Conductance of a subdiffusive random weighted tree

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May 2, 2019

Abstract

We work on a Galton–Watson tree with random weights, in the so-called "subdiffusive" regime. We study the rate of decay of the conductance between the root and the n-th level of the tree, as n goes to infinity, by a mostly analytic method. It turns out the order of magnitude of the expectation of this conductance can be less than $1/n$ (in contrast with the results of Addario-Berry–Broutin–Lugosi and Chen–Hu–Lin), depending on the value of the second zero of the characteristic function associated to the model.

We also prove the almost sure (and in L^p for some $p > 1$) convergence of this conductance divided by its expectation towards the limit of the additive martingale.

Keywords. Random walks in random environments, Galton–Watson trees, conductance.

2010 Mathematics Subject Classification. 60J80, 60G50, 60F25, 60F15.

1 Introduction

The strong links between electric networks and reversible random walks on graphs have emerged during the second half of the last century and were popularized in the seminal book [9]. The special cases of random walks on (random) trees have been thouroughly studied during the 90's by Lyons ([17, 18]) and Lyons, Pemantle and Peres ([21, 22]), making good use of the electric networks theory.

More recent works on *transient* λ-biased random walks on (Galton–Watson) trees show that the effective conductance of the tree is key to understanding the asymptotic behavior of the walk (see [3] for the speed and [15, 27] for the dimension of the harmonic measure).

In this paper, we deal with a *null-recurrent* model of random walk on a Galton-Watson random weighted tree in a regime called "subdiffusive" (see below for definitions). The effective conductance of the whole tree is zero by recurrence of the walk and we are interested in the rate of decay of the conductance between the root of the tree and the vertices at height n, as n goes to infinity. This gives the order of magnitude of the probability that the walk hits level n before returning to the root (see below for details).

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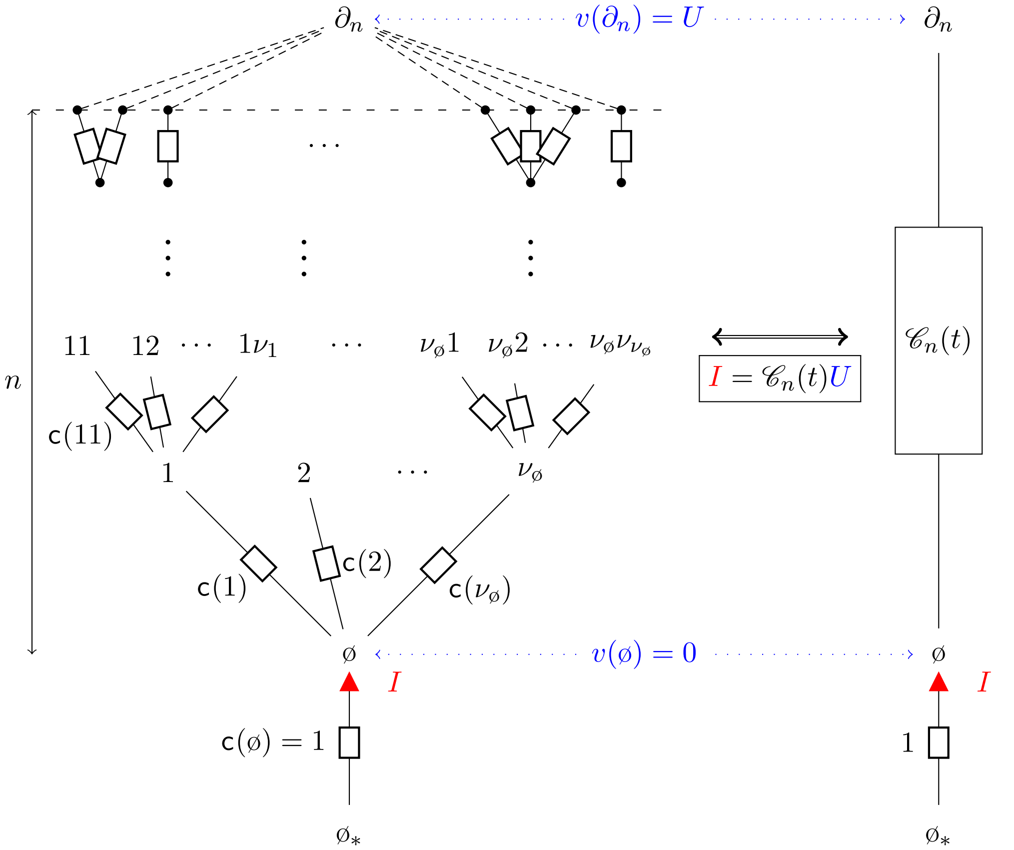

Figure 1: On the left, a rooted tree of height n with an artificial parent of the root and equipped with conductances. The potential is U at height n and 0 at the root. On the right, the equivalent reduced electrical network.

1.1 Conductance of a tree

We first briefly recall some notions of electric networks in the case where the network is a locally finite tree t, rooted at some vertex $\varnothing$. For more detailed and general statements about this theory, see [9] or [23]. For any vertex x of t, associate to the edge between x and its parent x_* a *conductance* $\mathsf{c}(x) \in (0, \infty)$, or alternatively a *resistance* $\mathsf{r}(x)$ equal to the inverse of the conductance. For convenience, we add an artificial parent of the root, denoted by $\varnothing_*$ and let $\mathsf{c}(\varnothing) = 1$ (this is to make the root "less special").

Now we fix some positive integer n and assume that the height of t is at least n. We impose a certain fixed electric potential U at the vertices at height n in t, while the potential at the root is 0. As another point of view, we may connect the vertices at height n to a new vertex ∂_n, the new edges having infinite conductance (zero resistance), and impose $v(\partial_n) = U$. This defines an electric potential v on the vertices of t between $\varnothing$ and the n-th level of t. To be more formal, we need some notations. For a vertex x

of t, let us denote by $|x|$ its height, by ν_x its number of children, by $x1$, $x2$, $\dots$, $x\nu_x$ its children and by $\pi(x)$ the sum of the conductances of the edges that are incident to x. The potential v defined on the n first levels of the tree satisfies:

$$v(x) = \begin{cases} 0 & \text{if } x = \varnothing; \\ U & \text{if } |x| = n; \\ \frac{1}{\pi^t(x)}\left(\mathsf{c}(x)v(x_*) + \sum_{j=1}^{\nu_x} \mathsf{c}(xi)v(xj)\right) & \text{if } 1 \le |x| \le n-1. \end{cases}$$

The last case in the previous equality is called *harmonicity* of v at x. Such a potential is well-known to exist and to be unique. For x in t, the electric current $i(x)$ flowing in the edge between x and its parent x_* is defined by Ohm's law as

$$i(x) = \mathsf{c}(x)(v(x) - v(x_*)).$$

(The harmonicity condition is the same as Kirchhoff's current law.) Now let

$$I = \sum_{j=1}^{\nu_\varnothing} i(j)$$

be the total current entering the tree. It is clear that the function $U \mapsto I$ is linear. The constant ratio I/U is called the *effective conductance* of t between $\varnothing$ and its n-th level and is denoted by $\mathscr{C}_n(t)$. See Figure 1 for a summary of this discussion.

