On pointed Hopf algebras associated to some conjugacy classes in $\mathbb{S}_n$.

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Hopf algebras

$\Bbbk$ alg. closed field char. 0

(H, m, Δ), Hopf algebra:

- (H, m) alg. with unit 1,
- (H, Δ) coassociative coalg. with counit ε,
- $\Delta : H \to H \otimes H$ preserves product and unit,
- exists $\mathcal{S} : H \to H$ "antipode" such that

  $m(\mathcal{S} \otimes \mathrm{id})\Delta = \mathrm{id}_H = m(\mathrm{id} \otimes \mathcal{S})\Delta$.

Examples: • Group algebras.

Γ group $\rightsquigarrow \Bbbk[\Gamma]$, vector sp. with basis e_g, $g \in \Gamma$,
$\Delta(e_g) = e_g \otimes e_g$

• Universal enveloping algebras.

$\mathfrak{g}$ Lie algebra $\rightsquigarrow U(\mathfrak{g})$, enveloping algebra,
$\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g}$

• Quantized enveloping algebras.

$\mathfrak{g}$ simple f. d. Lie algebra, $q \in \Bbbk - 0 \rightsquigarrow U_q(\mathfrak{g})$

$$U_q(\mathfrak{sl}(2)) = \Bbbk < E, F, K^{\pm 1} | K^{-1}K = KK^{-1} = 1,$$
$$KE = q\, EK, \qquad KF = q^{-1}FK,$$
$$EF - q^{-1}FE = 1 - K^2 >.$$

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes 1 + K \otimes E,$$
$$\Delta(F) = F \otimes 1 + K \otimes F.$$

Some invariants.

$G(H) := \{x \in H - 0 : \Delta(x) = x \otimes x\}$.
$=$ group of *group-likes*.

$H_0 := \sum C$, C simple subcoalgebras of H

$=$ largest cosemisimple subcoalgebra of H

$=:$*coradical* of H.

$\mathbb{k}G(H) \subseteq H_0$; H is *pointed non coss.*
$=$ *"pointed"* if $\mathbb{k}G(H) = H_0 \neq H$.

$H_{j+1} := \{x \in H : \Delta(x) \in H_j \otimes H + H \otimes H_0\}$.

$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_j \subseteq H_{j+1} \subseteq \dots$

is the *coradical filtration* of H.

Problem. Classify finite dim. Hopf algs.

Approach: relative position of the coradical.

$$\begin{tikzcd}
 & \text{semisimple} & \\
\text{Finite dim. Hopf alg.} \arrow[ur] \arrow[dr] & & \\
 & & H_0 \text{ Hopf subalg.} \\
 & \text{non semisimple} \arrow[ur] \arrow[dr] & \\
 & & \text{other}
\end{tikzcd}$$

$$
\begin{tikzcd}
 & & \begin{array}{c} G(H) \\ \text{abelian} \end{array} \\
 & \begin{array}{c} H_0 = \Bbbk G(H) \\ \text{(pointed)} \end{array} \arrow[ur] \arrow[dr] & \\
 & & \boxed{\begin{array}{c} G(H) \quad \text{non} \\ \text{abelian} \end{array}} \\
\begin{array}{c} H_0 \quad \text{Hopf} \\ \text{subalg.} \end{array} \arrow[uur] \arrow[dr] & & \\
 & \text{other} &
\end{tikzcd}
$$

Lifting method. N. A. and H.-J. Schneider, *Pointed Hopf Algebras*, MSRI Publications **43** (2002), 168, Cambridge Univ. Press.

Essential step: H Hopf algebra, H_0 Hopf subalg.

$\rightsquigarrow \mathfrak{B}(V) = \oplus_{n\geq 0}\mathfrak{B}^n(V)$ (Nichols algebra)

$\dim H < \infty \implies \dim \mathfrak{B}(V) < \infty$

(V, c) braided vector space: $c \in GL(V \otimes V)$

$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$

$\rightsquigarrow \mathfrak{B}(V) = \oplus_{n\geq 0}\mathfrak{B}^n(V)$

Nichols algebra: $\mathfrak{B}(V) = \oplus_{n\geq 0}\mathfrak{B}^n(V)$
graded algebra with extra structure

- $\mathfrak{B}^0(V) = \Bbbk$, $\mathfrak{B}^1(V) = V$.

- $\mathfrak{B}(V)$ generated by V as algebra.

