1}{\Delta\tau_n}\sum_{1\leqslant i<j<k\leqslant\Delta\tau_n}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{k-j}} + \frac{1}{\Delta\tau_n^2}\left(\sum_{1\leqslant i<j\leqslant\Delta\tau_n}\frac{1}{\sqrt{j-i}}\right)^2\right]$$

$$\leqslant \frac{c''}{L\log L}\Delta\tau_n. \quad (2.3.64)$$

Altogether, we see that

$$\sum_{i\neq j=1}^{b}\sum_{\substack{k\neq l=1\\ \{i,j\}\neq\{k,l\}}}^{b} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]\mathbf{1}_{\{\exists n\leqslant R_b: i,j\in\{\tau_{n-1}+1,\ldots,\tau_n\}\}}$$

$$= \sum_{n=1}^{R_b}\sum_{\substack{\tau_{n-1}<i,j,k,l\leqslant\tau_n\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}] \leqslant \frac{c}{L\log L}\sum_{n=1}^{R_b}\Delta\tau_n \leqslant \frac{c}{\log L}. \quad (2.3.65)$$

Finally, we consider the contribution to $\mathbb{E}_\tau[S_1^2]$ coming from the terms of point (3). We have (recall that $n < m$)

$$\sum_{\substack{\tau_{n-1}<i,k\leqslant\tau_n\\ \tau_{m-1}<j,l\leqslant\tau_m\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}] \leqslant \frac{c}{L\log L}\frac{1}{\Delta\tau_n\Delta\tau_m}\sum_{\substack{\tau_{n-1}<i\neq k\leqslant\tau_n\\ \tau_{m-1}<j\neq l\leqslant\tau_m}}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{l-k}}$$

$$+ \frac{c}{L\log L}\frac{1}{\Delta\tau_n}\sum_{\substack{\tau_{n-1}<i\neq k\leqslant\tau_n\\ \tau_{m-1}<j\leqslant\tau_m}}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{j-k}} + \frac{c}{L\log L}\frac{1}{\Delta\tau_m}\sum_{\substack{\tau_{n-1}<i\leqslant\tau_n\\ \tau_{m-1}<j\neq l\leqslant\tau_m}}\frac{1}{\sqrt{j-i}}\frac{1}{\sqrt{l-i}}. \quad (2.3.66)$$

But as $j > \tau_{m-1}$

$$\sum_{\tau_{n-1}<i\leqslant\tau_n}\frac{1}{\sqrt{j-i}} \leqslant \sum_{\tau_{n-1}<i\leqslant\tau_n}\frac{1}{\sqrt{\tau_{m-1}-i+1}} \leqslant c\left(\sqrt{\tau_{m-1}-\tau_{n-1}} - \sqrt{\tau_{m-1}-\tau_n}\right), \quad (2.3.67)$$

and as $k \leqslant \tau_n$

$$\sum_{\tau_{m-1}<l\leqslant\tau_m}\frac{1}{\sqrt{l-k}} \leqslant \sum_{\tau_{m-1}<l\leqslant\tau_m}\frac{1}{\sqrt{l-\tau_n}} \leqslant c\left(\sqrt{\tau_m-\tau_n} - \sqrt{\tau_{m-1}-\tau_n}\right), \quad (2.3.68)$$

so that

$$\sum_{\substack{\tau_{n-1}<i,k\leqslant\tau_n\\ \tau_{m-1}<j,l\leqslant\tau_m\\ \{i,j\}\neq\{k,l\}}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]$$

$$\leqslant \frac{c}{L\log L}\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right), \quad (2.3.69)$$

where we noted $T_{nm}=\tau_{m-1}-\tau_n$. Recalling (2.3.65) and the definition (2.3.49) of S_1, we can finally write

$$\widehat{\mathbf{E}}\left[\mathbb{E}_\tau[S_1^2]\right] \leqslant c\left(1+\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-1}\sum_{n<m\leqslant R_b}\sum_{\substack{\tau_{n-1}<i,k\leqslant\tau_n\\ \tau_{m-1}<j,l\leqslant\tau_m}} M_{ij}M_{kl}\mathbb{E}_\tau[D_{ij}D_{kl}]\right]\right)$$

$$\leqslant c+\frac{c}{L\log L}\widehat{\mathbf{E}}\left[\sum_{1\leqslant n<m\leqslant R_b}\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right)\right].$$

The remaining average can be estimated via the following Lemma.

Lemma 2.3.7. *There exists a constant $c>0$ depending only on $\mathrm{K}(\cdot)$, such that*

$$\widehat{\mathbf{E}}\left[\sum_{1\leqslant n<m\leqslant R_b}\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right)\right]\leqslant cL(\log L)^{7/4}. \quad (2.3.70)$$

Of course this implies that $\widehat{\mathbf{E}}\mathbb{E}_\tau[S_1^2]\leqslant c(\log L)^{3/4}$, which together with (2.3.57) implies the claim of Lemma 2.3.3. □

Proof of Lemma 2.3.7. One has the inequality

$$\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right)\leqslant\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_m}, \quad (2.3.71)$$

which is a good approximation when T_{nm} is not that large compared with $\Delta\tau_n$ and $\Delta\tau_m$, and

$$\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right)\leqslant c\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}, \quad (2.3.72)$$

which is accurate when T_{nm} is large. We use these bounds to cut the expectation (2.3.70) into two parts, a term where $m-n\leqslant H_L$ and one where $m-n>H_L$, with H_L to be chosen later:

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{R_b}\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right)\right]$$

$$\leqslant\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{(n+H_L)\wedge R_b}\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_m}\right]+c\,\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right]. \quad (2.3.73)$$

We claim that there exists a constant c such that for every $l \geqslant 1$,

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-l} \sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\right] \leqslant c\sqrt{L}(\log L)^{2+\frac{1}{12}} \tag{2.3.74}$$

(the proof is given later). Then the first term in the right-hand side of (2.3.73) is

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{(n+H_L)\wedge R_b} \sqrt{\Delta\tau_n}\sqrt{\Delta\tau_m}\right] = \sum_{l=1}^{H_L}\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-l} \sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\right]$$
$$\leqslant cH_L\sqrt{L}(\log L)^{2+1/12}.$$

If we choose $H_L = \sqrt{L}(\log L)^{-1/3}$, we get from (2.3.73)

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+1}^{R_b}\left(\sqrt{T_{nm}+\Delta\tau_n}-\sqrt{T_{nm}}\right)\left(\sqrt{T_{nm}+\Delta\tau_m}-\sqrt{T_{nm}}\right)\right]$$
$$\leqslant cL(\log L)^{7/4} + c\,\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right]. \tag{2.3.75}$$

As for the second term in (2.3.73), recall that $T_{nm} = \tau_{m-1} - \tau_n$ and decompose the sum in two parts, according to whether T_{nm} is larger or smaller than a certain $K_L > 1$ to be fixed:

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right]$$
$$\leqslant \widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\mathbf{1}_{\{T_{nm}>K_L\}}\right] + \widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\Delta\tau_n\Delta\tau_m\mathbf{1}_{\{T_{nm}\leqslant K_L\}}\right]$$
$$\leqslant \frac{1}{K_L}\widehat{\mathbf{E}}\left[\left(\sum_{n=1}^{R_b}\Delta\tau_n\right)^2\right] + L^2\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\mathbf{1}_{\{\tau_{n+H_L}-\tau_n\leqslant K_L\}}\right]$$
$$\leqslant \frac{L^2}{K_L} + L^4\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right). \tag{2.3.76}$$

We now set $K_L = L(\log L)^{-7/4}$, so that we get in the previous inequality

$$\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b}\sum_{m=n+H_L+1}^{R_b}\frac{\Delta\tau_n\Delta\tau_m}{T_{nm}}\right] \leqslant L(\log L)^{7/4} + L^4\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right), \tag{2.3.77}$$

and we are done if we prove for instance that $\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right) = o(L^{-4})$. Indeed,

$$\widehat{\mathbf{P}}\left(\tau_{H_L}\leqslant K_L\right) = \widehat{\mathbf{P}}\left(R_{K_L}\geqslant H_L\right) \leqslant c\mathbf{P}\left(R_{K_L}\geqslant H_L\right) \tag{2.3.78}$$

where we used Lemma 2.A.1 to take the conditioning off from $\widehat{\mathbf{P}} := \mathbf{P}(\cdot|b\in\tau)$ (in fact, $K_L \leqslant b/2$ since $b \geqslant \varepsilon L$). Recalling the choices of H_L and K_L, we get that

$H_L/\sqrt{K_L} = (\log L)^{13/24}$ and, combining (2.3.78) with Lemma 2.A.2, we get

$$\widehat{\mathbf{P}}\left(\tau_{H_L} \leqslant K_L\right) \leqslant c'\, e^{-c(\log L)^{13/12}} = o(L^{-4}) \tag{2.3.79}$$

which is what we needed.

To conclude the proof of Lemma 2.3.7, we still have to prove (2.3.74). Note that

$$\begin{aligned}\widehat{\mathbf{E}}\left[\sum_{n=1}^{R_b-l} \sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\mathbf{1}_{\{R_b>l\}}\right] &= \widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b>l\}} \sum_{n=1}^{R_b-l} \widehat{\mathbf{E}}\left[\sqrt{\Delta\tau_n}\sqrt{\Delta\tau_{n+l}}\,|R_b\right]\right] \\ &= \widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b>l\}}(R_b-l)\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\,|R_b\right]\right] \leqslant \widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\,\geqslant\,2\}}\right],\end{aligned} \tag{2.3.80}$$

where we used the fact that, under $\widehat{\mathbf{P}}(\cdot|R_b = p)$ for a fixed p, the law of the jumps $\{\Delta\tau_n\}_{n\,\leqslant\,p}$ is exchangeable. We first bound (2.3.80) when R_b is large:

$$\begin{aligned}\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\,\geqslant\,\kappa\sqrt{L\log L}\}}\right] &\leqslant L^2\widehat{\mathbf{P}}\left(R_b \geqslant \kappa\sqrt{L\log L}\right) \\ &\leqslant L^2\mathbf{P}(b\in\tau)^{-1}\mathbf{P}\left(R_b \geqslant \kappa\sqrt{L\log L}\right).\end{aligned} \tag{2.3.81}$$

In view of (2.1.10), we have $\mathbf{P}(b\in\tau)^{-1} = O(\sqrt{L})$. Thanks to Lemma 2.A.2 in the Appendix, and choosing κ large enough, we get

$$\mathbf{P}\left(R_b \geqslant \kappa\sqrt{L\log L}\right) \leqslant e^{-c\kappa^2\log L + o(\log L)} = o(L^{-5/2}), \tag{2.3.82}$$

and therefore

$$\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\,\geqslant\,\kappa\sqrt{L\log L}\}}\right] = o(1). \tag{2.3.83}$$

As a consequence,

$$\begin{aligned}\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\,\geqslant\,2\}}\right] &= \widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{2\,\leqslant\,R_b<\kappa\sqrt{L\log L}\}}\right] + o(1) \\ &\leqslant \sqrt{L}(\log L)^{1/12}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b\,\geqslant\,2\}}\right] \\ &\quad + \kappa\sqrt{L\log L}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b>\sqrt{L}(\log L)^{1/12}\}}\right] + o(1).\end{aligned} \tag{2.3.84}$$

Let us deal with the second term:

$$\begin{aligned}&\widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b>\sqrt{L}(\log L)^{1/12}\}}\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\right] \\ &\quad= \frac{1}{\mathbf{P}(b\in\tau)}\sum_{i=1}^{b}\sum_{j=1}^{b-i}\sqrt{i}\sqrt{j}\mathbf{P}\left(\tau_1=i, \tau_2-\tau_1=j, b\in\tau, R_b>\sqrt{L}(\log L)^{1/12}\right) \\ &= \frac{1}{\mathbf{P}(b\in\tau)}\sum_{i=1}^{b}\sum_{j=1}^{b-i}\sqrt{i}\sqrt{j}\mathrm{K}(i)\mathrm{K}(j)\mathbf{P}\left(b-i-j\in\tau, R_{b-i-j}>\sqrt{L}(\log L)^{1/12}-2\right).\end{aligned} \tag{2.3.85}$$

But we have

$$\begin{aligned}\mathbf{P}\left(R_{b-i-j} > \sqrt{L}(\log L)^{1/12} - 2 \,|b-i-j \in \tau\right) & \\ \leqslant 2\mathbf{P}\left(R_{(b-i-j)/2} > \frac{1}{2}\sqrt{L}(\log L)^{1/12} - 1 \,|b-i-j \in \tau\right) & \\ \leqslant c\mathbf{P}\left(R_{(b-i-j)/2} > \frac{1}{2}\sqrt{L}(\log L)^{1/12} - 1\right) & \\ \leqslant c\mathbf{P}\left(R_L > \frac{1}{2}\sqrt{L}(\log L)^{1/12} - 1\right) \leqslant c'\, e^{-c(\log L)^{1/6}} & \qquad (2.3.86)\end{aligned}$$

where we first used Lemma 2.A.1 to take the conditioning off, and then Lemma 2.A.2. Putting (2.3.85) and (2.3.86) together, we get

$$\begin{aligned}\widehat{\mathbf{E}}\left[\mathbf{1}_{\{R_b > \sqrt{L}(\log L)^{1/12}\}}\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\right] & \\ \leqslant c' e^{-c(\log L)^{1/6}} \frac{1}{\mathbf{P}(b\in\tau)} \sum_{i=1}^{b}\sum_{j=1}^{b-i} \sqrt{i}\sqrt{j}\mathrm{K}(i)\mathrm{K}(j)\mathbf{P}\left(b-i-j\in\tau\right) & \\ = c' e^{-c(\log L)^{1/6}} \widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b \geqslant 2\}}\right]. & \qquad (2.3.87)\end{aligned}$$

So, recalling (2.3.84), we have

$$\widehat{\mathbf{E}}\left[R_b\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b \geqslant 2\}}\right] \leqslant 2\sqrt{L}(\log L)^{1/12}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b \geqslant 2\}}\right] + o(1) \qquad (2.3.88)$$

and we only have to estimate (recall (2.1.10))

$$\begin{aligned}\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b \geqslant 2\}}\right] &= \sum_{p=1}^{b-1}\sum_{q=1}^{b-p} \sqrt{p}\sqrt{q}\mathrm{K}(p)\mathrm{K}(q)\frac{\mathbf{P}(b-p-q\in\tau)}{\mathbf{P}(b\in\tau)} \\ &\leqslant c\sqrt{b}\sum_{p=1}^{b-1}\sum_{q=1}^{b-p}\frac{1}{p\,q}\frac{1}{\sqrt{b+1-p-q}}. \qquad (2.3.89)\end{aligned}$$

Using twice the elementary estimate

$$\sum_{k=1}^{M-1}\frac{1}{k}\frac{1}{\sqrt{M-k}} \leqslant c\frac{1}{\sqrt{M}}\log M, \qquad (2.3.90)$$

we get

$$\widehat{\mathbf{E}}\left[\sqrt{\tau_1}\sqrt{\tau_2-\tau_1}\mathbf{1}_{\{R_b \geqslant 2\}}\right] \leqslant c\sqrt{b}\sum_{p=1}^{b-1}\frac{1}{p}\frac{1}{\sqrt{b-p+1}}\log(b-p+1) \leqslant c\sqrt{b}\frac{1}{\sqrt{b}}(\log L)^2. \qquad (2.3.91)$$

Together with (2.3.88), this proves the desired estimate (2.3.74).

□

2.3.4. Dimension $d = 4$ (a sketch). As we mentioned just after Theorem 2.1.5, it is possible to adapt the change-of-measure argument to prove non-coincidence of quenched and annealed critical points in dimension $d \geqslant 4$ for the general walks of Assumption 2.1.1, while the method of Birkner and Sun [BS10] does not seem to adapt easily much beyond the simple random walk case. In this section, we only deal with the case $d = 4$, since the Theorem 2.1.5 is obtained for $d \geqslant 5$ in [BGdH10], with more general condition than Assumption 2.1.1. We will not give details, but for the interested reader we hint at the "right" change of measure which works in this case.

The "change of measure function" $g_{\mathcal{I}}(Y)$ is still of the form (2.2.12), factorized over the blocks which belong to $\mathcal{I}$, but this time M is a matrix with a finite bandwidth:

$$F_k(Y) = -\frac{1}{\sqrt{L}} \sum_{i=L(k-1)+1}^{kL-p_0} \Delta_i \cdot \Delta_{i+p_0}, \tag{2.3.92}$$

where p_0 is an integer. The role of the normalization $L^{-1/2}$ is to guarantee that $\|M\| < \infty$. The integer p_0 is to be chosen such that $A(p_0) > 0$, where $A(\cdot)$ is the function defined in Lemma 2.3.4. The existence of such p_0 is guaranteed by the asymptotics (2.3.17), whose proof for $d = 4$ is the same as for $d = 3$.

For the rest, the scheme of the proof of $\beta_c^{\mathrm{que}} \neq \beta_c^{\mathrm{a}}$ (in particular, the coarse-graining procedure) is analogous to that we presented for $d = 3$, and the computations involved are considerably simpler.

2.A. Some technical estimates on Random Walks

Lemma 2.A.1. *(Lemma A.2 in [GLT10b]) Let* $\mathbf{P}$ *be the law of a recurrent renewal whose inter-arrival law satisfies* $\mathrm{K}(n) \stackrel{n\to\infty}{\sim} c_K n^{-3/2}$ *for some* $c_K > 0$*. There exists a constant* $c > 0$*, that depends only on* $\mathrm{K}(\cdot)$*, such that for any non-negative function* $f_N(\tau)$ *which depends only on* $\tau \cap \{1, \ldots, N\}$*, one has*

$$\sup_{N>0} \frac{\mathbf{E}[f_N(\tau)\,|2N \in \tau]}{\mathbf{E}[f_N(\tau)]} \leqslant c. \tag{2.A.1}$$

Lemma 2.A.2. *Under the same assumptions as in Lemma 2.A.1, and with* $R_N := |\tau \cap \{1, \ldots, N\}|$*, there exists a constant* $c > 0$*, such that for any positive function* $\alpha(N)$ *which diverges at infinity and such that* $\alpha(N) = o(\sqrt{N})$*, we have*

$$\mathbf{P}\left(R_N \geqslant \sqrt{N}\alpha(N)\right) \leqslant e^{-c\alpha(N)^2 + o\left(\alpha(N)^2\right)}. \tag{2.A.2}$$

Proof. For every $\lambda > 0$

$$\begin{aligned}\mathbf{P}\left(R_N \geqslant \sqrt{N}\alpha(N)\right) &= \mathbf{P}\left(\tau_{\sqrt{N}\alpha(N)} \leqslant N\right) = \mathbf{P}\left(\lambda\alpha(N)^2 \frac{\tau_{\sqrt{N}\alpha(N)}}{N} \leqslant \lambda\alpha(N)^2\right) \\ &\leqslant e^{\lambda\alpha(N)^2}\mathbf{E}\left[e^{-\lambda\frac{\alpha(N)^2}{N}\tau_{\sqrt{N}\alpha(N)}}\right] = e^{\lambda\alpha(N)^2}\mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right]^{\sqrt{N}\alpha(N)}. \end{aligned}\tag{2.A.3}$$

The asymptotic behavior of $\mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right]$ is easily obtained:

$$1 - \mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right] = \sum_{n\in\mathbb{N}} \mathrm{K}(n)\left(1 - e^{-n\lambda\alpha(N)^2/N}\right)$$
$$\overset{N\to\infty}{\sim} c\frac{\sqrt{\lambda}\alpha(N)}{\sqrt{N}}, \qquad c = c_K \int_0^\infty \frac{1-e^{-x}}{x^{3/2}}\,\mathrm{d}x,$$

where the condition $\alpha(N)^2/N \to 0$ was used to transform the sum into an integral. Therefore, we get

$$\mathbf{E}\left[e^{-\lambda\alpha(N)^2\frac{\tau_1}{N}}\right]^{\sqrt{N}\alpha(N)} = \left(1 - c\frac{\sqrt{\lambda}\alpha(N)}{\sqrt{N}} + o\left(\frac{\alpha(N)}{\sqrt{N}}\right)\right)^{\sqrt{N}\alpha(N)}$$
$$= e^{-c\sqrt{\lambda}\alpha(N)^2 + o\left(\alpha(N)^2\right)}. \quad (2.A.4)$$

Then, for any $\lambda > 0$,

$$\mathbf{P}\left(R_N \geqslant \sqrt{N}\alpha(N)\right) \leqslant e^{(\lambda - c\sqrt{\lambda})\alpha(N)^2 + o\left(\alpha(N)^2\right)} \quad (2.A.5)$$

and taking $\lambda = c^2/4$ we get the desired bound. □

We need also the following standard result (cf. for instance [GLT11, Sec.5]):

Lemma 2.A.3. *Under the same hypothesis as in Lemma 2.A.1, we have the following convergence in law:*

$$\frac{c_K}{\sqrt{N}} R_N \overset{N\to\infty}{\Rightarrow} \frac{1}{\sqrt{2\pi}}\,|\mathcal{Z}| \qquad (\mathcal{Z} \sim \mathcal{N}(0,1)). \quad (2.A.6)$$

2.A.1. Proof of (2.3.60). We wish to show that for distinct i, j, l smaller than r,

$$\mathbb{E}^X\mathbb{E}^Y[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)|X_r = Y_r] \leqslant \frac{c}{r}. \quad (2.A.7)$$

We use the same method as in Section 2.3.2: we fix $x \in \mathbb{Z}^d$, and we use the notation $\mathbb{E}^Y_{r,x}[\cdot] = \mathbb{E}^Y[\cdot\,|Y_r = x]$. Then,

$$\mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)(\Delta_i \cdot \Delta_l)\right] = \mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)\left(\Delta_i \cdot \mathbb{E}^Y_{r,x}\left[\Delta_l\,|\Delta_i, \Delta_j\right]\right)\right]$$
$$= \frac{1}{r-2}\mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)\left(\Delta_i \cdot (x - \Delta_i - \Delta_j)\right)\right]$$
$$= \frac{1}{r-2}\mathbb{E}^Y_{r,x}\left[(\Delta_i \cdot \Delta_j)\left((x\cdot\Delta_i) - \|\Delta_i\|^2\right) - (\Delta_i\cdot\Delta_j)^2\right]$$
$$\leqslant \frac{1}{r-2}\left(\mathbb{E}^Y_{r,x}\left[(x\cdot\Delta_i)\left(\Delta_i \cdot \mathbb{E}^Y_{r,x}\left[\Delta_j\,|\Delta_i\right]\right)\right] + \mathbb{E}^Y_{r,x}[\|\Delta_i\|^3\,\|\Delta_j\|]\right)$$
$$= \frac{1}{r-2}\left(\frac{1}{r-1}\mathbb{E}^Y_{r,x}\left[(x\cdot\Delta_i)^2 - (x\cdot\Delta_i)\,\|\Delta_i\|^2\right] + \mathbb{E}^Y_{r,x}[\|\Delta_i\|^3\,\|\Delta_j\|]\right)$$
$$\leqslant \frac{c}{r}\left(\mathbb{E}^Y_{r,x}\left[\frac{\|x\|^2}{r}\,\|\Delta_i\|^2 + \frac{\|x\|}{r}\,\|\Delta_i\|^3 + \|\Delta_i\|^3\,\|\Delta_j\|\right]\right)$$

and we can take by symmetry $i = 1$, $j = 2$. Therefore,

$$\begin{aligned}
&\mathbb{E}^X\mathbb{E}^Y\left[\,(\Delta_i\cdot\Delta_j)(\Delta_i\cdot\Delta_l)|\,X_r=Y_r\right]\\
&\qquad= \mathbb{E}^X\left[\mathbb{E}^Y\left[(\Delta_i\cdot\Delta_j)(\Delta_i\cdot\Delta_l)\,|Y_r=X_r\right]\frac{\mathbb{P}^Y(Y_r=X_r)}{\mathbb{P}^{X-Y}(Y_r=X_r)}\right]\\
&\qquad\leqslant \frac{c}{r}\mathbb{E}^X\mathbb{E}^Y\left[Q\left(\frac{\|X_r\|}{r^{1/2}},\|\Delta_1\|,\|\Delta_2\|\right)\middle|\,Y_r=X_r\right], \qquad (2.A.8)
\end{aligned}$$

where

$$Q\left(\frac{\|X_r\|}{r^{1/2}},\|\Delta_1\|,\|\Delta_2\|\right)=\frac{\|X_r\|^2}{r}\|\Delta_1\|^2+\frac{\|X_r\|}{\sqrt{r}}\|\Delta_1\|^3+\|\Delta_1\|^3\|\Delta_1\|\,.$$

At this point, one can apply directly the result of the Remark 2.3.5. □

2.A.2. Proof of (2.3.61). We wish to prove that, for distinct $i, j, k, l \leqslant r$,

$$\mathbb{E}_\tau[(\Delta_i\cdot\Delta_j)(\Delta_k\cdot\Delta_l)]\leqslant\frac{c}{r^2}. \qquad (2.A.9)$$

The proof is very similar to that of (2.A.7), so we skip details. What one gets is that

$$\begin{aligned}
&\mathbb{E}_\tau\left[(\Delta_i\cdot\Delta_j)(\Delta_k\cdot\Delta_l)\right]\\
&\qquad\leqslant\frac{c}{r^2}\frac{\mathbb{E}^X\left[\mathbb{E}^Y\left[Q'\left(\frac{\|X_r\|}{r^{1/2}},\{\|\Delta_i\|\}_{i=1,2,3}\right)\middle|\,Y_r=X_r\right]\mathbb{P}^Y(Y_r=X_r)\right]}{\mathbb{P}^{X-Y}(Y_r=X_r)}, \qquad (2.A.10)
\end{aligned}$$

where Q' is a polynomial of degree 4 in the variable $\|X_r\|/\sqrt{r}$ and of degree at most 3 in each of the $\|\Delta_i\|$. Again, like after (2.A.8), one uses the Remark 2.3.5 to get the desired result.

2.A.3. Proof of (2.3.62)-(2.3.63). In view of (2.3.14), in order to prove (2.3.62) it suffices to prove that for $0 < i \neq k \leqslant r, 0 < j \neq l \leqslant s$

$$\sum_{\nu,\mu=1}^{3}\mathbb{E}^X\mathbb{E}^Y[\Delta_i^{(\nu)}\Delta_k^{(\mu)}|X_r=Y_r]\mathbb{E}^X\mathbb{E}^Y[\Delta_j^{(\nu)}\Delta_l^{(\mu)}|X_s=Y_s]\leqslant\frac{c}{rs}. \qquad (2.A.11)$$

Both factors in the left-hand side have already been computed in (2.3.55)-(2.3.56). Using these two expressions and once more the Local Limit Theorem, one arrives easily to (2.A.11). The proof of (2.3.63) is essentially identical.

CHAPTER 3

The effect of disorder on the free energy

3.1. Introduction

3.1.1. Reminder of the random walk pinning model. Let $X = (X_s)_{s \geqslant 0}$ and $Y = (Y_s)_{s \geqslant 0}$ be two independent continuous time random walks on $\mathbb{Z}^d$, $d \geqslant 1$, starting from 0, with jump rates 1 and $\rho \geqslant 0$ respectively, and which have identical irreducible symmetric jump probability kernels. We also make the assumption that the increments X and Y on $\mathbb{Z}^d$ have finite second moments. We denote by $\mathbb{P}^X$, $\mathbb{P}^{Y,\rho}$ the associated probability laws.

For $\beta \in \mathbb{R}$ and $t \in \mathbb{R}_+$, and for a fixed realization of Y, we define a Gibbs transformation of the path measure $\mathbb{P}^X$: the polymer path measure $\mu^Y_{t,\beta}$ defined by

$$\frac{\mathrm{d}\mu^Y_{t,\beta}}{\mathrm{d}\mathbb{P}^X}(X) = \frac{e^{\beta L_t(X,Y)}\ \mathbf{1}_{\{X_t=Y_t\}}}{Z^Y_{t,\beta}}, \tag{3.1.1}$$

where $L_t(X,Y) := \int_0^t \mathbf{1}_{\{X_s=Y_s\}}\,\mathrm{d}s$ is the intersection time between X and Y, and

$$Z^Y_{t,\beta} := \mathbb{E}^X\left[e^{\beta L_t(X,Y)}\ \mathbf{1}_{\{X_t=Y_t\}}\right] \tag{3.1.2}$$

is the partition function of the system. Under the measure $\mu^Y_{t,\beta}$, an energy reward β is given to the walk X for staying in touch with Y. More physical motivations for the model were already given in Introduction and Section 1.2.1.

Given a trajectory $Y = (Y_s)_{s \geqslant 0}$, we also define the partition function along a time interval $[t_1, t_2]$ as

$$Z^Y_{[t_1,t_2],\beta} := Z^{\theta_{t_1}Y}_{t_2-t_1,\beta}, \tag{3.1.3}$$

where $\theta_t Y := (Y_{s+t} - Y_t)_{s \geqslant 0}$ (θ_t is the shift operator along time, it preserves the law of Y). One fundamental property of the *pinned* partition function, is the stochastic superadditivity of $\log Z^Y_{t,\beta}$. Indeed, Remark 1.2.1 gives

$$Z^Y_{[u,w],\beta} \geqslant Z^Y_{[u,v],\beta} Z^Y_{[v,w],\beta}, \quad \text{for any } u \leqslant v \leqslant w. \tag{3.1.4}$$

This property allows to prove the existence of the quenched and annealed free energy and critical points (cf. Proposition 1.2.2).

Definition 3.1.1 (Free energy and critical points)**.** *The quenched free energy*

$$\mathtt{F}(\beta,\rho) := \lim_{t\to\infty} \frac{1}{t}\log Z^Y_{t,\beta} = \sup_{t>0} \frac{1}{t}\mathbb{E}^{Y,\rho}\left[\log Z^Y_{t,\beta}\right] \tag{3.1.5}$$

exists and is non-random $\mathbb{P}^{Y,\rho}$ almost surely. There exists a quenched critical point $\beta_c^{\mathrm{que}}(\rho)$ such that $\mathtt{F}(\beta,\rho) > 0 \Leftrightarrow \beta > \beta_c(\rho)$.

The annealed free-energy is $\mathtt{F}^{\mathrm{a}}(\beta,\rho) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{Y,\rho}\left[Z_{t,\beta}^Y\right]$*, and the annealed critical point is* $\beta_c^{\mathrm{a}}(\rho) := \inf\{\beta\,;\ \mathtt{F}^{\mathrm{a}}(\beta,\rho) > 0\}$*. We have by Jensen inequality that* $\mathtt{F}(\beta,\rho) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,\rho)$*, and* $\beta_c(\rho) \geqslant \beta_c^{\mathrm{a}}(\rho)$*.*

The critical value $\beta_c(\rho)$ identifies the phase transition between the localized and the delocalized phase, as noticed in Section 1.2.1. Therefore X sticks to Y when $\beta > \beta_c(\rho)$ and $L_t(X,Y)$ is asymptotically of order t in the localized phase.

It is natural to compare the quenched free energy curve with the annealed one, and one needs to give some accurate description about the annealed free energy curve. As it was remarked in [BS10] and in Remark 1.2.3, the annealed partition function $\mathbb{E}^{Y,\rho}[Z_{t,\beta}^Y]$ is simply the partition function of a homogeneous pinning model

$$\mathbb{E}^{Y,\rho}\left[Z_{t,\beta}^Y\right] = \mathbb{E}^X\mathbb{E}^{Y,\rho}\left[e^{\beta L_t(X-Y,0)}\mathbf{1}_{\{(X-Y)_t=0\}}\right].$$

Under $\mathbb{P}^X \times \mathbb{P}^{Y,\rho}$, $X-Y$ is a symmetric random walk with jump rate $(1+\rho)$. By rescaling time so that the random walk $X-Y$ has jump rate 1, one obtains that

$$\mathtt{F}^{\mathrm{a}}(\beta,\rho) = (1+\rho)\mathtt{F}(\beta/(1+\rho),0). \tag{3.1.6}$$

We write $\mathtt{F}(\beta)$ for $\mathtt{F}(\beta,0)$.

The model is in fact *exactly solvable* in the sense that one has an explicit formula for the free energy. We give a complete description of the pure model in Section 1.1.3 (and Section 1.1.2 for the discrete version of the model).

Let $p_t(\cdot) := \mathbb{P}^X(X_t = \cdot)$ denote the transition probability kernel of X at time t, and set $G := \int_0^\infty p_t(0)\,\mathrm{d}t$ ($G < \infty$ when $d \geqslant 3$). We recall the result on the pure model, Proposition 1.1.11, that apply to the annealed model.

Proposition 3.1.2. *For* $d \geqslant 1$*, and in view of Proposition 1.1.11 and (3.1.6), the annealed critical point is* $\beta_c^{\mathrm{a}}(\rho) = (1+\rho)/G$ *(we use the convention that* $G^{-1} = 0$ *if* $G = \infty$*, for* $d = 1,2$*). One has also the critical behavior of the annealed free energy, given by Proposition 1.1.11:*

- *for* $d = 1,3$*,*
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c^{\mathrm{a}}}{\sim} c_0(\beta-\beta_c^{\mathrm{a}})^2. \tag{3.1.7}$$
- *for* $d = 2$*,*
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c^{\mathrm{a}}}{=} \exp\left(-c_0\,\frac{1+o(1)}{\beta}\right). \tag{3.1.8}$$
- *for* $d = 4$*,*
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c^{\mathrm{a}}}{\sim} c_0(\beta-\beta_c^{\mathrm{a}})/\log(\beta-\beta_c^{\mathrm{a}}). \tag{3.1.9}$$
- *for* $d \geqslant 5$
$$\mathtt{F}(\beta) \overset{\beta\downarrow\beta_c^{\mathrm{a}}}{\sim} c_0(\beta-\beta_c^{\mathrm{a}}). \tag{3.1.10}$$

(c_0 *is a constant that can be made explicit, and that depends on* G*, the dimension and the second moment of the jump kernel).*
In any dimension, we also have

$$\lim_{\beta\to\infty} \mathtt{F}(\beta) - \beta + 1 = 0. \tag{3.1.11}$$

Part of the above result (namely, the value of β_c), was proved in [BS10]. We have included here also the asymptotic behavior near β_c in order to know the specific heat exponent in any dimension. The knowledge of the annealed specific heat exponent (the free energy behaves like $(\beta-\beta_c)^{2-\mathtt{C}}$ when $\beta \to \beta_c^+$ where $\mathtt{C}$ is the specific heat exponent) allows to make predictions concerning disorder relevance.

3.1.2. Harris criterion and disorder relevance. We already mentioned the Harris criterion in Introduction, based on the specific heat exponent of the pure system: if the specific heat exponent is negative then disorder should be *irrelevant*, if it is positive, disorder should be *relevant* (this corresponds to $d \geqslant 4$ in our model). It however gives no prediction for the marginal case when the specific heat exponent vanishes (and in that case, it is believed that disorder relevance depends on the model which is considered).

For the Random Walk Pinning Model, various pieces of work have brought this prediction on rigorous grounds [BT10, BS10, BS11], especially in terms of critical point shift: the question of the relevance or irrelevance of disorder for the RWPM is solved, also for the marginal case $d=3$, both for the continuous time model, see [BS10, BS11], and for the discrete time model, see Chapter 2, [BT10, BS10].

Theorem 3.1.3 ([BS10, BS11], Continuous time RWPM)**.** *In dimension $d=1$ and $d=2$, one has $\beta_c(\rho)=\beta_c^{\mathrm{a}}(\rho)=0$ for any positive ρ: disorder is irrelevant. In dimension $d \geqslant 3$, one has $\beta_c > \beta_c^{\mathrm{a}} > 0$ for each $\rho>0$: disorder is relevant. Moreover, we have a bound on the shift of the critical point, as given in Theorem 1.2.4*

Let us also mention that the picture of disorder relevance/irrelevance for the renewal pinning model (widely recalled in the introductive Chapter 1) is mostly complete, thanks to a series of recent articles [Ale08, DGLT09, GLT10b, GLT11], see Section 1.4.2. It has been showed that the Harris criterion is verified, and that in the marginal case, disorder is relevant.

Another issue that has been given much attention is the so called *smoothing* of the free energy curve. It is believed that for many systems, the presence of disorder makes the free energy curve more regular: the phase transition is at least of second order (there is no discontinuity in the derivative). In particular this means that if the annealed specific heat exponent is negative, quenched and annealed exponent have to differ. This underlines disorder relevance, and gives further justification for the Harris criterion.

Smoothing type results have been shown by Aizenman and Wehr for disordered Ising model [AW90], and more recently by Giacomin and Toninelli for the random pinning model based on renewal process [GT06] (and also for a hierarchical version of the same model [LT09]).

We also mention that there exist some peculiar pinning models for which there is no smoothing phenomenon and the quenched and annealed systems have always the same behavior, even if the critical points are different (see e.g. [Ale09]).

3.1.3. Smoothing of the phase transition. The first result we present for the disordered model is the smoothing of the free-energy curve around the phase transition. This phenomenon occurs in dimension $d \geqslant 3$. For $d = 1, 2$, the model is a bit different because of recurrence of the random walk in these dimensions, see later.

Theorem 3.1.4. *For all $d \geqslant 3$, $\rho > 0$, $\beta > 0$, we have*

$$\mathtt{F}(\beta, \rho) \leqslant \frac{dG^2}{4\rho}(\beta - \beta_c(\rho))_+^2. \tag{3.1.12}$$

We stress that the constant $\frac{dG^2}{4\rho}$ has been improved (and optimized) with respect to [BL11, Th.1.6], in the spirit of other smoothing results, as [Gia07, Th.5.6].

This shows that if $d \geqslant 4$, the disorder makes the phase transition at least of second order, whereas it is of first order for the *annealed* model (see Proposition 3.1.2). The methods that has been used to prove the previous smoothing results [AW90, GT06] have been a strong source of inspiration for our proof, but, as the nature of the disorder is very different here, some new ideas are necessary. A crucial point is to use an estimate on how $\mathtt{F}(\beta, \rho)$ varies with ρ, which is present in [BS11].

It has been proved for the renewal pinning model that the critical exponent for the free-energy is related to the asymptotics of the number of contacts at the critical point [Lac10, Prop. 1.3]. For the random walk pinning model an analogous relation holds (where the number of contact is replaced by $L_t(X, Y)$) and gives the following result. We include also its proof, which is very similar to what is done in [Lac10], for the sake of completeness.

Corollary 3.1.5. *Let us fix $\rho > 0$, $d \geqslant 3$ and $\varepsilon > 0$. Then, under $\mathbb{P}^{Y,\rho}$,*

$$\lim_{t\to\infty} \mu^Y_{t,\beta_c(\rho)} \left(L_t(X, Y) \geqslant t^{1/2+\varepsilon} \right) = 0, \tag{3.1.13}$$

in probability.

Remark 3.1.6. This result contrasts with what happens for the pure model ($\rho = 0$), where typically $L_t(X, 0) \asymp t$ at β_c for $d \geqslant 5$ (as shown in Corollary 3.A.5). One also has that at the critical temperature, $L_t(X, 0)/\log t$ converges in law to an exponential random variable [ET60].

Proof Suppose there exists some $c > 0$ such that one can find an arbitrarily large value of t for which

$$\mathbb{P}^Y \left\{ \mu^Y_{t,\beta_c(\rho)} \left(L_t(X, Y) \geqslant t^{1/2+\varepsilon} \right) \geqslant c \right\} \geqslant c. \tag{3.1.14}$$

Then we define t_0 large enough such that the above holds, $u := t_0^{-1/2}$ and $\beta_c := \beta_c(\rho)$. One has

$$Z^Y_{t_0,\beta_c+u} = \mathbb{E}^X \left[e^{(\beta_c+u)L_{t_0}(X,Y)} \mathbf{1}_{\{X_{t_0}=Y_{t_0}\}} \right] = Z^Y_{t_0,\beta_c}\, \mu^Y_{t_0,\beta_c} \left(e^{uL_{t_0}(X,Y)} \right), \tag{3.1.15}$$

so that

$$\mathbb{E}^{Y,\rho}\left[\log Z^Y_{t_0,\beta_c+u}\right] = \mathbb{E}^{Y,\rho}\left[\log Z^Y_{t_0,\beta_c} + \log \mu^Y_{t_0,\beta_c}\Big(\exp\big(uL_{t_0}(X,Y)\big)\Big)\right]$$
$$\geqslant \mathbb{E}^{Y,\rho}\left[\log p_{t_0}(Y_{t_0})\right] + \mathbb{E}^{Y,\rho}\left[\log\Big(c\exp(ut_0^{1/2+\varepsilon})\Big)\mathbf{1}_{\left\{\mu^Y_{t_0,\beta_c}\left(L_{t_0}(X,Y)\geqslant t_0{}^{1/2+\varepsilon}\right)\geqslant c\right\}}\right]$$
$$\geqslant -d\log t_0 + c(t_0^{\varepsilon}+\log c) \geqslant t_0^{\varepsilon/2}, \quad (3.1.16)$$

where in the first inequality we used that $Z^Y_{t_0,\beta_c} \geqslant p_{t_0}(Y_{t_0})$ (recall the notation introduced just before Proposition 3.1.2, this is just using the fact that $\beta_c \geqslant 0$). The second inequality uses an estimate from [BS10, Lemma 3.1] which is valid if t_0 is large enough for the first term, and (3.1.14) for the second term. The last inequality is valid if t_0 is large enough. This implies, by (3.1.4)

$$\mathtt{F}\big(\beta_c(\rho)+u,\rho\big) \geqslant \frac{1}{t_0}\mathbb{E}^{Y,\rho}\left[\log Z^Y_{t_0,\beta_c+u}\right] \geqslant t_0^{\varepsilon/2-1} \geqslant u^{2-\varepsilon}. \quad (3.1.17)$$

This contradicts Theorem 3.1.4, therefore (3.1.14) cannot hold. □

In dimension 1 or 2, the situation is a bit different due to recurrence of the random walk. In dimension $d=2$, the coincidence of quenched and annealed critical point, and the fact that the phase transition is of infinite order of the annealed system implies that the phase transition is also of infinite order (*i.e.* smoother than any power of $(\beta-\beta_c)$) for the quenched system. In dimension $d=1$, one also shows that disorder does not change the nature of the phase transition (or at least not in a significant way).

Proposition 3.1.7 (Quenched free-energy at high temperature for $d=1$)**.** *There exist a constant $c>0$ such that for any ρ there exists β_0 such that*

$$\mathtt{F}(\beta,\rho) \geqslant \frac{c}{1+\rho}\beta^2\log(1/\beta)^{-1}, \quad \forall\beta\in[0,\beta_0]. \quad (3.1.18)$$

Thus, $\mathtt{F}$ and $\mathtt{F}^{\mathrm{a}}$ have the same critical exponent.

Proof By Jensen inequality one has (for some constant C_1) that for every t and $\rho \geqslant 0$

$$\mathbb{E}^{Y,\rho}\left[\log\mathbb{E}^X\left[e^{\beta L_t(X,Y)}\,\middle|\, X_t=Y_t\right]\right] \geqslant \beta\mathbb{E}^{Y,\rho}\mathbb{E}^X\left[L_t(X,Y)\,\middle|\,X_t=Y_t\right] \geqslant C_1\frac{\beta\sqrt{t}}{\sqrt{1+\rho}}. \quad (3.1.19)$$

The last inequality can be obtain by integrating the local central limit Theorem (see [Spi01, Prop.7.9, Ch.II] for the discrete time version, the proof being identical for continuous time). Therefore

$$\begin{aligned}\mathbb{E}^{Y,\rho}\left[\log Z^Y_{t,\beta}\right] &= \mathbb{E}^{Y,\rho}\left[\log\mathbb{E}^X\left[e^{\beta L_t(X,Y)}\middle|X_t=Y_t\right]\right] + \mathbb{E}^{Y,\rho}\left[\log\mathbb{P}^X\left(X_t=Y_t\right)\right]\\ &\geqslant C_1\frac{\beta\sqrt{t}}{\sqrt{1+\rho}} - \log t, \quad (3.1.20)\end{aligned}$$

where we also used [BS10, Lemma 3.1] to bound the second term (the bound being valid for t large enough, say $t \geqslant t_0(\rho)$). Now, if we set $T := C_2(1+\rho)\beta^{-2}[\log(1/\beta)]^2$, the previous inequality holds for all $\beta \leqslant t_0^{-1/2}$, and gives

$$\mathbb{E}^{Y,\rho}\left[\log Z^Y_{T,\beta}\right] \geqslant C_1\sqrt{C_2}\log(1/\beta) + 2\log\beta + O\big(\log\log(1/\beta)\big) \geqslant \log(1/\beta) \quad (3.1.21)$$

if C_2 is large enough. From (3.1.5), we finally have

$$\mathtt{F}(\beta) \geqslant \frac{1}{T}\mathbb{E}^{Y,\rho}\left[\log Z^Y_{T,\beta}\right] \geqslant \frac{1}{C_2(1+\rho)}\beta^2\log(1/\beta)^{-1}. \quad (3.1.22)$$

□

Remark 3.1.8. We believe that the factor $\log(1/\beta)^{-1}$ above is an artifact of the proof and that in reality $\mathtt{F}(\beta,\rho) \sim c(\rho)\beta^2$. A clear reason to believe so is to consider an alternative Brownian model where $(Y_t)_{t\geqslant 0}$ is a realization Brownian motion with covariance function $\mathbf{E}^Y[Y_s, Y_t] = \rho(s\wedge t)$. The partition function is given by

$$\mathcal{Z}^Y_{t,\beta} = \mathbf{E}^X\left[e^{\beta L_t(X,Y)}\right], \quad (3.1.23)$$

where under $\mathbf{P}^X$, X is a standard Brownian Motion (independent of Y) and $L_t(X,Y)$ the intersection local time between X and Y. For this model, Brownian scaling implies that $L_{t\beta^2}(X,Y) \overset{\text{(law)}}{=} \beta L_t(X,Y)$, which implies that there exists a constant $c(\rho)$ $(= \bar{\mathtt{F}}(1,\rho))$ such that for all $\beta \geqslant 0$,

$$\bar{\mathtt{F}}(\beta,\rho) := \lim_{t\to\infty}\frac{1}{t}\mathbf{E}^Y\log\mathcal{Z}^Y_{t,\beta} = c(\rho)\beta^2. \quad (3.1.24)$$

This model should be the high-temperature scaling-limit of our random walk pinning model and hence share the same critical properties.

3.1.4. Low temperature asymptotics. The quenched low-temperature asymptotic also exhibits contrasts with the annealed one. The reason is that to optimize the local-time, X has to follow Y closely, which has an extra entropic cost. In the annealed case, one can force X not to jump. For the sake of simplicity, we present the result only in the case of the simple symmetric random walk in $\mathbb{Z}^d$ (for any $d \geqslant 1$) but the result holds in the more general framework given in Section 3.1.1. This result gives again a contrasts with the pure model, see (3.1.11).

Theorem 3.1.9. *When Y is the simple symmetric random walk in $\mathbb{Z}^d$, one has*

$$\mathtt{F}(\beta,\rho) = \beta - \rho\log(d\beta) - 1 + o(1) \quad \text{as } \beta\to\infty. \quad (3.1.25)$$

In general, for a walk Y with a kernel jump p_Y which as finite second moment, the result also holds with $\log d$ replaced by $-\sum_{x\in\mathbb{Z}^d} p_Y(x)\log(p_Y(x)/2)$.

Remark 3.1.10. The proof of Theorem 3.1.9 does not only gives the result but also a clear idea of how a typical path X behaves under the polymer measure at high temperature. Essentially X follows every jump of Y, and the distance between jumps of X and Y are *i.i.d.* exponential variables of mean $1/\beta$. In particular the asymptotic contact fraction is close to $1-\rho\beta^{-1}$ (whereas it is more of order $1-\beta^{-2}$ for the pure model).

The sequel of the Chapter is organized as follows:

- In Section 3.2 we prove the smoothing result, Theorem 3.1.4,

- In Section 3.3 we prove the low temperature asymptotics, Theorem 3.1.9,
- In Appendix, we recall some statements for the pure model, from Section 1.1.3 and [BL11, Appendix].

Section 3.2 and Section 3.3 are independent.

3.2. The smoothing phenomenon, proof of Theorem 3.1.4

In the proof, we make use of the following three statements. The first two are extracted from Proposition 3.1.2 and Theorem 3.1.3, the third one is extracted from [BS11].

Proposition 3.2.1. *For $d \geqslant 3$, we have*

(i) $\beta_c^{\mathrm{a}}(\rho) = \frac{1+\rho}{G}$,
(ii) *for any $\rho > 0$, one has $\beta_c^{\mathrm{a}}(\rho) < \beta_c(\rho)$,*
(iii) *the function $\rho \mapsto \beta_c(\rho)/(1+\rho)$ is non-decreasing [BS11, Thm 1.3].*

Let ρ be fixed and $d \geqslant 3$ be fixed. Given $\beta > \beta_c(\rho)$, we define $\rho' = \rho'(\beta)$ by

$$\rho' := \rho + G(\beta - \beta_c(\rho)). \tag{3.2.1}$$

Note that $\mathtt{F}(\rho', \beta) = 0$. Indeed, we have

$$\frac{1+\rho'}{1+\rho} = 1 + G\frac{\beta - \beta_c(\rho)}{1+\rho} = 1 + \frac{\beta - \beta_c(\rho)}{\beta_c^{\mathrm{a}}(\rho)} \geqslant \frac{\beta}{\beta_c(\rho)}, \tag{3.2.2}$$

so that by (iii) of the above proposition, $\beta \leqslant \beta_c(\rho')$.

Our strategy to prove Theorem 3.1.4 is to find a lower bound for $\mathtt{F}(\rho', \beta)$ that involves $\mathtt{F}(\rho, \beta)$, by considering the contribution of exceptional (under $\mathbb{P}^{Y,\rho'}$) stretches where the empirical jump rate of Y is of order ρ.

First we bound from below the probability that under $\mathbb{P}^{Y,\rho'}$, the partition function $Z^Y_{L,\beta}$ is greater than $\exp\left(L(1-\varepsilon)\mathtt{F}(\beta,\rho)\right)$.

Lemma 3.2.2. *For any $\varepsilon > 1/2$, one can find L_0 (depending on β, ρ and ε) such that for all $L \geqslant L_0$*

$$\log\left(\mathbb{P}^{Y,\rho'}\left\{\log Z^Y_{L,\beta} > L(1-\varepsilon)\mathtt{F}(\beta,\rho)\right\}\right) \geqslant -(1+2\varepsilon)L\frac{(\rho'-\rho)^2}{2\rho} - 1 - \log 2. \tag{3.2.3}$$

Proof From the definition of the free-energy one can find L_0 such that for all $L \geqslant L_0$

$$\mathbb{P}^{Y,\rho}(A) := \mathbb{P}^{Y,\rho}\left\{\log Z^Y_{L,\beta} > L(1-\varepsilon)\mathtt{F}(\beta,\rho)\right\} \geqslant 1 - \varepsilon. \tag{3.2.4}$$

By a classical entropy inequality, one has

$$\mathbb{P}^{Y,\rho'}(A) \geqslant \mathbb{P}^{Y,\rho}(A)\exp\left(-\frac{1}{\mathbb{P}^{Y,\rho}(A)}\left(\mathbb{H}(\mathbb{P}^{Y,\rho}|\mathbb{P}^{Y,\rho'}) + e^{-1}\right)\right), \tag{3.2.5}$$

where $\mathbb{H}(\mathbb{P}^{Y,\rho}|\mathbb{P}^{Y,\rho'}) := \mathbb{E}^{Y,\rho'}\left[\frac{\mathrm{d}\mathbb{P}^{Y,\rho}}{\mathrm{d}\mathbb{P}^{Y,\rho'}}\log\frac{\mathrm{d}\mathbb{P}^{Y,\rho}}{\mathrm{d}\mathbb{P}^{Y,\rho'}}\right]$ is the relative entropy of $\mathbb{P}^{Y,\rho}$ with respect to $\mathbb{P}^{Y,\rho'}$, that are mutually absolutely continuous. Let κ^Y_L denote he number of jumps of the walk Y_t in $[0,L]$. Under $\mathbb{P}^{Y,\rho}$, it is a Poisson variable of mean ρL. One has

$$\frac{\mathrm{d}\mathbb{P}^{Y,\rho}}{\mathrm{d}\mathbb{P}^{Y,\rho'}} = e^{L(\rho'-\rho)}\left(\frac{\rho L}{\rho' L}\right)^{\kappa_L^Y}, \tag{3.2.6}$$

and therefore

$$\begin{aligned}\mathbb{H}(\mathbb{P}^{Y,\rho}|\mathbb{P}^{Y,\rho'}) &= L(\rho'-\rho) + \log(\rho/\rho')\mathbb{E}^{Y,\rho}\left[\kappa_L^Y\right] \\ &= L\left[(\rho'-\rho) - \rho\log\left(1+\frac{\rho'-\rho}{\rho}\right)\right] \leqslant L\frac{(\rho'-\rho)^2}{2\rho},\end{aligned} \tag{3.2.7}$$

which inserted in (3.2.5) gives the result, using also that $\varepsilon \geqslant 1/2$ and so $(1-\varepsilon)^{-1} \leqslant 1+2\varepsilon$ and $(1-\varepsilon)^{-1}e^{-1} \leqslant 1$. ☐

We keep the notation $A := \left\{\log Z_{L,\beta}^Y > L(1-\varepsilon)\mathtt{F}(\beta,\rho)\right\}$ and write $q := \mathbb{P}^{Y,\rho'}(A)$. Let an arbitrary $\varepsilon > 0$ be fixed, consider L large enough so that Lemma 3.2.2 is valid, and that some conditions later mentioned in the proof are fulfilled.

We divide the the system into blocks of size L, $B_i := [(i-1)L, iL]$, for $i \in \mathbb{N}$. With this definition, under $\mathbb{P}^{Y,\rho'}$, the random variables $(Z_{B_i,\beta}^Y)$ are *i.i.d.* distributed with the same distribution as $Z_{L,\beta}^Y$. We define

$$\mathcal{A} = \mathcal{A}(Y) := \left\{i \in \mathbb{N},\ \log Z_{B_i,\beta}^Y \geqslant (1-\varepsilon)L\mathtt{F}(\beta,\rho)\right\} \tag{3.2.8}$$

the indexes of the blocks B_i which are favorable, that is to say where the empirical jump rate of Y is $\rho > \rho'$. If one orders the elements of $\mathcal{A} = \{i_1, i_2, \ldots\}$, $i_1 < i_2 < \ldots$, and sets $i_0 = 0$, one has that under $\mathbb{P}^{Y,\rho'}$ the sequence $(i_k - i_{k-1})_{k\geqslant 1}$ is i.i.d, i_1 being a geometric random variable of parameter q.

For our purpose we find a lower bound involving $\mathtt{F}(\beta,\rho)$ for $\mathbb{E}^{Y,\rho'}\left[\log Z_{i_n,\beta}^Y\right]$, the partition function of a system of length i_nL for $n \in \mathbb{N}$. With the strategy of targeting only the blocks in $\mathcal{A}$, by (3.1.4), one gets that

$$\frac{1}{i_nL}\log Z_{i_nL,\beta}^Y \geqslant \frac{n}{i_n}\frac{1}{n}\sum_{k=1}^n \frac{1}{L}\log Z_{[i_{k-1}L,i_kL],\beta}^Y, \tag{3.2.9}$$

so that letting n goes to infinity, one has with the law of large numbers

$$0 \geqslant \mathtt{F}(\beta,\rho') \geqslant \frac{1}{\mathbb{E}^{Y,\rho'}[i_1]}\frac{1}{L}\mathbb{E}^{Y,\rho'}\left[\log Z_{i_1L,\beta}^Y\right]. \tag{3.2.10}$$

One then estimates $\mathbb{E}^{Y,\rho'}\left[\log Z_{i_1L,\beta}^Y\right]$, using again superadditivity

$$0 \geqslant \mathbb{E}^Y\left[\log Z_{i_1L,\beta}^Y\right] \geqslant \mathbb{E}^Y\left[\log Z_{(i_1-1)L,\beta}^Y\right] + (1-\varepsilon)L\mathtt{F}(\beta,\rho), \tag{3.2.11}$$

where we used that $\mathbb{E}^Y\left[\log Z^Y_{B_{i_1},\beta}\right] \geqslant (1-\varepsilon)L\mathtt{F}(\beta,\rho)$. Since one has $Z^Y_{t,\beta} \geqslant p_t(Y_t)$ for all $t\in\mathbb{R}_+$, one gets

$$\mathbb{E}^Y\left[\log Z^Y_{(i_1-1)L,\beta}\right] \geqslant \sum_{i=1}^{\infty}(1-q)q^{i-1}\mathbb{E}^Y\left[\log p_{(i-1)L}(Y_{(i-1)L})\right]$$

$$\geqslant -(1+\varepsilon)\frac{d}{2}\sum_{i=2}^{\infty}(1-q)q^{i-1}\log((i-1)L) \geqslant -(1+\varepsilon)\frac{d}{2}\log((q^{-1}-1)L), \quad (3.2.12)$$

where the second inequality comes from [BS10, Lemma 3.1], provided that L is large enough and $i>1$, and the last one comes from Jensen's inequality applied to $\mathbb{E}^Y[\log((i_1-1)L)]$. Then combining this with (3.2.11) one has

$$\mathtt{F}(\beta,\rho) \leqslant \frac{1}{(1-\varepsilon)L}(1+\varepsilon)\frac{d}{2}\log(q^{-1}L). \quad (3.2.13)$$

From Lemma 3.2.2 one has (if L is large enough)

$$\frac{\log(q^{-1}L)}{L} \leqslant (1+3\varepsilon)\frac{(\rho'-\rho)^2}{2\rho} \leqslant (1+3\varepsilon)\frac{G^2(\beta-\beta_c(\rho))^2}{2\rho}, \quad (3.2.14)$$

which, as ε is arbitrary, gives the result. □

3.3. Low temperature asymptotics, proof of Theorem 3.1.9

Our bounds are obtained by decomposing the partition function into a product, each term of the product corresponding to a time interval.

To describe our decomposition, we need some definitions. We fix a typical realization of Y. Let T_i be the time of the i-th jump. For some β (large) fixed and $i\geqslant 1$, we define the times

$$\begin{aligned} T_i^- &= T_i - \varepsilon_i^- \quad \text{with } \varepsilon_i^- = \beta^{-2/3}\wedge\frac{T_i-T_{i-1}}{2},\\ T_i^+ &= T_i + \varepsilon_i^+ \quad \text{with } \varepsilon_i^+ = \beta^{-2/3}\wedge\frac{T_{i+1}-T_i}{2}, \end{aligned} \quad (3.3.1)$$

where we used the convention that $T_0=0$, and set also $T_0^+=0$. The value $\beta^{-2/3}$ in the definition is an *ad hoc* choice for the proof of the upper bound, and has no deep signification. We also define $\varepsilon_i := \varepsilon_i^- + \varepsilon_i^+ = T_i^+ - T_i^-$. We bound $Z^Y_{T_k^+,\beta}$ by bounding the contributions of the intervals $[T_{i-1}^+, T_i^-)$ and $[T_i^-, T_i^+)$, $i=1\dots k$.

Proof Lower Bound. To lower bound $\log Z^Y_{T_k^+,\beta}$, we use superadditivity:

$$\log Z^Y_{T_k^+,\beta} \geqslant \sum_{i=1}^{k}\left(\log Z^Y_{[T_{i-1}^+,T_i^-]} + \log Z^Y_{[T_i^-,T_i^+]}\right). \quad (3.3.2)$$

Let us note that Y makes no jump on $[T_{i-1}^+, T_i^-]$, so that

$$Z^Y_{[T_{i-1}^+,T_i^-]} = \mathbb{E}^X\left[e^{L_{T_i^- - T_{i-1}^+}(X,0)}\mathbf{1}_{\{X_{T_i^- - T_{i-1}^+}=0\}}\right]. \quad (3.3.3)$$

Then, constraining X not to jump either, we get that for all $t \geqslant 0$

$$\mathbb{E}^X\left[e^{L_t(X,0)}\mathbf{1}_{\{X_t=0\}}\right] \geqslant e^{\beta t}\mathbb{P}^X\left(X_s = 0 \text{ for all } 0 \leqslant s \leqslant t\right) = e^{(\beta-1)t}. \tag{3.3.4}$$

To bound $\log Z^Y_{[T_i^-,T_i^+]}$, let use notice that for $i \geqslant 1$, Y makes one jump (and only one) in the interval $[T_i^-, T_i^+)$ (of length ε_i). Hence

$$Z^Y_{[T_i^-,T_i^+]} = \mathbb{E}_0^X\left[e^{L_{\varepsilon_i}(X,Y^{(i)})}\mathbf{1}_{\{X_{\varepsilon_i}=Y_{\varepsilon_i}\}}\right], \tag{3.3.5}$$

where $Y^{(i)} = (Y_s^{(i)})_{s\in[0,\varepsilon_i)}$ is defined by $Y_s^{(i)} = 0$ for $s \in [0, \varepsilon_i^-)$, $Y_s^{(i)} = e_1 = (1, 0, \ldots, 0)$ for $s \in [\varepsilon_i^-, \varepsilon_i)$ (by symmetry and the fact that the random walk is nearest neighbor the direction of the jump has no importance). We will compute the contribution of the terms in which X does one and only one jump, furthermore in the right direction e_1. We have

$$\begin{aligned}\mathbb{E}_0^X\left[e^{L_{\varepsilon_i}(X,Y^{(i)})}\mathbf{1}_{\{X \text{ makes one jump, } X_{\varepsilon_i}=e_1\}}\right] &= \frac{1}{2d}e^{-\varepsilon_i}\int_0^{\varepsilon_i} e^{\beta(\varepsilon_i - |s-\varepsilon_i^-|)}\,\mathrm{d}s \\ &= \frac{1}{2d}e^{-\varepsilon_i}\int_{-\varepsilon_i^-}^{\varepsilon_i^+} e^{\beta(\varepsilon_i-|s'|)}\,\mathrm{d}s' = \frac{e^{(\beta-1)\varepsilon_i}}{d\beta}\left[1 - \frac{e^{-\beta\varepsilon_i^-}}{2} - \frac{e^{-\beta\varepsilon_i^+}}{2}\right].\end{aligned} \tag{3.3.6}$$

The term $\frac{1}{2d}e^{-\varepsilon_i}\,\mathrm{d}s$ is the probability of having only one jump in $[0,\varepsilon_i]$ located in the time increment $[s, s+\mathrm{d}s]$, that goes in the right direction (cf. factor $(2d)^{-1}$), $\varepsilon_i - |s-\varepsilon_i^-|$ is the value of the intersection time of X and Y on $[0,\varepsilon_i)$ if X jumps at time s. Combining (3.3.4)-(3.3.6) with the inequality (3.3.2), we obtain

$$\frac{1}{T_k^+}\log Z^Y_{T_k^+,\beta} \geqslant (\beta-1) + \frac{k}{T_k^+}\left[-\log(d\beta) + \frac{1}{k}\sum_{i=1}^k \log\left(1 - \frac{e^{-\beta\varepsilon_i^-} + e^{-\beta\varepsilon_i^+}}{2}\right)\right]. \tag{3.3.7}$$

The sequence $\log\left(1 - \frac{1}{2}\left(e^{-\beta\varepsilon_i^-} + e^{-\beta\varepsilon_i^+}\right)\right)$ is ergodic (the dependence between terms has range only one). Then using the ergodic theorem one obtains that $\mathbb{P}^{Y,\rho}$-a.s.

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^k \log\left(1 - \frac{e^{-\beta\varepsilon_i^-} + e^{-\beta\varepsilon_i^+}}{2}\right) = \mathbb{E}^{Y,\rho}\left[\log\left(1 - \frac{e^{-\beta\varepsilon_1^-} + e^{-\beta\varepsilon_1^+}}{2}\right)\right] = o(1), \tag{3.3.8}$$

where $o(1)$ is with respect to $\beta \to \infty$. The last inequality is easy to get, as $\varepsilon_1^\pm$ are truncated exponential variables of mean $1/2$ and that the truncation at $\beta^{-1/3}$ is harmless. Moreover, by the law of large numbers, we have that $\mathbb{P}^{Y,\rho}$-a.s,

$$\lim_{k\to\infty}\frac{k}{T_k^+} = \lim_{k\to\infty}\frac{k}{T_k} = \rho. \tag{3.3.9}$$

This gives us

$$\lim_{k\to\infty}\frac{1}{T_k^+}\log Z^Y_{T_k^+,\beta} \geqslant (\beta-1) - \rho\log d\beta + o(1). \tag{3.3.10}$$

Upper Bound. We are now ready to prove the upper bound. We cut the trajectory X on the intervals $[T_{i-1}^+, T_i^-)$ and $[T_i^-, T_i^+)$ for $i \geqslant 1$ and use the properties of Y on these intervals, the way we did for the lower bound. In order to get an upper bound, we have to maximize over the contribution of intermediate points,

$$Z^Y_{T_k^+,\beta} \leqslant \prod_{i=1}^k \max_{x_1\in\mathbb{Z}^d} \mathbb{E}^X_{x_1}\left[e^{\beta L_{T_i^- - T_{i-1}^+}(X,0)}\right] \max_{x_2\in\mathbb{Z}^d} \mathbb{E}^X_{x_2}\left[e^{\beta L_{\varepsilon_i}(X,Y^{(i)})}\right]. \tag{3.3.11}$$

We can bound the first part of the terms by using Lemma 3.A.7 (that we prove later on):

$$\max_{x\in\mathbb{Z}^d} \mathbb{E}^X_x\left[e^{\beta L_t(0,X)}\right] = \mathbb{E}^X_0\left[e^{L_t(0,X)}\right] \leqslant e^{\left(\beta-1+\frac{1}{\beta}\right)t}\left(1+\frac{1}{\beta}\right), \tag{3.3.12}$$

where the first equality is due to Markov property for X applied at the first hitting time of zero, and the fact that $\mathbb{E}^X_0\left[e^{L_t(0,X)}\right]$ is a non-decreasing function of t.

For the other terms one has to analyze the contributions of all possible trajectories of X. The main contribution is given by paths X starting from zero that make one jump and such that $X_{\varepsilon_i} = e_1$: we already computed the value of this contribution in (3.3.6). If X makes no jump or one jump but not in the right direction (or if X makes at most one jump but does not start from zero), it spends some portion of the time away from Y and then $L_{\varepsilon_i}(X,Y^{(i)}) \leqslant \varepsilon_1^- \vee \varepsilon_i^+ \leqslant \beta^{-2/3}$. Therefore the total contribution of such paths is bounded by $e^{\beta^{1/3}}$. The probability that X makes more than two jumps is bounded by $4\beta^{-4/3}$ if β is large enough (the number of jump is a Poisson variable of parameter ε_i, which is at most $2\beta^{-2/3}$). In addition $e^{\beta L_{\varepsilon_i}(X,Y^{(0)})} \leqslant e^{\beta\varepsilon_i}$ so that the total contribution of paths making more than two jumps is bounded by $4\beta^{-4/3}e^{\beta\varepsilon_i}$. Hence we have

$$\begin{aligned}\max_{x\in\mathbb{Z}^d} \mathbb{E}^X_x\left[e^{\beta L_{\varepsilon_i}(X,Y^{(i)})}\right] &\leqslant \frac{1}{d\beta}e^{(\beta-1)\varepsilon_i} + e^{\beta^{1/3}} + 4\beta^{-4/3}e^{\beta\varepsilon_i}\\ &\leqslant \frac{1}{d\beta}e^{(\beta-1)\varepsilon_i}\left(1+C(\beta e^{\beta^{1/3}-\beta\varepsilon_i}+\beta^{-1/3})\right).\end{aligned} \tag{3.3.13}$$

Combining all these inequalities one finally gets

$$Z^Y_{T_k^+,\beta} \leqslant e^{(\beta-1+\beta^{-1})T_k^+}(1+\beta^{-1})^k(d\beta)^{-k}\prod_{i=1}^k\left(1+C(\beta e^{\beta^{1/3}-\beta\varepsilon_i}+\beta^{-1/3})\right), \tag{3.3.14}$$

and hence

$$\begin{aligned}&\frac{1}{T_k^+}\log Z^Y_{T_k^+,\beta} \leqslant \beta - 1 + \beta^{-1}\\ &+\frac{k}{T_k^+}\left[\log(1+\beta^{-1}) - \log(d\beta) + \frac{1}{k}\sum_{i=1}^k \log\left(1+C(\beta e^{\beta^{1/3}-\beta\varepsilon_i}+\beta^{-1/3})\right)\right].\end{aligned} \tag{3.3.15}$$

Applying the ergodic theorem, one obtains that $\mathbb{P}^{Y,\rho}$-a.s.

$$\begin{aligned}&\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^k \log\left(1+C(\beta e^{\beta^{1/3}-\beta\varepsilon_i}+\beta^{-1/3})\right)\\ &\qquad = \mathbb{E}^{Y,\rho}\left[\log\left(1+C(\beta e^{\beta^{1/3}-\beta\varepsilon_1}+\beta^{-1/3})\right)\right] = o(1),\end{aligned} \tag{3.3.16}$$

where we used the definition of ε_1 to estimate the last expectation (ε_i is equal to $2\beta^{-2/3}$ with probability $1 - O(\beta^{-2/3})$ when β is large).

Furthermore, as already noticed, k/T_k^+ converges almost surely to ρ, so that we have

$$\lim_{k\to\infty} \frac{1}{T_k^+} \log Z^Y_{T_k^+,\beta} \leqslant \beta - 1 - \rho \log \beta d + o(1). \tag{3.3.17}$$

□

3.A. The homogeneous case

We give in this section several results on the pure model, which are to be compared with the results on the quenched system. In the homogeneous case, (when $\rho = 0$) the model is just the pinning of a random walk on a deterministic defect line $\mathbb{R}_+ \times \{0\}$. It turns out here that a more general view point makes the problem easier to solve, and that is the reason why we introduce now a more general version of our pinning model.

We consider two increasing sequences $(\tau'_i)_{i\geqslant 1}$ and $(\tau_i)_{i\geqslant 0}$ such that $\tau_0 = 0$, and $(\tau'_{i+1} - \tau_i)_{i\geqslant 0}$ and $(\tau_i - \tau'_i)_{i\geqslant 1}$ are two independent *i.i.d.* sequences, where τ'_1 is a mean 1 exponential variable, and where the distribution of $\tau_1 - \tau'_1$ has support in $\mathbb{R}_+ \cup \{\infty\}$ (if $\tau_n = \infty$ for some n, we choose by convention τ'_k and $\tau_k = \infty$ for all $k \geqslant n$). We further assume that the distribution of $(\tau_1 - \tau'_1)_{1\geqslant 0}$, when restricted to $\mathbb{R}_+$, is absolutely continuous with respect to the Lebesgue measure, with density that we denote by $K(\cdot)$, and that $\int_0^\infty \exp(\varepsilon t)K(t)\,\mathrm{d}t = \infty$ for all $\varepsilon > 0$. We denote by μ the joint law of the two sequences, and we remark that under μ, both sequences $(\tau_i)_{i\geqslant 0}$ and $(\tau'_i - \tau'_1)$ are renewal sequences. We may use the notation $K(\infty) = \mu(\tau_1 - \tau'_1 = \infty) = \mu(\tau_1 = \infty)$.

Set $\mathcal{T} := \bigcup_{i=0}^\infty [\tau_i, \tau'_i)$, and call it the set of contact.

Given $\beta \in \mathbb{R}$, we now modify the law of the sequences $(\tau'_i)_{i\geqslant 0}$ and $(\tau_i)_{i\geqslant 0}$ by introducing a Gibbs transform $\mu^{\mathrm{pin}}_{t,\beta}$ of the measure μ

$$\frac{\mathrm{d}\mu^{\mathrm{pin}}_{t,\beta}}{\mathrm{d}\mu} = \frac{e^{\beta|\mathcal{T}\cap[0,t]|}}{Z^{\mathrm{pin}}_{t,\beta}} \mathbf{1}_{\{t\in\mathcal{T}\}}, \tag{3.A.1}$$

where $|A|$ stands for the Lebesgue measure of a set $A \subset \mathbb{R}$, and where

$$Z^{\mathrm{pin}}_{t,\beta} = \mu\left[e^{\beta|\mathcal{T}\cap[0,t]|}\mathbf{1}_{\{t\in\mathcal{T}\}}\right]. \tag{3.A.2}$$

We also define

$$\mathtt{F}(\beta) := \lim_{t\to\infty} \frac{1}{t} \log Z^{\mathrm{pin}}_{t,\beta} \tag{3.A.3}$$

which is well defined, by superadditivity.

Remark 3.A.1. In the case of a continuous time random walk X with jump rate 1 (of law $\mathbb{P}^X$), we set $\tau_0 = 0$ and for all $i \geqslant 1$

$$\begin{aligned} \tau'_i &:= \inf\{t > \tau_{i-1}, X_t \neq 0\}, \\ \tau_i &:= \inf\{t > \tau'_i, X_t = 0\}. \end{aligned} \tag{3.A.4}$$

One can check that $(\tau_i - \tau'_i)_{i \geqslant 1}$ and $(\tau'_{i+1} - \tau_i)_{i \geqslant 0}$ are independent *i.i.d.* sequences (for $d \geqslant 3$, there are only finitely many terms in the sequences) that satisfies the assumptions given above. Therefore, our definition (3.1.1) of $\mu^{\text{pin}}_{t,\beta}$ (with Y replaced with 0) coincides with the one of (3.A.1). This underlines two things:

- The pinning model we present in this section is indeed a generalization of the pure (or annealed) model for the random walk-pinning.
- In annealed random-walk pinning, the Gibbs transformation changes only the return time to zero and the time X spends on zero. Conditionally on these times, the law of the excursions out of the origin remains the same that under $\mathbb{P}^X$.

We can describe the measure $\mu^{\text{pin}}_{t,\beta}$ in a very simple way, because we are interested only in the law of $\mathcal{T} \cap [0,t]$ (as it is the only part that is modified by the Gibbs transformation). We introduce some definitions to describe the measure.

If $(1-\beta)^{-1}\int_0^\infty K(t)\,\mathrm{d}t \geqslant 1$ or $\beta \geqslant 1$, let $b \geqslant 0$ be defined by

$$(1-\beta+b)^{-1}\int_0^\infty e^{-bt}K(t)\,\mathrm{d}t = 1 \tag{3.A.5}$$

and $b = 0$ if $(1-\beta)^{-1}\int_0^\infty K(t)\,\mathrm{d}t < 1$.
For notational reasons, define $\lambda := (1-\beta+b)$. Then, we define $\widetilde{K}^\beta(t) := \lambda^{-1}e^{-bt}K(t)$ for $t \in (0,\infty)$, and $\widetilde{K}^\beta(\infty) = 1 - \int_0^\infty \widetilde{K}^\beta(t)\,\mathrm{d}t$. Finally, let $\widetilde{\mu}^\beta$ be another probability law for (τ,τ') defined by:

- $\tau_0 = 0$ $\widetilde{\mu}^\beta$-a.s.
- $(\tau'_{i+1} - \tau_i)_{i \geqslant 0}$ and $(\tau_i - \tau'_i)_{i \geqslant 1}$ are independent *i.i.d.* sequences,
- τ'_1 is an exponential variable of mean λ^{-1},
- $\tau_1 - \tau'_1$ has support $\mathbb{R}_+ \cup \{\infty\}$. On $\mathbb{R}_+$, its law is absolutely continuous w.r.t. Lebesgue measure with density $\widetilde{K}^\beta(\cdot)$, and $\widetilde{\mu}^\beta(\tau_1 - \tau'_1 = \infty) = \widetilde{K}^\beta(\infty)$.

Let $\mathcal{F}_t$ denote the sigma algebra generated by $\mathcal{T} \cap [0,t]$. We have the following lemma, describing the measure $\mu^{\text{pin}}_{t,\beta}$.

