Koszul algebras
and
MacMahon’s Master Theorem

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Overview

- MacMahon’s "Master Theorem": statement, some background, references, ...
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- Koszul algebras: a quick introduction
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- Koszul algebras: a quick introduction

- Application: a new proof of the quantum Master Theorem
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The manuscript, joint with **Phùng Hồ Hải** (University of Duisburg-Essen), has been submitted to the LMS

preprint arXiv: math.QA/0603169
Objective

Doron Zeilberger
Rutgers University
The precise formulation of qMMT will be given later

– according to [GLZ], qMMT is "a key ingredient in a finite non-commutative formula for the colored Jones polynomial of a knot" – 

but here is the original MMT . . .
MacMahon’s Master Theorem (original version, 1917):

Given a matrix \( A = (a_{ij})_{n \times n} \) over some commutative ring \( R \) and commuting indeterminates \( x_1, \ldots, x_n \) over \( R \). For each \((m_1, \ldots, m_n) \in \mathbb{Z}_+^n\), the \( R \)-coefficient of \( x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \) in

\[
\left( \sum_{j=1}^{n} a_{1j} x_j \right)^{m_1} \left( \sum_{j=1}^{n} a_{2j} x_j \right)^{m_2} \ldots \left( \sum_{j=1}^{n} a_{nj} x_j \right)^{m_n}
\]

is identical to the corresponding coefficient in

\[
\det \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)^{-1}
\]
Background

Percy Alexander MacMahon
1854 - 1929
Background

Percy Alexander MacMahon
1854 - 1929

- 1882 - 1888: Instructor in Mathematics at the Royal Military Academy
- 1891 - 1898: Instructor/Professor of Physics at the Artillery College
- 1894 - 1896: President of the London Math Society
Title page of MacMahon’s book containing the "Master Theorem"

(originally published at Cambridge, 1917)
... and here is where the name comes from:

wherein the denominator is in symbolic form in such wise that on multiplication the factors $a_1 b_2, a_1 b_2 c_3, \ldots$ are to be placed in determinant brackets $|a_1 b_2|, |a_1 b_2 c_3|, \ldots$ and denote the co-axial minors of the determinant $|a_1 b_2 \ldots n_n|$, which appertains to the matricular relation.

This is a master theorem in the Theory of Permutations.

We will write

$$V_n$$

for the expression

$$|(1 - a_1 x_1)(1 - b_2 x_2) \ldots (1 - n_n x_n)|,$$

and the reader will be able to verify that $V_n$ has also the expression

$$\begin{vmatrix}
a_1 - 1/x_1 & a_2 & \cdots & a_n \\
b_1 & b_2 - 1/x_2 & \cdots & b_n \\
\vdots & \vdots & \ddots & \vdots \\
n_1 & n_2 & \cdots & n_n - 1/x_n
\end{vmatrix}$$

Applications of the Theorem.
Some later proofs:

  - analysis: contour integration

  - combinatorics: cycle-generating functions, binomial identities

  - algebra: Grothendieck duality
Andrews' Problem:

5. MacMahon's Master Theorem and the Dyson Conjecture.

PROBLEM 5. Are there q-analogs of MacMahon's Master Theorem and the Dyson Conjecture?

First let us recall:

MacMahon's Master Theorem (MacMahon (1894), (1915)). The coefficient of $X_1^{p_1}X_2^{p_2} \ldots X_n^{p_n}$ in
... and some motivation:

\[
\delta_{ij} = 1 \text{ if } i = j \quad \text{and} \quad \delta_{ij} = 0 \text{ if } i \neq j.
\]

This theorem has remarkable consequences in the theory of permutations and has been used to provide solutions to generalized Rencontre problems (MacMahon (1915)), the Menage problem (Percus (1971)), and numerous other results. Section 3 of MacMahon's book (MacMahon (1915)) as well as pages 18-31 of J.K. Percus (1971) provide ample evidence of this assertion.

I.J. Good (1962) utilized this...
GLZ proved their qMMT in response to this problem.