- $\mathfrak{B}(V)$ is a braided Hopf algebra.

- $P(\mathfrak{B}(V)) = V$.

rank of $\mathfrak{B}(V) = \dim V$

Braided vector space of diagonal type.

$\exists$ basis $v_1, \dots, v_\theta$, $(q_{ij})_{1 \le i,j \le \theta}$ in $\Bbbk^\times$:

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad \forall i, j$$

Braided vector space of Cartan type.

$\exists$ $(a_{ij})_{1 \le i,j \le \theta}$ <u>generalized</u> Cartan matrix

$$q_{ij} q_{ji} = q_{ii}^{a_{ij}}.$$

Theorem. (V, c) Cartan type, $1 \neq q_{ii}$ root of 1.
$\dim \mathfrak{B}(V) < \infty \iff (a_{ij})$ of finite type.

N. A. & H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), 1-45.

I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, `math.QA/0411477`; Inventiones Math., to appear.

Theorem. $1 \neq q_{ii}$ root of 1.

(V, c) of rank 2 or 3

$\Longrightarrow \dim \mathfrak{B}(V) < \infty$ classified.

I. Heckenberger, *Classification of arithmetic root systems of rank 3*, `math.QA/0509145`.

See the next tables from *loc. cit.*

	generalized Dynkin diagrams	fixed parameters
1	$\overset{q}{\circ}\qquad\overset{r}{\circ}$	$q,r\in k^*$
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$	$q\in k^*\setminus\{1\}$
3	$\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{-1}{\circ}\overset{q}{\text{———}}\overset{-1}{\circ}$	$q\in k^*\setminus\{-1,1\}$
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$	$q\in k^*\setminus\{-1,1\}$
5	$\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-1}{\circ}\quad\overset{-q^{-1}}{\circ}\overset{q^2}{\text{———}}\overset{-1}{\circ}$	$q\in k^*\setminus\{-1,1\}$, $q\notin R_4$
6	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{———}}\overset{\zeta q^{-1}}{\circ}$	$\zeta\in R_3$, $q\in k^*\setminus\{1,\zeta,\zeta^2\}$
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_3$
8	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-\zeta^2}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^3}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^3}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_{12}$
9	$\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-\zeta^2}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_{12}$
10	$\overset{-\zeta}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{\zeta^3}{\circ}\quad\overset{\zeta^3}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_9$
11	$\overset{q}{\circ}\overset{q^{-3}}{\text{———}}\overset{q^3}{\circ}$	$q\in k^*\setminus\{-1,1\}$, $q\notin R_3$
12	$\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}\quad\overset{\zeta^2}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_8$
13	$\overset{\zeta^6}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-\zeta^{-4}}{\circ}\quad\overset{\zeta^6}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}\quad\overset{-\zeta^{-4}}{\circ}\overset{\zeta^5}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_{24}$
14	$\overset{\zeta}{\circ}\overset{\zeta^2}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_5$
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_{20}$
16	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{\zeta^5}{\circ}\quad\overset{\zeta^3}{\circ}\overset{-\zeta^4}{\text{———}}\overset{-\zeta^{-4}}{\circ}\quad\overset{\zeta^5}{\circ}\overset{-\zeta^{-2}}{\text{———}}\overset{-1}{\circ}\quad\overset{\zeta^3}{\circ}\overset{-\zeta^2}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_{15}$
17	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{———}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$	$\zeta\in R_7$