Lemma 3.A.2. *For any $A \in \mathcal{F}_t$, one has*

$$\mu\left[\mathbf{1}_A\, e^{\beta|\mathcal{T}\cap[0,t]|}\mathbf{1}_{\{t\in\mathcal{T}\}}\right] = e^{bt}\widetilde{\mu}^\beta(A \cap \{t \in \mathcal{T}\}). \tag{3.A.6}$$

As a consequence

$$\mu^{\text{pin}}_{t,\beta}(A) := \widetilde{\mu}^\beta(A \mid t \in \mathcal{T}). \tag{3.A.7}$$

Proof We write $Z^{\text{pin}}_{t,\beta}(A) := \mu\left[\mathbf{1}_A\, e^{\beta|\mathcal{T}\cap[0,t]|}\mathbf{1}_{\{t\in\mathcal{T}\}}\right]$, and we decompose $Z^{\text{pin}}_{t,\beta}(A)$ according to the number of jumps made before t. As $A \in \mathcal{F}_t$, $\mathbf{1}_A$ can be written as a function of $(\{\tau_i \mid \tau_i < t\}, \{\tau'_i \mid \tau'_i < t\})$ and one has the following integral form for

$Z^{\mathrm{pin}}_{t,\beta}(A)$,

$$\begin{aligned}
Z^{\mathrm{pin}}_{t,\beta}(A) &= \sum_{n=0}^{\infty} \int_{0\leqslant t'_1 \leqslant t_1 \leqslant \ldots \leqslant t'_n \leqslant t_n < t} \mathbf{1}_A\, e^{(\beta-1)(t-t_n)} \prod_{i=1}^{n} e^{(\beta-1)(t'_i - t_{i-1})} K(t_i - t'_i)\,\mathrm{d}t'_i\,\mathrm{d}t_i \\
&= e^{bt} \sum_{n=0}^{\infty} \int_{0\leqslant t'_1 \leqslant t_1 \leqslant \ldots \leqslant t'_n \leqslant t_n < t} \mathbf{1}_A\, \lambda e^{\lambda(t_n - t)} \prod_{i=1}^{n} e^{\lambda(t'_i - t_{i-1})} \widetilde{K}^{\beta}(t_i - t'_i)\,\mathrm{d}t'_i\,\mathrm{d}t_i \\
&= e^{bt} \widetilde{\mu}^{\beta}(A \,\cap\, \{t \in \mathcal{T}\}). \quad (3.A.8)
\end{aligned}$$

□

We can now prove some statements from Proposition 3.1.2,

Proposition 3.A.3. *We have, for b defined as above in (3.A.5), $b = \mathtt{F}(\beta)$. Moreover, b can alternatively be defined by*

$$\int_0^{\infty} e^{-bt} \mu(t \in \mathcal{T})\,\mathrm{d}t := \beta^{-1}, \quad (3.A.9)$$

if the equation has a solution and $b = 0$ if not. Let $\beta_c = \inf\{\beta, \mathtt{F}(\beta) > 0\}$, then

$$\beta_c := \left(\int_0^{\infty} \mu(t \in \mathcal{T})\,\mathrm{d}t \right)^{-1}. \quad (3.A.10)$$

Moreover, if $b > 0$, or if $\beta = \beta_c$ and $\int_0^\infty tK(t)\,\mathrm{d}t < \infty$, then

$$\lim_{t\to\infty} \widetilde{\mu}^{\beta}(t \in \mathcal{T}) = \frac{1}{1 + \int_0^\infty e^{-bt} tK(t)\,\mathrm{d}t}. \quad (3.A.11)$$

Remark 3.A.4. In the case of the homogeneous Random Walk Pinning Model, one can get the asymptotics of $\mathtt{F}(\beta)$ around β_c given in Proposition 3.1.2, by using the local central limit Theorem for X_t (see [Spi01, Prop.7.9, Ch.II] for the discrete time version, the proof being identical for continuous time). We have

$$\mu(t \in \mathcal{T}) = p_t(0) = (cst. + o(1)) t^{-d/2}. \quad (3.A.12)$$

Then, Proposition 3.1.2 follows from (3.A.9), and an application of an Abelian theorem (see [Gia07, Th.2.1] for the discrete case).

Proof We start with the proof of the last item. Thanks to the Markov property, one has the following recursion equation

$$\begin{aligned}
\widetilde{\mu}^{\beta}(t \in \mathcal{T}) &= \widetilde{\mu}^{\beta}(\tau'_1 \geqslant t) + \int_0^t \widetilde{\mu}^{\beta}\left(t \in \mathcal{T},\ \tau_1 \in [s, s+\mathrm{d}s)\right)\mathrm{d}s \\
&= \exp(-\lambda t) + \int_0^t \widetilde{\mu}^{\beta}(\tau_1 \in [s, s+\mathrm{d}s))\widetilde{\mu}^{\beta}(t - s \in \mathcal{T})\,\mathrm{d}s. \quad (3.A.13)
\end{aligned}$$

By the key renewal theorem, [Asm03, Th.4.7, Ch.V], one has

$$\lim_{t\to\infty} \widetilde{\mu}^{\beta}(t \in \mathcal{T}) := \frac{\int_0^\infty e^{-\lambda t}\,\mathrm{d}t}{\lambda^{-1} + \int_0^\infty t\widetilde{K}^{\beta}(t)\,\mathrm{d}t} = \frac{1}{1 + \int_0^\infty e^{-bt} tK(t)\,\mathrm{d}t}. \quad (3.A.14)$$

When $b > 0$ or $K(t)$ is integrable, the limit is positive. In that case equation (3.A.6) with A equals Ω to the full space gives

$$Z^{\mathrm{pin}}_{t,\beta} = (cst. + o(1)) \exp(bt), \tag{3.A.15}$$

so that $b = \mathtt{F}(\beta)$. For all the other cases, we have necessarily $\mathtt{F}(\beta) \leqslant 0$ as $Z^{\mathrm{pin}}_{t,\beta} \leqslant \lambda^{-1}$. To get that $\mathtt{F}(\beta) = 0$ it is therefore sufficient to prove that $\mathtt{F}(\beta)$ is non-negative. This is done for the random-walk pinning in [BS10], here it could be done using the assumption $\int_0^\infty e^{\varepsilon t} K(t)\,\mathrm{d}t = \infty$ for all ε (which is also necessary).

Now, we turn to the proof of (3.A.9). Let $K_1(t) = e^{-t}$ be the density with respect to the Lebesgue measure of τ_1' (under μ). For $t > 0$,

$$\mu(t \in \mathcal{T}) = e^{-t} + \int_0^t \sum_{n=1}^\infty \mu(\tau_n \in [s, s+\,\mathrm{d}s)) e^{-(t-s)}\,\mathrm{d}s = \sum_{n=0}^\infty [(K_1 * K)^{*n} * K_1](t). \tag{3.A.16}$$

Therefore using the fact that Laplace transform transforms convolutions into products, one obtains, for all $b > 0$

$$\begin{aligned} \int_0^\infty e^{-bt} \mu(t \in \mathcal{T})\,\mathrm{d}t &= \sum_{n=0}^\infty \left(\int_0^\infty e^{-(b+1)t}\,\mathrm{d}t \right)^{n+1} \left(\int_0^\infty e^{-bt} K(t)\,\mathrm{d}t \right)^n \\ &= \frac{1}{1+b} \frac{1}{1 - \frac{1}{1+b}\int_0^\infty e^{-bt} K(t)\,\mathrm{d}t}, \end{aligned} \tag{3.A.17}$$

which with (3.A.5) gives us the right result (the case $\mathtt{F}(\beta) = 0$ is obtained by continuity and non-negativity of the free-energy). The value of β_c is then an easy consequence. □

We now give a Corollary that describes the local intersection time $L_t(X, 0)$ under $\mu^{\mathrm{pin}}_{t,\beta}$.

Corollary 3.A.5. *When $b > 0$ or when $\beta = \beta_c$ and $\int_0^\infty tK(t)\,\mathrm{d}t < \infty$, $\frac{|\mathcal{T} \cap [0,t]|}{t}$ under $\mu^{\mathrm{pin}}_{t,\beta}$ converges in probability to*

$$\frac{1}{1 + \int_0^\infty e^{-bt} tK(t)\,\mathrm{d}t} > 0. \tag{3.A.18}$$

Proof As $\int_0^\infty tK(t)\,\mathrm{d}t < \infty$, the law of large numbers (applied first for the renewal process τ and then to the sum of independent exponential times) tells us that

$$\lim_{t\to\infty} \frac{|\mathcal{T} \cap [0,t]|}{t} = \frac{1}{1 + \int_0^\infty e^{-bt} tK(t)\,\mathrm{d}t}, \qquad \widetilde{\mu}^\beta - a.s, \tag{3.A.19}$$

and therefore the convergence also holds in probability.

Restricted on $\mathcal{F}_t$, the measure $\mu^{\mathrm{pin}}_{t,\beta}$ is equal to $\widetilde{\mu}^\beta(\cdot\ | t \in \mathcal{T})$ and we also have that $\widetilde{\mu}^\beta(t \in \mathcal{T})$ is bounded away from zero by (3.A.11). This gives us that the law of $\frac{|\mathcal{T} \cap [0,t]|}{t}$ under $\mu^{\mathrm{pin}}_{t,\beta}$ converges in probability to the same limit. □

Remark 3.A.6. In dimension d, as noted in Remark 3.A.4, the local central limit Theorem gives

$$\mu(t \in \mathcal{T}) = \mathbb{P}^X(X_t = 0) = (cst. + o(1))t^{-d/2}. \tag{3.A.20}$$

For $d \geqslant 5$ this implies that $\int_0^\infty tK(t)\,\mathrm{d}t < \infty$. Indeed for t large enough

$$\mathbb{P}^X(X_t = 0) \geqslant \int_{0<t_1<t_2<t} e^{-t_1} K(t_2 - t_1) e^{-(t-t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 \geqslant cst. \int_{t-2}^{t-1} K(s)\,\mathrm{d}s, \tag{3.A.21}$$

so that $\int_{t-2}^{t-1} K(s)\,\mathrm{d}s = O(t^{-d/2})$. Therefore, one can apply Corollary 3.A.5 to get that $L_t(X,0)$ is of order t for $\beta = \beta_c$.

We present here an advanced version of (3.1.11), which was used for the proof of the upper bound in Theorem 3.1.9.

Lemma 3.A.7. *For any value of t, for any random walk X with jump rate 1 one has*

$$e^{(\beta-1)t} \leqslant \mathbb{E}^X\left[e^{L_t(X,0)}\right] \leqslant e^{\left(\beta-1+\frac{1}{\beta}\right)t}\left(1+\frac{1}{\beta}\right). \tag{3.A.22}$$

Proof The left hand side inequality is simply obtained by considering the contribution of trajectories that never jumps. To obtain the other inequality we decompose the partition function according to the time t_i', t_i that are respectively the i-th jump out of zero, and the i-th return to zero. We write $Z_{t,\beta} = \mathbb{E}^X\left[e^{L_t(X,0)}\right]$,

$$Z_{t,\beta} = e^{(\beta-1)t} \sum_{n=0}^{\infty} \int_{0<t_1'<t_1<\dots<t_n<t} \prod_{i=1}^{n} e^{-\beta(t_i'-t_{i-1})} K(t_i - t_i')\,\mathrm{d}t_i'\,\mathrm{d}t_i$$
$$\left[1 + \int_{t_n}^{t} e^{-\beta(t-t_{n+1}')}\,\mathrm{d}t_{n+1}' \int_t^\infty K(t_{n+1} - t_{n+1}')\,\mathrm{d}t_{n+1}\right]. \tag{3.A.23}$$

Then one remark that for the random walk $K(t) \leqslant 1$ for all t. Indeed the probability that after the first jump, the first excursion returns within a time in the interval $[t, t+\,\mathrm{d}t]$ (which is equal to $K(t)\,\mathrm{d}t$) is smaller than the probability that X makes a jump in the interval $[t,\,\mathrm{d}t]$ (which is equal to $\,\mathrm{d}t$). Hence

$$\begin{aligned} \int_{t_{i-1}}^{t_i} e^{-\beta(t_i'-t_{i-1})} K(t_i - t_i')\,\mathrm{d}t_i' &\leqslant \frac{1}{\beta}, \\ \int_{t_n}^{t} e^{-\beta(t-t_{n+1}')}\,\mathrm{d}t_{n+1}' &\leqslant \frac{1}{\beta} \\ \int_t^\infty K(t_{n+1} - t_{n+1}')\,\mathrm{d}t_{n+1} &\leqslant 1. \end{aligned} \tag{3.A.24}$$

Therefore

$$\begin{aligned} Z_{t,\beta} &\leqslant e^{(\beta-1)t}\sum_{n=0}^{\infty}\int_{0<t_1<\dots<t_n}\frac{1}{\beta^n}\left(1+\frac{1}{\beta}\right)\,\mathrm{d}t_1\dots\mathrm{d}t_n \\ &= e^{(\beta-1)t}\left(1+\frac{1}{\beta}\right)\sum_{n=0}^{\infty}\frac{t^n}{\beta^n n!} \end{aligned} \tag{3.A.25}$$

which is exactly the result. □

Part 2

Modèle d'accrochage en environnement corrélé

CHAPTER 4

Hierarchical pinning model in correlated random environment

4.1. Introduction

It is widely expected, on general grounds, that correlations in the environment may change qualitatively the Harris criterion (discussed in Introduction and in the introductive Chapter 1): it is replaced by the Weinrib and Halperin prediction that has already been mentioned in Section 1.4.3. The study of the random pinning model with correlated disorder is still in a rudimentary form. In [Poi11, Poi12] a case with finite-range correlations was studied, and no modification of the Harris criterion was found. On the other extreme, in Chapter 6 and 7, not only correlations decay in a power-law way, but potentials are so strongly correlated that in a system of length N there are typically favorable regions (in a sense we discuss in Chapters 6-7) of size $\gg \log N$. In this case, we are able to compute the critical point and to give sharp estimates on the critical behavior for $\beta > 0$. In particular, we find that an arbitrarily small amount of disorder *does* change the critical exponent, irrespective of the value of the non-disordered critical exponent ν^{pur}.

Hierarchical models on diamond lattices, homogeneous or disordered [Ble89, CEGM84, DG84], are a powerful tool in the study of the critical behavior of statistical mechanics models, especially because real-space renormalization group transformations à la Migdal-Kadanoff are exact in this case. In this spirit, in the present work we consider the hierarchical version of the pinning model introduced in the *i.i.d.* setting in [DHV92] and discussed in Section 1.3.4. We consider the case where disorder is Gaussian and its correlation structure respects the hierarchical structure of the model: the correlation between the potential at i and j is given by $\kappa^{d(i,j)}$, where $0 < \kappa < 1$ and $d(i,j)$ is the tree distance between i and j on a binary tree. The Weinrib-Halperin criterion in this context would say that the following equivalence holds

$$\text{irrelevance} \Longleftrightarrow \max(\kappa, 1/2) < B^2/4, \tag{4.1.1}$$

where we recall that $B \in (1,2)$ is a parameter that defines the geometry of the diamond lattice.

A closer inspection of the model, however, shows easily that *the phase transition does not survive* for $\kappa > 1/2$ (cf. Section 4.3.1). When instead correlations are summable (which corresponds to $\kappa < 1/2$) we find, in agreement with (4.1.1), irrelevance if $B > \sqrt{2}$ (see Theorem 4.3.3 and Proposition 4.5.1). As for $B \leqslant \sqrt{2}$, again we find agreement with the Weinrib-Halperin criterion: disorder is relevant (see Proposition 4.3.5) and if in addition $\kappa < B^2/4$, the model behaves like in the *i.i.d.* case as far as the difference between quenched and annealed critical points is concerned, see Theorem 4.3.3. The crucial step in proving Theorem 4.3.3 (and Proposition 4.5.1) is to show that for $\kappa < \min(1/2, B^2/4)$ the Gibbs measure of the annealed system near the annealed critical point is close (in a suitable sense) to the Gibbs measure of the homogeneous system near its critical point (cf. Theorem 4.3.1 and Proposition 4.3.2). This is the step that requires the most technical work, in particular because the annealed critical point is not known explicitly for $\kappa \neq 0$. Once this is done, the proof of disorder relevance/irrelevance according to $B \lessgtr \sqrt{2}$ can be obtained generalizing the ideas that were developed for the *i.i.d.* model.

Finally, the region $B^2/4 < \kappa < 1/2, B < \sqrt{2}$ reserves somewhat of a surprise: while we are not able to capture sharply the behavior of the annealed model and of the difference between quenched and annealed critical points (as we do for $\kappa < \min(1/2, B^2/4)$, see Theorem 4.3.1, Proposition 4.3.2 and Theorem 4.3.3), we can prove that the annealed model has a different critical behavior than the homogeneous model with the same value of B. In particular, the contact fraction at the annealed critical point scales qualitatively differently (as a function of the system size) than for the homogeneous model, see Equation (4.4.33). In view of Theorem 4.3.1 mentioned above, this means that if we fix $B < \sqrt{2}$ and we increase κ starting from 0, at $\kappa = B^2/4$ the annealed system has a "phase transition" where its critical properties change. As we discuss in Section 4.3.1, this suggests that, while for $\kappa < B^2/4$ the annealed free energy near the annealed critical point $h_c^{\mathrm{a}}(\beta)$ has a singularity of type $(h - h_c^{\mathrm{a}}(\beta))^{\nu^{\mathrm{pur}}}$ and $\nu^{\mathrm{pur}} = \log_2 / \log(2/B)$, for $B^2/4 < \kappa < 1/2$ the annealed free energy should vanish as $h \searrow h_c^{\mathrm{a}}(\beta)$ with a larger exponent.

As a side remark, let us recall that Dyson [Dys69] used a *hierarchical* ferromagnetic Ising model (which, at least formally, resembles very much our annealed pinning model, cf. (4.3.2)) plus the Griffiths correlation inequalities, to derive criteria for existence of a ferromagnetic phase transition for a *non-hierarchical*, one-dimensional Ising ferromagnet with couplings decaying as $J_{i-j} \sim |i-j|^{-\xi}$. We stress that, in contrast, in our case there are no available correlation inequalities which would allow to infer directly results on the non-hierarchical pinning model starting from the hierarchical one. It is however possible to give reliable conjecture on the disordered (non-hierarchical) pinning model

Let us now give an overview of the organization of the Chapter:

• In Section 4.2 we define the model and give preliminary results, in particular on the homogeneous case, and we state our main results in Section 4.3;

• In Section 4.3.1 we discuss the case $\kappa > 1/2$, showing that the phase transition does not survive;

• In Section 4.4, we study in detail the annealed model, giving first some preliminary tools (Section 4.4.1), then looking at the case $\kappa < 1/2 \wedge B^2/4$ and proving Theorem 4.3.1 and Proposition 4.3.2 (Section 4.4.2), and finally focusing on the case $B^2/4 < \kappa < 1/2$ (Section 4.4.3);

• In Section 4.5 we prove disorder irrelevance for $\kappa < 1/2$, $B > \sqrt{2}$, and in section 4.6 we prove disorder relevance for $\kappa < 1/2 \wedge B^2/4$, $B \leqslant \sqrt{2}$.

4.2. Model and preliminaries

4.2.1. Hierarchical pinning model with hierarchically correlated disorder. We recall the hierarchical pinning model, introduced in Section 1.3. Let $1 < B < 2$. We consider the following iteration

$$Z_{n+1}^{(i)} = \frac{Z_n^{(2i-1)} Z_n^{(2i)} + B - 1}{B}, \tag{4.2.1}$$

for $n \in \mathbb{N} \cup \{0\}$ and $i \in \mathbb{N}$. We study the case in which the initial condition is random and given by $Z_0^{(i)} = e^{\beta\omega_i + h}$, with $h \in \mathbb{R}$, $\beta \geqslant 0$ and where $\omega := \{\omega_i\}_{i\in\mathbb{N}}$ is a sequence of centered Gaussian variables, whose law is denoted by $\mathbb{P}$. One defines the law $\mathbb{P}$ thanks to the correlations matrix K and note $\kappa_{ij} := \mathbb{E}[\omega_i\omega_j]$. We interpret $Z_n^{(i)}$ as the partition function on the i^{th} block of size 2^n.

We make the very natural choice of restricting to a correlation structure of hierarchical type. For $p \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, recall the definition

$$I_{k,p} := \{(k-1)2^p + 1, \dots, k2^p\} \tag{4.2.2}$$

of the k^{th} block of size 2^p. We define the hierarchical distance $d(\cdot,\cdot)$ on $\mathbb{N}$ by establishing that $d(i,j) = p$ if i,j are contained in the same block of size 2^p but not in the same block of size 2^{p-1}. In other words, $d(i,j)$ is just the tree distance between i and j, if $\mathbb{N}$ is seen as the set of the leaves of an infinite binary tree.

We assume that κ_{ij} depends only on $d(i,j)$ and for $d(i,j) = p$ we write $\kappa_{ij} =: \kappa_p$ ($\kappa_0 = 1$), with $\kappa_p \geqslant 0$ for every p. Actually, we make the explicit choice

$$\kappa_p = \kappa^p \qquad \text{for some} \qquad 0 < \kappa < 1/2. \tag{4.2.3}$$

It is standard that such a Gaussian law actually exists, as seen in Section 1.3.4.

We see in Section 4.3.1 that the reason why we exclude the case $\kappa \geqslant 1/2$ is that the model becomes less interesting: there is no phase transition for the quenched model and the annealed model is not well defined. For $\kappa = 0$, one recovers the model with *i.i.d.* disorder.

We point out that all our results can be easily extended to the case where $\kappa := \lim_{p\to\infty} |\kappa_p|^{1/p}$ exists and is in $(0, 1/2)$.

We already mentioned the Galton-Watson interpretation of the partition function in Section 1.3, and we recall it here briefly. We take $1 < B < 2$, and we set $\mathbf{P}_n$ the law of a Galton-Watson tree $\mathcal{T}_n$ of depth $n+1$, where the offspring distribution concentrates on 0 with probability $\frac{B-1}{B}$ and on 2 with probability $\frac{1}{B}$. We define the set $\mathcal{R}_n \subset \{1, \dots, 2^n\}$ of individuals that are present at the n^{th} generation (which are the leaves of $\mathcal{T}_n$).

One has the useful following Proposition

Proposition 4.2.1 ([GLT10b], Proposition 4.1)**.** *For any $n \geqslant 0$ and given a subset $I \subset \{1, \dots, 2^n\}$, one defines $\mathcal{T}_I^{(n)}$ to be the subtree of the standard binary tree of depth $n+1$, obtained by deleting all the edges, except those which link leaves $i \in I$ to the root. We note $v(n, I)$ the number of nodes of $\mathcal{T}_I^{(n)}$, with the convention that* *leaves are not counted as nodes, while the root is. Then one has*

$$\mathbf{E}_n\left[\delta_I\right] = B^{-v(n,I)}, \tag{4.2.4}$$

where $\delta_I := \prod_{i\in I}\delta_i$ *and where* $\delta_i = 1$ *if the individual* i *is present at generation* n *(i.e. if* $i \in \mathcal{R}_n$*), and* $\delta_i = 0$ *otherwise. In particular* $\mathbf{E}_n[\delta_i] = B^{-n}$ *for every* $i \in \{1, \dots, 2^n\}$.

Using the recursive structure of the Galton-Watson tree $\mathcal{T}_n$, one can rewrite the partition function as

$$Z_n^{(i)} = \mathbf{E}_n\left[\exp\left(\sum_{k=1}^{2^n}(\beta\omega_{2^n(i-1)+k} + h)\delta_k\right)\right], \tag{4.2.5}$$

since it satisfies the iteration (4.2.1) and the correct initial condition $Z_0^{(i)} = \exp(\beta\omega_i + h)$. In the sequel, we write $Z_{n,h}^{\omega,(i)}$ instead of $Z_n^{(i)}$ (we keep explicitly the dependence on h and on ω, the dependence on β being implicit to get simpler notations), and $Z_{n,h}^{\omega}$ for $Z_{n,h}^{\omega,(1)}$ if there is no ambiguity.

It is convenient to define

$$H_{n,h}^{\omega,(i)} = \sum_{k\in I_{i,n}}(\beta\omega_k + h)\delta_k \tag{4.2.6}$$

as the Hamiltonian on the i^{th} block of size 2^n (we also write $H_{n,h}^{\omega}$ for $H_{n,h}^{\omega,(1)}$ if there is no ambiguity). This allows to introduce the polymer measure

$$\frac{\mathrm{d}\mathbf{P}_{n,h}^{\omega}}{\mathrm{d}\mathbf{P}_n} := \frac{1}{Z_{n,h}^{\omega}}\exp\left(H_{n,h}^{\omega}\right). \tag{4.2.7}$$

We then define the *quenched* free energy of the model,

Theorem 4.2.2. *The limit*

$$\mathtt{F}(\beta, h) := \lim_{n\to\infty}\frac{1}{2^n}\log Z_{n,h}^{\omega} \overset{\mathbb{P}-a.s}{=} \lim_{n\to\infty}\frac{1}{2^n}\mathbb{E}[\log Z_{n,h}^{\omega}]. \tag{4.2.8}$$

exists $\mathbb{P}$*-almost surely and in* $L^1(\mathrm{d}\mathbb{P})$*, is almost surely constant and non-negative. It is called the quenched free energy. Moreover,* $\mathtt{F}_n(\beta,h) := \frac{1}{2^n}\mathbb{E}[\log Z_{n,h}^{\omega}]$ *converges exponentially fast to* $\mathtt{F}(\beta,h)$*, and more precisely one has for all* $n \geqslant 1$

$$\mathtt{F}_n(\beta,h) - \frac{1}{2^n}\log B \leqslant \mathtt{F}(\beta,h) \leqslant \mathtt{F}_n(\beta,h) + \frac{1}{2^n}\log\left(\frac{B^2+B-1}{B(B-1)}\right). \tag{4.2.9}$$

One has that $\mathtt{F}(\beta,h) < \infty$ trivially from $Z_{n,h}^{\omega} \leqslant \exp(\sum_{i=1}^{2^n}(\beta|\omega_i| + h))$.

Proof The convergence of $\mathtt{F}_n(\beta,h)$ is classical, and the proof is similar to the one of Proposition 4.2.3. Then, for any $\varepsilon > 0$, we define

$$\Psi_{n,\varepsilon} := \mathbb{P}\left(|\log Z_{n,h}^{\omega,\beta} - \mathbb{E}\log Z_{n,h}^{\omega,\beta}| \geqslant \varepsilon 2^n\right),$$

and we show that $\sum_{n\in\mathbb{N}}\Psi_{n,\varepsilon} < \infty$, so that we are able to use Borel-Cantelli Lemma to prove the almost-sure convergence in (4.2.8). We mention that this concentration condition (of $\log Z_{n,h}^{\omega,\beta}$ around its mean) is actually very general to prove the existence of the quenched free energy.

Let us briefly estimate $\Psi_{n,\varepsilon}$, recalling the following inequality for functions of a Gaussian vector $\widehat{\omega} \in \mathbb{R}^m$ of *i.i.d.* standard $\mathcal{N}(0,1)$ with law $\widehat{\mathbb{P}}$: if $f : \mathbb{R}^m \to \mathbb{R}$ is a function with Lipschitz norm L, then $\widehat{\mathbb{P}}(|f(\widehat{\omega}) - \widehat{\mathbb{E}}[f(\widehat{\omega})]| \geqslant \varepsilon) \leqslant 2e^{-t^2/(4L^2)}$.

We adapt the construction (1.3.18) for the (finite) vector $\omega^{(n)} := \{\omega_i\}_{1 \leqslant i \leqslant 2^n}$. Set $\mathcal{I}^{(n)} := \{I_{k,p}, p \in \{0,\dots,n\}, k \in \{1,\dots,2^{n-p}\}\}$ (the set of blocks in a system of size 2^n), with $I_{k,p}$ defined in (4.2.2), and let $\widehat{\omega}^{(n)} = \{\widehat{\omega}_I\}_{I \in \mathcal{I}^{(n)}}$ be a Gaussian vector of *i.i.d.* standard $\mathcal{N}(0,1)$ variables. Then, one has the following equality in law

$$\omega_i = \sum_{\substack{I \in \mathcal{I}^{(n)} \\ I \ni i}} \widehat{\kappa}_I \widehat{\omega}_I, \qquad \text{for all } i \in \{1,\dots,2^n\}, \tag{4.2.10}$$

where $\widehat{\kappa}_{I_{k,p}} = \widehat{\kappa}_p$, with $\widehat{\kappa}_n = \sqrt{\kappa^n}$ and $\widehat{\kappa}_p = \sqrt{\kappa^p - \kappa^{p+1}}$ for $p < n$ (one only has to check that the vector constructed as above has the right correlation structure).

From this construction, we set $f(\widehat{\omega}^{(n)}) := 2^{-n} \log Z_{N,h}^{\omega,\beta}$, and we remark that f is a function of Lipschitz norm L, with L^2 being bounded from above by

$$\begin{aligned} \sup_{\widehat{\omega}^{(n)}} \|\nabla f(\widehat{\omega}^{(n)})\|^2 &:= \sup_{\widehat{\omega}^{(n)}} \sum_{I \in \mathcal{I}^{(n)}} \left| \frac{\partial f}{\partial \widehat{\omega}_I} \right|^2 \\ &= \sup_{\omega} \sum_{p=0}^{n} \sum_{k=1}^{2^{n-p}} \frac{\beta^2}{2^{2n}} \mathbf{E}_{n,h}^{\omega,\beta} \left[\widehat{\kappa}_p \sum_{i \in I_{k,p}} \delta_i \right]^2 \leqslant \frac{\beta^2}{2^n} \sum_{p=0}^{n} 2^p \kappa^p. \end{aligned} \tag{4.2.11}$$

Therefore, the inequality mentioned above leads to $\Psi_{n,\varepsilon} \leqslant 2e^{-cst.\varepsilon^2\beta^{-2}(2\wedge 1/\kappa)^n}$, which is summable for all $\varepsilon > 0$ if $\kappa < 1$. □

We define also the *annealed* partition function $Z_{n,h}^{\mathrm{a}} := \mathbb{E}[Z_{n,h}^{\omega}]$, and the *annealed* free energy:

$$\mathtt{F}^{\mathrm{a}}(\beta,h) := \lim_{n\to\infty} \frac{1}{2^n} \log \mathbb{E}[Z_{n,h}^{\omega}]. \tag{4.2.12}$$

Proposition 4.2.3. *The limit in* (4.2.12) *exists, is non-negative and finite, and* $\mathtt{F}_n^{\mathrm{a}}(\beta,h) := \frac{1}{2^n}\log \mathbb{E}[Z_{n,h}^{\omega}]$ *converges exponentially fast to* $\mathtt{F}^{\mathrm{a}}(\beta,h)$*, and more precisely one has for all* $n \geqslant 1$

$$\mathtt{F}_n^{\mathrm{a}}(\beta,h) - \frac{1}{2^n}\log B \leqslant \mathtt{F}^{\mathrm{a}}(\beta,h) \leqslant \mathtt{F}_n^{\mathrm{a}}(\beta,h) + O((2\kappa)^n). \tag{4.2.13}$$

Note that the error terms in the upper bounds in (4.2.9)-(4.2.13) are not of the same order.

Finiteness of the annealed free energy would fail if the correlations were not summable, *i.e.* if $\sum_j \kappa_{ij} = \infty$, which would be the case for $\kappa \geqslant 1/2$. The proof of Proposition 4.2.3 is postponed to Section 4.4.1.

The properties of $\mathtt{F}^{\mathrm{a}}$ are well known in the non-correlated case, since in this case the annealed model is just the hierarchical homogeneous pinning model (see

Section 1.3.2). In the correlated case, the analysis of the annealed model is much more complex, see Section 4.4.2.

We also have the existence of critical points for both *quenched* and *annealed* models, thanks to the convexity and the monotonicity of the free energies with respect to h.

Proposition 4.2.4 (Critical points). *Let $\beta > 0$ being fixed. There exist critical values $h_c^{\mathrm{a}}(\beta), h_c(\beta)$ such that*

- $\mathtt{F}^{\mathrm{a}}(\beta,h) = 0$ *if* $h \leqslant h_c^{\mathrm{a}}(\beta)$ *and* $\mathtt{F}^{\mathrm{a}}(\beta,h) > 0$ *if* $h > h_c^{\mathrm{a}}(\beta)$
- $\mathtt{F}(\beta,h) = 0$ *if* $h \leqslant h_c(\beta)$ *and* $\mathtt{F}(\beta,h) > 0$ *if* $h > h_c(\beta)$.

One has $-c_\kappa\beta^2 \leqslant h_c^{\mathrm{a}}(\beta) \leqslant h_c(\beta) \leqslant 0$ *for some constant* $c_\kappa < \infty$.

The inequality $h_c^{\mathrm{a}}(\beta) \leqslant h_c(\beta)$ is a direct consequence of the Jensen inequality, that allows us to compare the annealed and the quenched free energy, since it gives $\mathtt{F}(\beta,h) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,h)$. The fact that $h_c^{\mathrm{a}}(\beta) \geqslant -c_\kappa\beta^2$ is discussed after (4.3.2). The bound $h_c(\beta) \leqslant 0$ follows from $\mathtt{F}(\beta,h) \geqslant \mathtt{F}(0,h)$, which is proven in [Gia07, Prop.5.1] (the proof is given there for the *i.i.d.* disorder model but it works identically for the correlated case, since it simply requires that $\mathbb{E}(\omega_i) = 0$).

In the sequel, we often write h_c^{a} instead of $h_c^{\mathrm{a}}(\beta)$ for brevity.

We note that as already noticed (see Section 1.3), the critical point $h_c(\beta)$ marks the transition from a delocalized to a localized regime. If $h < h_c(\beta)$, $\mathtt{F}(\beta,h) = 0$ and the density of contact goes to 0: we are in the delocalized regime. On the other hand, if $h > h_c(\beta)$, we have $\mathtt{F}(\beta,h) > 0$, and there is a positive density of contacts: this is the localized regime. Such a remark applies also naturally to the annealed model.

4.2.2. Critical behavior of the pure model. We recall our notation

$$S_n^{(i)} = \sum_{k\in I_{i,n}} \delta_k \tag{4.2.14}$$

the number of contact points on the block $I_{i,n}$, and write $S_n = S_n^{(1)}$ if there is no ambiguity. We then have of course $S_n^{(i)} = S_{n-1}^{(2i-1)} + S_{n-1}^{(2i)}$.

The pure model is the model in which $\beta = 0$: its partition function is $Z_{n,h}^{\mathrm{pur}} = \mathbf{E}_n\left[\exp\left(hS_n\right)\right]$ and we let $\mathtt{F}(h)$ denote its free energy. It is well known that the pure model exhibits a phase transition at the critical point $h_c(\beta = 0) = 0$:

Theorem 4.2.5 ([GLT10a], Theorem 1.2). *For every $B \in (1,2)$, there exist two constants $c_0 := c_0(B) > 0$ and $c_0' := c_0'(B) > 0$ such that for all $0 \leqslant h \leqslant 1$, we have*

$$c_0 h^{\nu^{\mathrm{pur}}} \leqslant \mathtt{F}(h) \leqslant c_0' h^{\nu^{\mathrm{pur}}} \tag{4.2.15}$$

with

$$\nu^{\mathrm{pur}} = \frac{\log 2}{\log(2/B)} > 1. \tag{4.2.16}$$

The exponent ν^{pur} is the pure critical exponent (in the following, we often note $\nu = \nu^{\mathrm{pur}}$). Note that ν is an increasing function of B, and that we have $\nu = 2$ for

$B = B_c := \sqrt{2}$. We give other useful estimates on the pure model in Section 1.3.2 and Appendix 4.A.

4.3. Main results

In this section we frequently write h_c^a instead of $h_c^{\mathrm{a}}(\beta)$.

It turns out that the effect of correlations is extremely different according to whether $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ or not. In the former case, our first result says that, the correlations decaying fast enough, the critical properties of the annealed model are very close to those of the pure one.

First, let us write down more explicitly what $Z_{n,h}^{\mathrm{a}} = \mathbb{E}[Z_{n,h}^{\omega}]$ is. Note that the Gaussian structure of the disorder is very helpful, to be able to give an explicit formula for the annealed partition function, only in terms of two points correlations. The computation gives

$$Z_{n,h}^{\mathrm{a}} = \mathbf{E}_n \left[\exp \left(\left(\frac{\beta^2}{2} + h \right) \sum_{k=1}^{2^n} \delta_k + \beta^2/2 \sum_{p=1}^{n} \kappa_p \sum_{\substack{1 \leqslant i,j \leqslant 2^n \\ d(i,j)=p}} \delta_i \delta_j \right) \right] =: \mathbf{E}_n \left[e^{H_{n,h}^{\mathrm{a}}} \right]. \tag{4.3.1}$$

One easily realizes that

$$H_{n,h}^{\mathrm{a}} \; = \; h \sum_{k=1}^{2^n} \delta_k + \frac{\beta^2}{2} \sum_{i,j=1}^{2^n} \kappa_{ij} \delta_i \delta_j \; = \; \left(\frac{\beta^2}{2} + h \right) S_n + \beta^2 \sum_{p=1}^{n} \kappa_p \sum_{i=1}^{2^{n-p}} S_{p-1}^{(2i-1)} S_{p-1}^{(2i)}. \tag{4.3.2}$$

In particular note that

$$(h + \beta^2/2) \sum_{k=1}^{2^n} \delta_k \leqslant H_{n,h}^{\mathrm{a}} \leqslant (h + c_\kappa \beta^2) \sum_{k=1}^{2^n} \delta_k := \left(h + \frac{\beta^2}{2} \sum_{p \geqslant 0} 2^{p-1} \kappa^p \right) \sum_{k=1}^{2^n} \delta_k,$$

which together with the fact that $h_c(\beta = 0) = 0$, implies $-c_\kappa \beta^2 \leqslant h_c^{\mathrm{a}}(\beta) \leqslant -\beta^2/2$.

We also use the notation $H_n^{\mathrm{a},(k)}$ for the "annealed Hamiltonian" on the k^{th} block of size 2^n

$$H_{n,h}^{\mathrm{a},(k)} = h \sum_{l \in I_{k,n}} \delta_l + \frac{\beta^2}{2} \sum_{i,j \in I_{k,n}} \kappa_{ij} \delta_i \delta_j.$$

and the following relation holds:

$$H_{n+1,h}^{\mathrm{a}} = H_{n,h}^{\mathrm{a},(1)} + H_{n,h}^{\mathrm{a},(2)} + \beta^2 \kappa_{n+1} S_n^{(1)} S_n^{(2)}. \tag{4.3.3}$$

If we set $h = h_c^{\mathrm{a}} + u$, so that the phase transition is at $u = 0$, one has

$$Z_{n,h}^{\mathrm{a}} = \mathbf{E}_n \left[\exp(u S_n) e^{H_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}} \right] = Z_{n,h_c^{\mathrm{a}}}^{\mathrm{a}} \mathbf{E}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}} \left[\exp(u S_n) \right], \tag{4.3.4}$$

where

$$\frac{\mathrm{d}\mathbf{P}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}}{\mathrm{d}\mathbf{P}_n} := \frac{1}{Z_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}} \exp\left(H_{n,h_c^{\mathrm{a}}}^{\mathrm{a}} \right). \tag{4.3.5}$$

The measure $\mathbf{P}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}$ is the annealed polymer measure at the critical point h_c^{a}.

We can finally formulate our first result:

Theorem 4.3.1. *Let $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$. There exist some $\beta_0 > 0$ and constants $c_1, c_2 > 0$ such that for every $\beta \leqslant \beta_0$ and $u \in [0,1]$, one has*

$$\mathbf{E}_n\Big[\exp\Big(e^{-c_1\beta^2} u S_n\Big)\Big] - c_2\beta^2\left(\frac{4\kappa}{B^2}\right)^n \leqslant \mathbf{E}_n\Big[\exp\left(uS_n\right) e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\Big] \leqslant \mathbf{E}_n\Big[\exp\Big(e^{c_1\beta^2} u S_n\Big)\Big] \tag{4.3.6}$$

so that, for any $u \in [0,1]$,

$$\mathtt{F}\left(e^{-c_1\beta^2}u\right) \leqslant \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}} + u) \leqslant \mathtt{F}\left(e^{c_1\beta^2}u\right). \tag{4.3.7}$$

Theorem 4.3.1 is saying that the critical behavior of the annealed free energy around h_c^{a} is the same as that of the pure model around $h = 0$ (in particular, same critical exponent ν).

The essential tool is to prove that the measures $\mathbf{P}_n$ and $\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ are close. This is the contents of the following Proposition:

Proposition 4.3.2. *If $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$, then there exist some $\beta_0 > 0$ and a constant $c_1 > 0$ such that, for every $\beta \leqslant \beta_0$, for any non-empty subset I of $\{1, \dots, 2^n\}$ one has*

$$\left(e^{-c_1\beta^2}\right)^{|I|} \mathbf{E}_n\left[\delta_I\right] \leqslant \mathbf{E}_n\left[\delta_I e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\right] \leqslant \left(e^{c_1\beta^2}\right)^{|I|} \mathbf{E}_n\left[\delta_I\right], \tag{4.3.8}$$

where $\delta_I := \prod_{i\in I}\delta_i$. The case $I = \emptyset$ is dealt with by Lemma 4.4.1 below, that says that the partition function at the critical point approaches 1 exponentially fast:

$$e^{-c_2\beta^2(4\kappa/B^2)^n} \leqslant Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \leqslant 1. \tag{4.3.9}$$

Observe that (4.3.9) says that if $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ the partition function of the annealed model at h_c^{a} is very close to that of the pure model at its critical point $h = 0$ (which equals identically 1). We will see in Theorem 4.3.6 that (4.3.8) fails, even for $\beta > 0$ small, if $\kappa > \frac{B^2}{4} \wedge \frac{1}{2}$.

With the crucial Proposition 4.3.2 in hand, it is not hard to prove that for $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ the Harris criterion for disorder relevance is not modified by the presence of disorder correlations:

Theorem 4.3.3. *Let $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$.*

- *If $1 < B \leqslant B_c = \sqrt{2}$, then disorder is relevant: the quenched and annealed critical points differ for every $\beta > 0$, and:*
  - *if $B < B_c$, there exist a constant $c_3 > 0$ such that for every $0 \leqslant \beta \leqslant 1$*

$$(c_3)^{-1}\beta^{\frac{2}{2-\nu}} \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant c_3\beta^{\frac{2}{2-\nu}} \; ; \tag{4.3.10}$$

  - *if $B = B_c$, there exist a constant $c_4 > 0$ and some $\beta_0 > 0$ such that for every $0 \leqslant \beta \leqslant \beta_0$*

$$\exp\left(-\frac{c_4}{\beta^4}\right) \leqslant h_c(\beta) - h_c^{\mathrm{a}}(\beta) \leqslant \exp\left(-\frac{c_4^{-1}}{\beta^{2/3}}\right). \tag{4.3.11}$$

- *If $B_c < B < 2$, then disorder is irrelevant: there exists some $\beta_0 > 0$ such that $h_c(\beta) = h_c^{\mathrm{a}}(\beta)$ for any $0 < \beta \leqslant \beta_0$. More precisely, for every $\eta > 0$ and choosing $u > 0$ sufficiently small, $\mathtt{F}(\beta, h_c^{\mathrm{a}}(\beta) + u) \geqslant (1-\eta)\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}(\beta) + u)$.*

With some extra effort one can presumably improve the upper bound (4.3.11) to $\exp(-c_2^{-1}/\beta^2)$ and the lower bound to $\exp(-c_2(\epsilon)/\beta^{2+\epsilon})$ for every $\epsilon > 0$, as is known for the uncorrelated case $\kappa = 0$ [GLT10b, GLT11]. We will not pursue this line.

Remark 4.3.4. It is important to note that Theorems 4.3.1 and 4.3.3 do not require the knowledge of the value of h_c^{a} (in general there is no hope to compute it exactly). This makes the analysis of the quenched model considerably more challenging than in the *i.i.d.* disorder case $\kappa = 0$, where it is immediate to see that $h_c^{\mathrm{a}}(\beta) = -\beta^2/2$.

We mentioned in the introduction that for the *i.i.d.* model one can prove that, when the free-energy critical exponent ν of the homogeneous model is smaller than 2, such exponent is modified by an arbitrarily small amount of disorder (more precisely, the result is that the exponent is at least 2 as soon as $\beta > 0$). The same holds for the model with correlated disorder:

Proposition 4.3.5. *If $\kappa < 1/2$, for every $B \in (1,2)$ there exists a constant $c(B) < \infty$ such that for all $\beta > 0$ and $h \in \mathbb{R}$, we have*

$$\mathtt{F}(\beta, h) \leqslant \frac{c(B)}{\beta^2} \left(h - h_c(\beta)\right)_+^2 . \tag{4.3.12}$$

We restrict to $\kappa < 1/2$ since otherwise there is no phase transition.

We do not give here the proof of this Proposition since, thanks to summability of the correlations, it is very similar to the one for the *i.i.d.* hierarchical model [LT09]. We refer also to Section 5.4.1, where a similar result is proved in the non-hierarchical case.

In the case $1/2 > \kappa \geqslant B^2/4$ correlations have a much more dramatic effect on critical properties and in particular we expect them to change the value of the annealed critical exponent from the value $\nu = \log 2/\log(2/B)$ to a larger one. Partial results in this direction are collected in the following Theorem, which shows that (some) critical properties of the annealed model differ from those of the homogeneous one.

Theorem 4.3.6. *Let $B^2/4 < \kappa < 1/2$ and $\beta > 0$. In contrast with (4.3.9), the partition function at the critical point does not converge to 1. Rather, one has*

$$\prod_{p=0}^{n-1} Z_{p,h_c^{\mathrm{a}}}^{\mathrm{a}} \leqslant \frac{1}{\beta\sqrt{\kappa}} \left(\frac{B}{2\sqrt{\kappa}}\right)^n . \tag{4.3.13}$$

Also, the average number of individuals at generation n at the critical point satisfies

$$\mathbf{E}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}\left[S_n\right] = \mathbf{E}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}\left[\sum_{i=1}^{2^n} \delta_i\right] \leqslant \frac{c(B)}{\beta} \frac{1}{\kappa^{(n+1)/2}}. \tag{4.3.14}$$

When proving Theorem 4.3.6 we will actually prove that the m^{th} moment of S_n under $\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ is at most of order $\kappa^{-mn/2}$. Therefore, with high probability S_n is much smaller than $(2/B)^n$, which would be the order of magnitude of S_n for $\kappa < B^2/4 \wedge 1/2$, as can be deduced from Propositions 4.3.2 and 4.2.1.

In other words, if we fix $B < \sqrt{2}$ and we let κ grow but tuning h so that we are always at the annealed critical point, there is a phase transition in the behavior of the finite-volume contact fraction when crossing the value $\kappa = B^2/4$, cf. also Figure 4.1.

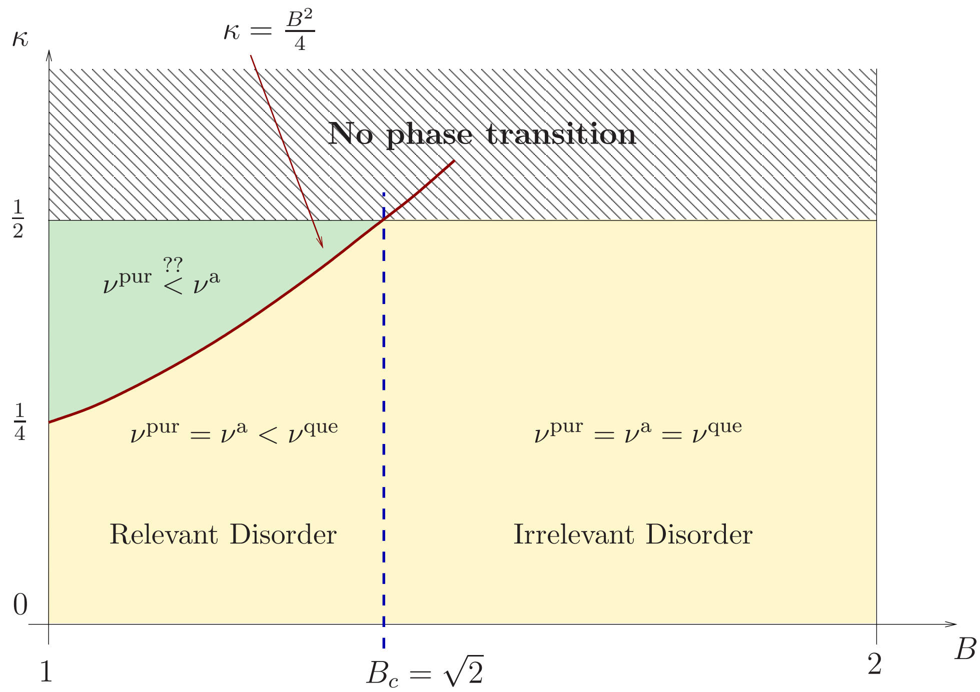

FIGURE 4.1. Overview of the qualitative behavior of the model. One takes $\kappa < 1/2$, otherwise neither annealed nor quenched model have any phase transition. For $\kappa < 1/2 \wedge B^2/4$ the annealed model exhibits the same critical behavior as the pure one, and so the critical exponent is $\nu^{\mathrm{a}} = \nu = \log 2 \big/ \log(2/B)$. Moreover, the measures $\mathbf{P}_n$ and $\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ are similar (in the sense of Proposition 4.3.2) and the criterion relevance/irrelevance of disorder is the same as in the *i.i.d.* disorder case: disorder is irrelevant for $B > B_c := \sqrt{2}$, marginally relevant at $B = B_c$ and relevant for $B < B_c$ (cf. Theorem 4.3.3). The region above the parabola $\kappa = B^2/4$ remains to be understood, but partial results (Theorem 4.3.6) suggest that the critical behavior of the annealed model is different from the one of the pure model, in particular the annealed critical exponent should be larger. Note that disorder is proven to be relevant for all $B < B_c$, $\kappa < 1/2$ through the "smoothing result" of Proposition 4.3.5, showing that the quenched critical exponent is strictly larger than the pure one.

Remark 4.3.7. In view of Remark 1.3.4, the choice of the hierarchical structure (4.2.3) for the correlations, corresponds to a power law decay $\kappa_{ij} \sim |i-j|^{-\zeta}$ in the non-hierarchical framework, where $\zeta = \log(1/\kappa)/\log 2 > 0$ (or $\kappa = 2^{-\zeta}$). Note that $\zeta > 0$ for all $\kappa < 1$, and that $\zeta > 1$ if $\kappa < 1/2$ (summable correlations) and $\zeta < 1$ if $\kappa > 1/2$ (non-summable correlations).

Moreover, the pure critical exponent is $\nu^{\mathrm{pur}} = \log 2/\log(2/B)$ in the hierarchical case. Therefore, the condition $\kappa < B^2/4$ (with the additional condition of correlation summability, $\kappa < 1/2$) that one find in the crucial Proposition 4.3.2, translates into $\zeta\nu^{\mathrm{pur}} > 2$. This condition is found in the Weinrib-Halperin criterion, and should be the pertinence/non-pertinence criterion in the $\zeta < 1$ case.

4.3.1. A note on the case $\kappa > 1/2$. As we saw in the previous Chapter, the case $\kappa > 1/2$ (non-summable correlations) is somehow particular, and correspond to power-law decaying correlations with an exponent $\xi < 1$ in the non-hierarchical case. Restricting to the event where all the δ_n are equal to 1 and using Proposition 4.2.1, one sees that

$$Z^{\mathrm{a}}_{n,h} \geqslant \left(\frac{1}{B}\right)^{2^n} \exp\left(\left((h+\beta^2/2) + \beta^2/2\sum_{p=1}^{n}\kappa_p 2^{p-1}\right)2^n\right). \tag{4.3.15}$$

Thus, we see that $\mathtt{F}^{\mathrm{a}}(\beta,h) = \infty$ unless

$$K_\infty := \sum_{p=0}^{\infty} \kappa_p 2^p < +\infty. \tag{4.3.16}$$

For $\kappa > 1/2$, not only the annealed free energy is ill-defined. One can also prove that the quenched free energy is strictly positive for every value of $h \in \mathbb{R}$: the quenched system does not have a localization/delocalization phase transition.

Theorem 4.3.8. *If $\kappa > 1/2$, then $\mathtt{F}(\beta,h) > 0$ for every $\beta > 0, h \in \mathbb{R}$, so that $h_c(\beta) = -\infty$. There exists some constant $c_5 > 0$ such that for all $h \leqslant -1$ and $\beta > 0$*

$$\mathtt{F}(\beta,h) \geqslant \exp\left(-c_5|h|(|h|/\beta^2)^{\log 2/\log(2\kappa)}\right). \tag{4.3.17}$$

The proof of $h_c(\beta) = -\infty$ can be presumably extended to the case $\kappa = 1/2$. To avoid technicalities, we do not develop this case here.

Proof In this proof (and in the sequel), we do not keep track of the constants $c, C, \ldots$, and therefore they can change from line to line.

The proof is identical to what is done in Section 5.4.2, and we do not repeat it here completely. However, we give the hierarchical substitute for the bound 4.3.20 on the free energy, which is the bound obtained with the strategy of targeting all the "good" blocks.

Let us fix some $l \in \mathbb{N}$ (that is chosen large later on), take $n > l$ and let $\mathcal{I} \subset \{1,\ldots,2^{n-l}\}$, which is supposed to denote the set of indexes corresponding to "good blocks" of size 2^l. Then for any fixed ω, targeting only the blocks in $\mathcal{I}$ gives (a similar inequality was proven in [LT09])

$$Z^{\omega}_{n,h} \geqslant \left(\frac{B-1}{B^2}\right)^{v(n-l,\mathcal{I})} \prod_{k\in\mathcal{I}} Z^{\omega,(k)}_{l,h}, \tag{4.3.18}$$

where $v(n-l,\mathcal{I}_n)$ is the number of nodes in the subtree $\mathcal{T}_{\mathcal{I}}^{(n-l)}$ defined in Proposition 4.2.1 and $Z_{l,h}^{\omega,(i)}$ is the partition function on $I_{k,l}$, the i^{th} block of size 2^l, cf. (4.2.2). The term $\left(\frac{B-1}{B^2}\right)^{v(n-l,\mathcal{I})}$ is a lower bound on the probability that the node $1 \leqslant i \leqslant 2^{n-l}$ at generation $n-l$ has at least one descendant at level $n-l+1$ if and only if $i \in \mathcal{I}$ (see Figure 4.2).

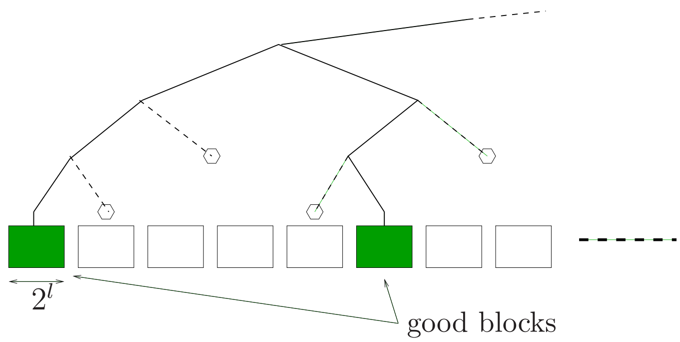

FIGURE 4.2. The strategy of aiming exactly at the good (colored) blocks is represented above. One first places the subtree $\mathcal{T}_{\mathcal{I}}^{(n-l)}$, which is present with probability $(1/B)^{v(n-l,\mathcal{I})}$, and then forces all the leaves that do not lead to any good block (the hexagons in the figure) not to have any children, which happens with probability larger than $((B-1)/B)^{v(n-l,\mathcal{I})}$. The maximal amount of nodes that such a tree can contain is reached when all the good blocks are all equally distant one from another, and is thus bounded as in (4.3.19).

It was shown in [LT09] that

$$v(n,\mathcal{I}) \leqslant |\mathcal{I}| \left(2+n-l-\lfloor \log_2 |\mathcal{I}| \rfloor\right) \tag{4.3.19}$$

so that

$$\frac{1}{2^n}\log Z_{n,h}^{\omega} \geqslant \frac{1}{2^n}\sum_{k\in\mathcal{I}} \log Z_{l,h}^{\omega,(k)} - \log\left(\frac{B^2}{B-1}\right)\frac{|\mathcal{I}|}{2^n}\left(2+n-l-\lfloor \log_2 |\mathcal{I}| \rfloor\right). \tag{4.3.20}$$

From this point, the proof is identical to what is done in Section 5.4.2 to prove Theorem 5.2.5. The only point that changes is the estimation of the entropic cost of the change of measure procedure, used to bound the probability for a block to be favorable. Indeed, the correlation matrix in the hierarchical case is not a Toeplitz matrix anymore, and the estimate of $\langle K^{-1}\mathbf{1}_{2^l}, \mathbf{1}_{2^l}\rangle$ is dealt with Lemma 4.B.1, that stresses that $\mathbf{1}_{2^l}$ is an eigenvector of K, with eigenvalue $\lambda_0 = 1 + \sum_{p=1}^{l} 2^{p-1}\kappa_p$. □

4.4. Study of the annealed model

Let us remark first of all that since $\kappa_n \geqslant 0$ for all $n \geqslant 0$, one has thanks to (4.3.3) that $H_{n+1,h}^{\mathrm{a}} \geqslant H_{n,h}^{\mathrm{a},(1)} + H_{n,h}^{\mathrm{a},(2)}$, and therefore

$$Z_{n+1,h}^{\mathrm{a}} \geqslant \frac{(Z_{n,h}^{\mathrm{a}})^2 + B - 1}{B}. \tag{4.4.1}$$

From this one deduces that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \leqslant 1$. Indeed, the map $x \mapsto (x^2+(B-1))/B$ has an unstable fixed point at 1, and $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} > 1$ would imply that $\mathtt{F}^{\mathrm{a}}(\beta, h^{\mathrm{a}}_c) > 0$.

4.4.1. An auxiliary partition function, proof of Proposition 4.2.3. It is very convenient for the following to introduce a modified partition function, both for the quenched case and for the annealed one, defining

$$\bar{Z}^{\omega}_{n,h} = \mathbf{E}_n\left[\exp\left(H^{\omega}_{n,h} + \theta\beta^2\kappa_n(S_n)^2\right)\right], \qquad \text{with } \theta := \frac{\kappa}{2(1-2\kappa)} \tag{4.4.2}$$

and

$$\bar{Z}^{\mathrm{a}}_{n,h} = \mathbb{E}[\bar{Z}^{\omega}_{n,h}] = \mathbf{E}_n\left[\exp(\bar{H}^{\mathrm{a}}_{n,h})\right], \tag{4.4.3}$$

with

$$\bar{H}^{\mathrm{a}}_{n,h} = H^{\mathrm{a}}_{n,h} + \theta\beta^2\kappa_n(S_n)^2. \tag{4.4.4}$$

Note that θ vanishes for $\kappa \to 0$ (no need of the auxiliary partition function for the non-correlated model) and that it diverges for $\kappa \to 1/2$, where the annealed model is not well-defined.

We also naturally define $\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h) := \lim_{n\to\infty} 2^{-n}\log \bar{Z}^{\mathrm{a}}_{n,h}$ (the existence of the limit will be shown in the course of the proof of Proposition 4.2.3) and, using $\delta_k \leqslant 1$, one gets that $Z^{\mathrm{a}}_{n,h} \leqslant \bar{Z}^{\mathrm{a}}_{n,h} \leqslant e^{\theta\beta^2(4\kappa)^n} Z^{\mathrm{a}}_{n,h}$, so that $\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h) = \mathtt{F}^{\mathrm{a}}(\beta,h)$ (recall we chose $\kappa < 1/2$). Similarly, if $\bar{\mathtt{F}}(\beta,h) := \lim_{n\to\infty} 2^{-n}\log \bar{Z}^{\omega}_{n,h}$ then $\bar{\mathtt{F}}(\beta,h) = \mathtt{F}(\beta,h)$.

Then, from (4.3.3), one gets that (recall $\kappa_n = \kappa^n$ and (4.2.14))

$$\begin{aligned}
\bar{H}^{\mathrm{a}}_{n+1,h} \leqslant H^{\mathrm{a},(1)}_{n,h} + H^{\mathrm{a},(2)}_{n,h} &+ \frac{\beta^2}{2}\kappa^{n+1}(S^{(1)}_n)^2 + \frac{\beta^2}{2}\kappa^{n+1}(S^{(2)}_n)^2 \\
&+ 2\theta\beta^2\kappa^{n+1}(S^{(1)}_n)^2 + 2\theta\beta^2\kappa^{n+1}(S^{(1)}_n)^2 \\
= H^{\mathrm{a},(1)}_{n,h} + \theta\beta^2\kappa^n(S^{(1)}_n)^2 &+ H^{\mathrm{a},(2)}_{n,h} + \theta\beta^2\kappa^n(S^{(2)}_n)^2 = \bar{H}^{\mathrm{a},(1)}_{n,h} + \bar{H}^{\mathrm{a},(2)}_{n,h}
\end{aligned} \tag{4.4.5}$$

where we used the self-explanatory notation $\bar{H}^{\mathrm{a},(i)}_{n,h}$ for the auxiliary Hamiltonian in the block $I_{i,n}$. We used the bounds $ab \leqslant 1/2(a^2+b^2)$ and $(a+b)^2 \leqslant 2(a^2+b^2)$ and then the definition of θ.

This gives in particular that

$$\bar{Z}^{\mathrm{a}}_{n+1,h} \leqslant \frac{(\bar{Z}^{\mathrm{a}}_{n,h})^2 + B - 1}{B}, \tag{4.4.6}$$

from which one deduces that $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \geqslant 1$ for all $n \in \mathbb{N}$. Indeed, otherwise, for some $n_0 \in \mathbb{N}$ one has $\bar{Z}^{\mathrm{a}}_{n_0,h^{\mathrm{a}}_c} < 1$, and then one can find some $h > h^{\mathrm{a}}_c$ such that $\bar{Z}^{\mathrm{a}}_{n_0,h} \leqslant 1$, which combined with (4.4.6) gives that $\bar{Z}^{\mathrm{a}}_{n,h} \leqslant 1$ for all $n \geqslant n_0$. Therefore one would have $\mathtt{F}^{\mathrm{a}}(\beta,h) = \bar{\mathtt{F}}^{\mathrm{a}}(\beta,h) = 0$, which is a contradiction with the definition of h^{a}_c.

Proof of Proposition 4.2.3 One has from (4.4.1)

$$\frac{Z^{\mathrm{a}}_{n+1,h}}{B} \geqslant \left(\frac{Z^{\mathrm{a}}_{n,h}}{B}\right)^2, \tag{4.4.7}$$

and from (4.4.6) and the fact that $\bar{Z}^{\mathrm{a}}_{n,h} \geqslant (B-1)/B$

$$K_B \bar{Z}^{\mathrm{a}}_{n+1,h} \leqslant (K_B \bar{Z}^{\mathrm{a}}_{n,h})^2 \qquad \text{with} \qquad K_B = \frac{B^2+B-1}{B(B-1)}. \tag{4.4.8}$$

Therefore, the sequence $\{2^{-n}\log(Z^{\mathrm{a}}_{n,h}/B)\}_{n\geqslant 1}$ and $\{2^{-n}\log(K_B\bar{Z}^{\mathrm{a}}_{n,h})\}_{n\geqslant 1}$ are non-decreasing and non-increasing respectively, so that both converge to a limit, $\mathtt{F}^{\mathrm{a}}(\beta,h)$ and $\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h)$ respectively, but we have already remarked earlier in this section that $\mathtt{F}^{\mathrm{a}}(\beta,h) = \bar{\mathtt{F}}^{\mathrm{a}}(\beta,h)$. One finally has

$$\begin{aligned} \mathtt{F}^{\mathrm{a}}(\beta,h) &\geqslant \mathtt{F}^{\mathrm{a}}_n(\beta,h) - 2^{-n}\log B \\ \mathtt{F}^{\mathrm{a}}(\beta,h) = \bar{\mathtt{F}}^{\mathrm{a}}(\beta,h) &\leqslant \bar{\mathtt{F}}^{\mathrm{a}}_n(\beta,h) + 2^{-n}\log K_B, \end{aligned} \tag{4.4.9}$$

so that since $\bar{\mathtt{F}}^{\mathrm{a}}_n(\beta,h) \leqslant \mathtt{F}^{\mathrm{a}}_n(\beta,h) + \theta\beta^2(2\kappa)^n$, one gets the desired result.

□

4.4.2. Proof of Theorem 4.3.1 and Proposition 4.3.2. The really crucial point is to prove that, provided that $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$, the annealed partition function (and the auxiliary one $\bar{Z}^{\mathrm{a}}_{n,h}$) at the annealed critical point converges exponentially fast to 1.

Lemma 4.4.1. *If $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ then there exist some constant $c_2 > 0$ and some $\beta_0 > 0$ such that for any $n \geqslant 0$ and every $\beta \leqslant \beta_0$, one has*

$$\begin{aligned} \exp\left(-c_2\beta^2(4\kappa/B^2)^n\right) \leqslant \quad & Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \quad \leqslant 1, \\ 1 \leqslant \quad & \bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \quad \leqslant \exp\left(c_2\beta^2(4\kappa/B^2)^n\right). \end{aligned}$$

Proof of Theorem 4.3.1 given Lemma 4.4.1 and Proposition 4.3.2 We expand $\exp(uS_n)$, to get

$$\mathbf{E}_n\left[\exp(uS_n)\, e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] = \sum_{k=0}^{\infty} \frac{u^k}{k!}\mathbf{E}_n\left[(S_n)^k\, e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right]. \tag{4.4.10}$$

Thanks to Proposition 4.3.2, we have that for any $k \geqslant 1$

$$\left(e^{-c_1\beta^2}\right)^k \mathbf{E}_n\left[(S_n)^k\right] \leqslant \mathbf{E}_n\left[(S_n)^k\, e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \leqslant \left(e^{c_1\beta^2}\right)^k \mathbf{E}_n\left[(S_n)^k\right], \tag{4.4.11}$$

and with (4.4.10) we have then

$$\mathbf{E}_n\left[\exp(uS_n)\, e^{H_{n,h^{\mathrm{a}}_c}}\right] \leqslant Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} + \mathbf{E}_n\left[\sum_{k=1}^{\infty}\frac{\left(u\, e^{c_1\beta^2}\right)^k}{k!}(S_n)^k\right] \leqslant \mathbf{E}_n\left[\exp\left(e^{c_1\beta^2}uS_n\right)\right] \tag{4.4.12}$$

where we used that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \leqslant 1$. We naturally get the other inequality in the same way

$$\mathbf{E}_n\left[\exp(uS_n)\, e^{H_{n,h^{\mathrm{a}}_c}}\right] \geqslant \mathbf{E}_n\left[\exp\left(e^{-c_1\beta^2}uS_n\right)\right] - c_2\beta^2\left(\frac{4\kappa}{B^2}\right)^n, \tag{4.4.13}$$

where we used Lemma 4.4.1 to get that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \geqslant 1 - c_2\beta^2(4\kappa/B^2)^n$. □

Remark 4.4.2. Using the same type of expansion, Proposition 4.3.2 gives more general results: for example, one can get

$$\mathbf{E}_n\left[\exp\left(e^{-pc_1\beta^2}u(S_n)^p\right)\right] - c_2\beta^2\left(\frac{4\kappa}{B^2}\right)^n \leqslant \mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\exp\left(u(S_n)^p\right)\right]$$

$$\mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\exp\left(u(S_n)^p\right)\right] \leqslant \mathbf{E}_n\left[\exp\left(e^{pc_1\beta^2}u(S_n)^p\right)\right]. \quad (4.4.14)$$

In the sequel, we refer to this Remark to avoid repeating this kind of computation.

Before proving Proposition 4.3.2 and Lemma 4.4.1, we prove the following result, valid for any $\kappa < 1/2$. Given $I \subset \{1,\dots,2^n\}$ we say that I is *complete* if $2i-1 \in I$ for some $i \in \mathbb{N}$ if and only if $2i \in I$.

Lemma 4.4.3. *For every $n \geqslant 1$, and any non-empty and complete subset I of $\{1,\dots,2^n\}$, one has*

$$\left(\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h^{\mathrm{a}}_c}\right)^{|I|}\mathbf{E}_n[\delta_I] \leqslant \mathbf{E}_n\left[\delta_I e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \quad (4.4.15)$$

$$\leqslant \mathbf{E}_n\left[\delta_I e^{\bar{H}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \leqslant \left(\prod_{p=0}^{n-1} \bar{Z}^{\mathrm{a}}_{p,h^{\mathrm{a}}_c}\right)^{|I|}\mathbf{E}_n[\delta_I]. \quad (4.4.16)$$

Note that if $I=\emptyset$, these inequalities are false, since $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \leqslant 1 \leqslant \bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}$.

Proof of Lemma 4.4.3 As the two bounds rely on a similar argument, that is $H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c} \geqslant H^{\mathrm{a},(1)}_{n,h^{\mathrm{a}}_c} + H^{\mathrm{a},(2)}_{n,h^{\mathrm{a}}_c}$ in one case, and $\bar{H}^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c} \leqslant \bar{H}^{\mathrm{a},(1)}_{n,h^{\mathrm{a}}_c} + \bar{H}^{\mathrm{a},(2)}_{n,h^{\mathrm{a}}_c}$ in the other case, we focus only on the lower bound.

We prove it by iteration, the case $n=1$ being trivial (the only non-empty complete subset is $I=\{1,2\}$ and the inequalities can be checked by hand). Now assume that the assumption is true for some $n \geqslant 1$ and take I a non-empty complete subset of $\{1,\dots,2^{n+1}\}$. We decompose I into two subsets $I_1 = I \cap [1,2^n]$ and $I_2 = I \cap [2^n+1, 2^{n+1}]$ and we define $\widetilde{I}_2$ to be the subset obtained by shifting I_2 to the left by 2^n. It is easy to realize that both I_1 and $\widetilde{I}_2$ are complete subsets of $\{1,\dots,2^n\}$ and one has $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$.

Now, using that $H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c} \geqslant H^{\mathrm{a},(1)}_{n,h^{\mathrm{a}}_c} + H^{\mathrm{a},(2)}_{n,h^{\mathrm{a}}_c}$, one has

$$\mathbf{E}_{n+1}\left[\delta_I e^{H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c}}\right] \geqslant \frac{1}{B}\mathbf{E}_n\left[\delta_{I_1} e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right]\mathbf{E}_n\left[\delta_{\widetilde{I}_2} e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \quad (4.4.17)$$

and two cases can occur.

(1) $\widetilde{I}_2 = \emptyset$, $|I_1| = |I|$ (or $I_1 = \emptyset$, $|\widetilde{I}_2| = |I|$). Then, (4.4.17) plus the induction step gives

$$\mathbf{E}_{n+1}\left[\delta_I e^{H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c}}\right] \geqslant \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}] Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}\left(\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h^{\mathrm{a}}_c}\right)^{|I|}. \quad (4.4.18)$$

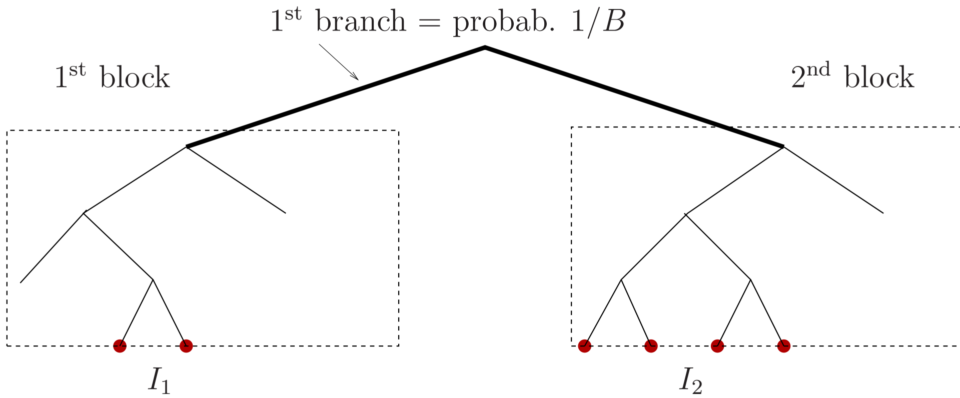

FIGURE 4.3. Decomposition of a non-empty complete set I into two subsets I_1 and I_2. If I is non empty, the first generation must be non-empty (this has probability $1/B$). Conditionally on this, the occurrence of I_1 and I_2 are independent events.

Since $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \leqslant 1$, one has $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \geqslant \left(Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\right)^{|I|}$, and obtains the claim at level $n+1$.

(2) $I_1, I_2 \neq \emptyset$. In this case, from (4.4.17), the recurrence assumption directly gives

$$\mathbf{E}_{n+1}\left[\delta_I e^{H^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}}}\right] \geqslant \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]\left(\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}}\right)^{|I_1|+|I_2|}. \tag{4.4.19}$$

This gives the result at level $n+1$, using that $|I| = |I_1| + |I_2|$, and bounding again $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \leqslant 1$.

□

Proof of Proposition 4.3.2 Given $I \subset \{1, \dots, 2^n\}$, let I' be the smallest complete subset of $\{1, \dots, 2^n\}$ that contains I, and note that $|I'| \leqslant 2|I|$. Note that

$$\mathbf{E}_n[\delta_I \exp(H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}})] = \mathbf{E}_n[\delta_{I'} \exp(H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}})], \qquad \mathbf{E}_n[\delta_I] = \mathbf{E}_n[\delta_{I'}],$$

simply because of the offspring distribution of the Galton-Watson tree: if the individual $2i-1$ is present at generation n, so is the individual $2i$. This immediately implies that the statement of Lemma 4.4.3 holds for every I (not necessarily complete), if $|I|$ is replaced by $2|I|$.

Then, Lemmas 4.4.3 and 4.4.1 imply Proposition 4.3.2 with $c_1 = 2c_2 \sum_{p=0}^{\infty} \left(\frac{4\kappa}{B^2}\right)^n = 2c_2 \frac{B^2}{B^2 - 4\kappa}$.

□

Proof of Lemma 4.4.1 One would like to use a result analogue to Proposition 4.3.2 to bound $\bar{Z}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} = \mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}} \exp\left(\theta\kappa^n (S_n)^2\right)\right]$. So we first prove a weaker upper bound. The proof relies strongly on the pure model estimates presented in Appendix 4.A, which show that the term $\theta\kappa_n (S_n)^2$ in $\bar{Z}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ has little effect if $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$.

Take $\varphi := (2\kappa) \vee \frac{4\kappa}{B^2} < 1$ and C the constant c associated to $A = 1$ in Corollary 4.A.4, and fix some $\beta \leqslant \beta_0$, with $\beta_0 := \left(\prod_{p=0}^{\infty} e^{C(p+2)\varphi^p}\right)^{-2} \leqslant 1$. We prove iteratively on n that for all subsets I of $\{1,\dots,2^n\}$ one has

$$\mathbf{E}_n\left[\delta_I e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \leqslant (x_n)^{|I|}\mathbf{E}_n[\delta_I], \qquad \text{with } x_n := \prod_{p=0}^{n} e^{C(p+1)\beta\varphi^p}. \tag{4.4.20}$$

Note that with our choice of β_0 one has $(x_n)^2 \leqslant \beta_0^{-1}$ for all $n \geqslant 0$.

The case $n = 0$ is trivial (just use that $h^{\mathrm{a}}_c \leqslant -\beta^2/2$, as discussed after (4.3.2)). Now assume that (4.4.20) is true for some $n \geqslant 0$ and take I a subset of $\{1,\dots,2^{n+1}\}$.