The effective conductance has a pleasant and useful interpretation in terms of random walks on the vertices of t. We associate to the conductances of the edges a probability kernel P on t in the following way:

$$\mathrm{P}(x, xi) = \frac{\mathsf{c}(xi)}{\pi(x)} \quad \text{if } 1 \le i \le \nu_x \quad \text{and} \quad \mathrm{P}(x, x_*) = \frac{\mathsf{c}(x)}{\pi(x)}.$$

For x in t, we write P_x for a probability measure under which the random sequence $(X_k)_{k\ge 0}$ is a random walk starting from x with probability kernel P and consider the stopping times

$$\tau_x = \inf\{s \ge 0 : X_s = x\}, \quad \tau_x^+ = \inf\{s \ge 1 : X_s = x\} \quad \text{and} \quad \tau^{(n)} = \inf\{s \ge 0 : |X_s| = n\}.$$

Then, by the Markov property, the function

$$v(x) = \mathrm{P}_x(\tau^{(n)} < \tau_\varnothing)$$

is the electric potential when the vertices at height n have potential 1 and the root has potential 0. As a consequence, by definition of the current i and, again, the Markov property,

$$\mathscr{C}_n(t) = I = \sum_{j=1}^{\nu_\varnothing} \mathsf{c}(j)\mathrm{P}_j(\tau^{(n)} < \tau_\varnothing) = \pi(\varnothing)\mathrm{P}_\varnothing(\tau^{(n)} < \tau_\varnothing^+).$$

The conductance $\mathscr{C}(t)$ of the whole tree, equal to the limit of the non-increasing sequence $(\mathscr{C}_n(t))_{n\ge 0}$, is then positive if and only if the associated random walk is transient.

A typical choice for the conductances is $\mathsf{c}(x) = \lambda^{-|x|}$, for some fixed $\lambda > 0$. It corresponds to the λ-biased random walk on the vertices of t, introduced in [17]. In words, the walker jumps with *weight* λ to the parent of its current position, and with weight 1 to one of its children. If t is the tree in which every vertex has $d \geq 2$ children, this random walk is transient if and only if $\lambda < d$, and the special recurrent case $\lambda = d$ may be seen as critical.

In [1], this infinite d-ary tree is considered with the set of conductances $\mathsf{c}(x) = d^{-|x|}X_x$, where the positive random variables $(X_x)_{x\in t}$ are i.i.d, which corresponds in some way to a (still recurrent) perturbation around this critical regime. The authors prove that the expectation $\mathbf{E}[\mathscr{C}_n(t)]$ is of order $1/n$ as n goes to infinity[1]. This result has been recently extended in [7] to the case of an infinite, random *Galton–Watson* trees.

In this work, we investigate the rate of decay of the sequence of random variables $(\mathscr{C}_n)$ in another "critical" setting known as the *subdiffusive* ([13]) regime for Galton-Watson trees with random weights.

1.2 Subdiffusive random weighted trees

What we call an (edge-)weighted tree is a (rooted, planar) tree t together with a *weight function* $\mathsf{A}^t : t \setminus \{\varnothing\} \to (0, \infty)$. For a vertex $x \neq \varnothing$ in t, $\mathsf{A}^t(x)$ represents the weight of the edge connecting x to its parent.

We naturally associate to this weighted tree the following probability kernel: for x in t,

$$\mathrm{P}^t(x, xi) = \frac{\mathsf{A}^t(xi)}{1 + \sum_{j=1}^{\nu_x} \mathsf{A}^t(xj)} \quad \text{if } 1 \leq i \leq \nu_x \quad \text{and} \quad \mathrm{P}^t(x, x_*) = \frac{1}{1 + \sum_{j=1}^{\nu_x} \mathsf{A}^t(xj)},$$

that is, if a random walker is at vertex x, it may jump to the i-th child of x with weight $\mathsf{A}^t(xi)$ and to the parent of x with weight 1 (if the weights are all constant equal to λ^{-1}, we recover the λ-biased random walk on t). Recall that we also add a vertex $\varnothing_*$ to serve as an artificial parent to the root (and the walk is reflected at $\varnothing_*$).

This random walk is easily seen to correspond to the conductance c^t defined by

$$\forall x \in t, \quad \mathsf{c}^t(x) = \prod_{\varnothing \prec y \preceq x} \mathsf{A}^t(y),$$

where the product above is indexed by the ancestors of x (including x) which are distinct from $\varnothing$.

To define a *Galton-Watson tree with random weights* consider a random sequence of positive real numbers

$$\mathbf{A} = (A(1), \dots, A(\nu)),$$

whose length $\nu \in \{0, 1, 2, \dots\}$ may also be random. Define the free monoid

$$\mathcal{U} = \bigsqcup_{k \geq 0} \mathbb{N}^k$$

[1] Actually their result is much more precise, but this suffices for the purpose of this introduction.

of all the finite words on the alphabet $\mathbb{N} = \{1, 2, \dots\}$, with the convention that $\mathbb{N}^0$ contains only the empty word $\varnothing$. Now consider the family $(\mathbf{A}_x)_{x \in \mathcal{U}}$ of i.i.d. random sequences indexed by $\mathcal{U}$, whose common distribution is the law of $\mathbf{A}$. We build a random weighted tree T in the following way: the root $\varnothing$ of T has $\nu_\varnothing$ children labelled 1, 2, ..., $\nu_\varnothing$, where $\nu_\varnothing$ is the length of the random sequence

$$\mathbf{A}_\varnothing = (A_\varnothing(1), \dots, A_\varnothing(\nu_\varnothing))$$

and, for $1 \leq i \leq \nu_\varnothing$, the weight $\mathsf{A}^T(i)$ of the edge between $\varnothing$ and its child i is $A_\varnothing(i)$. Then, proceed in the same way for the children 1, 2, ..., $\nu_\varnothing$ of the root, in order to pick their numbers of children ν_1, ν_2, ..., $\nu_{\nu_\varnothing}$ and the weights of the corresponding edges :

$$\mathsf{A}^T(11) = A_1(1), \dots, \mathsf{A}^T(1\nu_1) = A_1(\nu_1), \mathsf{A}^T(21) = A_2(1), \dots, \mathsf{A}^T(2\nu_2) = A_2(\nu_2), \dots,$$

and so on, so that the weight of the edge between a vertex xi in T and its parent x is $\mathsf{A}^T(xi) = A_x(i)$. Notice that T, without its weights, is a Galton-Watson tree whose reproduction law is the distribution of ν. For this reason we denote by $\mathbf{GW}$ the distribution of T, seen as a random variable in the space of weighted trees.

This very rich family of random walk in a random environment was introduced in [20] and generalized in [10]. The random walk of probability kernel P^t may be transient or recurrent, for $\mathbf{GW}$-almost every weighted tree t, depending on whether the convex characteristic function

$$\psi(s) = \log \mathbf{E} \sum_{i=1}^{\nu} A(i)^s, \qquad \forall s \in \mathbb{R},$$

stays positive on the interval $[0, 1]$.[2] Since [20], this model has attracted a lot of attention. For the transient case, see for instance [2] or [26]. The recurrent case in general is studied in [11] or [6], among many others. The recent article [5] focusses on the slow null-recurrent regime.