The GLZ-quantization is **not** the first non-commutative version of MMT – Foata proved one as early as 1965 –

GLZ claim their quantization to be **natural**
This talk is to support this claim
Reformulation of MMT

Recall: \[ A = (a_{ij}) \in \text{Mat}_{n \times n}(R) \]
\[ x_1, \ldots, x_n \text{ commuting indeterminates over } R \]

For each \( m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n \), let the \( R \)-coefficient of \( x^m = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \) in \( \prod_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} x_j \right)^{m_i} \in R[x_1, \ldots, x_n] \) be denoted by \( c_A(m) \)

**MMT:** \[ 1 = \det \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \cdot \sum_{m} c_A(m)x^m \]

This is an identity in \( R[[x_1, \ldots, x_n]] \)
Reformulation of MMT

\[ \text{MMT:} \quad 1 = \det \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \cdot \sum_{m} c_A(m) x^m \]

\[ \iff \quad \text{all } x_i \mapsto t \]

\[ \text{MMT':} \quad 1 = \det (1_{n \times n} - At) \cdot \sum_{d=0}^{\infty} \left( \sum_{|m|=d} c_A(m) \right) t^d = \sum_i m_i \]

an identity in \( R[t] \)
Reformulation of MMT

**MMT:** \[ 1 = \det \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) \cdot \sum_{m} c_A(m) x^m \]

\[ \uparrow \quad \text{choose } A \text{ "generic"} \]

**MMT':** \[ 1 = \det (1_{n \times n} - At) \cdot \sum_{d=0}^{\infty} \left( \sum_{|m|=d} c_A(m) \right) t^d \]
Reformulation of MMT

**MMT**: \[ 1 = \det \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} \right) \cdot \sum_m c_A(m) x^m \]

\[ \uparrow \]

**MMT'**: \[ 1 = \det (1_{n \times n} - At) \cdot \sum_{d=0}^{\infty} \left( \sum_{|m|=d} c_A(m) \right) t^d \]

\[ = \sum_{d=0}^{n} \text{trace}(\wedge^d A)(-t)^d \]

\[ = \text{trace}(S^d A) \]
Reformulation of MMT

To summarize, we have the following modern interpretation of MMT:

$$1 = \left( \sum_{d=0}^{n} \text{trace}(-t^d A) \right) \cdot \left( \sum_{d=0}^{\infty} \text{trace}(S^d A) t^d \right)$$

All this is a well-known
Part II

Next: Koszul algebras

**Def:** A **quadratic algebra** is a factor of the tensor algebra $T(V)$ of some finite-dimensional $k$-vector space $V$ modulo quadratic relations:

$$A \cong T(V) / (R(A)), \quad R(A) \subseteq T(V)_2 = V \otimes^2$$
Def: A quadratic algebra is a factor of the tensor algebra $T(V)$ of some finite-dimensional $\mathbb{k}$-vector space $V$ modulo quadratic relations:

$$A \cong T(V)/(R(A)), \quad R(A) \subseteq T(V)_2 = V \otimes^2$$

The natural grading of $T(V)$ descends to a grading of $A$:

$$A = \bigoplus_{d \geq 0} A_d \text{ with } A_0 = \mathbb{k}, \ A_1 \cong V$$
**Def:** A *quadratic algebra* is a factor of the tensor algebra $\mathcal{T}(V)$ of some finite-dimensional $\mathbb{k}$-vector space $V$ modulo quadratic relations:

\[ A \cong \mathcal{T}(V)/(R(A)), \quad R(A) \subseteq \mathcal{T}(V)_2 = V \otimes^2 \]

In short: \[ A \leftrightarrow \{A_1, R(A)\} \]
Def: A quadratic algebra is a factor of the tensor algebra $T(V)$ of some finite-dimensional $k$-vector space $V$ modulo quadratic relations:

$$A \cong T(V)/ (R(A)),$$  \hspace{1cm}  $$R(A) \subseteq T(V)_2 = V \otimes 2$$

Notation: $\tilde{x}_1, \ldots, \tilde{x}_n$ will be a $k$-basis of $V$

$\Rightarrow$ $T(V) = k\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle$, the free algebra