If $I = \emptyset$, then we simply use that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \leqslant 1$. If $I \neq \emptyset$ decompose it as in the proof of Lemma 4.4.3 into two subsets I_1, I_2 and let $\widetilde{I}_2$ be obtained by translating I_2 to the left by 2^n, so that $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$ (see Figure 4.3). Then, from the iteration (4.3.3) on $H^{\mathrm{a}}_{n,h}$ one has

$$H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c} \leqslant H^{\mathrm{a},(1)}_{n,h^{\mathrm{a}}_c} + \frac{\beta^2}{2}\kappa^{n+1}\left(S^{(1)}_n\right)^2 + H^{\mathrm{a},(1)}_{n,h^{\mathrm{a}}_c} + \frac{\beta^2}{2}\kappa^{n+1}\left(S^{(2)}_n\right)^2, \tag{4.4.21}$$

so that one gets

$$\begin{aligned}\mathbf{E}_{n+1}\left[\delta_I e^{H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c}}\right] \leqslant \frac{1}{B}\mathbf{E}_n&\left[\delta_{I_1} e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\exp\left(\frac{\beta^2}{2}\kappa^{n+1}(S_n)^2\right)\right] \\ &\times \mathbf{E}_n\left[\delta_{\widetilde{I}_2} e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\exp\left(\frac{\beta^2}{2}\kappa^{n+1}(S_n)^2\right)\right].\end{aligned} \tag{4.4.22}$$

Now one can use the inductive assumption to estimate each part of (4.4.22). Expanding the exponential term and recalling that $\beta_0(x_n)^2 \leqslant 1$, one has for instance

$$\begin{aligned}\mathbf{E}_n\left[\delta_{I_1} e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\exp\left(\frac{\beta^2}{2}\kappa^{n+1}(S_n)^2\right)\right] &= \sum_{k=0}^{\infty}\frac{(\beta^2\kappa^{n+1}/2)^k}{k!}\mathbf{E}_n\left[\delta_{I_1} e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}(S_n)^{2k}\right] \\ &\leqslant \sum_{k=0}^{\infty}(x_n)^{|I_1|+2k}\frac{(\beta^2\kappa^{n+1}/2)^k}{k!}\mathbf{E}_n\left[\delta_{I_1}(S_n)^{2k}\right] \\ \leqslant (x_n)^{|I_1|}\,\mathbf{E}_n\left[\delta_{I_1} e^{(x_n)^2\frac{\beta^2}{2}\kappa^{n+1}(S_n)^2}\right] &\leqslant (x_n)^{|I_1|}\,\mathbf{E}_n\left[\delta_{I_1} e^{\frac{\beta\kappa}{2}\kappa^n(S_n)^2}\right].\end{aligned} \tag{4.4.23}$$

We now use Corollary 4.A.4 to get that

$$\mathbf{E}_n\left[\delta_{I_1} e^{\frac{\beta\kappa}{2}\kappa^n(S_n)^2}\right] \leqslant \exp\left(C\frac{\beta\kappa}{2}\varphi^{-1}\varphi^{n+1}\right)^{n|I_1|+1}\mathbf{E}_n[\delta_{I_1}]. \tag{4.4.24}$$

Combining this with (4.4.22)-(4.4.23) and the definition of $\varphi \geqslant 2\kappa$ one gets

$$\mathbf{E}_{n+1}\left[\delta_I e^{H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c}}\right] \leqslant (x_n)^{|I|}\left(e^{C\frac{\beta}{4}\varphi^{n+1}}\right)^{n|I|+2}\frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]. \tag{4.4.25}$$

Using that $n|I| + 2 \leqslant (n+2)|I|$ (because $I \neq \emptyset$) and the definition of $x_{n+1} = x_n e^{C(n+2)\beta\varphi^{n+1}}$, one gets equation (4.4.20) at level $n+1$.

We have performed a first crucial step: there exist some $\beta_0 > 0$ and a constant $x := \lim_{n\to\infty} x_n$, such that for every $n \in \mathbb{N}$ and every $\beta \leqslant \beta_0$ one has

$$\mathbf{E}_n\left[\delta_I e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \leqslant x^{|I|}\mathbf{E}_n[\delta_I] \quad \text{for every } I \subset \{1,\dots,2^n\}. \tag{4.4.26}$$

Then using the idea of Remark 4.4.2, one has from the definition of $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}$ (and expanding the exponential term)

$$\begin{aligned}\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} &= \mathbf{E}_n\left[e^{H_{n,h^{\mathrm{a}}_c}}\right] + \sum_{k=1}^{\infty}\frac{(\theta\beta^2\kappa^n)^k}{k!}\mathbf{E}_n\left[e^{H_{n,h^{\mathrm{a}}_c}}(S_n)^{2k}\right]\\ &\leqslant Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} + \mathbf{E}_n\left[\exp\left(x^2\theta\beta^2\kappa^n(S_n)^2\right) - 1\right]\\ &\leqslant Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} + \exp\left(c\beta^2(4\kappa/B^2)^n\right) - 1,\end{aligned} \tag{4.4.27}$$

where we used (4.4.26) for the first inequality and Theorem 4.A.3 for the second one. Then using that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \leqslant 1$, one has the desired upper bound for $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}$. On the other hand, with $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \geqslant 1$ one gets that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \geqslant 1 - c'\beta^2(4\kappa/B^2)^n$, which concludes the proof. □

Remark 4.4.4. Adapting the proof of Proposition 4.3.2 to the auxiliary partition function $\bar{Z}^{\omega}_{n,h}$, one gets under the same hypothesis that there exists a constant c_1' such that for any non-empty subset I of $\{1,\dots,2^n\}$ one has

$$\left(e^{-c_1'\beta^2}\right)^{|I|}\mathbf{E}_n[\delta_I] \leqslant \mathbf{E}_n\left[\delta_I e^{\bar{H}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \leqslant \left(e^{c_1'\beta^2}\right)^{|I|}\mathbf{E}_n[\delta_I]. \tag{4.4.28}$$

This implies, together with Lemma 4.4.1, an analog of Theorem 4.3.1: there exist some $\beta_0 > 0$ and constants $c_1', c_2' > 0$ such that for every $\beta \leqslant \beta_0$ and $u \in [0,1]$, one has

$$\mathbf{E}_n\left[\exp\left(e^{-c_1'\beta^2}uS_n\right)\right] \leqslant \mathbf{E}_n\left[\exp(uS_n)\, e^{\bar{H}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\right] \leqslant \mathbf{E}_n\left[\exp\left(e^{c_1'\beta^2}uS_n\right)\right] + c_2'\beta^2\left(\frac{4\kappa}{B^2}\right)^n. \tag{4.4.29}$$

4.4.3. The case $B^2/4 < \kappa < 1/2$: proof of Theorem 4.3.6. Using the identity (4.3.3), one has for all $n \in \mathbb{N}$ and $h \in \mathbb{R}$

$$Z^{\mathrm{a}}_{n+1,h} = \frac{1}{B}\mathbf{E}_n^{\otimes 2}\left[e^{H^{\mathrm{a},(1)}_{n,h}}e^{H^{\mathrm{a},(2)}_{n,h}}\exp\left(\beta^2\kappa^{n+1}S_n^{(1)}S_n^{(2)}\right)\right] + \frac{B-1}{B} \tag{4.4.30}$$

$$= \frac{1}{B}\sum_{m=0}^{\infty}\frac{(\beta^2\kappa^{n+1})^m}{m!}\mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h}}(S_n)^m\right]^2 + \frac{B-1}{B}. \tag{4.4.31}$$

If one takes $h = h^{\mathrm{a}}_c$ and uses the bound $Z^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c} \leqslant 1$, one gets

$$\sum_{m=0}^{\infty}\frac{(\beta^2\kappa^{n+1})^m}{m!}\mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}(S_n)^m\right]^2 \leqslant 1, \tag{4.4.32}$$

so that bounding each term of the sum by 1, one gets that for all $m \geqslant 0$

$$\mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}(S_n)^m\right] \leqslant \sqrt{m!}\left(\frac{1}{\beta}\left(\frac{1}{\sqrt{\kappa}}\right)^{n+1}\right)^m. \tag{4.4.33}$$

For $m = 1$ (using $Z^{\mathrm{a}}_{n,h} \geqslant (B-1)/B$) we obtain (4.3.14), but also an estimate for all the moments of S_n.

Using Lemma 4.4.3 one has

$$\left(\frac{2}{B}\right)^n \prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} \leqslant \mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}} S_n\right] \leqslant \frac{1}{\beta}\left(\frac{1}{\sqrt{\kappa}}\right)^{n+1}, \tag{4.4.34}$$

which implies (4.3.13). Another observation is that, writing $h = h_c^{\mathrm{a}} + u$, one gets from (4.4.33) that

$$\mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h}}\right] = \mathbf{E}_n\left[e^{uS_n} e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\right] \leqslant \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}}\left(\frac{u}{\beta}\left(\frac{1}{\sqrt{\kappa}}\right)^{n+1}\right)^m. \tag{4.4.35}$$

Thus if $u \leqslant (\sqrt{\kappa})^n$, one has that $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}+u} = \mathbf{E}_n\left[e^{H^{\mathrm{a}}_{n,h}}\right]$ does not grow with n. This is in contrast with the pure model where

$$Z^{\mathrm{pur}}_{n,u} = \mathbf{E}_n[\exp(uS_n)] \geqslant \exp(u\mathbf{E}_n(S_n)) = \exp(u(2/B)^n)$$

which diverges with n if $u = (\sqrt{\kappa})^n$ (recall we are considering $\kappa > B^2/4$).

All these facts lead us to conjecture that the phase transition of the annealed model for $B^2/4 < \kappa < 1/2$ is smoother than that of the pure model, and actually of order $\nu^{\mathrm{a}} = \log 2/\log(1/\sqrt{\kappa})$ in view of (4.4.33) (if one links the mean number of points at the critical point to the critical exponent).

4.5. Disorder relevance: control of the Variance

To prove disorder irrelevance for $B > B_c$ and the upper bounds on the difference between quenched and annealed critical points in Theorem 4.3.3, we use the following Proposition:

Proposition 4.5.1. *Let $\kappa < (B^2/4 \wedge 1/2)$. If $B > B_c$, there exists a $\beta_0 > 0$ such that for $\beta \leqslant \beta_0$ and for every $\eta \in (0,1)$ one can find $\varepsilon > 0$ such that for all $u \in (0,\varepsilon)$*

$$\mathtt{F}(\beta, h_c^{\mathrm{a}} + u) \geqslant (1-\eta)\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}} + u). \tag{4.5.1}$$

If $B < B_c$, then for every $\eta \in (0,1)$ one can find constants $c, \beta_0, \epsilon > 0$ such that if $\beta \leqslant \beta_0$, for all $u \in (c\beta^{\frac{2}{2-\nu}}, \epsilon(\eta))$

$$\mathtt{F}(\beta, h_c^{\mathrm{a}} + u) \geqslant (1-\eta)\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}} + u) \tag{4.5.2}$$

with ν as in (4.2.16).

If $B = B_c$, then for every $\eta \in (0,1)$ one can find $\beta_0 > 0$ and a constant $c > 0$ such that if $\beta \leqslant \beta_0$, for all $u \in \left(c\exp\left(-c\beta^{-2/3}\right), 1\right)$

$$\mathtt{F}(\beta, h_c^{\mathrm{a}} + u) \geqslant (1-\eta)\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}} + u). \tag{4.5.3}$$

Proof This is based on the study of the variance $\mathcal{V}_n := \mathbb{E}[(\bar{Z}^{\omega}_{n,h})^2] - \mathbb{E}[\bar{Z}^{\omega}_{n,h}]^2$.

Fix some $B \in (1,2)$. One has

$$\mathbb{E}\left[\left(\bar{Z}^{\omega}_{n,h}\right)^2\right] = \mathbf{E}_n^{\otimes 2}\left[\exp\left(\bar{H}^{\mathrm{a}}_{n,h}(\delta) + \bar{H}^{\mathrm{a}}_{n,h}(\delta') + \beta^2 \sum_{i,j=1}^{2^n} \kappa_{ij}\delta_i\delta'_j\right)\right] \tag{4.5.4}$$

with δ and δ' two independent copies of the same Galton-Watson process. We also have $\mathbb{E}\left[\bar{Z}^{\omega}_{n,h}\right]^2 = \mathbf{E}_n^{\otimes 2}\left[\exp\left(\bar{H}^{\mathrm{a}}_{n,h}(\delta)+\bar{H}^{\mathrm{a}}_{n,h}(\delta')\right)\right]$. To simplify notations, we write $h = h_c^{\mathrm{a}} + u$ and we define

$$D_n := \sum_{i,j=1}^{2^n} \kappa_{ij}\delta_i\delta'_j. \tag{4.5.5}$$

Then,

$$\mathcal{V}_n = \mathbf{E}_n^{\otimes 2}\left[e^{uS_n}e^{uS'_n}\left(e^{\beta^2 D_n}-1\right)e^{\bar{H}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}(\delta)}e^{\bar{H}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}(\delta')}\right] \leqslant \widetilde{\mathcal{V}}_n := \mathbf{E}_n^{\otimes 2}\left[e^{CuS_n}e^{CuS'_n}\left(e^{C\beta^2 D_n}-1\right)\right], \tag{4.5.6}$$

where we expanded the exponential and used Remark 4.4.2 and Eq. (4.4.28).

Using the Cauchy-Schwarz inequality in (4.5.6),

$$\widetilde{\mathcal{V}}_n \leqslant \mathbf{E}_n\left[e^{2CuS_n}\right]\sqrt{\mathbf{E}_n^{\otimes 2}\left[\left(e^{C\beta^2 D_n}-1\right)^2\right]} \leqslant \mathbf{E}_n\left[e^{2CuS_n}\right]\sqrt{\mathbf{E}_n^{\otimes 2}\left[e^{2C\beta^2 D_n}-1\right]}. \tag{4.5.7}$$

We define $Q_n := \mathcal{V}_n/\mathbb{E}[\bar{Z}^{\omega}_{n,h}]^2 \leqslant \mathcal{V}_n$, (recall that $h \geqslant h_c^{\mathrm{a}}$ and that $\mathbb{E}\bar{Z}^{\omega}_{n,h_c^{\mathrm{a}}} \geqslant 1$). Then one also uses Proposition 4.A.1 to get that $\mathbf{E}_n\left[e^{2CuS_n}\right] \leqslant c\exp\left(c2^n u^{\nu}\right)$. Therefore, one has

$$Q_n \leqslant c\exp\left(c2^n u^{\nu}\right)\sqrt{\mathbf{E}_n^{\otimes 2}\left[e^{2C\beta^2 D_n}-1\right]}. \tag{4.5.8}$$

Defining

$$n_1 = n_1(u) := \log(1/u)/\log(2/B) = \nu\log(1/u)/\log 2, \tag{4.5.9}$$

which is the value of n at which $\mathbf{E}_n[\exp(uS_n)]$ starts getting large, one has for $p \geqslant 0$

$$Q_{n_1+p} \leqslant ce^{c2^p}\sqrt{\mathbf{E}_{n_1+p}^{\otimes 2}\left[e^{2C\beta^2 D_{n_1+p}}-1\right]}. \tag{4.5.10}$$

Thus it is left to estimate the last term, with Proposition 4.A.5.

4.5.1. The case $B > B_c$. Thanks to Proposition 4.A.5 there exists some $\beta_0 > 0$ such that for $\beta < \beta_0$ and for all $n \in \mathbb{N}$

$$\mathbf{E}_n^{\otimes 2}\left[e^{2C\beta^2 D_n}-1\right] \leqslant c\beta_0^2\Phi^n, \tag{4.5.11}$$

for some $\Phi < 1$. Choose $p_1 = p_1(n_1)$ such that $e^{c2^{p_1}}\sqrt{\Phi^{n_1}} = 1$ (note that p_1 diverges with n_1) and then

$$Q_{n_1+p_1} \leqslant c'\sqrt{\Phi^{p_1}} \overset{n_1\to\infty}{\longrightarrow} 0. \tag{4.5.12}$$

Then we use that

$$\mathbb{E}\left[\log\bar{Z}^{\omega}_{n,h}\right] \geqslant \log\left(\frac{\mathbb{E}[\bar{Z}^{\omega}_{n,h}]}{2}\right)\mathbb{P}\left(\bar{Z}^{\omega}_{n,h} \geqslant \frac{\mathbb{E}[\bar{Z}^{\omega}_{n,h}]}{2}\right) + \log\left(\frac{B-1}{B}\right), \tag{4.5.13}$$

where $\mathbb{P}\left(\bar{Z}^{\omega}_{n,h} \geqslant \mathbb{E}[\bar{Z}^{\omega}_{n,h}]/2\right) \geqslant 1 - 4Q_n$ from the Tchebyshev inequality. We apply this with $n = n_1 + p_1(n_1)$ to get (using also Theorem 4.2.2 and (4.4.9))

$$\begin{aligned}\mathtt{F}(\beta,h) &\geqslant \frac{1}{2^n}\mathbb{E}\left[\log \bar{Z}^{\omega}_{n,h}\right] - \frac{\log B}{2^n} \geqslant (1-4\eta)\frac{1}{2^n}\log\left(\mathbb{E}[\bar{Z}^{\omega}_{n,h}]\right) - \frac{c}{2^n} \\ &\geqslant (1-4\eta)\mathtt{F}^{\mathrm{a}}(\beta,h) - \frac{c'}{2^{p_1(n_1)}}2^{-n_1} \geqslant (1-5\eta)\mathtt{F}^{\mathrm{a}}(\beta,h), \end{aligned} \tag{4.5.14}$$

provided that n_1 is large enough to ensure both

$$Q_{n_1+p_1} \leqslant c'\Phi^{p_1(n_1)/2} \leqslant \eta \tag{4.5.15}$$

$$\text{and} \quad c' 2^{-p_1(n_1)} u^{\nu} \leqslant \eta \mathtt{F}^{\mathrm{a}}(\beta,h) \qquad \text{for all} \quad u \in (0,1). \tag{4.5.16}$$

Note that the requirement on n_1 in (4.5.16) also depends only on η, cf. Theorem 4.3.1. Since n_1 is related to u via (4.5.9), one has actually to assume that $u \leqslant \epsilon(\eta)$ with ϵ sufficiently small, as required in Proposition 4.5.1.

4.5.2. The case $B < B_c$. Given $\eta > 0$ and $\beta \leqslant 1$, fix some $p_1 = p_1(\eta)$ such that (4.5.16) holds and assume that $c_1\beta^{2/(2-\nu)} \leqslant u \leqslant \epsilon(\eta)$ with $c_1 = c_1(\eta)$ to be chosen sufficiently large later (observe that if $\epsilon(\eta)$ is small one has that n_1 and p_1 are large, so the above requirement on p_1 is coherent). The definition of $n_1(u)$ (which gives $u = (B/2)^{n_1}$) and of ν (which gives $(2/B)^{\nu} = 2$) imply that

$$\beta^2 \leqslant c_1^{-1}\left(\frac{2}{B^2}\right)^{p_1(\eta)}\left(\frac{B^2}{2}\right)^{n_1+p_1(\eta)} \leqslant c_2\left(\frac{B^2}{2}\right)^{n_1+p_1(\eta)} \tag{4.5.17}$$

where $c_2 = c_2(\eta)$ can be made arbitrarily small by choosing c_1 large. Then, again provided that c_2 is small enough (*i.e.* c_1 large enough), we can apply Proposition 4.A.5 to get from (4.5.10)

$$Q_{n_1+p_1(\eta)} \leqslant c\, e^{c2^{p_1(\eta)}}\sqrt{c\beta^2\left(\frac{2}{B^2}\right)^{n_1+p_1(\eta)}} \leqslant c'\, e^{c2^{p_1(\eta)}}\sqrt{c_2(\eta)} \leqslant \eta. \tag{4.5.18}$$

From this point on, the proof proceeds like in the case $B > B_c$, starting from (4.5.13).

4.5.3. The case $B = B_c$. This is similar to the case $B < B_c$. The value of β_0 has to be chosen small enough to guarantee that Proposition 4.A.5 is applicable. We skip details.

□

4.6. Disorder relevance: critical point shift lower bounds

To prove disorder relevance, we give a finite size condition for delocalization, adapting the fractional moment method, first used in [DGLT09], and then in [GLT10b, GLT11] for the pinning model with *i.i.d.* disorder.

4.6.1. Fractional moment iteration. For $\gamma < 1$ let x_γ to be the largest solution of

$$x = \frac{x^2 + (B-1)^\gamma}{B^\gamma}.$$

One can easily see that for γ sufficiently close to 1 (which we assume to be the case in what follows) x_γ actually exists and is strictly less than 1. Moreover one has that x_γ increases to 1 as γ increases to 1. Then we have:

Proposition 4.6.1. *Take $\kappa < 1/2$. Then, setting $A_n := \mathbb{E}\left[\left(\bar{Z}^\omega_{n,h}\right)^\gamma\right]$ with $\bar{Z}^\omega_{n,h}$ defined in (4.4.2), one has*

$$A_{n+1} \leqslant \frac{A_n^2 + (B-1)^\gamma}{B^\gamma}. \tag{4.6.1}$$

If there exists some n_0 such that $A_{n_0} \leqslant x_\gamma$, then $\mathtt{F}(\beta, h) = 0$.

Proof If for some n_0 one has $A_{n_0} \leqslant x_\gamma$, then iterating (4.6.1) one gets $A_n \leqslant x_\gamma \leqslant 1$ for all $n \geqslant n_0$. Using the Jensen's inequality one has

$$\frac{1}{n}\mathbb{E}[\log \bar{Z}^\omega_{n,h}] = \frac{1}{\gamma n}\mathbb{E}[\log(\bar{Z}^\omega_{n,h})^\gamma] \leqslant \frac{1}{\gamma n}\log A_n \tag{4.6.2}$$

which gives $\mathtt{F}(\beta, h) = \bar{\mathtt{F}}(\beta, h) = 0$ (equality of the two free energies was noted after (4.4.2)).

We now turn to the proof of (4.6.1). We define $Z^\mu_{n,h} = \mathbf{E}_n\left[e^{H^\omega_{n,h}} e^{\mu\kappa_n\beta^2(S_n)^2}\right]$ and use that $(S_{n+1})^2 \leqslant 2(S_n)^2 + 2(S_n)^2$ to get the iteration

$$Z^\mu_{n+1,h} \leqslant \frac{1}{B} Z^{2\kappa\mu,(1)}_{n,h} Z^{2\kappa\mu,(2)}_{n,h} + \frac{B-1}{B} \tag{4.6.3}$$

where as usual the two partition functions in the r.h.s. refer to the first and second sub-system of size 2^n. From this, and using the inequality $(a+b)^\gamma \leqslant a^\gamma + b^\gamma$ for any $a, b \geqslant 0$ and $\gamma \leqslant 1$, one has

$$\mathbb{E}\left[(Z^\mu_{n+1,h})^\gamma\right] \leqslant \frac{1}{B^\gamma}\mathbb{E}\left[\left(Z^{2\kappa\mu,(1)}_{n,h} Z^{2\kappa\mu,(2)}_{n,h}\right)^\gamma\right] + \frac{(B-1)^\gamma}{B^\gamma}. \tag{4.6.4}$$

One then shows the following

Lemma 4.6.2. *If $\mu \geqslant \theta$ with $\theta = \frac{\kappa}{2(1-2\kappa)}$ as in (4.4.2),*

$$\mathbb{E}\left[\left(Z^{2\mu\kappa,(1)}_{n,h} Z^{2\mu\kappa,(2)}_{n,h}\right)^\gamma\right] \leqslant \mathbb{E}\left[\left(Z^\mu_{n,h}\right)^\gamma\right]^2. \tag{4.6.5}$$

This gives directly (4.6.1), taking $\mu = \theta$ so that $Z^\mu_{n,h} = \bar{Z}^\omega_{n,h}$. □

Proof of Lemma 4.6.2 One sets

$$\Phi(t, \mu) := \log \mathbb{E}_t\left[\left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^\gamma\right], \tag{4.6.6}$$

where one defines $\mathbb{P}_t$ to be the law of a Gaussian vector $(\omega_1, \ldots, \omega_{2^{n+1}})$ with correlations $\kappa_{ij}(t) = \kappa_p$ if $d(i,j) = p \leqslant n$, and $\kappa_{ij}(t) = t\kappa_{n+1}$ if $d(i,j) = n+1$. Then one can compute the derivatives of Φ. Using the definition of $Z^{\mu}_{n,h}$ one has for $t \geqslant 0$, $\mu \in \mathbb{R}$

$$\frac{\partial \Phi}{\partial \mu}(t,\mu) = \frac{\gamma \kappa_n \beta^2}{\mathbb{E}_t\left[\left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma}\right]} \times \mathbb{E}_t\left[\mathbf{E}_n^{\otimes 2}\left[\left((S_n^{(1)})^2 + (S_n^{(2)})^2\right) e^{H^{\omega,(1)}_{n,h} + H^{\omega,(2)}_{n,h}} e^{\mu \kappa_n ((S_n^{(1)})^2 + (S_n^{(2)})^2)}\right] \left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma - 1}\right]. \tag{4.6.7}$$

Thanks to Proposition 4.B.3 one gets

$$\frac{\partial \Phi}{\partial t}(t,\mu) = \frac{\kappa_{n+1}}{\mathbb{E}_t\left[\left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma}\right]} \sum_{i=1}^{2^n} \sum_{j=2^n+1}^{2^{n+1}} \mathbb{E}_t\left[\frac{\partial^2}{\partial \omega_i \partial \omega_j}\left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma}\right]. \tag{4.6.8}$$

For the values of i, j under consideration one has

$$\frac{\partial}{\partial \omega_i \partial \omega_j}\left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma} = \gamma^2 \beta^2 \mathbf{E}_n^{\otimes 2}\left[\delta_i \delta_j e^{H^{\omega,(1)}_{n,h} + H^{\omega,(2)}_{n,h}} e^{\mu \kappa_n ((S_n^{(1)})^2 + (S_n^{(2)})^2)}\right] \times \left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma - 1}. \tag{4.6.9}$$

Therefore,

$$\sum_{i=1}^{2^n} \sum_{j=2^n+1}^{2^{n+1}} \frac{\partial^2}{\partial \omega_i \partial \omega_j}\left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma} \leqslant \frac{\gamma^2 \beta^2}{2} \mathbf{E}_n^{\otimes 2}\left[\left((S_n^{(1)})^2 + (S_n^{(2)})^2\right) e^{H^{\omega,(1)}_{n,h} + H^{\omega,(2)}_{n,h}} e^{\mu \kappa_n ((S_n^{(1)})^2 + (S_n^{(2)})^2)}\right] \left(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\right)^{\gamma - 1}, \tag{4.6.10}$$

and as a consequence, since we chose $\kappa_n = \kappa^n$

$$\frac{\partial \Phi}{\partial t}(t,\mu) \leqslant \frac{\kappa}{2} \frac{\partial \Phi}{\partial \mu}(t,\mu). \tag{4.6.11}$$

Thus, the function $t \mapsto \Phi(t, \mu - \kappa t/2)$ is non-increasing and

$$\log \mathbb{E}\left[\left(Z^{\mu-\kappa/2,(1)}_{n,h} Z^{\mu-\kappa/2,(2)}_{n,h}\right)^{\gamma}\right] = \Phi(1, \mu - \kappa/2) \leqslant \Phi(0,\mu) = 2 \log \mathbb{E}_t\left[\left(Z^{\mu}_{n,h}\right)^{\gamma}\right]. \tag{4.6.12}$$

Then, one uses that for $\mu \geqslant \frac{\kappa}{2(1-2\kappa)}$ one has $2\mu\kappa \leqslant \mu - \kappa/2$, which allows us to conclude. □

4.6.2. Change of measure. In this section we prove the lower bounds of Theorem 4.3.3 on the critical point shift for $B \leqslant B_c$.

One fixes γ close to 1 such that x_γ is also close to 1, and proves that if $h = h_c^{\mathrm{a}} + u$ with $u > 0$ small enough, one has $A_{n_0} := \mathbb{E}\left[\left(\bar{Z}^{\omega}_{n_0,h}\right)\right] \leqslant x_\gamma$ for some $n_0 \in \mathbb{N}$. To this purpose, we introduce a change of measure in the spirit of [GLT11]. Define

$$\begin{aligned} g(\omega) &:= \mathbf{1}_{\{F(\omega) \leqslant R\}} + \varepsilon_R \mathbf{1}_{\{F(\omega)>R\}}, \\ F(\omega) &:= \langle V\omega,\omega\rangle - \mathbb{E}[\langle V\omega,\omega\rangle], \end{aligned} \tag{4.6.13}$$

where the choices of the symmetric $2^n \times 2^n$ matrix V, of $R \in \mathbb{R}$ and $\varepsilon_R > 0$ will be made later. Note that we have chosen F to be centered. Then using the Hölder inequality, one has

$$\mathbb{E}\left[(\bar{Z}^{\omega}_{n,h})^{\gamma}\right] = \mathbb{E}\left[g(\omega)^{-\gamma}(g(\omega)\bar{Z}^{\omega}_{n,h})^{\gamma}\right] \leqslant \mathbb{E}\left[(g(\omega))^{-\frac{\gamma}{1-\gamma}}\right]^{1-\gamma} \mathbb{E}\left[g(\omega)\bar{Z}^{\omega}_{n,h}\right]^{\gamma}. \tag{4.6.14}$$

Remark 4.6.3. The original idea [GLT10b] is to take $g(\omega) = \frac{\mathrm{d}\check{\mathbb{P}}}{\mathrm{d}\mathbb{P}}$ where $\check{\mathbb{P}}$ is a new probability measure on $\{\omega_1,\dots,\omega_{2^n}\}$ such that $\check{\mathbb{P}}$ and $\mathbb{P}$ are mutually absolutely continuous. Then, to control both terms in (4.6.14), one has to choose $\check{\mathbb{P}}$ in a certain sense close enough to $\mathbb{P}$, such that the first term is close to 1, but also such that under the measure $\check{\mathbb{P}}$ the annealed partition function $\mathbb{E}\left[g(\omega)\bar{Z}_{n,h_c^{\mathrm{a}}}\right] = \check{\mathbb{E}}\left[\bar{Z}_{n,h_c^{\mathrm{a}}}\right]$ is small.

The choice of g and F in (4.6.13) has the same effect of the change of measure in [GLT10b], that is inducing negative correlations between different ω_i, and the specific form (4.6.13) is chosen for technical reasons, to deal more easily with the case in which $\langle V\omega,\omega\rangle$ is large.

Let us first deal with the Radon-Nikodym part of (4.6.14): we make here the choice $\varepsilon_R := \mathbb{P}(F(\omega) \geqslant R)^{1-\gamma}$. Then one has

$$\mathbb{E}\left[(g(\omega))^{-\frac{\gamma}{1-\gamma}}\right] \leqslant 1 + (\varepsilon_R)^{-\frac{\gamma}{1-\gamma}}\mathbb{P}(F(\omega) \geqslant R) = 1 + \mathbb{P}(F(\omega) \geqslant R)^{1-\gamma} = 1 + \varepsilon_R. \tag{4.6.15}$$

We now use the following lemma to estimate ε_R in terms of R. We let $\|V\|^2 = \sum_{i,j} V_{ij}^2$ and K denote the covariance matrix $(\kappa_{ij})_{1 \leqslant i,j \leqslant 2^n}$.

Lemma 4.6.4. *If V is such that V_{ij} depends only on $d(i,j)$ and $\|V\|^2 = 1$, then one has $\mathbb{V}\mathrm{ar}(F) < 2K_\infty^2$ with K_∞ defined in (4.3.16), so that*

$$\mathbb{P}(F(\omega) \geqslant R) \leqslant \frac{2K_\infty}{R^2} \xrightarrow{R\to\infty} 0. \tag{4.6.16}$$

Thus one gets that $\varepsilon_R \leqslant const \times R^{-2(1-\gamma)}$, which can be made arbitrarily small choosing R large.

Proof We have that $\mathbb{V}\mathrm{ar}(F) = \mathbb{E}\left[\langle V\omega,\omega\rangle^2\right] - \mathbb{E}\left[\langle V\omega,\omega\rangle\right]^2$, and we can compute

$$\begin{aligned} \mathbb{E}\left[\langle V\omega,\omega\rangle^2\right] &= \sum_{i,j=1}^{2^n}\sum_{k,l=1}^{2^n} V_{ij}V_{kl}\mathbb{E}[\omega_i\omega_j\omega_k\omega_l] = \sum_{i,j=1}^{2^n}\sum_{k,l=1}^{2^n} V_{ij}V_{kl}(\kappa_{ij}\kappa_{kl}+\kappa_{ik}\kappa_{jl}+\kappa_{il}\kappa_{jk}) \\ &= \mathbb{E}\left[\langle V\omega,\omega\rangle\right]^2 + 2Tr\left((VK)^2\right). \end{aligned} \tag{4.6.17}$$

We now use Lemma 4.B.1, which says that V and K can be co-diagonalized, and that the eigenvalues of K are bounded by K_∞, to get that $Tr\left((VK)^2\right) \leqslant K_\infty^2 Tr(V^2) = K_\infty^2$ (recall that $Tr(V^2) = \|V\|^2 = 1$, as V is symmetric). One finally gets that

$\mathbb{V}\mathrm{ar}(F) \leqslant 2K_\infty^2$, and as F is centered, using Tchebyshev's inequality gives the result. □

Next, we study the second factor in the r.h.s. of (4.6.14):

$$\mathbb{E}\left[g(\omega)\bar{Z}^{\omega}_{n,h}\right] \leqslant \mathbb{E}\left[\mathbf{1}_{\{F(\omega)\leqslant R\}}\bar{Z}^{\omega}_{n,h}\right] + \varepsilon_R \mathbb{E}\left[\bar{Z}^{\omega}_{n,h}\right]. \tag{4.6.18}$$

To study the first term we define the measure $\widetilde{\mathbb{P}}$ on $\{\omega_1,\dots,\omega_{2^n}\}$ to be absolutely continuous with respect to $\mathbb{P}$, with Radon-Nikodym derivative given by $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\bar{Z}^{\omega}_{n,h}}{\bar{Z}^{\mathrm{a}}_{n,h}}$. One then has

$$\mathbb{E}\left[\mathbf{1}_{\{F(\omega)\leqslant R\}}\bar{Z}^{\beta,\omega}_{n,h}\right] = \bar{Z}^{\mathrm{a}}_{n,h}\widetilde{\mathbb{P}}\left(F(\omega)\leqslant R\right). \tag{4.6.19}$$

We are now ready to choose $V = V_n$, and we do so as in [GLT10b]. We take V to be zero on the diagonal ($V_{ii}=0$), and for $i,j\in\{1,\dots,2^n\}$

$$V_{ij} := \frac{\mathbf{E}_n[\delta_i\delta_j]}{Y_n}, \qquad \text{if } i\neq j, \tag{4.6.20}$$

where

$$Y_n := \left(\sum_{\substack{i,j=1\\ i\neq j}}^{2^n} \mathbf{E}_n[\delta_i\delta_j]^2\right)^{1/2} \tag{4.6.21}$$

is used to normalize V. We stress that V satisfy the conditions of Lemma 4.6.4.

One can compute easily Y_n , since from Proposition 4.2.1 we have $\mathbf{E}_n[\delta_i\delta_j] = B^{-n-d(i,j)+1}$, and one finds (cf. [GLT10b, Eq. (8.23)])

$$Y_n = \begin{cases} \sqrt{n} & \text{if } B = B_c := \sqrt{2}, \\ \Theta\left(\left(\frac{2}{B^2}\right)^n\right) & \text{if } B < B_c \end{cases} \tag{4.6.22}$$

where $X = \Theta(Y)$ means that $X \geqslant cY$ for some positive constant c.

Proposition 4.6.5. *We choose $V = V_n$ as in (4.6.20)-(4.6.21), and $R = R_n := \frac{1}{2}\widetilde{\mathbb{E}}[F(\omega)]$. Then there exists some $\delta > 0$ small such that, if $u(2/B)^n \leqslant \delta$, one has*

$$R := \frac{1}{2}\widetilde{\mathbb{E}}[F(\omega)] \geqslant c\beta^2 Y_n. \tag{4.6.23}$$

Therefore, from (4.6.22), R can be made arbitrarily large with n. Moreover there exists a constant $\zeta > 0$ which does not depend on n, such that

$$\widetilde{\mathbb{P}}\left(F(\omega)\geqslant R\right) = \widetilde{\mathbb{P}}\left(F(\omega)\geqslant \frac{1}{2}\widetilde{\mathbb{E}}[F(\omega)]\right) \geqslant \zeta. \tag{4.6.24}$$

Combining this Proposition to (4.6.18) and (4.6.19), one gets that

$$\mathbb{E}\left[g(\omega)\bar{Z}^{\omega}_{n,h}\right] \leqslant \bar{Z}^{\mathrm{a}}_{n,h}\left(1-\zeta+\varepsilon_R\right). \tag{4.6.25}$$

Recalling the equality (4.4.29) (which is the analog of Theorem 4.3.1 for the alternative partition function $\bar{Z}^{\mathrm{a}}_{n,h}$), one has for $\kappa < B^2/4 \wedge 1/2$

$$\bar{Z}^{\mathrm{a}}_{n,h} \leqslant \mathbf{E}_n\left[e^{c_1' u S_n}\right] + c_2'\beta^2\left(\frac{4\kappa}{B^2}\right)^n \leqslant e^{c\delta} + \delta, \tag{4.6.26}$$

provided that $u \leqslant \delta(B/2)^n$ with δ small (to be able to apply Proposition 4.A.1 to $\mathbf{E}_n\left[e^{c_1' u S_n}\right]$), and that $n \geqslant n_\delta$ to deal with the term $(4\kappa/B^2)^n$. Therefore, if δ and ε_R was chosen small enough (that is smaller than some constant $c = c(\zeta)$), one has for $n \geqslant n_\delta$ that $\mathbb{E}\left[g(\omega)\bar{Z}_{n,h}^{\beta,\omega}\right] \leqslant 1-\zeta/2$ for all $u \leqslant \delta(B/2)^n$. This and (4.6.15) bound the two terms in (4.6.14), so that one has

$$A_n := \mathbb{E}\left[(\bar{Z}_{n,h}^{\omega})^{\gamma}\right] \leqslant (1+\varepsilon_R)\,(1-\zeta/2)^{\gamma} \leqslant 1-\zeta/3 \leqslant x_\gamma, \tag{4.6.27}$$

where the two last inequalities hold if ε_R is small and γ close to 1. To sum up, for δ, β small and R large enough, one has that $A_n \leqslant x_\gamma$ for all $u \leqslant \delta(B/2)^n$, and so $\mathtt{F}(\beta, h_c^{\mathrm{a}} + u) = 0$.

Then, let us check how large has to be n so that our choice of $R := \frac{1}{2}\widetilde{\mathbb{E}}[F(\omega)]$ becomes large. From Proposition 4.6.5 one has that $R \geqslant c\beta^2 Y_n$ so that one has to take $\beta^2 Y_n \geqslant C$ for some constant C large enough. From (4.6.22), in order to have $\beta^2 Y_n \geqslant C$,

- if $B < B_c$, it is enough to take n larger than $n_0 := \log(C'\beta^{-2})/\log(2/B^2)$;
- if $B = B_c$, one has to take n larger than $n_0 := c'\beta^{-4}$.

Then for $n = n_0$ one gets that R is large, but one also needs to take $u \leqslant \delta(2/B)^{n_0}$ to ensure that $A_{n_0 \vee n_\delta} \leqslant x_\gamma$. Notice that from the choice of n_0 above, the condition on u translates into

$$u \leqslant \begin{cases} c'\beta^{2\log(2/B)/\log(2/B^2)} = c'\beta^{\frac{2}{2-\nu}} & \text{if } B < B_c, \\ e^{-c\beta^{-4}} & \text{if } B = B_c, \end{cases} \tag{4.6.28}$$

where we also used that $\nu = \log 2/\log(2/B)$. One then gets the desired bounds (4.3.10)-(4.3.11) on the difference between quenched and annealed critical points. □

4.6.3. Proof of Proposition 4.6.5. To compute $\widetilde{\mathbb{E}}[F(\omega)]$, we define for any $1 \leqslant i,j \leqslant 2^n$

$$U_{ij} := \widetilde{\mathbb{E}}[\omega_i\omega_j] = \frac{1}{\bar{Z}_{n,h}^{\mathrm{a}}}\mathbf{E}_n\mathbb{E}\left[\omega_i\omega_j\, e^{\bar{H}_{n,h}^{\omega}}\right]. \tag{4.6.29}$$

A Gaussian integration by parts gives easily

$$U_{ij} = \kappa_{ik} + u_{ij} := \kappa_{ij} + \beta^2\sum_{k,l=1}^{2^n}\kappa_{ik}\kappa_{jl}\bar{\mathbf{E}}_{n,h}^{\mathrm{a}}[\delta_k\delta_l], \tag{4.6.30}$$

where $\bar{\mathbf{E}}_{n,h}^{\mathrm{a}}$ denotes expectation w.r.t. the measure whose density with respect to $\mathbf{P}_n$ is $\exp(\bar{H}_{n,h}^{\mathrm{a}})/\bar{Z}_{n,h}^{\mathrm{a}}$. We then compare $\bar{\mathbf{E}}_{n,h}^{\mathrm{a}}[\delta_k\delta_l]$ with $\mathbf{E}_n[\delta_k\delta_l]$, using that $h = h_c^{\mathrm{a}} + u$, $0 \leqslant u \leqslant \delta(B/2)^n$:

$$\bar{\mathbf{E}}_{n,h}^{\mathrm{a}}[\delta_k\delta_l] = \frac{1}{\bar{Z}_{n,h}^{\mathrm{a}}}\mathbf{E}_n\left[\delta_k\delta_l e^{\bar{H}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}} e^{uS_n}\right] \leqslant e^{2c_1\beta^2}\mathbf{E}_n\left[\delta_k\delta_l e^{e^{c_1\beta^2}uS_n}\right] \leqslant c'\mathbf{E}_n\left[\delta_k\delta_l\right] \tag{4.6.31}$$

where in the first inequality we used Remark 4.4.4 and the fact that $\bar{Z}_{n,h}^{\mathrm{a}} \geqslant \bar{Z}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}} \geqslant 1$, and in the second inequality we used that $u(2/B)^n \leqslant \delta$ to apply Corollary 4.A.2.

The same argument easily gives $\bar{\mathbf{E}}^{\mathrm{a}}_{n,h}[\delta_k\delta_l] \geqslant c\mathbf{E}_n[\delta_k\delta_l]$ in the range of u considered, so that $c\beta^2 a_{ij} \leqslant u_{ij} \leqslant c'\beta^2 a_{ij}$, where

$$a_{ij} := \sum_{k,l=1}^{2^n} \kappa_{ik}\kappa_{jl}\mathbf{E}_n[\delta_k\delta_l] \geqslant Y_n(KVK)_{ij} \tag{4.6.32}$$

(the inequality is due to the fact that V is zero on the diagonal). We finally get

$$\widetilde{\mathbb{E}}[F(\omega)] = \widetilde{\mathbb{E}}[\langle V\omega,\omega\rangle] - \mathbb{E}[\langle V\omega,\omega\rangle] = \sum_{i,j=1}^{2^n} V_{ij}(\kappa_{ij}+u_{ij}) - \mathbb{E}[\langle V\omega,\omega\rangle] = \sum_{i,j=1}^{2^n} V_{ij}u_{ij}, \tag{4.6.33}$$

so that we only have to compute $\sum_{i,j=1}^{2^n} V_{ij}a_{ij} \geqslant Y_n\mathrm{Tr}(VKVK)$. Since $\|V\|^2 = 1$ and all eigenvalues of K are between 1 and K_∞, one has $\mathrm{Tr}\left((VK)^2\right) = \Theta(1)$. Altogether, we get (4.6.23).

We now prove (4.6.24). Using the Paley-Zygmund inequality, we get that

$$\widetilde{\mathbb{P}}(F(\omega) \geqslant R) = \widetilde{\mathbb{P}}\left(F(\omega) \geqslant \frac{1}{2}\widetilde{\mathbb{E}}[F(\omega)]\right) \geqslant \frac{\widetilde{\mathbb{E}}[F(\omega)]^2}{4\widetilde{\mathbb{E}}[F(\omega)^2]}, \tag{4.6.34}$$

so that we only have to prove the following:

$$\widetilde{\mathbb{V}\mathrm{ar}}(F(\omega)) = \widetilde{\mathbb{E}}[\langle V\omega,\omega\rangle^2] - \widetilde{\mathbb{E}}[\langle V\omega,\omega\rangle]^2 = O(\widetilde{\mathbb{E}}[F(\omega)]^2). \tag{4.6.35}$$

Indeed from this it follows immediately that there exists some constant $\zeta > 0$ such that $\widetilde{\mathbb{E}}[F(\omega)]^2/\widetilde{\mathbb{E}}[F(\omega)^2] \geqslant \zeta$.

We now prove (4.6.35), studying $\widetilde{\mathbb{E}}\left[\langle V\omega,\omega\rangle^2\right] = \sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}\widetilde{\mathbb{E}}[\omega_i\omega_j\omega_k\omega_l]$, starting with the computation, for any $1 \leqslant i,j,k,l \leqslant 2^n$, of

$$\widetilde{\mathbb{E}}[\omega_i\omega_j\omega_k\omega_l] = \frac{1}{\bar{Z}^{\mathrm{a}}_{n,h}}\mathbf{E}_n\mathbb{E}\left[\omega_i\omega_j\omega_k\omega_l\ e^{\bar{H}^{\omega}_{n,h}}\right]. \tag{4.6.36}$$

Again, a Gaussian integration by parts gives, after elementary computations,

$$\begin{aligned}\widetilde{\mathbb{E}}\left[\omega_i\omega_j\omega_k\omega_l\right] = A_{ijkl} + B_{ijkl} := [\kappa_{ij}U_{kl} + \kappa_{ik}U_{jl} + \kappa_{il}U_{jk} + \kappa_{jk}u_{il} + \kappa_{jl}u_{ik} + \kappa_{kl}u_{ij}] \\ + \beta^4 \sum_{r,s,t,v=1}^{2^n} \kappa_{ir}\kappa_{js}\kappa_{kt}\kappa_{lv}\bar{\mathbf{E}}^{\mathrm{a}}_{n,h}\left[\delta_r\delta_s\delta_t\delta_v\right].\end{aligned} \tag{4.6.37}$$

We estimate $\widetilde{\mathbb{E}}\left[\langle V\omega,\omega\rangle^2\right]$ by analyzing separately A_{ijkl} and B_{ijkl}.

Contribution from B_{ijkl}: we have

$$B_{ijkl} \leqslant c\beta^4 \sum_{r,s,t,v=1}^{2^n} \kappa_{ir}\kappa_{js}\kappa_{kt}\kappa_{lv}\mathbf{E}_n\left[\delta_r\delta_s\delta_t\delta_v\right], \tag{4.6.38}$$

where we used again Proposition 4.3.2 and Corollary 4.A.2 as in (4.6.31) (recall that we consider $u \leqslant \delta(B/2)^n$). Then defining

$$W_{ij} := \frac{\mathbf{E}_n[\delta_i\delta_j]}{Y_n} = V_{ij} + \frac{\mathbf{1}_{\{i=j\}}}{Y_nB^n}. \tag{4.6.39}$$

we get

$$\sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}B_{ijkl} \leqslant c\beta^4 \sum_{r,s,t,v=1}^{2^n} (KWK)_{rs}(KWK)_{tv}\mathbf{E}_n\left[\delta_r\delta_s\delta_t\delta_v\right]$$
$$\leqslant c'\beta^4 \sum_{\substack{r,s,t,v=1\\ r\neq s,t\neq v}}^{2^n} W_{rs}W_{tv}\mathbf{E}_n\left[\delta_r\delta_s\delta_t\delta_v\right] + c''\beta^4 \sum_{r,t,v=1}^{2^n} W_{rr}W_{tv}\mathbf{E}_n\left[\delta_r\delta_t\delta_v\right], \quad (4.6.40)$$

where we used the following claim:

Claim 4.6.6. *There exists a constant* $c' > 0$ *such that for every* $1 \leqslant i,j \leqslant 2^n$, $(WK)_{ij} \leqslant c'W_{ij}$ *and* $(KW)_{ij} \leqslant c'W_{ij}$.

Proof of the Claim We write $q = d(i,j)$, so $W_{ij} =: W_q$, and

$$(WK)_{ij} = \sum_{l=1}^{2^n} W_{il}\kappa_{lj} = \sum_{p=0}^{q-1} 2^{p-1}W_p\kappa_q + \sum_{p=0}^{q-1} 2^{p-1}W_q\kappa_p + \sum_{p=q+1}^{n} 2^{p-1}W_p\kappa_p, \quad (4.6.41)$$

where we decomposed the sum according to the positions of l ($d(i,l) = p < q$, $d(i,l) = q$ or $d(i,l) > q$). Using that W_p is decreasing with p, we get that the second and the third term are both smaller than $(\sum 2^p\kappa_p)W_q$. We only have to deal with the first term, using the explicit expression of W_p, together with Proposition 4.2.1:

$$\sum_{p=0}^{q-1} 2^{p-1}W_p = \frac{1}{Y_n}B^{-n}\sum_{p=0}^{q-1}\left(\frac{2}{B}\right)^{p-1} \leqslant c\,\frac{1}{Y_n}B^{-n}\left(\frac{2}{B}\right)^{q} = c\,2^qW_q, \quad (4.6.42)$$

so that the first term in (4.6.41) is smaller than $c2^q\kappa_qW_q$. One then has that $(WK)_{ij} \leqslant c'W_{ij}$, and the same computations also gives that $(KW)_{ij} \leqslant c'W_{ij}$. □

The main term in the r.h.s. of (4.6.40) is the first one, for which we have

Lemma 4.6.7. *Let* $B \leqslant B_c$. *There exists a constant* $c > 0$ *such that*

$$\sum_{\substack{r,s,t,v=1\\ r\neq s,t\neq v}}^{2^n} V_{rs}V_{tv}\mathbf{E}_n\left[\delta_r\delta_s\delta_t\delta_v\right] = \frac{1}{Y_n^2}\sum_{\substack{r,s,t,v=1\\ r\neq s,t\neq v}}^{2^n} \mathbf{E}_n[\delta_r\delta_s]\mathbf{E}_n[\delta_t\delta_v]\mathbf{E}_n[\delta_r\delta_s\delta_t\delta_v] \leqslant cY_n^2. \quad (4.6.43)$$

This can be found in the proof of Lemma 4.4 of [GLT10b] for $B = B_c$; the proof is easily extended to the case $B < B_c$.

As for the remaining terms in (4.6.40), it is not hard to see, using repeatedly Proposition 4.2.1, that they give a contribution of order $o(Y_n^2)$. For instance, one has

$$\beta^4 \sum_{\substack{r,t,v=1\\ t\neq v}}^{2^n} W_{rr}W_{tv}\mathbf{E}_n\left[\delta_r\delta_t\delta_v\right]$$
$$\leqslant c\beta^4\frac{2^n}{B^nY_n^2}\sum_{p=0}^{n} 2^pB^{-n-p}\sum_{q=0}^{n} 2^qB^{-n-p-q} = \beta^4 o(Y_n^2). \quad (4.6.44)$$

Altogether one has

$$\sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}B_{ijkl} = \beta^4 O\left(Y_n^2\right) = O\left(\widetilde{\mathbb{E}}[F(\omega)]^2\right), \tag{4.6.45}$$

cf. (4.6.23).

Contribution of A_{ijkl}: recalling that $U_{ij}=\kappa_{ij}+u_{ij}$, one has $\kappa_{ij}U_{kl}+\kappa_{kl}u_{ij} \leqslant U_{ij}U_{kl}$. Thus, we get

$$\sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}(\kappa_{ij}U_{kl}+\kappa_{kl}u_{ij}) \leqslant \left(\sum_{i,j=1}^{2^n} V_{ij}U_{ij}\right)^2 = \widetilde{\mathbb{E}}\left[\langle V\omega,\omega\rangle\right]^2, \tag{4.6.46}$$

that we recall is not $O(\widetilde{\mathbb{E}}[F(\omega)]^2)$, but will be canceled in the variance. The other contributions are, thanks to symmetry of V, all equal to (or smaller than)

$$\sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}\kappa_{ik}U_{jl} = \sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}\kappa_{ik}\kappa_{jl} + \sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}\kappa_{ik}u_{jl}, \tag{4.6.47}$$

where the first term is $\mathrm{Tr}\left((VK)^2\right)$ which is bounded as remarked before. Thanks to the estimate $u_{jl} \leqslant c'\beta^2 a_{jl} = c'\beta^2 Y_n (KWK)_{jl}$, the second term is bounded above by a constant times

$$\begin{aligned}\beta^2 Y_n \sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}\kappa_{ik}(KWK)_{jl} &\leqslant \beta^2 Y_n \mathrm{Tr}\left((WK)^3\right)\\ &\leqslant c\beta^2 Y_n \mathrm{Tr}(W^2) \leqslant 2c\beta^2 Y_n = O(\widetilde{\mathbb{E}}[F(\omega)]).\end{aligned} \tag{4.6.48}$$

We used Lemma 4.B.1 to co-diagonalize W and K and to bound the eigenvalues of K by a constant, and then the fact that the eigenvalues λ_i of W are also bounded, so that $\sum|\lambda_i|^3 \leqslant c\sum|\lambda_i|^2 = c\mathrm{Tr}(W^2) = O(1)$. Indeed, $\mathrm{Tr}(W^2) = \mathrm{Tr}(V^2) + \sum_i W_{ii}^2 = 1 + (2/B^2)^n Y_n^{-2} = 1 + o(1)$. Putting together (4.6.37) with the estimates (4.6.45), (4.6.46) and (4.6.48) we have

$$\begin{aligned}\widetilde{\mathbb{V}\mathrm{ar}}(F(\omega)) = \widetilde{\mathbb{E}}(\langle V\omega,\omega\rangle^2) - \left(\widetilde{\mathbb{E}}\langle V\omega,\omega\rangle\right)^2 &= \sum_{ijkl}(A_{ijkl}+B_{ijkl})V_{ij}V_{kl} - \left(\widetilde{\mathbb{E}}\langle V\omega,\omega\rangle\right)^2\\ &= O\left(\widetilde{\mathbb{E}}[F(\omega)]^2\right)\end{aligned} \tag{4.6.49}$$

and (4.6.35) is proven.

4.A. Pure model estimates

We give here estimates on the pure system, that rely on the results of Section 1.3.2, mainly Proposition 4.A.1, that control the polymer measure of the homogeneous hierarchical pinning model.

We first give some estimates on the partition function of a system of size n.

Proposition 4.A.1. (1) *There exist constants $a_0 > 0$ and $c_0 > 0$ such that for any $n \geqslant 0$, if $u \leqslant a_0\,(B/2)^n$ one has*

$$\mathbf{E}_n\left[\exp\left(uS_n\right)\right] \leqslant \exp\left(c_0 u (2/B)^n\right). \tag{4.A.1}$$

(2) *There exists a constant $c > 0$ such that for any $n \geqslant 0$ and $u \geqslant 0$ one has*

$$\mathbf{E}_n\left[\exp\left(uS_n\right)\right] \leqslant c\exp\left(cu^{\nu}2^n\right), \tag{4.A.2}$$

where ν is as in (4.2.16).

Proof For the first inequality, the same type of computation was already done in [GLT10a], and we give here only an outline of the proof. The partition function R_k of the pure model satisfies the iteration

$$\begin{cases} R_0 = e^u, \\ R_{k+1} = \frac{R_k^2+B-1}{B}. \end{cases} \tag{4.A.3}$$

Defining $P_k := R_k - 1$, we can linearize the iteration equation verified by P_k to obtain that $P_{k+1} = \frac{2}{B}P_k + cst.P_k^2$. Therefore, we get by iteration that $P_k \leqslant c_0 u \left(\frac{2}{B}\right)^k$ for every $k \leqslant n$ (because we stay in the linear regime for the chosen value of u), so that for $k = n$ we get the result.

For the second inequality, we use that for any $n \geqslant 0$ and $u \geqslant 0$,

$$\frac{1}{2^n}\log \mathbf{E}_n\left[\exp\left(uS_n\right)\right] \leqslant \mathtt{F}(u) + \frac{c(B)}{2^n}, \tag{4.A.4}$$

from [GLT10a, Th.1.1], and this gives immediately the result, using (4.2.15). ☐

Defining for any subset $I \subset \{1, \dots, 2^n\}$ $\delta_I := \prod_{i\in I}\delta_i$, and $\delta_I = 1$ if $I = \emptyset$, one wants to compare $\mathbf{E}_n[\delta_I e^{uS_n}]$ and $\mathbf{E}_n[\delta_I]$ when the partition function $Z_{n,h}^{\text{pure}}$ is still in the linear regime $0 \leqslant u \leqslant a_0\ (B/2)^n$, the bound $\mathbf{E}_n[\delta_I e^{uS_n}] \geqslant \mathbf{E}_n[\delta_I]$ being trivial.

Corollary 4.A.2. *There exist constants $a_0 > 0$ and $c' > 0$ such that for any $n \geqslant 0$ and any non-empty subset $I \subset \{1, \dots, 2^n\}$, if $0 \leqslant u \leqslant a_0\,(B/2)^n$ one has*

$$\mathbf{E}_n\left[\delta_I \exp\left(uS_n\right)\right] \leqslant \exp\left(c'u\left(\frac{2}{B}\right)^n\right)^{|I|}\mathbf{E}_n\left[\delta_I\right]. \tag{4.A.5}$$

Proof We prove by iteration on n that for all non-empty subsets $I \subset \{1, \dots, 2^n\}$, if $u \leqslant a_0\,(B/2)^n$ one has

$$\mathbf{E}_n\left[\delta_I \exp\left(uS_n\right)\right] \leqslant \exp\left(c_0 u\sum_{k=0}^{n}\left(\frac{2}{B}\right)^k\right)^{|I|}\mathbf{E}_n\left[\delta_I\right], \tag{4.A.6}$$

where c_0 is the constant obtained in Proposition 4.A.1.

The case $n = 0$ is trivial. Let us assume that we have the assumption for $n \geqslant 0$ and prove it for $n + 1$. Take I a non-empty subset of $\{1, \dots, 2^{n+1}\}$. As in the proof of Lemma 4.4.3, one decomposes I into its "left" and "right" part and writes $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$ and $|I| = |I_1| + |\widetilde{I}_2|$.

If $I_1, \widetilde{I}_2 \neq \emptyset$, using the induction hypothesis, one easily has

$$\mathbf{E}_{n+1}\left[\delta_I \exp\left(uS_{n+1}\right)\right] = \frac{1}{B}\mathbf{E}_n\left[\delta_{I_1}\exp\left(uS_n\right)\right]\mathbf{E}_n\left[\delta_{\widetilde{I}_2}\exp\left(uS_n\right)\right] \leqslant \exp\left(c_0 u \sum_{k=0}^{n}\left(\frac{2}{B}\right)^k\right)^{|I_1|+|\widetilde{I}_2|}\frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}], \quad (4.A.7)$$

which gives the right bound.

If $I_1 = \emptyset$ (or analogously if $\widetilde{I}_2 = \emptyset$), one has $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{\widetilde{I}_2}]$ and

$$\mathbf{E}_{n+1}\left[\delta_I \exp\left(uS_{n+1}\right)\right] = \frac{1}{B}\mathbf{E}_n\left[\exp\left(uS_n\right)\right]\mathbf{E}_n\left[\delta_{\widetilde{I}_2}\exp\left(uS_n\right)\right] \leqslant e^{c_0 u (2/B)^{n+1}}\exp\left(c_0 u \sum_{k=0}^{n}\left(\frac{2}{B}\right)^k\right)^{|\widetilde{I}_2|}\frac{1}{B}\mathbf{E}_n[\delta_{\widetilde{I}_2}], \quad (4.A.8)$$

where the first part is dealt with Proposition 4.A.1, and the second one with the induction hypothesis. □

The following two results allow us to control the strength of the correlations in the annealed system.

Theorem 4.A.3. *Let $B \in (1,2)$. Let $(b_n)_{n \geqslant 0}$ be a sequence that goes to 0 as n goes to infinity. There exists a constant $c_b > 0$ such that for all $n \geqslant 0$ and every $0 \leqslant u \leqslant b_n(\frac{B^2}{4} \wedge \frac{1}{2})^n$ one has*

$$\mathbf{E}_n\left[\exp\left(u(S_n)^2\right)\right] \leqslant \exp\left(c_b u \left(\frac{4}{B^2}\right)^n\right). \quad (4.A.9)$$

Corollary 4.A.4. *Let $B \in (1,2)$, $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ and note $\varphi := (2\kappa) \vee \frac{4\kappa}{B^2} < 1$. Then for every $A > 0$ there exists a constant $c_A > 0$ such that for any $n \geqslant 0$, any $u \in [0, A]$ and any subset I of $\{1, \dots, 2^n\}$, one has*

$$\mathbf{E}_n\left[\delta_I \exp\left(u\kappa^n (S_n)^2\right)\right] \leqslant \left(e^{c_A u \varphi^n}\right)^{n|I|+1}\mathbf{E}_n[\delta_I]. \quad (4.A.10)$$

Note that if $I = \emptyset$, the statement is implied by Theorem 4.A.3.

Proof of Theorem 4.A.3 The proof relies on Proposition 4.A.1. Let us consider $u \leqslant b_n(\frac{B^2}{4} \wedge \frac{1}{2})^n$. One writes

$$J := \mathbf{E}_n\left[\exp\left(\frac{1}{2}u(S_n)^2\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-z^2/2}\mathbf{E}_n\left[\exp\left(z\sqrt{u}S_n\right)\right] \mathrm{d}z. \quad (4.A.11)$$

One sets $\Delta := \frac{a}{\sqrt{u}}\left(\frac{B}{2}\right)^n$, where a is a constant that will be chosen small. Note that thanks to our choice of u, one has $\Delta \geqslant a\, b_n^{-1/2}$ that goes to infinity as n grows to infinity. Then one decomposes the integral J according to the values of z, and writes $J = J_1 + J_2$, where

$$\begin{aligned} J_1 &:= \frac{1}{\sqrt{2\pi}} \int_{z \leqslant \Delta} e^{-z^2/2} \mathbf{E}_n \left[\exp\left(z\sqrt{u} S_n \right) \right] \mathrm{d}z \\ J_2 &:= \frac{1}{\sqrt{2\pi}} \int_{z \geqslant \Delta} e^{-z^2/2} \mathbf{E}_n \left[\exp\left(z\sqrt{u} S_n \right) \right] \mathrm{d}z. \end{aligned} \tag{4.A.12}$$

To bound J_1, one chooses $a \leqslant a_0$ with a_0 as in Proposition 4.A.1, such that for the values of z considered one has $z\sqrt{u} \leqslant a_0(B/2)^n$ and then one applies Proposition 4.A.1-(1) to get

$$J_1 \leqslant \frac{1}{\sqrt{2\pi}} \int_{z \leqslant \Delta} e^{-z^2/2} \exp\left(cz\sqrt{u}(2/B)^n \right) \mathrm{d}z \leqslant \exp\left(\frac{c^2}{2} u \left(4/B^2 \right)^n \right). \tag{4.A.13}$$

We deal with the term J_2, decomposing again according to the values of z. Let us first introduce some notations: we define the sequence $(\Delta_k)_{k \geqslant 0}$ by the iteration

$$\begin{cases} \Delta_0 = \Delta \\ \Delta_{k+1} = \Delta (\Delta_k)^{2/\nu} \, (> \Delta_k > 1), \end{cases} \tag{4.A.14}$$

and define also $m = \inf\{k,\, \Delta_k \geqslant A\sqrt{u}2^n\}$, for some A chosen large enough later. We point out that m is finite. Indeed for a fixed large n, if $\nu \leqslant 2$, then $\Delta_k \geqslant \Delta^{k+1}$ and goes to infinity as k goes to infinity. Otherwise, if $\nu > 2$, Δ_k goes to $\Delta^{\nu/(\nu-2)}$ as k goes to infinity. Then, we just need to check that $\Delta^{\nu/(\nu-2)} \geqslant A\sqrt{u}2^n$ if n is large. Using the value of $\nu = \log 2 / \log(2/B)$ one has $2^{1/\nu} = 2/B$, so that $\Delta^\nu = a^\nu u^{-\nu/2} 2^{-n}$. Then

$$\frac{\Delta^\nu}{(\sqrt{u}2^n)^{\nu-2}} = a^\nu \frac{u^{-\nu/2} 2^{-n}}{u^{\nu/2-1} 2^{n(\nu-2)}} = a^\nu \left(u 2^n \right)^{-(\nu-1)} \geqslant a^\nu b_n^{1-\nu}, \tag{4.A.15}$$

where we used that $u2^n \leqslant b_n$. As $\nu > 1$, it remains only to take n large.

One decomposes J_2 as follows:

$$J_2 = \sum_{k=0}^{m-1} \frac{1}{\sqrt{2\pi}} \int_{\Delta_k}^{\Delta_{k+1}} e^{-z^2/2} \mathbf{E}_n \left[\exp\left(z\sqrt{u} S_n \right) \right] \mathrm{d}z + \frac{1}{\sqrt{2\pi}} \int_{\Delta_m}^{+\infty} e^{-z^2/2} \mathbf{E}_n \left[\exp\left(z\sqrt{u} S_n \right) \right] \mathrm{d}z. \tag{4.A.16}$$

Each term of the sum in (4.A.16) can be dealt with Proposition 4.A.1-(2). One gets

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{\Delta_k}^{\Delta_{k+1}} e^{-z^2/2} \mathbf{E}_n \left[\exp\left(z\sqrt{u} S_n \right) \right] \mathrm{d}z &\leqslant \mathbf{E}_n \left[\exp\left(\Delta_{k+1} \sqrt{u} S_n \right) \right] P\left(\mathcal{N} \geqslant \Delta_k \right) \\ &\leqslant c_1 \exp\left(c_2 2^n u^{\nu/2} (\Delta_{k+1})^\nu \right) \exp\left(-c(\Delta_k)^2 \right), \end{aligned} \tag{4.A.17}$$

where $\mathcal{N}$ stands for a standard centered Gaussian. Now recall the definition of Δ_k and Δ, that gives $(\Delta_{k+1})^\nu = \Delta^\nu (\Delta_k)^2 = a^\nu u^{-\nu/2} 2^{-n} (\Delta_k)^2$, so that one can bound the term in (4.A.17) by

$$c_1 \exp\left((c_2 a^\nu - c)(\Delta_k)^2 \right) \leqslant c_1 \exp\left(-c(\Delta_k)^2/2 \right), \tag{4.A.18}$$

where the inequality is valid provided one has chosen a sufficiently small.

Let us now deal with the last term in (4.A.16), trivially bounding $S_n \leqslant 2^n$:

$$\frac{1}{\sqrt{2\pi}}\int_{\Delta_m}^{\infty} e^{-z^2/2}\mathbf{E}_n\left[\exp\left(z\sqrt{u}S_n\right)\right]\,\mathrm{d}z \leqslant \frac{1}{\sqrt{2\pi}}\int_{\Delta_m}^{\infty} e^{-z^2/2}e^{z\sqrt{u}2^n}\,\mathrm{d}z$$
$$= e^{u4^n/2}P\left(\mathcal{N}\geqslant \Delta_m-\sqrt{u}2^n\right) \leqslant e^{A^{-2}(\Delta_m)^2}e^{-c(1-A^{-1})^2(\Delta_m)^2} \leqslant e^{-c(\Delta_m)^2/2}, \quad (4.A.19)$$

where we used that $\sqrt{u}2^n \leqslant A^{-1}\Delta_m$, and supposed that A was chosen large enough for the last inequality.

We finally get that for n large one has

$$J_2 \leqslant c_1\sum_{k=0}^{m} e^{-c(\Delta_k)^2/2} \leqslant \begin{cases} Ce^{-c\Delta^2/2} & \text{if } \nu\leqslant 2,\\ Cme^{-c\Delta^2/2} & \text{if } \nu>2,\end{cases} \quad (4.A.20)$$

where in the case $\nu \leqslant 2$ we used that $\Delta_k \geqslant \Delta^{k+1}$. Note that for $\nu > 2$, using (4.A.15), one also can bound m from above as follows: since $\Delta_k = \Delta^{\frac{1-(2/\nu)^{k+1}}{1-2/\nu}}$,

$$\frac{\Delta_k}{\sqrt{u}2^n} = \frac{\Delta^{\nu/(\nu-2)}}{\sqrt{u}2^n}\Delta^{-\frac{\nu}{\nu-2}(2/\nu)^{k+1}} \geqslant a^{\nu/(\nu-2)}b_n^{(1-\nu)/(\nu-2)}\Delta^{-c'(2/\nu)^k}. \quad (4.A.21)$$

If one takes $k \geqslant -\log\log\Delta/\log(2/\nu)$ one gets that $\Delta_k \geqslant a^{\nu/(\nu-2)}b_n^{(1-\nu)/(\nu-2)}e^{-c'}\sqrt{u}2^n$. If n is large enough this implies that $m \leqslant \mathit{const}\times\log\log\Delta$.

Then one easily gets that $J_2 = o\left(\Delta^{-2}\right)$, with $\Delta^{-2} = O\left(u(4/B^2)^n\right)$, so that combining with the bound on J_1 one has

$$J \leqslant \exp\left(\frac{c_0^2}{2}u\left(4/B^2\right)^n\right) + o\left(u\left(4/B^2\right)^n\right). \quad (4.A.22)$$

□

Proof of Corollary 4.A.4 We proceed by induction. Fix $A > 0$ and $u \leqslant A$, and take the constant c_A obtained in Theorem 4.A.3 for the sequence $b_n = A\left(\frac{4\kappa}{B^2}\wedge 2\kappa\right)^n$. The case $n = 0$ is trivial. Suppose now that the assumption is true for some n, and take I a subset of $\{1,\ldots,2^{n+1}\}$.
Suppose $I \neq \emptyset$ (otherwise one already has the result from Theorem 4.A.3). As in the proof of Lemma 4.4.3, one decomposes I into its "left" and "right" part and $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$. Using that $(S_{n+1})^2 \leqslant 2(S_n^{(1)})^2 + 2(S_n^{(2)})^2$ one gets

$$\mathbf{E}_{n+1}\left[\delta_I\exp\left(u\kappa^{n+1}(S_{n+1})^2\right)\right]$$
$$\leqslant \frac{1}{B}\mathbf{E}_n\left[\delta_{I_1}\exp\left((2\kappa)u\kappa^n(S_n)^2\right)\right]\mathbf{E}_n\left[\delta_{\widetilde{I}_2}\exp\left((2\kappa)u\kappa^n(S_n)^2\right)\right]$$
$$\leqslant \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]\left(e^{c_A u(2\kappa)\varphi^n}\right)^{n|I_1|+n|\widetilde{I}_2|+2} \leqslant \mathbf{E}_{n+1}[\delta_I]\left(e^{c_A u2\kappa\varphi^n}\right)^{(n+1)|I|+1}, \quad (4.A.23)$$

where for the second inequality we used the recursion assumption and for the last one the assumption $|I| \geqslant 1$. Now one just uses that $2\kappa \leqslant \varphi$ to conclude. □

From Corollary 4.A.4 one can deduce the following Proposition, useful to control the variance of the partition function (see Section 4.5). Define as in (4.5.5) $D_n :=$

$\sum_{i,j=1}^{2^n} \kappa_{ij}\delta_i\delta'_j$, where δ and δ' are the populations at generation n of two independent GW trees.

Proposition 4.A.5. *Let $B \in (1,2)$, $\kappa < \frac{1}{2} \wedge \frac{B^2}{4}$ and set $\varphi = (2\kappa) \wedge (4\kappa/B^2) < 1$.*

- *If $B > B_c$, then for every $\Phi \in \left(\frac{2}{B^2} \vee \varphi, 1\right)$ there exist some $u_0 > 0$ and some constant $c > 0$, such that for every $n \in \mathbb{N}$, $u \in [0, u_0]$ one has*

$$\mathbf{E}_n^{\otimes 2}\left[\exp\left(uD_n\right)\right] \leqslant 1 + cu\Phi^n. \tag{4.A.24}$$

- *If $B < B_c$ there exist some $a_1 > 0$ and some constant $c > 0$, such that for every $n \in \mathbb{N}$, if $u \leqslant a_1\left(\frac{B^2}{2}\right)^n$ one has*

$$\mathbf{E}_n^{\otimes 2}\left[\exp\left(uD_n\right)\right] \leqslant 1 + cu\left(\frac{2}{B^2}\right)^n. \tag{4.A.25}$$

- *If $B = B_c$, there exists some u_0 such that if $u \leqslant u_0$ then for all $n \leqslant \frac{1}{2}u^{-1/3}$ one has*

$$\mathbf{E}_n^{\otimes 2}\left[\exp\left(uD_n\right)\right] \leqslant 1 + 2u^{1/3}. \tag{4.A.26}$$

Proof One has

$$\begin{aligned} D_{n+1} &= D_n^{(1)} + D_n^{(2)} + \kappa_{n+1}\left(S_n^{(1)}{S'}_n^{(2)} + {S'}_n^{(1)}{S'}_n^{(2)}\right) \\ &\leqslant D_n^{(1)} + D_n^{(2)} + \frac{\kappa_n}{2}\left(\left(S_n^{(1)}\right)^2 + \left({S'}_n^{(2)}\right)^2 + \left(S_n^{(2)}\right)^2 + \left({S'}_n^{(1)}\right)^2\right). \end{aligned} \tag{4.A.27}$$

Since clearly D_{n+1} vanishes when either of the two GW trees is empty, one has for every $v \in [0,1]$

$$\begin{aligned} \mathbf{E}_{n+1}^{\otimes 2}\left[e^{vD_{n+1}}\right] &\leqslant \frac{1}{B^2}\mathbf{E}_n^{\otimes 2}\left[e^{vD_n}\exp\left(\frac{v}{2}\kappa_n\left((S_n)^2 + (S'_n)^2\right)\right)\right]^2 + \frac{B^2-1}{B^2} \\ &\leqslant \frac{1}{B^2}e^{c_0 v\varphi^n}\mathbf{E}_n^{\otimes 2}\left[\exp\left(ve^{c_0 v(\varphi')^n}D_n\right)\right]^2 + \frac{B^2-1}{B^2}, \end{aligned} \tag{4.A.28}$$

where in the second inequality we expanded e^{vD_n} as in Remark 4.4.2 and used Corollary 4.A.4 to get the constant $c_0 > 0$ for $\varphi := (2\kappa) \vee \frac{4\kappa}{B^2}$ and some $\varphi' \in (\varphi, 1)$. Then we set $v_0 \leqslant 1$ and for $n \geqslant 0$ define $v_{n+1} := v_n e^{-c_0 v_n(\varphi')^n} \leqslant v_0$. Define $X_n := \mathbf{E}_n^{\otimes 2}\left[\exp\left(v_nD_n\right)\right] - 1$, so that using the previous inequality one has

$$X_{n+1} \leqslant \frac{1}{B^2}e^{c_0 v_n\varphi^n}(X_n+1)^2 - \frac{1}{B^2} \leqslant \frac{2e^{c_0v_0\varphi^n}}{B^2}X_n\left(1 + \frac{X_n}{2}\right) + cv_0\varphi^n. \tag{4.A.29}$$

We consider the different cases $B < B_c$, $B = B_c$ and $B > B_c$ separately, but each time we estimate from above $\mathbf{E}_n^{\otimes 2}\left[e^{v_nD_n}\right]$. One then easily deduces Proposition 4.A.5 using that there exists a constant c_1 such that $v_n \geqslant c_1v_0$, and then $\mathbf{E}_n^{\otimes 2}\left[e^{c_1v_0D_n}\right] \leqslant 1 + X_n$. One concludes taking $u := c_1v_0$.

In the sequel we actually study the iteration

$$\widehat{X}_{n+1} = \frac{2e^{w_n}}{B^2}\widehat{X}_n\left(1 + \frac{\widehat{X}_n}{2}\right) + (c/c_0)w_n, \quad \widehat{X}_0 = X_0 \tag{4.A.30}$$

where we defined $w_n := c_0 v_0 \varphi^n$. Clearly, $X_n \leqslant \widehat{X}_n$ for every n.