Table 1: Weyl equivalence for rank 2 arithmetic root systems

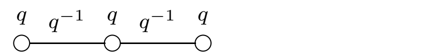
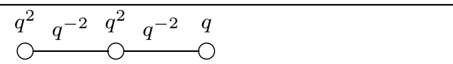
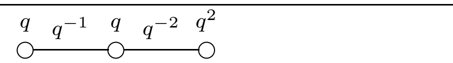
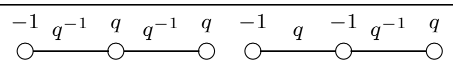
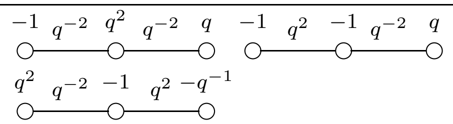
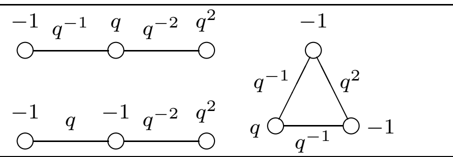
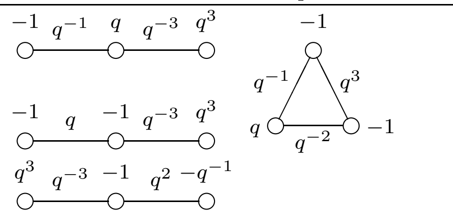
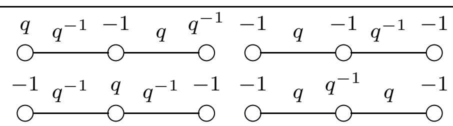
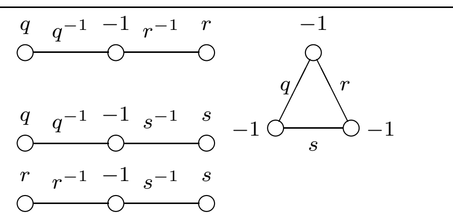
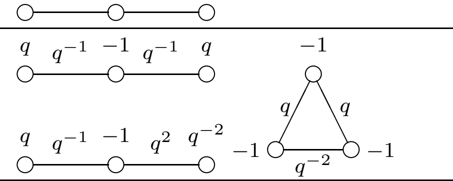
row	gener. Dynkin diagrams	fixed parameters	$\mathcal{G}_{\chi,E}$
1		$q \in k^* \setminus \{1\}$	$[3,4]$
2		$q \in k^* \setminus \{-1,1\}$	$[3,4]$
3		$q \in k^* \setminus \{-1,1\}$	$[3,4]$
4		$q \in k^* \setminus \{-1,1\}$	$\mathbb{Z}_2 \times [3]$
5		$q \in k^* \setminus \{-1,1\}$ $q \notin R_4$	$\mathbb{Z}_2 \times [4]$
6		$q \in k^* \setminus \{-1,1\}$	$\mathbb{Z}_2 \times [4]$
7		$q \in k^* \setminus \{-1,1\}$ $q \notin R_3$	$\mathbb{Z}_2 \times [6]$
8		$q \in k^* \setminus \{-1,1\}$	$\mathbb{Z}_2^3$
9		$q, r, s \in k^* \setminus \{1\}$ $qrs = 1$, $q \neq r$, $q \neq s$, $r \neq s$	$\mathbb{Z}_2^3$
10		$q \in k^* \setminus \{-1,1\}$ $q \notin R_3$	$\mathbb{Z}_2 \times [4]$

Table 2: Weyl equivalence for connected rank 3 arithmetic root systems

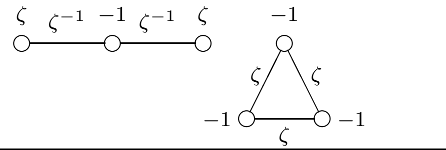
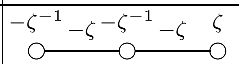
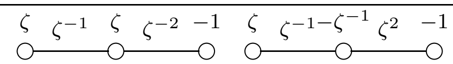
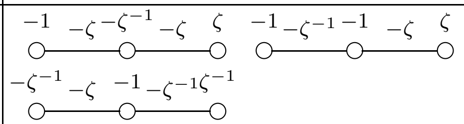
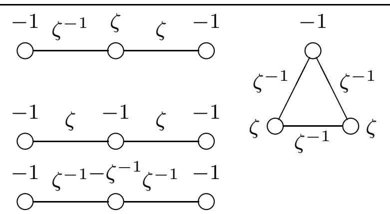
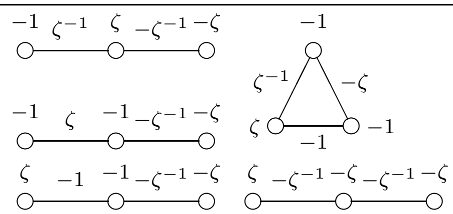
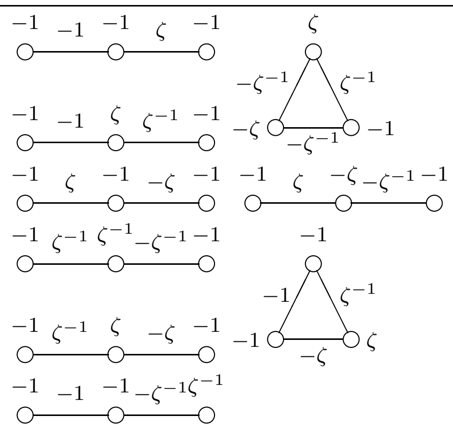
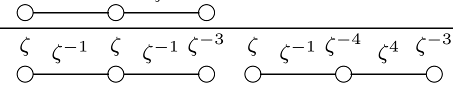
row	gener. Dynkin diagrams	fixed param.	$\mathcal{G}_{\chi,E}$
11		$\zeta \in R_3$	$[3,4]$
12		$\zeta \in R_3$	$[3,4]$
13		$\zeta \in R_3 \cup R_6$	$[3,4]$
14		$\zeta \in R_3$	$\mathbb{Z}_2 \times [4]$
15		$\zeta \in R_3$	$\mathbb{Z}_2 \times [6]$
16		$\zeta \in R_3$	$[6]$
17		$\zeta \in R_3$	$\mathbb{Z}_2^3$
18		$\zeta \in R_9$	$[3,4]$