$x_i := \tilde{x}_i \mod R(A)$, algebra generators for $A$
Example: Quantum affine $n$-space

For a fixed $0 \neq q \in \mathbb{k}$, define

$$A_q^n|_0 := \mathbb{k}\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle / (\tilde{x}_j \tilde{x}_i - q \tilde{x}_i \tilde{x}_j | 1 \leq i < j \leq n)$$

Thus, $A_q^n|_0$ is generated $x_1, \ldots, x_n$ subject to the relations

$$x_j x_i = q x_i x_j \quad \text{for } i < j.$$
Def: The **quadratic dual** of $A \leftrightarrow \{V, R(A)\}$ is defined by

$$A^\dagger \leftrightarrow \{V^*, R(A)\}$$

with $R(A)^\perp = \{f \in (V \otimes 2)^* \mid f(R(A)) = 0\}$ and $(V \otimes 2)^* \cong (V^*) \otimes 2$ as usual.
Def: The **quadratic dual** of $A \leftrightarrow \{V, R(A)\}$ is defined by

$$A^! \leftrightarrow \{V^*, R(A)^\perp\}$$

with $R(A)^\perp = \{f \in (V \otimes 2)^* \mid f(R(A)) = 0\}$ and $(V \otimes 2)^* \cong (V^*) \otimes 2$ as usual

**Notation:** $\tilde{x}_1, \ldots, \tilde{x}_n$ is a $k$-basis of $V$, as before
Denote the dual basis of $V^*$ by $\tilde{x}_1, \ldots, \tilde{x}_n$
$\leadsto$ generators $x^i = \tilde{x}^i \mod R(A)^\perp$ for $A^!$
The quadratic dual of quantum plane $A_q^{n|0}$ is denoted by $A_q^{0|n}$.

The procedure described yields algebra generators $x^1, \ldots, x^n$ for $A_q^{0|n}$ satisfying the defining relations

$$x^\ell x^\ell = 0 \quad \text{for all } \ell$$

and

$$x^i x^j + q x^j x^i = 0 \quad \text{for } i < j$$
The category QA

objects: quadratic algebras $\text{k}$
morphisms: graded algebra maps
The category QA

objects: quadratic algebras/k
morphisms: graded algebra maps

Some further **operations on** QA:

- ordinary tensor product $A \otimes B$
- Segre product $A \circ B = \bigoplus_n A_n \otimes B_n$
- $A \bullet B \leftrightarrow \{ A_1 \otimes B_1, S_{23}(R(A) \otimes R(B)) \}$

where $S_{23} : A_1^\otimes 2 \otimes B_1^\otimes 2 \to (A_1 \otimes B_1)^\otimes 2$ switches the 2nd and 3rd factors.
The category QA

objects: quadratic algebras $/k$
morphisms: graded algebra maps

$\mathcal{Q}A^{\text{op}}$ is the category of "quantum linear spaces" over $k$

Analogies:

- $\bullet \leftrightarrow \circ$: tensor product of quantum spaces
- $\otimes$: direct sum of quantum spaces
- $!$: dualization plus parity change
The bialgebra $\text{end } A$

**Def:** For a given quadratic algebra $A$, one defines

$$\text{end } A = A^l \bullet A$$

So

$$\text{end } A \leftrightarrow \{ V^* \otimes V, S_{23}(R(A)^\perp \otimes R(A)) \}$$
The bialgebra $\text{end } A$

**Def:** For a given quadratic algebra $A$, one defines

$$\text{end } A = A^! \bullet A$$

So $\text{end } A \leftrightarrow \{V^* \otimes V, S_{23}(R(A)^\perp \otimes R(A))\}$

**Notation:** $\tilde{x}_i, \tilde{x}^j$ dual bases for $V$ and $V^*$ as before

$\leadsto \tilde{z}_i^j = \tilde{x}^j \otimes \tilde{x}_i$ a basis of $V^* \otimes V$

$\leadsto z_i^j = \tilde{z}_i^j \mod R(\text{end } A)$ generate $\text{end } A$
The bialgebra $\text{end} \ A$