- Take $B > B_c := \sqrt{2}$. Let us fix some $\Phi \in \left(\frac{2}{B^2} \vee \varphi, 1\right)$. One has that $X_0 \leqslant C_0 v_0$ and one shows easily by iteration, using (4.A.30) and the definition of w_n, that $\widehat{X}_n \leqslant C_n \Phi^n v_0$, with $(C_n)_{n\in\mathbb{N}}$ an increasing sequence satisfying

$$C_{n+1} = C_n e^{w_n} \left(1 + \frac{1}{2} C_n v_0 \Phi^n\right) + c' \varphi^n \Phi^{-(n+1)} \tag{4.A.31}$$

(use that $\Phi > (2/B^2)$). Then we show that provided that v_0 has been chosen small enough, $(C_n)_{n\in\mathbb{N}}$ is a bounded sequence. Indeed, using that $C_n \geqslant C_0$ one has

$$\begin{aligned} C_{n+1} &\leqslant C_n e^{w_n} \left(1 + \frac{1}{2} C_n v_0 \Phi^n + c' \Phi^{-1} C_n^{-1} (\varphi/\Phi)^n\right) \\ &\leqslant C_n e^{w_n} \exp\left(\frac{1}{2} C_n v_0 \Phi^n\right) \exp\left(c''(\varphi/\Phi)^n\right) \leqslant A \exp\left(\frac{1}{2} v_0 \sum_{k=0}^{n} C_k \Phi^k\right). \end{aligned} \tag{4.A.32}$$

where we noted $A := \prod_{n=0}^{\infty} e^{w_n} e^{c''(\varphi/\Phi)^n}$, with $A < +\infty$ thanks to the definition of w_n and using that $\Phi > \varphi$. It is then not difficult to see that if v_0 is chosen small enough, more precisely such that $A \exp\left(v_0 C_0 \sum_{k=0}^{n} \Phi^k\right) \leqslant 2C_0$, then C_n remains smaller than $2C_0$ for every $n \in \mathbb{N}$. From this, one gets that $X_n \leqslant 2C_0 \Phi^n v_0$ for every n.

- Take $B < B_c$. The idea is that if X_0 is small enough, (4.A.30) can be approximated by the iteration $X_{n+1} \leqslant \frac{2}{B^2} X_n$ while X_n remains small. For any fixed $n \geqslant 0$, one chooses $v_0 = a\left(B^2/2\right)^n$ with a small (chosen in a moment), and one has $X_0 \leqslant C_0 a \left(\frac{B^2}{2}\right)^n$. Then one shows by iteration that

$$\widehat{X}_k \leqslant C_k a \left(B^2/2\right)^{n-k} \tag{4.A.33}$$

for some increasing sequence $(C_k)_{k\in\mathbb{N}}$ verifying

$$C_{k+1} = e^{w_k} C_k \left(1 + \frac{C_k}{2} a \left(\frac{B^2}{2}\right)^{n-k}\right) + a^{-1} \left(\frac{B^2}{2}\right)^{k+1-n} w_k. \tag{4.A.34}$$

One then shows with the same method as in the case $B > B_c$ that C_n is bounded by some constant C uniformly in n, provided that a had been chosen small enough. Thus taking $k = n$ one has $X_n \leqslant ca = cv_0 \left(2/B^2\right)^n$.

- Take $B = B_c = \sqrt{2}$. The iteration (4.A.30) gives

$$X_{n+1} \leqslant e^{w_n} X_n \left(1 + \frac{X_n}{2}\right) + (c/c_0) w_n, \tag{4.A.35}$$

and we recall that $w_n = c_0 v_0 \varphi^n$. Take $v_0 = \varepsilon^3$, so that $X_0 \leqslant \varepsilon$ for ε small. We now show that if $\varepsilon \leqslant \varepsilon_0$ with ε_0 chosen small enough, one has for all $n \leqslant \frac{1}{2}\varepsilon^{-1}$ that $X_n \leqslant \varepsilon\,(1 + n\varepsilon)$. We prove this by induction. For $n = 0$ this is just because one chose $X_0 \leqslant \varepsilon$. If $X_n \leqslant \varepsilon(1+n\varepsilon)$ and $n\varepsilon \leqslant 1/2$, one has (note that $w_n \leqslant c_0\varepsilon^3$ for all n)

$$\begin{aligned} X_{n+1} \leqslant e^{c_0\varepsilon^3}\varepsilon(1+n\varepsilon)\left(1+\frac{1}{2}\varepsilon(1+n\varepsilon)\right)+c\varepsilon^3, \\ \leqslant \varepsilon\left[(1+c_0'\varepsilon^3)(1+n\varepsilon)(1+3\varepsilon/4)+c\varepsilon^2\right] \\ \leqslant \varepsilon\left[1+\varepsilon\left(n+3/4+c_0'\varepsilon^2+c\varepsilon\right)\right] \leqslant \varepsilon(1+(n+1)\varepsilon), \end{aligned} \tag{4.A.36}$$

provided that $\varepsilon \leqslant \varepsilon_0$ with ε_0 small enough. This concludes the induction step. Thus one has that for all $n \leqslant \frac{1}{2}\varepsilon^{-1}$, $X_n \leqslant 2\varepsilon$, with $\varepsilon = v_0^{1/3}$. □

4.B. Hierarchically correlated Gaussian vectors

Lemma 4.B.1. *Let $m(\cdot)$ be a function from $\mathbb{N}$ to $\mathbb{R}$ and for $n \in \mathbb{N}$ let Let $M := M^{(n)} = (M_{ij})_{1 \leqslant i,j \leqslant 2^n}$ be the $2^n \times 2^n$ matrix with entries $M_{ij} := m(d(i,j))$. Then, the eigenvectors of such a matrix do not depend on the function $m(\cdot)$, and the eigenvalues are*

$$\lambda_0 = m(0) + \sum_{k=1}^{n} 2^{k-1} m(k) \quad \text{withmultiplicity} 1, \tag{4.B.1}$$

and for $1 \leqslant p \leqslant n$

$$\lambda_p = m(0) + \sum_{k=1}^{n-p} 2^{k-1} m(k) - 2^{n-p} m(n+1-p) \quad \text{withmultiplicity} 2^{p-1}. \tag{4.B.2}$$

This comes directly from the fact that

$$M^{(n)} = \begin{pmatrix} & M^{(n-1)} & & \begin{matrix} m(n) & \cdots & m(n) \\ \vdots & & \vdots \\ m(n) & \cdots & m(n) \end{matrix} \\ \begin{matrix} m(n) & \cdots & m(n) \\ \vdots & & \vdots \\ m(n) & \cdots & m(n) \end{matrix} & & & M^{(n-1)} \end{pmatrix}, \tag{4.B.3}$$

where each block is of size 2^{n-1}. One computes the eigenvalues: the eigenvector $(1,\dots,1)$ gives λ_0, the eigenvector $(1,\dots,1,-1,\dots,-1)$ gives λ_1. Then the eigenvectors $(X,0)$ and $(0,X)$ with $X \neq (1,\dots,1)$ being an eigenvector of $M^{(n-1)}$ give all the others eigenvalues, which are the eigenvalue associated to X with $M^{(n-1)}$, but with multiplicity multiplied by 2.

Remark 4.B.2. Lemma 4.B.1 shows that the spectral radius of $M^{(n)}$ is upper bounded by $\sum_{p=0}^{\infty} 2^p |m(p)|$. Also, two matrices with entries depending only on the distances $d(i,j)$ can be co-diagonalized, as the eigenvectors do not depend on the values of the entries, and one can describe the diagonalizing orthogonal matrix Ω

$$\Omega = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & 1 & -\sqrt{2} & 0 & \\ 1 & -1 & 0 & \sqrt{2} & \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & -1 & 0 & -\sqrt{2} & \end{pmatrix} \tag{4.B.4}$$

such that $\Omega^t K \Omega = Diag\,(\lambda_0, \lambda_1, \lambda_2, \lambda_2, \ldots)$ with λ_i given in Lemma 4.B.1.

Let $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ be the centered Gaussian family with correlation structure $\mathbb{E}[\omega_i\omega_j] = \kappa_{d(i,j)}$. The following Proposition gives the dependence on κ_n of a smooth function of $\omega_1, \ldots, \omega_{2^n}$:

Proposition 4.B.3. *If $f : \mathbb{R}^{2^n} \mapsto \mathbb{R}$ is twice differentiable and grows at most polynomially at infinity, one has*

$$\frac{\partial}{\partial \kappa_n}\mathbb{E}\left[f(\omega_1, \ldots, \omega_{2^n})\right] = \sum_{i=1}^{2^{n-1}} \sum_{j=2^{n-1}+1}^{2^n} \mathbb{E}\left[\frac{\partial^2 f}{\partial\omega_i \partial\omega_j}(\omega)\right]. \tag{4.B.5}$$

Proof Thanks to Remark 4.B.2, one has

$$\mathbb{E}\left[f(\omega_1, \ldots, \omega_{2^n})\right] = \widetilde{\mathbb{E}}\left[f(\Omega\omega)\right], \tag{4.B.6}$$

with Ω defined in (4.B.4), and where $\widetilde{\mathbb{P}}$ stands for the law of a centered Gaussian vector of covariance matrix $\Delta := Diag\,(\lambda_0, \lambda_1, \lambda_2, \lambda_2, \ldots)$. The eigenvalues λ_i and their multiplicity are given in Lemma 4.B.1. Then, as only $\lambda_0 = \kappa_0 + \sum_{k=1}^{n} 2^{k-1}\kappa_k$ and $\lambda_1 = \kappa_0 + \sum_{k=1}^{n-1} 2^{k-1}\kappa_k - 2^{n-1}\kappa_n$ depend on κ_n one gets

$$\frac{\partial}{\partial \kappa_n}\mathbb{E}\left[f(\omega)\right] = 2^{n-1}\frac{\partial}{\partial\lambda_0}\widetilde{\mathbb{E}}\left[f(\Omega\omega)\right] - 2^{n-1}\frac{\partial}{\partial\lambda_1}\widetilde{\mathbb{E}}\left[f(\Omega\omega)\right]. \tag{4.B.7}$$

Then one uses the classical Gaussian fact that if ω is a centered Gaussian variable of variance σ^2 and g is a differentiable function which grows at most polynomially at infinity,

$$\frac{\partial}{\partial\sigma^2}\mathbb{E}\left[g(\omega)\right] = \frac{1}{2}\mathbb{E}\left[\frac{\partial^2 g}{\partial\omega^2}(\omega)\right]. \tag{4.B.8}$$

Plugging this result in (4.B.7) one gets

$$\frac{1}{2^{n-1}}\frac{\partial}{\partial\kappa_n}\mathbb{E}[f(\omega_1,\dots,\omega_{2^n})]$$
$$= \frac{1}{2}\sum_{i,j=1}^{2^n}\Omega_{i1}\Omega_{j1}\widetilde{\mathbb{E}}\left[\frac{\partial^2 f}{\partial x_i\partial x_j}\bigg|_{x=\Omega\omega}\right] - \frac{1}{2}\sum_{i,j=1}^{2^n}\Omega_{i2}\Omega_{j2}\widetilde{\mathbb{E}}\left[\frac{\partial^2 f}{\partial x_i\partial x_j}\bigg|_{x=\Omega\omega}\right]$$
$$= \frac{1}{2^n}\sum_{\substack{i,j=1\\ d(i,j)=n}}^{2^n}\mathbb{E}\left[\frac{\partial^2 f}{\partial\omega_i\partial\omega_j}(\omega)\right], \quad (4.B.9)$$

where in the second equality we used the values of Ω_{k1} and Ω_{k2}. □

Remark 4.B.4. With the same type of computations, since Ω is explicit, one can also compute the derivative with respect to κ_p for $p \leqslant n$, and after some computations, one gets

$$\frac{\partial}{\partial\kappa_p}\mathbb{E}\left[f(\omega_1,\dots,\omega_{2^n})\right] = \frac{1}{2}\sum_{\substack{i,j=1\\ d(i,j)=p}}^{2^n}\mathbb{E}\left[\frac{\partial^2 f}{\partial\omega_i\partial\omega_j}(\omega)\right]. \quad (4.B.10)$$

CHAPTER 5

Pinning model in long-range correlated Gaussian environment

5.1. Introduction

The problem we investigate in this Chapter is analogous to the one of Chapter 4, that is the study of the (non-hierarchical) pinning model in random correlated environment of Gaussian type. In the case where the disorder is *i.i.d.*, the question of relevance/irrelevance of disorder is predicted by the *Harris criterion* (recall Section 1.4.2). As already discussed in Section 1.4.2, it has also been mathematically settled in the past few years in many articles [Ale08, DGLT09, GLT11, Ton08b]: disorder is relevant if and only if $\nu^{\mathrm{pur}} \leqslant 2$ (ν^{pur} is the critical exponent of the homogeneous model). We raise here the question of the influence of correlations on this criterion. Following the reasoning of Weinrib and Halperin [WH83] one could argue that, introducing correlations with power-law decay $r^{-\zeta}$ (where $\zeta > 0$, and r the distance between the points), disorder should be relevant if $\nu^{\mathrm{pur}} < 2/\zeta$, and irrelevant if $\nu^{\mathrm{pur}} > 2/\zeta$. Therefore, the Harris prediction for disorder relevance/irrelevance should be modified only if $\zeta < 1$.

In this Chapter, we give partial results that confirm the Weinrib-Halperin criterion for $\zeta > 1$, together with various estimates on the disordered annealed systems. We also show that the case $\zeta < 1$ is somehow special, and that the behavior of the system does not fit the criterion in that case.

5.1.1. Reminder of the disordered pinning model. In this Chapter, we consider the usual pinning model defined in Section 1.4. We recall briefly the framework we are dealing with.

Consider $\tau := \{\tau_n\}_{n \geqslant 0}$ a recurrent renewal process as defined in Section 1.1.2, with law denoted by $\mathbf{P}$. The set $\tau = \{\tau_n\}_{n \geqslant 0}$ represents the set of contact points between the polymer and the defect line (cf. Introduction). We assume that the inter-arrival distribution $\mathrm{K}(\cdot)$ verifies

$$\mathrm{K}(n) \overset{n\to\infty}{=} (1+o(1))\frac{c_K}{n^{1+\alpha}}, \qquad \text{for } n \in \mathbb{N}, \tag{5.1.1}$$

for some $c_K > 0$ and $\alpha > 0$. The fact that the renewal is recurrent simply means that $\mathrm{K}(\infty) = \mathbf{P}(\tau_1 = +\infty) = 0$. We also assume for simplicity that $\mathrm{K}(n) > 0$ for all $n \in \mathbb{N}$.

Given a sequence $\omega = (\omega_n)_{n\in\mathbb{N}}$ of real numbers (the environment), and parameters $h \in \mathbb{R}$ and $\beta \geqslant 0$, we define the polymer measure $\mathbf{P}^{\omega,\beta}_{N,h}$, $N \in \mathbb{N}$, as follows

$$\frac{\mathrm{d}\mathbf{P}^{\omega,\beta}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z^{\omega,\beta}_{N,h}} \exp\left(\sum_{n=1}^{N}(h+\beta\omega_n)\delta_n\right)\delta_N, \tag{5.1.2}$$

where we noted $\delta_n := \mathbf{1}_{\{n\in\tau\}}$, and where $Z^{\omega,\beta}_{N,h} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(h+\beta\omega_n)\delta_n\right)\delta_N\right]$ is the *partition function* of the system.

In what follows, we take ω a random ergodic sequence, with law denoted by $\mathbb{P}$ (more assumptions on the law $\mathbb{P}$ are made in the next section). The ergodicity of the environment allows us to collect the results concerning the quenched free energy and the quenched critical point:

Proposition 5.1.1. *The limit*

$$\mathtt{F}(\beta,h) := \lim_{N\to\infty}\frac{1}{N}\log Z^{\omega,\beta}_{N,h} = \sup_{N\in\mathbb{N}}\frac{1}{N}\mathbb{E}\log Z^{\omega,\beta}_{N,h}, \tag{5.1.3}$$

exists and is constant $\mathbb{P}$ a.s. It is called the quenched free energy. *There exist a quenched critical point $h_c^{\mathrm{que}}(\beta) \in \mathbb{R}$, such that $\mathtt{F}(\beta,h) > 0$ if and only if $h > h_c^{\mathrm{que}}(\beta)$.*

We recall that the critical point $h_c^{\mathrm{que}}(\beta)$ marks the transition between the delocalized phase (for $h < h_c^{\mathrm{que}}$) and the localized phase (for $h > h_c^{\mathrm{que}}$), as noticed in Section 1.4.1.

We study (as usual) the behavior of the disordered system close to the critical point, which gives many informations on the trajectories under the polymer measure $\mathbf{P}^{\omega,\beta}_{N,h}$ close to criticality. We compare the disordered system with the pure one, *i.e.* with no disorder ($\beta = 0$), to know how the presence of inhomogeneities modifies the localization phase transition. We recall here the main result on the pure system, whose partition function is denoted by $Z_{N,h}$ see Section 1.1.2 (recall Theorem 1.1.6, Propositions 1.1.4 and 1.1.10).

Proposition 5.1.2 ([Gia07], Chapter 2)**.** *The pure free energy is defined by $\mathtt{F}(h) := \lim_{N\to\infty}\frac{1}{N}\log Z_{N,h}$, and exhibits a phase transition at the critical point $h_c = 0$ (recall we have a recurrent renewal τ). One has the following asymptotics of $\mathtt{F}(h)$ around $h = 0_+$:*

$$\mathtt{F}(h) = \begin{cases} \frac{\alpha}{\Gamma(1-\alpha)c_K}\, h^{1/\alpha}\,(1+o(1)) \text{ if } \alpha < 1, \\ (\mathbf{E}[\tau_1])^{-1}\, h\,(1+o(1)) \text{ if } \alpha > 1. \end{cases} \tag{5.1.4}$$

The pure critical exponent is therefore $\nu^{\mathrm{pur}} := 1 \vee 1/\alpha$, and it encodes the critical behavior of the homogeneous model. We left aside the case $\alpha = 1$ which brings some technicalities, and in the sequel we do not treat this case, since the computations require more care (even if they use exactly the same techniques).

5.1.2. Assumptions on the environment. Up to now, the pinning model defined above was studied only in an *i.i.d.* environment, or in the case of a Gaussian environment with only finite-range correlations [Poi11, Poi12]. In this latter case, it is shown that the features of the system are the same as with an *i.i.d.* environment, and in particular, the criterion for relevance/irrelevance of disorder is the same as in the *i.i.d.* case, according to the Weinrib-Halperin prediction. We consider here the case of a Gaussian environment with long-range correlations.

We define the sequence $\omega = (\omega_n)_{n\in\mathbb{N}}$ to be a Gaussian stationary process with zero mean and unitary variance, whose law is denoted by $\mathbb{P}$, and with correlation function $(\rho_n)_{n\geqslant 0}$. We denote the covariance matrix $\Upsilon = (\Upsilon_{ij})_{i,j\in\mathbb{N}}$ (with the notation $\Upsilon_{ij} := \mathbb{E}[\omega_i\omega_j]$), which is symmetric definite positive (so that ω is well-defined). Note that we have $\Upsilon_{ij} = \rho_{|i-j|}$ thanks to to stationarity, so that Υ is an infinite Toeplitz matrix. We also assume that $\lim_{n\to\infty} |\rho_n| = 0$, so that the sequence ω is ergodic (see [CFS82, Ch.14 §2, Th.2]).

The Weinrib-Halperin criterion suggests to consider a power-law decaying correlation function, $\rho_n = n^{-\zeta}$, and we recall Remark 1.4.3, which tells that this is a valid choice for a correlation function. The criterion for relevance/irrelevance of disorder, in view of the Weinrib-Halperin prediction, should be modified with respect to the *i.i.d.* case only if $\zeta < 1$. In what follows, we make two different assumptions on the disorder, in order to distinguish the cases $\zeta > 1$ and $\zeta < 1$ in a more general way.

Assumption 5.1.3 (Summable correlations)**.** *Correlations are said to be Summable if $\sum |\rho_k| < +\infty$, which corresponds to a power-law decay $\zeta > 1$ of the correlations. This means that Υ is a bounded operator, and we make the additional assumption that Υ^{-1} is also a bounded operator, so that the spectrum of Υ is contained in an interval $[\mathsf{a}, \mathsf{A}]$, with $0 < \mathsf{a} < \mathsf{A} < \infty$.*

Assumption 5.1.4 (Non-Summable correlations)**.** *Correlations are said to be Non-Summable if $\sum |\rho_k| = +\infty$. We make the additional assumption that $\rho_k \geqslant 0$ for all $k \geqslant 0$, and that there exists some $\zeta \in (0, 1)$ and a constant $c_0 > 0$ such that*

$$\rho_k \overset{k\to\infty}{\sim} c_0 k^{-\zeta}. \tag{5.1.5}$$

These assumptions are natural, and the additional conditions we make (Υ^{-1} is a bounded operator in the summable case, and correlations are non-negative and have power-law decay in the non-summable case) are essentially imposed for technical reasons. We often refer to the different assumptions directly in terms of the power-law decay $\zeta > 0$ of the correlations, for the clarity of the statements.

Remark 5.1.5. Chapter 4 is devoted to the study of the hierarchical version of this model, and we believe that all the results one gets in Chapter 4 should have an analogue in the non-hierarchical framework. The Remark 1.3.4 tells us that the choice of the hierarchical structure (4.2.3) for the correlations corresponds to a power law decay $\kappa_{ij} \sim |i-j|^{-\zeta}$, where $\zeta = \log(1/\kappa)/\log 2$. Moreover, as Remark 4.3.7 shows, there is a condition $\kappa < B^2/4$ (with the additional condition of having summable correlations $\kappa < 1/2$, *i.e.* $\zeta > 1$) that appears in the crucial Proposition 4.3.2 (to be compared with Proposition 5.3.1). This translates in a more readable way in $\nu_{\rm hier}^{\rm pur} > 2/\zeta$, where $\nu_{\rm hier}^{\rm pur} = \log 2/\log(2/B)$ is the pure critical exponent in the hierarchical case, this condition being more easily transposed in the non-hierarchical framework.

We therefore compare our model with the hierarchical one, and give more predictions on the behavior of the system, and on the influence of correlated disorder on its critical properties: see Figure 5.1, in comparison with Figure 4.1.

We now explain briefly how this Chapter is organized: in Section 5.2 we present our main results on the model and comment them, as well for the annealed system (Theorem 5.2.2) as for the disordered one (Theorems 5.2.3-5.2.5); in Section 5.3 we collect some crucial observations on the annealed model in the correlated case, and prove Theorem 5.2.2; in Section 5.4 we prove the results on the disordered system.

5.2. Main results

The presence of correlations has a strong influence on the analysis of the disordered system, in particular for the annealed model, and we are not able to transpose all the known results of the *i.i.d.* case, even when correlations are summable. Our goal is to identify, in the (α,ζ)-plane (α the parameter of the renewal process, ζ the exponent of the power-law decay of the correlations), the regions where disorder is relevant, and the regions where it is irrelevant.

5.2.1. Preliminary results on the annealed model. We first focus on the study of the *annealed* model, which is often, as shown before, the first step towards the understanding of the disordered model. The annealed partition function is given, thanks to a Gaussian computation, by

$$Z_{N,h}^{\rm a,\Upsilon} := \mathbb{E}[Z_{N,h}^{\omega,\beta}] = \mathbf{E}\left[e^{H_{N,h}^{\rm a,\Upsilon}}\,\delta_N\right], \quad \text{with} \quad H_{N,h}^{\rm a,\Upsilon} := (\beta^2/2+h)\sum_{n=1}^{N}\delta_n + \beta^2\sum_{n=1}^{N}\delta_n\sum_{k=1}^{N-n}\rho_k\delta_{n+k}. \tag{5.2.1}$$

We keep the superscript Υ in $Z_{N,h}^{\rm a,\Upsilon}$, that recalls that the correlation structure has a very strong influence on the annealed partition function. It avoids many confusions, since in the sequel, we compare two annealed systems with different correlation structures. However, if there is no ambiguity, we drop this superscript.

The first Remark one makes is that (5.2.1) is far from being the partition function of the standard homogeneous pinning model introduced in Section 1.1.2. This is why studying the pinning model in correlated random environment is so complicated: even annealing techniques, that give simple and non-trivial bounds in the case of an *i.i.d.* environment (where the annealed model is the standard homogeneous one), are not easy to apply.

The annealed model is actually interesting in itself, since it gives an example of a homogeneous pinning model (there is no disorder), in which the rewards correlate according to the position of the renewal points: the closer renewal points are, the bigger the reward they get is. We can also consider the annealed model as a "standard" homogeneous pinning model, and by standard we mean that a reward h is given to each contact point, but with an underlying correlated renewal process, that is with non-*i.i.d.* inter-arrivals.

We first show the existence of the annealed free energy, and of the annealed critical point, provided that correlations are summable (recall Assumption 5.1.3).

Proposition 5.2.1. *Under Assumption 5.1.3, the limit*

$$\mathtt{F}^{a,\Upsilon}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z^{\mathrm{a},\Upsilon}_{N,h} \tag{5.2.2}$$

exists, is non-negative and finite. There exists a critical point $h_c^{\mathrm{a},\Upsilon}(\beta) \in \mathbb{R}$ *(we give bounds on* $h_c^{\mathrm{a},\Upsilon}(\beta)$ *in Section 5.3.1), such that* $\mathtt{F}^{\mathrm{a},\Upsilon}(\beta,h) > 0$ *if and only if* $h > h_c^{\mathrm{a},\Upsilon}(\beta)$.

Proof The non-negativity, convexity and monotonicity of $\mathtt{F}^{\mathrm{a},\Upsilon}$ are classical results and note that finiteness is proved thanks to the comparison of the annealed model with the standard homogeneous one (see (5.3.1)), thanks to summability of the correlations. That is why we focus here only on the existence of the limit in (5.2.2), which uses a property of $\log Z^{\mathrm{a}}_{N,h}$ which is close to super-additivity (there would be super-additivity with positive correlations).

In view of the expression of $H^{\mathrm{a},\Upsilon}_{N,h}$ in (5.2.1), one has the following inequality, for all integers N, M

$$H^{\mathrm{a},\Upsilon}_{N+M,h} \geqslant H^{\mathrm{a},\Upsilon}_{N,h}+H^{\mathrm{a},\Upsilon}_{M,h}+\beta^2\sum_{i=1}^{N}\sum_{j=N+1}^{M} 0\wedge\rho_{|i-j|} \geqslant H^{\mathrm{a},\Upsilon}_{N,h}+H^{\mathrm{a},\Upsilon}_{M,h}-\beta^2\sum_{k=1}^{N+M} k|\rho_k|, \tag{5.2.3}$$

where we only used that there are at most k integers $(i,j) \in [0,N]\times[N+1,N+M]$ that are at distance k. We define $\mathsf{R}_p := \sum_{k=1}^p k|\rho_k|$, so that one has that $\log Z^{\mathrm{a}}_{N+M,h} \geqslant \log Z^{\mathrm{a}}_{N,h}+\log Z^{\mathrm{a}}_{M,h}-\beta^2\mathsf{R}_{N+M}$. Then, if we show that $\sum_{p\in\mathbb{N}}\frac{1}{p(p+1)}\mathsf{R}_p < +\infty$, we have from Hammersley's generalized super-additive Theorem, see [Ham62, Th.2], that the limit in (5.2.2) exists. Since we have

$$\frac{1}{p(p+1)}\mathsf{R}_p = \frac{1}{p}\mathsf{R}_p - \frac{1}{p+1}\mathsf{R}_{p+1} + |\rho_{p+1}|, \tag{5.2.4}$$

and that $\mathsf{R}_p/p \leqslant \sum_{k=1}^p |\rho_k|$, we are done thanks to the summability of the correlations. □

For simplicity, we often write h_c^{a} instead of $h_c^{\mathrm{a},\Upsilon}(\beta)$ (if there is no possible confusion), and also $h := h_c^{\mathrm{a}} + u$, so that we now study the critical behavior of the free energy for $u \searrow 0$. Then we can write

$$Z^{\mathrm{a}}_{n,h} = \mathbf{E}\left[e^{u\sum_{n=1}\delta_n} e^{H^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}}\right] = Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\mathbf{E}^{\mathrm{a}}_{n,h_c}\left[\exp\left(h\sum_{n=0}^{N}\delta_n\right)\right], \tag{5.2.5}$$

where the measure $\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ is the *annealed* polymer measure at the critical point h_c^{a}. We would like to be able to compare $\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ with the measure $\mathbf{P}$ (in the pure case), as we do in Chapter 4 (see Proposition 4.3.2) for the hierarchical version of this model. In the present case, since there is no iterative structure for the partition function, there are many technicalities that are harder to deal with, but we have results in this direction, such as Propositions 5.3.1-5.3.5.

The following Theorem states that if the correlations decay sufficiently fast, more precisely if $m_\Upsilon := \sum_{k\in\mathbb{N}} k|\rho_k| < \infty$, (that corresponds to a power-law decay $\zeta > 2$ of the correlations), the annealed free energy has the same critical exponent as the pure free energy.

Theorem 5.2.2. *We suppose that $m_\Upsilon < \infty$. Then there exist some $\beta_0 > 0$ and a constant $c_1 > 0$, such that for all $\beta \leqslant \beta_0$ one has*

$$\mathtt{F}(c_1^{-1}u) \leqslant \mathtt{F}^{\mathrm{a},\Upsilon}(\beta, h_c^{\mathrm{a},\Upsilon}(\beta)+u) \leqslant \mathtt{F}(c_1 u), \tag{5.2.6}$$

as long as $\mathtt{F}^{\mathrm{a},\Upsilon}(\beta, h_c^{\mathrm{a},\Upsilon}(\beta)+u) \leqslant 1$.

We prove this result in Section 5.3.3. Note that we do not need the exact value of $h_c^{\mathrm{a},\Upsilon}(\beta)$ to get the critical exponent for the annealed free energy, and we are in general not able to compute it, even if we are able to give some estimates in the Section 5.3.1.

It is difficult to go beyond the condition $m_\Upsilon < \infty$, since without it, the correlations spread easily from one block to another (see (5.3.14)-(5.3.13) in Section 5.3.2, that do not necessarily hold if $m_\Upsilon = \infty$). In Chapter 4, we study the hierarchical version of this model, and we are able in that case to get much more precise estimates thanks to its recursive structure (see Section 4.3). In view of Theorem 4.3.1 and Remark 5.1.5, we believe that Theorem 5.2.2 should hold if $\zeta > 2/\nu^{\mathrm{pur}} = 2(\alpha\wedge 1)$ (and of course with the additional assumption of summable correlations, $\zeta > 1$). Moreover, Section 4.3.1 in the hierarchical case also suggests that the annealed critical exponent should be larger than the pure one in the region $\zeta > 2(\alpha\wedge 1), \zeta > 1$. We refer to Figure 5.1 that collects results and predictions on the annealed and quenched models.

5.2.2. Influence of disorder in the case of summable correlations. We now turn to the analysis of the influence of disorder on the phase transition of the disordered system. According to the Weinrib-Halperin prediction, one should find that with summable correlations, *i.e.* if $\zeta > 1$, the criterion for disorder relevance/irrelevance should not be modified with respect to the *i.i.d.* case. We now give some results confirming this criterion, and we prove them in Section 5.4. We stress that, if $\zeta < 1$, our system exhibits a degenerate behavior cf. Theorem 5.2.5, which contrasts with the Weinrib-Halperin criterion, as discussed below.

5.2.2.1. *Smoothing of the phase transition.* We give here a first result that shows the effect of disorder on the phase transition. We show that in presence of disorder, the phase transition is always at least of order 2, as in the *i.i.d.* case (see [Gia07, Th.5.6]).

Theorem 5.2.3. *Under Assumption 5.1.3 of summable correlations, for every $\alpha > 0$ one has that, for all $\beta > 0$ and $h \in \mathbb{R}$*

$$\mathtt{F}(\beta, h) \leqslant \frac{1+\alpha}{2\Upsilon_\infty \beta^2}\,(h - h_c(\beta))_+^2\,, \tag{5.2.7}$$

where we defined $\Upsilon_\infty := \left(1 + 2\sum_{k\in\mathbb{N}} \rho_k\right) \in (0, +\infty)$.

This stresses the relevance of disorder in the case $\alpha > 1/2$, where the pure model exhibits a phase transition of order $\nu^{\mathrm{pur}} := 1 \vee 1/\alpha < 2$. Therefore, with summable correlations, we already have identified a region of the (α, ζ)-plane where disorder is relevant: it corresponds to the relevant disorder region in the *i.i.d.* case, as predicted by the Weinrib-Halperin criterion.

The quantity Υ_∞ is of interest, and is widely use in the sequel. Let us explain briefly where it comes from. Set $\mathbf{1}_l$ the vector constituted of l 1s and then of 0s. One has $\langle \Upsilon \mathbf{1}_l, \mathbf{1}_l \rangle = \sum_{i,j=1}^{l} \rho_{ij} > 0$ (where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean scalar product). One has

$$\Upsilon_\infty := \lim_{l\to\infty} \frac{\langle \Upsilon \mathbf{1}_l, \mathbf{1}_l \rangle}{\langle \mathbf{1}_l, \mathbf{1}_l \rangle} = 1 + 2 \sum_{k\in\mathbb{N}} \rho_k > 0, \tag{5.2.8}$$

where the positivity comes from the fact that the lowest eigenvalue of Υ is bounded away from 0 (see Assumption 5.1.4). Note that Υ_∞ is an increasing function of the correlations, and that Υ_∞ becomes infinite when correlations are no longer summable.

5.2.2.2. *Shift of the critical points at low temperature.* The techniques used in [Poi11, Ton08a] to give an asymptotic expansion of the quenched critical point when β is large (*i.e.* at low temperature) are also valid in the case of non-negative correlations. These results are not difficult to derive, but we mention them here briefly, to get a more complete picture of the disordered system.

Proposition 5.2.4. *Under Assumption 5.1.3 of summability of the correlations, and if $\rho_k \geqslant 0$ for all $k \in \mathbb{N}$, one has*

$$h_c(\beta) \overset{\beta\to\infty}{\sim} -\frac{\Upsilon_\infty}{2(1+\alpha)}\beta^2, \quad \textit{and} \quad h_c^{\mathrm{a},\Upsilon}(\beta) \overset{\beta\to\infty}{\sim} -\frac{\Upsilon_\infty}{2}\beta^2. \tag{5.2.9}$$

From this Proposition, proven in Section 5.4.1 (and in (5.3.8) for the annealed critical point), one therefore has that for all values of $\alpha > 0$, the annealed and quenched critical points differ of $\frac{\alpha}{2(1+\alpha)}\Upsilon_\infty\beta^2$, asymptotically as β goes to infinity. One sees the influence of the presence of inhomogeneities in this case, and can compare this result with [Ton08a, Equation (3.8)], showing how the correlations modify the behavior of the system. Indeed, when correlations increase, the asymptotic difference between the annealed and the quenched critical point increases, and in addition the two critical points are also shifted towards $-\infty$.

5.2.3. The effect of non-summable correlations. The first piece of evidence that correlations may change the critical properties of the system with respect to the *i.i.d.* case, is when correlations are not summable. If the correlations are such that $\sum_{k\in\mathbb{N}} \rho_k = +\infty$, the annealed model is actually ill-defined. Indeed, we have the bound (5.3.7) on the annealed partition function (imposing that there is a contact at every site in $\{1, \dots, N\}$), so that $\frac{1}{N} \log Z_{N,h}^{\mathrm{a},\Upsilon} \geqslant \log \mathrm{K}(1) + \beta^2/2 + h + \beta^2 \sum_{k=1}^{N} \rho_k$, and letting N go to infinity, we see that the annealed free energy is infinite.

Under Assumption 5.1.4 (non-summable, power-law decaying correlations), the annealed model is therefore not well-defined. But not only the annealed free energy is ill-defined: we also prove that the quenched free energy is strictly positive for every value of $h \in \mathbb{R}$: the disordered system does not have a localization/delocalization phase transition and is always localized.

Theorem 5.2.5. *Under Assumption 5.1.4, one has that* $\mathtt{F}(\beta, h) > 0$ *for every* $\beta > 0, h \in \mathbb{R}$, *so that* $h_c^{\mathrm{que}}(\beta) = -\infty$. *There exists some constant* $c_2 > 0$ *such that for all* $h \leqslant -1$ *and* $\beta > 0$

$$\mathtt{F}(\beta, h) \geqslant \exp\left(-c_2 |h| \left(|h|/\beta^2\right)^{1/(1-\zeta)}\right). \tag{5.2.10}$$

This shows that the phase transition disappears when correlations are too strong. Proposition 5.2.4 (and (5.3.5)) suggests that the critical points (both quenched and annealed) are "pushed" towards $-\infty$ when correlations increase. We now have a clearer picture of the behavior of the disordered system, and of its dependence on the strength of the correlations, that we collect in the Figure 5.1.

This provides an example where strongly correlated disorder always modifies (in an extreme way) the behavior of the system, for every value of the renewal parameter α. However the fact that $h_c(\beta) = -\infty$ (which comes from non-boundedness of the ω_i) does not allow us to study sharply how the phase transition is modified by the presence of disorder, and therefore we cannot verify nor contradict the Weinrib-Halperin prediction. That is why we introduce in Chapters 6-7 a strongly-correlated, $\{0, -1\}$-valued random environment. where the critical point is $h_c = 0$.

In general, with very strong correlations (in a sense that has to be precised, see Chapter 7), one should actually have that $h_c(\beta) = -\beta \operatorname{ess\,sup}(\omega_1)$ (with the definition $\operatorname{ess\,sup}(\omega_1) = \inf\{a \in \mathbb{R}, \mathbb{P}(\omega_1 > a) = 0\}$), since one should be able to find long stretches where ω is very close to the maximum of its support.

5.3. The annealed model

5.3.1. Discussion on the annealed critical point. We first give some easy bounds on the critical point $h_c^{\mathrm{a},\Upsilon}(\beta)$: for example, using (5.2.1) and bounding $\delta_i \leqslant 1$, we have

$$H_{N,h}^{\mathrm{a},\Upsilon} \leqslant \left(\frac{\beta^2}{2} + h + \beta^2 \sum_{k=1}^{N} \rho_k \vee 0\right) \sum_{n=1}^{N} \delta_n. \tag{5.3.1}$$

Using a similar lower bound, one can compare the annealed model to two standard homogeneous ones, and one gets that

$$-\frac{\beta^2}{2}\left(1 + 2\sum(\rho_k \vee 0)\right) \geqslant h_c^{\mathrm{a},\Upsilon}(\beta) \geqslant -\frac{\beta^2}{2}\left(1 + 2\sum(\rho_k \wedge 0)\right). \tag{5.3.2}$$

From the definition (5.2.1), we also observe that $Z_{N,h}^{\mathrm{a},\Upsilon}$ is increasing in every ρ_k, so that when the correlations are increased, the annealed critical point decreases. One can more generally compare two annealed systems with different correlation structures. Consider two Gaussian sequences ω and $\widetilde{\omega}$, with correlation matrices $\Upsilon = (\rho_{|i-j|})_{i,j \geqslant 0}$, respectively $\widetilde{\Upsilon} = (\widetilde{\rho}_{|i-j|})_{i,j \geqslant 0}$ (with the same condition that $\rho_0 = \widetilde{\rho}_0 = 1$), and look at the two corresponding annealed partition functions $Z_{n,h}^{\mathrm{a},\Upsilon}$, respectively

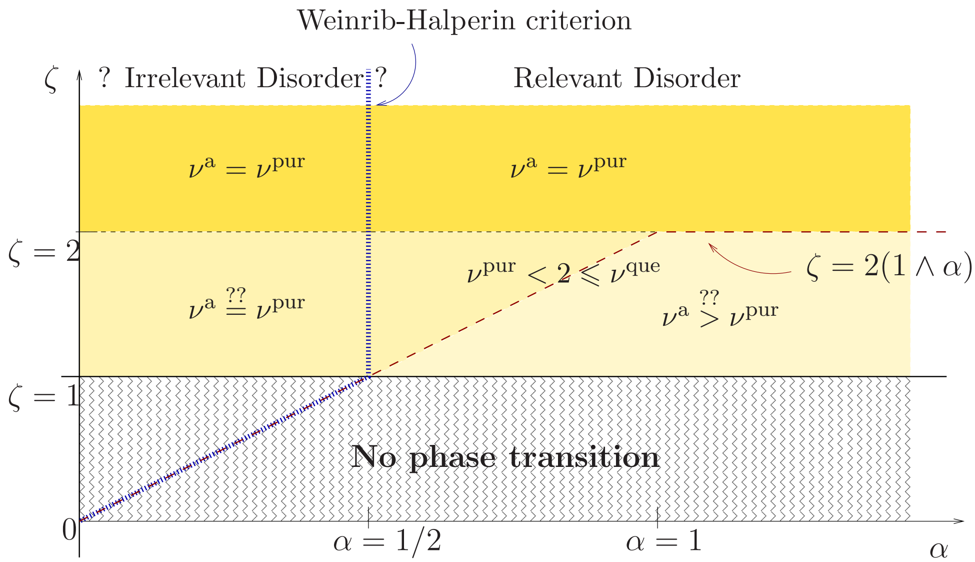

FIGURE 5.1. Overview of the annealed behavior and of disorder relevance/irrelevance in the (α, ζ)-plane for the non-hierarchical model, in analogy with Figure 4.1. In the region $\zeta < 1$ (non-summable correlations), the annealed model is not well-defined, and Theorem 5.2.5 tells us that the quenched free energy is always positive, so that $h_c^{\mathrm{que}}(\beta) = -\infty$ for all $\beta > 0$. It is therefore not possible to apply the Weinrib-Halperin criterion in this region. In the region $\zeta > 1$, the annealed model is well-defined, and Theorem 5.2.2 implies that the annealed critical behavior is the same that the pure one if $\zeta > 2$. For $\zeta \in (1, 2)$, comparison with the hierarchical case, (see Remark 5.1.5) allows us to predict that the annealed critical exponent ν^{a} should be equal to the pure one ν^{pur} if $\zeta\nu^{\mathrm{pur}} > 2$ (*i.e.* $\zeta > 2(1 \wedge \alpha)$), and that it should be strictly larger if $\zeta\nu^{\mathrm{pur}} < 2$. Theorem 5.2.3 shows that disorder is relevant for $\alpha > 1/2$, in agreement with the Weinrib-Halperin prediction, but we still have no proof of disorder irrelevance for $\alpha < 1/2$, that we believe to hold.

$Z_{n,h}^{\mathrm{a},\widetilde{\Upsilon}}$. One has easily, thanks to the form of the annealed partition function (see (5.2.1)), that

$$Z_{n,h-\beta^2\Delta}^{\mathrm{a},\widetilde{\Upsilon}} \leqslant Z_{n,h}^{\mathrm{a},\Upsilon} \leqslant Z_{n,h+\beta^2\Delta}^{\mathrm{a},\widetilde{\Upsilon}}, \qquad \text{with } \Delta := \sum_{k\in\mathbb{N}} |\rho_k - \widetilde{\rho}_k|, \tag{5.3.3}$$

which gives for example that $h_c^{\mathrm{a},\widetilde{\Upsilon}}(\beta) - \beta^2\Delta \leqslant h_c^{\mathrm{a},\Upsilon}(\beta) \leqslant h_c^{\mathrm{a},\widetilde{\Upsilon}}(\beta) + \beta^2\Delta$.

We set $\Upsilon^{(n)} = (\rho_{|i-j|}^{(n)})_{i,j\geqslant 0}$ the n-truncation of the Υ correlation matrix: set $\rho_k^{(n)} = \rho_k$ if $k \leqslant n$ and $\rho_k^{(n)} = 0$ if $k > n$. The matrix $\Upsilon^{(n)}$ defined this way is then symmetric and positive definite. In this case, one has that $\Delta_n = \sum_{k\in\mathbb{N}} |\rho_k - \rho_k^{(n)}| = \sum_{k>n} |\rho_k|$. From the assumption $\sum |\rho_k| < \infty$, one has that Δ_n goes to 0 as n goes to ∞, so that for all $\beta \geqslant 0$

$$h_c^{\mathrm{a},\Upsilon}(\beta) = h_c^{\mathrm{a},\Upsilon^{(n)}}(\beta) + \beta^2 o_{n\to\infty}(1), \tag{5.3.4}$$

This does not provide a way of computing the critical point, but combining (5.3.4) with the small disorder asymptotic of the annealed critical point in systems with finite-range correlations (see [Poi12, Prop.4.2]), one gets

$$h_c^{\mathrm{a},\Upsilon}(\beta) \overset{\beta\searrow 0}{\sim} -\frac{\beta^2}{2}\left(1+2\sum_{k\in\mathbb{N}}\rho_k \mathbf{P}(k\in\tau)\right). \tag{5.3.5}$$

This result suggests that the quenched critical point is shifted towards $-\infty$ when correlations increase, and when they are not summable anymore, it should be equal to $-\infty$ for all $\beta>0$. This is confirmed by Theorem 5.2.5.

Let us also briefly discuss the case of non-negative correlations, that brings some simplifications. If correlations ρ_k are positive, one has a better upper bound than $h_c^{\mathrm{a},\Upsilon}(\beta) \leqslant -\beta^2/2$, using the following inequality

$$Z_{n,h}^{\mathrm{a},\Upsilon} \geqslant \sum_{m=1}^{N} \sum_{0=:\tau_0<\ldots<\tau_m=N} \prod_{i=1}^{m} e^{h+\beta^2/2}\mathrm{K}(\tau_i-\tau_{i-1})e^{\beta^2\rho_{\tau_i-\tau_{i-1}}}. \tag{5.3.6}$$

Therefore, the annealed critical point is $h_c^{\mathrm{a},\Upsilon}(\beta) \leqslant -\beta^2/2 - \log\left(\sum_{n\in\mathbb{N}}\mathrm{K}(n)e^{\beta^2\rho_n}\right)$, which stresses that for positive correlations, one has $h_c^{\mathrm{a},\Upsilon}(\beta) < -\beta^2/2$ for all $\beta>0$.

In the case of non-negative correlations, it is also easy to derive low-temperature asymptotics for the annealed critical point. Indeed, we have the following bound on the annealed partition function:

$$Z_{N,h}^{\mathrm{a},\Upsilon} \geqslant \mathrm{K}(1)^N \exp((\beta^2/2+h)N)\exp\left(N\beta^2\sum_{k=1}^{N}\rho_k\right), \tag{5.3.7}$$

which yields that $h_c^{\mathrm{a},\Upsilon}(\beta) \leqslant -\frac{\beta^2}{2}\Upsilon_\infty - \log \mathrm{K}(1)$ (recall the definition (5.2.8) of Υ_∞). As β goes to ∞, this matches the upper bound in (5.3.2),

$$\text{if } \rho_k \geqslant 0\ \forall k\in\mathbb{N}, \quad \text{one has} \qquad h_c^{\mathrm{a},\Upsilon}(\beta) \overset{\beta\to\infty}{\sim} -\frac{\Upsilon_\infty}{2}\beta^2. \tag{5.3.8}$$

5.3.2. Observations on the annealed partition function. We now give the reason why the condition $m_\Upsilon := \sum k|\rho_k| < \infty$ simplifies the analysis of the annealed system, in particular in Theorem 5.2.2. Given two arbitrary disjoint blocks B_1 and B_2, the contribution to the Hamiltonian of these two blocks can be divided into:

- two *internal* contributions $(\beta^2/2+h)\sum_{i\in B_a}\delta_i + \beta^2\sum_{i,j\in B_a, i<j}\delta_i\delta_j\rho_{|i-j|}$ for $a=1,2$,
- an *interaction* contribution $\beta^2\sum_{i\in B_1, j\in B_2}\delta_i\delta_j\rho_{|i-j|}$.

We also refer to the latter term as the correlation term. Then we can use uniform bounds to control the interactions between B_1 and B_2, since there are at most k points at distance k between B_1 and B_2:

$$-m_\Upsilon = -\sum_{k=1}^{\infty} k|\rho_k| \leqslant \sum_{k=1}^{\infty}\rho_k \sum_{\substack{i\in B_1, j\in B_2\\ |i-j|=k}} \delta_i\delta_j \leqslant \sum_{k=1}^{\infty} k|\rho_k| = m_\Upsilon \tag{5.3.9}$$

Thanks to this remark, if $m_\Upsilon < \infty$, we have "quasi super-multiplicativity" (super-multiplicativity would hold if all of the ρ_k were non-negative): for any $N \geqslant 1$ and $0 \leqslant k \leqslant N$, one has

$$Z^{\mathrm{a}}_{n,h} \geqslant e^{-\beta^2 m_\Upsilon} Z^{\mathrm{a}}_{k,h} Z^{\mathrm{a}}_{N-k,h}. \tag{5.3.10}$$

We also get the two following bounds, which can be seen as substitutes for the renewal property (property that we do not have in our annealed system because of the two-body $\delta_i\delta_j$ term). Decomposing according to the last renewal before some integer $M \in [0, N]$, and the first after it, one gets

$$Z^{\mathrm{a}}_{n,h} \geqslant \sum_{i=0}^{M} \sum_{j=M+1}^{N} e^{-\beta^2 m_\Upsilon} Z^{\mathrm{a}}_{i,h}\, \mathrm{K}(j-i) e^{\beta^2/2+h-\beta^2\sum|\rho_k|}\, Z^{\mathrm{a}}_{N-j,h}, \tag{5.3.11}$$

and

$$Z^{\mathrm{a}}_{n,h} \leqslant \sum_{i=0}^{M} \sum_{j=M+1}^{N} e^{\beta^2 m_\Upsilon} Z^{\mathrm{a}}_{i,h}\, \mathrm{K}(j-i) e^{\beta^2/2+h+\beta^2\sum|\rho_k|}\, Z^{\mathrm{a}}_{N-j,h}. \tag{5.3.12}$$

Note that the terms $e^{\beta^2/2+h-\beta^2\sum|\rho_k|}$ and $e^{\beta^2/2+h+\beta^2\sum|\rho_k|}$ come from bounding uniformly the contribution of the point j to the partition function. If we write $h = h^{\mathrm{a}}_c + u$, and using that h^{a}_c is of order β^2 (see Section 5.3.1), we get a constant $c > 0$ such that

$$e^{-c\beta^2} e^u \sum_{i=0}^{M} \sum_{j=M+1}^{N} Z^{\mathrm{a}}_{i,h} \mathrm{K}(j-i) Z^{\mathrm{a}}_{N-j,h} \leqslant Z^{\mathrm{a}}_{n,h} \leqslant e^{c\beta^2} e^u \sum_{i=0}^{M} \sum_{j=M+1}^{N} Z^{\mathrm{a}}_{i,h} \mathrm{K}(j-i) Z^{\mathrm{a}}_{N-j,h}. \tag{5.3.13}$$

Note that one has also uniform bounds for $u \in [-1, 1]$ (we are interested in the critical behavior, *i.e.* for u close to 0): one replaces the constant $e^{c\beta^2} e^u$ by $C_1 := e^{c\beta^2+1}$, and the constant $e^{-c\beta^2} e^u$ by C_1^{-1}.

In a general way, for any indexes $0 = i_0 < i_1 < i_2 < \cdots < i_m = N$, we also get

$$\left(e^{-c\beta^2} e^u\right)^m \prod_{k=1}^{m} Z^{\mathrm{a}}_{i_k - i_{k-1},h} \leqslant \mathbf{E}\left[\prod_{k=1}^{m} \delta_{i_k} e^{H^{\mathrm{a}}_{N,h}}\right] \leqslant \left(e^{c\beta^2} e^u\right)^m \prod_{k=1}^{m} Z^{\mathrm{a}}_{i_k - i_{k-1},h}. \tag{5.3.14}$$

When β is small, (5.3.13)-(5.3.14) are close to the renewal equation verified by $Z^{\mathrm{pur}}_{N,h}$ which is the same as (5.3.13)-(5.3.14) with $\beta = 0$. In the sequel, we refer to (5.3.13)-(5.3.14) as the *quasi-renewal property*. We can actually show Theorem 5.2.2 provided that these inequalities hold. Therefore if one is able to get (5.3.13)-(5.3.14) with a weaker condition than $m_\Upsilon < \infty$ (which could be $\zeta > 2(\alpha \wedge 1)$, as the comparison with the hierarchical model suggests, see Remark 5.1.5), such a Theorem would follow.

5.3.3. The annealed critical behavior, proof of Theorem 5.2.2. In this Section, we drop the superscript Υ in $Z^{\mathrm{a},\Upsilon}_{N,h}$, and write h^{a}_c instead of $h^{\mathrm{a},\Upsilon}_c(\beta)$, to keep notations simple.

The essential tool is to prove that with small correlations, the partition function at $h = h^{\mathrm{a}}_c$ is close to the homogeneous partition function without the two-body interaction at $h = 0$, $Z^{\mathrm{pur}}_{N,h=0} = \mathbf{P}(n \in \tau)$.

Proposition 5.3.1. *We assume that the quasi-renewal property (5.3.14)-(5.3.13) holds. Define for all $\lambda > 0$ $\widehat{Z}_{h_c^{\mathrm{a}}}(\lambda) := \sum_{n=0}^{\infty} e^{-\lambda n} Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$. Then there exists a constant $c_1 > 0$, such that for every $0 < \lambda \leqslant 1$ one has*

$$c_1^{-1}\widehat{\mathbf{P}}(\lambda) \leqslant \widehat{Z}_{h_c^{\mathrm{a}}}(\lambda) \leqslant c_1\widehat{\mathbf{P}}(\lambda). \tag{5.3.15}$$

This Proposition says that, increasing β and tuning h so that we stay at the annealed critical point, we control the behavior of the Laplace Transform $\widehat{Z}_{h_c^{\mathrm{a}}}(\lambda)$ of $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$, which is of the same order as $\widehat{\mathbf{P}}(\lambda)$. We then are able to adapt the proof of Proposition 1.1.10 and Theorem 1.1.6, using the same methods than in section 1.1.2.

Proof of Theorem 5.2.2 given Proposition 5.3.1 Recall that we define $u := h - h_c^{\mathrm{a}}$, so that we only work with $u > 0$, $u \in [0,1]$, as we already know that for $u \leqslant 0$, $\mathtt{F}^{\mathrm{a}}(\beta, u) = 0 = \mathtt{F}(u)$. Using (5.2.5) and the same expansion as in (1.1.23), we get that

$$Z^{\mathrm{a}}_{n,h} = \mathbf{E}\left[e^{u\sum_{n=1}^{N}\delta_n} e^{H^a_{n,h_c^{\mathrm{a}}}}\delta_N\right] = \frac{e^u}{e^u-1}\sum_{m=1}^{N}\sum_{0<i_1<\ldots<i_m=N}(e^u-1)^m\mathbf{E}\left[\delta_{i_1}\ldots\delta_{i_m}e^{H^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}}\right]. \tag{5.3.16}$$

Note that as there is no renewal structure for $\mathbf{E}\left[\cdot\ e^{H^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}}\right]$, one cannot factorize the quantity $\mathbf{E}\left[\delta_{i_1}\ldots\delta_{i_m}e^{H^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}}\right]$ easily. However, since we have the quasi-renewal property (5.3.14), we get the two following bounds, valid for any $m \in \mathbb{N}$ and subsequence $0 < i_1 < \ldots < i_m = N$, uniformly for $u \in [0,1]$:

$$(C_1^{-1})^m\prod_{k=1}^{m} Z^{\mathrm{a}}_{i_k-i_{k-1},h_c^{\mathrm{a}}} \leqslant \mathbf{E}\left[\delta_{i_1}\ldots\delta_{i_m}\delta_N e^{H^a_{n,h_c^{\mathrm{a}}}}\right] \leqslant (C_1)^m\prod_{k=1}^{m} Z^{\mathrm{a}}_{i_k-i_{k-1},h_c^{\mathrm{a}}}, \tag{5.3.17}$$

where $C_1 := e^{c\beta+1}$ is defined in Section 5.3.2. Now, we define

$$\begin{aligned}\bar{Z}^{\mathrm{a}}_{N,h} &:= \frac{e^u}{e^u-1}\sum_{m=1}^{N}\left(C_1^{-1}(e^u-1)\right)^m\sum_{0<i_1<\ldots<i_m=N}\prod_{k=1}^{m} Z^{\mathrm{a}}_{i_k-i_{k-1},h_c^{\mathrm{a}}}\\ \text{and}\quad \widetilde{Z}^{\mathrm{a}}_{N,h} &:= \frac{e^u}{e^u-1}\sum_{m=1}^{N}\left(C_1(e^u-1)\right)^m\sum_{0<i_1<\ldots<i_m=N}\prod_{k=1}^{m} Z^{\mathrm{a}}_{i_k-i_{k-1},h_c^{\mathrm{a}}},\end{aligned} \tag{5.3.18}$$

so that $\bar{Z}^{\mathrm{a}}_{N,h} \leqslant Z^{\mathrm{a}}_{n,h} \leqslant \widetilde{Z}^{\mathrm{a}}_{N,h}$. For $u > 0$, we can define $\bar{b} > 0$ and $\widetilde{b} > 0$ such that

$$\widehat{Z}_{h_c^{\mathrm{a}}}(\bar{b}) = C_1(e^u-1)^{-1}, \qquad \text{and} \qquad \widehat{Z}_{h_c^{\mathrm{a}}}(\widetilde{b}) = C_1^{-1}(e^u-1)^{-1}, \tag{5.3.19}$$

if the equations have a solution and otherwise set $\bar{b} = 0$, or $\widetilde{b} = 0$. Such definitions give as in the proof of Proposition 1.1.10, that $\lim \frac{1}{N}\log \bar{Z}^{\mathrm{a}}_{N,h} = \bar{b}$ and $\lim \frac{1}{N}\log \widetilde{Z}^{\mathrm{a}}_{N,h} = \widetilde{b}$. Then we have that $\bar{b} \leqslant \mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}} + u) \leqslant \widetilde{b}$, from the fact that $\bar{Z}^{\mathrm{a}}_{N,h} \leqslant Z^{\mathrm{a}}_{n,h} \leqslant \widetilde{Z}^{\mathrm{a}}_{N,h}$. Using that $\widehat{\mathbf{P}}(\cdot)$ is decreasing one therefore gets that

$\widehat{\mathbf{P}}(\widetilde{b}) \leqslant \widehat{\mathbf{P}}(\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}+u)) \leqslant \widehat{\mathbf{P}}(\bar{b})$. The definitions (5.3.19), combined with Proposition 5.3.1, gives that for every $u>0$ such that $\bar{b} \leqslant 1$ one has

$$(c_1 C_1)^{-1}(e^u-1)^{-1} \leqslant \widehat{\mathbf{P}}(\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}+u)) \leqslant c_1 C_1 (e^u-1)^{-1}. \tag{5.3.20}$$

We finally have that for $u \geqslant 0$ small enough

$$(e^{cu}-1)^{-1} \leqslant \widehat{\mathbf{P}}(\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}+u)) \leqslant (e^{c'u}-1)^{-1}. \tag{5.3.21}$$

Applying the inverse of $\widehat{\mathbf{P}}$ (which is also decreasing), one gets the result from the fact that $\mathtt{F}(u) = \widehat{\mathbf{P}}((e^u-1)^{-1})$ for all positive u (see (1.1.26)). □

Proof of Proposition 5.3.1 Let us first prove a preliminary result, that will be useful, both in the case $\alpha<1$, and in the case $\alpha>1$.

Claim 5.3.2. *For every $\alpha>0$, if the quasi-renewal property* (5.3.14)-(5.3.13) *holds, then for all $N \in \mathbb{N}$ one has $Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}} \leqslant C_1$, where $C_1 = e^{c\beta^2+1}$ is defined above.*

Indeed, the l.h.s. inequality in (5.3.14) yields that for all $u \in [-1,1]$, one has

$$C_1^{-1} Z^{\mathrm{a}}_{M+N,h} \geqslant (C_1^{-1} Z^{\mathrm{a}}_{N,h})(C_1^{-1} Z^{\mathrm{a}}_{M,h})$$

for all $M, N \geqslant 0$. Therefore one gets that if $C_1^{-1} Z^{\mathrm{a}}_{n_0,h} > 1$ for some n_0, then the partition function grows exponentially, and $\mathtt{F}(\beta,h)>0$. This gives directly that $C_1^{-1} Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}} \leqslant 1$ for all $N \in \mathbb{N}$. □

We now focus only on the case $\alpha<1$, since Proposition 5.3.5 gives a better result in the case $\alpha>1$. We know that $\widehat{\mathbf{P}}(\lambda) \sim c\lambda^{-\alpha}$ when λ goes to 0 (recall the assumption on $K(\cdot)$). Then we only have to show that $\widehat{Z}^{\mathrm{a}}_{h_c^{\mathrm{a}}}(\lambda)$ is of order $\lambda^{-\alpha}$ as $\lambda \searrow 0$, or equivalently that $\sum_{n=1}^{N} Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ is of order N^α for large N, thanks to an Abelian Theorem [BGT87, Th.1.7.1].

Upper bound. We prove the following Lemma

Lemma 5.3.3. *For $\alpha<1$, there exists a constant $C_0>0$ such that for any $N \geqslant 1$*

$$\sum_{n=0}^{N} Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \leqslant C_0 N^\alpha. \tag{5.3.22}$$

Proof If the Lemma were not true, then for any constant $A>0$ arbitrarily large, there would exist some $n_0 \geqslant 1$ such that

$$\sum_{n=0}^{n_0} Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \geqslant A n_0^\alpha. \tag{5.3.23}$$

But in this case, using the l.h.s. inequality of (5.3.13), we get for any $2n_0 \leqslant p \leqslant 4n_0$

$$\begin{aligned} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} &\geqslant C_1^{-1} \sum_{i=0}^{\lfloor p/2 \rfloor} \sum_{j=\lfloor p/2\rfloor+1}^{p} Z^{\mathrm{a}}_{i,h_c^{\mathrm{a}}} \mathrm{K}(j-i) Z^{\mathrm{a}}_{p-j,h_c^{\mathrm{a}}} \\ &\geqslant C_1^{-1} \left(\sum_{i=0}^{n_0} \sum_{j=p-n_0}^{p} Z^{\mathrm{a}}_{i,h_c^{\mathrm{a}}} Z^{\mathrm{a}}_{p-j,h_c^{\mathrm{a}}} \right) \min_{n \leqslant p} \mathrm{K}(n) \geqslant C_1^{-1} A^2 n_0^{2\alpha} \min_{n \leqslant 4n_0} \mathrm{K}(n), \end{aligned} \tag{5.3.24}$$

where we restricted the sum to i and $p-j$ smaller than n_0 to be able to use the inequality (5.3.23). On the other hand, with the assumption that $\mathrm{K}(n) \sim c_K n^{-(1+\alpha)}$, there exists a constant $c>0$ (not depending on n_0) such that one has that $\min_{n \leqslant 4n_0} \mathrm{K}(n) \geqslant c n_0^{-(1+\alpha)}$. And thus for any $2n_0 \leqslant p \leqslant 4n_0$ one has that

$$Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} \geqslant c' A^2 n_0^{\alpha-1}.$$

Then, summing over p, we get an inequality similar to (5.3.23):

$$\sum_{p=0}^{4n_0} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} \geqslant \sum_{p=2n_0}^{4n_0} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} \geqslant cA^2 n_0^{\alpha} =: \bar{c} A^2 (4n_0)^{\alpha}. \tag{5.3.25}$$

Now, we are able to repeat this argument with n_0 replaced with $4n_0$ and A with $\bar{c}A^2$. By induction, we finally have for any $k \geqslant 0$

$$\sum_{n=0}^{4^k n_0} Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \geqslant (\bar{c})^{2^k-1} A^{2^k} (4^k n_0)^{\alpha}. \tag{5.3.26}$$

To find a contradiction, we choose $A > (\bar{c})^{-1}$, so that $(\bar{c})^{2^k-1} A^{2^k} \geqslant \gamma^{2^k}$ with $\gamma > 1$. Now, we can choose $k \in \mathbb{N}$ such that $\gamma^{2^k} (4^k n_0)^{\alpha-1} \geqslant 2C_1$ (C_1 being the constant in Claim 5.3.2). Thanks to (5.3.26), we get that at least one of the terms $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ for $n \leqslant 4^k n_0$ is bigger than $(4^k n_0)^{\alpha-1}\gamma^{2^k} \geqslant 2C_1$, which contradicts the Claim 5.3.2. □

Lower Bound. We use the following Lemma

Lemma 5.3.4. *If $\alpha < 1$, there exists some $\eta > 0$, such that if for some $n_0 \geqslant 1$ one has*

$$\sum_{i=0}^{n_0} Z^{\mathrm{a}}_{i,h} \sum_{j=n_0}^{\infty} \mathrm{K}(j-i) \leqslant \eta \qquad \text{and} \qquad \sum_{i=0}^{n_0} Z^{\mathrm{a}}_{i,h} \leqslant \eta n_0^{\alpha}, \tag{5.3.27}$$

then $\mathtt{F}^{\mathrm{a}}(\beta,h) = 0$.

This Lemma comes easily from [GLT10b, Lemma 5.2] where the case $\alpha = 1/2$ was considered, and gives a finite-size criterion for delocalization. It comes from cutting the system into blocks of size n_0, and then using a coarse-graining argument in order to reduce ourselves to finite-size estimates (on segments of size $\leqslant n_0$). It is therefore not difficult to extend it to every $\alpha < 1$, in particular thanks to the quasi-renewal property (5.3.14)-(5.3.13), that allows us to proceed to the coarse-graining decomposition of the system.

From this Lemma, one deduces that at $h = h_c^{\mathrm{a}}$, for all $n \in \mathbb{N}$ one has

$$\textstyle\sum_{i=1}^{n} \sum_{j=n}^{\infty} Z^{\mathrm{a}}_{i,h_c^{\mathrm{a}}} \mathrm{K}(j-i) \geqslant \frac{\eta}{2} \tag{5.3.28}$$

$$\text{or } \textstyle\sum_{i=1}^{n} Z^{\mathrm{a}}_{i,h_c^{\mathrm{a}}} \geqslant \frac{\eta}{2} n^{\alpha}. \tag{5.3.29}$$

Indeed, otherwise, one could find some $n_0 \geqslant 0$ such that both of these assumptions fail, and then one picks some $\varepsilon > 0$ such that $Z^{\mathrm{a}}_{n_0,h_c^{\mathrm{a}}+\varepsilon}$ verifies the conditions of Lemma 5.3.4, so that $\mathtt{F}^{\mathrm{a}}(\beta, h_c^{\mathrm{a}}+\varepsilon) = 0$. This contradicts the definition of h_c^{a}.

We now try to deduce the behavior of $\widehat{Z}^{\mathrm{a}}_{h_c^{\mathrm{a}}}(\lambda)$ from (5.3.28)-(5.3.29). We define the sets

$$\begin{aligned} E_1 &:= \{n \geqslant 0, \text{ such that (5.3.28) holds}\}, \\ E_2 &:= \{n \geqslant 0, \text{ such that (5.3.29) holds}\}. \end{aligned} \tag{5.3.30}$$

For $\lambda > 0$, we define $f(\lambda) = \sum_{n=0}^{\infty} e^{-\lambda n}(n+1)^{-\alpha}$. We know that $f(\lambda) \sim cst.\lambda^{\alpha-1}$ thanks to an Abelian Theorem [BGT87, Th.1.7.1]. Then, using the assumption on $\mathrm{K}(\cdot)$ to find some constant $c > 0$ such that for all $i \leqslant n$ one has $\sum_{j=n}^{\infty} \mathrm{K}(j-i) \leqslant c(n+1-i)^{-\alpha}$, one gets

$$\begin{aligned} \widehat{Z}_{h_c^{\mathrm{a}}}(\lambda) f(\lambda) &= \sum_{n=0}^{\infty} e^{-\lambda n} \sum_{i=1}^{n} Z^{\mathrm{a}}_{i,h_c^{\mathrm{a}}} (n+1-i)^{-\alpha} \geqslant \sum_{n=0}^{\infty} c^{-1} e^{-\lambda n} \sum_{i=1}^{n} Z^{\mathrm{a}}_{i,h_c^{\mathrm{a}}} \sum_{j=n}^{\infty} \mathrm{K}(j-i) \\ &\geqslant c^{-1}\eta/2 \sum_{n\in \mathbf{E}_1} e^{-\lambda n} \geqslant c^{-1} e^{-1} \eta/2 \, |E_1 \cap \{1,\dots,\lfloor 1/\lambda \rfloor\}|, \end{aligned} \tag{5.3.31}$$

where in the second inequality we used the definition of E_1, and in the last one we cut the sum at $\lfloor 1/\lambda \rfloor$. Thus we get from our estimate on $f(\lambda)$, that for any $\lambda \leqslant 1$

$$\widehat{Z}_{h_c^{\mathrm{a}}} \geqslant c' \lambda^{-\alpha} \big(\lambda \, |E_1 \cap \{1,\dots,\lfloor 1/\lambda \rfloor\}| \big). \tag{5.3.32}$$

Using the definition of E_2, we also have

$$\begin{aligned} \widehat{Z}_{h_c^{\mathrm{a}}}(\lambda) &\geqslant e^{-1} \sum_{i=0}^{\lfloor 1/\lambda \rfloor} Z^{\mathrm{a}}_{i,h_c^{\mathrm{a}}} \geqslant e^{-1} \frac{\eta}{2} \left[\max(E_2 \cap \{1,\dots,\lfloor 1/\lambda \rfloor\})\right]^{\alpha} \\ &\geqslant c' \lambda^{-\alpha} \big(\lambda \, |E_2 \cap \{1,\dots,\lfloor 1/\lambda \rfloor\}| \big)^{\alpha}. \end{aligned} \tag{5.3.33}$$

Let us now notice that from (5.3.28)-(5.3.29), for all $n \geqslant 0$ we have $n \in E_1 \cup E_2$, so that $\max\big(\frac{1}{n} |E_1 \cap \{1,\dots,n\}|, \frac{1}{n} |E_2 \cap \{1,\dots,n\}|\big) \geqslant 1/2$. Then, combining (5.3.32) and (5.3.33), we get that $\widehat{Z}^{\mathrm{a}}_{h_c^{\mathrm{a}}}(\lambda) \geqslant c\lambda^{-\alpha}$ for $\lambda \leqslant 1$. □

5.3.3.1. *Improvement of Proposition 5.3.1 in the case $\alpha > 1$.* In this case, we can estimate $Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}$ more precisely, and estimate not only the Laplace transform of $Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}$ (cf. Proposition 5.3.1), but $Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}$ itself.

Proposition 5.3.5. *Let $\alpha > 1$. Assume that the quasi-renewal property (5.3.14)-(5.3.13) holds. Then there exist two constants c_1 and c_2 such that, for any $N \geqslant 2$ and any sequence of indexes $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_m = N$ with $m \geqslant 1$, we have*

$$(c_1)^m \mathbf{E}(\delta_{i_1} \dots \delta_{i_m}) \leqslant \mathbf{E}\left[\delta_{i_1} \dots \delta_{i_m} e^{H^a_{N,h_c^{\mathrm{a}}}}\right] \leqslant (c_2)^m \mathbf{E}(\delta_{i_1} \dots \delta_{i_m}). \tag{5.3.34}$$

In particular, if $m = 1$ one has that $c_1 \mathbf{P}(N \in \tau) \leqslant Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}} \leqslant c_2 \mathbf{P}(N \in \tau)$.

This Proposition tells that the annealed polymer measure at the critical point is "close" to the renewal measure $\mathbf{P}$, so that the behavior of the annealed model is very close to the one of the homogeneous model. In Proposition 5.3.1 we only had the behavior of the Laplace transform of the sequence $(Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}})_{n\in\mathbb{N}}$, which lead to control the sum $\sum_{0<i_1<\ldots<i_k=N}\mathbf{E}\left[\delta_{i_1}\ldots\delta_{i_m}e^{H^a_{n,h_c^{\mathrm{a}}}}\right]$. In the case $\alpha>1$, we therefore control every term of this sum.

We have $\mathbf{P}(\delta_{i_1}\ldots\delta_{i_m})=\prod_{k=1}^m\mathbf{P}(i_k-i_{k-1}\in\tau)$, so that recalling (5.3.17), we only have to compare $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ with $\mathbf{P}(n\in\tau)$. If we get two constants c_1 and c_2 such that $c_1\mathbf{P}(N\in\tau)\leqslant Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}\leqslant c_2\mathbf{P}(N\in\tau)$ for all $N\geqslant 0$, then we are done.

For $\alpha>1$, we have $\lim_{N\to\infty}\mathbf{P}(N\in\tau)=\mathbf{E}[\tau_1]^{-1}$ (see Theorem 1.1.7). Thus, we only have to show that $Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}$ is bounded away from 0 and $+\infty$, which is provided by the following lemma.

Lemma 5.3.6. *If* (5.3.14)-(5.3.13) *hold, and if $\alpha>1$, there exist constants $c_0>0$ and $C_1>0$, such that for all $N\geqslant 0$*

$$c_0\leqslant Z^{\mathrm{a}}_{N,h_c^{\mathrm{a}}}\leqslant C_1 \tag{5.3.35}$$

Proof The upper bound is already given by Claim 5.3.2, thanks to quasi super-multiplicativity. For the other bound, we show the following claim.

Claim 5.3.7. *If* (5.3.14)-(5.3.13) *hold, and if $\alpha>1$, let $\varepsilon>0$ (small) and $A>0$ (large) be fixed according to the conditions* (5.3.41)-(5.3.43) *below. Then for every $N\geqslant 0$, there exists some $n_1\in[N-A,N]$ such that $Z^{\mathrm{a}}_{n_1,h_c^{\mathrm{a}}}\geqslant\varepsilon$.*

From this Claim and inequality (5.3.13) with the choice $M=N-1$, we have

$$Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\geqslant C_1^{-1}\sum_{n=0}^{N-1}Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\mathrm{K}(N-n)e^{\beta^2/2+h_c^{\mathrm{a}}}\geqslant C'Z^{\mathrm{a}}_{n_1,h_c^{\mathrm{a}}}\mathrm{K}(N-n_1), \tag{5.3.36}$$

where we only kept the term $n=n_1$ in the sum, n_1 being given by the Claim 5.3.7. We get that for every $N\geqslant 0$,

$$Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\geqslant\varepsilon C'\left(\min_{i\leqslant A}\mathrm{K}(i)\right)e^{\beta^2/2+h_c^{\mathrm{a}}}=:c_0, \tag{5.3.37}$$

which ends the proof of Lemma 5.3.6. □

Now, we prove the Claim 5.3.7 by contradiction. The idea is to prove that if the claim were not true, we can increase a bit the parameter h and still be in the delocalized phase.

Proof of Claim 5.3.7 Let us suppose that the claim is not true. Then we can find some n_0, such that for any $k\in[n_0-A,n_0]$ one has $Z^{\mathrm{a}}_{k,h_c^{\mathrm{a}}}\leqslant\varepsilon$. The integer n_0 being fixed, we choose some $h>h_c^{\mathrm{a}}$ close enough to h_c^{a} such that for this n_0, we have (recall $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\leqslant C_1$)

$$Z^{\mathrm{a}}_{n,h}\leqslant 2C_1\quad\text{for all}\quad n\leqslant n_0, \tag{5.3.38}$$

$$\text{and}\quad Z^{\mathrm{a}}_{k,h}\leqslant 2\varepsilon\quad\text{for all}\quad k\in[n_0-A,n_0]. \tag{5.3.39}$$

We will now see that the properties (5.3.38)-(5.3.39) are kept when we consider bigger systems: we show that we have $Z^{\mathrm{a}}_{n,h}\leqslant 2C_1$ for all $n\leqslant 2n_0$, and $Z^{\mathrm{a}}_{k,h}\leqslant 2\varepsilon$ for all $k\in[2n_0-A,2n_0]$. By induction one therefore gets that $Z^{\mathrm{a}}_{N,h}\leqslant 2C_1$ for all N, such that $\mathtt{F}^{\mathrm{a}}(\beta,h)=0$, which gives a contradiction with the definition of h_c^{a}.