With a slight abuse of terminology (see below), we call "subdiffusive" this model (and by extension the random tree we work on) when the following hypotheses are satisfied:

$$\psi(1) = \log \mathbf{E} \sum_{i=1}^{\nu} A(i) = 0 \quad \text{and} \tag{H_{norm}}$$

$$\psi'(1) := \mathbf{E}\left[\sum_{i=1}^{\nu} A(i) \log A(i)\right] \in [-\infty, 0). \tag{$H_{\text{derivative}}$}$$

To state our main result about the conductance in this case, we need to introduce the additive martingale, also called Mandelbrot's multiplicative cascade, or Biggins' martingale, $(M_n(T))_{n \geq 0}$, defined by

$$M_n(T) = \sum_{|x|=n} \prod_{\varnothing \prec y \preceq x} \mathsf{A}^T(y) = \sum_{|x|=n} \mathsf{c}^T(x).$$

[2]For a necessary and sufficient condition, additional integrability conditions are needed, see [10].

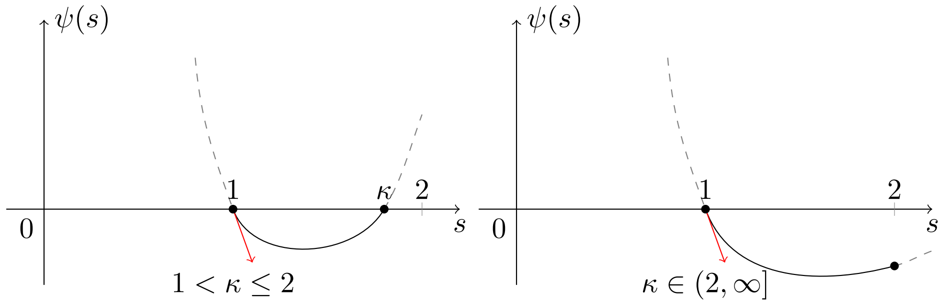

Figure 2: Schematic behavior of ψ under our hypotheses

It is easily seen to be a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$ defined by

$$\mathcal{F}_n = \sigma\{\mathbf{A}_x : |x| \leq n-1\}.$$

By Biggins' theorem (see also [14, 19]) it converges almost surely and in L^1 to a random variable $M_\infty(T)$ which is positive on the event of non-extinction, provided we also assume the following integrability hypothesis:

$$\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu} A(i)\Big)\log^+\Big(\sum_{i=1}^{\nu} A(i)\Big)\Big] < \infty. \tag{$H_{X\log X}$}$$

The non-degeneracy of $M_\infty(T)$ also allows to prove that, under our assumptions, the random walk on T is almost surely *null-recurrent* (we provide a short proof of this well-known fact in the appendix for completeness).

Now, we may expect results similar to [1, 7]. This is not exactly true: there will be different behaviors depending on the value of

$$\kappa = \inf\{s > 1 : \psi(s) = 0\} \in (1, \infty].$$

Our work uses the tail probabilities and some moments of the random variable $M_\infty(T)$, which depend on the value of κ.

For two positive functions f and g defined on a neighborhood of $+\infty$, we write $f(x) \asymp g(x)$ as x goes to infinity, when, for some constants c and C in $(0, \infty)$, for x large enough,

$$cg(x) \leq f(x) \leq Cg(x).$$

Under the following integrability hypothesis:

$$\begin{aligned} &\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu} A(i)\Big)^{\kappa}\Big] + \mathbf{E}\Big[\sum_{i=1}^{\nu} A(i)^{\kappa}\log^+ A(i)\Big] < \infty, \quad \text{if } 1 < \kappa \leq 2, \\ &\mathbf{E}\Big[\Big(\sum_{i=1}^{\nu} A(i)\Big)^{2}\Big] < \infty, \quad \text{if } \kappa \in (2, \infty], \end{aligned} \tag{H_κ}$$

we owe to [16, Theorem 2.1, Theorem 2.2] the following fact:

Fact 1. *If* (H_{norm}), ($H_{\text{derivative}}$) *and* (H_κ) *are satisfied, then the random variable* M_∞ *has finite moments of order* p *for all* p *in* $[1, \kappa)$ *if* $\kappa \le 2$ *and for all* p *in* $[1, 2]$ *if* $\kappa > 2$.

If $\kappa \le 2$*, the asymptotic tail probability of* M_∞ *satisfies*

$$\mathbf{P}(M_\infty > s) \asymp_{s\to\infty} s^{-\kappa}. \tag{1}$$

In the previous statement, as in the rest of this work, we feel free to omit T as an argument or as a superscript and write $\mathscr{C}_n$ for $\mathscr{C}_n(T)$, A for A^T, ν for ν^T, ..., when there is no risk of confusion.

Throughout this work, we will assume that (H_{norm}), ($H_{\text{derivative}}$) and (H_κ) (which supersedes ($H_{X \log X}$)) hold. These assumptions are summed up in Figure 2.

One of the most striking result about this regime is given (under some additional assumptions) in [13]: for **GW**-almost every infinite t, P^t-almost surely,

$$\lim_{n\to\infty} \frac{\log \max_{0\le i\le n} |X_i|}{\log n} = 1 - \frac{1}{\kappa \wedge 2},$$

hence the name "subdiffusive" in the case $\kappa < 2$, that we improperly (but conveniently) extend to this whole "fast, null-recurrent" case. A central limit theorem may be found in [10]. More recently, Aïdékon and de Raphélis ([4, 8]) have proved the joint convergence of the renormalized height of the walk together with its trace towards a continuous-time process and the real forest coded by this process.

Regarding the conductance, our main result is the following:

Theorem 2. *Under the hypotheses* (H_{norm}), ($H_{\text{derivative}}$) *and* (H_κ)*, as* n *goes to infinity,*

$$\mathbf{E}[\mathscr{C}_n] \asymp \frac{1}{n^{1/(\kappa-1)}} \quad \textit{if } 1 < \kappa < 2 \ ;$$

$$\mathbf{E}[\mathscr{C}_n] \asymp \frac{1}{n \log n} \quad \textit{if } \kappa = 2 \ \textit{and}$$

$$\mathbf{E}[\mathscr{C}_n] \sim \frac{1}{n\mathbf{E}[M_\infty^2]} = \frac{1 - \mathbf{E}\left[\sum_{i=1}^{\nu} A(i)^2\right]}{n\mathbf{E}\left[\sum_{1\le i\ne j\le \nu} A(i)A(j)\right]} \quad \textit{if } \kappa > 2,$$

and, in any case, almost surely,

$$\lim_{n\to\infty} \mathscr{C}_n / \mathbf{E}[\mathscr{C}_n] = M_\infty.$$

Moreover, the above convergence also holds in L^p *for* $p \in [1, \kappa)$ *if* $1 < \kappa \le 2$ *and in* L^2 *if* $\kappa > 2$.