Properties:

1. $\text{end} \ A$ is a bialgebra over $k$, with comultiplication

$$\Delta: \text{end} \ A \to \text{end} \ A \otimes \text{end} \ A, \quad \Delta(z^j_i) = \sum_l z^l_i \otimes z^j_l$$

and counit

$$\epsilon: \text{end} \ A \to k, \quad \epsilon(z^j_i) = \delta_{i,j}$$

2. $A$ is a left $\text{end} \ A$-comodule algebra; the coaction is

$$\delta_A: A \to \text{end} \ A \otimes A, \quad \delta_A(x_i) = \sum_j z^j_i \otimes x_j$$
The bialgebra $A_n^{\ell|0}$

Example: Right quantum matrices

This is the algebra $A_q^{n|0}$. Defining relations:

**column relations:** $z_j^\ell z_i^\ell = q z_i^\ell z_j^\ell$ (all $\ell, i < j$)

**cross relations:** $z_i^k z_j^\ell - z_j^\ell z_i^k = q^{-1} z_j^k z_i^\ell - q z_i^\ell z_j^k$ (i < j, k < $\ell$)

Note the "missing" row relations. The algebra $A_q^{n|0}$ is non-commutative even for $q = 1$!
The relations for the generators $z^j_i$ of $\text{end} A_q^{n|0}$ are exactly those used by GLZ to define "right quantum matrices".

(GLZ do not arrive at these relations via Manin’s construction)
The bialgebra $\text{end } A$

The relations for the generators $z_j^i$ of $\text{end } A_q^{n|0}$ are exactly those used by GLZ to define "right quantum matrices".

(GLZ do not arrive at these relations via Manin’s construction)

$\leadsto$ "generic right quantum matrix" $Z = (z_j^i)_{n \times n}$

Any algebra map $\varphi : \text{end } A_q^{n|0} \rightarrow R$ ("$R$-point" of the space defined by $\text{end } A_q^{n|0}$) yields a right quantum matrix $\varphi Z$ over $R$
For any quadratic algebra $A$, one has Koszul complexes

$$K^{\ell,\bullet}(A): 0 \to A_{\ell}^* \to A_{\ell-1}^* \otimes A_1 \to \cdots \to A_1^* \otimes A_{\ell-1} \to A_\ell \to 0$$

for all $\ell \geq 0$; for details see [Manin].
Koszul complexes

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for all $\ell \geq 0$; for details see [Manin].

**Example:** For $A = S(V) = A_{n|0}^{q=1}$, these are the familiar Koszul complexes

$$\ldots \longrightarrow \wedge^{\ell-i+1}(V) \otimes S^{i-1}(V) \longrightarrow \wedge^{\ell-i}(V) \otimes S^i(V) \longrightarrow \ldots$$
Koszul complexes

For any quadratic algebra $A$, one has Koszul complexes

$$K^{\ell, \bullet}(A) : 0 \to A_{\ell}^! \to A_{\ell-1}^! \otimes A_1 \to \cdots \to A_1^! \otimes A_{\ell-1} \to A_{\ell} \to 0$$

for all $\ell \geq 0$; for details see [Manin].

**Lemma 1**

All $K^{\ell, \bullet}(A)$ are complexes of $\text{end } A$-comodules.
Koszul complexes

For any quadratic algebra $A$, one has Koszul complexes

$$K^{\ell, \bullet}(A) : 0 \to A^!_\ell \to A^!_{\ell-1} \otimes A_1 \to \cdots \to A^!_1 \otimes A_{\ell-1} \to A_{\ell} \to 0$$

for all $\ell \geq 0$; for details see [Manin].