• We first start to show that for any $p \in [n_0+1, 2n_0]$, one has $Z^{\mathrm{a}}_{p,h} \leqslant 2C_1$. We use the r.h.s. inequality of (5.3.13) with $M = n_0$, and we divide the sum into two parts:

$$Z^{\mathrm{a}}_{p,h} \leqslant C_1 \sum_{i=n_0-A}^{n_0} \sum_{j=n_0+1}^{p} Z^{\mathrm{a}}_{i,h} \mathrm{K}(j-i) Z^{\mathrm{a}}_{p-j,h} + C_1 \sum_{i=0}^{n_0-A-1} \sum_{j=n_0+1}^{p} Z^{\mathrm{a}}_{i,h} \mathrm{K}(j-i) Z^{\mathrm{a}}_{p-j,h}$$
$$\leqslant 4\varepsilon C_1^2 \sum_{n \geqslant 1} n\mathrm{K}(n) + 4C_1^3 \sum_{n \geqslant A} n\mathrm{K}(n), \quad (5.3.40)$$

where we used the properties (5.3.38)-(5.3.39), and the fact that $\mathrm{K}(j-i)$ appears at most $j-i$ times. Thus we have $Z_{p,h} \leqslant 2C_1$ for $p \in [n_0+1, 2n_0]$ provided that

$$\varepsilon \leqslant (4C_1 \mathbf{E}[\tau_1])^{-1} \qquad \text{and} \qquad \sum_{n \geqslant A} n\mathrm{K}(n) \leqslant (4C_1^2)^{-1}, \quad (5.3.41)$$

and we have the property (5.3.38) with n_0 replaced by $2n_0$.

• We now show that $Z^{\mathrm{a}}_{p,h} \leqslant 2\varepsilon$ for all $p \in [2n_0 - A, 2n_0]$. Again, we use the r.h.s. inequality of (5.3.13) with $M = \lfloor p/2 \rfloor$, and the properties (5.3.38)-(5.3.39) to get

$$Z^{\mathrm{a}}_{p,h} \leqslant C_1 \sum_{i=\lfloor p/2\rfloor - A/2}^{\lfloor p/2 \rfloor} \sum_{j=\lfloor p/2\rfloor+1}^{\lfloor p/2\rfloor + A/2} Z^{\mathrm{a}}_{i,h} \mathrm{K}(j-i) Z^{\mathrm{a}}_{p-j,h} + C_1 \sum_{\substack{i<\lfloor p/2\rfloor - A/2 \\ or\ j>\lfloor p/2\rfloor + A/2}} Z^{\mathrm{a}}_{i,h} \mathrm{K}(j-i) Z^{\mathrm{a}}_{p-j,h}$$
$$\leqslant 4\varepsilon^2 C_1 \sum_{n \geqslant 1} n\mathrm{K}(n) + 4C_1^2 \sum_{n \geqslant A/2} n\mathrm{K}(n), \quad (5.3.42)$$

where we also used that we have $i, p-j \in [n_0 - A, n_0]$ in the first sum (since $p \in [2n_0 - A, 2n_0]$), and $j - i \geqslant A/2$ in the second sum. Thus we have $Z_{p,h} \leqslant 2\varepsilon$ for $p \in [2n_0 - A, 2n_0]$ provided that

$$\varepsilon \leqslant (4C_1 \mathbf{E}[\tau_1])^{-1} \qquad \text{and} \qquad \sum_{n \geqslant A/2} n\mathrm{K}(n) \leqslant (4C_1^2)^{-1}\varepsilon, \quad (5.3.43)$$

and we have the property (5.3.39) with n_0 replaced by $2n_0$. □

Claim 5.3.7 controls directly the partition function (it is the analogue of Lemma 4.4.1), instead of its Laplace transform as in Proposition 5.3.1. We emphasize that this improvement can be very useful, because it allows us to compare $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \mathbf{E}^{\mathrm{a}}_{n,h_c}[\delta_i]$ with $\mathbf{P}(i \in \tau)$, as we did in the hierarchical case (Chapter 4), thanks to Proposition 4.3.2. For example an easy computation (expanding the exponential) gives that

$$\mathbf{E}\left[e^{c_2 u \sum_{n=1}^N \delta_n} \mathbf{1}_{\{N\in\tau\}}\right] \leqslant Z^{\mathrm{a}}_{n,h} = \mathbf{E}\left[\exp\left(u \sum_{n=1}^N \delta_n\right) e^{H^{\mathrm{a}}_{N,h^{\mathrm{a}}_c}}\right] \leqslant \mathbf{E}\left[e^{c_2 u \sum_{n=1}^N \delta_n} \mathbf{1}_{\{N\in\tau\}}\right], \quad (5.3.44)$$

which gives more directly Theorem 5.2.2. Proposition 5.3.5 could be the key to a future study of the disordered system via annealed bounds, as it was the case in Chapter 4, with Proposition 4.3.2.

Propositions 5.3.5-5.3.1 and Theorem 5.2.2, in view of the results in Chapter 4, give the hope of proving that the Harris criterion (irrelevance if $\alpha < 1/2$, relevance if $\alpha > 1/2$) holds if $m_\Upsilon < \infty$, especially in terms of critical point shifts. We do not develop the analysis in the direction of the study of the critical point shift, which is still open and would require a stronger knowledge of the annealed system, even if Chapter 4 give some clues to solve this problem.

5.4. Proof of the results on the disordered system

5.4.1. The case of summable correlations. As we saw in Section 5.3, the annealed model is well-defined only under the Assumption 5.1.3 of summable correlations.

5.4.1.1. *The smoothing phenomenon, proof of Theorem 5.2.3.* We give here the proof of this Theorem for the sake of completeness, but it is very similar to what is done in [GT06] for the case of independent variable. The main idea is to stand at $h_c(\beta)$ ($h_c(\beta) \geqslant h_c^{\mathrm{a}}(\beta) > -\infty$ since the correlations are summable), and to get a lower bound for $\mathtt{F}(\beta, h_c(\beta))$ involving $\mathtt{F}(\beta, h)$, by choosing a suitable localization strategy for the polymer to adopt, and computing the contribution to the free energy of this strategy. This is inspired by what is done in [Gia07, Chapter 6] to bound the critical point of the random copolymer model. More precisely one gives a definition of a "good block", supposed to be favorable to localization in that the ω_i are sufficiently positive, and analyses the contribution of the strategy of aiming only at the good blocks.

Let us fix some $l \in \mathbb{N}$ (to be optimized later), take $n \in \mathbb{N}$ and let $\mathcal{I} \subset \{1, \dots, n\}$, which is supposed to denote the set of indexes corresponding to "good blocks" of size l, and we order its elements: $\mathcal{I} = \{i_p\}_{p\in\mathbb{N}}$ with $i_1 < i_2 < \cdots$. We then divide a system of size nl into n blocks of size l, and denote $Z_{l,h}^{\omega,(k)}$ the (pinned) partition function on the k^{th} block of size l, that is $Z_{l,h}^{\omega,(k)} = Z_{l,h}^{\theta^{(k-1)l}\omega,\beta}$ (θ being the shift operator, *i.e.* $\theta^p\omega := (\omega_{n+p})_{n\geqslant 0}$).

For any fixed ω and $n \in \mathbb{N}$, we denote $\mathcal{I}_n = \mathcal{I} \cap [0, n]$, so that targeting only the blocks in $\mathcal{I}_n$ gives

$$Z_{nl,h}^{\omega,\beta} \geqslant \mathrm{K}((n - i_{|\mathcal{I}_n|})l) \prod_{k=1}^{|\mathcal{I}_n|} \mathrm{K}((i_k - i_{k-1} - 1)l) \prod_{k\in\mathcal{I}_n} Z_{l,h}^{\omega,\beta,(k)}, \tag{5.4.1}$$

with the convention that $\mathrm{K}(0) := 1$. Then if $\varepsilon > 0$ is fixed (meant to be small), taking l large enough so that $\log \mathrm{K}(kl) \geqslant -(1+\varepsilon)(1+\alpha)\log(kl)$ for all $k \geqslant 0$, one has

$$\frac{1}{nl}\log Z_{nl,h}^{\omega,\beta}$$
$$\geqslant \frac{1}{nl}\sum_{k\in\mathcal{I}_n}\log Z_{l,h}^{\omega,(k)} - (1+\varepsilon)(1+\alpha)\frac{1}{nl}\left(\log((n-i_{|\mathcal{I}_n|})l) + \sum_{k=1}^{\mathcal{I}_n}\log((i_k-i_{k-1}-1)l)\right)$$
$$\geqslant \frac{1}{n}\sum_{k\in\mathcal{I}_n}\frac{1}{l}\log Z_{l,h}^{\omega,(k)} - (1+\varepsilon)(1+\alpha)\frac{1}{l}\frac{|\mathcal{I}_n|+1}{n}\log\left(\frac{n}{|\mathcal{I}_n|+1}-1\right), \quad (5.4.2)$$

where we used Jensen inequality in the last inequality (which only means that the entropic cost of targeting the blocks of $\mathcal{I}_n$ is maximal when all its elements are equally distant). Note that (5.4.2) is very general, and it is useful to derive some results on the free energy, choosing the appropriate definition for an environment to be favorable (and thus the blocks to be aimed), and the appropriate size of the blocks (see Section 5.4.2 for another example of application).

We fix $\beta > 0$, and set $u := h - h_c(\beta)$. Then, fix $\varepsilon > 0$, and define the events

$$\mathcal{A}_l^{(k)} = \left\{ Z_{l,h_c(\beta)}^{\omega,(k)} \geqslant \exp\left((1-\varepsilon)l\,\mathtt{F}(\beta,h_c(\beta)+u)\right)\right\}, \quad (5.4.3)$$

and define $\mathcal{I}_n$ the set of favorable blocks

$$\mathcal{I}(\omega) := \{k \in \mathbb{N} \ : \ \mathcal{A}_l^{(k)} \text{ is verified}\}. \quad (5.4.4)$$

Then taking l large enough so that (5.4.2) is valid for the ε chosen above, one has

$$\frac{1}{nl}\log Z_{nl,h}^{\omega,\beta} \geqslant \frac{|\mathcal{I}_n|}{n}(1-\varepsilon)\mathtt{F}(\beta,h_c(\beta)+u) - (1+\varepsilon)(1+\alpha)\frac{1}{l}\frac{|\mathcal{I}_n|+1}{n}\log\left(\frac{n}{|\mathcal{I}_n|+1}-1\right). \quad (5.4.5)$$

We also note $p_l := \mathbb{P}(\mathcal{A}_l^{(1)}) = \mathbb{P}(1 \in \mathcal{I}_n)$, so that one has that $\mathbb{P}$-a.s. $\lim_{n\to\infty}\frac{1}{n}|\mathcal{I}_n| = p_l$, thanks to Birkhoff's Ergodic Theorem (cf. [Nad98, Chap. 2]). Then, letting n go to infinity, one has

$$0 = \mathtt{F}(\beta,h_c(\beta)) \geqslant p_l(1-\varepsilon)\mathtt{F}(\beta,h_c(\beta)+u) - (1+\varepsilon)(1+\alpha)p_l\frac{1}{l}\log(p_l^{-1}-1)$$
$$\geqslant p_l\left((1-\varepsilon)\mathtt{F}(\beta,h_c(\beta)+u) + (1+2\varepsilon)(1+\alpha)\frac{1}{l}\log(p_l)\right), \quad (5.4.6)$$

the second inequality coming from the fact that p_l^{-1} is large for large l.

We now give a bound on p_l, with the same change of measure technique used in the proof of Lemma A.2.1. We consider the measure $\bar{\mathbb{P}}$ on $\{\omega_1,\dots,\omega_l\}$ which is absolutely continuous with respect to $\mathbb{P}$, and consists in translating the ω_i's of u/β, without changing the correlation matrix Υ. Then, using that $l^{-1}\log Z_{l,h_c(\beta)}^{\omega,\beta}$ converges to $\mathtt{F}(\beta,h_c(\beta)+u)$ in $\bar{\mathbb{P}}$-probability as l goes to infinity, we have that $\bar{\mathbb{P}}(\mathcal{A}_l^{(1)}) \geqslant 1-\varepsilon$, for l sufficiently large. We recall the classic entropy inequality

$$\mathbb{P}(\mathcal{A}) \geqslant \bar{\mathbb{P}}(\mathcal{A})\exp\left(-\frac{1}{\bar{\mathbb{P}}(\mathcal{A})}(\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P})+e^{-1})\right), \quad (5.4.7)$$

with $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P})$ the relative entropy of $\bar{\mathbb{P}}$ w.r.t. $\mathbb{P}$. As in Appendix A-(A.2.3), one computes $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) = \frac{u^2}{2\beta^2}\langle \Upsilon^{-1}\mathbf{1}_l, \mathbf{1}_l\rangle$, where $\mathbf{1}_l$ is the vector whose l elements are all equal to 1.

From Lemma A.1.1 one directly has that $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) = (1+o(1))\frac{u^2}{2\Upsilon_\infty\beta^2}\, l$, so that for l large one gets that

$$\frac{1}{l}\log p_l \geqslant -(1+\varepsilon)\frac{1}{l}(1-\varepsilon)^{-1}\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) \geqslant -\frac{1+2\varepsilon}{1-\varepsilon}\frac{u^2}{2\Upsilon_\infty\beta^2}. \tag{5.4.8}$$

This inequality, combined with (5.4.6), gives

$$\mathtt{F}(\beta, h_c(\beta)+u) \leqslant -\frac{1+2\varepsilon}{1-\varepsilon}(1+\alpha)\frac{1}{l}\log p_l \leqslant \left(\frac{1+2\varepsilon}{1-\varepsilon}\right)^2 \frac{1+\alpha}{2\Upsilon_\infty\beta^2}u^2, \tag{5.4.9}$$

which, thanks to the arbitrariness of ε, concludes the proof. □

5.4.1.2. *On the critical points at low temperature, proof of Proposition 5.2.4.* We now estimate the quenched critical point when β goes to ∞, in the case of non-negative summable correlations. The techniques are very similar to what is done in [Ton07], and we include it here for the sake of completeness.

As far as the lower bound is concerned, one uses a fractional moment method. Indeed, if one shows that $\frac{1}{N}\log\mathbb{E}[(Z_{N,h}^{\omega,\beta})^\gamma] \stackrel{N\to\infty}{\to} 0$, then using Jensen inequality, one gets that

$$\frac{1}{N}\mathbb{E}\log Z_{N,h}^{\omega,\beta} = \frac{1}{\gamma N}\mathbb{E}\log(Z_{N,h}^{\omega,\beta})^\gamma \leqslant \frac{1}{\gamma N}\log\mathbb{E}[(Z_{N,h}^{\omega,\beta})^\gamma], \tag{5.4.10}$$

and therefore $\mathtt{F}(\beta,h)=0$, giving a lower bound on $h_c(\beta)$.

We fix $1>\gamma>\frac{1}{1+\alpha}$ (this choice will be clear in a moment), and we estimate $(Z_{N,h}^{\omega,\beta})^\gamma$, using that for any $\gamma<1$ one has $(\sum a_i)^\gamma \leqslant \sum (a_i)^\gamma$, if the a_i's are non-negative:

$$(Z_{N,h}^{\omega,\beta})^\gamma \leqslant \sum_{m=1}^{N}\sum_{i_1<i_2<\ldots<i_m=N}\prod_{k=1}^{m} e^{\gamma h+\beta\gamma\omega_{i_k}}\mathrm{K}(i_k-i_{k-1})^\gamma. \tag{5.4.11}$$

Then we write $C_\gamma := \sum_{n\in\mathbb{N}}\mathrm{K}(i_k-i_{k-1})^\gamma < \infty$ (finiteness is provided by the assumption $\gamma(1+\alpha)>1$), so that defining $\mathrm{K}_\gamma(n) := C_\gamma^{-1}\mathrm{K}(n)^\gamma$, one gets a recurrent renewal process τ_γ, with inter-arrival distribution given by $\mathrm{K}_\gamma(\cdot)$. One obtains the bound

$$\begin{aligned}\mathbb{E}(Z_{N,h}^{\omega,\beta})^\gamma &\leqslant \mathbb{E}\mathbf{E}_\gamma\left[\exp\left(\sum_{n=1}^{N}(\gamma h+\log(C_\gamma)+\gamma\beta\omega_i)\delta_n\right)\right]\\ &\leqslant \mathbf{E}_\gamma\left[\exp\left(\left(\gamma h+\log(C_\gamma)+\frac{\gamma^2\beta^2}{2}+\gamma^2\beta^2\sum_{k\in\mathbb{N}}\rho_k\right)\sum_{n=1}^{N}\delta_n\right)\right],\end{aligned} \tag{5.4.12}$$

where we used a uniform bound on the correlations for the second inequality, as in (5.3.1). Therefore, thanks to (5.4.10), one has that $h_c(\beta)\geqslant -\frac{\gamma\beta^2}{2}\Upsilon_\infty - \log(C_\gamma)$, which gives the lower bound for Proposition 5.2.4, since γ can be chosen arbitrarily close to $(1+\alpha)^{-1}$.

To get the upper bound, one uses a localization strategy for the polymer, targeting regions where the empirical mean of the ω_i's is large. One uses the same strategy as in Section 5.4.1, that is to get a lower bound on $\mathtt{F}(\beta,h)$ involving $\mathtt{F}(\beta,h+u)$, and using (5.4.2). For any fixed $\varepsilon>0$, provided that l is large enough, and analogously with (5.4.6) one has

$$\mathtt{F}(\beta,h)\geqslant p_l\left((1-\varepsilon)\mathtt{F}(\beta,h+u)+(1+2\varepsilon)(1+\alpha)\frac{1}{l}\log p_l\right), \tag{5.4.13}$$

with

$$p_l:=\mathbb{P}\left(Z^{\omega,\beta}_{l,h}\geqslant e^{(1-\varepsilon)l\mathtt{F}(\beta,h+u)}\right)\;\geqslant\;\exp\left(-l\frac{1+2\varepsilon}{1-\varepsilon}\frac{u^2}{2\Upsilon_\infty\beta^2}\right), \tag{5.4.14}$$

as shown in (5.4.8). If one sets $u_0:=(1-\eta)(1+\alpha)^{-1}\Upsilon_\infty\beta^2$, with $1-\eta:=\frac{1-\varepsilon}{1+2\varepsilon}$ (that can be made arbitrarily close to 1 by taking ε small), one finally has

$$\mathtt{F}(\beta,h)\geqslant(1-\varepsilon)p_l\left(\mathtt{F}(\beta,h+u_0)-\frac{\Upsilon_\infty}{2(1+\alpha)}\beta^2\right). \tag{5.4.15}$$

Then for any $t\in\mathbb{R}$, by imposing all contacts, one gets that $\frac{1}{N}\log Z^{\omega}_{N,t}\geqslant\log \mathrm{K}(1)+t+\frac{1}{N}\sum_{i=1}^N\omega_i$, and in the end $\mathtt{F}(\beta,t)\geqslant t+\log\mathrm{K}(1)$. Therefore, recalling the definition of u_0, one has

$$\mathtt{F}(\beta,h)\geqslant(1-\varepsilon)p_l\left(h+(1-2\eta)\frac{\Upsilon_\infty}{2(1+\alpha)}\beta^2+\log\mathrm{K}(1)\right), \tag{5.4.16}$$

which yields that $h_c(\beta)\leqslant -(1-3\eta)\frac{\Upsilon_\infty}{2(1+\alpha)}\beta^2$ if β is large. As η can be made as small as desired, it gives the upper bound for Proposition 5.2.4. □

Note that the proof shows that for β large, if h is close to $-\frac{\beta^2}{2(1+\alpha)}\Upsilon_\infty$, the right strategy in order to localize is to target large regions where the empirical mean of the ω_i's is close to $\frac{\beta^2}{1+\alpha}\Upsilon_\infty$.

5.4.2. The case of non-summable correlations, proof of Theorem 5.2.5. This Theorem is the non-hierarchical analogue of Theorem 1.3.9, which is proved in [BT11, Sec.4]. But because there are some technical differences, we include the proof here for the sake of completeness.

Proof The idea is to lower bound the partition function by exhibiting a suitable localization strategy for the polymer, that consists in aiming at "good" blocks, *i.e.* blocks where ω_i is very large. We then compute the contribution to the free energy of this strategy, in the spirit of (5.4.2). For $\zeta<1$ (non-summable correlations), it is a lot easier to find such large block (see Lemma 5.4.1 to be compared with the independent case). In this sense, the behavior of the system is qualitatively different from the $\zeta>1$ case.

Clearly, it is sufficient to prove the claim for h negative and large enough in absolute value. Let us fix h negative with $|h|$ large and take $l=l(h)\in\mathbb{N}$, to be chosen later. Recall (5.4.2), and define

$$\mathcal{A}_l^{(k)}:=\left\{\text{for all } i\in[(k-1)l,kl]\cap\mathbb{N},\ \text{one has } \beta\omega_i+h\geqslant|h|\right\}, \tag{5.4.17}$$

and as in Section 5.4.1 the set of favorable blocks $\mathcal{I}_n$, and $p_l := \mathbb{P}(\mathcal{A}_l^{(1)}) = \mathbb{P}(1 \in \mathcal{I}_n)$.

One notices that $Z_{l,h}^{\omega,(k)} \geqslant Z_{l,|h|}^{\mathrm{pur}}$ for all $k \in \mathcal{I}_n$, so that provided that l is large enough, one has $l^{-1} \log Z_{l,|h|}^{\mathrm{pin,pure}} \geqslant \frac{1}{2}\mathtt{F}(|h|)$. Therefore, from (5.4.2), if l is large enough so that the above inequality is valid, and letting n goes to infinity, we get $\mathbb{P}$-a.s.

$$\mathtt{F}(\beta,h) \geqslant \frac{p_l}{2}\mathtt{F}(|h|) - Cp_l\frac{1}{l}\log(p_l^{-1}-1) \geqslant p_l\left(c|h| + c'\frac{1}{l}\log p_l\right), \tag{5.4.18}$$

where we used that $\mathbb{P}$-a.s. $\lim_{n\to\infty}\frac{1}{n}|\mathcal{I}_n| = p_l$, because of Birkhoff's Ergodic Theorem (cf. [Nad98, Chap. 2]). The second inequality comes from the fact that, for $|h| \geqslant 1$, one has $\mathtt{F}(|h|) \geqslant cst.\,|h|$, and that p_l^{-1} is large if l is large.

It then remains to estimate the probability p_l.

Lemma 5.4.1. *Under Assumption 5.1.4, there exist two constants $c, C > 0$ such that for every $l \in \mathbb{N}$ and $A \geqslant C(\log l)^{1/2}$ one has*

$$\mathbb{P}\left(\forall i \in \{1,\dots,l\},\ \omega_i \geqslant A\right) \geqslant c^{-1}\exp\left(-cA^2 l^{\zeta}\right). \tag{5.4.19}$$

From this Lemma, that we prove in Appendix A (Lemma A.2.1), and choosing l such that $\sqrt{\log l} \leqslant 2|h|/(C\beta)$, one gets that

$$p_l = \mathbb{P}\big(\forall i \in \{1,\dots,l\},\ \omega_i \geqslant 2|h|/\beta\big) \geqslant c^{-1}\exp\left(-cl^{\zeta}h^2/\beta^2\right). \tag{5.4.20}$$

Then in view of (5.4.18) one chooses $l = (\bar{C}|h|/\beta^2)^{1/(1-\zeta)}$ (this is compatible with the condition $\sqrt{\log l} \leqslant 2|h|/(C\beta)$ if $|h|$ is large enough) so that one gets $c|h| + c'l^{-1}\log p_l \geqslant c|h|/2 \geqslant c/2$, provided that $\bar{C}$ is large enough. And (5.4.18) finally gives with this choice of l

$$\mathtt{F}(\beta,h) \geqslant cst.\,\exp\left(-cl^{\zeta}h^2/\beta^2\right) \geqslant cst.\,\exp\left(-c'|h|\left(|h|/\beta^2\right)^{1/(1-\zeta)}\right). \tag{5.4.21}$$

□

CHAPTER 6

Sharp critical behavior in random correlated $\{-1,0\}$-environment

6.1. Description of the model and preliminary results

6.1.1. The model. We recall briefly the definition of the polymer measure. Let $\tau := \{\tau_n\}_{n\geqslant 0}$ be a recurrent renewal sequence (as defined in Section 1.1.2), whose law is denoted by $\mathbf{P}$, and with inter-arrival distribution denoted by $\mathrm{K}(\cdot)$. We assume that $\mathrm{K}(\cdot)$ satisfies

$$\mathrm{K}(n) := \mathbf{P}(\tau_1 = n) = (1+o(1))\frac{c_K}{n^{1+\alpha}}, \tag{6.1.1}$$

for some $\alpha > 0$, $\alpha \neq 1$ (again, the assumption $\alpha \neq 1$ does not hide anything). We assume also for simplicity that $\mathrm{K}(n) > 0$ for all $n \in \mathbb{N}$. We use the notation

$$\bar{\mathrm{K}}(n) := \mathbf{P}(\tau_1 > n) = \sum_{i=n+1}^{\infty} \mathrm{K}(i). \tag{6.1.2}$$

With a slight abuse of notation, τ also denotes the set $\{k \in \mathbb{N} \mid \tau_n = k \text{ for some } n\}$. As explained in the Introduction, the set τ can be thought of as the set of return times to its departure point (call it 0) of some random walk X on some state space, say $\mathbb{Z}^d$. The graph of the random walk $(k, X_k)_{k\in[0,N]}$ is interpreted as a one-dimensional polymer chain living in a $(d+1)$-dimensional space, and interacting with the defect line $[0,N]\times\{0\}$.

Given a sequence $\omega = (\omega_n)_{n\in\mathbb{N}}$ of real numbers (the environment), $h \in \mathbb{R}$ (the pinning parameter) and $\beta \geqslant 0$ (the inverse temperature), we define the sequence of polymer measures $\mathbf{P}^{\omega,\beta}_{N,h}$, $N \in \mathbb{N}$ as follows

$$\frac{\mathrm{d}\mathbf{P}^{\omega,\beta}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z^{\omega,\beta}_{N,h}} \exp\left(\sum_{n=1}^{N} (h+\beta\omega_n)\mathbf{1}_{\{n\in\tau\}}\right), \tag{6.1.3}$$

where $Z^{\omega,\beta}_{N,h} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(h+\beta\omega_n)\mathbf{1}_{\{n\in\tau\}}\right)\right]$ is the *partition function* of the system.

This definition is slightly different of the one used up to now, because we left aside the condition $\{N\in\tau\}$. This is the polymer measure with *free* boundary condition, whereas the one defined in Section 1.4, is with *pinned* boundary condition. Both the *free* and *pinned* partition functions are useful, and we keep the superscript "pin" to distinguish the *pinned* one.

We study the properties of $\tau \cap [0,N]$ under the polymer measure $\mathbf{P}^{\omega,\beta}_{N,h}$ for large values of N. In this Chapter, we focus on a particular type of environment ω, constructed as follows:

Let $\widehat{\tau} = (\widehat{\tau}_n)_{n \geqslant 0}$, $\widehat{\tau}_0 = 0$ be a recurrent renewal process (let $\widehat{\mathbf{P}}$ denote its law), with inter-arrival law $\widehat{\mathrm{K}}(\cdot)$ that satisfies

$$\widehat{\mathrm{K}}(n) := \widehat{\mathbf{P}}(\widehat{\tau}_1 = n) = (1+o(1))\frac{\widehat{c}_K}{n^{1+\widetilde{\alpha}}}, \tag{6.1.4}$$

for some $\widetilde{\alpha} > 1$. These conditions ensure that $\widehat{\mathbf{E}}[\widehat{\tau}_1] < \infty$ which is crucial to ensure that the free energy is self-averaging (see below). Then let $(X_i)_{i \geqslant 1}$ be a sequence of *i.i.d.* random variables (with law $\mathbb{P}$ independent of $\widehat{\mathbf{P}}$) satisfying

$$\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = -1) = 1/2 \tag{6.1.5}$$

and set

$$\omega_n = X_i, \quad \forall n \in (\tau_{i-1}, \tau_i]. \tag{6.1.6}$$

For later convenience we may use another construction to get ω. We start from the renewal process $\widetilde{\tau}$ (let $\widetilde{\mathbf{P}}$ denote its law), with inter-arrival law $\widetilde{\mathrm{K}}(\cdot)$ given by

$$\widetilde{\mathbf{P}}(\widetilde{\tau}_1 = n) := \widetilde{\mathrm{K}}(n) := \sum_{k=1}^{\infty} 2^{-k}\widehat{\mathbf{P}}(\widehat{\tau}_k = n). \tag{6.1.7}$$

One can check (using Proposition 6.A.2 in the appendix), that

$$\widetilde{\mathrm{K}}(n) = (1+o(1))\frac{2\widehat{c}_K}{n^{1+\widetilde{\alpha}}}. \tag{6.1.8}$$

Then one sets

$$\omega_i = \begin{cases} 0 & \text{if there exists some } n \geqslant 0 \text{ such that } i \in (\widetilde{\tau}_{2n}, \widetilde{\tau}_{2n+1}], \\ -1 & \text{if there exists some } n \geqslant 0 \text{ such that } i \in (\widetilde{\tau}_{2n+1}, \widetilde{\tau}_{2n+2}]. \end{cases} \tag{6.1.9}$$

This construction gives an environment with the same law as the first one, conditioned to $X_1 = 0$, and this conditioning is harmless for our purpose.

Remark 6.1.1. The reason to choose such an environment is that it is a simple framework to study the influence of long-range power-law correlations for disordered pinning models. One can compute the correlation easily: for any $i \in \mathbb{N}$, $k \geqslant 0$

$$\mathbb{C}\mathrm{ov}(\omega_i, \omega_{i+k}) = \frac{1}{4}\widehat{\mathbf{P}}\left(\exists n \in \mathbb{N},\ (i; i+k) \in (\widehat{\tau}_{n-1}, \widehat{\tau}_n]^2\right). \tag{6.1.10}$$

The latter term is equal to

$$\frac{1}{4}\sum_{l=1}^{i-1}\widehat{\mathbf{P}}(l \in \widehat{\tau})\widehat{\mathbf{P}}(\widehat{\tau}_1 > k+i-l) \overset{k\to\infty}{\sim} \frac{\widehat{c}_K}{4\widetilde{\alpha}}\sum_{l=1}^{i-1}\widehat{\mathbf{P}}(l \in \widehat{\tau})(i-l+k)^{-\widetilde{\alpha}}. \tag{6.1.11}$$

One uses the renewal theorem to get that $\widehat{\mathbf{P}}(l \in \widehat{\tau}) \sim_{l\to\infty} \widehat{\mathbf{E}}[\widehat{\tau}_1]^{-1}$, so that taking i large, one has that $\mathrm{Cov}(\omega_i, \omega_{i+k})$ is of order $k^{1-\widetilde{\alpha}}$, which decays slower and slower as $\widetilde{\alpha}$ is taken close to 1.

The reason why we impose $\widetilde{\alpha} > 1$ is that for $\widetilde{\alpha} < 1$ the model is somewhat trivial. Indeed, in that case, the infinite-volume quenched (averaged) free energy has the same critical behavior as for the non-disordered model. Moreover, in this case one loses the ergodicity of the environment sequence and the free energy is no more a self-averaging quantity (*i.e.* the almost sure limit in (6.1.15) does not exist).

6.1.2. Reminder of the homogeneous model. Before giving our results, we recall some facts about the easier case $\beta = 0$, that is the *homogeneous pinning model* of Section 1.1.2. Recall the definition of the polymer measure with *free* boundary condition (different from the one with *pinned* boundary condition given in Section 1.1.2):

$$\frac{\mathrm{d}\mathbf{P}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z_{N,h}} \exp\left(\sum_{n=1}^{N} h\mathbf{1}_{\{n\in\tau\}}\right), \tag{6.1.12}$$

where $Z_{N,h} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} h\mathbf{1}_{\{n\in\tau\}}\right)\right]$ is the *partition function* with free boundary condition.

As far as the free energy is concerned, it is equivalent to work with the *free* or *pinned* boundary condition. We recall that the free energy (of the homogeneous system) is $\mathtt{F}(h) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h}$, and has a phase transition at the critical point $h_c = 0$, with critical exponent $\nu^{\mathrm{pur}} = 1 \vee 1/\alpha$ (recall Theorem 1.1.6).

We know from Section 1.1.2 that the number of contact points $|\tau \cap [0,N]|$ under the polymer measure is of order N for $h > 0$ (and also for $h = 0$, $\alpha > 1$ by the renewal Theorem 1.1.7, [Asm03, Chapter 1, Theorem 2.2]), and $o(N)$ in the other cases. In fact one can get a precise statement on the number of contacts under the homogeneous polymer measure, see Proposition 1.1.9, that we recall here

Proposition 6.1.2. *(Asymptotic behavior of the path measure)*

- *When $h < 0$, for all k one has*

$$\lim_{N\to\infty} \mathbf{P}_{N,h}(|\tau \cap [0,N]| = k+1) = (1-e^h)e^{kh}. \tag{6.1.13}$$

- *When $h = 0$ and $\alpha \in (0,1)$, one has that under $\mathbf{P} = \mathbf{P}_{N,h=0}$*

$$N^{-\alpha}|\tau \cap [0,N]| \Rightarrow \mathcal{A}_\alpha, \tag{6.1.14}$$

  where A_α is the inverse of an α-stable law.

6.1.3. Preliminary results on the disordered model. This Chapter presents results for our inhomogeneous model that exhibits sharp contrast with Theorem 1.1.6 and Proposition 6.1.2. We show that disorder modifies the phase transition between the localized phase (order N contacts, positive free energy), and the delocalized phase, ($O(1)$ contacts, zero free energy). Due to the correlations present in the environment, this phenomenon is very different from what was observed for the *i.i.d.* environment case.

In order to state our results, we first need to show the existence of the free energy for the inhomogeneous model.

Proposition 6.1.3. *The limit*

$$\mathtt{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z^{\omega,\beta}_{N,h}, \tag{6.1.15}$$

exists $\mathbb{P}\times\widehat{\mathbf{P}}$ *almost surely. One has* $\mathtt{F}(\beta,h) = \lim_{N\to\infty} \frac{1}{N}\widehat{\mathbf{E}}\,\mathbb{E}\left[\log Z^{\omega,\beta}_{N,h}\right]$.

Proof One (re-)introduces (see Section 1.4) the partition function with *pinned* boundary condition:

$$Z^{\omega,\beta,\mathrm{pin}}_{N,h} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(h+\beta\omega_n)\mathbf{1}_{\{n\in\tau\}}\right)\mathbf{1}_{\{N\in\tau\}}\right]. \tag{6.1.16}$$

Note that (see in [Gia07, Equation (4.25)] and its proof) there exists a constant $c>0$ such that

$$cN^{-1}e^{-\beta+h}Z^{\omega,\beta}_{N,h} \leqslant Z^{\omega,\beta,\mathrm{pin}}_{N,h} \leqslant Z^{\omega,\beta}_{N,h}, \tag{6.1.17}$$

so that it is equivalent to work with Z or Z^{pin} as far as $\mathtt{F}$ is concerned. Then one notices that

$$Z^{\omega,\beta,\mathrm{pin}}_{N+M,h} \geqslant \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(h+\beta\omega_n)\mathbf{1}_{\{n\in\tau\}}\right)\mathbf{1}_{\{N\in\tau,\ (N+M)\in\tau\}}\right] = Z^{\omega,\beta,\mathrm{pin}}_{N,h} Z^{\theta^N\omega,\beta,\mathrm{pin}}_{M,h}, \tag{6.1.18}$$

where θ is the shift operator, *i.e.* $\theta^N\omega := (\omega_{n+N})_{n\,\geqslant\,0}$. So that in particular

$$\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_{N+M},h} \geqslant \log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h} + \log Z^{\theta^{\widehat{\tau}_N}\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_{M+N}-\widehat{\tau}_N,h}. \tag{6.1.19}$$

Note that from the renewal construction of the environment, the two terms on the right hand-side are independent and that the law of the second one is the same as the law of $\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_M,h}$. Therefore one can use Kingman's superadditive ergodic Theorem [Kin73, Th.1], or simply the law of large numbers (like it is done in [Gia07, Sec.4.2]) to conclude that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h} &= \lim_{N\to\infty}\frac{1}{N}\widehat{\mathbf{E}}\,\mathbb{E}\left[\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h}\right]\\ &= \sup_{N\,\geqslant\,0}\frac{1}{N}\widehat{\mathbf{E}}\,\mathbb{E}\left[\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h}\right] =: \bar{\mathtt{F}}(\beta,h).\end{aligned} \tag{6.1.20}$$

Then the law of large numbers for $\widehat{\tau}$ gives that

$$\mathtt{F}(\beta,h) = \lim_{N\to\infty}\frac{1}{\widehat{\tau}_N}\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h} = \lim_{N\to\infty}\frac{N}{\widehat{\tau}_N}\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\widehat{\mathbf{E}}\left[\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h}\right] = \frac{1}{\widehat{\mathbf{E}}\left[\widehat{\tau}_1\right]}\bar{\mathtt{F}}(\beta,h). \tag{6.1.21}$$

Note that we have proved only convergence almost surely along the random subsequence $\widehat{\tau}$. Then one can use standard arguments to show that convergence holds for the whole sequence and also in $\mathbb{L}_1$ (details are omitted).

□

A matter of interest for disordered pinning models in the *i.i.d.* environment case is how the free-energy compares with the annealed free-energy defined by

$$\mathtt{F}^{\mathrm{a}}(\beta,h) := \lim_{N\to\infty}\frac{1}{N}\log\widehat{\mathbf{E}}\,\mathbb{E}\left[Z_{N,h}^{\omega,\beta}\right]. \tag{6.1.22}$$

Jensen's inequality gives that $\mathtt{F}(\beta,h) \leqslant \mathtt{F}^{\mathrm{a}}(\beta,h)$. In our case this bound does not give much information. Indeed,

$$Z_{N,h} \geqslant \widehat{\mathbf{E}}\,\mathbb{E}\left[Z_{N,h}^{\omega,\beta}\right] \geqslant \frac{1}{2}\widehat{\mathbf{P}}(\widehat{\tau}_1 > N)Z_{N,h}. \tag{6.1.23}$$

As $\widehat{\mathbf{P}}[\widehat{\tau}_1 > N]$ behaves like $N^{-\widetilde{\alpha}}$ for N large, this factor does not affect the limit after taking the log and dividing by N. Therefore, $\mathtt{F}^{\mathrm{a}}(\beta,h) = \mathtt{F}(h)$ and the annealed bound for the free-energy becomes simply

$$\mathtt{F}(\beta,h) \leqslant \mathtt{F}(h), \tag{6.1.24}$$

which is obvious from monotonicity in ω of $Z_{N,h}^{\omega,\beta}$. This contrasts with the case of *i.i.d.* environment, for which the annealed bound gives a non-trivial upper-bound on the free-energy.

6.2. Main results

6.2.1. Statement of the results and comparison with the previous literature. What we show concerning the free-energy of our disordered model is that it is positive for every positive h (*i.e.* that the presence of negative ω is not sufficient to repel the trajectories from the defect line). Moreover, we are able to compute the asymptotics of the free-energy around $h = 0_+$ up to a constant.

Theorem 6.2.1. *There exist two constants $C_1 > 0$ and $C_2 > 0$ (depending on β), such that for any $h \in (0,1)$, one has*

$$C_1 h^{\frac{\widetilde{\alpha}}{(1\wedge\alpha)}}|\log h|^{1-\widetilde{\alpha}} \leqslant \mathtt{F}(\beta,h) \leqslant C_2 h^{\frac{\widetilde{\alpha}}{(1\wedge\alpha)}}|\log h|^{1-\widetilde{\alpha}}. \tag{6.2.1}$$

Remark 6.2.2. Note that in the statement of the theorem the constants depend on β. This will be the case of many constants introduced during the proof, and we may not mention it, as in the sequel we always consider β as a fixed positive parameter.

Our second result is that at the critical point $h = 0$, the trajectories are strictly delocalized, in the sense that typical trajectories have only finitely many returns to zero.

Theorem 6.2.3. *The sequence of law $(\nu_N)_{N\geqslant 0}$ on $\mathbb{N}$ defined by*

$$\nu_N(A) := \mathbf{P}_{N,h=0}^{\beta,\omega}(|\tau\cap[0,N]| \in A), \tag{6.2.2}$$

(the laws of the number of contact under $\mathbf{P}_{N,h=0}^{\beta,\omega}$ is tight for almost every realization of ω.

We prove this result in Section 6.6.1, and actually we get a more precise result in Corollary 6.6.2 and Proposition 6.6.3 that we sum up as follows.

Proposition 6.2.4. *For almost every ω, for any $\varepsilon > 0$ there exists $a_0 = a_0(\omega,\beta,\varepsilon) \in \mathbb{R}$ such that for $a \geqslant a_0$ and $a \leqslant N^{\frac{(1/\alpha)\wedge\alpha}{\widetilde{\alpha}}-\varepsilon}$ one has*

$$a^{-\varepsilon-\frac{\widetilde{\alpha}(\alpha+1)-1}{1\wedge\alpha}} \leqslant \mathbf{P}^{\omega,\beta}_{N,h=0}\left(|\tau\cap[0,N]| = a\right) \leqslant a^{\varepsilon-\widetilde{\alpha}(1\vee\alpha)}. \tag{6.2.3}$$

Remark 6.2.5. Proposition 6.2.4 indicates that the asymptotic law of the number of contacts under $\mathbf{P}^{\omega,\beta}_{N,h=0}$ has a power-decaying tail. This power-law behavior contrasts with what happens for $h < 0$, where the law of $|\tau\cap[0,N]|$ has an exponential tail. In view of how our results are obtained, we conjecture that it is the lower-bound given in Proposition 6.2.4 that is sharp.

It is instructive to compare the sharp estimates of Theorems 6.2.1 and 6.2.3 with the results available in the literature on other pinning models.

The first important remark is that the free energy critical exponent (call it ν, so that $\nu = \widetilde{\alpha}/(1\wedge\alpha)$, cf. (6.2.1)) is different both from the critical exponent of the homogeneous model: $\nu = 1/(1\wedge\alpha)$ (cf. Theorem 1.1.6) and from that of the disordered model with *i.i.d.* disorder. In the latter the critical exponent equals $\nu = 1/\alpha$ if $\alpha < 1/2$ and β small (regime of irrelevant disorder [Ale08, Ton08b]), and in all cases (every $\alpha,\beta > 0$) one observes a disorder induced *smoothing* of the free-energy curve near the critical point, that implies $\nu \geqslant 2$ when it exists [GT06] (in contrast, remark that the critical exponent in (6.2.1) can be smaller than 2 for our correlated model). Always concerning the critical exponent, let us also add that, up to now, precise asymptotics of the free-energy (close to the critical point) for pinning models had been proved only for the case of homogeneous (or weakly inhomogeneous, *i.e.* periodic) environment (Theorem 1.1.6), and for the mentioned case of *i.i.d.* environment, $\alpha < 1/2$ and β small [Ale08, Ton08b, GT09] (we let aside [Ale09] where it is proved that first order transition occurs for a very special model).

A second important observation concerns the value of the critical point. In our model, it equals zero for the homogeneous model (and therefore for the annealed one) but also for the quenched model (for every $\alpha,\widetilde{\alpha},\beta$). This is in contrast with what happens for *i.i.d.* random environment: in that case, the critical point of the annealed model equals $h_c^{\mathrm{a}}(\beta) = -\log\mathbf{E}[e^{\beta\omega_1}]$. Also, for *i.i.d.* environment it is a crucial issue to know whether the critical point $h_c^{\mathrm{que}}(\beta)$ of the quenched model coincides or not with $h_c^{\mathrm{a}}(\beta)$: one has $h_c^{\mathrm{que}}(\beta) = h_c^{\mathrm{a}}(\beta)$ if $\alpha < 1/2$, β small [Ale08, Ton08b], and $h_c^{\mathrm{que}}(\beta) < h_c^{\mathrm{a}}(\beta)$ if $\alpha \geqslant 1/2$ (every $\beta > 0$, with sharp bounds on their difference in the limit of β small [AZ09, DGLT09, GLT10b, GLT11]); another situation where $h_c^{\mathrm{que}}(\beta) < h_c^{\mathrm{a}}(\beta)$ is $\alpha < 1/2$, β large [Ton08a].

Finally, we make some observations concerning the behavior of the trajectories at the critical point, given by Theorem 6.2.3. The exact behavior is known for the pure model (cf. Proposition 6.1.2), in the irrelevant disorder regime for *i.i.d.* disorder (see [Lac10]), but very little is known in the other cases (in [Lac10] it is shown that there are at most $N^{1/2+\varepsilon}$ contacts with large probability, this result being linked to the above mentioned free energy critical exponent bound $\nu \geqslant 2$). In contrast, in our model the number of contacts at the critical point is not directly related to the critical behavior of the free energy (see however Proposition 6.2.4). Note that up to now, for *i.i.d.* disordered pinning models, the best general bound one has for the number of contact points in the delocalized phase is $O(\log N)$ [Gia07, Sec.8.2] or [Mou], but in our case one has that it is $O(1)$.

We stress that the results we have are in complete contradiction with the Weinrib-Halperin criterion, which predicts disorder relevance/irrelevance for a large class of correlated systems, in the case where the correlation function decays like $r^{-\zeta}$, r being the displacement and $\zeta > 0$. Weinrib and Halperin figured in that case an extension of the so-called Harris criterion, which predicts that disorder is relevant if $\nu^{\text{pur}} < 2/(\zeta \wedge 1)$, and irrelevant if $\nu^{\text{pur}} < 2/(\zeta \wedge 1)$. Contrary to this prediction, we find in the present case that disorder is relevant, for every value of α (and thus of ν^{pur}) and for every decay of the correlation function $\zeta = \widetilde{\alpha} - 1$ (see Remark 6.1.1). As argued in details in Chapter 7, the important quantity is actually not the parameter ζ, but the size of the regions with $\omega \equiv 0$.

Concerning previous results on pinning models with correlated random environment, the only work we are aware of is [Poi12, Poi11], where a model with *finite-range* disorder correlations is studied. Let us also mention that the authors of [BL11, BT10, BS10, BS11] consider a random walk that is pinned on a second (quenched) random walk: this can also be seen as an example of a pinning model in a correlated environment. In both of this cases, however, the results one finds are similar to the ones of the *i.i.d.* environment case.

Remark 6.2.6. We have chosen to constrain ourselves only to a very particular setup for the sake on simplicity, however our results should hold with much greater generality for correlated environment $\omega \in \{-1, 0\}^{\mathbb{N}}$, see Chapter 7.

6.2.2. Strategy of the polymer under $\mathbf{P}^{\omega,\beta}_{N,h}$, ideas of the proofs. We give in this section an idea on the strategy the polymer adopts under the measure $\mathbf{P}^{\omega,\beta}_{N,h}$, this understanding clarifying the schemes of the proofs of Theorems 6.2.1 and 6.2.3.

The proof of Theorem 6.2.1 gives the right bounds on the free energy, but also a heuristic understanding of the typical behavior of the trajectories under the measure $\mathbf{P}^{\omega,\beta}_{N,h}$. The idea is that the polymer tends to pin on the regions where $\omega \equiv 0$, but only those of length larger than $h^{-\frac{1}{1\wedge\alpha}}|\log h|$, whereas they are repelled from the interface by any other region. Thus the idea to prove Theorem 6.2.1 is to estimate the contribution of all these different kinds of regions to the partition function. For the lower bound the strategy of targeting only regions of length larger than $h^{-\frac{1}{1\wedge\alpha}}|\log h|$ already gives the right result. To get the upper bound, one has to control the contribution of all the possible trajectories. Roughly, the argument is that one uses a coarse-graining argument to cut the system into blocks of finite size, and sees that if one block does not contain a region of length larger than $h^{-\frac{1}{1\wedge\alpha}}|\log h|$, then it does not contribute to the partition function.

A consequence of this observation is that the behavior of the free-energy near the critical point depends on the frequency of occurrence of regions of length $h^{-\frac{1}{1\wedge\alpha}}|\log h|$ where $\omega \equiv 0$. When $\widetilde{\alpha}$ is close to one, these regions occur relatively frequently, and for this reason the critical exponent for the free-energy in our model is close to the one of the homogeneous model. The two exponents get more and more different when $\widetilde{\alpha}$ grows and this type of regions becomes more rare.

Now, let us explain how we intend to prove Theorem 6.2.3 and Proposition 6.2.4. We bound from above the probability of having exactly a contacts before N under the measure $\mathbf{P}^{\omega,\beta}_{N,h=0}$ by considering the contribution of the different strategies for the polymer trajectory. For a trajectory τ, let $V_N^{\widehat{\tau}}(\tau)$ be the number of $\widehat{\tau}$-renewal stretches (we call $\widehat{\tau}$-stretch a segment of the type $(\widehat{\tau}_i,\widehat{\tau}_{i+1}]$) visited by τ:

$$V_N^{\widehat{\tau}}(\tau) := |\{i \in \mathbb{N} \mid \exists\, j \in \tau \cap [0,N], j \in (\widehat{\tau}_i,\widehat{\tau}_{i+1}]\}|. \tag{6.2.4}$$

We split the set of trajectories such that $\{|\tau \cap [0,N]| = a\}$ into two cases

- The trajectory τ visits a lot of $\widehat{\tau}$-stretches (say $V_N^{\widehat{\tau}}(\tau) \geqslant a^{\varepsilon}$),
- The trajectory τ visits only a few $\widehat{\tau}$-stretches ($V_N^{\widehat{\tau}}(\tau) < a^{\varepsilon}$).

One remarks that for any trajectory τ

$$\mathbb{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n\in\tau\}}}\right] \leqslant \left(\frac{1+e^{-\beta}}{2}\right)^{V_N^{\widehat{\tau}}(\tau)}, \tag{6.2.5}$$

where we recall that $\mathbb{E}$ denotes the average only on the values of $\{X_i\}_{i\in\mathbb{N}}$, *i.e.* on the disorder ω *conditionally on the realization of* $\widehat{\tau}$. Equation (6.2.5) tells us that visiting a lot of stretches has, in average, a strong energetic cost, and that therefore these trajectories do not contribute a lot to the partition function (this is formalized in the proof of Lemmas 6.6.4 and 6.6.8). In order to have a result that holds almost surely, however, one has to be careful in the way of using Borel-Cantelli Lemma.

For the second type of trajectories, on the other hand, we observe that in order not to visit many $\widehat{\tau}$-stretches, one has to put a lot of contacts in very few $\widehat{\tau}$-stretches, and this strategy has a large entropic cost (which is a priori not that easy to control). The most convenient way of doing this is to target sufficiently large stretches and put the contacts there. The key idea to estimate this is to realize that in order to visit the long stretches without having too many contacts before, τ has to grow much faster that it would typically do, in the sense that τ_x has to be larger than $x^{\widetilde{\alpha}(1\wedge\alpha)}$ (cf. Lemmas 6.6.5 and 6.6.9), which is much larger than what it would typically be, that is, $x^{(1\wedge\alpha)}$. We get this thanks to Lemma 6.A.4 which says that the first $\widehat{\tau}$-stretch of size $l \gg 1$ occurs at distance approximately $l^{\widetilde{\alpha}}$ from the origin. One also notices that targeting at the first jump a sufficiently large $\widehat{\tau}$-stretch and putting all the contacts in it already gives the right lower bound in Proposition 6.2.4, and we believe this is the right strategy for the polymer to adopt.

6.3. Lower bound on the free energy

We prove in this section the easier half of Theorem 6.2.1. Here and later we choose h small enough (then one can say that the results hold for all $h \in (0,1)$ by modifying the constant C_1). For practical reasons we compute a lower bound for $\bar{\mathtt{F}}(\beta,h)$, which according to (6.1.21) is equal to $\mathtt{F}(\beta,h)$ up to a multiplicative constant. Then, according to (6.1.20), it is sufficient to estimate $\widehat{\mathbf{E}}\,\mathbb{E}[\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h}]$ for a given N to get a lower bound.

We define M_N to be the size of the longest inter-arrival among the N first of the renewal $\widehat{\tau}$:

$$M_N := \max_{i\in[1,N]} (\widehat{\tau}_i - \widehat{\tau}_{i-1}), \tag{6.3.1}$$

and $i_{\max}$ to be the smallest index such that $\widehat{\tau}_i - \widehat{\tau}_{i-1} = M_N$. In order to get an explicit lower bound on $Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h}$, we consider the contribution of trajectories τ that have contacts with the defect line only in the interval $(\widehat{\tau}_{i_{\max}-1}, \widehat{\tau}_{i_{\max}}]$.

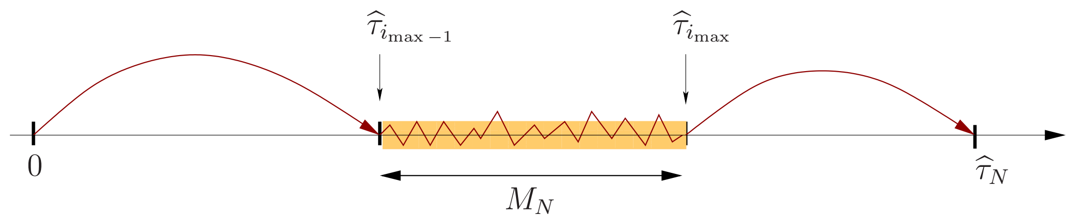

FIGURE 6.1. The strategy to get the lower bound is to target only the longest $\widehat{\tau}$-stretch, which is of size M_N, starting at $\widehat{\tau}_{i_{\max}-1}$ and ending at $\widehat{\tau}_{i_{\max}}$.

If $X_{i_{\max}} = 0$, then one has

$$Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h} \geqslant \mathrm{K}(\widehat{\tau}_{i_{\max}-1}) e^{h-\beta} Z^{\mathrm{pin}}_{M_N,h} \mathrm{K}(\widehat{\tau}_N - \widehat{\tau}_{i_{\max}}) e^{h-\beta}, \tag{6.3.2}$$

where $Z^{\mathrm{pin}}_{N,h}$ denotes the partition function of the homogeneous pinning model with pinned boundary condition (similar to (6.1.16) but with $\beta = 0$). Now note that our assumptions on $\mathrm{K}(\cdot)$ ensures that for N sufficiently large one has

$$\min(\mathrm{K}(\widehat{\tau}_{i_{\max}-1}), \mathrm{K}(\widehat{\tau}_N - \widehat{\tau}_{i_{\max}})) \geqslant \frac{1}{2} \widehat{c}_K (\widehat{\tau}_N)^{-(1+\alpha)}. \tag{6.3.3}$$

From all this one gets that there exists a constant C_3 (depending on β) such that

$$\widehat{\mathbf{E}}\mathbb{E}\left[\log Z^{\omega,\beta,\mathrm{pin}}_{\widehat{\tau}_N,h}\right] \geqslant \mathbb{P}(X_{i_{\max}} = 0)\left(C_3 - 2(1+\alpha)\widehat{\mathbf{E}}[\log \widehat{\tau}_N] + \widehat{\mathbf{E}} \log Z^{\mathrm{pin}}_{M_N,h}\right). \tag{6.3.4}$$

Then, one must estimate $\widehat{\mathbf{E}} \log Z^{\mathrm{pin}}_{M_N,h}$. We use the following estimate for Z^{pin}_N

Lemma 6.3.1. *There exists a constant C_4 such that for every $h \in (0,1)$, and every N,*

$$Z^{\mathrm{pin}}_{N,h} \geqslant C_4 N^{-1} e^{N\mathtt{F}(h)}. \tag{6.3.5}$$

Proof We first observe that for every pair of integers (n_1, n_2), decomposing over the first return time after n_1, one has

$$e^h Z_{n_1+n_2,h} \leqslant e^h Z_{n_1,h} e^h Z_{n_2,h}, \tag{6.3.6}$$

so that the sequence $\{\log(e^h Z_{N,h})\}_{N\in\mathbb{N}}$ is subadditive. Then one has that $\mathtt{F}(h)$ verifies $\mathtt{F}(h) = \inf_{N\in\mathbb{N}} \frac{1}{N} \log e^h Z_{N,h}$ and $Z_{N,h} \geqslant e^{-h} e^{N\mathtt{F}(h)}$ for all N. And therefore, one gets the result by using (6.1.17) which gives

$$Z^{\mathrm{pin}}_{N,h} \geqslant cN^{-1} Z_{N,h}. \tag{6.3.7}$$

□

Plugging the above result into (6.3.4) one has

$$\begin{aligned}\widehat{\mathbf{E}}\,\mathbb{E}\left[\log Z_{N,h}^{\omega,\beta,\text{pin}}\right] &\geqslant \frac{C_3}{2} - (1+\alpha)\widehat{\mathbf{E}}[\log\widehat{\tau}_N] + \frac{1}{2}\widehat{\mathbf{E}}\left[M_N\mathtt{F}(h) - \log M_N\right] + \frac{\log C_4}{2}\\ &\geqslant \frac{1}{2}\widehat{\mathbf{E}}[M_N]\mathtt{F}(h) - C_5\log\widehat{\mathbf{E}}[\widehat{\tau}_N] + C_6 \geqslant \frac{1}{2}\widehat{\mathbf{E}}[M_N]\mathtt{F}(h) - C_5\log N - C_7, \quad (6.3.8)\end{aligned}$$

where we used in the second inequality that $M_N \leqslant \widehat{\tau}_N$ and Jensen inequality so that $C_5 = \frac{3}{2}+\alpha$, and in the second one that $\widehat{\mathbf{E}}[\widehat{\tau}_N] = N\widehat{\mathbf{E}}[\widehat{\tau}_1]$ so that $C_7 = C_5\log\widehat{\mathbf{E}}[\widehat{\tau}_1] - C_6$. From the assumption we have on $\widehat{K}$, one has, uniformly for all $n \gg N^{\varepsilon}$,

$$\widehat{\mathbf{P}}\left[M_N \leqslant n\right] = \widehat{\mathbf{P}}(\widehat{\tau}_1 \leqslant n)^N = \exp\left(-\frac{\widehat{c}_K}{\widetilde{\alpha}}Nn^{-\widetilde{\alpha}}(1+o(1))\right). \quad (6.3.9)$$

So that using Riemann sum as approximation of integral, one gets that $\widehat{\mathbf{E}}[M_N] = (C_8 + o(1))N^{\widetilde{\alpha}^{-1}}$, where

$$C_8 = \int_0^{\infty}\left(1 - \exp\left(-\frac{\widehat{c}_K}{\alpha}x^{-\widetilde{\alpha}}\right)\right). \quad (6.3.10)$$

Now we choose N to be equal to $N_h := C_9 h^{-\frac{\widetilde{\alpha}}{(1\wedge\alpha)}}|\log h|^{\widetilde{\alpha}}$, so that if h is small enough

$$\frac{1}{2}\mathtt{F}(h)\widehat{\mathbf{E}}[M_{N_h}] - C_6\log N_h \geqslant \frac{C_7}{2}\mathtt{F}(h)N_h^{1/\widetilde{\alpha}} - C_6\log N_h \geqslant |\log h|, \quad (6.3.11)$$

where the last inequality holds provided C_9 (entering in the definition of N_h) is large enough, using the behavior of $\mathtt{F}(h)$ as h goes to 0. This combined with (6.3.8) gives the lower inequality in (6.2.1) as

$$\mathtt{F}(\beta,h) \geqslant \frac{1}{N_h}\widehat{\mathbf{E}}\,\mathbb{E}\left[\log Z_{N_h,h}^{\omega,\beta,\text{pin}}\right]. \quad (6.3.12)$$

6.4. Upper bound on the free energy when $\alpha > 1$

The next two sections are devoted to the proof of the upper bound for the free-energy. This is much more complicated than the lower bound, as one has to control the contribution of all possible trajectories for τ.

Somehow, things get technically simpler if one does not try to capture the $(\log h)^{1-\widetilde{\alpha}}$ factor. Therefore we prove first a rougher result, to give a clear presentation of the strategy we use. For the two next sections, we use the alternative construction for the environment ω based on the renewal $\widetilde{\tau}$, and presented in equation (6.1.9).

For this section we introduce the following notation

$$\begin{aligned}\widetilde{T}_n &= \widetilde{\tau}_{2n},\ \forall n \geqslant 0,\\ \xi_n &= \widetilde{T}_n - \widetilde{T}_{n-1},\ \forall n \geqslant 1.\end{aligned} \quad (6.4.1)$$

6.4.1. Rough bound.

Proposition 6.4.1. *When $\alpha > 1$, one can find a constant C_2 such that*

$$\mathtt{F}(\beta, h) \leqslant C_2 h^{\widetilde{\alpha}}. \tag{6.4.2}$$

Proof The idea of the proof is to say that only the long stretches of ω with $\omega \equiv 0$ can contribute to the free energy, and that others cannot. The first step is to perform a kind of coarse-graining procedure in order to treat the contribution of each segment $(T_n, T_{n+1}]$, separately (Lemma 6.4.2 below), and then to show that the contribution of segments that are too short is zero.

It turns out that the coarse graining we present here is not optimal, and this is the reason why a log factor is lost. An improved coarse graining method is presented in the next subsection.

We introduce a new notation to describe the contribution of a given segment: for a and $b \in \mathbb{N}$, one defines (recall that θ is the shift operator defined just before (6.1.19))

$$Z^{\omega,\beta}_{[a,b],h} := \exp(\beta\omega_a + h) Z^{\theta^a\omega,\beta}_{(b-a),h}. \tag{6.4.3}$$

Here is our coarse graining Lemma

Lemma 6.4.2. *For every $N \in \mathbb{N}$*

$$Z^{\omega,\beta}_{\widetilde{T}_N,h} \leqslant \prod_{i=1}^{N} \left[\left(\max_{x \in (\widetilde{T}_{i-1}, \widetilde{T}_i]} Z^{\omega,\beta}_{[x,\widetilde{T}_i],h} \right) \vee 1 \right]. \tag{6.4.4}$$

Proof We proceed by induction. The claim is obvious for $N = 1$. For the process τ, define $\tau^{(N)}_{\text{next}} := \inf\{n > \widetilde{T}_N, n \in \tau\}$, then one has (using the Markov property for τ)

$$\begin{aligned} \frac{Z^{\omega,\beta}_{\widetilde{T}_{N+1},h}}{Z^{\omega,\beta}_{\widetilde{T}_N,h}} &= \mathbf{E}^{\omega,\beta}_{\widetilde{T}_N,h}\left[\exp\left(\sum_{n=\widetilde{T}_N+1}^{\widetilde{T}_{N+1}} (\beta\omega_n + h) \mathbf{1}_{\{n\in\tau\}} \right) \right] \\ &= \sum_{x=\widetilde{T}_N+1}^{\widetilde{T}_{N+1}} \mathbf{P}^{\omega,\beta}_{\widetilde{T}_N,h}\left(\tau^{(N)}_{\text{next}} = x\right) Z^{\omega,\beta}_{[x,\widetilde{T}_{N+1}],h} + \mathbf{P}^{\omega,\beta}_{\widetilde{T}_N,h}\left(\tau^{(N)}_{\text{next}} > \widetilde{T}_{N+1}\right). \end{aligned} \tag{6.4.5}$$

And the above sum is smaller than $\left(\max_{x\in(\widetilde{T}_N,\widetilde{T}_{N+1}]} Z^{\omega,\beta}_{[x,\widetilde{T}_{N+1}],h}\right) \vee 1$ as it is a convex combination of the terms in the maximum. $\square$

Now we remark that by definition $\omega_{\widetilde{T}_i} = -1$. Therefore, for any $x \in (\widetilde{T}_{i-1}+1, \widetilde{T}_i]$ one has

$$\begin{aligned} Z^{\omega,\beta}_{[x,\widetilde{T}_i],h} = \mathbf{E}\left[e^{\sum_{n=x}^{\widetilde{T}_i} (\beta\omega_{n+x}+h)\mathbf{1}_{\{n\in\tau\}}} \right] &\leqslant e^{h(\widetilde{T}_i - x)} \mathbf{E}\left[e^{\beta\omega_{\widetilde{T}_i} \mathbf{1}_{\{\widetilde{T}_i - x\in\tau\}}} \right] \\ &= e^{h(\widetilde{T}_i - x)} \left[1 - (1-e^{-\beta}) \mathbf{P}(\widetilde{T}_i - x \in \tau) \right] \\ &\leqslant e^{h\xi_i} \left(1 - (1-e^{-\beta}) \inf_{n\geqslant 1} \mathbf{P}(n\in\tau) \right). \end{aligned} \tag{6.4.6}$$

As $\mathbf{E}[\tau_1]<\infty$, the renewal Theorem [Asm03, Chapter 1, Theorem 2.2] ensures that $\inf_{n\geqslant 1}\mathbf{P}(n\in\tau)>0$. From this one obtains the following result that we record as a Lemma

Lemma 6.4.3. *One can find a constant $C_{10}>0$ (depending on β) such that the following bounds hold*

$$\begin{aligned} &\max_{x\in(\widetilde{T}_{i-1},\widetilde{T}_i]} Z^{\omega,\beta}_{[x,\widetilde{T}_i],h}\leqslant(1-C_{10}) && \text{if } \xi_i<C_{10}h^{-1},\\ &\max_{x\in(\widetilde{T}_{i-1},\widetilde{T}_i]} Z^{\omega,\beta}_{[x,\widetilde{T}_i],h}\vee 1\leqslant e^{h\xi_i} && \text{if } \xi_i\geqslant C_{10}h^{-1}. \end{aligned} \tag{6.4.7}$$

Then, the only segments that contribute to the free energy are the segments longer than $C_{10}h^{-1}$. From Lemma 6.4.2 and 6.4.3 one gets that

$$\log Z^{\omega,\beta}_{\widetilde{T}_N,h}\leqslant h\sum_{i=1}^{N}\xi_i\mathbf{1}_{\{\xi_i>C_{10}h^{-1}\}}. \tag{6.4.8}$$

Now using (twice) the law of large numbers one gets that

$$\mathtt{F}(\beta,h)=\lim_{N\to\infty}\frac{N}{\widetilde{T}_N}\frac{1}{N}\log Z^{\omega,\beta}_{\widetilde{T}_N,h}\leqslant\frac{1}{\widetilde{\mathbf{E}}\left[\xi_1\right]}h\widetilde{\mathbf{E}}\left[\xi_1\mathbf{1}_{\{\xi_1>C_{10}h^{-1}\}}\right]. \tag{6.4.9}$$

From the definition of ξ and the properties (6.1.8) of the renewal $\widetilde{\tau}$ one gets that $\widetilde{\mathbf{E}}\left[\xi_1\right]$ is a positive constant, and that

$$\mathbf{E}\left[\xi_1\mathbf{1}_{\{\xi_1>C_{10}h^{-1}\}}\right]\leqslant C_{11}h^{\widetilde{\alpha}-1}. \tag{6.4.10}$$

This finishes the proof. □

6.4.2. Finer bound. The reason why we lose a power of $\log h$ in the previous proof is that our coarse graining Lemma does not take into account the cost for τ to do long jumps between the segments contributing to the free energy. We present in this section a method to control this. This is rather technical but allows to get an upper bound matching the lower bound proved in Section 6.3.

Proposition 6.4.4. *When $\alpha>1$, one can find a constant C_2 such that*

$$\mathtt{F}(\beta,h)\leqslant C_2h^{\widetilde{\alpha}}|\log h|^{1-\widetilde{\alpha}} \tag{6.4.11}$$

Proof We define the sequence $(J_i)_{i\geqslant 0}$ as $J_0:=0$, and

$$J_{i+1}:=\inf\{n>J_i,\xi_{n+1}\geqslant C_{10}h^{-1}\}, \tag{6.4.12}$$

with the constant C_{10} given in Lemma 6.4.3. Furthermore one sets

$$\mathcal{T}_N:=\widetilde{T}_{J_N}. \tag{6.4.13}$$

We have cut the system in *metablocks* composed of one block bigger than $C_{10}h^{-1}$, and then other smaller blocks. As the free-energy is a limit in the almost sure sense, conditioning to an event of positive probability (for the environment) is harmless. For matters of translation invariance (we want the sequence $\left\{(\omega_n)_{n\in(\mathcal{T}_N,\mathcal{T}_{N+1}]}\right\}_{N\geqslant 0}$ to be *i.i.d.*) we choose to observe an environment conditioned to satisfy $\xi_1\geqslant C_{10}h^{-1}$. We denote this conditioned probability by $\widetilde{\mathbf{P}}^{(1)}$.

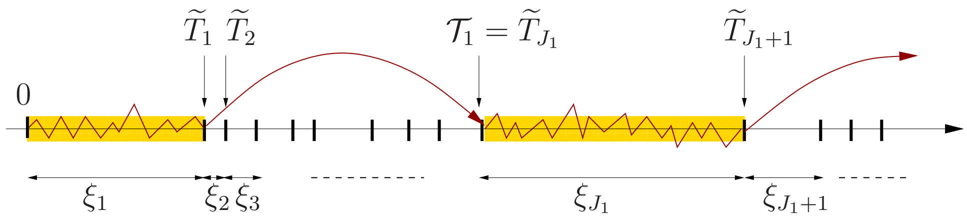

FIGURE 6.2. The above figure represents the decomposition of our environment according to the metablocks $(\mathcal{T}_{i-1}, \mathcal{T}_i]$, constituted of $J_i - J_{i-1}$ unit blocks entities. A metablock is composed of a unit block larger than $C_{10}h^{-1}$, followed by a sequence of $J_i - J_{i-1} - 1$ (possibly equal to zero but that is quite rare) smaller unit blocks. As we explained in Section 6.2.2, the trajectory of the polymer targets the blocks with $\xi_i \geqslant h^{-1}|\log h|$. Our proof, and in particular Lemma 6.4.5 confirms this idea. It also says that regions of smaller length but located close to each other could possibly contribute to the free energy, but the quantitative estimates in equation (6.4.24) show that this contribution is negligible.

In analogy with Lemma 6.4.2 one has the following decomposition for the partition function

$$Z^{\omega,\beta}_{\mathcal{T}_N,h} \leqslant \prod_{i=1}^{N} \left(\max_{x \in (\mathcal{T}_{i-1}, \mathcal{T}_i]} Z^{\omega,\beta}_{[x,\mathcal{T}_i],h} \right) \vee 1. \tag{6.4.14}$$

(the proof being exactly the same). This allows to treat the contribution to $Z^{\omega,\beta}_{\mathcal{T}_N,h}$ of the different segments $(\mathcal{T}_i, \mathcal{T}_{i+1}]$ separately.

Now what we show is that the segment $(\mathcal{T}_i, \mathcal{T}_{i+1}]$ gives a contribution to the free energy only if one of the two following condition is satisfied:

- ξ_{J_i+1} is much larger than $C_{10}h^{-1}$ (by a factor $|\log h|$),
- $J_{i+1} - J_i$ is unusually small.

In the other cases, we show that the energy gain that one has on the block $(\mathcal{T}_i, \widetilde{T}_{J_i+1}]$ is overcome by the entropic cost of aiming at the segment $(\widetilde{T}_{J_i+1}, \mathcal{T}_{i+1}]$.

Lemma 6.4.5. *For any $n \geqslant 0$, any $\delta > 0$ there exists a constant C_{12} depending on β and δ such that if $\xi_{J_n+1} < C_{12}h^{-1}|\log h|$ and $J_{n+1} - J_n \geqslant h^{-1-\delta}$, then*

$$\max_{x \in (\mathcal{T}_n, \mathcal{T}_{n+1}]} Z^{\omega,\beta,}_{[x,\mathcal{T}_{n+1}]} \leqslant 1. \tag{6.4.15}$$

If $\xi_{J_n+1} \geqslant C_{12}h^{-1}|\log h|$ or $J_{n+1} - J_n \leqslant h^{-1-\delta}$ then

$$\max_{x \in (\mathcal{T}_n, \mathcal{T}_{n+1}]} Z^{\omega,\beta}_{[x,\mathcal{T}_{n+1}]} \leqslant e^{h\xi_{J_n+1}}. \tag{6.4.16}$$

We postpone the proof of the Lemma to the end of the section and prove Proposition 6.4.4 now.

Combining Lemma 6.4.5 and the decomposition (6.4.14) one gets that

$$\log Z^{\omega,\beta}_{\mathcal{T}_N,h} \leqslant h \sum_{n=0}^{N-1} \xi_{J_n+1} \mathbf{1}_{\{\xi_{J_n+1} \geqslant C_{12}h^{-1}|\log h| \text{ or } J_{n+1}-J_n \leqslant h^{-1-\delta}\}} \tag{6.4.17}$$

Note that the terms in the sum of right-hand side are *i.i.d.* distributed and have finite mean. Therefore using twice the law of large numbers, one gets

$$\begin{aligned} \mathtt{F}(\beta,h) &\leqslant \lim_{N\to\infty} \frac{N}{\mathcal{T}_N}\frac{1}{N} h \sum_{n=0}^{N-1} \xi_{J_n+1} \mathbf{1}_{\{\xi_{J_{n+1}} \,\geqslant\, C_{12}h^{-1}|\log h| \text{ or } J_{n+1}-J_n \,\leqslant\, h^{-1-\delta}\}} \\ &= \frac{h}{\widetilde{\mathbf{E}}^{(1)}\left[\mathcal{T}_1\right]} \widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{12}h^{-1}|\log h| \text{ or } J_1 \,\leqslant\, h^{-1-\delta}\}}\right]. \end{aligned} \tag{6.4.18}$$

From its definition one has

$$\widetilde{\mathbf{E}}^{(1)}\left[\mathcal{T}_1\right] = \frac{\widetilde{\mathbf{E}}[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{10}h^{-1}\}}]}{\widetilde{\mathbf{P}}[\xi_1 \geqslant C_{10}h^{-1}]} + \widetilde{\mathbf{E}}[J_1-1]\frac{\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 < C_{10}h^{-1}\}}\right]}{\widetilde{\mathbf{P}}[\xi_1 < C_{10}h^{-1}]} = \frac{\widetilde{\mathbf{E}}[\xi_1]}{\widetilde{\mathbf{P}}\left[\xi_1 \geqslant C_{10}h^{-1}\right]}, \tag{6.4.19}$$

where the last equality comes from the fact that J_1 is a geometric variable of parameter $\widetilde{\mathbf{P}}(\xi_1 \geqslant C_{10}h^{-1})$. It remains to estimate

$$\begin{aligned} &\widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{12}h^{-1}|\log h| \text{ or } J_1 \,\leqslant\, h^{-1-\delta}\}}\right] \\ &\qquad \leqslant \widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{12}h^{-1}|\log h|\}}\right] + \widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{J_1 \,\leqslant\, h^{-1-\delta}\}}\right]. \end{aligned} \tag{6.4.20}$$

The first term gives the main contribution, it is equal to

$$\frac{\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{12}h^{-1}|\log h|\}}\right]}{\widetilde{\mathbf{P}}\left[\xi_1 \geqslant C_{10}h^{-1}\right]}. \tag{6.4.21}$$

The second one is equal to

$$\frac{\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{10}h^{-1}\}}\right]}{\widetilde{\mathbf{P}}\left[\xi_1 \geqslant C_{10}h^{-1}\right]} \widetilde{\mathbf{P}}\left[J_1 \leqslant h^{-1-\delta}\right], \tag{6.4.22}$$

so that overall

$$\begin{aligned} &\mathtt{F}(\beta,h) \leqslant h(\widetilde{\mathbf{E}}[\xi_1])^{-1} \\ &\qquad \left(\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{11}h^{-1}|\log h|\}}\right] + \widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{10}h^{-1}\}}\right] \widetilde{\mathbf{P}}\left[J_1 \leqslant h^{-1-\delta}\right]\right). \end{aligned} \tag{6.4.23}$$

Then one can check, using (6.1.8), that there exists C_{13} such that

$$\begin{aligned} \widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{10}h^{-1}\}}\right] &\leqslant C_{13} h^{\widetilde{\alpha}-1}, \\ \widetilde{\mathbf{E}}[\xi_1 \mathbf{1}_{\{\xi_1 \,\geqslant\, C_{13}h^{-1}|\log h|\}}] &\leqslant C_{13}|\log h|^{1-\widetilde{\alpha}} h^{\widetilde{\alpha}-1}, \\ \widetilde{\mathbf{P}}[J_1 \leqslant h^{-1-\delta}] &\leqslant h^{-1-\delta}\widetilde{\mathbf{P}}(\xi_1 \geqslant C_{10}h^{-1}) \leqslant C_{13} h^{\widetilde{\alpha}-1-\delta}, \end{aligned} \tag{6.4.24}$$

which is enough to conclude. □

Proof of Lemma 6.4.5 We start by remarking that by translation invariance (from our choice to impose that $\xi_1 \geqslant C_{10}h^{-1}$) it is sufficient to prove the result in the case $n = 0$.