Table 2: Weyl equivalence for connected rank 3 arithmetic root systems

H pointed, $G(H)$ abelian

Theorem.

Let H be a finite-dimensional *pointed* Hopf algebra with $G(H)$ abelian.

If all prime divisors of $|G(H)|$ are $> 7 \implies H$ is a variation of a Frobenius-Lusztig kernel.

N. A. and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, `http://arxiv.org/abs/math.QA/0502157`.

- Proof: uses results by Heckenberger on Cartan type and rank 2.

- Any H like this has an explicit presentation.

- There are H with prime divisors ≤ 7 NOT variations of Frobenius-Lusztig kernels.

H pointed, $G = G(H)$ not abelian

Braided vector spaces attached to G

$\mathcal{C} = \{t_1 = s, \dots, t_M\}$ a conjugacy class in G,

(ρ, V) irred. repr. of G^s, fixed $s \in \mathcal{C}$.

$g_i \in G$ such that $g_i s g_i^{-1} = t_i$ for all $1 \le i \le M$.

$M(\mathcal{C}, \rho) = \mathrm{Ind}_{G^s}^{G} = \oplus_{1 \le i \le M} g_i \otimes V$.

$g_i v := g_i \otimes v \in M(\mathcal{C}, \rho)$, $1 \le i \le M$, $v \in V$.

action and coaction $\delta(g_i v) = t_i \otimes g_i v$,

$g \cdot (g_i v) = g_j(\gamma \cdot v)$, $g g_i = g_j \gamma$, $1 \le j \le M$, $\gamma \in G^s$.

Braiding:

$$c(g_i v \otimes g_j w) = t_i \cdot (g_j w) \otimes g_i v = g_h(\gamma \cdot v) \otimes g_i v.$$

$t_i g_j = g_h \gamma$ for unique h, $1 \le h \le M$ and $\gamma \in G^s$.

Since $s \in Z(G^s)$, by the Schur Lemma,

s acts by a scalar q_{ss} on V.

Problem. To classify finite dim. pointed Hopf algs. with $G(H)$, the first step is when $\dim \mathfrak{B}(M(\mathcal{C}_1,\rho_1) \oplus \cdots \oplus M(\mathcal{C}_N,\rho_N)) < \infty$?

Rack	**rk**	**Relations**	$\dim \mathfrak{B}(V)$	**top**
$(\mathbb{Z}/3, \triangleright^g), g = 2$ (Transpositions in $\mathbb{S}_3$)	3	5 relations in degree 2	$12 = 3.2^2$	$4 = 2^2$
$(\mathbb{Z}/5, \triangleright^g), g = 2$	5	10 relations in degree 2 1 relation in degree 4	$1280 = 5.4^4$	$16 = 4^2$
$(\mathbb{Z}/7, \triangleright^g), g = 3$	7	21 relations in degree 2 1 relation in degree 6	$326592 = 7.6^6$	$36 = 6^2$
$(\mathbb{Z}/2 \times \mathbb{Z}/2, \triangleright^g)$, $g = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	4	8 relations in degree 2 1 relation in degree 6	72	$9 = 3^2$
Transpositions in $\mathbb{S}_4$	6	16 relations in degree 2	576	12
Faces of the cube	6	16 relations in degree 2	576	12
Transpositions in $\mathbb{S}_5$	10	45 relations in degree 2	8294400	40

- Examples related to $\mathbb{S}_3$, $\mathbb{S}_4$

A. Milinski and H.-J. Schneider, Contemp. Math. **267** (2000), pp. 215–236.

S. Fomin and K. Kirillov, Progr. Math. **172**, Birkhauser, (1999), pp. 146–182.

- Some more examples:

M. Graña, J. Algebra **231** (2000), pp. 235-257.

N. A. and M. Graña, Adv. in Math. **178**, 177–243 (2003).

- Example related to $\mathbb{S}_5$:

web page of M. Graña.