**Def:** The quadratic algebra $A$ is said to be **Koszul** iff the complexes $K^{\ell, \bullet}(A)$ are exact for $\ell > 0$. 
Some Koszul facts

- Koszul algebras were introduced by Stewart Priddy in connection with his investigation of Yoneda algebras

\[ \text{Ext}_A(k, k) \ (\text{Trans AMS, 1970}) \]
Some Koszul facts

- Koszul algebras were introduced by Stewart Priddy in connection with his investigation of Yoneda algebras $\text{Ext}_A(\mathbb{k}, \mathbb{k})$ (Trans AMS, 1970)

- There are many equivalent definitions; e.g.,
Some Koszul facts

- Koszul algebras were introduced by Stewart Priddy in connection with his investigation of Yoneda algebras
  \[ \text{Ext}_A(\mathbb{k}, \mathbb{k}) \ (\text{Trans AMS, 1970}) \]

- There are many equivalent definitions; e.g.,
  a graded algebra $A$ is Koszul iff the minimal graded $A$-resolution of $\mathbb{k}$ is linear.
Some Koszul facts

- Koszul algebras were introduced by Stewart Priddy in connection with his investigation of Yoneda algebras
  \[ \text{Ext}_A(k, k) \] (Trans AMS, 1970)

- There are many equivalent definitions; e.g.,
  
  a graded algebra \( A \) is Koszul iff the minimal graded \( A \)-resolution of \( k \) is linear.

- The class of Koszul algebras is quite robust: it is stable under the operations \(!, \otimes, \circ, \bullet, \text{end}, \ldots\)
Some Koszul facts

- Koszul algebras were introduced by Stewart Priddy in connection with his investigation of Yoneda algebras
  \[ \text{Ext}_A(\mathbb{k}, \mathbb{k}) \text{ (Trans AMS, 1970)} \]

- There are many equivalent definitions; e.g.,
  a graded algebra \( A \) is Koszul iff the minimal graded \( A \)-resolution of \( \mathbb{k} \) is linear.

- The class of Koszul algebras is quite robust: it is stable under the operations \(!, \otimes, \circ, \bullet, \text{end}, \ldots\)

- A sufficient condition for \( A \) to be Koszul is the existence of a PBW-basis.
Some Koszul facts

Example: $A_q^{n|0}$ and right quantum matrices

Recall that $A_q^{n|0}$ is generated $x_1, \ldots, x_n$ subject to the relations

$$x_j x_i = q x_i x_j \quad \text{for } i < j.$$  

Therefore, $A$ has a $\mathbb{k}$-basis consisting of the ordered monomials $x^m = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$; this is a PBW-basis.

$\implies$ $A_q^{n|0}$ is Koszul, and hence $\text{end } A_q^{n|0}$ as well.
Notation: $B$ some bialgebra over $\mathbb{k}$ (later: $B = \text{end } A$)  
$\mathcal{R}_B$ Grothendieck ring of all left $B$-comodules that are finite-dimensional/$\mathbb{k}$ (or f.g. projective)
Notation: $B$ some bialgebra over $\mathbb{k}$ (later: $B = \text{end } A$)

$\mathcal{R}_B$ Grothendieck ring of all left $B$-comodules that are finite-dimensional/$\mathbb{k}$ (or f.g. projective)

In more detail:

- $B$-comodule $V \rightsquigarrow [V] \in \mathcal{R}_B$
- $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ exact $\rightsquigarrow [V] = [U] + [W]$ in $\mathcal{R}_B$
- Multiplication in $\mathcal{R}_B$ is given by the tensor product of $B$-comodules
**Def:** Let $V$ be a $B$-comodule; so have $\delta_V : V \to B \otimes V$

Consider the map

\[
\text{Hom}_{\mathbb{k}}(V, B \otimes V) \cong B \otimes V \otimes V^* \xrightarrow{\text{Id}_B \otimes \langle \cdot , \cdot \rangle} B \otimes \mathbb{k} \cong B
\]

**evaluation** $V \otimes V^* \to \mathbb{k}$
**Def:** Let $V$ be a $B$-comodule; so have $\delta_V : V \to B \otimes V$.

Consider the map

$$\text{Hom}_{\mathbb{k}}(V, B \otimes V) \cong B \otimes V \otimes V^* \xrightarrow{\text{Id}_B \otimes \langle \cdot, \cdot \rangle} B \otimes \mathbb{k} \cong B$$

The image of $\delta_V$ under this map will be denoted by $\chi_V$ and called the **character** of $V$. 