We have to control the value of $Z^{\omega,\beta}_{[x,\mathcal{T}_1],h}$ for every $x \in (0, \mathcal{T}_1]$. We start with the easier case $x > \widetilde{T}_1$. In that case we can use the strategy of the previous section:

supposing that $x \in (\widetilde{T}_a, \widetilde{T}_{a+1}]$ then one has (exactly like in the proof of Lemma 6.4.2),

$$Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant \prod_{i=a+1}^{J_1} \left[\left(\max_{y \in (\widetilde{T}_{i-1}, \widetilde{T}_i]} Z^{\omega,\beta}_{[y,\widetilde{T}_i],h} \right) \vee 1 \right]. \tag{6.4.25}$$

and one can show that all the terms in the product on the right hand-side are equal to one, since all blocks $[\widetilde{T}_{i-1}, \widetilde{T}_i]$ are smaller than $C_{10}h^{-1}$ (cf. Lemma 6.4.3).

To prove (6.4.16) one also uses equation (6.4.25), and then Lemma 6.4.3 to bound the different factors of the product on the right-hand side.

Now we turn to the case $x \in (0, \widetilde{T}_1]$, $\xi_1 \leqslant C_{11}h^{-1}|\log h|$, $J_1 \geqslant h^{-(1+\delta)}$. We use the following refinement of our block decomposition

Lemma 6.4.6. *For any $x \in (0, \widetilde{T}_1]$ there exists a constant $C_{14} \in (0,1)$ (depending on β, but not on C_{12}) such that*

$$Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant e^{\xi_1 h} \prod_{i=2}^{J_1} \left(1 - C_{14} \frac{\xi_i}{\widetilde{T}_{i-1}} \right). \tag{6.4.26}$$

Proof For notational convenience we also restrict to the case $x = 0$, but the proof works the same for all values of x.

We prove by induction on j, that for any $j \in [1, J_1]$,

$$Z^{\omega,\beta}_{\widetilde{T}_j,h} \leqslant \exp(\xi_1 h) \prod_{i=2}^{j} \left(1 - C_{14} \frac{\xi_i}{\widetilde{T}_{i-1}} \right). \tag{6.4.27}$$

The case $j = 1$ is just the second point of Lemma 6.4.3. Then for the induction step one remarks that

$$\frac{Z^{\omega,\beta}_{\widetilde{T}_{j+1},h}}{Z^{\omega,\beta}_{\widetilde{T}_j,h}} = \mathbf{E}^{\omega,\beta}_{\widetilde{T}_j,h} \left[\exp\left(\sum_{k=\widetilde{T}_j+1}^{\widetilde{T}_{j+1}} (\beta\omega_n + h) \mathbf{1}_{\{k \in \tau\}} \right) \right]. \tag{6.4.28}$$

Define

$$\begin{aligned} \tau^{(j)}_{\text{prev}} &:= \max\{\tau_k \mid \tau_k \leqslant \widetilde{T}_j\}, \\ \tau^{(j)}_{\text{next}} &:= \min\{\tau_k \mid \tau_k > \widetilde{T}_j\}. \end{aligned} \tag{6.4.29}$$

One can notice that the distribution of $\tau^{(j)}_{\text{next}}$ knowing $\tau^{(j)}_{\text{prev}}$ under $\mathbf{P}^{\omega,\beta}_{\widetilde{T}_j,h}$ does not depend on ω nor β, and that one has (recall $\bar{\mathrm{K}}(n) := \mathbf{P}(\tau_1 > n)$)

$$\mathbf{P}^{\omega,\beta}_{\widetilde{T}_j,h}(\tau^{(j)}_{\text{next}} = y | \tau^{(j)}_{\text{prev}} = z) = \frac{\mathrm{K}(y-z)}{\bar{\mathrm{K}}(\widetilde{T}_j - z)}. \tag{6.4.30}$$

Therefore

$$\begin{aligned}
&\mathbf{E}^{\omega,\beta}_{\widetilde{T}_j,h}\left[\exp\left(\sum_{k=\widetilde{T}_j+1}^{\widetilde{T}_{j+1}}(\beta\omega_n+h)\mathbf{1}_{\{k\in\tau\}}\right)\right]\\
&\qquad=\sum_{y=\widetilde{T}_j+1}^{\widetilde{T}_{j+1}}\mathbf{P}^{\omega,\beta}_{\widetilde{T}_j,h}\left(\tau^{(j)}_{\text{next}}=y\right)Z^{\omega,\beta}_{[y,\widetilde{T}_{j+1}],h}+\mathbf{P}^{\omega,\beta}_{\widetilde{T}_j,h}\left(\tau^{(j)}_{\text{next}}>\widetilde{T}_{j+1}\right)\\
&\qquad\leqslant\max_{z\in[0,\widetilde{T}_j]}\left[\frac{\sum_{t=\widetilde{T}_j+1}^{\widetilde{T}_{j+1}}\mathrm{K}(t-z)}{\bar{\mathrm{K}}(\widetilde{T}_j-z)}\max_{y\in(\widetilde{T}_j,\widetilde{T}_{j+1}]}Z^{\omega,\beta}_{[y,\widetilde{T}_{j+1}],h}+\frac{\bar{\mathrm{K}}(\widetilde{T}_{j+1}-z)}{\bar{\mathrm{K}}(\widetilde{T}_j-z)}\right].
\end{aligned}\tag{6.4.31}$$

From our definitions, we know that $\xi_{j+1}\leqslant C_{10}h^{-1}$ for all $j\in[1,J_1-1)$, and therefore Lemma 6.4.3 gives an upper bound to the partition functions $Z^{\omega,\beta}_{[y,\widetilde{T}_{j+1}],h}$, for $y\in(\widetilde{T}_j,\widetilde{T}_{j+1}]$.

$$\mathbf{E}^{\omega,\beta}_{\widetilde{T}_j,h}\left[\exp\left(\sum_{k=\widetilde{T}_j+1}^{\widetilde{T}_{j+1}}(\beta\omega_n+h)\mathbf{1}_{\{k\in\tau\}}\right)\right]\leqslant 1-C_{10}\min_{z\in[0,\widetilde{T}_j]}\frac{\sum_{t=\widetilde{T}_j+1}^{\widetilde{T}_{j+1}}\mathrm{K}(t-z)}{\bar{\mathrm{K}}(\widetilde{T}_j-z)}.\tag{6.4.32}$$

From there, we finish the proof by remarking that from our assumption on $\mathrm{K}(\cdot)$ (and using the change of variable $z'=\widetilde{T}_j-z$), there exist constants C_{14} and C_{15} such that

$$\min_{z\in[0,\widetilde{T}_j]}\frac{\sum_{t=\widetilde{T}_j+1}^{\widetilde{T}_{j+1}}\mathrm{K}(t-z)}{\bar{\mathrm{K}}(\widetilde{T}_j-z)}\geqslant C_{15}\min_{z'\in[0,\widetilde{T}_j]}(z'+1)^{\alpha}\sum_{u=1}^{\xi_{j+1}}(z'+u)^{-(1+\alpha)}\geqslant\frac{C_{14}}{C_{10}}\frac{\xi_{j+1}}{\widetilde{T}_j}.\tag{6.4.33}$$

where the last inequality comes from a straightforward computation. □

We can now finish the proof of Lemma 6.4.5. Note that for all $j\in[2,J_1]$ one has $\xi_j/\widetilde{T}_{j-1}\leqslant\xi_j/\xi_1\leqslant 1$ so that if $C_{14}<1$ one has

$$\log\prod_{j=2}^{J_1}\left(1-C_{14}\frac{\xi_j}{\widetilde{T}_{j-1}}\right)\leqslant -C_{14}\sum_{j=2}^{J_1}\frac{\xi_j}{\widetilde{T}_{j-1}}.\tag{6.4.34}$$

Then one remarks that

$$\sum_{j=2}^{J_1}\frac{\xi_j}{\widetilde{T}_{j-1}}\geqslant\sum_{j=2}^{J_1}\sum_{i=\widetilde{T}_{j-1}+1}^{\widetilde{T}_j}\frac{1}{i}\geqslant\frac{1}{2}\log(\mathcal{T}_1/\xi_1).\tag{6.4.35}$$

Given our assumptions $\mathcal{T}_1\geqslant J_1\geqslant h^{-(1+\delta)}$ and $\xi_1\leqslant C_{12}h^{-1}|\log h|$, one has that $\log(\mathcal{T}_1/\xi_1)$ is larger than $\frac{\delta}{2}|\log h|$ if C_{12} is small enough. Then using Lemma 6.4.6 one gets that

$$\log Z^{\omega,\beta}_{[x,\mathcal{T}_1]}\leqslant(C_{12}-C_{14}\frac{\delta}{4})|\log h|\leqslant 0\tag{6.4.36}$$

if C_{12} has been chosen small enough. □

6.5. Upper bound on the free energy when $\alpha < 1$

The case $\alpha < 1$ is a bit more difficult than the case $\alpha > 1$. The reason is that one has not $\inf_{n \in \mathbb{N}} \mathbf{P}(n \in \tau) > 0$ (which was really crucial to prove Lemma 6.4.3), and one has to replace this by technical estimates on the renewal (for example Lemma 6.5.5), that are a bit more difficult to work with.

We have to change the length of the blocks in our coarse graining procedure, and therefore we renew our definition of $\widetilde{T}$ and ξ for this section. Let C_{16} be a fixed (small) constant (how small is to be decided in the proof). Set $L = L(h) := \lfloor C_{16} h^{-1/\alpha} \rfloor$.

In analogy with the previous section, define

$$\begin{aligned} \widetilde{T}_i &:= \widetilde{\tau}_{iL}, \ \forall i \geqslant 0, \\ \xi_i &:= \widetilde{T}_i - \widetilde{T}_{i-1}, \ \forall i \geqslant 1. \end{aligned} \tag{6.5.1}$$

As for the case $\alpha > 1$, the proof simplifies considerably if one drops the $|\log h|$ factor in the result. We expose first this simpler proof in the next Section. Then in Section 6.5.2 we refine the argument in order to get the exact upper bound in (6.2.1).

6.5.1. Rough bound. The result we prove in this section is

Proposition 6.5.1. *When $\alpha < 1$, one can find a constant C_2 such that*

$$\mathtt{F}(\beta, h) \leqslant C_2 h^{\frac{\widetilde{\alpha}}{\alpha}}. \tag{6.5.2}$$

In order to do so, we prove an asymptotic upper bound for $Z^{\omega,\beta}_{\widetilde{T}_N,h}$. The first step is a coarse-graining decomposition of $Z^{\omega,\beta}_{\widetilde{T}_N,h}$ that allows to treat the contribution of each segment $(\widetilde{T}_n, \widetilde{T}_{n+1}]$ separately. It turns out that we need something a bit more sophisticated than Lemma 6.4.2.

Lemma 6.5.2. *For every $N \in \mathbb{N}$*

$$Z^{\omega,\beta}_{\widetilde{T}_N,h} \leqslant \prod_{n=1}^{N} \max_{y \in [0, \widetilde{T}_n]} \left[\sum_{x=1}^{\xi_n} \frac{\mathrm{K}(x+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\widetilde{T}_{n-1}+x, \widetilde{T}_n],h} + \frac{\bar{\mathrm{K}}(\xi_n + y)}{\bar{\mathrm{K}}(y)} \right]. \tag{6.5.3}$$

The second ingredient we need is that segments $(\widetilde{T}_{n-1}, \widetilde{T}_n]$ that are short do not contribute to the free energy, or more precisely that only uncommonly long segments $(\widetilde{T}_{n-1}, \widetilde{T}_n]$ contribute effectively to the free-energy. Set $m := \mathbf{E}[\widetilde{\tau}_1]$.

Lemma 6.5.3. *If $\xi_n < 2mL(h)$ then*

$$\max_{y \geqslant 0} \left[\sum_{x=1}^{\xi_n} \frac{\mathrm{K}(x+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\widetilde{T}_{n-1}+x, \widetilde{T}_n],h} + \frac{\bar{\mathrm{K}}(\xi_n + y)}{\bar{\mathrm{K}}(y)} \right] \leqslant 1, \tag{6.5.4}$$

more precisely there exists a constant $C_{17}>0$ such that for every $y \geqslant 0$

$$\sum_{x=1}^{\xi_n} \mathrm{K}(x+y) Z^{\omega,\beta}_{[\widetilde{T}_{n-1}+x,\widetilde{T}_n],h} \leqslant (1-C_{17}) \sum_{x=1}^{\xi_n} \mathrm{K}(x+y). \qquad (6.5.5)$$

There exists a constant C_{18} such that if $\xi_n \geqslant 2mL(h)$, then

$$\max_{y \geqslant 0} \left[\sum_{x=1}^{\xi_n} \frac{\mathrm{K}(x+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\widetilde{T}_{n-1}+x,\widetilde{T}_n],h} + \frac{\bar{\mathrm{K}}(\xi_n+y)}{\bar{\mathrm{K}}(y)} \right] \leqslant e^h Z_{\xi_n,h} \leqslant e^{C_{18} h^{1/\alpha} \xi_n}. \qquad (6.5.6)$$

Proof of Proposition 6.5.1 .

Combining Lemma 6.5.2 and Lemma 6.5.3 (inequalities (6.5.4) and (6.5.6)), one obtains

$$\log Z^{\omega,\beta}_{\widetilde{T}_N,h} \leqslant C_{18} h^{1/\alpha} \sum_{n=1}^{N} \xi_n \mathbf{1}_{\{\xi_n \geqslant 2mL(h)\}}. \qquad (6.5.7)$$

Using (as in the previous sections) twice the law of large numbers one gets that

$$\mathtt{F}(\beta,h) \leqslant \frac{1}{\widetilde{\mathbf{E}}[\xi_1]} C_{18} h^{1/\alpha} \widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant 2mL(h)\}}\right]. \qquad (6.5.8)$$

By definition, $\widetilde{\mathbf{E}}[\xi_1] = mL(h)$. Using Proposition 6.A.2, one can estimate

$$\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant 2mL(h)\}}\right] \leqslant C_{19} \sum_{x=2mL(h)}^{\infty} xLx^{-(1+\widetilde{\alpha})} \leqslant C_{20} L^{2-\widetilde{\alpha}}. \qquad (6.5.9)$$

Replacing $L(h)$ by its value gives the result. □

We turn to the proof of the Lemmas,

Proof of Lemma 6.5.2 We prove this once again by induction on N. The result is obvious for $N=1$. As in Section 6.4, we use the notation

$$\begin{aligned} \tau^{(N)}_{\text{next}} &:= \min\{\tau_k \mid \tau_k > \widetilde{T}_N\}, \\ \tau^{(N)}_{\text{prev}} &:= \max\{\tau_k \mid \tau_k \leqslant \widetilde{T}_N\}. \end{aligned} \qquad (6.5.10)$$

Decomposing on the different possible values for $\tau^{(N)}_{\text{next}}$ one obtains

$$\begin{aligned} \frac{Z^{\omega,\beta}_{\widetilde{T}_{N+1},h}}{Z^{\omega,\beta}_{\widetilde{T}_N,h}} &= \mathbf{E}^{\omega,\beta}_{\widetilde{T}_N,h}\left[\exp\left(\sum_{n=\widetilde{T}_N+1}^{\widetilde{T}_N} (\beta\omega_n + h)\mathbf{1}_{\{n\in\tau\}}\right)\right] \\ &= \sum_{x=\widetilde{T}_N+1}^{\widetilde{T}_{N+1}} \mathbf{P}^{\omega,\beta}_{\widetilde{T}_N,h}\left(\tau^{(N)}_{\text{next}} = x\right) Z^{\omega,\beta}_{[x,\widetilde{T}_{N+1}]} + \mathbf{P}^{\omega,\beta}_{\widetilde{T}_N,h}\left(\tau^{(N)}_{\text{next}} > \widetilde{T}_{N+1}\right). \end{aligned} \qquad (6.5.11)$$

Recall that

$$\mathbf{P}^{\omega,\beta}_{\widetilde{T}_N,h}\left(\tau^{(N)}_{\text{next}} = x | \tau^{(N)}_{\text{prev}} = y\right) = \frac{\mathrm{K}(x-y)}{\bar{\mathrm{K}}(\widetilde{T}_N - y)}. \qquad (6.5.12)$$

Taking the maximum over all possibilities for $\tau_{\text{prev}}^{(N)}$ we have

$$\frac{Z^{\omega,\beta}_{\widetilde{T}_{N+1},h}}{Z^{\omega,\beta}_{\widetilde{T}_N,h}} \leqslant \max_{y \leqslant \widetilde{T}_N} \left[\sum_{x=\widetilde{T}_N+1}^{\widetilde{T}_{N+1}} \frac{\mathrm{K}(x-y)}{\bar{\mathrm{K}}(\widetilde{T}_N - y)} Z^{\beta,\omega}_{[x,\widetilde{T}_{N+1}],h} + \frac{\bar{\mathrm{K}}(\widetilde{T}_{N+1}-y)}{\bar{\mathrm{K}}(\widetilde{T}_N - y)} \right], \tag{6.5.13}$$

and we get the result by making the change of variables $x \to x - \widetilde{T}_N$ and $y \to \widetilde{T}_N - y$. □

The statement of Lemma 6.5.3 is translation invariant; therefore it is enough to prove it for $N = 1$. The core of the proof consists of proving two technical estimates.

Lemma 6.5.4. *If $\widetilde{T}_1 = \xi_1 < 2mL(h)$, then one can find $h_0(\beta) > 0$ and two constants $C_{21} > 0$ and $C_{22} > 0$ (depending on β), such that for all $h \leqslant h_0(\beta)$ one has*

$$\max_{x \in [0, L(h)/4]} Z^{\omega,\beta}_{[x,\widetilde{T}_1],h} \leqslant 1 - C_{21}, \tag{6.5.14}$$

and

$$\max_{x \geqslant L(h)/4} Z^{\omega,\beta}_{[x,\widetilde{T}_1],h} \leqslant Z_{2mL(h),h} \leqslant 1 + C_{22}. \tag{6.5.15}$$

where C_{22} can be made arbitrarily small by choosing C_{16} (entering in the definition of L) small. On the contrary C_{21} can be chosen independently of C_{16}.

Proof of Lemma 6.5.4 The second point is standard and we include it here for the sake of completeness. We notice that

$$\mathbf{P}\left(|\tau \cap [1,N]| \geqslant n\right) \leqslant \mathbf{P}\left(\nexists i \in [1,n],\ \tau_i - \tau_{i-1} > N\right) \leqslant (1 - \bar{\mathrm{K}}(N))^n. \tag{6.5.16}$$

Therefore

$$\begin{aligned} Z_{N,h} &= 1 + \sum_{n=1}^{N} (e^{nh} - e^{(n-1)h}) \mathbf{P}\left[\tau \cap [1,N] \geqslant n\right] \\ &\leqslant 1 + \sum_{n=1}^{N} h \left[e^h \left(1 - \bar{\mathrm{K}}(N)\right)\right]^n \leqslant 1 + \frac{h}{1 - e^h(1 - \bar{\mathrm{K}}(N))}, \end{aligned} \tag{6.5.17}$$

where the last inequality holds only if $e^h(1 - \bar{\mathrm{K}}(N)) < 1$. Now one uses that for h small

$$1 - e^h(1 - \bar{\mathrm{K}}(N)) \geqslant 1 - (1 + 2h)(1 - \bar{\mathrm{K}}(N)) \geqslant \bar{\mathrm{K}}(N) - 2h, \tag{6.5.18}$$

and also that $\bar{\mathrm{K}}(N) \geqslant (2\alpha)^{-1} c_K N^{-\alpha}$ for N large enough (from the definition of $\mathrm{K}(\cdot)$). Then plugging $N = 2mL(h)$, and recalling our definition of $L(h)$, one has (for h small enough)

$$1 - e^h(1 - \bar{\mathrm{K}}(N)) \geqslant h\left((2\alpha)^{-1} c_K (2mC_{16})^{-\alpha} - 2\right). \tag{6.5.19}$$

Then the result holds, setting $C_{22} := \left((2\alpha)^{-1} c_K (2mC_{16})^{-\alpha} - 2\right)^{-1}$.

The first point is more delicate and we focus on it now. Take $x \leqslant L/4$, and note that $[\widetilde{\tau}_{L/2}, \widetilde{\tau}_L] \subset [x, \widetilde{T}_1]$, so that

$$|\{i \in [x, \widetilde{T}_1] \mid \omega_i = -1\}| \geqslant L/4. \tag{6.5.20}$$

As $\widetilde{T}_1 = \xi_1 \leqslant 2mL$, this means that the proportion of ω equal to -1 in $[x, \widetilde{T}_1]$ is at least $1/(8m)$. We use this fact to prove that the renewal τ starting from x has to hit one of these -1 with positive probability. This is the content of the following Lemma whose proof is postponed at the end of the section.

Lemma 6.5.5. *There exists some constant $C_{23} > 0$ such that for any $M > 0$, $a > 0$, if one takes A a subset of $[1, M]$ of cardinality at least aM, one has*

$$\mathbf{P}\left(\tau \cap A \neq \emptyset\right) \geqslant C_{23} a^{1+\alpha}. \tag{6.5.21}$$

Set $a = 1/(8m)$, $M = \widetilde{T}_1 - x$ and $A := \{n \in [1, \widetilde{T}_1 - x] \mid \omega_{x+n} = -1\}$. Using translation invariance of τ, one gets

$$\begin{aligned} e^{-h} Z^{\omega,\beta}_{[x,\widetilde{T}_1],h} &\leqslant \mathbf{E}\left[e^{\sum_{n=1}^{\widetilde{T}_1 - x} h \mathbf{1}_{\{n\in\tau\}}} \mathbf{1}_{\{\tau\cap A = \emptyset\}}\right] + e^{-\beta} \mathbf{E}\left[e^{\sum_{n=1}^{\widetilde{T}_1 - x} h \mathbf{1}_{\{n\in\tau\}}} \mathbf{1}_{\{\tau\cap A \neq \emptyset\}}\right] \\ &\leqslant Z_{\widetilde{T}_1 - x,h} - (1-e^{-\beta}) \mathbf{P}\left(\tau \cap A \neq \emptyset\right) \leqslant Z_{2mL,h} - C_{23}(1-e^{-\beta})(8m)^{-(1+\alpha)}, \end{aligned} \tag{6.5.22}$$

where in the last line we used Lemma 6.5.5. This allows us to conclude using (6.5.15): provided that C_{22} is sufficiently small (which is ensured by choosing C_{16} small) one can take $C_{21} = \frac{C_{23}}{2}(1-e^{-\beta})(8m)^{-(1+\alpha)}$, provided also that h is small enough to absorb the e^h factor. $\square$

Proof of Lemma 6.5.3 We leave to the reader to check that (6.5.4) is a consequence of (6.5.5) and focus on the proof of the latter. For $\widetilde{T}_1 = \xi_1 \leqslant 2mL(h)$ and for any $y \geqslant 0$, Lemma 6.5.4 gives us

$$\sum_{x=1}^{\xi_1} \mathrm{K}(x+y) Z^{\omega,\beta}_{[x,\widetilde{T}_1]} \leqslant (1 - C_{21}) \sum_{x=1}^{L/4} \mathrm{K}(x+y) + (1 + C_{22}) \sum_{x=L/4+1}^{\xi_1} \mathrm{K}(x+y) \tag{6.5.23}$$

And therefore (6.5.5) holds if for all $y \geqslant 0$

$$\frac{\sum_{x=1}^{L/4} \mathrm{K}(x+y)}{\sum_{x=1}^{\xi_1} \mathrm{K}(x+y)} \geqslant \frac{\sum_{x=1}^{L/4} \mathrm{K}(x+y)}{\sum_{x=1}^{2mL} \mathrm{K}(x+y)} \geqslant \frac{C_{17} + C_{22}}{C_{21} - C_{17}}. \tag{6.5.24}$$

The middle term above is bounded away from zero uniformly in L and in y. Therefore (6.5.5) holds if C_{17} and C_{22} are small enough (and from Lemma 6.5.4, one can make C_{22} as small as needed by adjusting C_{16}).

For (6.5.6), first notice that for every value of y

$$\sum_{x=1}^{\xi_1} \frac{\mathrm{K}(x+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[x,\widetilde{T}_1],h} + \frac{\bar{\mathrm{K}}(\xi_n + y)}{\bar{\mathrm{K}}(y)} \leqslant \max_{x\in(0,\xi_1]} Z^{\omega,\beta}_{[x,\widetilde{T}_1],h} \leqslant \max_{x\in(0,\xi_1]} Z_{[x,\widetilde{T}_1],h} \leqslant e^h Z_{\xi_1,h} \tag{6.5.25}$$

which gives the first inequality.

Then from equation (6.5.15) one has that $e^h Z_{2mL,h}$ is bounded above by a constant, so that one can write $e^h Z_{2mL,h} \leqslant e^{\frac{C_{18}}{2} h^{1/\alpha} 2mL}$, choosing C_{18} sufficiently large. Then using the observation (6.3.6), one has that for every pair of integers (n_1, n_2)

$$e^h Z_{n_1+n_2,h} \leqslant e^h Z_{n_1,h} e^h Z_{n_2,h}, \tag{6.5.26}$$

which allows us to say that for every $k \in \mathbb{N}$

$$e^h Z_{2mkL,h} \leqslant e^{k\frac{C_{18}}{2}h^{1/\alpha}2mL}, \tag{6.5.27}$$

so that (by monotonicity of $Z_{N,h}$ in N), (6.5.6) holds for every $\xi_1 \geqslant 2mL$. □

Proof of Lemma 6.5.5 First notice that

$$\mathbf{P}(\tau \cap A \neq \emptyset) \geqslant \sum_{n=1}^{(aM)^\alpha} \mathbf{P}(\tau_{n-1} \leqslant aM/2, \tau_n \in A \cap (aM/2, M]). \tag{6.5.28}$$

Now for every $x \leqslant aM/2$, one has

$$\mathbf{P}(\tau_n \in A \cap (aM/2, M] \mid \tau_{n-1} = x) = \sum_{y \in A \cap (aM/2,M]} \mathrm{K}(y-x) \geqslant$$

$$|A \cap (aM/2, M]| \min_{m \leqslant M} \mathrm{K}(m) \geqslant \frac{aM}{2} C_{24} M^{-(1+\alpha)}, \tag{6.5.29}$$

and therefore

$$\mathbf{P}(\tau \cap A \neq \emptyset) \geqslant \frac{a}{2} M^{-\alpha} C_{24} \sum_{n=1}^{(aM)^\alpha} \mathbf{P}[\tau_{n-1} \leqslant aM/2]. \tag{6.5.30}$$

As $\mathbf{P}[\tau_{n-1} \leqslant aM/2]$ is bounded away from zero uniformly for all $n \leqslant (aM)^\alpha$ (see for example (1.8) in [Don97]), on can find C_{23} such that

$$\mathbf{P}(\tau \cap A \neq \emptyset) \geqslant C_{23} a^{1+\alpha}. \tag{6.5.31}$$

□

6.5.2. Finer bound. As in Section 6.4, to get the $|\log h|^{1-\widetilde{\alpha}}$ factor, one needs a new coarse graining procedure which takes into account the cost for τ of doing long jumps between blocks that effectively contribute to the free energy. We are then able to get an upper bound on the free energy that matches the lower bound proved in Section 6.3.

Proposition 6.5.6. *When $\alpha < 1$, one can find a constant C_2 such that*

$$\mathtt{F}(\beta, h) \leqslant C_2 h^{\widetilde{\alpha}/\alpha} |\log h|^{1-\widetilde{\alpha}}. \tag{6.5.32}$$

The method is quite similar to the one used in the case $\alpha > 1$. Define the sequence $(J_i)_{i \geqslant 0}$ as $J_0 := 0$, and

$$J_{i+1} := \inf\{n > J_i, \xi_{n+1} \geqslant 2mL(h)\}. \tag{6.5.33}$$

Set $\mathcal{T}_N := \widetilde{T}_{J_N}$. Note that we used for $\widetilde{T}_i$ and ξ_i the definitions (6.5.1).

Our system is decomposed in metablocks made of one block bigger than $2mL$, and then other smaller blocks. This is the same type of decomposition as shown in Figure 6.2, except that the blocks that constitute one metablock are already composed of L $\widehat{\tau}$-jumps (instead of 2 in the case $\alpha > 1$), so that their typical size is mL.

We proceed as in Section 6.4.2, conditioning the environment to satisfy $\xi_1 \geqslant 2mL$. We denote this conditioned probability $\widetilde{\mathbf{P}}^{(1)}$, and underline that as far as the free energy is concerned, conditioning the environment to an event of positive probability is harmless. This is done for a matter of translation invariance: thanks to this trick, the sequence $\{(\omega_n)_{n\in(\mathcal{T}_N,\mathcal{T}_{N+1}]}\}_{N\,\geqslant\,0}$ is *i.i.d.* under $\mathbf{P}^{(1)}$.

As we did in Lemma 6.5.2, we can get an upper bound on the free-energy that factorizes the contribution of the different blocks

$$Z^{\omega,\beta}_{\mathcal{T}_N,h} \leqslant \prod_{n=0}^{N-1} \max_{y\in[0,\mathcal{T}_{n+1}]} \left[\sum_{x=1}^{\mathcal{T}_{n+1}-\mathcal{T}_n} \frac{\mathrm{K}(x+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\mathcal{T}_n+x,\mathcal{T}_{n+1}],h} + \frac{\bar{\mathrm{K}}(\mathcal{T}_{n+1}-\mathcal{T}_n+y)}{\bar{\mathrm{K}}(y)} \right]. \tag{6.5.34}$$

The proof being exactly the same that for Lemma 6.5.2, we leave it to the reader (we will use this kind of coarse graining repeatedly in the remaining of this Chapter).

Now, we show a Lemma analogue of Lemma 6.4.5, which tells that a block $(\mathcal{T}_i,\mathcal{T}_{i+1}]$ contributes to the free energy only if ξ_{J_i+1} is much larger than $2mL$ (by a factor $\log L$), or if $J_{i+1}-J_i$ is relatively small.

Lemma 6.5.7. *There exists a constant C_{16} (entering in the definition of $L(h)$), such that for any $n \geqslant 0$:*
If $\xi_{J_n+1} < L\log L$ and $J_{n+1}-J_n \geqslant L^{(\tilde\alpha+1)/2}$, then

$$\max_{y\,\geqslant\,0} \left[\sum_{x=1}^{\mathcal{T}_{n+1}-\mathcal{T}_n} \frac{\mathrm{K}(x+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\mathcal{T}_n+x,\mathcal{T}_{n+1}],h} + \frac{\bar{\mathrm{K}}(\mathcal{T}_{n+1}-\mathcal{T}_n+y)}{\bar{\mathrm{K}}(y)} \right] = 1. \tag{6.5.35}$$

If $\xi_{J_n+1} \geqslant L\log L$ or $J_{n+1}-J_n < L^{(\tilde\alpha+1)/2}$, then

$$\max_{y\,\geqslant\,0} \left[\sum_{x=1}^{\mathcal{T}_{n+1}-\mathcal{T}_n} \frac{\mathrm{K}(x+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\mathcal{T}_n+x,\mathcal{T}_{n+1}],h} + \frac{\bar{\mathrm{K}}(\mathcal{T}_{n+1}-\mathcal{T}_n+y)}{\bar{\mathrm{K}}(y)} \right] \leqslant e^{C_{18}h^{1/\alpha}\xi_{J_n+1}}. \tag{6.5.36}$$

(For the same constant C_{18} as in Lemma 6.5.3).

We postpone the proof of the Lemma to the end of the section.

Proof of Proposition 6.5.6 From the decomposition (6.5.34) and Lemma 6.5.7, one has

$$\log Z^{\omega,\beta}_{\mathcal{T}_N,h} \leqslant C_{18}h^{1/\alpha} \sum_{n=0}^{N-1} \xi_{J_n+1}\mathbf{1}_{\{\xi_{J_n+1}\,\geqslant\,L\log L \text{ or } J_{n+1}-J_n<L^{(\tilde\alpha+1)/2}\}}. \tag{6.5.37}$$

Using twice the law of large numbers one gets as a consequence

$$\mathtt{F}(\beta,h) \leqslant \frac{C_{18}}{\widetilde{\mathbf{E}}^{(1)}[\mathcal{T}_1]} h^{1/\alpha} \widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{\xi_1\,\geqslant\,L\log L \text{ or } J_1<L^{(\tilde\alpha+1)/2}\}}\right]. \tag{6.5.38}$$

Then in analogy with (6.4.19), one gets from the definition of $\mathcal{T}_1$ that

$$\widetilde{\mathbf{E}}^{(1)}[\mathcal{T}_1] = \frac{\widetilde{\mathbf{E}}[\xi_1\mathbf{1}_{\{\xi_1\,\geqslant\,2mL\}}]}{\widetilde{\mathbf{P}}[\xi_1\geqslant 2mL]} + \widetilde{\mathbf{E}}[J_1-1]\frac{\widetilde{\mathbf{E}}\left[\xi_1\mathbf{1}_{\{\xi_1<2mL\}}\right]}{\widetilde{\mathbf{P}}[\xi_1<2mL]} = \frac{\widetilde{\mathbf{E}}[\xi_1]}{\widetilde{\mathbf{P}}[\xi_1\geqslant 2mL]}. \tag{6.5.39}$$

One also has

$$\widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant L\log L \text{ or } J_1 < L^{(\tilde\alpha+1)/2}\}}\right] \leqslant \widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant L\log L\}}\right] + \widetilde{\mathbf{E}}^{(1)}\left[\xi_1 \mathbf{1}_{\{J_1 < L^{(\tilde\alpha+1)/2}\}}\right] = \frac{\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant L\log L\}}\right]}{\widetilde{\mathbf{P}}(\xi_1 \geqslant 2mL)} + \frac{\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant 2mL\}}\right]\widetilde{\mathbf{P}}(J_1 < L^{(\tilde\alpha+1)/2})}{\widetilde{\mathbf{P}}(\xi_1 \geqslant 2mL)}, \quad (6.5.40)$$

and hence

$$\mathtt{F}(\beta,h) \leqslant (mL)^{-1} h^{1/\alpha}\left(\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant L\log L\}}\right] + \widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant 2mL\}}\right]\widetilde{\mathbf{P}}(J_1 < L^{(\tilde\alpha-1)/2})\right), \quad (6.5.41)$$

where we also used that $\widetilde{\mathbf{E}}[\xi_1] = mL$. Then Proposition 6.A.1 allows us to bound the right-hand side of the above equation: one can check that there exists a constant C_{25} such that

$$\begin{aligned} \widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant 2mL\}}\right] &\leqslant C_{25} L^{2-\tilde\alpha}, \\ \widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant L\log L\}}\right] &\leqslant C_{25} L^{2-\tilde\alpha}(\log L)^{1-\tilde\alpha}, \\ \widetilde{\mathbf{P}}(J_1 < L^{(\tilde\alpha+1)/2}) &\leqslant L^{(\tilde\alpha+1)/2}\widetilde{\mathbf{P}}(\xi_1 \geqslant 2mL) \leqslant C_{25} L^{(1-\tilde\alpha)/2}, \end{aligned} \quad (6.5.42)$$

which is enough to conclude, recalling the definition of $L(h)$. □

Proof of Lemma 6.5.7 By using translation invariance, it is sufficient (and notationally more convenient) to prove the result only in the case $n = 0$.

We first prove that in all cases

$$\sum_{x \in (\widetilde{T}_1, \mathcal{T}_1]} \mathrm{K}(x+y) Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant \sum_{x \in (\widetilde{T}_1, \mathcal{T}_1]} \mathrm{K}(x+y), \quad (6.5.43)$$

which is the easy part, and then prove that for every $x \in (1, \xi_1]$

$$\begin{aligned} Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} &\leqslant 1 \quad \text{when } \xi_1 < L\log L \text{ and } J_1 \geqslant L^{(\tilde\alpha+1)/2}, \\ Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} &\leqslant e^{C_{18}\xi_1 h^{\frac{1}{\alpha}}} \quad \text{in every other cases.} \end{aligned} \quad (6.5.44)$$

Combining of (6.5.43), (6.5.44) we prove both (6.5.35) and (6.5.36).

If $x \in (\widetilde{T}_a, \widetilde{T}_{a+1}]$ with $a \in \{1, \dots, J_1 - 1\}$, one uses a coarse graining argument similar to the one of Lemma 6.5.2 to factorize $Z^{\omega,\beta}_{[x,\mathcal{T}_1],h}$, and also equation (6.5.4) in Lemma 6.5.3 to show that since all blocks we consider are of size $\xi_i \leqslant 2mL$, most of the terms in the factorization are smaller than 1:

$$Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant Z^{\omega,\beta}_{[x,\widetilde{T}_{a+1}],h} \prod_{i=a+1}^{J_1} \max_{y \geqslant 0}\left[\sum_{t=1}^{\xi_i} \frac{\mathrm{K}(t+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\widetilde{T}_{i-1}+t,\widetilde{T}_i],h} + \frac{\bar{\mathrm{K}}(\xi_n + y)}{\bar{\mathrm{K}}(y)}\right] \leqslant Z^{\omega,\beta}_{[x,\widetilde{T}_{a+1}],h}. \quad (6.5.45)$$

Then from this and equation (6.5.5) in Lemma 6.5.3, one has

$$\sum_{x\in(\widetilde{T}_a,\widetilde{T}_{a+1}]} \mathrm{K}(x+y) Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant \sum_{x=1}^{\xi_{a+1}} \mathrm{K}(x+y+\widetilde{T}_a) Z^{\omega,\beta}_{[\widetilde{T}_a+x,\widetilde{T}_{a+1}],h} \leqslant \sum_{x\in(\widetilde{T}_a,\widetilde{T}_{a+1}]} \mathrm{K}(x+y), \tag{6.5.46}$$

which ends the proof of (6.5.43).

Let us deal with the case $x \in (0,\xi_1]$. One needs a statement analogue to the one of Lemma 6.4.6, that is

Lemma 6.5.8. *There exists a constant $C_{26} < 1$ such that for any $x \in (0,\xi_1]$,*

$$Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant e^{C_{18}h^{1/\alpha}\xi_1} \prod_{b=2}^{J_1} \left(1 - C_{26}\frac{\xi_b}{\widetilde{T}_{b-1}}\right). \tag{6.5.47}$$

Note that the second line of (6.5.44) is an immediate consequence of this Lemma.

Proof of Lemma 6.5.8 One uses the coarse graining procedure similar to the one of Lemma 6.5.2 to get

$$Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant Z_{\xi_1,h} \prod_{b=2}^{J_1} \max_{y\in[0,\widetilde{T}_{b-1}]} \left[\sum_{t=1}^{\xi_b} \frac{\mathrm{K}(t+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\widetilde{T}_{b-1}+t,\widetilde{T}_b],h} + \frac{\bar{\mathrm{K}}(\xi_b+y)}{\bar{\mathrm{K}}(y)}\right], \tag{6.5.48}$$

One uses equation (6.5.6) to bound $Z_{\xi_1,h}$. As for the other factors of the product, one already has good bounds on them thanks to Lemma 6.5.4. Indeed, equation (6.5.5) gives directly

$$\sum_{t=1}^{\xi_b} \frac{\mathrm{K}(t+y)}{\bar{\mathrm{K}}(y)} Z^{\omega,\beta}_{[\widetilde{T}_{b-1}+t,\widetilde{T}_b],h} + \frac{\bar{\mathrm{K}}(\xi_b+y)}{\bar{\mathrm{K}}(y)} \leqslant (1-C_{17}) \sum_{t=1}^{\xi_b} \frac{\mathrm{K}(t+y)}{\bar{\mathrm{K}}(y)} \leqslant 1 - C_{26}\frac{\xi_b}{\widetilde{T}_{b-1}}, \tag{6.5.49}$$

where the last inequality holds for all $y \in [0,\widetilde{T}_{b-1}]$ and is obtained in the same way that (6.4.33). □

We are now ready to prove (6.5.44). If $\xi_1 \leqslant L\log L$ and $J_1 \geqslant L^{(\widetilde{\alpha}+1)/2}$, then from Lemma 6.5.8,

$$\log Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant C_{18}h^{1/\alpha}L\log L - C_{26}\sum_{b=2}^{J_1}\frac{\xi_b}{\widetilde{T}_{b-1}}, \tag{6.5.50}$$

where we used that $\xi_b/\widetilde{T}_{b-1} \leqslant \xi_b/\xi_1 \leqslant 1$, and $C_{26} < 1$. Moreover, one also has

$$\sum_{b=2}^{J_1}\frac{\xi_b}{\widetilde{T}_{b-1}} \geqslant \frac{1}{2}\log\left(\mathcal{T}_1/\xi_1\right) \tag{6.5.51}$$

(see (6.4.35)), so that with our assumptions $\mathcal{T}_1 \geqslant J_1 \geqslant L^{(\widetilde{\alpha}+1)/2}$ and $\xi_1 \leqslant L\log L$, the inequality (6.5.50) gives (recall also that $L = \lfloor C_{16}h^{-1/\alpha}\rfloor$)

$$\log Z^{\omega,\beta}_{[x,\mathcal{T}_1],h} \leqslant C_{18}C_{16}\log L - \frac{C_{26}}{2}\log\left(L^{(\widetilde{\alpha}-1)/2}/\log L\right), \tag{6.5.52}$$

which is negative if one chooses C_{16} small enough, and h sufficiently small (so that $L(h)$ is large). □

6.6. Number of contacts under $\mathbf{P}^{\omega,\beta}_{N,h=0}$, proof of Theorem 6.2.3

As for Theorem 6.2.1, the cases $\alpha < 1$ and $\alpha > 1$ present some dissimilarities and therefore the details for them will be treated separately. However, in the first part of this section, we give the ideas behind the proof and its first step for the two cases. As we always have in this section $h = 0$, we drop dependence in h in the notation.

Recall the definition (6.1.6) of our environment ω. For any event A, define

$$Z^{\omega,\beta}_N(A) := \mathbf{E}\left[e^{\sum_{n=1}^N \beta\omega_n \mathbf{1}_{\{n\in\tau\}}}\mathbf{1}_{\{\tau\in A\}}\right]. \tag{6.6.1}$$

We prove Theorem 6.2.3 (in fact a finer result that gives an estimate on the asymptotic of the tail behavior of $|\tau\cap[0,N]|$).

Proposition 6.6.1. *For almost every ω, for every $\varepsilon > 0$ there exists some a_0 (depending on ω, β and ε), and some $\delta = \delta(\varepsilon)$ which can be made arbitrarily small, such that for all $a \geqslant a_0$ and for all $N \in \mathbb{N}$ one has:*
if $\alpha > 1$

$$\begin{aligned} &Z^{\omega,\beta}_N(|\tau\cap[0,N]| = a) \leqslant a^{\varepsilon} N^{-\alpha}\max(a^{-\widetilde{\alpha}\alpha}, N^{-1}), \qquad \text{if } a \leqslant N^{\frac{1}{\widetilde{\alpha}}+\delta},\\ &Z^{\omega,\beta}_N(|\tau\cap[0,N]| = a) \leqslant e^{-N^{\delta^4}} \qquad \text{if } a > N^{\frac{1}{\widetilde{\alpha}}+\delta}; \end{aligned} \tag{6.6.2}$$

and if $\alpha < 1$

$$\begin{aligned} &Z^{\omega,\beta}_N(|\tau\cap[0,N]| = a) \leqslant a^{\varepsilon} N^{-\alpha} a^{-\widetilde{\alpha}}, \qquad \text{if } a \leqslant N^{\frac{\alpha}{\widetilde{\alpha}}+\delta},\\ &Z^{\omega,\beta}_N(|\tau\cap[0,N]| = a) \leqslant e^{-N^{\delta^4}} \qquad \text{if } a > N^{\frac{\alpha}{\widetilde{\alpha}}+\delta}. \end{aligned} \tag{6.6.3}$$

From the above proposition, that we prove in Section 6.6.1, we get the following result that is stronger than Theorem 6.2.3, and gives an upper tail for the number of contacts points.

Corollary 6.6.2. *For almost every ω, for every ε, there exist some $\delta > 0$ and a constant $\mathrm{C} = \mathrm{C}(\omega,\beta,\varepsilon)$ such that for all N, for every $a \leqslant N^{\frac{1\wedge\alpha}{\widetilde{\alpha}}+\delta}$*

$$\mathbf{P}^{\omega,\beta}_N\left(|\tau\cap[0,N]| = a\right) \leqslant \begin{cases} \mathrm{C}a^{\varepsilon-\widetilde{\alpha}} & \text{if } \alpha < 1,\\ \mathrm{C}a^{\varepsilon}\max(a^{-\widetilde{\alpha}\alpha}, N^{-1}) & \text{if } \alpha > 1, \end{cases} \tag{6.6.4}$$

and

$$\mathbf{P}^{\omega,\beta}_N\left(|\tau\cap[0,N]| \geqslant N^{\frac{1\wedge\alpha}{\widetilde{\alpha}}+\delta}\right) \leqslant \mathrm{C}e^{-N^{\delta^4/2}}. \tag{6.6.5}$$

Moreover

$$Z^{\omega,\beta}_N \leqslant \mathrm{C}N^{-\alpha}. \tag{6.6.6}$$

Proof We prove everything in the case $\alpha < 1$, the other case being similar. Let us start with the last statement. Fix $\varepsilon > 0$ small, and then some $\delta \leqslant \varepsilon$ and a_0 such that Proposition 6.6.1 holds for ε. Then, a_0 being fixed, there exist a constant $C(a_0)$ such that for all $a \leqslant a_0$

$$Z^{\omega,\beta}_N(|\tau\cap[0,N]| = a) \leqslant \mathbf{P}(|\tau\cap[0,N]| = a) \leqslant aC(a_0)N^{-\alpha}, \tag{6.6.7}$$

where we used Proposition 6.A.1 to get the last inequality. This, together with the estimates (6.6.3), implies that

$$Z_N^{\omega,\beta} = Z_N^{\omega,\beta}(|\tau\cap[0,N]|<a_0) + \sum_{a=a_0}^{\infty} Z_N^{\omega,\beta}(|\tau\cap[0,N]|=a)$$
$$\leqslant C(a_0)N^{-\alpha}\sum_{a=0}^{a_0-1} a + N^{-\alpha}\sum_{a=a_0}^{\infty} a^{\varepsilon-\widetilde{\alpha}} \leqslant CN^{-\alpha}. \quad (6.6.8)$$

For the first two statements, one uses that

$$Z_N^{\omega,\beta} \geqslant \mathbf{P}(\tau_1 > N) \geqslant C_{27}N^{-\alpha}, \quad (6.6.9)$$

for some constant $C_{27}>0$. Combined with (6.6.3) (or with (6.6.7) for $a<a_0$), this gives the right bound for the first statement for $a \leqslant N^{\frac{\alpha}{\widetilde{\alpha}}+\delta}$. The second statement is also an easy consequence of (6.6.9) and (6.6.3), writing

$$\mathbf{P}_N^{\omega,\beta}\left(|\tau\cap[0,N]| \geqslant N^{\frac{\alpha}{\widetilde{\alpha}}+\delta}\right) \leqslant \frac{1}{Z_N^{\omega,\beta}} Z_N^{\omega,\beta}\left(|\tau\cap[0,N]| \geqslant N^{\frac{\alpha}{\widetilde{\alpha}}+\delta}\right)$$
$$\leqslant (C_{27})^{-1}N^{\alpha}\sum_{k=N^{\frac{\alpha}{\widetilde{\alpha}}+\delta}}^{N} e^{-N^{\delta^4}} \leqslant Ce^{-N^{\delta^4/2}}. \quad (6.6.10)$$

□

At the end of the Section, we prove the following result that complements the above, and gives a lower tail for the number of contact points under $\mathbf{P}_N^{\omega,\beta}$.

Proposition 6.6.3. *For almost every ω, for any ε, there exists a_0 such that for $a \geqslant a_0$, and $a \leqslant N^{\frac{1\wedge\alpha}{\widetilde{\alpha}}-\varepsilon}$ one has*

$$Z_N^{\omega,\beta}(|\tau\cap[0,N]|=a) \geqslant a^{-\varepsilon}N^{-\alpha}a^{-\frac{\widetilde{\alpha}(\alpha+1)-1}{1\wedge\alpha}}. \quad (6.6.11)$$

and

$$\mathbf{P}_N^{\omega,\beta}(|\tau\cap[0,N]|=a) \geqslant a^{-\varepsilon-\frac{\widetilde{\alpha}(\alpha+1)-1}{1\wedge\alpha}}. \quad (6.6.12)$$

Note that Corollary 6.6.2 and Proposition 6.6.3 give respectively the upper and the lower bound in Proposition 6.2.4.

We recall briefly here Section 6.2.2, which describes the strategy to adopt to prove Proposition 6.6.1. Recall the definition (6.2.4) of $V_N^{\widehat{\tau}}(\tau)$, the number of $\widehat{\tau}$-stretches visited by τ, and inequality (6.2.5)

$$\mathbb{E}\left[e^{\sum_{n=1}^{N}\beta\omega_n\mathbf{1}_{\{n\in\tau\}}}\right] \leqslant \left(\frac{1+e^{-\beta}}{2}\right)^{V_N^{\widehat{\tau}}(\tau)}, \quad (6.2.5)$$

where $\mathbb{E}$ denotes the average only on the values of $\{X_i\}_{i\in\mathbb{N}}$, *i.e.* on the disorder ω conditionally on the realization of $\widehat{\tau}$. One estimates in Lemmas 6.6.4 and 6.6.8 the contribution of trajectories of τ that visit many $\widehat{\tau}$-stretches, and in Lemmas 6.6.5 and 6.6.9 the contribution of trajectories of τ that visit few $\widehat{\tau}$-stretches.

6.6.1. Proof of Proposition 6.6.1 in the case $\alpha > 1$. We prove the Proposition from the two following Lemmas.

Lemma 6.6.4. *Given $\delta > 0$, there exists some $x_0(\omega, \beta, \varepsilon)$ such that for every $x \geqslant x_0$, and every $y \in [x, x^{\widetilde{\alpha}(1-\delta)}]$ one has*

$$Z_y^{\omega,\beta}(\tau_x = y) \leqslant e^{-x^{\delta/2}}. \tag{6.6.13}$$

Lemma 6.6.5. *If $\alpha > 1$, for any $\varepsilon > 0$ there exists some $\delta > 0$ and $a_0 \in \mathbb{N}$ such that for all $a \geqslant a_0$, and for all $N \in \mathbb{N}$ one has*

$$\mathbf{P}\left[|\tau \cap [0,N]| = a \ ; \ \forall x \in [a^\delta, a-1], \tau_x > x^{\widetilde{\alpha}(1-\delta)}\right] \leqslant a^\varepsilon N^{-\alpha} \max(a^{-\alpha\widetilde{\alpha}}, N^{-1})/2. \tag{6.6.14}$$

Proof of Proposition 6.6.1 Let us fix $\varepsilon > 0$. As ω is non-positive, the definition of $Z_N^{\omega,\beta}(A)$ implies that for every A

$$Z_N^{\omega,\beta}(A) \leqslant \mathbf{P}(A). \tag{6.6.15}$$

Therefore, Lemma 6.6.5 gives us directly that one can find δ such that for a large enough one has

$$Z_N^{\omega,\beta}(|\tau \cap [0,N]| = a \ ; \ \forall x \in [a^\delta, a-1], \tau_x > x^{\widetilde{\alpha}(1-\delta)}) \leqslant a^\varepsilon N^{-\alpha} \max(a^{-\alpha\widetilde{\alpha}}, N^{-1})/2. \tag{6.6.16}$$

Let us show now that

$$Z_N^{\omega,\beta}(|\tau \cap [0,N]| = a, \ \exists x \in [a^\delta, a-1], \tau_x \leqslant x^{\widetilde{\alpha}(1-\delta)}) \leqslant a^\varepsilon N^{-\alpha} \max(a^{-\alpha\widetilde{\alpha}}, N^{-1})/2, \tag{6.6.17}$$

(which combined with (6.6.16) gives the first part of (6.6.2)). We do so by decomposing over all possible values for x and τ_x.

$$\begin{aligned} Z_N^{\omega,\beta}(|\tau \cap [0,N]| = a, \ \exists x \in [a^\delta, a-1], \tau_x \leqslant x^{\widetilde{\alpha}(1-\delta)}) &\leqslant \sum_{x=a^\delta}^{a-1} \sum_{y=x}^{x^{\widetilde{\alpha}(1-\delta)}} Z_N^{\omega,\beta}(\tau_x = y; \tau_a > N) \\ &\leqslant \sum_{x=a^\delta}^{a-1} \sum_{y=x}^{x^{\widetilde{\alpha}(1-\delta)} \wedge N} Z_y^{\omega,\beta}(\tau_x = y) \mathbf{P}(\tau_{a-x} > N - y). \end{aligned} \tag{6.6.18}$$

Using Lemma 6.6.4, one gets that the above is smaller than

$$\sum_{x=a^\delta}^{a-1} \sum_{y=x}^{x^{\widetilde{\alpha}(1-\delta)} \wedge N} e^{-x^{\delta/2}} \mathbf{P}(\tau_{a-x} > N - y). \tag{6.6.19}$$

If $a \geqslant N^\delta$ then $e^{-x^{\delta/2}} \leqslant e^{-N^{\delta^3/2}}$ so that (6.6.19) is smaller than $N^2 e^{-N^{\delta^3/2}}$ and (6.6.17) holds. If $a \leqslant N^\delta$ and δ is small enough, from Proposition 6.A.2, $\mathbf{P}(\tau_{a-x} > N - y) \leqslant 2a\bar{\mathrm{K}}(N)$, and therefore one has

$$\sum_{x=a^\delta}^{a-1} \sum_{y=x}^{x^{\widetilde{\alpha}(1-\delta)} \wedge N} e^{-x^{\delta/2}} \mathbf{P}(\tau_{a-x} > N - y) \leqslant 2a a^{\widetilde{\alpha}(1-\delta)+1} e^{-a^{\delta^2}/2} \bar{\mathrm{K}}(N) \tag{6.6.20}$$

which implies (6.6.17).

For the case $a > N^{\frac{1}{\tilde{\alpha}}+\delta}$, the left-hand side of (6.6.16) is equal to zero for δ small enough, since the condition $\tau_a > a^{\tilde{\alpha}(1-\delta)} > N^{1+\delta(\tilde{\alpha}-1-\delta)}$ would contradict the event $\{|\tau \cap [0,N]| = a\}$. Moreover the left-hand side of (6.6.17) is smaller than $N^2 e^{-N^{\delta^3/2}} \leqslant e^{-N^{\delta^4}}$ for N large enough, so that Proposition 6.6.1 is proved. $\square$

Proof of Lemma 6.6.4 Note that if one wants to visit only a few $\widehat{\tau}$-stretches, one has to put a lot of contacts in very few $\widehat{\tau}$-stretches. One then notices that, according to Lemma 6.A.4, if y is larger than some $N_0(\widehat{\tau})$, the longest $\widehat{\tau}$-stretch in the interval $[0,y]$ is of length smaller than $y^{1/\tilde{\alpha}}\log y \leqslant \tilde{\alpha} x^{1-\delta}\log x$ for any $y \leqslant x^{(1-\delta)\tilde{\alpha}}$. For that reason if $\tau_x = y$, with $x \geqslant N_0(\widehat{\tau})$ and for the values of y considered, there cannot be a $\widehat{\tau}$-stretch longer than $x^{1-3\delta/4}$, so that

$$V_x^{\widehat{\tau}}(\tau) \geqslant \frac{x}{\max\{\tau_{i+1}-\tau_i \mid \tau_i \leqslant y\}} \geqslant x^{3\delta/4}, \tag{6.6.21}$$

and from (6.2.5) one gets that for x large enough

$$\mathbb{E}\left[\sum_{y=x}^{x^{(1-\delta)\tilde{\alpha}}} Z_y^{\omega,\beta}(\tau_x = y)\right] \leqslant \sum_{y=x}^{x^{(1-\delta)\tilde{\alpha}}} \mathbf{P}(\tau_x = y)\left(\frac{1+e^{-\beta}}{2}\right)^{x^{3\delta/4}} \leqslant \left(\frac{1+e^{-\beta}}{2}\right)^{x^{3\delta/4}} \tag{6.6.22}$$

Using the Markov inequality and the Borel-Cantelli Lemma, one gets that there exists a (random) $x_0(\omega)$ such that for all $x \geqslant x_0(\omega)$

$$\sum_{y=x}^{x^{(1-\delta)\tilde{\alpha}}} Z_y^{\beta,\omega}(\tau_x = y) \leqslant \exp\left(-x^{\delta/2}\right). \tag{6.6.23}$$

$\square$

The condition $\forall x \geqslant a^\delta, \tau_x > x^{(1-\delta)\tilde{\alpha}}$ implies that τ_x is stretched out at all scales, and one has to sum over the different ways of stretching τ. Thus Lemma 6.6.5 requires a multi-scale analysis and for the sake of clarity, we restate it in an apparently more complicated version. One reason for doing so is that it allows to do a proof by induction.

Lemma 6.6.6. *For all values of $l \in \mathbb{N}$, if $\delta_2 \leqslant \delta(l)$ there exists a constant $C(l)$ such that for all $N \in \mathbb{N}$ and $a \in \mathbb{N}$ large enough with $a \leqslant N^{\frac{1}{\tilde{\alpha}-\delta_2}}$, one has*

$$\max_{d\in[0,a^{\alpha^{-l}(\tilde{\alpha}-\delta_2)^{-l+1}}/2]} \mathbf{P}\left[|\tau \cap [0,N-d]| = a\ ;\ \forall x \in [a^{\delta_2}, a-1], \tau_x > x^{\tilde{\alpha}-\delta_2} - d\right]$$
$$\leqslant C(l) N^{-\alpha} a^{(\alpha(\tilde{\alpha}-\delta_2))^{-l}} \max(a^{-\alpha(\tilde{\alpha}-\delta_2)}, N^{-1}). \tag{6.6.24}$$

Remark 6.6.7. The probability of the event on the left hand side of (6.6.24) is zero when $a > N^{\frac{1}{\tilde{\alpha}-\delta_2}}$ as $|\tau \cap [0,N-d]| = a$ implies $\tau_{a-1} \leqslant N-d$. Therefore the result holds in fact for all a. Using (6.6.7) one notices that the result holds for all a and N (after eventually changing the constant $C(l)$).

One gets Lemma 6.6.5 from this by taking δ_2 small enough and l, a large enough and $d=0$. The reason we prove the result for all $d \in [0, a^{\alpha^{-l}(\tilde{\alpha}-\delta_2)^{-l+1}}/2]$ and not only for $d=0$ is to make the induction step in the proof work.

Proof We introduce some additional notations that will make the proof more readable. We define for all $j \geqslant 0$

$$\begin{aligned} x_j &:= a^{(\alpha(\tilde{\alpha}-\delta_2))^{-j}} \quad (x_0 = a), \\ y_j &:= \frac{1}{2} x_j^{\tilde{\alpha}-\delta_2} = \frac{1}{2} a^{\alpha^{-j}(\tilde{\alpha}-\delta_2)^{-j+1}}. \end{aligned} \tag{6.6.25}$$

With these notations, (6.6.24) reads

$$\max_{d\in[0,y_l]} \mathbf{P}\left[|\tau \cap [0, N-d]| = a \; ; \; \forall x \in [a^{\delta_2}, a-1], \tau_x \geqslant x^{\tilde{\alpha}-\delta_2} - d\right] \\ \leqslant C(l) N^{-\alpha} x_l \max(y_0^{-\alpha}, N^{-1}). \tag{6.6.26}$$

Note that x_j and y_j are decreasing in j, and also that provided that δ_2 is small enough, one has for any $j \geqslant 0$ that both x_j and y_j tends to infinity with a, and

$$y_j \gg x_j. \tag{6.6.27}$$

Let us start with the proof of the case $l=0$. On the event we consider, τ_{a-1} has to be larger than $(a-1)^{\tilde{\alpha}-\delta_2} - d$, *i.e.* larger than what it would typically be under $\mathbf{P}$. We use Proposition 6.A.2 to bound from above the probability of this event. The quantity we have to bound is smaller than

$$\begin{aligned} \mathbf{P}&\left[\tau_a > N-d \; ; \; \tau_{a-1} \in \left((a-1)^{\tilde{\alpha}-\delta_2} - d, N-d\right]\right] \\ &= \sum_{y=(a-1)^{\tilde{\alpha}-\delta_2}+1-d}^{N-d} \mathbf{P}(\tau_{a-1} = y)\mathbf{P}(\tau_1 > N-y-d) \\ &= (1+o(1)) \sum_{y=(a-1)^{\tilde{\alpha}-\delta_2}+1-d}^{N-d} a\mathrm{K}(y)\bar{\mathrm{K}}(N-d-y) \\ &\leqslant C(0) a N^{-\alpha}(a^{-\alpha(\tilde{\alpha}-\delta_2)} \vee N^{-1}), \end{aligned} \tag{6.6.28}$$

where here (and later in the proof) $o(1)$ denotes a quantity that goes to zero when both a and N are large. Proposition 6.A.2 was used to get from the second to the third line, the last inequality coming from a straightforward computation, using the assumption on $\mathrm{K}(\cdot)$.

We assume now that (6.6.26) holds for all $l' < l$ and prove it for l. Fix $d \leqslant y_l$. Assume that $\delta_2 = \delta_2(l)$ is small enough, so that $x_l \geqslant a^{\delta_2}$. We decompose over all the possible values for τ_{x_l}, and use the Markov property for the renewal process. The l.h.s. of (6.6.26) is smaller than

$$\mathbf{P}\left[|\tau\cap[0,N-d]|=a\ ;\ \forall x\in[x_l,a-1],\tau_x>x^{\widetilde{\alpha}-\delta_2}-d\right]=\sum_{d_1=2y_l+1}^{N}\mathbf{P}\left(\tau_{x_l}=d_1-d\right)$$
$$\times\,\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\ ;\forall x\in[0,a-x_l-1],\tau_x\geqslant(x+x_l)^{\widetilde{\alpha}-\delta_2}-d_1\right]. \tag{6.6.29}$$

On the event we are considering in (6.6.26), τ_{x_l} has to be larger than $2y_l-d\geqslant y_l$, *i.e.* larger than what it would typically be under $\mathbf{P}$ (cf. (6.6.27)). Therefore $\mathbf{P}\left(\tau_{x_l}=d_1-d\right)$ can always be estimated by using Proposition 6.A.2. If $d_1\leqslant y_i$, the quantity

$$\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\ ;\forall x\in[0,a-x_l],\tau_x\geqslant(x+x_l)^{\widetilde{\alpha}-\delta_2}-d_1\right]$$
$$\leqslant\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\ ;\forall x\in[0,a-x_l],\tau_x\geqslant x^{\widetilde{\alpha}-\delta_2}-d_1\right] \tag{6.6.30}$$

can be estimated by using the induction hypothesis (6.6.26) for $i<l$. For this reason we decompose the sum in the right hand side of (6.6.29) in l terms, corresponding to $d_1\in(2y_l,y_{l-1}]$, $d_1\in(y_j,y_{j-1}]$ ($j\in\{1,\dots l-1\}$) and $d_1\in(y_0,N]$. When $d_1>y_0$ one cannot use the induction step and for this reason the contribution from $d_1\in(y_0,N]$ is dealt with separately.

Notice that

$$\sum_{d_1\in(y_j,y_{j-1}]}\mathbf{P}\left(\tau_{x_l}=d_1-d\right)$$
$$\times\,\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\ ;\forall x\geqslant 0,\tau_x\geqslant(x+x_l)^{\widetilde{\alpha}-\delta_2}-d_1\right]$$
$$\leqslant(1+o(1))\sum_{d_1\in(y_j,y_{j-1}]}x_l\mathrm{K}(d_1-d)C(j-1)N^{-\alpha}x_{j-1}\max(y_0^{-\alpha},N^{-1})$$
$$\leqslant C'(j)x_ly_j^{-\alpha}x_{j-1}\max(y_0^{-\alpha},N^{-1}). \tag{6.6.31}$$

From the definitions of x_j and y_j one has $y_j^{\alpha}=\frac{1}{2^{\alpha}}x_{j-1}$, so that $y_j^{-\alpha}x_{j-1}=2^{\alpha}$ for all $j\geqslant 1$. The term corresponding to $d_1\in(2y_l,y_{l-1}]$ can be dealt with in the same manner.

Now we estimate the sum on $d_1\in(y_0,N]$. By Proposition 6.A.2 one has

$$\sum_{d_1\in(y_0,N]}\mathbf{P}\left(\tau_{x_l}=d_1-d\right)\mathbf{P}\left(|\tau\cap[0,N-d_1]|=a-x_l\right)$$
$$\leqslant(1+o(1))\sum_{d_1\in(y_0,N]}x_l\mathrm{K}(d_1-d)\mathbf{P}\left(|\tau\cap[N-d_1]|=a-x_l\right). \tag{6.6.32}$$

If d_1 is less that $N/2$, then choosing δ small enough

$$\mathbf{P}\left(|\tau\cap[N-d_1]|=a-x_l\right)=\sum_{x=1}^{N-d_1}\mathbf{P}\left(\tau_{a-x_l-1}=x\right)\bar{\mathrm{K}}(N-d_1-x)$$

$$=\sum_{x=1}^{N^{1-\delta}}\mathbf{P}\left(\tau_{a-x_l-1}=x\right)\bar{\mathrm{K}}(N-d_1-x)+\sum_{x=N^{1-\delta}+1}^{N-d_1}\mathbf{P}\left(\tau_{a-x_l-1}=x\right)\bar{\mathrm{K}}(N-d_1-x)$$

$$=(1+o(1))\left[\bar{\mathrm{K}}(N-d_1)+\sum_{x=N^{1-\delta}+1}^{N-d_1}a\mathrm{K}(x)\bar{\mathrm{K}}(N-d_1-x)\right]\leqslant c(l)N^{-\alpha},\quad(6.6.33)$$

where we made use of $\bar{\mathrm{K}}(N-d_1-x)=\bar{\mathrm{K}}(N-d_1)(1+o(1))$ for $x\leqslant N^{1-\delta}$, and of Proposition 6.A.2 for $x>N^{1-\delta}$. Note that we also used the restriction $a\leqslant N^{\frac{1}{\tilde{\alpha}-\delta_2}}$ for the last inequality, to get that $aN^{-\alpha(1-\delta)}\leqslant 1$. Hence one has

$$\sum_{d_1\in(y_0,N/2]}x_l\mathrm{K}(d_1-d)\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\right]\leqslant c'(l)x_ly_0^{-\alpha}N^{-\alpha},\qquad(6.6.34)$$

for $c'(l)$ large enough. To estimate the contribution of $d_1\in(N/2,N]$, one notices that

$$\sum_{L=0}^{\infty}\mathbf{P}\left(|\tau\cap[0,L]|=a-x_l\right)=\mathbf{E}\Big[\#\{L\in\mathbb{R},\ L\in[\tau_{a-x_l},\tau_{a-x_l+1})\}\Big]$$

$$=\mathbf{E}[\tau_{a-x_l+1}-\tau_{a-x_l}]=\mathbf{E}[\tau_1].\quad(6.6.35)$$

so that

$$\sum_{d_1\in(N/2,N]}x_l\mathrm{K}(d_1-d)\mathbf{P}\left(|\tau\cap[N-d_1]|=a-x_l\right)$$

$$\leqslant c(l)x_lN^{-(1+\alpha)}\sum_{d_1\in(N/2,N]}\mathbf{P}\left[|\tau\cap[N-d_1]|=a-x_l\right]\leqslant c'(l)x_lN^{-(1+\alpha)}.\quad(6.6.36)$$

This, together with (6.6.31) and (6.6.34) gives the result.

□

6.6.2. Proof of Proposition 6.6.1 in the case $\alpha<1$. One has to adapt Lemmas 6.6.4 and 6.6.5 to this new case. The difference lies in the following fact: as here the renewal does not have finite mean, one needs a stretch of length much longer than x to set x contacts on the defect line.

Lemma 6.6.8. *Given $\delta>0$, there exists some $x_0(\omega,\beta,\varepsilon)$ such that for every $x\geqslant x_0$, every $y\in[x,x^{\frac{\tilde{\alpha}}{\alpha}(1-\delta)}]$ one has*

$$Z_y^{\omega,\beta}(\tau_x=y)\leqslant e^{-x^{\delta/8}}.\qquad(6.6.37)$$

Lemma 6.6.9. *If $\alpha<1$, for any $\varepsilon>0$ there exists $\delta>0$ and $a_0\in\mathbb{N}$ such that for all $a\geqslant a_0$, for all N one has*

$$\mathbf{P}\left[|\tau\cap[0,N]|=a\ ;\ \forall x\in[a^{\delta},a-1],\tau_x>x^{\frac{\tilde{\alpha}}{\alpha}(1-\delta)}\right]\leqslant a^{\varepsilon-\tilde{\alpha}}N^{-\alpha}/2.\qquad(6.6.38)$$

The proof from the two Lemmas of the case $\alpha < 1$ in Proposition 6.6.1 is exactly the same as in the case $\alpha > 1$, and therefore we leave it to the reader.

Proof of Lemma 6.6.8 First note that if one wants to visit only a limited number of stretches after x jumps (say less than $x^{\delta/2}$), one must do at least $x^{1-\delta/2}$ jumps in the same stretch. On the other hand, note that provided x is large enough, from Lemma 6.A.4 the longest $\widehat{\tau}$-stretch in $[0,y]$ for $y \leqslant x^{\frac{\widetilde{\alpha}}{\alpha}(1-\delta)}$ has length smaller than $x^{\frac{1-(3\delta/4)}{\alpha}}$. For these reasons if x is large enough, and for the values of y that we consider

$$\{V_y^{\widehat{\tau}}(\tau) \leqslant x^{\delta/2}; \tau_x = y\} \subset \left\{\exists t \in [0,x), (\tau_{t+x^{1-\delta/2}} - \tau_t) \leqslant x^{\frac{1-(3\delta/4)}{\alpha}}\right\}. \qquad (6.6.39)$$

As a consequence

$$\begin{aligned} Z_y^{\omega,\beta}(V_y^{\widehat{\tau}}(\tau) \leqslant x^{\delta/2}\ ;\ \tau_x = y) &\leqslant \mathbf{P}\left(\exists t \in [0,x), (\tau_{t+x^{1-\delta/2}} - \tau_t) \leqslant x^{\frac{1-(3\delta/4)}{\alpha}}\right) \\ &\leqslant x\mathbf{P}\left(\tau_{x^{1-\delta/2}} \leqslant x^{\frac{1-(3\delta/4)}{\alpha}}\right) \leqslant x\left[\mathbf{P}\left(\tau_1 \leqslant x^{\frac{1-(3\delta/4)}{\alpha}}\right)\right]^{x^{1-\delta/2}} \leqslant \frac{1}{2}e^{-x^{\delta/8}} \qquad (6.6.40) \end{aligned}$$

if x is large enough. On the other hand according to (6.2.5)

$$\mathbb{E}\left[\sum_{y=x}^{x^{\frac{\widetilde{\alpha}}{\alpha}(1-\delta)}} Z_y^{\omega,\beta}(V_y^{\widehat{\tau}}(\tau) > x^{\delta/2}\ ; \tau_x = y)\right] \leqslant \left(\frac{1+e^{-\beta}}{2}\right)^{x^{-\delta/2}}. \qquad (6.6.41)$$

Using the Markov inequality and the Borel-Cantelli Lemma, one gets that there exists a (random) integer x_0 such that for all $x \geqslant x_0$

$$\sum_{y=x}^{x^{\frac{\widetilde{\alpha}}{\alpha}(1-\delta)}} Z_y^{\omega,\beta}\left(V_y^{\widehat{\tau}}(\tau) > x^{\delta/2}\ ; \tau_x = y\right) \leqslant e^{x^{-\delta/8}}/2, \qquad (6.6.42)$$

which together with (6.6.40) ends the proof.

□

For Lemma 6.6.9 one proceeds as for Lemma 6.6.5, and prove a recursive statement.

Lemma 6.6.10. *For all values of l, if $\delta_2 \leqslant \delta(l)$ there exists a constant $C(l)$ such that for all $N \in \mathbb{N}$ and $a \in \mathbb{N}$ large enough with $a \leqslant N^{\alpha(\widetilde{\alpha}-\alpha\delta_2)^{-1}}$, one has*

$$\max_{d \in [0, a^{\alpha^{-1}(\widetilde{\alpha}-\alpha\delta_2)^{-l+1}}/2]} \mathbf{P}\left[|\tau \cap [0, N-d]| = a\ ;\ \forall x \in [a^{\delta_2}, a-1], \tau_x > x^{\frac{\widetilde{\alpha}}{\alpha}-\delta_2} - d\right] \leqslant C(l) N^{-\alpha} a^{(\widetilde{\alpha}-\alpha\delta_2)^{-l}} a^{-(\widetilde{\alpha}-\alpha\delta_2)}. \qquad (6.6.43)$$

Note that Remark 6.6.7 made for Lemma 6.6.6 applies also here.

Proof This is very similar to the $\alpha > 1$ case. One uses some different notations this time:

$$\begin{aligned} x_j &:= a^{(\widetilde{\alpha}-\alpha\delta_2)^{-j}} \quad (x_0 = a), \\ y_j &:= \frac{1}{2} x_j^{\frac{\widetilde{\alpha}}{\alpha}-\delta_2} = \frac{1}{2} a^{\alpha^{-1}(\widetilde{\alpha}-\alpha\delta_2)^{-j+1}}. \end{aligned} \qquad (6.6.44)$$

With these notations, (6.6.43) reads

$$\max_{d\in[0,y_l]} \mathbf{P}\left[|\tau\cap[0,N-d]|=a\ ;\ \forall x\in[a^{\delta_2},a-1],\tau_x\geqslant x^{\frac{\tilde{\alpha}}{\alpha}-\delta_2}-d\right]\leqslant C(l)N^{-\alpha}x_l y_0^{-\alpha}. \tag{6.6.45}$$

We also have that x_j and y_j are decreasing in j, and that provided that δ_2 is small enough, one has for any $j\geqslant 0$ that both x_j and y_j tends to infinity with a, and that

$$y_j \gg x_j^{\alpha}. \tag{6.6.46}$$

We prove the statement first in the case $l=0$. Note that on the event we consider, $\tau_{a-1}\gg a^{\alpha}$ *i.e.* τ_{a-1} has to be much larger than what it would typically be under $\mathbf{P}$. Therefore one can use Proposition 6.A.1 to estimate its probability. We get that the l.h.s. of (6.6.45) is smaller than

$$\begin{aligned}
&\mathbf{P}\left[\tau_a>N-d\ ;\ \tau_{a-1}\in((a-1)^{\frac{\tilde{\alpha}}{\alpha}-\delta_2}-d,N-d]\right]\\
&=\sum_{y=(a-1)^{\frac{\tilde{\alpha}}{\alpha}-\delta_2}+1-d}^{N-d}\mathbf{P}[\tau_{a-1}=y]\mathbf{P}\left[\tau_1>N-d-y\right]=(1+o(1))\sum_{y=(a-1)^{\frac{\tilde{\alpha}}{\alpha}-\delta_2}+1-d}^{N-d}a\mathrm{K}(y)\bar{\mathrm{K}}(N-d-y)\\
&\leqslant C(0)aN^{-\alpha}\max\left(a^{(\tilde{\alpha}-\alpha\delta_2)},N^{-\alpha}\right)=C(0)a^{1-(\tilde{\alpha}-\alpha\delta_2)}N^{-\alpha}.
\end{aligned} \tag{6.6.47}$$

Proposition 6.A.1 was used to get the third line. The last equality comes from the fact that we consider only $a\leqslant N^{\alpha(\tilde{\alpha}-\delta_2\alpha)^{-1}}$. Here (and later in the proof) $o(1)$ denotes a quantity that tends to zero when both a and N gets large.