Our method is almost entirely analytic and inspired by [12].

Remark 1. One could expect that in the "non-lattice case" (see [16, p.270]) we may also obtain asymptotic equivalences of $\mathbf{E}[\mathscr{C}_n]$, for $\kappa \in (1, 2]$. Unfortunately, our method was not powerful enough to obtain a more precise result: our lower bound of the tail probabilities of $\mathscr{C}_n/\mathbf{E}[\mathscr{C}_n]$ (see Lemma 13) is not sharp enough.

Acknowledgements: This work was part of the author's PhD thesis. The author would like to warmly thank Yueyun Hu, one of his supervisors, for introducing him to this problem and for constant guidance.

2 Algebraic identities and lower bound

In order to use the branching property, we introduce for any weighted tree t, for any vertex x of t, the reindexed subtree of t starting from x:

$$t[x] = \{y \in \mathcal{U} : xy \in t\} \quad \text{with weights given by} \quad \mathsf{A}^{t[x]}(y) = \mathsf{A}^t(xy), \quad \forall y \in t[x].$$

We denote, for any weighted tree t and $n \geq 0$,

$$\beta_n(t) = \mathrm{P}_{\varnothing}^t(\tau^{(n)} < \tau_{\varnothing_*}),$$

which is the conductance between $\varnothing_*$ and the n-th level of t.

Now, by the Markov property (or by the law of conductances in parallel), for any $n \geq 1$,

$$\mathscr{C}_n(t) = \sum_{i=1}^{\nu_\varnothing^t} \mathsf{A}^t(i)\beta_{n-1}(t), \tag{2}$$

while by the law of resistances in series,

$$\beta_n(t) = \frac{\mathscr{C}_n(t)}{1 + \mathscr{C}_n(t)}. \tag{3}$$

Combining these identities gives, for all $n \geq 2$,

$$\mathscr{C}_n(t) = \sum_{i=1}^{\nu_\varnothing^t} \mathsf{A}^t(i) \frac{\mathscr{C}_{n-1}(t[i])}{1 + \mathscr{C}_{n-1}(t[i])}. \tag{4}$$

By the branching property, and the hypothesis (H_{norm}) we already obtain the recursive equation:

$$\mathbf{E}[\mathscr{C}_n] = \mathbf{E}\Big[\sum_{i=1}^{\nu} A(i)\Big]\mathbf{E}\Big[\frac{\mathscr{C}_{n-1}}{1 + \mathscr{C}_{n-1}}\Big] = \mathbf{E}\Big[\frac{\mathscr{C}_{n-1}}{1 + \mathscr{C}_{n-1}}\Big]. \tag{5}$$

From now on, we let, for $n \geq 1$,

$$u_n = \mathbf{E}[\mathscr{C}_n] \quad \text{and} \quad a_n = 1/u_n.$$

Moreover, for any random variable ξ such that $\mathbf{E}[\xi]$ exists in $(0, \infty)$, we define the renormalized random variable $\langle \xi \rangle$ by $\langle \xi \rangle = \xi / \mathbf{E}[\xi]$.

Going back to (5), we obtain

$$u_n = \mathbf{E}\Big[\frac{\mathscr{C}_{n-1}}{1 + \mathscr{C}_{n-1}}\Big] = u_{n-1} - \mathbf{E}\Big[\frac{\mathscr{C}_{n-1}^2}{1 + \mathscr{C}_{n-1}}\Big] = u_{n-1} - u_{n-1}\mathbf{E}\Big[\frac{\langle \mathscr{C}_{n-1} \rangle^2}{a_{n-1} + \langle \mathscr{C}_{n-1} \rangle}\Big].$$

Introducing, for $a > 0$ the (convex) function $\phi_a : x \mapsto x^2/(a + x)$, the previous equality becomes

$$u_{n-1} - u_n = u_{n-1}\mathbf{E}[\phi_{a_{n-1}} \langle \mathscr{C}_{n-1} \rangle], \tag{6}$$

which is key in the rest of this work.

The rough idea here, is that we expect $\langle \mathscr{C}_n \rangle$ to be "close" to M_∞ so that, as n is large,

$$\mathbf{E}[\phi_{a_{n-1}} \langle \mathscr{C}_{n-1} \rangle] \approx \mathbf{E}[\phi_{a_{n-1}}(M_\infty)].$$

Indeed, at least one of the inequalities is correct in this heuristics:

Lemma 3. *Let $\phi : \mathbb{R}_+ \to (0, \infty)$ be* any *convex, continuously differentiable function, regularly varying at infinity. Then, for any $n \geq 1$,*

$$\mathbf{E}[\phi\langle\mathscr{C}_n\rangle] \leq \mathbf{E}[\phi(M_n)].$$

Moreover, whenever for all $x \geq 0$, $\phi(x) \leq Cx^p$, for some constant $C > 0$, for $p \in (1, \kappa)$ if $\kappa \leq 2$, and for $p = 2$ if $\kappa > 2$, one has

$$\mathbf{E}[\phi\langle\mathscr{C}_n\rangle] \leq \mathbf{E}[\phi(M_n)] \leq \mathbf{E}[\phi(M_\infty)].$$

The proof of this lemma uses the following fact:

Fact 4. *Let ξ be a non-negative random variable such that $\mathbb{E}[\xi]$ is in $(0, \infty)$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable, regularly varying at infinity, convex function. Then,*

$$\mathbb{E}\Big[\phi\Big\langle\frac{\xi}{1+\xi}\Big\rangle\Big] \leq \mathbb{E}[\phi\langle\xi\rangle]. \tag{7}$$

This fact itself is already stated in [12, Proof of Lemma 3.1] and is mostly a consequence of [13, Formula 3.3]. However, since the statement of this formula is rather long, we give a more direct but less general proof in the appendix (as will be clear in the proof, the assumption that ϕ is regularly varying at infinity can be weakened).

Proof of Lemma 3. We prove by induction that, for any $n \geq 1$, for any ϕ as in the first statement of the lemma,

$$\mathbf{E}[\phi\langle\mathscr{C}_n\rangle] \leq \mathbf{E}[\phi(M_n)].$$

Notice that $\mathscr{C}_1(T) = M_1(T)$ and for $n \geq 1$, by (5) and (4), observe that

$$\langle\mathscr{C}_n(T)\rangle = \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i)\Big\langle\frac{\mathscr{C}_n(T[i])}{1+\mathscr{C}_n(T[i])}\Big\rangle$$

By independence, it suffices to show that, for any $k \geq 0$, for any $(a_1, a_2, ..., a_k) \in (0, \infty)^k$,

$$\mathbf{E}\Big[\phi\Big(\sum_{i=1}^k a_i\Big\langle\frac{\mathscr{C}_n(T[i])}{1+\mathscr{C}_n(T[i])}\Big\rangle\Big)\Big] \leq \mathbf{E}\Big[\phi\Big(\sum_{i=1}^k a_i M_n(T[i])\Big)\Big].$$