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**Characters**
**Def:** Let $V$ be a $B$-comodule; so have $\delta_V : V \to B \otimes V$

Consider the map

$$\text{Hom}_k(V, B \otimes V) \cong B \otimes V \otimes V^* \xrightarrow{\text{Id}_B \otimes \langle \cdot, \cdot \rangle} B \otimes k \cong B$$

The image of $\delta_V$ under this map will be denoted by $\chi_V$ and called the **character** of $V$.

**Explicitly:** If $\delta_V(v_j) = \sum_i b_{i,j} \otimes v_i$ for some $k$-basis $\{v_i\}$ of $V$ then

$$\chi_V = \sum_i b_{i,i}$$
Def: Let $V$ be a $B$-comodule; so have $\delta_V : V \to B \otimes V$
Consider the map

$$\text{Hom}_{k}(V, B \otimes V) \cong B \otimes V \otimes V^* \xrightarrow{\text{Id}_B \otimes \langle . , . \rangle} B \otimes k \cong B$$

The image of $\delta_V$ under this map will be denoted by $\chi_V$ and called the character of $V$.

Lemma 2 The map $[V] \mapsto \chi_V$ yields a well-defined ring homomorphism $\chi : \mathcal{R}_B \to B$. 
Recall the modern interpretation of the original MMT:

\[ 1 = \left( \sum_{d=0}^{n} \text{trace}(\bigwedge^d A)(-t)^d \right) \cdot \left( \sum_{d=0}^{\infty} \text{trace}(S^d A)t^d \right) \]

for any $n \times n$-matrix $A$ over some commutative ring.
Here is the version for Koszul algebras:

**Theorem 1**  \((\text{PHH & L})\) 

*Let \(A\) be a Koszul algebra and \(B = \text{end} A\). Then the following identity holds in \(B[t]\):*

\[
1 = \left( \sum_{m \geq 0} \chi_{A^!_*} \left( -t^m \right) \right) \cdot \left( \sum_{\ell \geq 0} \chi_{A_\ell} t^\ell \right)
\]
Proof: Recall the exact Koszul complex

\[ K^{\ell, \bullet}(A) : 0 \to A_\ell^! \to A_{\ell-1}^! \otimes A_1 \to \cdots \to A_1^! \otimes A_{\ell-1} \to A_\ell \to 0 \]

By Lemma 1, this gives equations in \( \mathcal{R}_B \):

\[ \sum_i (-1)^i [A_i^!] [A_{\ell-i}] = 0 \quad (\ell > 0) \]
Proof: Recall the exact Koszul complex

\[ K^{\ell, *} (A) : 0 \rightarrow A^{\ell}_{\ell} \rightarrow A^{\ell}_{\ell-1} \otimes A_1 \rightarrow \cdots \rightarrow A^{\ell}_1 \otimes A_{\ell-1} \rightarrow A_{\ell} \rightarrow 0 \]

By Lemma 1, this gives equations in \( R_B \):

\[ \sum_{i} (-1)^i [A^{\ell}_{i} *] [A_{\ell-i}] = 0 \quad (\ell > 0) \]

Defining \( P_A(t) = \sum_i [A_i] t^i \), \( P_{A^*}(t) = \sum_i [A^{\ell}_{i} *] t^i \in R_B[t] \), this becomes

\[ 1 = P_{A^*}(-t) \cdot P_A(t) \]
Proof: Recall the exact Koszul complex

\[ K^{\ell, \bullet}(A) : \ 0 \to A^1_\ell^* \to A^1_{\ell-1} \otimes A_1 \to \cdots \to A^1_1 \otimes A_{\ell-1} \to A_\ell \to 0 \]

By Lemma 1, this gives equations in \( R_B \):

\[ \sum_i (-1)^i [A^1_i^*][A_{\ell-i}] = 0 \quad (\ell > 0) \]

Defining \( P_A(t) = \sum_i [A_i] t^i \), \( P_{A^*}(t) = \sum_i [A^1_i^*] t^i \in R_B[t] \), this becomes

\[ 1 = P_{A^*}(-t) \cdot P_A(t) \]