We now assume the statement for all $l'<l$ and prove it for l. Fix $d\leqslant y_l$. Assume that $\delta_2=\delta_2(l)$ is small enough, so that $x_l\geqslant a^{\delta_2}$. We decompose over all the possible values for τ_{x_l} and use the Markov property for the renewal process, so that the l.h.s. of (6.6.45) is smaller than

$$\begin{aligned}
&\sum_{d_1=2y_l+1}^{N}\mathbf{P}\left(\tau_{x_l}=d_1-d\right)\\
&\qquad\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\ ;\ \forall x\in[0,a-x_l-1],\tau_x>(x+x_l)^{\frac{\tilde{\alpha}}{\alpha}-\delta_2}-d_1\right].
\end{aligned} \tag{6.6.48}$$

Note that in the above sum, one always has $\tau_{x_l}\geqslant 2y_l-d\geqslant y_l$, *i.e.* is much larger than the value it typically takes under $\mathbf{P}$ (cf. (6.6.46)). Therefore one can use Proposition 6.A.1 to estimate the term $\mathbf{P}\left(\tau_{x_l}=d_1-d\right)$. As for the second term

$$\begin{aligned}
&\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\ ;\ \forall x\in[0,a-x_l-1],\tau_x>(x+x_l)^{\frac{\tilde{\alpha}}{\alpha}-\delta_2}-d_1\right]\\
&\leqslant\mathbf{P}\left[|\tau\cap[0,N-d_1]|=a-x_l\ ;\ \forall x\in[0,a-x_l-1],\tau_x>x^{\frac{\tilde{\alpha}}{\alpha}-\delta_2}-d_1\right],
\end{aligned} \tag{6.6.49}$$

and it can be bounded from above by using the induction hypothesis when $d_1\leqslant y_i$, $i<l$.

For this reason, we separate the contribution of the different terms $d_1 \in (2y_l, y_{l-1}]$, $d_1 \in (y_j, y_{j-1}]$ ($j \in \{1, \dots l-1\}$) and $d_1 \in (y_0, N]$ in the sum (6.6.48). We just focus on the last one, as the computation for $d_1 \leqslant y_0$ is exactly the same as in Lemma 6.6.6 (see (6.6.31)), using Proposition 6.A.1 instead of Proposition 6.A.2. For $d_1 \in (y_0, N]$, one cannot use the induction hypothesis. Using Proposition 6.A.1, one gets

$$\sum_{d_1 \in (y_0, N]} \mathbf{P}\left(\tau_{x_l} = d_1 - d\right) \mathbf{P}\left(|\tau \cap [0, N-d_1]| = a - x_l\right) \\ \leqslant (1+o(1)) \sum_{d_1 \in (y_0, N]} x_l \mathrm{K}(d_1) \mathbf{P}\left(|\tau \cap [0, N-d_1]| = a - x_l\right). \quad (6.6.50)$$

As in (6.6.33) one shows that for $d_1 \leqslant N/2$, uniformly on the choice of $a \leqslant N^{\alpha(\widetilde{\alpha}-\alpha\delta_2)^{-1}}$, one has

$$\mathbf{P}\left(|\tau \cap [0, N-d_1]| = a - x_l\right) \leqslant c(l) N^{-\alpha}, \quad (6.6.51)$$

so that

$$\sum_{d_1 \in (y_0, N/2]} x_l \mathrm{K}(d_1) \mathbf{P}\left(|\tau \cap [0, N-d_1]| = a - x_l\right) \leqslant c'(l) x_l y_0^{-\alpha} N^{-\alpha}. \quad (6.6.52)$$

For the case $d_1 > N/2$, one remarks that

$$\sum_{L=0}^{N/2-1} \mathbf{P}\left(|\tau \cap [0, L]| = a - x_l\right) = \mathbf{E}\Big[\left|\{L \in [0, N/2-1] \ : L \in [\tau_{a-x_l}, \tau_{a-x_l+1})\}\right|\Big] \\ \leqslant \mathbf{E}\left[\max(\tau_1, N/2)\right]. \quad (6.6.53)$$

Therefore

$$\sum_{d_1 \in (N/2, N]} x_l \mathrm{K}(d_1) \mathbf{P}\left[|\tau \cap [0, N-d_1]| = a - x_l\right] \\ \leqslant c(l) x_l N^{-(1+\alpha)} \mathbf{E}\left[\max(\tau_1, N/2)\right] \leqslant c'(l) x_l N^{-2\alpha} \leqslant c'(l) x_l (y_0 N)^{-\alpha}. \quad (6.6.54)$$

The last inequality comes from the fact that $y_0 \leqslant N$ for the range of a that we consider.

□

6.6.3. Proof of Proposition 6.6.3. Here the strategy consists in targeting directly the first $\widehat{\tau}$-stretch with $\omega \equiv 0$, of size larger than $2C_{28} a^{\frac{1}{1\wedge\alpha}}$ (with C_{28} a constant to be determined, depending only on $\mathrm{K}(\cdot)$), and then getting a contacts in that stretch before exiting the system. Define $i_a := \min\{i \mid \widehat{\tau}_{i+1} - \widehat{\tau}_i \geqslant 2C_{28} a^{\frac{1}{1\wedge\alpha}},\ \omega_{\widehat{\tau}_{i+1}} = 0\}$, so that $\omega \equiv 0$ on $(\widehat{\tau}_{i_a}, \widehat{\tau}_{i_a+1}]$.

One wants to estimate i_a and $\widehat{\tau}_{i_a}$. Let us define

$$M_N^* := \max_{1 \leqslant i \leqslant N} \{\widehat{\tau}_{i+1} - \widehat{\tau}_i \mid \omega_{\widehat{\tau}_i} = 0\}. \quad (6.6.55)$$

Adapting the proof of Lemma 6.A.4, one gets a random integer N_0 such that for all $N \geqslant N_0$

$$M_N^*(\widehat{\tau}) \geqslant N^{1/\widetilde{\alpha}} (\log\log N)^{-1}. \quad (6.6.56)$$

So that if a is large enough

$$2C_{28}a^{\frac{1}{1\wedge\alpha}} \geqslant M^*_{i_a} \geqslant i_a^{1/\widetilde{\alpha}}(\log\log i_a)^{-1}, \tag{6.6.57}$$

and hence

$$i_a \leqslant a^{\frac{\widetilde{\alpha}}{1\wedge\alpha}}(\log a). \tag{6.6.58}$$

By the law of large numbers for $\widehat{\tau}$, the above inequality transfers to $\widehat{\tau}_{i_a}$: one also has for a large enough $\widehat{\tau}_{i_a} \leqslant a^{\frac{\widetilde{\alpha}}{1\wedge\alpha}}(\log a)$. Note that under the assumption $a \leqslant N^{\frac{1\wedge\alpha}{\widetilde{\alpha}}-\varepsilon}$, one has $\widehat{\tau}_{i_a} \ll N$.

Then, decomposing $Z_N^{\omega,\beta}(|\tau\cap[0,N]| = a)$ according to the position of τ_1 and τ_{a-1}, and restricting to the event $\tau_1 \in (\widehat{\tau}_{i_a}, \widehat{\tau}_{i_a} + C_{28}a^{\frac{1}{1\wedge\alpha}}]$, one gets

$$\begin{aligned} Z_N^{\omega,\beta}(|\tau\cap[0,N]| = a) &\geqslant \sum_{d=\widehat{\tau}_{i_a}}^{\widehat{\tau}_{i_a}+C_{28}a^{\frac{1}{1\wedge\alpha}}} \mathrm{K}(d) \sum_{f=d}^{\widehat{\tau}_{i_a}+2C_{28}a^{\frac{1}{1\wedge\alpha}}} \mathbf{P}(\tau_{a-2} = f-d)\mathbf{P}(\tau_1 > N-f) \\ &\geqslant C_{29}a^{\frac{1}{1\wedge\alpha}}\left(\widehat{\tau}_{i_a} \vee a^{\frac{1}{1\wedge\alpha}}\right)^{-(1+\alpha)} \mathbf{P}\left(\tau_{a-2} \leqslant C_{28}a^{\frac{1}{1\wedge\alpha}}\right)(N - \widehat{\tau}_{i_a} - 2C_{28}a^{\frac{1}{1\wedge\alpha}})^{-\alpha}, \end{aligned} \tag{6.6.59}$$

where we used the asymptotic properties of $\mathrm{K}(\cdot)$ and the fact that $\widehat{\tau}_{i_a} \ll N$. Then one chooses the constant C_{28} such that $\mathbf{P}(\tau_{a-2} \leqslant C_{28}a^{\frac{1}{1\wedge\alpha}})$ is bounded away from 0 (take $C_{28} = 2\mathbf{E}[\tau_1]$ if $\alpha > 1$ and $C_{28} = 1$ if $\alpha < 1$), and use our bound on $\widehat{\tau}_{i_a}$ to get the result. □

6.A. Renewal results

We gather here a set of technical results concerning renewal processes. They are used throughout this Chapter for the different renewals $\widehat{\tau}, \widetilde{\tau}$ and τ, and therefore we state them for a generic renewal $\sigma = \{\sigma_n\}_{n\geqslant 0}$, starting from $\sigma_0 = 0$, whose law is denoted $\mathbf{P}$, and whose inter-arrival law satisfies

$$K(n) := \mathbf{P}(\sigma_1 = n) = (1+o(1))c_\sigma n^{-(1+\vartheta)}, \tag{6.A.1}$$

where $\vartheta > 0$ and $\vartheta \neq 1$. We also assume that σ is recurrent, that is $K(\infty) = \mathbf{P}(\sigma_1 = +\infty) = 0$. The results would stand still if c_σ was replaced by a slowly varying function but for the sake of simplicity, we restrict to the pure power-law case. We have two subsections concerning respectively results for positive recurrent renewals ($\vartheta > 1$), and null-recurrent renewals ($\vartheta < 1$).

6.A.1. Null recurrent case, $\vartheta < 1$. We present a result of Doney concerning local-large deviation above the median for renewal processes.

Proposition 6.A.1 ([Don97], Theorem A)**.** *If $\vartheta < 1$, then one has that uniformly for $x \gg N^\vartheta$*

$$\mathbf{P}(\sigma_N = x) = (1+o(1))NK(x). \tag{6.A.2}$$

More precisely, for any sequence a_N such that $N^\vartheta = o(a_N)$ one has

$$\lim_{N\to\infty} \sup_{x\geqslant a_N} \left|\frac{\mathbf{P}(\sigma_N = x)}{NK(x)} - 1\right| = 0. \tag{6.A.3}$$

6.A.2. Positive recurrent case, $\vartheta > 1$. In this case we introduce $m = \mathbf{E}[\tau_1] < \infty$. We first prove the following equivalent of Proposition 6.A.1. The proof present some similarities as well as some crucial differences with the one in [Don97].

Proposition 6.A.2. *For all $\delta > 0$, one has uniformly for all $x \geqslant (m+\delta)N$.*

$$\mathbf{P}\left(\sigma_N = x\right) = (1+o(1))NK\left(x - mN\right), \tag{6.A.4}$$

or more precisely

$$\lim_{N\to\infty} \sup_{x \,\geqslant\, (m+\delta)N} \left| \frac{\mathbf{P}(\sigma_N = x)}{NK(x-mN)} - 1 \right| = 0. \tag{6.A.5}$$

A simple consequence is that uniformly for $x \gg N$,

$$\mathbf{P}\left(\sigma_N = x\right) = (1+o(1))NK\left(x\right). \tag{6.A.6}$$

Remark 6.A.3. The idea behind this result (like for Proposition 6.A.1) is that if σ_N has to be way above its median, the reasonable way to do it is to take all the excess in one big jump, the rest of the trajectory being typical. Other strategies with several long jumps are proved to be comparatively unlikely. This is an important point to understand what is going on in Sections 6.4,6.5 and 6.6.

Proof Given δ, we set $\varepsilon > 0$ that is meant to be arbitrarily small. Take some $x \geqslant (m+\delta)N$.

Let us start with the lower bound,

$$\begin{aligned}
\mathbf{P}\left(\sigma_N = x\right) &\geqslant \mathbf{P}\left(\sigma_N = x\ ;\ \exists!\ i \in [1,N],\ \sigma_i - \sigma_{i-1} \geqslant \varepsilon x\right) \\
&= N \sum_{y=\varepsilon x}^{x} \mathbf{P}(\sigma_1 = y)\mathbf{P}\left(\sigma_{N-1} = x - y\ ;\ \forall i \in [1,N-1],\ \sigma_i - \sigma_{i-1} \leqslant \varepsilon x\right) \\
&\geqslant N \min_{y\in[\varepsilon x, x-(m-\varepsilon)N]} K(y)\mathbf{P}\big(\sigma_{N-1} \in [(m-\varepsilon)N, (1-\varepsilon)x]\ ;\ \forall i \in [1,N-1],\ \sigma_i - \sigma_{i-1} \leqslant \varepsilon x\big).
\end{aligned} \tag{6.A.7}$$

The second line is obtained by using independence and exchangability of the increments (decomposing over all N possibilities for i), and the third line by restricting to the values $y \in [\varepsilon x, x-(m-\varepsilon)N]$. Then the assumption one has on $K(\cdot)$ guarantees that

$$\min_{y\in[\varepsilon x, x-(m-\varepsilon)N]} K(y) = (1+o(1))K(x-(m-\varepsilon)N). \tag{6.A.8}$$

Using the law of large numbers for σ_{N-1}, one has that for ε sufficiently small

$$\begin{aligned}
&\mathbf{P}\left(\sigma_{N-1} \in [(m-\varepsilon)N, (1-\varepsilon)x]\ ;\ \forall i \in [1,N-1],\ \sigma_i - \sigma_{i-1} \leqslant \varepsilon x\right) \\
&\quad\geqslant \mathbf{P}\left(\sigma_{N-1} \in [(m-\varepsilon)N, (1-\varepsilon)x]\right) - \mathbf{P}\left(\exists i \in [1,N-1],\ \sigma_i - \sigma_{i-1} \geqslant \varepsilon x\right) \\
&\quad= 1 + o(1) + NO((\varepsilon x)^{-\xi}) = 1 + o(1).
\end{aligned} \tag{6.A.9}$$

One gets the result by taking ε arbitrarily close to zero.

For the upper bound, it is easy to control the contribution of trajectories that make at least one large jump of order x. We start with the more delicate part of controlling the contribution of trajectories that do not. We prove it to be negligible.

$$\begin{aligned}
&\mathbf{P}\left(\sigma_N = x \; ; \; \forall i \in [1,N], \; \sigma_i - \sigma_{i-1} \leqslant \varepsilon x\right) \\
&\quad \leqslant \mathbf{P}\left(\sigma_N = x \; ; \; \exists n_1, n_2 \in [1,N]^2, \; \sigma_{n_i} - \sigma_{n_i - 1} \in [x^{1-\varepsilon}, \varepsilon x] \text{ for } i = 1,2\right) \\
&\quad + \mathbf{P}\left(\sigma_N = x \; ; \; \exists i \in [1,N], \; \sigma_i - \sigma_{i-1} \in [x^{1-\varepsilon}, \varepsilon x] \; ; \; \forall j \neq i \; \sigma_j - \sigma_{j-1} < x^{1-\varepsilon}\right) \\
&\quad + \mathbf{P}\left(\sigma_N = x \; ; \; \forall j \in [1,N], \; \sigma_j - \sigma_{j-1} < x^{1-\varepsilon}\right). \qquad (6.A.10)
\end{aligned}$$

We can bound the first term by using the union bound on the different possibilities for n_1 and n_2, to get some constant C_{30}

$$\begin{aligned}
&\mathbf{P}\left(\sigma_N = x \; ; \; \exists i,j \in [1,N]^2, \; \sigma_i - \sigma_{i-1} \geqslant x^{1-\varepsilon}, \sigma_j - \sigma_{j-1} \geqslant x^{1-\varepsilon}\right) \\
&\leqslant \binom{N}{2} \sum_{y,z = x^{1-\varepsilon}}^{x} \mathbf{P}\left(\sigma_1 = y\right) \mathbf{P}\left(\sigma_1 = z\right) \mathbf{P}\left(\sigma_{N-2} = x - y - z\right) \leqslant C_{30} N^2 x x^{-2(1-\varepsilon)(1+\vartheta)},
\end{aligned} \qquad (6.A.11)$$

which smaller than $N x^{-2\zeta + 2\varepsilon(1+\vartheta)}$ uniformly in $x \geqslant N$. Hence this term is negligible compared to the bound $N x^{-(1+\vartheta)}$, if ε is strictly smaller than $(\vartheta - 1)/(2(1+\vartheta))$.

To estimate the other terms in (6.A.10), define a renewal process $\bar{\sigma}$ with $\bar{\sigma}_0 := 0$, and $\bar{\sigma}_i - \bar{\sigma}_{i-1} := (\sigma_i - \sigma_{i-1}) \mathbf{1}_{\{\sigma_i - \sigma_{i-1} < x^{1-\varepsilon}\}}$. One can bound the second and third term in the r.h.s. of (6.A.10) from above by $\mathbf{P}\left(\bar{\sigma}_{N-1} \geqslant (1-\varepsilon)x\right)$. Now we estimate this term by using Chernov bounds. For any positive λ, one has

$$\mathbf{P}\left(\bar{\sigma}_N \geqslant (1-\varepsilon)x\right) \leqslant \mathbf{E}\left[e^{\lambda \bar{\sigma}_1}\right]^N e^{-\lambda(1-\varepsilon)x}. \qquad (6.A.12)$$

Using the trivial bound $\mathbf{E}[(\bar{\sigma}_1)^k] \leqslant (x^{1-\varepsilon})^{k-1} m$, one finds that

$$\mathbf{E}\left[e^{\lambda \bar{\sigma}_1}\right] \leqslant 1 + \frac{m}{x^{1-\varepsilon}} (e^{\lambda x^{1-\varepsilon}} - 1). \qquad (6.A.13)$$

If one chooses $\lambda = o(x^{-1+\varepsilon})$, one gets as N goes to infinity

$$\mathbf{E}\left[e^{\lambda \bar{\sigma}_1}\right]^N \leqslant \exp\left(\lambda m N (1 + o(1))\right), \qquad (6.A.14)$$

such that for N large enough,

$$\mathbf{P}\left(\bar{\sigma}_N \geqslant (1-\varepsilon)x\right) \leqslant \exp\left(\lambda (mN - (1-\varepsilon)x)(1 + o(1))\right) \leqslant e^{-C_{31} x^{\varepsilon/2}}, \qquad (6.A.15)$$

where the last inequality comes from taking ε small enough, and $\lambda = x^{-1+\varepsilon/2}$ (the constant C_{31} depends only the choice of δ). This is negligible compared to the bound one must obtain.

Then, we estimate the main contribution, using the union bound and exchangeability of the increments

$$\begin{aligned}\mathbf{P}\left(\sigma_N = x\ ;\ \exists i \in [1,N],\ \sigma_i - \sigma_{i-1} \geqslant \varepsilon x\right) &\leqslant N \sum_{y=\varepsilon x+1}^{x} \mathbf{P}(\sigma_1 = y)\mathbf{P}\left(\sigma_{N-1} = x-y\right) \\ &\leqslant N \Bigg[\max_{y\in[x-(m+\varepsilon)N,x]} K(y)\mathbf{P}\left(\sigma_{N-1} \leqslant (m+\varepsilon)N\right) \\ &\qquad + \max_{y\in(\varepsilon x, x-(m+\varepsilon)N)} K(y)\mathbf{P}\left(\sigma_{N-1} > (m+\varepsilon)N\right)\Bigg]. \end{aligned} \tag{6.A.16}$$

The law of large numbers gives

$$\mathbf{P}\left(\sigma_{N-1} > (m+\varepsilon)N\right) = o(1). \tag{6.A.17}$$

On the other hand, one has from the assumption on $K(\cdot)$ that

$$\begin{aligned} \max_{y\in[x-(m+\varepsilon)N,x]} K(y) &= (1+o(1))K(x-(m+\varepsilon)N), \\ \max_{y\in[\varepsilon x, x-(m+\varepsilon)N]} K(y) &= (1+o(1))K(\varepsilon x) = O(x^{-(1+\alpha)}). \end{aligned} \tag{6.A.18}$$

This, together with the fact that ε can be chosen arbitrarily close to zero, gives the result.

□

We finish with giving a result on the size of the longest inter-arrival interval up to the N^{th} jump,

$$M_N := \max_{1\leqslant i\leqslant N} \{\sigma_i - \sigma_{i-1}\}. \tag{6.A.19}$$

Lemma 6.A.4. *If $\vartheta > 1$, there exists a random integer $N_0(\sigma)$, such that for all $N \geqslant N_0$*

$$N^{1/\vartheta}(\log\log N)^{-1} \leqslant M_N \leqslant N^{1/\vartheta}\log N. \tag{6.A.20}$$

Proof We use the fact that increments are *i.i.d.* to get

$$\mathbf{P}\left(M_N \leqslant A\right) = \mathbf{P}(\sigma_1 \leqslant A)^N. \tag{6.A.21}$$

Then, using that $\mathbf{P}(\sigma_1 > A)$ is of order $A^{-\vartheta}$, one has that there exist constants $C_{32}, C_{33} > 0$ such that

$$\mathbf{P}\left(M_N > N^{1/\vartheta}\log N\right) \leqslant 1 - \exp\left(-C_{32}(\log N)^{-\vartheta}\right) = (1+o(1))C_{32}(\log N)^{-\vartheta}, \tag{6.A.22}$$

and

$$\begin{aligned}\mathbf{P}\left(M_N < N^{1/\vartheta}(\log\log N)^{-1}\right) &\leqslant \left(1 - C_{33}N^{-1}(\log\log N)^{\vartheta}\right)^N \\ &= \exp\left(-(1+o(1))C_{33}(\log\log N)^{\vartheta}\right) \end{aligned} \tag{6.A.23}$$

Since $\vartheta > 1$, one has from (6.A.22) that the sequence $\mathbf{P}\left(M_{2^k} > 2^{(k-1)/\vartheta}\log 2^{k-1}\right)$ for $k \geqslant 1$ is summable, and from (6.A.23) that the sequence $\mathbf{P}\left(M_{2^k} < 2^{(k+1)/\vartheta}(\log\log 2^{k+1})^{-1}\right)$ is also summable.

The Borel-Cantelli Lemma gives that there exists a random integer k_0, such that for all $k \geqslant k_0$

$$2^{(k+1)/\vartheta}(\log\log 2^{k+1})^{-1} \leqslant M_{2^k} \leqslant 2^{(k-1)/\vartheta}\log 2^{k-1}. \tag{6.A.24}$$

One notices that $(M_N)_{N\geqslant 0}$ is a non decreasing sequence. Thus, taking $N \geqslant N_0 := 2^{k_0+1}$, and choosing k such that $2^{k-1} < N \leqslant 2^k$, then one has $k-1 \geqslant k_0$ and so

$$M_N \leqslant M_{2^k} \leqslant 2^{(k-1)/\vartheta}\log 2^{k-1} \leqslant N^{1/\vartheta}\log N, \tag{6.A.25}$$

and

$$M_N \geqslant M_{2^{k-1}} \geqslant 2^{k/\vartheta}(\log\log 2^k)^{-1} \geqslant N^{1/\vartheta}\log\log N. \tag{6.A.26}$$

□

CHAPTER 7

On the appearance of a *strongly relevant* regime

7.1. Introduction

In this thesis, we extensively studied the influence of correlated disorder on the phase transition, to decide whether the presence of inhomogeneities affects the critical properties of the system. In the case of an *i.i.d.* environment, the *Harris criterion* was discussed (disorder is relevant if $\nu^{\text{pur}} < 2$, irrelevant if $\nu^{\text{pur}} > 2$), and in the case of a correlated environment, with power-law decaying correlation function, of type $r^{-\zeta}$, $\zeta > 0$, one refers to the *Weinrib-Halperin prediction*.

We have seen previously that this prediction is valid in the hierarchical version of this model (see Chapter 4), when correlations are of Gaussian type and summable, *i.e.* if $\zeta > 1$. The case $\zeta < 1$ turns out to be much more problematic: we have already seen in Chapters 4-5, that strong correlations are in a way shifting the critical point $h_c(\beta)$ towards $-\beta \operatorname{ess\,sup}(\omega_1)$. That is why, in Chapter 6 and in the present one, we consider only the case of a bounded correlated environment, to be able to study the phase transition, that occurs at some finite $h_c(\beta)$.

Chapter 6 deals with the case where the sequence $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ (the environment) is $\{-1, 0\}$-valued, and constructed by concatenating blocks of random size, in which ω is constant equal to 0 or -1, see construction (6.1.6). We consider in this Chapter a more general case, where ω is ergodic (this is needed to ensure the existence of the free energy), and $\{-1, 0\}$-valued for the simplicity of the statements.

7.1.1. Reminder on the pinning model. Let τ be a recurrent renewal process, as defined in Section 1.1.2, with law $\mathbf{P}$, that represents the contact points of the polymer trajectory with the defect line. We make the assumption that the inter-arrival distribution, denoted by $\mathrm{K}(\cdot)$, satisfies

$$\mathrm{K}(n) := \mathbf{P}(\tau_1 = n) \overset{n\to\infty}{=} (1 + o(1))\frac{c_K}{n^{1+\alpha}}, \tag{7.1.1}$$

for some $\alpha > 0$ and $c_K > 0$ (we take $\alpha \neq 1$ to avoid some technicalities).

We define as in Chapter 6 the *quenched* polymer measure with *free boundary condition* (analogously with Section 1.4, where there is a *pinned boundary condition*), as a Gibbs transformation of the law $\mathbf{P}$. Given the environment $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ which is an ergodic sequence, and parameters $h \in \mathbb{R}, \beta \geqslant 0$, we define

$$\frac{\mathrm{d}\mathbf{P}^{\omega,\beta}_{N,h}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z^{\omega,\beta}_{N,h}} \exp\left(\sum_{n=1}^{N} (h + \beta\omega_n)\delta_n\right), \tag{7.1.2}$$

with the notation $\delta_n := \mathbf{1}_{\{n\in\tau\}}$, and where $Z_{N,h}^{\omega,\beta} := \mathbf{E}\left[\exp\left(\sum_{n=1}^N (h+\beta\omega_n)\delta_n\right)\right]$ is the partition function of the disordered system.

One defines the quenched free energy of the system

$$\mathtt{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h}^{\omega,\beta}, \tag{7.1.3}$$

which is known to exist and be $\mathbb{P}$-a.s. constant, thanks to the ergodicity of the sequence ω (see Proposition 5.1.1). There exists a (*quenched*) critical point $h_c(\beta)$, for which one has that $\mathtt{F}(\beta,h) > 0$ if and only if $h > h_c(\beta)$. As noticed in Chapter 1, there are two distinct phases: a delocalized phase for $h < h_c(\beta)$, where the trajectories are wandering away from the defect line, and a localized phase for $h > h_c(\beta)$, where the trajectories stick to the line.

Moreover, we define the *annealed* system as in Chapter 1: the *annealed* partition function is $\mathbb{E}[Z_{N,h}^{\omega,\beta}]$, the *annealed* free energy $\mathtt{F}^{\mathrm{a}}(\beta,h) := \lim_{N\to\infty} \frac{1}{N}\log \mathbb{E} Z_{N,h}^{\omega,\beta}$, and there is an *annealed* critical point $h_c^{\mathrm{a}}(\beta)$. It is classical to get from the Jensen inequality that $\mathtt{F}^{\mathrm{a}}(\beta,h) \geqslant \mathtt{F}(\beta,h)$, and therefore that $h_c^{\mathrm{a}}(\beta) \leqslant h_c(\beta)$.

Note that, as in Chapter 6, (7.1.2) is the definition of the polymer measure with free boundary conditions at the right endpoint. We also consider the *pinned* version of the measure (the one introduced in Section 1.4.1), denoted $\mathbf{P}_{N,h}^{\omega,\beta,\mathrm{pin}}$, where a trajectory is constrained to return to 0 at its endpoint,: the *pinned* partition function is $Z_{N,h}^{\omega,\beta,\mathrm{pin}} := \mathbf{E}\left[\exp\left(\sum_{n=1}^N (h+\beta\omega_n)\delta_n\right)\delta_N\right]$. As far as the free energy is concerned, working with the *free* or *pinned* partition function are equivalent, see [Gia07, Ch.4].

We compare the characteristics of the phase transition of the disordered system with the homogeneous ones (with no disorder, $\beta = 0$), in order to decide if disorder is relevant, *i.e.* if the presence of homogeneities changes the critical behavior of the system. We recall Theorem 1.1.6, which says that the pure free energy $\mathtt{F}(0,h) =: \mathtt{F}(h)$ exhibits a phase transition at the critical point $h_c(0) = 0$ (since τ is recurrent), with a critical exponent $\nu^{\mathrm{pur}} = 1 \vee 1/\alpha$ (there is a logarithmic correction in the case $\alpha = 1$).

7.2. A general $\{-1,0\}$-disordered model

7.2.1. Definition of the model. We consider an ergodic sequence $\omega := (\omega_n)_{n\in\mathbb{N}}$ with values in $\{-1,0\}$, whose law is denoted by $\mathbb{P}$.

Since $\omega \in \{-1,0\}^{\mathbb{N}}$, the environment we consider is either neutral or repulsive, and one trivially has $Z_{N,h}^{\omega,\beta} \leqslant Z_{N,h}^{\mathrm{pur}}$, so that $\mathtt{F}(\beta,h) \leqslant \mathtt{F}(0,h)$ for all $\beta \geqslant 0$. Therefore, the localization phase transition occurs at some $h_c(\beta) \geqslant 0$. As suggested in Chapter 5, when the correlations are sufficiently strong, the critical point is shifted towards its minimal possible value $-\beta\,\mathrm{ess\,sup}(\omega_1)$, and is therefore equal to $h_c(0) = 0$. The possibility to know exactly the critical point allows us to analyze the system close to criticality in detail, so that one gets precise statements, the fact that the critical point $h_c(\beta)$ is equal to the homogeneous critical point $h_c(0) = 0$ making these statements more readable.

We comment briefly our choice to restrict to sequences ω that are $\{-1,0\}$-valued, the results we present in this Chapter being very general and holding for any sequence of bounded (discrete) random variables. The assumption that ω can only take two values -1 and 0 could be weakened, assuming that the largest value M that ω can take is bounded, and that the second largest is strictly smaller than M. If ω is a discrete random variable with law $\mathbb{P}$, it means $M := \operatorname{ess\,sup}(\omega_1) < \infty$, and that there exists some $\theta > 0$ such that $\mathbb{P}(\omega \in (M-\theta, M)) = 0$. Thanks to a scaling and translation argument of the parameters h and β, one can come down to the case where $M = 0$ and $\theta = 1$, and results for more general (discrete) environments ω could then be derived from our analysis. One could also possibly extend this to continuous variables, the important assumption being that $M := \operatorname{ess\,sup}(\omega_1) < \infty$. In what follows, we restrict ourselves to the case $\omega \in \{-1,0\}^{\mathbb{N}}$ to keep our analysis and the statement of our results simpler.

7.2.2. Notations and preliminary results. It is discussed in Section 1.4.4 that the correlation function is not the main object that quantifies the appearance of a new type of behavior. The right quantity to look at is the length of the "favorable" blocks (*i.e.* zones where the value of ω is high), and how they are spread along the defect line.

The main result below is a sufficient condition, that we conjecture to be also necessary in the case $\alpha > 1$, on the sequence ω to get localization as soon as $h > 0$, and bounds on the free energy in the case where $h_c(\beta) = 0$ (see Theorem 7.3.2). This is analogous to Theorem 6.2.1, where the sharp critical behavior of the free energy is given, but the result we give here is more general, since there is no special renewal construction of the environment, the proofs relying however on similar ideas.

From Theorem 7.3.2, we observe that if correlations are strong enough (in a sense we explain in Section 7.3), disorder can be *strongly relevant*: it always modifies (*i.e.* for all $\alpha > 0$ and all $\beta > 0$) the characteristics of the order transition with respect to the homogeneous one. Moreover we provide natural examples of correlated $\{-1,0\}$ environment on which results can be derived from Theorem 7.3.2. In particular in Section 7.3.2, we give Theorem 7.3.4 in the case of an environment based on the sign of a correlated Gaussian process, that show that in the case of non-summable correlations, disorder is relevant for all $\alpha > 0$.

The appearance of a *strongly relevant* regime was not expected, and is in contrast with the Weinrib-Halperin prediction. In the case where $h_c(\beta) > 0$, which occurs when correlations are weak enough (in a sense we comment in a moment), we believe that the Weinrib-Halperin criterion should hold.

We consider a sequence $\omega \in \{\omega_i\}_{i \geqslant -1}$ (we choose $i \geqslant -1$ instead of $i \in \mathbb{N}$ for notational convenience, see the following definitions), and we assume that ω is ergodic and $\{-1,0\}$-valued (we note abusively $\omega \in \{-1,0\}^{\mathbb{N}}$). We also take ω non-trivial, in the sense that $\mathbb{P}(\omega_1 = 0) > 0$ and $\mathbb{P}(\omega_1 = -1) > 0$.

As it is suggested by the construction of the environment in Chapter 6, we divide our system into blocks where the sequence ω is constant valued. Given the environment $\omega = \{\omega_i\}_{i \geqslant -1}$, we condition it to have $\omega_{-1} = -1$, $\omega_0 = 0$ (which has positive probability, so that the free energy is not affected). We then define the sequences $(T_n)_{n \geqslant 0}$ and $(\xi_n)_{n \geqslant 1}$ iteratively, setting $T_0 := 0$, and for all $n \geqslant 1$

$$\begin{aligned} T_n &:= \inf\{i > T_{n-1} \ ; \ \omega_{i+1} \neq \omega_i\}, \\ \xi_n &:= T_n - T_{n-1} \end{aligned} \tag{7.2.1}$$

Thus we have divided our system into blocks of size ξ_n, on which ω is constant valued, equal alternatively to 0 and to -1 (we write $\omega \equiv 0$ and $\omega \equiv -1$). The choice of conditioning to $\omega_{-1} = -1, \omega_1 = 0$ enables us to identify the blocks with odd indexes $[T_{2k}, T_{2k+1})$ for $k \geqslant 0$ (therefore of size ξ_{2k+1}), as the blocks on which $\omega \equiv 0$.

Remark 7.2.1. The ergodicity and non triviality of the sequence ω implies that $\mathbb{E}[\xi_1] < +\infty$ and $\mathbb{E}[\xi_2] < +\infty$. Indeed, one considers the (ergodic) sequence $\bar{\omega} = \{\bar{\omega}_n\}_{n \geqslant 0}$ defined by $\bar{\omega}_n := (\omega_{n-1}, \omega_n)$ for all $n \geqslant 0$. This sequence is, according to our notations, conditioned to start with $\bar{\omega}_0 = (-1, 0)$. Then in [Shi96, Ch.I.2.c], the *generalized renewal process* of the set $\{(-1,0)\}$ is defined as the sequence of indexes $\{k \geqslant 0, \bar{\omega}_k = (-1,0)\} = \{T_{2j}\}_{j \geqslant 0}$, and the *return-time process* to the set $\{(-1,0)\}$ (of positive measure) is defined as the sequence $(T_{2j} - T_{2(j-1)})_{j\in\mathbb{N}} = (\xi_{2j-1} + \xi_{2j})_{j\in\mathbb{N}}$. It is shown that the return-time process is ergodic (see [Shi96, Th.I.2.19]), and [Shi96, Eq. (14) Ch.I.2] states that the first return time to $(-1, 0)$ (*i.e.* $T_2 = \xi_1 + \xi_2$) has expectation $\mathbb{P}(\omega_{-1} = -1, \omega_0 = 0)^{-1} < +\infty$ (since ω is non trivial).

We now introduce a notion of "good" block: we call "*A-block*" a block $(T_i, T_{i+1}]$, on which $\omega \equiv 0$ (take i even), and whose size is larger than A. We denote

$$\mathcal{B}_N(A) := \#\{i \ ; \ T_{2i+1} \leqslant N, \xi_{2i+1} \geqslant A\}, \tag{7.2.2}$$

the number of *A-blocks* before N.

One remarks that, applying Birkhoff's Ergodic Theorem (see [Nad98, Chap. 2]) to the sequence $(\xi_{2j-1}, \xi_{2j})_{j\in\mathbb{N}}$, one has $\mathbb{P}$-a.s.

$$\lim_{N\to\infty} \frac{\mathcal{B}_N(A)}{N} = \frac{1}{\mathbb{E}[\xi_1 + \xi_2]} \mathbb{P}\left(\xi_1 \geqslant A\right). \tag{7.2.3}$$

Note that the ergodicity of the sequence $(\xi_{2j-1}, \xi_{2j})_{j\in\mathbb{N}}$ comes from Remark 7.2.1, with a slightly different version of the sequence ω (that do not change the main properties of our system). We should condition our environment to $\bar{\omega}_0 \in \{(-1,0),(0,-1)\}$: then, in the definition (7.2.1), we do not impose the first block to be constituted of 0's, but we have that the sequence $(\xi_j)_{j\in\mathbb{N}}$ is the return-time process to the set $\{(-1,0),(0,-1)\}$ (of positive measure), and is therefore ergodic (see [Shi96, Th.I.2.19]).

The limit in (7.2.3) denotes the asymptotic density of regions of 0 larger than A. In the sequel, we make the assumption that $\mathbb{P}\left(\xi_1 \geqslant A\right) > 0$ for all $A \in \mathbb{N}$. Otherwise there is some $a_0 > 0$ such that $\mathbb{P}$-*a.s.*, all the blocks of 0's are of size smaller than a_0. In this case, it is easy to see that $h_c(\beta) > 0$, from (7.3.5) (one actually would have $h_c(\beta) \geqslant c\beta$).

We define $J_0 := 0$, and then by iteration

$$J_{n+1} = J_{n+1}(A) := \min\{j > J_n \ ; \ j \text{ is odd}, \xi_j \geqslant A\} \quad \text{for } n \geqslant 0, \tag{7.2.4}$$

so that the k^{th} *A-block* is $[T_{J_k-1}, T_{J_k})$, of size ξ_{J_k} (we do not write the dependence on A if there is no ambiguity).

We also define $\mathcal{T}_k(A)$ as the position of the endpoint of the k^{th} *A-block*, and $d_k(A)$ the (approximate) distance between the k^{th} and the $(k+1)^{\text{th}}$ *A-block*:

$$\begin{aligned} \mathcal{T}_k(A) &:= T_{J_k}, \\ d_k(A) &:= \mathcal{T}_k(A) - \mathcal{T}_{k-1}(A). \end{aligned} \tag{7.2.5}$$

We regroup in Figure 7.1 the above notations, *i.e.* the decomposition of our environment ω into blocks $[T_{i-1}, T_i)_{i\in\mathbb{N}}$ of size ξ_k, and for $A > 0$ fixed, in metablocks $[\mathcal{T}_{k-1}(A), \mathcal{T}_k(A))_{k\in\mathbb{N}}$ of size $d_k(A)$.

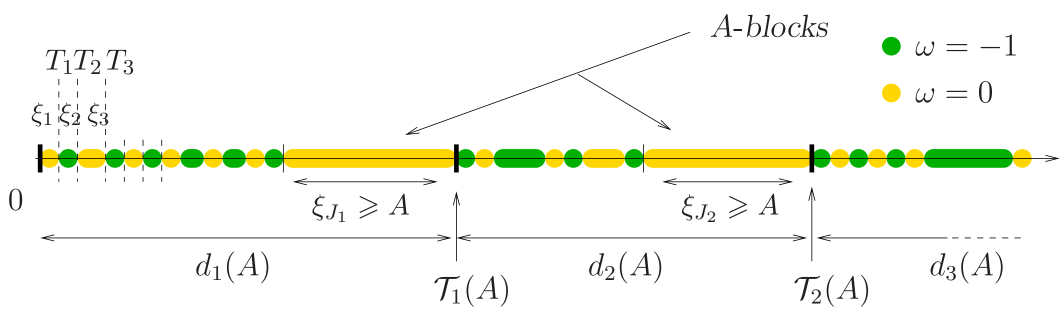

FIGURE 7.1. We have a decomposition of the system into segments $[T_{i-1}, T_i)_{i\in\mathbb{N}}$ of size ξ_i, in which the value of ω is constant. We define a *A-block* as a segment $[T_{i-1}, T_i)_{i\in\mathbb{N}}$ constituted of 0's, and which is larger than A. With a fixed parameter A, we divide our system into metablocks $[\mathcal{T}_{k-1}(A), \mathcal{T}_k(A))_{k\in\mathbb{N}}$, composed of blocks with $\omega \equiv -1$ or smaller than A, and then of one ending *A-block*. The integer J_k is the index of the k^{th} *A-block*, so that $\mathcal{T}_k(A) = T_{J_k}$, and we also denote $d_k(A)$ the size of the k^{th} metablock $[\mathcal{T}_{k-1}(A), \mathcal{T}_k(A))$, which represents also the (approximate) distance between two *A-blocks*, since $d_k(A)$ is guessed to be $\gg A$.

We remark that $\mathbb{P}(\xi_1 \geqslant A)$, linked to the limiting density of *A-blocks*, cf. (7.2.3), is also related to the expectation of $d_1(A)$, interpreted as the mean distance from the origin of the first *A-block*. Indeed, applying again Birkhoff's Ergodic Theorem for $\mathcal{T}_k(A)$, one gets that $\mathbb{P}$-a.s.

$$\mathbb{E}[d_1(A)] = \lim_{k\to\infty} \frac{\mathcal{T}_k(A)}{k} = \lim_{k\to\infty} \frac{\mathcal{T}_k(A)}{\mathcal{B}_{\mathcal{T}_k(A)}(A)} = \mathbb{E}[\xi_1 + \xi_2]\, \mathbb{P}\left(\xi_1 \geqslant A\right)^{-1}, \tag{7.2.6}$$

where we used Equation (7.2.3) for the last inequality.

7.3. Results on the disordered model

7.3.1. Main Theorem: bounds on the free energy in the general case.

Definition 7.3.1. One defines

$$\varepsilon(x) := \inf_{A \leqslant x} \frac{1}{A} \mathbb{E}[\log d_1(A)], \tag{7.3.1}$$

which is non-increasing, and one can therefore note ε^{-1} its generalized inverse, defined by

$$\varepsilon^{-1}(x) := \sup\{y, \, \varepsilon(y) \geqslant x\}. \tag{7.3.2}$$

We stress that the constants appearing in the following Theorem depend on β (as we consider β to be a fixed parameter, we often drop a priori the dependence of the constants on β).

We also stress that $\mathbb{P}(\xi_1 \geqslant n) \overset{n\to\infty}{=} o(1/n)$ since $\mathbb{P}(\xi_1 \geqslant n)$ is decreasing and summable (because $\mathbb{E}[\xi_1] < \infty$). Therefore, $n\mathbb{P}(\xi_1 \geqslant n)$ is asymptotically decreasing.

Theorem 7.3.2. 1. *We assume that $\mathbb{P}(\xi_1 \geqslant A) > 0$ for all $A \in \mathbb{N}$. If $\lim_{x\to\infty} \varepsilon(x) = 0$, then $\lim_{u\to 0} \varepsilon^{-1}(u) = +\infty$, and there exist two constants $c_0, c_0' > 0$ such that for all $h \in (0,1)$ and $\beta \in (0,1)$,*

$$\mathtt{F}(\beta, h) \geqslant c_0' A_h \mathtt{F}(h) \mathbb{P}(\xi_1 \geqslant A_h), \tag{7.3.3}$$

where we defined $A_h := \varepsilon^{-1}(c_0 \mathtt{F}(h))$, that goes to infinity as h goes to 0. In particular one has $h_c(\beta) = 0$, and the r.h.s. of (7.3.3) goes to 0 as h goes to 0 (thanks to the remark above).

2. *If there exists a constant $c > 0$ such that one has that for all indexes $1 \leqslant i_1 < \ldots < i_m$*

$$\mathbb{P}\left(\omega_{i_1} = 0, \ldots, \omega_{i_n} = 0\right) \leqslant e^{-cn}, \tag{7.3.4}$$

then there exists a constant $\eta > 0$ such that for all $\beta \in (0,1)$ one has $h_c(\beta) \geqslant h_c^{\mathrm{a}}(\beta) \geqslant \eta\beta$.

3. *In all cases, one also has a rough upper bound on the free energy: there exist two constants $C, c > 0$ (that do not depend on β) such that for all $\beta \in (0,1)$, one has some $h_0 > 0$ such that for all $h \in (0, h_0)$ and $\beta \in (0,1)$ one has*

$$\mathtt{F}(\beta, h) \leqslant Ch\mathbb{E}\left[\xi_1 \mathbf{1}_{\{\xi_1 > c\beta h^{-1}\}}\right]. \tag{7.3.5}$$

This theorem is actually very useful, because it applies to many types of environment, and we give possible applications in the sequel, in particular Theorem 7.3.4.

We remark that the asymptotic behavior of $\mathbb{E}[\log d_1(A)]$ captures a lot of informations on the behavior of the sequence ω, and in particular on the size and distribution of the blocks with $\omega \equiv 0$, but this quantity is often difficult to estimate. However, using Jensen inequality, one has

$$\varepsilon(x) \leqslant \bar{\varepsilon}(x) := \inf_{A \leqslant x} \frac{1}{A} \log \mathbb{E}[d_1(A)] = \inf_{A \leqslant x} -\frac{1}{A} \log cst.\mathbb{P}(\xi_1 \geqslant A), \tag{7.3.6}$$

where we already saw that $\mathbb{E}[d_1(A)] = cst.\mathbb{P}(\xi_1 \geqslant A)^{-1}$ in (7.2.6). Therefore if $\bar{\varepsilon}(x) \overset{x\to\infty}{\to} 0$ one also has $\varepsilon(x) \overset{x\to\infty}{\to} 0$, and using that $\bar{\varepsilon}^{-1} \geqslant \varepsilon^{-1}$ and that $n\mathbb{P}(\xi_1 \geqslant n)$ is asymptotically decreasing to 0 as n goes to infinity, one gets the same statement as in Theorem 7.3.2-(part **1**), with A_h replaced by $\bar{A}_h = \bar{\varepsilon}^{-1}(c_0 \mathtt{F}(h)^{-1})$. This is a much more handy formulation, since one only has to estimate $\log \mathbb{P}(\xi_1 \geqslant A)$, instead of $\mathbb{E}[\log d_1(A)]$.

Moreover, if one knows the behavior of $\mathbb{P}(\xi_1 \geqslant A)$, one is able to get an upper bound on the free energy of the system from Theorem 7.3.2-(part **3**). For example, a small computation gives that

$$\mathbb{E}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant A\}}\right] = \sum_{n \geqslant A} n\left(\mathbb{P}(\xi_1 \geqslant n) - \mathbb{P}(\xi_1 \geqslant n+1)\right) = A\mathbb{P}(\xi \geqslant A) + \sum_{n > A} \mathbb{P}(\xi_1 \geqslant n). \tag{7.3.7}$$

If $\mathbb{P}(\xi_1 \geqslant A)$ decays sufficiently fast (essentially, faster than $A^{-(1+\varepsilon)}$), one therefore gets that $\mathbb{E}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant A\}}\right] \leqslant cA\mathbb{P}(\xi \geqslant A)$, and from (7.3.5) one has

$$\mathtt{F}(\beta, h) \leqslant C'\mathbb{P}(\xi_1 \geqslant ch^{-1}), \tag{7.3.8}$$

which gives a very simple (but rough) upper bound. One already notices that $\mathtt{F}(\beta,h) \overset{h\to 0}{=} o(h)$, using the previous observation that $\mathbb{P}(\xi_1 \geqslant n) \overset{n\to\infty}{=} o(1/n)$. Disorder is therefore *relevant* for all $\alpha > 1$ as in the *i.i.d.* case, and thanks to (7.3.8), it is actually relevant for all $\alpha > 0$ if $\mathbb{P}(\xi_1 \geqslant n)$ decays faster than any power of n: we say that we are in the *strongly relevant regime*.

One can actually try to get better estimates on the upper bound, in order to track the real criterion that decides whether $h_c(\beta) = 0$ or not. As we explain and justify in Section 7.4.3, we can improve the upper bound on the free energy (and (7.3.3) should be sharp, at least for $\alpha > 1$, up to a constant in the definition of A_h). One should actually have the following criterion, valid for all $\alpha > 0$

Conjecture 7.3.3. *If $\omega \in \{-1, 0\}^{\mathbb{N}}$ is ergodic and non trivial, we have the following equivalence*

$$\liminf_{A\to\infty} \frac{1}{A}\mathbb{E}[\log d_1(A)] > 0 \quad \iff \quad h_c(\beta) > 0 \quad \textit{for all } \beta > 0. \tag{7.3.9}$$

If $\lim_{x\to\infty} \varepsilon(x) = 0$ (with $\varepsilon(\cdot)$ the function defined in (7.3.1)), then $h_c(\beta) = 0$ for all $\beta > 0$. Moreover, for all $\beta \in (0,1)$ there exists some $h_0 > 0$ such that for all $h \in (0, h_0)$ one has

$$\mathtt{F}(\beta, h) \leqslant ch\mathbb{E}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant A'_h\}}\right], \tag{7.3.10}$$

where we defined $A'_h := \varepsilon^{-1}(c'_0 h)$ with some constant $c'_0 > 0$ (that depends on β). Note that $A'_h \overset{h\to 0}{\to} \infty$, and that the upper bound in (7.3.10) goes to 0 as h goes to 0.

This bound does not match exactly the lower bound (7.3.3), but one has the upper bound

$$\mathbb{E}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant A'_h\}}\right] \leqslant cA'_h\mathbb{P}(\xi_1 \geqslant A'_h), \tag{7.3.11}$$

provided that $\mathbb{P}(\xi_1 \geqslant A)$ decays sufficiently fast (faster than $1/A$), the same way as (7.3.8) is obtained from (7.3.5). The bound (7.3.11) then matches (7.3.3) in the case $\alpha > 1$, the case $\alpha < 1$ requiring a more delicate analysis.

Therefore, the quantity $\mathbb{E}[\log d_1(A)]$ should encode the right properties of the sequence ω, that is to answer whether one has localization as soon as $h > 0$. As discussed in Section 7.4.3, one might need an additional assumption on the sequence ω to prove this conjecture, especially on the concentration of $\log d_1(A)$ around its mean value.

7.3.1.1. *A first example of application: the block-environment of Chapter 6.* In the previous Chapter, the environment ω is constructed by defining directly the law of the sequence $(\xi_k)_{k\in\mathbb{N}}$ as an *i.i.d.* sequence, or equivalently taking $(T_n)_{n\in\mathbb{N}}$ a renewal sequence with an inter-arrival law with heavy tail of exponent $1+\widetilde{\alpha}$, $\widetilde{\alpha} > 1$. One can actually generalize to the case where there is an asymmetry between $0's$ and $-1's$, and define $(\xi_{2k+1})_{k\geqslant 0}$ as an *i.i.d.* sequence with $\mathbb{P}(\xi_1 = n) = \widetilde{K}(n) \overset{n\to\infty}{\sim} \widetilde{c}_K n^{-(1+\widetilde{\alpha})}$, $\widetilde{\alpha} > 1$; and then set also $(\xi_{2k})_{k\in\mathbb{N}}$ another *i.i.d.* sequence, independent of $(\xi_{2k+1})_{k\geqslant 0}$, with $\mathbb{E}[\xi_2] < +\infty$.

In that case, one computes easily $\log \mathbb{P}(\xi_1 \geqslant A)$, and one is therefore able to get sharp bounds on the free energy, similar to the lower bound (7.3.3), which gives in that case $\mathtt{F}(\beta, h) \geqslant c|\log h|^{1-\widetilde{\alpha}}\mathtt{F}(h)^{\widetilde{\alpha}}$, see Theorem 6.2.1. The upper bound (7.3.8) gives that $\mathtt{F}(\beta, h) \leqslant c' h^{\widetilde{\alpha}}$, which is a rough bound, given in Sections 6.4.1 (for the case $\alpha > 1$, the case $\alpha < 1$ needing more work). Getting the sharp upper bound (Theorem 6.2.1) is more tricky, and one uses strongly the independence of the blocks' sizes. We only mention that the sharp upper bound matches the one of Conjecture 7.3.3 in the case $\alpha > 1$, the method sketched in Section 7.4.3 generalizing the one used in Chapter 6.

We now give an example of a very natural environment ω, and we give therefore a concrete and very interesting application of Theorem 7.3.2.

7.3.2. Case of an environment based on a correlated Gaussian sequence. In Chapter 5, we considered ω to be a Gaussian correlated sequence, but when the correlations were too strong, the phase transition disappeared ($h_c(\beta) = -\infty$). We consider here a sequence ω that is also based on a correlated Gaussian sequence, but with all the ω_i's bounded, so that the system always exhibits a phase transition.

Let $\mathsf{W} := (\mathsf{W}_n)_{n\geqslant 0}$ be a centered normalized stationary Gaussian process, with covariance matrix Υ, whose law is denoted $\mathbb{P}$. Because of the stationarity, the correlations depend only on the distance between i and j, that is $\Upsilon_{ij} := \mathbb{E}[\mathsf{W}_i\mathsf{W}_j] = \rho_{|i-j|}$, $\rho_0 = 1$. We also assume that correlations are non-negative, and that there exists some $\zeta > 0$ and some constants $c_0 > 0$ such that the correlation function $(\rho_n)_{n\geqslant 0}$ verifies

$$\rho_k \overset{k\to\infty}{\sim} c_0 k^{-\zeta}, \qquad \text{and } \rho_k \geqslant 0 \text{ for all } k \geqslant 0. \tag{7.3.12}$$

It is natural from the Gaussian sequence $\{\mathsf{W}_n\}_{n\in\mathbb{N}}$, to define the environment ω with values in $\{-1, 0\}$ by

$$\omega_i := -\mathbf{1}_{\{\mathsf{W}_i \leqslant 0\}}, \tag{7.3.13}$$

and we often refer in the sequel to this choice as the *Gaussian Signs* environment.

Thanks to a standard Gaussian computation, it is easy to check that for any $i, k \in \mathbb{N}$, if we define $\widehat{\mathsf{W}}_1 := \mathsf{W}_i$ and $\widehat{\mathsf{W}}_2 := \frac{1}{\sqrt{1-\rho_k^2}}(\mathsf{W}_{i+k} - \rho_k \mathsf{W}_i)$, then $\widehat{\mathsf{W}}_1$ and $\widehat{\mathsf{W}}_2$ are independent (centered and unitary) Gaussian variables $\mathcal{N}(0,1)$. One has that

$$\mathbb{E}[\omega_i\omega_{i+k}] = \mathbb{P}(\mathsf{W}_i \geqslant 0,\, \mathsf{W}_{i+k} \geqslant 0) = \mathbb{P}\left(\widehat{\mathsf{W}}_1 \geqslant 0,\, \widehat{\mathsf{W}}_2 \geqslant -\frac{\rho_k}{(1-\rho_k^2)^{1/2}}\widehat{\mathsf{W}}_1\right), \tag{7.3.14}$$

so that, substracting $\mathbb{P}(\mathsf{W}_i \geqslant 0)\mathbb{P}(\mathsf{W}_{i+k} \geqslant 0) = \mathbb{P}(\widehat{\mathsf{W}}_1 \geqslant 0, \widehat{\mathsf{W}}_2 \geqslant 0) = 1/4$ one gets

$$\begin{aligned}\mathbb{C}\mathrm{ov}(\omega_i, \omega_{i+k}) &= \mathbb{P}(\mathsf{W}_i \geqslant 0, \mathsf{W}_{i+k} \geqslant 0) - 1/4 \\ &= \mathbb{P}(\widehat{\mathsf{W}}_1 \geqslant 0)\mathbb{P}\left(0 \geqslant \widehat{\mathsf{W}}_2 \geqslant -\frac{\rho_k}{(1-\rho_k^2)^{1/2}}\widehat{\mathsf{W}}_1 \middle| \widehat{\mathsf{W}}_1 \geqslant 0\right) \overset{k\to\infty}{\sim} \frac{\rho_k}{2\pi}, \quad (7.3.15)\end{aligned}$$

where the last asymptotic comes after a short computation, using that $\rho_k \overset{k\to\infty}{\to} 0$.

The sequences W and ω are ergodic, so that $\mathtt{F}(\beta, h)$ exists, and one has the following bounds on the free energy.

Theorem 7.3.4. *For $\omega = (\omega_i)_{i\in\mathbb{N}}$ defined in (7.3.13), and with Assumption (7.3.12) on the sequence W, one has*

- *If $\zeta < 1$, then $h_c(\beta) = 0$ for all $\beta > 0$. One has some constant $c > 0$ such that*

$$\mathtt{F}(\beta, h) \geqslant \exp\left(-c|\log h|^{1/(1-\zeta)}\mathtt{F}(h)^{-\zeta/(1-\zeta)}\right). \quad (7.3.16)$$

Moreover, for all $\beta \in (0,1)$, there exists some $h_0 > 0$ and some $c' > 0$ such that, for all $h \in (0, h_0)$ one has

$$\mathtt{F}(\beta, h) \leqslant \exp\left(-c' h^{-\zeta}\right) \quad (7.3.17)$$

- *If $\zeta > 1$, then there is some $\eta > 0$ such that $\mathtt{F}(\beta, h) \leqslant \mathtt{F}^{\mathrm{a}}(\beta, h) \leqslant \mathtt{F}(h - \eta\beta)$, so that $h_c(\beta) \geqslant h_c^{\mathrm{a}}(\beta) \geqslant \eta\beta$.*

This is based on Theorem 7.3.2 and on the following Proposition that estimates the probability for a Gaussian correlated vector to be componentwise non-negative.

Proposition 7.3.5. *We suppose that the Gaussian stationary sequence W verifies Assumption (7.3.12). Then,*

- *If $\zeta < 1$, one has two constants $c_1, c_2 > 0$ such that one has*

$$\mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{1, \dots, n\}\right) \geqslant e^{-c_1 n^{\zeta} \log n} \quad (7.3.18)$$

and also that for all indexes $1 \leqslant i_1 < \ldots < i_n$ one has

$$\mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{i_1, \dots, i_n\}\right) \leqslant e^{-c_2 n^{\zeta}}. \quad (7.3.19)$$

- *If $\zeta > 1$, there exists some constant $c_2' > 0$ such that for all indexes $1 \leqslant i_1 < \ldots < i_n$ one has*

$$\mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{i_1, \dots, i_n\}\right) \leqslant e^{-c_2' n}. \quad (7.3.20)$$

We mention that [BDZ95, Th.1.1] gives a much sharper result in the case where the covariances of $\{\mathsf{W}_n\}_{n\in\mathbb{N}}$ are given by the Green function of some transient random walk on $\mathbb{Z}^d$. They note that one can construct a transient Random Walk on $\mathbb{Z}^d$ so that $\rho_n \sim c_\zeta n^{-\zeta}$ for $\zeta \in (0, 2\wedge d)$, with some explicit constant c_ζ (see [BD94, BDZ95] for more details). One then has, in dimension $d = 1$ and for $\zeta < 1$, that the lower bound in Proposition 7.3.5 is of the right order:

$$\lim_{n\to\infty} -\frac{1}{n^{\zeta}\log n}\log\mathbb{P}(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{1, \dots, n\}) = \mathsf{C}_\zeta, \quad (7.3.21)$$

where the constant C_ζ is explicit.

The limit (7.3.21) is therefore a great improvement of our Proposition in a particular case (refer to [BDZ95] for the explicit Assumptions one has to make), and would allow us to get a slightly more precise upper bound in Theorem 7.3.4. As we do not hunt for the sharp upper bound in Theorem 7.3.4, and as improving (7.3.18)-(7.3.19) to a result of the type (7.3.21) would only give a slightly better upper bound (that however would not match the lower bound (7.3.16)), we are satisfied with Proposition 7.3.5 which is valid with very weak assumptions, and that we prove in Appendix A.3.

The case $\zeta = 1$, being marginal, is more problematic, and we refer to the Remark A.3.2 in Appendix A: we believe that a statement similar to (7.3.20) holds also in that case, and that the system should stand in the "*classical* regime" where $h_c(\beta) > 0$ for all $\beta > 0$.

With Proposition 7.3.5 in hands, one estimates rather precisely $\mathbb{P}(\xi_1 \geqslant A)$ which is equal to $\mathbb{P}(\mathsf{W}_i \geqslant 0\ ;\ \forall i \in \{1,\dots,A\})$. If $\zeta > 1$ one has $\bar{\varepsilon}(A) \geqslant \delta$ uniformly in A, for some $\delta > 0$ (recall the definition (7.3.6) of $\bar{\varepsilon}$).

On the other hand if $\zeta < 1$, one has $\varepsilon(x) \leqslant \bar{\varepsilon}(x) \leqslant c' x^{\zeta-1} \log x$, and thus there exists a constant c such that we have $\varepsilon^{-1}(u) \leqslant \mathsf{c}(u^{-1}|\log u|)^{1/(1-\zeta)}$. From the definition $A_h := \varepsilon^{-1}(c_0 \mathtt{F}(h))$ in Theorem 7.3.2-(part **1**), one therefore has the bound $A_h \leqslant c(\mathtt{F}(h)^{-1}|\log h|)^{1/(1-\zeta)}$. One concludes using Proposition 7.3.5, which gives that $\mathbb{P}(\xi_1 \geqslant A_h) \geqslant \exp\left(-c|\log h|^{1/(1-\zeta)}\mathtt{F}(h)^{-\zeta/(1-\zeta)}\right)$, which combined with (7.3.3) brings (7.3.16). Moreover, Proposition 7.3.5 also implies that $\mathbb{P}(\xi_1 \geqslant A) \leqslant e^{-c_2' A^{\zeta}}$ so that (7.3.17) follows directly from the bound (7.3.8).

7.3.3. Observations on the effects of correlations. In analogy with Chapter 5, where the system had a different behavior according to whether correlations were summable or not, we have here two very different regimes. The first one, that we call *Classical* regime, is when $\lim_{x\to\infty} \varepsilon(x) \geqslant \delta > 0$ and hence $h_c(\beta) > 0$ (according to Conjecture 7.3.3), and where we believe that the Weinrib-Halperin prediction holds. The second one, that we call *Strongly Relevant* regime, is when $\lim_{x\to\infty} \varepsilon(x) = 0$ and $h_c(\beta) = 0$ (see Theorem 7.3.2), and where disorder smoothes the phase transition for all values of $\alpha > 0$, if $\mathbb{P}(\xi \geqslant n)$ decays faster than any power of n.

We comment our results more in terms of the exponential decay of $\mathbb{P}(\xi_1 \geqslant A)$ than in terms of $\lim_{x\to\infty} \varepsilon(x)$ (as it was the case in Theorem 7.3.2 or Conjecture 7.3.3), for the simplicity of the exposition.

7.3.3.1. *Classical regime: if* $\mathbb{P}(\xi_1 \geqslant A)$ *decays exponentially fast.* If $\bar{\varepsilon}(x) \geqslant \delta$ (which is the case in Theorem 7.3.2-(part **2**)), the decay of $\mathbb{P}(\xi_1 \geqslant A)$ is exponential. One could think that the system behave as if the ω_i's were *i.i.d.*, at least regarding the distribution and size of the regions of 0's and -1's. The situation is actually a bit more complicated. In particular, one could have that $\mathbb{P}(\xi_1 \geqslant A)$ decays exponentially fast (and $\bar{\varepsilon}(x) \overset{x\to\infty}{\to} \delta > 0$), but that $\varepsilon(x)$ still goes to zero: for example if the large regions where $\omega \equiv 0$ do aggregate too much, $\mathbb{E}[\log d_1(A)]$ would be much smaller than $\log \mathbb{E}[d_1(A)]$.

In the case where $\lim_{x\to\infty} \varepsilon(x) \geqslant \delta > 0$, the model should exhibit the same features as in the *i.i.d.* case, the blocks of -1's and 0's having (more or less) the same typical distribution and size. We therefore expect that, in this case, the presence of the correlations does not modify the Harris criterion for disorder relevance/irrelevance, as suggested by the Weinrib-Halperin prediction.

7.3.3.2. *Strongly relevant regime: if* $\mathbb{P}(\xi_1 \geqslant A)$ *decays sub-exponentially.* Thanks to the inequality (7.3.6), if $\bar{\varepsilon}(x) \stackrel{x\to\infty}{\to} 0$ one has $h_c(\beta) = 0$. In that case, $\mathbb{P}(\xi_1 \geqslant A)$ decays sub-exponentially, and the behavior of the sequence ω is then very different from the case where the ω_i's are *i.i.d.* We find that, if $\mathbb{P}(\xi_1 \geqslant A)$ decays faster than any power of A, disorder is *always relevant*. We actually believe that, in great generality, $\mathtt{F}(\beta, h)$ has a smoother smoother phase transition than $\mathtt{F}(0, h)$, for all values of $\alpha > 0$. This is a striking result, in view of the Weinrib-Halperin criterion, which does not predict such an atypical behavior.

In the case where $\mathbb{P}(\xi_1 \geqslant A)$ decays faster than any power of A (for example stretched-exponentially), inequality (7.3.8) shows that the phase transition is even of infinite order, whereas it is of order $\nu^{\mathrm{pur}} := \alpha \wedge 1$ in the homogeneous case: disorder is *strongly* relevant, in the sense that it makes the phase transition infinitely smooth, for every value of the renewal parameter α. This is the case with the Gaussian Signs environment in Section 7.3.2: in Theorem 7.3.4, even if the upper bound (7.3.17) on the free energy does not match the lower bound (7.3.16), the free energy has a stretched-exponential decay. It is the first example we are aware of in the pinning model framework, in which it is shown that the presence of disorder makes the phase transition of infinite order (the case $\alpha = 0$ where one already has $\nu^{\mathrm{pur}} = \infty$ being left aside).

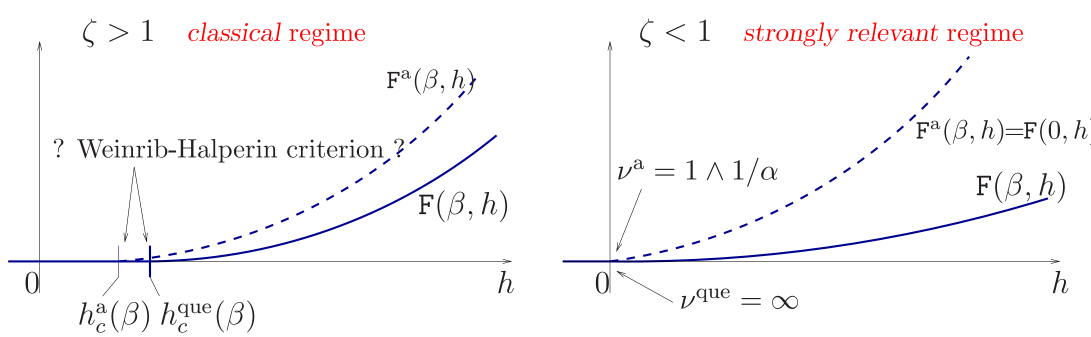

FIGURE 7.2. Difference of behavior of the system in a Gaussian Signs environment (Section 7.3.2, definition (7.3.13)), in the case $\zeta > 1$ (left) and in the case $\zeta < 1$ (right). In the case where $\zeta > 1$, the question of disorder relevance/relevance is still open, and we believe that the Weinrib-Halperin criterion is valid. There should be an extension of the Harris criterion to the correlated case, with no modification of it (as settled in the hierarchical framework, Chapter 4): it is the *classical* regime. In the case where $\zeta < 1$, one has a completely different picture. The critical point $h_c(\beta)$ is equal to 0 (its minimal possible value), and the phase transition of the disordered system is of infinite order: it is the *strongly relevant* regime.

The passage from a *classical* to a *strongly relevant* regime is due to the appearance of exceptional large stretches where $\omega \equiv 0$, that are responsible for the new behavior of the system. We also note that the closer ζ is to 0, the closer the behavior is to the homogeneous one: see (7.3.16) that gives a polynomial decay of the lower bound when $\zeta \to 0$, and Theorem 6.2.1, where the bounds on the free energy are matching the ones of the homogeneous case when $\widetilde{\alpha}$ goes to 1 (that corresponds to $\zeta \to 0$). This is due to the fact that the stretches of 0's occur more and more often, and are larger and larger, the system looking piecewise homogeneous.

7.3.3.3. *Strategy of localization.* We now give a hint on how the trajectories behave under the measure $\mathbf{P}_{N,h}^{\omega,\beta}$, in particular in the case where $\varepsilon(x) \overset{x\to\infty}{\to} 0$ and hence $h_c(\beta) = 0$. We underline that the blocks of 0's and of -1's play different roles. The blocks with $\omega \equiv 0$ are energetically rewarding (if $h > 0$), whereas the -1's regions cause entropic loss, because the trajectories avoid them as much as possible.

This intuition is confirmed by the proof of Theorem 7.3.2. Indeed, the proof of the lower and upper bounds (7.3.3)-(7.3.5) on the free energy (and Conjecture 7.3.3, see Section 7.4.3) shows that when $\varepsilon(x) \overset{x\to\infty}{\to} 0$, very large regions where $\omega \equiv 0$ appear, and the polymer targets these regions, in particular the very large ones, where the energy reward compensates the entropic cost of the targeting strategy.

To be more specific, for $h > 0$ fixed, the right localization strategy consists in aiming only at the A_h-*blocks*, with $A_h = \varepsilon^{-1}(c_0\mathtt{F}(h))$ for some constant $c_0 > 0$, and avoiding the other regions. The value of A_h is justified by the fact that the entropic cost of aiming at the first A_h-*block*, which is $\mathrm{K}(d_1(A_h)) \approx \exp(-(1+\alpha)A_h\varepsilon(A_h))$, is compensated by the energy reward one gets on this block, which is approximately $\exp(\mathtt{F}(h)A_h)$.

The idea of the proof is to divide our system into segments $[\mathcal{T}_{k-1}(A), \mathcal{T}_k(A)]$ for some $A = A(h)$ (recall the definition (7.2.5) of $\mathcal{T}_k(A)$, Figure 7.1), and to estimate the contribution of the different segments separately. For the lower bound (7.3.3), it is enough to take $A = A_h = \varepsilon^{-1}(c_0\mathtt{F}(h))$, and to use the strategy described above: one does a large jump up to the ending A_h-*block* of the segment $[\mathcal{T}_{k-1}(A_h), \mathcal{T}_k(A_h)]$, and collects the energetic reward of this A_h-*block*. For the upper bound on the free energy, the analysis is much more delicate, since one has to compare the contribution of all trajectories.

Section 7.4.3, thanks to a multiscale iterated argument, actually suggests that a block $[\mathcal{T}_{k-1}(A), \mathcal{T}_k(A)]$ do not contribute to the partition function, except if $A \geqslant A'_h = \varepsilon^{-1}(c'_0 h)$, the other ones being globally repulsive and therefore avoided by the renewal. In the case $\alpha > 1$ (where $\mathtt{F}(h) \overset{h\to 0}{\sim} ch$), this confirms the idea mentioned above that the trajectories jump from one A_h-*block* to another.

Thus, the behavior of the polymer trajectories under $\mathbf{P}_{N,h}^{\omega,\beta}$ is described in a sharp way if $\lim_{x\to\infty}\varepsilon(x) = 0$. We note that in the case where $\lim_{x\to\infty}\varepsilon(x) \geqslant \delta > 0$, the entropic cost of aiming at some A-*block* ($\approx \exp(-(1+\alpha)\delta A)$) is not balanced by the energy reward one gets on this block ($\approx \exp(A\mathtt{F}(h))$) if $\mathtt{F}(h)$ is smaller than η', no matter how A large is. If $\lim_{x\to\infty}\varepsilon(x) \geqslant \delta > 0$, then there should be some $h_0 = h_0(\delta)$, such that for $h \leqslant h_0(\delta)$ one cannot find segments $[\mathcal{T}_{k-1}(A), \mathcal{T}_k(A)]$ that contribute to the partition function, which explains Conjecture 7.3.3-(7.3.9).

7.3.4. Annealed estimates. To compare the disordered system to the homogeneous one, we usually study the annealed system, that in the present case shows some peculiar behavior.

7.3.4.1. *Triviality of the annealed system in the strongly relevant regime.* We note that under the assumption that $\liminf_{A\to\infty} A^{-1}\log\mathbb{P}(\xi_1\geqslant A)=0$ (strongly relevant regime), the annealed model is trivial. Indeed, imposing all ω_i's to be equal to 0 in a system of size N, one gets that $\mathbb{E}Z_{N,h}^{\omega,\beta}\geqslant\mathbb{P}(\xi_1\geqslant N)Z_{N,h}^{\mathrm{pur}}$. Thus, we readily have

$$\mathtt{F}^{\mathrm{a}}(\beta,h)=\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}Z_{N,h}^{\omega,\beta}\geqslant\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}(\xi_1\geqslant N)+\mathtt{F}(0,h)=\mathtt{F}(0,h). \tag{7.3.22}$$

Since one has the other trivial bound $Z_{n,h}^{\omega,\beta}\leqslant Z_{n,h}^{\mathrm{pur}}$, one gets $\mathtt{F}^{\mathrm{a}}(\beta,h)=\mathtt{F}(0,h)$. The bound $\mathtt{F}(\beta,h)\leqslant\mathtt{F}^{\mathrm{a}}(\beta,h)$ therefore gives no more information than the trivial one $\mathtt{F}(\beta,h)\leqslant\mathtt{F}(0,h)$ (see Figure 7.2).

On the correlation lengths. In the *i.i.d.* case, $\mathtt{F}(\beta,h)$ is $\mathbb{P}$-a.s. equal to the exponential decay rate of the two-point function $\mathbf{E}_{N,h}^{\omega,\beta}(\delta_i\delta_{i+k})-\mathbf{E}_{N,h}^{\omega,\beta}(\delta_i)\mathbf{E}_{N,h}^{\omega,\beta}(\delta_{i+k})$ when $k\to\infty$, as proven in [Ton07, Th.3.5], under particular assumptions on the renewal law (when $\mathbf{P}$ is the law of the return times to the origin of the simple random walk). This yields that $\mathtt{F}(\beta,h)^{-1}$ is the *quenched* correlation length in the *i.i.d.* case.

We also have the usual *quenched-averaged* correlation length, *i.e.* the inverse of the exponential decay rate of $\mathbb{E}[\mathbf{E}_{N,h}^{\omega,\beta}(\delta_i\delta_{i+k})-\mathbf{E}_{N,h}^{\omega,\beta}(\delta_i)\mathbf{E}_{N,h}^{\omega,\beta}(\delta_{i+k})]$. In the *i.i.d.* case, the *quenched-averaged* correlation length is shown to be equal to $\mu(\beta,h)^{-1}$ [Ton07, Th.3.5] (under particular assumptions), where

$$\mu(\beta,h):=-\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[\frac{1}{Z_{N,h}^{\omega,\beta}}\right]. \tag{7.3.23}$$

We believe that one has this correlation length(s) interpretation also in the correlated framework, and we compare the two quantities $\mathtt{F}(\beta,h)$ and $\mu(\beta,h)$. One easily gets from Jensen inequality that $\mu(\beta,h)\leqslant\mathtt{F}(\beta,h)$, and in the *i.i.d.* framework one actually has that $c_\beta\mathtt{F}(\beta,h)^2<\mu(\beta,h)<\mathtt{F}(\beta,h)$ for $h>h_c(\beta)$ (a better lower bound is given in [Ton07, Th.3.3]), which means that the *quenched* and *quenched-averaged* correlation lengths diverge at the same critical point, namely $h_c(\beta)$.

Now we apply the idea of (7.3.22) to $\mu(\beta,h)$. If we assume that the distribution of 0's and -1's are symmetric, then one also has that $\liminf_{A\to\infty}A^{-1}\log\mathbb{P}(\xi_2\geqslant A)=0$, and

$$\frac{1}{N}\log\mathbb{E}\left[\frac{1}{Z_{N,h}^{\omega,\beta}}\right]\geqslant\frac{1}{N}\log\mathbb{P}(\xi_1=1,\xi_2\geqslant N)-\frac{1}{N}\log Z_{N,h-\beta}^{\mathrm{pur}}, \tag{7.3.24}$$

which directly give that $\mu(\beta,h)\leqslant\mathtt{F}(0,h-\beta)$ by letting N go to infinity. The trivial bound $Z_{N,h}^{\omega,\beta}\geqslant Z_{N,h-\beta}^{\mathrm{pur}}$ then gives that $\mu(\beta,h)=\mathtt{F}(0,h-\beta)$.