Assume the result is true for $k \geq 0$. Then,

$$\begin{aligned}&\mathbf{E}\Big[\phi\Big(\sum_{i=1}^k a_i\Big\langle\frac{\mathscr{C}_n(T[i])}{1+\mathscr{C}_n(T[i])}\Big\rangle + a_{k+1}\Big\langle\frac{\mathscr{C}_n(T[k+1])}{1+\mathscr{C}_n(T[k+1])}\Big\rangle\Big)\Big]\\ &= \mathbf{E}\Big[\phi\Big(B + a_{k+1}\Big\langle\frac{\mathscr{C}_n(T[k+1])}{1+\mathscr{C}_n(T[k+1])}\Big\rangle\Big)\Big],\end{aligned}$$

where B is the sum in the first expectation and is independent of the last term. Reasoning conditionally with respect to B, we may use the previous fact with the function $x \mapsto \phi(a_{k+1}x + B)$ to obtain that this expectation is bounded from above by

$$\mathbf{E}[\phi(B + a_{k+1}\langle\mathscr{C}_n(T[k+1])\rangle)] \leq \mathbf{E}[\phi(B + a_{k+1}M_n(T[k+1]))],$$

where for the last inequality, we used the induction hypothesis on n. Now reason conditionnally on $M_n(T[k+1])$ to use the induction hypothesis on k.

For the last assertion, it suffices to see that (M_n) is bounded in L^p if $1 < p < 2$ for $\kappa \le 2$ and bounded in L^2 for $\kappa > 2$, which is certainly well-known. For a quick proof, let $n \ge 1$ and p as above ($p = 2$ if $\kappa > 2$). Reason conditionally with respect to $\mathcal{F}_1$ and use an inequality due to Neveu (stated later in this paper as (17)):

$$\mathbf{E}[M_{n+1}^p \,|\, \mathcal{F}_1] \le \sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i)^p \mathbf{E}[M_n^p] + \Big(\sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i)\mathbf{E}[M_n]\Big)^p.$$

As a consequence,

$$\mathbf{E}[M_{n+1}^p] \le e^{\psi(p)}\mathbf{E}[M_n^p] + \mathbf{E}\Big[\Big(\sum_{i=1}^{\nu} A(i)\Big)^p\Big]$$

Since $\psi(p) < 0$ and $\mathbf{E}\big[(\sum_{i=1}^{\nu} A(i))^p\big] < \infty$ by assumption, this is easily seen to imply that $\sup_{n\ge1} \mathbf{E}[M_n^p] < \infty$. □

Now we see that we need to study the asymptotics of $a \mapsto \mathbf{E}[\phi_a(M_\infty)]$ as a goes to infinity. For later use, consider in general, for $p > 0$ and $a > 0$,

$$\varphi_p(a) = \mathbf{E}\Big[\Big(\frac{M_\infty^2}{a + M_\infty}\Big)^p\Big]. \tag{8}$$

Using the tail probability estimate in Fact 1, for $1 < \kappa \le 2$, or the integrability of M_∞^2 for $\kappa > 2$, one obtains:

Lemma 5. *As a goes to infinity,*

$$\varphi_p(a) \begin{cases} \asymp a^{p-\kappa} & \text{if } \kappa/2 < p < \kappa \le 2; \\ \asymp a^{p-\kappa}\log a & \text{if } p = \kappa/2; \\ \le C a^{p-2} & \text{for some constant } C > 0, \text{ if } \kappa > 2 \text{ and } 1 < p < 2; \\ \sim \mathbf{E}[M_\infty^2]/a & \text{if } \kappa > 2 \text{ and } p = 2. \end{cases} \tag{9}$$

Proof. Write $\mathbf{P}_{M_\infty}$ for the distribution of M_∞. Differentiate the function $x \mapsto \big(\frac{x^2}{a+x}\big)^p$, to obtain

$$\varphi_p(a) = \int_0^\infty \Big(\int_0^s p\frac{x^2 + 2ax}{(a+x)^2}\Big(\frac{x^2}{a+x}\Big)^{p-1} \mathrm{d}x\Big)\mathbf{P}_{M_\infty}(\mathrm{d}s).$$

Using Tonelli's theorem together with the change of variable $y = x/a$ yields

$$\varphi_p(a) = pa^{p-\kappa}\int_0^\infty \frac{y^{2p-\kappa-1}(y+2)}{(1+y)^{p+1}}(ay)^\kappa \mathbf{P}(M_\infty > ay)\,\mathrm{d}y. \tag{10}$$

Let $f(y)$ be the integrand in the last equation.

Now assume that $1 < \kappa \le 2$ and write $\underline{\ell}$ (respectively $\overline{\ell}$) for the inferior (respectively superior) limit of $s^\kappa \mathbf{P}(M_\infty > s)$, as s goes to infinity. Consider $\varepsilon > 0$ so small that $\underline{\ell} - \varepsilon > 0$. Let $N > 0$ be so large that

$$\forall s \ge N, \qquad s^\kappa \mathbf{P}(M_\infty > s) \in (\underline{\ell} - \varepsilon, \overline{\ell} + \varepsilon).$$

Assume that $a > N$. On the interval $(0, N/a)$, dominating $\mathbf{P}(M_\infty > ay)$ by 1 yields

$$f(y) \le a^\kappa y^{2p-1} \max_{0\le y\le N/a} \frac{2+y}{(1+y)^{p+1}},$$

so that in any case,

$$pa^{p-\kappa} \int_0^{N/a} f(y)\,\mathrm{d}y \le \Big[pN^{2p} \max_{0\le y\le 1} \frac{2+y}{(1+y)^{p+1}}\Big] a^{-p},$$

which will be negligible. On the other hand, if y is in the interval $[N/a, \infty)$, then

$$f(y) \le (\bar{\ell} + \varepsilon) \frac{y^{2p-\kappa-1}(y+2)}{(1+y)^{p+1}} \tag{11}$$

and similarly for the lower bound. Those bounds are integrable on $(0, \infty)$ if $p > \kappa/2$ and in this case, we may conclude by applying the monotone convergence theorem.

Now assume that $p = \kappa/2$. The bound above is still integrable at the neighborhood of ∞, but not at the neighborhood of 0. As a consequence, the main contribution in the integral comes from the term

$$\int_{N/a}^1 f(y)\,\mathrm{d}y \le (\bar{\ell} + \varepsilon) \int_{N/a}^1 y^{-1} \frac{2+y}{(1+y)^{p+1}}\,\mathrm{d}y \asymp_{a\to\infty} \log(a),$$

and the same is true for the lower bound.