Now apply the ring homomorphism \( \chi[t] : R_B[t] \to B[t] \). QED
Garoufalidis, Lê and Zeilberger’s qMMT is exactly the special case of the Theorem where $A = A_q^{n|0}$

Spelled out in detail . . .
Notation: \[ A = A_q^{n|0} = \mathbb{k}[x_i \mid i = 1, \ldots, n] \]
\[ B = \text{end} A_q^{n|0} = \mathbb{k}[z_i^j \mid i, j = 1, \ldots, n] \]
\[ Z = (z_i^j)_{n \times n} \text{ are as before} \]

For each \( J \subseteq \{1, \ldots, n\} \), I will write \( Z_J = (z_i^j)_{i,j \in J} \).

Finally,
\[ \det_q(Z_J) = \sum_{\pi \in \mathcal{S}_m} (-q)^{-l(\pi)} z_i^{j_1}_{j_{\pi 1}} z_i^{j_2}_{j_{\pi 2}} \cdots z_i^{j_m}_{j_{\pi m}} \]

is the quantum determinant as defined by [FRT].
Theorem 2 (qMMT of GLZ) \( \text{in } B \otimes A = \bigoplus_{m \in \mathbb{Z}_{\geq 0}^n} B \otimes x^m \) put

\( X_i = \sum_j z_i^j \otimes x_j \) and define \( G(m) \) to be the \( B \)-coefficient of \( x^m \) in \( X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n} \). In \( B[t] \) put

\[
\begin{align*}
\text{Bos}(Z) &:= \sum_{\ell \geq 0} \sum_{|m| = \ell} G(m) t^\ell \\
\text{Ferm}(Z) &:= \sum_{m \geq 0} \sum_{J \subseteq \{1, \ldots, n\}} \det_q(Z_J)(-t)^m
\end{align*}
\]

Then:

\[
\text{Bos}(Z) \cdot \text{Ferm}(Z) = 1
\]
Sketch of proof: In view of Theorem 1 we must show

\[ \chi_{A_\ell} = \sum_{|m| = \ell} G(m) \]

\[ \chi_{A_m^*} = \sum_{\substack{J \subseteq \{1, \ldots, n\} \\mid |J| = m}} \det_q(Z_J) \]
For $\chi_{A_\ell} = \sum_{|m| = \ell} G(m)$, recall:

- the homogeneous component $A_\ell$ has a $k$-basis consisting of the ordered monomials $x^m$ with $|m| = \ell$;
- the coaction $\delta_A$ of $B = \text{end} A$ on $A$ is given by $\delta_A(x_i) = \sum_j z_i^j \otimes x_j = X_i$.

Therefore,

$$\delta_A(x^m) = X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} = \sum_{|r| = \ell} b_{r,m} \otimes x^r$$

for uniquely determined $b_{r,m} \in B$. In particular, $b_{m,m} = G(m)$. The desired equality follows.
The proof of $\chi_{A_m^!} = \sum_{J \subseteq \{1, \ldots, n\}} \det_q(Z_J)$ proceeds similarly using the $k$-basis of $A_m^!$ consisting of the elements

$$\wedge x_J := \sum_{\pi \in \mathfrak{S}_m} (-q)^{-l(\pi)} \tilde{x}_{j_{\pi_1}} \otimes \tilde{x}_{j_{\pi_2}} \otimes \ldots \otimes \tilde{x}_{j_{\pi_m}},$$

where $J = (j_1 < j_2 < \ldots < j_m)$ is an $m$-element subset of $\{1, \ldots, n\}$; for details, see the preprint by PHH&L.
Alternative proofs

The original proof in [GLZ] uses the calculus of difference operators developed by Zeilberger.
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Zeilberger has also written Maple programs QuantumMACMAHON and qMM that verify qMMT (available on Zeilberger’s web page).
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Independent work by Foata & Han gives an alternative new proof of qMMT using combinatorics on words. They also analyze the algebra of right quantum matrices in detail and give various modifications of qMMT (3 preprints, December 2005, available on arXiv).