Therefore one has that $h_c(\beta)=0$ and $\mathtt{F}(\beta,h)$ grows in a smoother way than $\mathtt{F}(0,h)=\mathtt{F}^{\mathrm{a}}(\beta,h)$, but $\mu(\beta,h)=\mathtt{F}(0,h-\beta)$ has a transition of the same order as $\mathtt{F}(0,h)$, at the critical point $h=\beta$. In particular, the *quenched* and *quenched-averaged* correlation lengths diverge at different points, 0 and β respectively. This stresses even more the abnormal behavior of the system, and gives the first (degenerate, in a way) example for pinning models in which one gets that $\mathtt{F}(\beta,h)$ and $\mu(\beta,h)$ have different critical points, and where $\mathtt{F}(\beta,h)$ possibly has a strictly larger critical exponent than $\mu(\beta,h)$ (for example in the Gaussian Signs example).

7.3.4.2. *Annealed bounds in the Classical regime.* We saw that in the *strongly relevant* regime, the annealed bound $\mathtt{F}(\beta,h)\leqslant \mathtt{F}^{\mathrm{a}}(\beta,h)$ is trivial. The following result shows that in the *classical* regime, easy bounds on the annealed free energy are useful, in particular to show that $h_c(\beta)\geqslant h_c^{\mathrm{a}}(\beta)>0$ (see Figure 7.2).

Proposition 7.3.6. *For $\omega\in\{-1,0\}^{\mathbb{N}}$ ergodic, if there exists a constant $c>0$ such that for all indexes $1\leqslant i_1<\ldots<i_m$ one has*

$$\mathbb{P}\left(\omega_{i_1}=0,\ldots,\omega_{i_n}=0\right)\leqslant e^{-cn}, \tag{7.3.25}$$

then there exists a constant $\eta>0$ such that $\mathtt{F}^{\mathrm{a}}(\beta,h)\leqslant \mathtt{F}(h-\eta\beta)$ for all $\beta\in(0,1)$.

In particular one has $h_c(\beta)\geqslant h_c^{\mathrm{a}}(\beta)\geqslant\eta\beta$.

We mention that this gives exactly Theorem 7.3.2-(part **2**).

Proof As ω is $\{-1,0\}$-valued, the sequence $(1+\omega_n)_{n\in\mathbb{N}}$ is $\{0,1\}$-valued. We can expand $e^{\beta\sum_{n=1}^N(1+\omega_n)\delta_n}$, using as in Sections 1.1.2 and 5.3.3 the binomial expansion of $(e^\beta-1+1)^{\sum_{n=1}^N(1+\omega_n)\delta_n}$, that gives

$$\exp\left(\beta\sum_{n=1}^N(1+\omega_n)\delta_n\right)=\sum_{m=0}^N(e^\beta-1)^m\sum_{1\leqslant i_1<\ldots<i_m\leqslant N}\prod_{k=1}^m(1+\omega_{i_k})\delta_k. \tag{7.3.26}$$

Thus from the assumption (7.3.25) one gets

$$\begin{aligned}\mathbb{E}\left[e^{\beta\sum_{n=1}^N(1+\omega_n)\delta_n}\right]&=\sum_{m=0}^N(e^\beta-1)^m\sum_{1\leqslant i_1<\ldots<i_m\leqslant N}\mathbb{P}\left(\omega_{i_1}=0,\ldots,\omega_{i_m}=0\right)\prod_{k=1}^m\delta_k\\&\leqslant\sum_{m=0}^N(e^\beta-1)^m e^{-cm}\sum_{1\leqslant i_1<\ldots<i_m\leqslant N}\prod_{k=1}^m\delta_k\leqslant\left(e^{-c}(e^\beta-1)+1\right)^{\sum_{n=1}^N\delta_n}.\end{aligned} \tag{7.3.27}$$

Using that there exists some $\eta=\eta(c)>0$ such that $e^{-c}(e^\beta-1)+1\leqslant e^{(1-\eta)\beta}$ for all $\beta\in(0,1)$ one has

$$\mathbb{E}Z_{N,h}^{\omega,\beta}=\mathbf{E}\left[e^{(h-\beta)\sum_{n=1}^N\delta_n}\mathbb{E}\left[e^{\beta\sum_{n=1}^N(1+\omega_n)\delta_n}\right]\right]\leqslant\mathbf{E}\left[e^{(h-\eta\beta)\sum_{n=1}^N\delta_n}\right]=Z_{N,h-\eta\beta}^{\mathrm{pur}}. \tag{7.3.28}$$

□

7.4. Bounds on the free energy, proof of Theorem 7.3.2

7.4.1. Lower Bound. For this part of the proof, we work with the pinned partition function, and we recall that, as far as the free energy is concerned, this is equivalent to working with the partition function with “free” boundary condition.

To get a lower bound on the free energy, we use a classical technique, that is to find a strategy of localization for the polymer, aiming only at favorable blocks. We consider that a block $(T_{n-1}, T_n]$ is favorable if it is a A_h-*block*, where $A_h \geqslant 2$ is chosen later, to optimize the lower bound. Note that the partition function on a good block is only a homogeneous partition function with parameter h (since $\omega \equiv 0$ on a A-*block*).

The parameter h and the quantity A_h being fixed, we consider a system of size $\mathcal{T}_N := \mathcal{T}_N(A_h)$ and collect all the N (good) A_h-*blocks* in it:

$$Z^{\omega,\beta,\mathrm{pin}}_{\mathcal{T}_N,h} \geqslant \prod_{k=1}^{N} K\left(d_k(A_h) - \xi_{J_k}\right) \prod_{k=1}^{N} e^{h-1} Z^{\mathrm{pin}}_{\xi_{J_k},h}. \tag{7.4.1}$$

Then Lemma 6.3.1 gives that there exists a constant C_4 such that for all $h \in (0,1)$ and all n, one has $Z^{\mathrm{pin}}_{n,h} \geqslant C_4 n^{-1} e^{n\mathtt{F}(h)}$, so that one has $Z^{\mathrm{pin}}_{\xi_{J_k},h} \geqslant C_4 \xi_{J_k}^{-1} e^{A_h \mathtt{F}(h)}$ for all $k \geqslant 0$. Moreover, the assumption (7.1.1) on $\mathrm{K}(\cdot)$ gives that there exists a constant c such that $\mathrm{K}(n) \geqslant c n^{-(1+\alpha)}$ for all $n \in \mathbb{N}$.

Finally one has

$$\log Z^{\omega,\beta,\mathrm{pin}}_{\mathcal{T}_N,h} \geqslant \sum_{k=1}^{N} -(1+\alpha)\log d_k(A_h) + A_h \mathtt{F}(h) - \log \xi_{J_k} - C, \tag{7.4.2}$$

so that using that $d_k(A_h) \geqslant \xi_{J_k}$ (and that $d_k(A_h) \geqslant 2$), one gets

$$\frac{1}{\mathcal{T}_N} \log Z^{\omega,\beta,\mathrm{pin}}_{\mathcal{T}_N,h} \geqslant \frac{N}{\mathcal{T}_N}\left(A_h \mathtt{F}(h) - (2+\alpha+C/\log 2)\frac{1}{N}\sum_{k=1}^{N} \log d_k(A_h)\right). \tag{7.4.3}$$

Then, letting N go to infinity and using Birkhoff's Ergodic Theorem, one gets that $\mathbb{P}$-a.s.

$$\begin{aligned}\mathtt{F}(\beta,h) \geqslant \frac{1}{\mathbb{E}[d_1(A_h)]} A_h \left(\mathtt{F}(h) - C_0 \frac{1}{A_h} \mathbb{E}[\log d_1(A_h)]\right) \\ \geqslant c\mathbb{P}(\xi_1 \geqslant A_h) A_h \left(\mathtt{F}(h) - C_0 \varepsilon(A_h)\right),\end{aligned} \tag{7.4.4}$$

where we used the definition (7.3.1) of ε, and (7.2.6).

We can now optimize the choice of A_h, by choosing

$$A_h := \varepsilon^{-1}(c_0 \mathtt{F}(h)), \qquad c_0 := (2C_0)^{-1} \tag{7.4.5}$$

so that using the definition of $\varepsilon^{-1}(\cdot)$ we have

$$\varepsilon(A_h) \leqslant c_0 \mathtt{F}(h) = \frac{1}{2C_0}\mathtt{F}(h). \tag{7.4.6}$$

All together, one gets (7.3.3) with $c_0' = c/2$.

7.4.2. Upper bound. The technique used to get the upper bound in Theorem 7.3.2 is very similar to the one used in Chapter 4 in the case $\alpha > 1$. We recall that the upper bound (7.3.5) is somewhat rough: for example in the case of Chapter 6, it does not give the right bound of Theorem 6.2.1 when $\alpha < 1$, and it does not capture the $|\log h|^{1-\widetilde{\alpha}}$ correction in Theorem 6.2.1 when $\alpha > 1$.

The idea is to use a coarse-graining argument in order to estimate the contribution of the different segments $[T_{i-1}, T_i)$ separately, and then identify the blocks that could actually contribute to the free energy. We show that only the blocks where ω is constant equal to 0, and that are sufficiently large (ξ_{2i+1} larger than some threshold value) have a non-zero contribution in our coarse-graining decomposition.

We (re-)introduce the notation used in Chapter 6 of the contribution of a given segment $[a, b]$, $a, b \in \mathbb{N}$, $a < b$ (recall (6.4.3))

$$Z^{\omega,\beta}_{[a,b],h} := \exp(\beta\omega_a + h) Z^{\theta^a\omega,\beta}_{(b-a),h}, \tag{7.4.7}$$

with θ the shift operator.

The following coarse-graining Lemma is directly extracted from Lemma 6.4.2

Lemma 7.4.1. *For every $N \in \mathbb{N}$*

$$Z^{\omega,\beta}_{T_N,h} \leqslant \prod_{i=1}^{N} \left[\left(\max_{x\in(T_{i-1},T_i]} Z^{\omega,\beta}_{[x,T_i],h} \right) \vee 1 \right]. \tag{7.4.8}$$

This coarse-graining Lemma is the first step of our analysis, and we already notice that the blocks $(T_{2i-1}, T_{2i}]$, $i \in \mathbb{N}$ (*i.e.* where $\omega \equiv -1$, except $\omega_{T_{2i}} = 0$) do not contribute to the free energy for h small. Indeed, since these blocks are composed of -1's, one has $\beta\omega_n + h \leqslant h - \beta \leqslant -\beta/2$ if $h \leqslant \beta/2$, for all $n \in (T_{2i-1}, T_{2i})$. Definition (7.4.7) therefore gives that for all $x \in (T_{2i-1}, T_{2i}]$ one has

$$Z^{\omega,\beta}_{[x,T_{2i}],h} \leqslant e^{-\beta/2} Z^{\omega,\beta}_{[x,T_{2i}-1],h} e^h \leqslant 1, \tag{7.4.9}$$

provided that $h \leqslant \beta/2$. In view of Lemma 7.4.1, the contribution of the blocks $(T_{2i-1}, T_{2i}]$ to the quantity $\log Z^{\omega,\beta}_{T_N,h}$ is therefore equal to 0.

We are left with the contribution of the blocks $(T_{2i}, T_{2i+1}]$, $i \geqslant 0$, that we treat with the following lemma.

Lemma 7.4.2. *For every $\beta \in (0,1)$, there exist some $h_0 > 0$ and some constant $c_1 > 0$ such that for any $h \in (0, h_0)$,*

$$\begin{aligned} &\max_{x\in(T_{2i},T_{2i+1}]} Z^{\omega,\beta}_{[x,T_{2i+1}],h} \leqslant 1 \qquad && \textit{if } \xi_{2i+1} \leqslant c_1\beta h^{-1}, \\ &\max_{x\in(T_{2i},T_{2i+1}]} Z^{\omega,\beta}_{[x,T_{2i+1}],h} \leqslant e^{\xi_{2i+1}h} \qquad && \textit{if } \xi_{2i+1} \geqslant c_1\beta h^{-1}. \end{aligned} \tag{7.4.10}$$

Proof The second inequality is trivial, using only that $\delta_n \leqslant 1$, and that $T_{2i+1} - x \leqslant \xi_{2i+1}$. We therefore focus on the first one, in the case $\alpha \in (0,1)$ (the case $\alpha > 1$ being treated in (6.4.7)).

On the blocks $(T_{2i}, T_{2i+1}]$, $i \geqslant 0$, one has $\omega \equiv 0$ except for $\omega_{T_{2i+1}} = -1$. Thus one gets for $x \in (T_{2i}, T_{2i+1}]$,

$$Z^{\omega,\beta}_{[x,T_{2i+1}],h} \leqslant e^h \mathbf{E}\left[\exp\left(h \sum_{n=1}^{T_{2i+1}-x} \delta_n \right) \exp(-\beta \mathbf{1}_{\{T_{2i+1}-x\in\tau\}}) \right]. \tag{7.4.11}$$

Then, setting $l = T_{2i+1} - x$, we notice that

$$\mathbf{E}\left[e^{h\sum_{n=1}^{l}\delta_n}e^{-\beta\mathbf{1}_{\{l\in\tau\}}}\right] \leqslant \mathbf{E}\left[e^{h\sum_{n=1}^{l}\delta_n}\right] - c\beta\mathbf{E}\left[e^{h\sum_{n=1}^{l}\delta_n}\mathbf{1}_{\{l\in\tau\}}\right] = Z_{l,h} - c\beta Z_{l,h}^{\mathrm{pin}}, \tag{7.4.12}$$

where we used that $\beta\mathbf{1}_{\{l\in\tau\}} \leqslant 1$ to expand $e^{-\beta\mathbf{1}_{\{l\in\tau\}}}$. We are therefore left with estimating the *pinned* and *free* partition function. When $l \leqslant h^{-1}$, we have easy bounds:

$$Z_{l,h} = 1 + \sum_{k=1}^{+\infty}\frac{h^k}{k!}\mathbf{E}\left[\left(\sum_{n=1}^{l}\delta_n\right)^k\right] \leqslant 1 + h\mathbf{E}\left[\sum_{n=1}^{l}\delta_n\right]\sum_{k=1}^{+\infty}\frac{(hl)^{k-1}}{k!} \leqslant 1 + c'hl^{\alpha}, \tag{7.4.13}$$

and also

$$Z_{l,h}^{\mathrm{pin}} \geqslant \mathbf{P}(l\in\tau) \geqslant c'' l^{\alpha-1}, \tag{7.4.14}$$

where we used twice Proposition 1.1.8, to estimate $\mathbf{P}(n\in\tau)$. Note that these bounds are actually sharp when l is smaller than the correlation length (which is $\mathtt{F}(h)^{-1}$, cf. [Gia08]).

All together, if $\xi_{2i+1} \leqslant h^{-1}$, one has that

$$Z_{[x,T_{2i+1}],h}^{\omega,\beta} \leqslant e^h\left(1 + c'(\xi_{2i+1})^{\alpha}\left(h - c\beta(\xi_{2i+1})^{-1}\right)\right). \tag{7.4.15}$$

If $\xi_{2i+1} \leqslant c_1 h^{-1}$ with $c_1 := c\beta/2$, one finally has

$$\max_{x\in(T_{2i},T_{2i+1}]} Z_{[x,T_{2i+1}],h}^{\omega,\beta} \leqslant e^h\left(1 - cst.\ \beta(\xi_{2i+1})^{\alpha-1}\right) \leqslant \exp\left(h - cst.\ \beta(\xi_{2i+1})^{\alpha-1}\right), \tag{7.4.16}$$

which is smaller than 1 provided that h is small enough, since $(\xi_{2i+1})^{\alpha-1} \geqslant ch^{1-\alpha} \gg h$. □

Lemma 7.4.2 gives that the contribution of a block $(T_{2i}, T_{2i+1}]$ is therefore null if $\xi_{2i+1} = T_{2i+1} - T_{2i} \leqslant c_1\beta h^{-1}$, and (possibly) non-zero otherwise. The bounds (7.4.10) are equivalent to Lemmas 6.4.3 (for $\alpha > 1$) and 6.5.3 (for $\alpha < 1$) in the previous Chapter, but in the present case, we deal with the cases $\alpha > 1$ and $\alpha < 1$ at the same time (with however some loss in the case $\alpha < 1$).

The coarse graining Lemma 7.4.1 therefore gives

$$\frac{1}{T_{2N}}\log Z_{T_{2N},h}^{\omega,\beta} \leqslant \frac{N}{T_{2N}}\frac{1}{N}\sum_{i=0}^{N-1} c_2 h\xi_{2i+1}\mathbf{1}_{\{\xi_{2i+1}\geqslant c_1' h^{-1}\}}. \tag{7.4.17}$$

Letting N go to infinity, one gets the upper bound (7.3.5), thanks to Birkhoff's Ergodic Theorem, that gives in particular $\lim_{N\to\infty}\frac{T_{2N}}{N} = \mathbb{E}[\xi_1+\xi_2] < +\infty$.

7.4.3. Conjecture on the appearance of a *strongly relevant* regime. In order to get a better upper bound on the free energy, it is necessary to take into account the cost for τ to do long jumps between the blocks that actually contribute to the free energy. Our coarse-graining Lemma 7.4.1 has therefore to be refined, as in Section 6.4.2 (and Section 6.5.2). We now give only a scheme on how one should proceed to get Conjecture 7.3.3, and in particular a precise bound on the free energy (that is sharp when $\alpha > 1$): it is based on a multistep improvement of the coarse graining argument presented in Section 7.4.2.

7.4.3.1. *First step of improvement of our coarse-graining.* The idea of Section 6.4.2, and especially of Lemma 6.4.5, is to say that blocks that are composed of -1's, or that are small (with $\omega \equiv 0$, $\xi \leqslant c_1 h^{-1}$ in view of Lemma 7.4.2) are globally repulsive: one would have on these segments $Z^{\omega,\beta}_{\text{block},h} \leqslant 1-\delta$, and trajectories of τ avoid them.

The method consists in cutting our system into metablocks $(\mathcal{T}_{k-1}, \mathcal{T}_k]_{k\in\mathbb{N}}$, where we denoted $\mathcal{T}_k := \mathcal{T}_k(c_1 h^{-1})$. Recall the definition (7.2.5) of $\mathcal{T}_k$, and Figure 7.1: each segment $(\mathcal{T}_{k-1}, \mathcal{T}_k]_{k\in\mathbb{N}}$ is composed of (many) small blocks (that are therefore repulsive), and finally of one $c_1 h^{-1}$-*block*. Then, analogously to the coarse-graining Lemma 7.4.1, one can decompose the partition function according to the segments $(\mathcal{T}_{k-1}, \mathcal{T}_k]_{k\in\mathbb{N}}$ (instead of $(T_{k-1}, T_k]$):

$$Z^{\omega,\beta}_{\mathcal{T}_N,h} \leqslant \prod_{i=1}^{N} \left[\left(\max_{x\in(\mathcal{T}_{i-1},\mathcal{T}_i]} Z^{\omega,\beta}_{[x,\mathcal{T}_i],h} \right) \vee 1 \right]. \tag{7.4.18}$$

This decomposition allows us to estimate the contribution of the different metablocks separately.

Let us compute briefly what should be the contribution of one of these metablocks. By translation invariance, we only study the contribution of the first metablock $(0, \mathcal{T}_1] = (0, d_1]$. Using again a coarse-graining argument on this system, according to a decomposition into inner blocks $(T_{k-1}, T_k]$, one sees that the only non-repulsive block is the last one, since it is the only one larger than $c_1 h^{-1}$, cf. Lemma 7.4.2. The strategy of τ is therefore to skip the first segments, and then to aim at the ending $c_1 h^{-1}$-*block*, the most favorable. One would approximately have that $Z^{\omega,\beta}_{\mathcal{T}_1,h} \leqslant \mathrm{K}(d_1) Z^{\text{pur}}_{[T_{J_1-1},T_{J_1}],h}$ (recall the definition (7.2.4) of $J_1 = J_1(c_1 h^{-1})$, see Figure 7.1), so that setting $A^{(0)} := c_1 h^{-1}$ one has

$$\log Z^{\omega,\beta}_{\mathcal{T}_1,h} \leqslant \xi_{J_1} h - c'' \log d_1(A^{(0)}(h)). \tag{7.4.19}$$

In view of the Definition 7.3.1, and provided that $\log d_k(A)$ is concentrated enough around its mean (this is the most problematic point of our argument), one would have that $\log d_k(A) \gtrsim A\varepsilon(A)$. In view of (7.4.19), this tells that the first metablock has a non-zero contribution only if $\xi_{J_k} \geqslant A^{(1)}(h)$, where we denoted $A^{(1)}(h) := c' h^{-1} A^{(0)}(h)\varepsilon(A^{(0)}(h))$. Equation (7.4.18) therefore tells that a metablock $(\mathcal{T}_{k-1}, \mathcal{T}_k]$ contributes to the free energy only if $\log Z^{\omega,\beta}_{(\mathcal{T}_{k-1},\mathcal{T}_k],h} > 0$, *i.e.* only if $\xi_{J_k} \geqslant A^{(1)}(h)$.

We now stress that, in view of the definition of $A^{(0)}(h)$, one has

$$A^{(1)}(h) \geqslant cA^{(0)}(h)\mathbb{E}[\log d_k(A^{(0)}(h))] \geqslant cA^{(0)}(h) \log A^{(0)}(h). \tag{7.4.20}$$

As a result one has that $A^{(1)}(h) \gg A^{(0)}(h)$ for h small, and we improved our coarse-graining argument since we eliminated the metablocks with too small ending segment, that give no contribution to the free energy.

After this first step, one gets a better upper bound for the free energy than (7.3.5). Indeed, in the spirit of (7.4.17) and using the estimate (7.4.19), we get that

$$\frac{1}{\mathcal{T}_N}\log Z^{\omega,\beta}_{\mathcal{T}_N,h} \leqslant \frac{N}{\mathcal{T}_N}\frac{1}{N}\sum_{k=0}^{N-1} h\xi_{J_k}\mathbf{1}_{\{\xi_{J_k}\geqslant A^{(1)}(h)\}} \quad \text{with } J_k = J_k(A^{(0)}(h)), \tag{7.4.21}$$

so that letting N goes to infinity (and recalling (7.2.6)), one has

$$\begin{aligned}\mathtt{F}(\beta,h) \leqslant c' h\mathbb{P}(\xi_1 \geqslant A^{(0)}(h))\mathbb{E}\left[\xi_1\mathbf{1}_{\{\xi_1\geqslant A^{(1)}(h)\}}|\xi_1 \geqslant A^{(0)}(h)\right] \\ = c'h\mathbb{E}[\xi_1\mathbf{1}_{\{\xi_1\geqslant A^{(1)}(h)\}}].\end{aligned} \tag{7.4.22}$$

This is much smaller than (7.3.5), since $A^{(1)}(h) \gg A^{(0)}(h) = c_1 h^{-1}$.

We mention that we skipped two technical points, making unjustified approximations. The first one is that $Z^{\omega,\beta}_{\mathcal{T}_1(A^{(0)}),h} \approx \mathrm{K}(d_1(A^{(0)}))Z^{\mathrm{pur}}_{[T_{J_1-1},T_{J_1}],h}$. One has to justify that all the "small" segments $(T_{k-1}, T_k]$ (smaller than $A^{(0)}(h)$), which give no contribution to the partition function, cf. (7.4.10), are actually repulsive, and that the trajectory of τ just skips them. This needs some work, as in Section 6.4.2, and essentially comes from a refinement of Lemma 7.4.1 that would keep track of the $\mathrm{K}(d_1)$ term, as done in Chapter 6 (see inequality (6.4.14) and Lemma 6.4.5).

The other approximation we made is that $\log d_1(A) \asymp A\varepsilon(A)$, and is more problematic. More should be known on the sequence ω, especially concerning the concentration of $\log d_1(A)$ around its mean. However we mention that in the case of the Gaussian Signs environment of Section 7.3.2, one could use Proposition 7.3.5 combined with Borel-Cantelli lemma (we skip details), to get that $\mathbb{P}$-a.s., there is a constant c such that $\max\{\xi_i,\ i\in\{1,\dots,N\}\} \leqslant c\log N^{1/(\zeta\wedge 1)}$. One deduces that $\log d_1(A) \geqslant e^{cA^{\zeta\wedge 1}}$ $\mathbb{P}$-*a.s.*, and one would be able to pursue the argument.

7.4.3.2. *A multiscale coarse-graining argument.* One can actually improve the previous estimate (7.4.22), the above argument being only the first step of an iterative improvement of our coarse-graining decomposition. The first step above tells, pushing a bit further the argument, that a metablock $(\mathcal{T}_{k-1}(A^{(0)}(h)), \mathcal{T}_k(A^{(0)}(h))]$ with ending $A^{(0)}(h)$-*block* smaller than $A^{(1)}(h)$ is actually repulsive, see (7.4.19). The second step of improvement consists in decomposing the system into meta-metablocks $(\mathcal{T}_{k-1}(A^{(1)}(h)), \mathcal{T}_k(A^{(1)}(h))]$ composed of (many) repulsive metablocks $(\mathcal{T}_{k-1}(A^{(0)}(h)), \mathcal{T}_k(A^{(0)}(h))]$, and then of one metablock $(\mathcal{T}_{k-1}(A^{(0)}(h)), \mathcal{T}_k(A^{(0)}(h))]$, with ending $A^{(1)}(h)$-*block*.

One should actually introduce a multiscale coarse-graining analysis. We fix $h > 0$ small, and introduce an increasing sequence $A^{(0)} \leqslant A^{(1)} \leqslant \cdots$, where $A^{(j)}$ is converging to some A'_h (possibly infinite) when j goes to infinity. We now decompose our system into segments at different scales. Define $\mathcal{T}^{(j)}_k := \mathcal{T}_k(A^{(j)})$: the segments $(\mathcal{T}^{(j)}_{k-1}, \mathcal{T}^{(j)}_k]_{k\in\mathbb{N}}$ constitute a partition (at scale j) of our system into metablocks with at their end some $A^{(j)}$-*block*. Each segment $(\mathcal{T}^{(j)}_{k-1}, \mathcal{T}^{(j)}_k]$ is itself divided into metablocks according to the decomposition at the scale $j-1$, see Figure 7.3.

We define (wisely) the sequence $(A^{(j)})_{j\geqslant 0}$ iteratively, in a way that one is able to reproduce the argument of the first step of improvement of our coarse-graining

(Section 7.4.3.1) at a higher scale. The idea is that, at scale j, a segment $(\mathcal{T}_{k-1}^{(j)}, \mathcal{T}_k^{(j)}]$ should be non-repulsive only if its ending $A^{(j)}$-*block* is actually larger than $A^{(j+1)}$. Therefore, when one factorizes the partition function into a product of partition functions over metablocks $(\mathcal{T}_{k-1}^{(j)}, \mathcal{T}_k^{(j)}]$ (see (7.4.18), that we use at scale j), one identifies the metablocks that actually contribute to the free energy as the ones with an ending $A^{(j+1)}$-*block*, in the spirit of Lemma 7.4.2, see Figure 7.3.

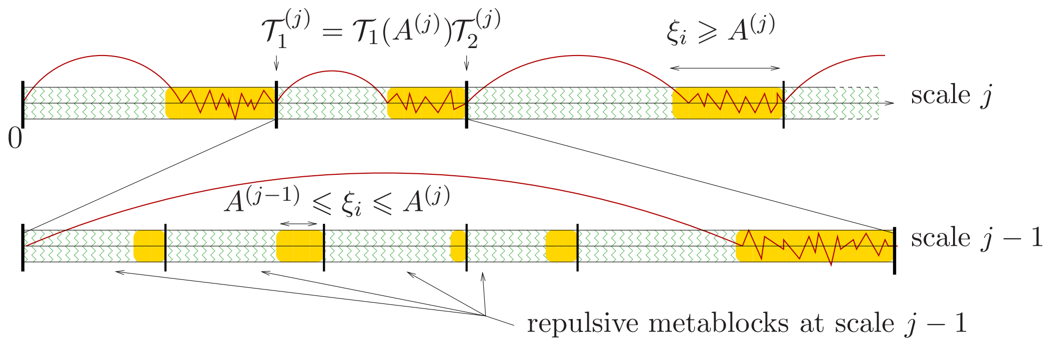

FIGURE 7.3. We use a coarse-graining argument at scale j, and estimate all the contributions of the segments $(\mathcal{T}_{k-1}^{(j)}, \mathcal{T}_k^{(j)}]$ separately. In order to estimate each one of these, we use a coarse-graining argument at scale $j-1$, to deduce that all blocks $(\mathcal{T}_{l-1}^{(j-1)}, \mathcal{T}_k^{(j-1)}]$ are repulsive (at scale $j-1$, only segments with ending $A^{(j)}$-*block* are contributing to the partition function), except the last one, that ends with a $A^{(j)}$-*block*. To sum up, we have identified repulsive zones at each scale (clearer zones in the figure), this identification relying on an iterative multiscale analysis: repulsive zones at scale j are recognized thanks to a coarse-graining argument, using that the repulsive zones at scale $j-1$ have already been singled out.

We now give the details of the choice of the sequence $(A^{(j)})_{j\in\mathbb{N}}$. We set $A^{(0)} = c_1 h^{-1}$, so that in view of Lemma 7.4.2, all blocks $(T_{k-1}, T_k]$, if they are not a $A^{(0)}$-*block*, are repulsive. We suppose that step j has been completed, $A^{(j)}$ being chosen so that the partition function over a segment $(\mathcal{T}_{k-1}^{(j-1)}, \mathcal{T}_k^{(j-1)}]$ is strictly smaller than 1 (and thus repulsive) if the ending $A^{(j-1)}$-*block* is smaller than $A^{(j)}$. We decompose the partition function at scale j, according to the segments $(\mathcal{T}_{k-1}^{(j)}, \mathcal{T}_k^{(j)}]_{k\in\mathbb{N}}$, and estimate the contribution of the different metablocks separately.

We restrict ourselves to study the contribution of the first segment $(0, \mathcal{T}_1^{(j)}] = (0, d_1(A^{(j)})]$, by translation invariance. The coarse-graining at scale $j-1$, tells us that thanks to our choice of $A^{(j)}$, all segments at scale $j-1$ inside of $(0, \mathcal{T}_1^{(j)}]$ are repulsive, except the last one that ends with a $A^{(j)}$-*block* (we used the informal iterative definition of $A^{(j)}$). For that reason, one should observe that τ avoids all the first blocks, landing into the ending $A^{(j)}$-*block*, the most favorable, giving

$$\log Z^{\omega,\beta}_{\mathcal{T}_1^{(j)},h} \leqslant \xi_{J_1^{(j)}} h - C' \log d_1(A^{(j)}) \approx \xi_{J_1^{(j)}} h - C'' A^{(j)} \varepsilon(A^{(j)}), \qquad (7.4.23)$$

where $J_k^{(j)} := J_k(A^{(j)})$ (recall the definition (7.2.4) of J_k): $\xi_{J_1^{(j)}}$ is the size of the first $A^{(j)}$-*block*. As a consequence, a metablock $(\mathcal{T}_k^{(j)}, \mathcal{T}_{k+1}^{(j)}]$ is contributing to the free energy only if, $\xi_{J_k^{(j)}} \geqslant \mathrm{C}h^{-1}A^{(j)}\varepsilon(A^{(j)})$. Thus we define

$$A^{(j+1)} := \mathrm{C}h^{-1}A^{(j)}\varepsilon(A^{(j)}), \tag{7.4.24}$$

which is the threshold for the ending $A^{(j)}$-*block* of a segment $(\mathcal{T}_{k-1}^{(j)}, \mathcal{T}_k^{(j)}]$ to really contribute to the partition function at scale j.

As done for the first step, one would get in the spirit of (7.4.22) and using estimate (7.4.23),

$$\begin{aligned} \mathtt{F}(\beta,h) \leqslant c' h \mathbb{P}(\xi_1 \geqslant A^{(j)}(h)) \mathbb{E}\left[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant A^{(j+1)}(h)\}} | \xi_1 \geqslant A^{(j)}(h)\right] \\ \leqslant c' h \mathbb{E}[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant A^{(j+1)}(h)\}}]. \end{aligned} \tag{7.4.25}$$

Note that from the definition of $A^{(j+1)}$, we have that $A^{(j+1)}(h) \gg A^{(j)}(h)$ only if $\varepsilon(A^{(j)}) \gg h$: the $(j+1)^{\mathrm{th}}$ step of the coarse-graining argument improves significantly the upper bound (7.4.25), only if $\varepsilon(A^{(j)}) \geqslant \mathbf{C}h$, for some large constant $\mathbf{C}$. Therefore we define $j_0 := \sup\{j,\ A^{(j)} \leqslant \varepsilon^{-1}(\mathbf{C}h)\}$ (we choose $\mathbf{C}$ in a moment), and we stop the iterative argument at $j = j_0$ (which is possibly infinite): pushing it further would not improve the bound significantly. Formally, we set $A^{(j)} = A^{(j_0)}$ for all $j \geqslant j_0$.

7.4.3.3. *Conclusions of the heuristic analysis.* Our previous observations suggest that a segment $(\mathcal{T}_{k-1}(A), \mathcal{T}_k(A)]$ does not contribute to the partition function if its ending A-*block* is smaller than $A^{(j_0)}$. A question is therefore to know whether j_0 is finite or not.

- If $\lim_{x\to\infty} \varepsilon(x) = \delta > 0$.

Then Equation (7.4.24) translates into $A^{(j+1)} \geqslant \mathrm{C}\delta A^{(j)} h^{-1}$, with $A^{(0)} = A^{(0)}(h) = c_1 h^{-1}$. As a result, one should have that $A^{(j)} \geqslant c_1(\mathrm{C}\delta h^{-1})^j$, and j_0 is infinite if h is chosen small enough ($h < \mathrm{C}\delta$). One would have in that case that $A^{(j)} \to \infty$ as $j \to \infty$. This argument concludes that for h positive and small, none of the segments $(\mathcal{T}_{k-1}(A), \mathcal{T}_k(A)]$ for $A \in \mathbb{N}$ has a positive contribution to the free energy, and then $\mathtt{F}(h,\beta) = 0$ (see (7.4.25) with $A^{(j)} \stackrel{j\to\infty}{\to} \infty$). We therefore conclude that if $\lim_{x\to\infty} \varepsilon(x) > 0$, we should have $h_c(\beta) > 0$.

- If $\lim_{x\to\infty} \varepsilon(x) = 0$.

We show that j_0 is finite for every $h > 0$. Indeed the iterative Equation (7.4.24) would give that while $j \leqslant j_0$ one has $\varepsilon(A^{(j)}) \geqslant \mathbf{C}h$, so that (7.4.24) gives

$$A^{(j+1)} \geqslant \mathrm{C}\mathbf{C}A^{(j)} \geqslant (\mathrm{C}\mathbf{C})^j A^{(0)}(h).$$

Then, provided that we choose $\mathbf{C} > \mathrm{C}^{-1}$, one has that $A^{(j)}$ overcomes the threshold $\varepsilon^{-1}(\mathbf{C}h)$ at some finite $j_0 = j_0(h)$, since $\varepsilon^{-1}(\mathbf{C}h)$ is defined and finite for all $h > 0$ (we use here that $\varepsilon(x) \stackrel{x\to\infty}{\to} 0$)

Considering the system at scale j_0, one therefore have from (7.4.25)

$$\mathtt{F}(\beta,h) \leqslant c' h \mathbb{E}[\xi_1 \mathbf{1}_{\{\xi_1 \geqslant A'_h\}}], \tag{7.4.26}$$

where $A'_h := A^{(j_0)} \geqslant \varepsilon^{-1}(\mathbf{C}h)$.

We stress that the argument sketched in this Section suggests that the renewal jumps from one A'_h block to another (see Figure (7.3)). This matches the strategy of the lower bound of Section 7.4.1 at least in the case $\alpha > 1$ (for which $\mathtt{F}(h) \overset{h\to 0}{\sim} ch$), where the right localization strategy was exactly to target A_h-*blocks*, with $A_h = \varepsilon^{-1}(c_0\mathtt{F}(h))$

APPENDIX A

Estimates on correlated Gaussian sequences

In this Appendix, we give some estimates on the probability for a strongly correlated Gaussian vector to be componentwise larger than some fixed value (for example Lemma A.2.1). These estimates lies on the study of the relative entropy of two translated correlated Gaussian vectors. Let $\mathsf{W} = \{\mathsf{W}_n\}_{n\in\mathbb{N}}$ be a stationary Gaussian process, centered and with unitary variance, and with covariance matrix denoted by Υ. The stationarity of the process tells that for any $i, j \geqslant 0$, $\Upsilon_{ij} = \mathbb{E}[\mathsf{W}_i\mathsf{W}_j] = \rho_{|i-j|}$ depends only on $|i-j|$, and one calls $(\rho_k)_{k\geqslant 0}$ the correlation function. Let Υ_l denote the restricted correlation matrix, that is the correlation matrix of the Gaussian vector $\mathsf{W}^{(l)} := (\mathsf{W}_1, \ldots, \mathsf{W}_l)$, which is symmetric positive definite.

We recall that two very different behaviors occur, according to whether correlations are summable or not, and we repeat once again the (slightly simplifying) assumptions we make in both cases. Note that we try to get the most general point of view possible, but we often assume that ρ_k is power-law decaying, *i.e.* that $\rho_k \sim ck^{-\zeta}$ for some $\zeta > 0$ (that clarify the statement of the results).

Assumption A.0.3 (Summable correlations)**.** *Correlations are said to be Summable if $\sum |\rho_k| < +\infty$, which corresponds to a power-law decay $\zeta > 1$ of the correlations. This means that Υ is a bounded operator, i.e. that $|||\Upsilon_l|||$ is uniformly bounded in $l \in \mathbb{N}$, where $|||\cdot|||$ denotes the operator norm. We make the additional assumption that Υ^{-1} is also a bounded operator: there exists some A such that $|||\Upsilon_l^{-1}||| \leqslant \mathsf{A}$ for all $l \in \mathbb{N}$.*

Assumption A.0.4 (Non-Summable correlations)**.** *Correlations are said to be Non-Summable if $\sum |\rho_k| = +\infty$. We make the additional assumption that $\rho_k \geqslant 0$ for all $k \geqslant 0$, and that there exists some $\zeta \in (0,1)$ and a constant $c_0 > 0$ such that*

$$\rho_k \overset{k\to\infty}{\sim} c_0 k^{-\zeta}. \tag{A.0.27}$$

A.1. Entropic cost of shifting a Gaussian vector.

In Section 5.4.1, and in Lemma A.2.1, one has to estimate the entropic cost of shifting the Gaussian correlated vector $\mathsf{W}^{(l)}$ by some vector V, V being chosen to be $\mathbf{1}_l$, the vector of size l constituted of only 1, or U, the Perron-Frobenius eigenvector of Υ (if the entries of Υ are non-negative). It appears after a short computation that the relative entropy of the two translated Gaussian vector is $\frac{1}{2}\langle \Upsilon^{-1}V, V\rangle$. We therefore give the two following Lemmas that estimate this quantity, one regarding the summable case, the other one the non-summable case.

Lemma A.1.1 (Case of summable correlations)**.** *Under the Assumption A.0.3 one has*

$$\langle \Upsilon_l^{-1} \mathbf{1}_l, \mathbf{1}_l \rangle \overset{l\to\infty}{=} (1+o(1))(\Upsilon_\infty)^{-1} l, \tag{A.1.1}$$

where $\Upsilon_\infty := 1 + 2\sum_{k\in\mathbb{N}} \rho_k$ *(and* $\mathbf{1}_l$ *was defined above).*

Now we give here a result which is the analogous of Lemma A.1.1, in the case of non-summable, non-negative correlations: we make Assumption A.0.4.

We then note λ the maximal (Perron-Frobenius) eigenvalue of Υ_l, so that thanks to the Perron-Frobenius theorem we can take U an eigenvector associated to this eigenvalue with $U_i > 0$ for all $i \in \{1,\dots,l\}$. Up to a multiplication, we can choose U such that $\min_{i\in\{1,\dots,l\}} U_i = 1$.

Lemma A.1.2 (Case of non-summable correlations)**.** *Under Assumption A.0.4, one has that* $\mathbf{1}_l \leqslant U \leqslant c\mathbf{1}_l$*, where the inequality is componentwise. Moreover, there exist two constants* $c_1, c_2 > 0$ *such that for all* $l \in \mathbb{N}$ *one has* $c_1 l^{1-\zeta} \leqslant \lambda \leqslant c_2 l^{1-\zeta}$*, and therefore*

$$c_2^{-1} l^{\zeta} \leqslant \langle \Upsilon_l^{-1} U, U \rangle \leqslant c c_1^{-1} l^{\zeta}. \tag{A.1.2}$$

Proof of Lemma A.1.1 The proof is classical, since we deal with Toeplitz matrices, and we include it here briefly, for the sake of completeness. The idea is to approximate Υ_l by the appropriate circulant matrix Λ_l

$$\Lambda_l := \begin{pmatrix} \rho_0 & \cdots & \rho_m & & & & \rho_m & \cdots & \rho_1 \\ \vdots & & & \ddots & & & & \ddots & \vdots \\ \rho_m & & & & & & 0 & & \rho_m \\ & \ddots & & & & \ddots & & & \\ & & \rho_m & \cdots & \rho_0 & \cdots & \rho_m & & \\ & & & \ddots & & & & \ddots & \\ \rho_m & & 0 & & & & & & \rho_m \\ \vdots & \ddots & & & & \ddots & & & \vdots \\ \rho_1 & \cdots & \rho_m & & & & \rho_m & \cdots & \rho_0 \end{pmatrix}, \quad \text{with } m = \lfloor \sqrt{l} \rfloor. \tag{A.1.3}$$

One has that Υ_l and Λ_l are asymptotically equivalent, in the sense that their respective operator norms are bounded, uniformly in l (thanks to the summability of the correlations), and that the Hilbert-Schmidt norm $||\cdot||_{\mathrm{HS}}$ of the difference $\Upsilon_l - \Lambda_l$ verifies

$$||\Upsilon_l - \Lambda_l||_{\mathrm{HS}}^2 := \frac{1}{l}\sum_{i,j}^{l} (\Upsilon_{ij} - \Lambda_{ij})^2 \leqslant \frac{c}{l}\left(\sum_{i=1}^{l} \sum_{k \geqslant m} \rho_k^2 + \sum_{i=1}^{m}\sum_{k=1}^{m} \rho_k^2 \right) \overset{l\to\infty}{\to} 0. \tag{A.1.4}$$

For the convergence, we used that $m \ll l$, and the summability of the correlations.

One notices that $\mathbf{1}_l$ is an eigenvector of Λ, and that $\Lambda_l \mathbf{1}_l = \upsilon_l \mathbf{1}_l$, where $\upsilon_l := 1 + 2\sum_{k=1}^m \rho_k$, which converges to Υ_∞. Then we use the idea that, as the operator norms of Υ_l^{-1} and of Λ_l^{-1} are asymptotically bounded, Υ_l^{-1} and Λ_l^{-1} are also asymptotically equivalent. One has

$$|\langle (\Upsilon_l^{-1} - \Lambda_l^{-1})\mathbf{1}_l, \mathbf{1}_l\rangle| = \upsilon_l^{-1} |\langle \Upsilon_l^{-1}(\Upsilon_l - \Lambda_l)\mathbf{1}_l, \mathbf{1}_l\rangle| \leqslant l\, \upsilon_l^{-1}\, |||\Upsilon_l^{-1}|||\, ||\Upsilon_l - \Lambda_l||_{\mathrm{HS}}. \tag{A.1.5}$$

Therefore $\langle \Upsilon_l^{-1}\mathbf{1}_l, \mathbf{1}_l\rangle = \langle \Lambda_l^{-1}\mathbf{1}_l, \mathbf{1}_l\rangle + o(l) = (1+o(1))\upsilon_l^{-1} l$, which concludes the proof since $b_l \overset{l\to\infty}{\to} \Upsilon_\infty$. □

Proof of Lemma A.1.2 We remark that the idea of the proof of Lemma A.1.1 would also work if $\zeta > 1/2$ (and without the assumption of non-negativity), because in that case $\sum \rho_k^2 < \infty$, and (A.1.4) is still valid. It is however difficult to adapt this proof to the $\zeta \leqslant 1/2$ case, and that is why we develop the following technique, that gives estimates on the eigenvector associated to the largest eigenvalue of Υ_l^{-1}.

We consider the Perron-Frobenius eigenvector U (not necessarily normalized) of Υ_l, with eigenvalue λ. We have that $U_i > 0$ for all $i \in \{1,\dots,l\}$, and as already mentioned, we choose U such that $\min_{i\in\{1,\dots,l\}} U_i = 1$. Let us stress that one has, in a classical way

$$\begin{aligned} \lambda &\geqslant \min_{i\in\{1,\dots,l\}} \sum_{j=1}^{l} \Upsilon_{ij} \geqslant c l^{1-\zeta}, \\ \lambda &\leqslant \max_{i\in\{1,\dots,l\}} \sum_{j=1}^{l} \Upsilon_{ij} \leqslant C l^{1-\zeta}, \end{aligned} \tag{A.1.6}$$

where we used the assumption (A.0.27) on the form of the correlations, and that $\zeta < 1$. Then one has $\langle \Upsilon_l^{-1} U, U\rangle = \lambda^{-1}\langle U, U\rangle$, so that we are left to show that the Perron-Frobenius eigenvector U is actually close to the vector $\mathbf{1}_l$. One actually shows that $\mathbf{1}_l \leqslant U \leqslant c\mathbf{1}_l$ where the inequality is componentwise, so that $cl \leqslant \langle U, U\rangle \leqslant c' l$, and it concludes the proof thanks to (A.1.6).

We now prove that $U_\infty := \max_{i\in\{1,\dots,n\}} U_i \leqslant c$ (we already have $\min_{i\in\{1,\dots,l\}} U_i = 1$). Let us show that for $i<j$

$$|U_i - U_j| \leqslant c \frac{|j-i|^{1-\zeta}}{n^{1-\zeta}} U_\infty. \tag{A.1.7}$$

One writes the relation $(\Upsilon_l U)_a = \lambda U_a$ for $a = i, j$, and gets

$$\begin{aligned} \lambda |U_i - U_j| &= \left| \sum_{k=1}^{l} (\Upsilon_{ik} - \Upsilon_{jk}) U_k \right| \\ &\leqslant U_\infty \sum_{k=1}^{l} (\Upsilon_{ik} - \Upsilon_{jk}) \mathbf{1}_{\{\Upsilon_{ik} > \Upsilon_{jk}\}} + U_\infty \sum_{k=1}^{l} (\Upsilon_{ik} - \Upsilon_{jk}) \mathbf{1}_{\{\Upsilon_{ik} > \Upsilon_{jk}\}}. \end{aligned} \tag{A.1.8}$$

From the assumption (A.0.27) on the form of the correlations, there is some constant $C > 0$ such that, if $|j-i| \geqslant C$, then one has $\rho_p > \rho_{p+|i-j|}$ for all $p \geqslant |j-i|$. Then one can write, in the case $i-j \geqslant C$, that

$$\sum_{k=0}^{l} (\Upsilon_{ik}-\Upsilon_{jk})\mathbf{1}_{\{\Upsilon_{ik}>\Upsilon_{jk}\}} \leqslant \sum_{p=j-i}^{i} (\rho_p-\rho_{p+j-i}) + 2\sum_{p=0}^{j-1} K_p + \sum_{p=j-i}^{2(j-i)} (\rho_p-\rho_{p-(j-i)})$$
$$\leqslant 2\sum_{p=0}^{2(j-i)} \rho_p \leqslant c|j-i|^{1-\zeta}. \quad \text{(A.1.9)}$$

The second term in (A.1.8) is dealt with the same way by symmetry, so that one finally has $\lambda|U_i-U_j| \leqslant cU_\infty |j-i|^{1-\zeta}$ for $|i-j| \geqslant C$. Inequality (A.1.7) follows for every $i,j\in\mathbb{N}$ by adjusting the constant.

Suppose that $U_\infty \geqslant 4$. The relation (A.1.7) gives that the components of the vector U cannot vary too much. One chooses i_0 such that $U_{i_0}=U_\infty$, and from (A.1.7) one gets that for all $j\in\mathbb{N}$

$$U_\infty - U_j \leqslant c\frac{|j-i_0|^{1-\zeta}}{n^{1-\zeta}} U_\infty. \quad \text{(A.1.10)}$$

There is therefore some $\delta>0$, such that having $|j-i_0| \leqslant \delta l$ implies that $U_j \geqslant \frac{1}{2}U_\infty (\geqslant 2)$. Then, take j_0 with $U_{j_0}=1$ so that from writing $(KU)_{j_0}=\lambda U_{j_0}$ one gets

$$\lambda = \sum_{k=1}^{l} \Upsilon_{j_0k}U_k \geqslant \sum_{\substack{k=1\\|k-k_0|\leqslant \delta l/2}}^{l} \Upsilon_{j_0k}\frac{U_\infty}{2} \geqslant \frac{U_\infty}{2}\frac{\delta}{2}cl^{1-\zeta}, \quad \text{(A.1.11)}$$

where we used in the last inequality that from (A.3.1) there exists a constant $c>0$ such that for all $k\in\{1,\dots,l\}$ one has $\Upsilon_{j_0k} \geqslant cl^{-\zeta}$, since $|j_0-k| \leqslant l$. One then concludes that $U_\infty \leqslant cst.$ thanks to (A.1.6). □

A.2. Probability for a Gaussian vector to be componentwise large

We prove the following Lemma

Lemma A.2.1. *Under Assumption A.0.4 of non-summable correlations, there exist two constants $c, C>0$ such that for every $l\in\mathbb{N}$, one has*

$$\mathbb{P}\left(\forall i\in\{1,\dots,l\},\ \mathsf{W}_i \geqslant A\right) \geqslant c^{-1}\exp\left(-c(A\vee C\sqrt{\log l})^2 l^\zeta\right). \quad \text{(A.2.1)}$$

This Lemma, taking $A \geqslant C\sqrt{\log l}$, gives directly Lemma 5.4.1. Setting $A=0$, one would also have an interesting statement, that is that, when $\zeta<1$, the probability that the Gaussian vector is componentwise non-negative does not decay exponentially fast in the size of the vector, but stretched-exponentially, see Proposition A.3.1.

Proof First of all, note $\mathcal{A} := \{\forall i\in\{1,\dots,l\},\ \mathsf{W}_i \geqslant A\}$. Set $\bar{\mathbb{P}}$ the law $\mathbb{P}$ on $\{\mathsf{W}_1,\dots,\mathsf{W}_l\}$, where the W_i's have been translated by $B := 2(A\vee C\sqrt{\log l})$ (the constant C is chosen later): under $\bar{\mathbb{P}}$, $\{\mathsf{W}_i\}_{i\in\{1,\dots,l\}}$ is a Gaussian vector of covariance matrix Υ_l, and such that $\bar{\mathbb{E}}\mathsf{W}_i = B$ for all $1 \leqslant i \leqslant l$. Then one uses the classical entropic inequality

$$\mathbb{P}(\mathcal{A}) \geqslant \bar{\mathbb{P}}(\mathcal{A}) \exp\left(-\bar{\mathbb{P}}(\mathcal{A})^{-1}(\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) + e^{-1})\right), \tag{A.2.2}$$

where $\mathbb{H}(\bar{\mathbb{P}}|\mathbb{P}) := \mathbb{E}\left[\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right]$ denotes the relative entropy of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Note that $\bar{\mathbb{P}}(\mathcal{A}) = \mathbb{P}\left(\min_{i=1,\dots,l} \mathsf{W}_i \geqslant A - B\right) = \mathbb{P}\left(\max_{i=1,\dots,l} \mathsf{W}_i \leqslant B - A\right)$, so that from the Claim A.2.2 below, and using that $B - A \geqslant C\sqrt{\log l}$, one gets that $\bar{\mathbb{P}}(\mathcal{A}) \geqslant 1/2$.

One is thus left with estimating the relative entropy $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P})$ in (A.2.2). A straightforward Gaussian computation gives

$$\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) = B^2 \langle \Upsilon_l^{-1} \mathbf{1}_l, \mathbf{1}_l \rangle, \tag{A.2.3}$$

so that from Lemma A.1.2 one has directly that $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) \leqslant cB^2 l^{-\zeta}$, which combined with (A.2.2) gives the right bound. □

Claim A.2.2. *Let $\{\mathsf{W}_i\}_{i\in\{1,\dots,l\}}$ be a centered Gaussian vector of law $\mathbb{P}$, with covariance matrix Υ_l such that all $\rho_{ij} \geqslant 0$ and $\rho_{ii} = 1$. There exists a constant $C > 0$ such that*

$$\mathbb{P}\left(\max_{i=1,\dots,l} \mathsf{W}_i \leqslant C\sqrt{\log l}\right) \geqslant 1/2. \tag{A.2.4}$$

It follows from the classical Slepian's Lemma that if $\{\widehat{\mathsf{W}}_i\}_{i\in\{1,\dots,l\}}$ is a vector of *i.i.d.* standard Gaussian variables (whose law is denoted $\widehat{\mathbb{P}}$), then one has

$$\mathbb{E}\left[\max_{i=1,\dots,l} \mathsf{W}_i\right] \leqslant \widehat{\mathbb{E}}\left[\max_{i=1,\dots,l} \widehat{\mathsf{W}}_i\right] \leqslant c\sqrt{\log l}, \tag{A.2.5}$$

where the second inequality is classical. Thus one gets

$$\mathbb{P}\left(\max_{i=1,\dots,l} \mathsf{W}_i \geqslant 2c\sqrt{\log l}\right) \leqslant \frac{1}{2c\sqrt{\log l}} \mathbb{E}\left[\max_{i=1,\dots,l} \mathsf{W}_i\right] \leqslant 1/2. \tag{A.2.6}$$

A.3. On the sign of a Gaussian sequence

In this Section, we make the assumption that the correlation are power-law decaying. We also suppose that the correlations are non-negative and that there exists some $\zeta > 0$ and some constants $c_0 > 0$ such that

$$\rho_k \overset{k\to\infty}{\sim} c_0 k^{-\zeta}. \tag{A.3.1}$$

Proposition A.3.1. • *If $\zeta < 1$, one has two constants $c_1, c_2 > 0$, such that for every $n \in \mathbb{N}$ one has*

$$\mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{1,\dots,n\}\right) \geqslant e^{-c_1 n^{\zeta} \log n}. \tag{A.3.2}$$

Moreover, for all subsequences $1 \leqslant i_1 < \dots < i_n$, one has

$$\mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{i_1,\dots,i_n\}\right) \leqslant e^{-c_2 n^{\zeta}}. \tag{A.3.3}$$

• *If $\zeta > 1$, there exists some constant $c_2' > 0$ such that for all subsequences $1 \leqslant i_1 < \ldots < i_n$ one has*

$$\mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{i_1, \ldots, i_n\}\right) \leqslant e^{-c_2' n}. \tag{A.3.4}$$

Proposition A.3.1 states that if $\zeta > 1$ the behavior is the same as in the *i.i.d.* case, that is that the probability of observing only non-negative values decays exponentially fast, whereas if $\zeta < 1$ this probability decays stretched-exponentially.

Let us mention that in the case $\zeta < 1$, [BDZ95, Th.1.1] improves in a significant way the above result. If the covariances of $(\mathsf{W}_n)_{n\in\mathbb{N}}$ are given by the Green function of some transient random walk on $\mathbb{Z}^d$ (one can construct such a random walk in a way that $\rho_n \sim c_\zeta n^{-\zeta}$, with some explicit constant c_ζ, see [BDZ95]), then the lower bound in Proposition A.3.1 is of the right order. More precisely one has

$$\lim_{n\to\infty} -\frac{1}{n^{-\zeta}\log n} \log \mathbb{P}(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{1, \ldots, n\}) = \mathsf{C}_\zeta, \tag{A.3.5}$$

where the constant C_ζ is explicit.

Since we actually do not need to know the constant in Proposition A.3.1, our result is enough, and we stress that it is true with very weak assumptions on the correlation structure.

Remark A.3.2. The case $\zeta = 1$ is more problematic because it is a marginal case, and our proofs would adapt to this case, giving

$$e^{-cn/\log n} \geqslant \mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{1, \ldots, n\}\right) \geqslant e^{-cn}. \tag{A.3.6}$$

In view of (A.3.5), and because the term n^ζ $(= \sum_{k=1}^n \rho_k)$ when $\zeta < 1$ would be replaced by $n/\log n$ if $\zeta = 1$, we believe that $\log \mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{1, \ldots, n\}\right)$ is of order n. When $\zeta = 1$, one would therefore have a statement similar to the case $\zeta > 1$.

Proof The lower bound in the case $\zeta < 1$ is a direct consequence (already remarked) of Lemma A.2.1. We therefore focus on the upper bounds.

To simplify notations, we prove the result for the specific sequence $i_k = k$ for all $k \in \{1, \ldots, n\}$, the general proof following the same reasoning. One first observes that for any subset $\{k_1, \ldots, k_m\} \subset \{1, \ldots, n\}$, $m \in \mathbb{N}$, one has

$$\mathbb{P}\left(\mathsf{W}_i \geqslant 0\,;\, \forall i \in \{1, \ldots, n\}\right) \leqslant \mathbb{P}\left(\mathsf{W}_{k_j} \geqslant 0\,;\, \forall j \in \{1, \ldots, m\}\right) \tag{A.3.7}$$

The idea is that if the k_j's are sufficiently far one from another, the Gaussian vector $(\mathsf{W}_{k_1}, \ldots, \mathsf{W}_{k_m})$ behaves like an independent Gaussian vector.

Claim A.3.3. • *If $\zeta < 1$, then there exists some $A > 0$ such that taking $k_j := j\lfloor An^{1-\zeta}\rfloor$ for $j \in \{0, \ldots, m := \lceil A^{-1} n^\zeta \rceil\}$, one has some constant $c > 0$ such that for all $n \in \mathbb{N}$*

$$\mathbb{P}\left(\mathsf{W}_{k_j} \geqslant 0\,;\, \forall j \in \{1, \ldots, m\}\right) \leqslant e^{-cm}. \tag{A.3.8}$$

• *If $\zeta > 1$, then there exists some integer $A > 0$ such that taking $k_j := jA$ for $j \in \{0, \ldots, m := \lceil A^{-1} n\rceil\}$, one has some constant $c > 0$ such that for all $n \in \mathbb{N}$*

$$\mathbb{P}\left(\mathsf{W}_{k_j} \geqslant 0\,;\, \forall j \in \{1, \ldots, m\}\right) \leqslant e^{-cm}. \tag{A.3.9}$$

This claim together with (A.3.7) gives the conclusion, and we now prove the claim.

Under $\mathbb{P}$, the vector $(\mathsf{W}_{k_1}, \dots, \mathsf{W}_{k_m})$ is a Gaussian vector with covariance matrix $\widetilde{\Upsilon}_m$, with $\widetilde{\Upsilon}_{ij} = \Upsilon_{k_i,k_j} = \rho_{k_{|j-i|}}$ for $i,j \in \{1,\dots,m\}$. We note $\widetilde{\mathbb{P}}$ the law of this m-dimensional vector. Then if $\widehat{\mathbb{P}}$ denotes the law of a m-dimensional independent standard $\mathcal{N}(0,\mathrm{Id})$ Gaussian vector, a change of measure procedure gives thanks to the Cauchy-Schwarz inequality

$$\widetilde{\mathbb{P}}\left(\mathsf{W}_j \geqslant 0\,;\,\forall j \in \{1,\dots,m\}\right) \leqslant \left(\frac{1}{2}\right)^{m/2} \widehat{\mathbb{E}}\left[\left(\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\widehat{\mathbb{P}}}\right)^2\right]^{1/2}. \tag{A.3.10}$$

One has $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\widehat{\mathbb{P}}}(X) = (\det \widetilde{\Upsilon}_m)^{-1/2} e^{-\frac{1}{2}\langle(\widetilde{\Upsilon}_m^{-1} - I)X,X\rangle}$ from the definitions of $\widetilde{\mathbb{P}}$ and $\widehat{\mathbb{P}}$, so that

$$\widehat{\mathbb{E}}\left[\left(\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\widehat{\mathbb{P}}}\right)^2\right]^{1/2} = (\det \widetilde{\Upsilon}_m)^{-1/2}(\det(2(\widetilde{\Upsilon}_m)^{-1} - I))^{-1/4} = \det(I - V^2)^{-1/4} \tag{A.3.11}$$

where we defined $V := \widetilde{\Upsilon}_m - I$.

We now estimate $\det(I - V^2)$ thanks to the study of its eigenvalues. Note that the maximal eigenvalue $\widetilde{\lambda}$ of $\widetilde{\Upsilon}_m$, as noticed in (A.1.6), verifies

$$\widetilde{\lambda} \leqslant \max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} \widetilde{\Upsilon}_{ij} \leqslant 1 + 2\sum_{p=1}^{m} \rho_{k_p} \tag{A.3.12}$$

Then we use the definition of k_p and m, and the assumption (A.3.1) on $(\rho_k)_{k\,\geqslant\,0}$. We get:

- if $\zeta < 1$ one has $\widetilde{\lambda} \leqslant 1 + cA^{-\zeta} n^{-\zeta(1-\zeta)} m^{1-\zeta} \leqslant 1 + cA^{-1}$,
- if $\zeta > 1$ one has $\widetilde{\lambda} \leqslant 1 + cA^{-\zeta}$.

In both cases one chooses A large enough so that $\widetilde{\lambda} \leqslant 3/2$. Thus the eigenvalues of $I - V^2$ are bounded from below by $1 - (\widetilde{\lambda} - 1)^2 \geqslant 3/4$, so that in the end one has $\det(I - V^2) \geqslant (3/4)^m$. Combining (A.3.11) and (A.3.10) one gets

$$\bar{\mathbb{P}}\left(\mathsf{W}_j \geqslant 0\,;\,\forall j \in \{1,\dots,m\}\right) \leqslant \left(\frac{1}{2}\right)^{m/2}\left(\frac{3}{4}\right)^{-m/4} \leqslant 3^{-m/4}. \tag{A.3.13}$$

□

Bibliography

[Ale08] K. S. Alexander. The effect of disorder on polymer depinning transitions. *Commun. Math. Phys.*, 279:117–146, 2008.

[Ale09] K. S. Alexander. Ivy on the ceiling: first-order polymer depinning transitions with quenched disorder. *Mark. Proc. and Relat. Fields*, 13:663–680, 2009.

[AS06] K. S. Alexander and V. Sidoravicius. Pinning of polymers and interfaces by random potentials. *Ann. Appl. Probab.*, 16(2):636–669, 2006.

[Asm03] S. Asmussen. *Applied probability and queues.* Applications of mathematics. Springer, second edition, 2003.

[AW90] M. Aizenman and J. Wehr. Rounding effects of quenched randomness on first-order phase transitions. *Commun. Math. Phys.*, 130:489–528, 1990.

[AZ09] K. S. Alexander and N. Zygouras. The effect of disorder on polymer depinning transitions. *Commun. Math. Phys.*, 291:659–689, 2009.

[AZ10] K. S. Alexander and N. Zygouras. Equality of critical points for polymer depinning transitions with loop exponent one. *Ann. Appl. Probab.*, 20(1):356–366, 2010.

[BD94] E. Bolthausen and J.-D. Deuschel. Critical large deviations for gaussian fields in the phase transition regime, i. *Ann. Probab.*, 21:1876–1920, 1994.

[BdH97] E. Bolthausen and F. den Hollander. Localization transition of random copolymers near selective interface. *Ann. Probab.*, 25:1334–1366, 1997.

[BdHO11] E. Bolthausen, F. den Hollander, and A. A. Opoku. A copolymer near a linear interface: variational characteriation of the free energy. *preprint, arXiv:1110.1315v2 [math.PR]*, 2011.

[BDZ95] E. Bolthausen, J.-D. Deuschel, and O. Zeitouni. Entropic repulsion of the lattice free field. *Commun. Math. Phys.*, 170(2):417–443, 1995.

[BGdH10] M. Birkner, A. Greven, and F. den Hollander. Quenched large deviation principle for words in a letter sequence. *Probab. Theory Relat. Fields*, 148:403–456, 2010.

[BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variations.* Cambridge University Press, Cambridge, 1987.

[BL] Q. Berger and H. Lacoin. Sharp critical behavior for random pinning model with correlated environment. *Stoch. Proc. Appl.*

[BL11] Q. Berger and H. Lacoin. The effect of disorder on the free-energy for the random walk pinning model: smoothing of the phase transition and low temperature asymptotics. *Jour. of Stat. Physics*, 42:322–341, 2011.

[Ble89] P.M. Bleher. The renormalization group on hierarchical lattices. In World Scientific Publications, editor, *Stochastic methods in mathematics and physics*, pages 171–201, Teaneck, NJ, 1989. Karpacz 1988.

[Bol02] E. Bolthausen. *Large deviations and Interacting random walks*, volume 1781 of *Lecture notes in Mathematics - École d'Été de Probabilités de Saint-Flour XXIX-1999*, pages 6–131. Springer Verlag, Berlin, 2002.

[BS10] M. Birkner and R. Sun. Annealed vs quenched critical points for a random walk pinning model. *Ann. Inst. H. Poincaré Probab. Stat.*, 46:414–441, 2010.

[BS11] M. Birkner and R. Sun. Disorder relevance for the random walk pinning model in dimension 3. *Ann. Inst. H. Poincaré Probab. Stat.*, 47:259–293, 2011.

[BT10] Q. Berger and F. L. Toninelli. On the critical point of the random walk pinning model in dimension $d = 3$. *Elec. Jour. Probab.*, 15:654–683, 2010.

[BT11] Q. Berger and F. L. Toninelli. Hierarchical pinning model in correlated random environment. *arXiv:1110.5781v1 [math.PR]*, 2011.

[CCFS86] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer. Finite-size scaling and correlation lengths for disordered systems. *Phys. Rev. Lett.*, 57(24), 1986.

[CdHar] D. Cheliotis and F. den Hollander. Variational characterization of the critical curve for pinning of random polymers. *Ann. Probab.*, to appear.

[CEGM84] P. Collet, J.-P. Eckmann, V. Glaser, and A. Martin. Study of the iterations of a mapping associated to a spin glass model. *Commun. Math. Phys.*, 94:353–370, 1984.

[CFS82] I. P. Cornfeld, S. V. Fomin, and I. G. Sinai. *Ergodic Theory.* Springer, New-York, 1982.

[CLM$^+$11] P. Caputo, H.t Lacoin, F. Martinelli, F. Simenhaus, and F. L. Toninelli. Polymer dynamics in the depinned phase: metastability with logarithmic barriers. *Probab. Theory Relat. Fields*, to appear, 2011.

[CMT08] P. Caputo, F. Martinelli, and F. L. Toninelli. On the approach to equilibrium for a polymer with adsorption and repulsion. *Elec. Jour. Probab.*, 13:213–258, 2008.

[CSY04] F. Comets, T. Shiga, and N. Yoshida. Probabilistic analysis of directed polymers in random environment: a review. *Adv. Stud. Pure Math.*, 39:115–142, 2004.

[De 79] P. G. De Gennes. *Scaling concepts in polymer physics*. Cornell University Press, Ithaca, NY, 1979.

[DE86] M. Doi and S. F. Edwards. *The theory of Polymer dynamics*. Oxford University Press, 1986.

[DG84] B. Derrida and E. Gardner. Renormalization group study of a disordered model. *J. Phys. A.: Math. Gen.*, 17:867–887, 1984.

[DGLT09] B. Derrida, G. Giacomin, H. Lacoin, and F. L. Toninelli. Fractional moment bounds and disorder relevance for pinning models. *Commun. Math. Phys.*, 287:867–887, 2009.

[dH09] F. den Hollander. *Random Polymers*. Lecture notes in Mathematics - École d'Été de Probabilités de Saint-Flour XXXVII-2007. Springer, 2009.

[DHV92] B. Derrida, V. Hakim, and J. Vannimenus. Effect of disorder on two-dimensional wetting. *J. Stat. Phys.*, 66:1189–1213, 1992.

[DK11] R. A. Doney and D. A. Korshunov. Local asymptotics for the time of first return to the origin of transient random walk. *Stat. Probab. Letters*, 81(5):363–365, september 2011.

[DM95] B. Davis and D. McDonald. An elementary proof of the local central limit theorem. *Jour. of Theor. Prob.*, 8(3):693–701, 1995.

[Don97] R. A. Doney. One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Relat. Fields*, 107:451–465, 1997.

[Dys69] F. J. Dyson. Existence of a phase-transition in a one-dimensional ising ferromagnet. *Commun. Math. Phys.*, 12(2):91–107, 1969.

[Edw65] S. F. Edwards. The statistical mechanics of polymers with excluded volume. *Proc. Phys. Sci.*, 85:613–624, 1965.

[ET60] P. Erdös and S. J. Taylor. Some problems concerning the structure of random walks. *Acta. Math. Acad. Sci. Hungar.*, 11:137–162, 1960.

[Fis66] M. E. Fisher. Effect of excluded volume on phase transitions in biopolymers. *J. Chem. Phys*, 45:1469–1473, 1966.

[Fis84] M. E. Fisher. Walks, walls, wetting, and melting. *Jour. of Stat. Physics*, 34:667–729, 1984.

[FLNO86] G. Forgacs, J. M. Luck, T. M. Nieuwenhuizen, and H. Orland. Wetting of a disordered substrate: exact critical behavior in two dimensionsfor. *Phys. Rev. Lett.*, 57:2184–2187, 1986.

[Flo53] P. Flory. *Principles of Polymer Chemistry.* Cornell University Press, 1953.

[Gia07] G. Giacomin. *Random Polymer models.* Imperial College Press, 2007.

[Gia08] G. Giacomin. Renewal convergence rates and correlation decay for homogeneous pinning models. *Elec. Jour. Probab.*, 13:513–529, 2008.

[Gia11] G. Giacomin. *Disorder and critical phenomena through basic probability models.* Lecture notes in Mathematics - École d'Été de Probabilités de Saint-Flour XL-2010. Springer, lecture notes in mathematics edition, 2011.

[GLT10a] G. Giacomin, H. Lacoin, and F. L. Toninelli. Hierarchical pinning model, quadratic maps and quenched disorder. *Probab. Theor. Rel. Fields*, 148:159–175, 2010.

[GLT10b] G. Giacomin, H. Lacoin, and F. L. Toninelli. Marginal relevance of disorder for pinning models. *Commun. Pure Appl. Math.*, 63:233–265, 2010.

[GLT11] G. Giacomin, H. Lacoin, and F. L. Toninelli. Disorder relevance at marginality and critical point shift. *Ann. Inst. H. Poincaré*, 47:148–175, 2011.

[GT06] G. Giacomin and F. L. Toninelli. Smoothing effect of quenched disorder on polymer depinning transitions. *Commun. Math. Phys.*, 266:1–16, 2006.

[GT09] G. Giacomin and F. L. Toninelli. On the irrelevant disorder regime of pinning models. *Ann. Probab.*, 37:1841–1875, 2009.

[Ham62] J. M. Hammersley. Generalization of the fundamental theorem on subadditive functions. *Math. Proc. Camb. Philos. Soc.*, 58:235–238, 1962.

[Har74] A. B. Harris. Effect of random defects on the critical behaviour of ising models. *J. Phys. C*, 7:1671–1692, 1974.

[HH85] D. A. Huse and C. L. Henley. Pinning and roughening of domain walls in ising systems due to random impurities. *Phys. Rev. Lett.*, 54:2708–2711, 1985.

[HJ85] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, 1985.

[Itô44] K. Itô. On the ergodicity of a certain stationary process. *Proc. Imp. Acad.*, 20(2):54–55, 1944.

[Kin73] J. F.C. Kingman. Subadditive ergodic theory. *Ann. Probab.*, 1:882–909, 1973.

[KK83] L. Kaup and B. Kaup. *Holomorphic functions of several variables.* Studies in Mathematics. Walter de Gruyter, 1983.

[Lac09] H. Lacoin. *Désordre et phénomène de localisation*. PhD thesis, Université Paris Diderot - Paris 7, 2009.

[Lac10] H. Lacoin. The martingale approach to disorder irrelevance for pinning models. *Elec. Comm. Probab.*, 15:418–427, 2010.

[Law96] G. F. Lawler. *Intersection of random walks*. Probability and its Applications. Springer, Boston, 1996.

[Le 97] J.-F. Le Gall. Marches aléatoires auto-évitantes et mesures de polymères. *Séminaire de Probabilité de Strasbourg*, 31:103–112, 1997.

[LK92] W. Li and K. Kaneko. Long-range correlation and partial $1/f^{\alpha}$ spectrum in a noncoding dna sequence. *Europhys. Lett.*, 17:655–660, 1992.

[LT09] H. Lacoin and F. L. Toninelli. A smoothing inequality for hierarchical pinning model. In A. B. de Monvel and A. Bovier, editors, *Spin Glasses: Statics and dynamics*, volume 62 of *Progress in Probability*, pages 271–278, 2009.

[Luk83] E. Lukacs. *Developments in characteristic function theory*. Charles Griffin & Company, London, 1983.

[MMO11] C. Micheletti, D. Marenduzzo, and E. Orlandini. Polymers with spatial or topological constraints: theoretical and computational results. *Physics Reports*, 504(1):1–73, 2011.

[Mou] J.-C. Mourrat. On the delocalized phase of the random pinning model. *Séminaire de probabilités*.

[Nad98] M. G. Nadkarni. *Basic Ergodic Theory*. Birkhäuser advanced texts. Springer, Berlin, 1998.

[OTW99] E. Orlandini, M. C. Tesi, and S. G. Whittington. A self-avoiding walk model of random copolymer adsorption. *J. Phys. A.: Math. Gen.*, 32:469–477, 1999.

[PBG+92] C.-K. Peng, S. V. Buldyrev, A. L. Goldberger, S. Havlin, F. Sciortino, M. Simons, and H. E. Stanley. Long-range correlations in nucleotide sequences. *Nature*, 365:168–170, 1992.

[Poi11] J. Poisat. Random pinning model with finite range correlations: disorder relevant regime. *arXiv:1111.2476v1 [math.PR]*, 2011.

[Poi12] J. Poisat. On quenched and annealed critical curves of random pinning model with finite range correlations. *Ann. Inst. H. Poincaré*, (arXiv:0903.3704v3 [math.PR]), to appear 2012.

[Pól49] G. Pólya. Remarks on characteristic functions. In *Proceedings of the [First] Berkeley Symposium in Mathematical Statistics and Probability*, pages 115–123, Berkeley & Los Angeles, 1949. University of California Press.

[PS66] D. Poland and H. A. Schegara. Occurence of a phase transition in nucleic acid models. *J. Chem. Phys*, 45:1464–1469, 1966.

[RG87] C. Richard and A. J. Guttmann. Poland-scherage models and the dna denaturation transition. *Jour. of Stat. Physics*, 58:2466–2469, 1987.

[Shi96] P. C. Shields. *The ergodic theory of discrete sample paths*. Graduate Studies in Mathematics. American Mathematical Society, 1996.

[Spi01] F. Spitzer. *Principles of Random Walks*. Springer, 2001.

[Ton07] F. L. Toninelli. Critical properties and finite-size estimates for the depinning transition of directed random polymers. *J. Stat. Phys.*, 126:1025–1044, 2007.

[Ton08a] F. L. Toninelli. Disordered pinning models and copolymers: beyond annealed bounds. *Ann. Appl. Probab.*, 18:1569–1587, 2008.

[Ton08b] F. L. Toninelli. A replica-coupling approach to disordered pinning models. *Commun. Math. Phys.*, 280:389–401, 2008.

[Ton09] F. L. Toninelli. Coarse graining, fractional moments and the critical slope of random copolymers. *Electron. Journal Probab.*, 14:531–547, 2009.

[WH83] A. Weinrib and B. I. Halperin. Critical phenomena in systems with long-range-correlated quenched disorder. *Phys. Rev. B*, 27:413–427, 1983.

[Whi02] S. G. Whittington. Random copolymers. *Physica A*, 314:214–219, 2002.

[YZ10] A. Yilmaz and O. Zeitouni. Differing averaged and quenched large deviations for random walks in random environments in dimensions two and three. *Commun. Math. Phys.*, 300:243–271, 2010.