Finally, assume that $\kappa > 2$ and recall that in this case, by our hypotheses, $\mathbf{E}[M_\infty^2]$ is finite, thus by Markov's inequality, for all $r > 0$, $\mathbf{P}(M_\infty > r) \le \mathbf{E}[M_\infty^2]/r^2$. Now, if $1 < p < 2$, the rest of the computations is exactly the same as in the first point, whereas if $p = 1$, by dominated convergence,

$$a\varphi_1(a) = \mathbf{E}\left[\frac{M_\infty^2}{1 + M_\infty/a}\right] \xrightarrow[a\to\infty]{} \mathbf{E}\Big[M_\infty^2\Big]. \qquad \square$$

Going back to (6) and using the two previous lemmas, we see that, for some constant C in $(0, \infty)$ and any $n \ge 2$,

$$u_{n-1} - u_n \le \begin{cases} Cu_{n-1}a_{n-1}^{1-\kappa} = Cu_{n-1}^\kappa & \text{if } 1 < \kappa < 2; \\ Cu_{n-1}a_{n-1}^{-1}\log a_{n-1} = Cu_{n-1}^2 \log(1/u_{n-1}) & \text{if } \kappa = 2; \\ \mathbf{E}[M_\infty^2]u_{n-1}^2 & \text{if } \kappa > 2. \end{cases} \tag{12}$$

To obtain our lower bound, it suffices to use one part of the following elementary analysis lemma:

Lemma 6. *Let (u_n) be a non-increasing sequence of positive real numbers going to 0 as n goes to infinity. Let $\alpha > 1$ and $C \in (0, \infty)$.*

1. *If for n large enough, $u_n - u_{n+1} \le Cu_n^\alpha$, then $\liminf n^{\frac{1}{\alpha-1}} u_n \ge [C(\alpha-1)]^{-\frac{1}{\alpha-1}}$.*

2. *If for n large enough, $u_n - u_{n+1} \geq C u_n^\alpha$, then* $\limsup n^{\frac{1}{\alpha-1}} u_n \leq [C(\alpha-1)]^{-\frac{1}{\alpha-1}}$.

3. *If for n large enough, $u_n - u_{n+1} \leq C u_n^2 \log(1/u_n)$, then* $\liminf u_n n \log n \geq C^{-1}$.

4. *If for n large enough, $u_n - u_{n+1} \geq C u_n^2 \log(1/u_n)$, then* $\limsup u_n n \log n \leq C^{-1}$.

Proof. To prove the first two assertions of the lemma, consider, for $x > 0$, $f(x) = x^{1-\alpha}/(\alpha-1)$. By the mean value theorem,

$$f(u_{n+1}) - f(u_n) = f'(\xi_n)(u_{n+1} - u_n) = \xi_n^{-\alpha}(u_n - u_{n+1}),$$

for some $\xi_n \in (u_{n+1}, u_n)$. The first case implies that $u_n \sim u_{n+1}$, therefore

$$f(u_{n+1}) - f(u_n) \leq C u_n^\alpha \xi_n^{-\alpha} \longrightarrow C,$$

and we may conclude by averaging this inequality.

In the second case, since $\xi_n \leq u_n$,

$$f(u_{n+1}) - f(u_n) \geq C u_n^\alpha \xi_n^{-\alpha} \geq C.$$

The method for the last two assertions is the same, except that we use the function g defined by

$$g(x) = \frac{1/x}{\log(1/x)},$$

whose derivative is

$$g'(x) = -\frac{1}{x^2 \log(1/x)}\Big(1 - \frac{1}{\log(1/x)}\Big).$$ □

Using the first and third points of this lemma with (12) finally yields the following lower bounds:

Proposition 7. *Under the hypotheses (H_{norm}), ($H_{\mathrm{derivative}}$) and (H_κ),*

1. $\liminf_{n\to\infty} n^{1/(\kappa-1)} \mathbf{E}[\mathscr{C}_n] > 0$, *if* $1 < \kappa < 2$;

2. $\liminf_{n\to\infty} n \log n \mathbf{E}[\mathscr{C}_n] > 0$, *if* $\kappa = 2$ *and*

3. $\liminf_{n\to\infty} n \mathbf{E}[\mathscr{C}_n] \geq 1/\mathbf{E}[M_\infty^2]$, *if* $\kappa > 2$.

Remark 2. If we assume that we are in the non-lattice case these lower bounds can also be made explicit (in terms on the distribution of M_∞) in the cases $1 < \kappa < 2$ and $\kappa = 2$. However, since our method does not provide explicit upper bounds, we chose not to make this additional assumption.

3 Upper bound and almost sure convergence

We start with an easy *a priori* upper bound.

Lemma 8. *In any case, one has*

$$\limsup_{n\to\infty} n u_n \le 1. \tag{13}$$

Proof. Let $n \ge 2$. We go back to (5) but this time, we write

$$u_{n-1} - u_n = \mathbf{E}\Big[\frac{\mathscr{C}_n^2}{1-\mathscr{C}_n}\Big] \ge \mathbf{E}\Big[\Big(\frac{\mathscr{C}_n}{1-\mathscr{C}_n}\Big)^2\Big] \ge \Big(\mathbf{E}\Big[\frac{\mathscr{C}_n}{1-\mathscr{C}_n}\Big]\Big)^2 = u_n^2.$$

This implies that

$$\frac{1}{u_n} - \frac{1}{u_{n-1}} \ge \frac{u_n}{u_{n-1}} = 1 - \mathbf{E}[\phi_{a_{n-1}}\langle \mathscr{C}_{n-1}\rangle],$$

by the identity (6). Using Lemma 3 yields

$$\frac{1}{u_n} - \frac{1}{u_{n-1}} \ge 1 - \varphi_1(a_{n-1}).$$

Since this lower bound goes to 1 as n goes to infinity, averaging the previous inequality and using Cesàro's lemma yields

$$\liminf_{n\to\infty} \frac{1}{n u_n} \ge 1,$$

hence the result. $\square$

To obtain more refined bounds, we first iterate (4). Using repeatedly the identity,

$$\frac{x}{1+x} = x - \frac{x^2}{1+x} = x - \phi_1(x), \quad \forall x > 0,$$

we obtain that, for any $n > k$,

$$\begin{aligned}\mathscr{C}_n(T) &= \sum_{|x|=1} \mathsf{c}(x)\mathscr{C}_{n-1}(T[x]) - \sum_{|x|=1} \mathsf{c}(x)\phi_1(\mathscr{C}_{n-1}(T[x])),\\ &= \sum_{|x|=k} \mathsf{c}(x)\mathscr{C}_{n-k}(T[x]) - \sum_{|x|\le k} \mathsf{c}(x)\phi_1(\mathscr{C}_{n-|x|}(T[x])).\end{aligned}$$

Dividing by $u_n = \mathbf{E}[\mathscr{C}_n]$, and using the equality

$$\phi_1(\mathscr{C}_{n-|x|}(T[x])) = \frac{u_{n-|x|}^2\langle \mathscr{C}_{n-|x|}(T[x])\rangle}{u_{n-|x|}(a_{n-|x|} + \langle \mathscr{C}_{n-|x|}(T[x])\rangle)} = \phi_{a_{n-|x|}}\langle \mathscr{C}_{n-|x|}(T[x])\rangle,$$

we finally obtain

$$\langle \mathscr{C}_n(T)\rangle = \frac{u_{n-k}}{u_n}\sum_{|x|=k} \mathsf{c}(x)\langle \mathscr{C}_{n-k}(T[x])\rangle - \frac{1}{u_n}\sum_{|x|\le k} \mathsf{c}(x) u_{n-|x|}\phi_{a_{n-|x|}}\langle \mathscr{C}_{n-|x|}(T[x])\rangle. \tag{14}$$