`http://mate.dm.uba.ar/~matiasg/`

- *No general approach up to now! But ...*

Strategy. *Given* $(\mathcal{C}, \rho)$*, find a braided subspace* U *of* $M(\mathcal{C}, \rho)$ *of diagonal type. Check if* $\dim \mathfrak{B}(U)$ *is infinite using the above mentioned results. If so, then* $\dim \mathfrak{B}(\mathcal{C}, \rho) = \infty$*.*

M. Graña, Contemp. Math. **267** (2000), pp. 111–134.

Lemma. Assume that there exists an involution

$$\sigma \in G \text{ such that } \sigma s \sigma = s^{-1}.$$

If $\dim \mathfrak{B}(\mathcal{C}, \rho) < \infty$ then $q_{ss} = -1$, $\operatorname{ord} s$ even.

Proof. Set $q = q_{ss}$.
If $s \neq s^{-1}$, take $U = \Bbbk s \oplus \Bbbk s^{-1}$.

Diagonal braiding: $\begin{pmatrix} q & q^{-1} \\ q^{-1} & q \end{pmatrix}$.

This is of Cartan type, with matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

Hence $\dim \mathfrak{B}(U) = \infty$ unless $q = -1$

Theorem 1. Let W be the Weyl group of a finite-dimensional semisimple Lie algebra. If $\pi \in W$ has odd order then $\dim \mathfrak{B}(\mathcal{C}_\pi, \rho) = \infty$ for any $\rho \in \widehat{W^\pi}$.

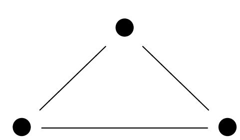
Proof. By R. W. Carter, *Conjugacy classes in the Weyl group*, Compositio Math. **25** (1972), 1–59, any element in W admits an involution such that $\sigma s \sigma = s^{-1}$.

Theorem 2. Let $\pi \in \mathbb{S}_n$. If all cycles in π have odd order except for exactly two transpositions, then $\dim \mathfrak{B}(\mathcal{C}_\pi, \rho) = \infty$ for any $\rho \in \widehat{\mathbb{S}_n^\pi}$.

Proof.

Step 1. Reduce to $\pi = (12)(34) \in \mathbb{S}_4$.

Step 2. Find U of Cartan type with Dynkin diagram



Application.

Classification of fin. dim. Nichols algebra with irreducible V for $\mathbb{S}_3$.

Orbit	**Isotropy group**	**Representation**	$\dim \mathfrak{B}(V)$	**Reference**
e	$\mathbb{S}_3$	any	∞	Trivial braiding
$\mathcal{C}_3$	$\mathbb{Z}_3$	any	∞	Theorem 1
$\mathcal{C}_2$	$\mathbb{Z}_2$	ε	∞	Trivial braiding
$\mathcal{C}_2$	$\mathbb{Z}_2$	sgn	12	[MS]

Classification of fin. dim. Nichols algebra with irreducible V for $\mathbb{S}_4$.

Orbit	**Isotropy group**	**Representation**	$\dim \mathfrak{B}(V)$	**Reference**
e	$\mathbb{S}_4$	any	∞	Trivial braiding
$\mathcal{C}_{2,2}$	$\mathbb{D}_4$	any	∞	Theorem 2
$\mathcal{C}_4$	$\mathbb{Z}_4$	ε, χ_4 or χ_4^3	∞	Trivial braiding
		χ_4^2	576	[AG2,6.12]
$\mathcal{C}_3$	$\mathbb{Z}_3$	any	∞	Theorem 1
$\mathcal{C}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	ε or $\varepsilon \oplus \mathrm{sgn}$	∞	Trivial braiding
$\mathcal{C}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathrm{sgn} \oplus \varepsilon$	576	[FK]
$\mathcal{C}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathrm{sgn} \oplus \mathrm{sgn}$	576	[MS]