On the other hand, by definition of M_n,

$$M_n(T) = \sum_{|x|=k} \mathsf{c}(x) M_{n-k}(T[x]), \tag{15}$$

therefore, for any $n > k$,

$$\langle \mathscr{C}_n(T) \rangle = \frac{a_n}{a_{n-k}} M_\infty(T) + \frac{a_n}{a_{n-k}} X_{k,n}(T) - \sum_{j=1}^{k} \frac{a_n}{a_{n-j}} Y_{j,n}(T), \tag{16}$$

where

$$X_{k,n}(T) = \sum_{|x|=k} \mathsf{c}(x) \left(\langle \mathscr{C}_{n-k}(T[x]) \rangle - M_\infty(T[x]) \right),$$

and for $j < n$,

$$Y_{j,n}(T) = \sum_{|x|=j} \mathsf{c}(x) \phi_{a_{n-j}} \langle \mathscr{C}_{n-j}(T[x]) \rangle.$$

Our next step is, for $p \in (1, \kappa \wedge 2)$, to estimate the L^p norms of $X_{k,n}$ and $Y_{j,n}$ in order to prove the convergence in L^p of $\langle \mathscr{C}_n \rangle$ towards M_∞. remark that, by the branching property, conditionally on $\mathcal{F}_k$, $X_{k,n}$ is a sum of independent, *centered* variables while conditionally on $\mathcal{F}_j$, $Y_{j,n}$ is a sum of independent, *non-negative* random variables. We may therefore use the following upper bounds:

Fact 9. *Let p be a real number in $[1, 2]$ and assume that $\xi_1, \ldots, \xi_k$ are independent real-valued random variables such that for all $1 \le i \le k$, $\mathbb{E}[|\xi_i|^p] < \infty$.*

1. *If $\xi_1, \ldots, \xi_k$ are non-negative, then*

$$\mathbb{E}[(\xi_1 + \cdots + \xi_k)^p] \le \sum_{i=1}^{k} \mathbb{E}[\xi_i^p] + \left(\sum_{i=1}^{k} \mathbb{E}\xi_i \right)^p. \tag{17}$$

2. *If $\xi_1, \ldots, \xi_k$ are centered, then*

$$\mathbb{E}[|\xi_1 + \cdots + \xi_k|^p] \le 2 \sum_{i=1}^{k} \mathbb{E}[|\xi_i|^p]. \tag{18}$$

The first inequality is due to Neveu ([24]) while the second is borrowed from von Bahr and Esseen ([28], see also [25, p. 83]).

Lemma 10. *Let p be in $(1, \kappa \wedge 2)$. Let $1 \le k \le n$, then, in any case,*

$$\|X_{k,n}\|_p \le 2^{1+1/p} \|M_\infty\|_p e^{k\psi(p)/p}; \tag{19}$$

$$\|Y_{j,n}\|_p \le e^{j\psi(p)/p} \varphi_p(a_{n-j})^{1/p} + \|M_\infty\|_p \varphi_1(a_{n-j}). \tag{20}$$

Proof. For $X_{k,n}$, we apply, conditionally on $\mathcal{F}_k$, the inequality (18), to obtain

$$\mathbf{E}[|X_{k,n}|^p \,|\, \mathcal{F}_k] \leq 2 \sum_{|x|=k} \mathsf{c}(x)^p \mathbf{E}[|\langle \mathscr{C}_{n-k} \rangle - M_\infty|^p]. \tag{21}$$

Now recall that, by convexity and Lemma 3, $\mathbf{E}[\langle \mathscr{C}_n \rangle^p] \leq \mathbf{E}[M_\infty^p]$, therefore,

$$\mathbf{E}(|\langle \mathscr{C}_{n-k} \rangle - M_\infty|)^p \leq 2^{p-1}(\mathbf{E}[\langle \mathscr{C}_{n-k} \rangle^p] + \mathbf{E}[M_\infty^p]) \leq 2^p \mathbf{E}[M_\infty^p].$$

On the over hand,

$$\begin{aligned}\mathbf{E}\Big[\sum_{|x|=k} \mathsf{c}(x)^p\Big] &= \mathbf{E}\mathbf{E}\Big[\sum_{|y|=k-1} \mathsf{c}(y)^p (\textstyle\sum_{i=1}^{\nu_y} \mathsf{A}(yi)) \Big| \mathcal{F}_{k-1}\Big] \\ &= e^{\psi(p)} \mathbf{E}\Big[\sum_{|x|=k-1} \mathsf{c}(x)^p\Big] = \cdots = e^{k\psi(p)}.\end{aligned}$$

Taking the expectation on both sides of (21) yields

$$\mathbf{E}[|X_{k,n}^p|] \leq 2^{p+1} e^{k\psi(p)} \mathbf{E}[M_\infty^p],$$

hence our inequality.

For $Y_{j,n}$, we use the inequality (17) conditionally on $\mathcal{F}_j$:

$$\mathbf{E}[Y_{j,n}^p \,|\, \mathcal{F}_j] \leq \sum_{|x|=j} \mathsf{c}(x)^p \mathbf{E}[(\phi_{a_{n-j}} \langle \mathscr{C}_{n-j} \rangle)^p] + \Big(\sum_{|x|=j} \mathsf{c}(x) \mathbf{E}\phi_{a_{n-j}} \langle \mathscr{C}_{n-j} \rangle\Big)^p$$

Therefore, using twice Lemma 3,

$$\mathbf{E}[{Y_{j,n}}^p] \leq e^{j\psi(p)} \varphi_p(a_{n-j}) + \varphi_1(a_{n-j})^p \mathbf{E}[M_n^p],$$

which implies our inequality. □

We now want to let $k = k_n$ in (16) but first we need to know when $a_{n-k_n} \sim a_n$.

Lemma 11. *Let $(k_n)_{n\geq 1}$ be a sequence of non-negative integers. If $\lim_{n\to\infty} k_n/n = 0$, then $a_{n-k_n} \sim a_n$.*

Proof. Recall that, by (6) and 3, for any $n \geq 2$,

$$1 \geq \frac{u_n}{u_{n-1}} = 1 - \mathbf{E}\phi_{a_{n-1}} \langle \mathscr{C}_{n-1} \rangle \geq 1 - \varphi_1(a_{n-1}).$$

Iterating this inequality yields, for any large enough n,

$$1 \geq \frac{u_n}{u_{n-k_n}} \geq \big(1 - \varphi_1(a_{n-k_n})\big)^{k_n}.$$

On the other hand, combining the lower bounds for (u_n) (Proposition 7) with (9) shows that, in any case, we may find $C \in (0, \infty)$ such that for all $n \geq 1$,

$$\varphi(a_n) \leq \frac{C}{n}.$$

Plugging this into the previous inequality gives

$$\frac{u_n}{u_{n-k_n}} \geq (1 - \frac{C}{n-k_n})^{k_n} \xrightarrow[n\to\infty]{} 1.$$ □

Lemma 12. *For any p in $(1, \kappa \wedge 2)$, the sequence $(\langle \mathscr{C}_n \rangle)$ converges towards M_∞ in L^p.*

Proof. Recall that $\psi(p) < 0$. Although it is not needed for this proof we let, for later use, $k_n = \lfloor (-2/\psi(p)) \log a_n \rfloor$, for $n \geq 1$, so that, for some constant $C_1 > 0$, for any $n \geq 1$,

$$\|X_{k_n,n}\|_p \leq C_1 a_n^{-2/p}.$$

It is clear from Proposition 7 that (k_n) satisfies the assumption of the previous lemma. Moreover,

$$\begin{aligned}\sum_{j=1}^{k_n} \|Y_{j,n}\|_p &\leq \frac{e^{\psi(p)/p}}{1 - e^{\psi(p)/p}} \varphi_p(a_{n-k_n})^{1/p} + \|M_\infty\|_p k_n \varphi_1(a_{n-k_n}) \\ &\leq C_2 \big(\varphi_p(a_{n-k_n})^{1/p} + k_n \varphi_1(a_{n-k_n})\big),\end{aligned}$$

for some constant $C_2 > 0$. Now, using (9), we see that, for some constants C_3, C_3', C_4 and C_4' in $(0, \infty)$, in any case, for any $n \geq 1$,

$$\begin{aligned} k_n \varphi_1(a_{n-k_n}) &\leq C_3 \log(a_{n-k_n})^2 a_{n-k_n}^{1-\kappa \wedge 2} \leq C_3' \log(a_n)^2 a_n^{1-\kappa \wedge 2} \\ \text{and} \quad \varphi_p(a_{n-k_n})^{1/p} &\leq C_4 a_{n-k_n}^{(p-\kappa \wedge 2)/p} \leq C_4' a_n^{1-(\kappa \wedge 2)/p}. \end{aligned}$$

Since $-2/p < -1 \leq 1 - \kappa \wedge 2 < 1 - (\kappa \wedge 2)/p < 0$, there exists $C > 0$ such that, for any $n \geq 1$,

$$\|X_{k_n,n}\|_p + \sum_{j=1}^{k_n} \|Y_{k_n,n}\|_p \leq C a_n^{1-(\kappa \wedge 2)/p}. \tag{22}$$

To conclude this proof, it remains to see that, by (16) and Minkowski's inequality,

$$\|\mathscr{C}_n - M_\infty\|_p \leq \Big(\frac{a_n}{a_{n-k_n}} - 1\Big) \|M_\infty\|_p + \frac{a_n}{a_{n-k_n}} \|X_{k_n,n}\|_p + \frac{a_n}{a_{n-k_n}} \sum_{j=1}^{k_n} \|Y_{j,n}\|_p$$

By our choice of (k_n) and the preceding lemma, this upper bound is asymptotically equivalent to $\|X_{k_n,n}\|_p + \sum_{j=1}^{k_n} \|Y_{j,n}\|_p$, which goes to 0 as n goes to infinity. □

From there, the almost sure convergence of $(\langle \mathscr{C}_n \rangle)$ towards M_∞ can be classically obtained by accelerating this L^p convergence and using a monotony argument.

Proof of the almost sure convergence in Theorem 2. Using (22), together with the *a priori* bound (13), shows the existence of $C > 0$ and $C' > 0$ such that for any $n \geq 1$,

$$\mathbf{E}\big[|\langle \mathscr{C}_n \rangle - M_\infty|^p\big] \leq C' a_n^{p-\kappa \wedge 2} \leq \frac{C}{n^{\kappa \wedge 2 - p}}.$$

Letting $\alpha = \lceil 2/(\kappa \wedge 2 - p) \rceil$, we obtain that

$$\sum_{n \geq 1} \mathbf{E}\big[|\langle \mathscr{C}_{n^\alpha} \rangle - M_\infty|^p\big] < \infty,$$

hence, by Borel-Cantelli's lemma, $(\langle \mathscr{C}_{n^\alpha}\rangle)$ converges almost surely to M_∞.

Now, let, for $n \geq 1$, $r_n = \lceil n^{1/\alpha} \rceil$. Then, for all $n \geq 1$,

$$(r_n - 1)^\alpha \leq n \leq r_n^\alpha,$$

and by the fact that the sequence $(\mathscr{C}_n)$ is non-increasing,

$$\mathscr{C}_{r_n^\alpha} \leq \mathscr{C}_n \leq \mathscr{C}_{(r_n-1)^\alpha}.$$

This implies that

$$\frac{u_{r_n^\alpha}}{u_{(r_n-1)^\alpha}} \langle \mathscr{C}_{r_n^\alpha} \rangle \leq \langle \mathscr{C}_n \rangle \leq \frac{u_{(r_n-1)^\alpha}}{u_{r_n^\alpha}} \langle \mathscr{C}_{(r_n-1)^\alpha} \rangle.$$

Now, write $(r_n - 1)^\alpha = r_n^\alpha - s_n$. Since $s_n / r_n^\alpha \to 0$, we may use Lemma 11 to see that

$$\frac{u_{r_n^\alpha}}{u_{(r_n-1)^\alpha}} \xrightarrow[n\to\infty]{} 1,$$

which concludes this part of the proof. □

We may now conclude in the case $\kappa > 2$.

End of the proof of Theorem 2 in the case $\kappa > 2$. We already know that, for all $n \geq 1$, by Lemma 3, $\mathbf{E}[\langle \mathscr{C}_n \rangle^2] \leq \mathbf{E}[M_\infty^2]$. Now, by the almost-sure convergence of $\langle \mathscr{C}_n \rangle$ to M_∞ and Fatou's lemma, $\mathbf{E}[M_\infty^2] \leq \liminf \mathbf{E}[\mathscr{C}_n^2]$, thus $\mathbf{E}[\langle \mathscr{C}_n \rangle^2] \to \mathbf{E}[M_\infty^2]$.

Finally, by dominated convergence,

$$\mathbf{E}\Big[\frac{\langle \mathscr{C}_{n-1}^2 \rangle}{a_n + \langle \mathscr{C}_{n-1} \rangle}\Big] \sim u_n \mathbf{E}\Big[M_\infty^2\Big],$$

and by the identity (6) and Lemma 6,

$$u_n \sim \frac{1}{n\mathbf{E}[M_\infty^2]}.$$

To obtain the last equality, proceed in the following way:

$$\begin{aligned}\mathbf{E}[M_\infty^2] &= \mathbf{E}[\sum_{i=1}^{\nu_\varnothing} \mathsf{A}(i)^2 M_\infty(T[i])^2] + \mathbf{E}\Big[\sum_{1\leq i\neq j\leq \nu_\varnothing} \mathsf{A}(i)\mathsf{A}(j) M_\infty(T[i]) M_\infty(T[j])\Big] \\ &= \mathbf{E}[\sum_{i=1}^{\nu} A(i)^2]\mathbf{E}[M_\infty^2] + \mathbf{E}[\sum_{1\leq i\neq j\leq \nu} A(i)A(j)],\end{aligned}$$

by the branching property. Finally notice that in the case $\kappa > 2$, $\mathbf{E}[\sum_{i=1}^{\nu} A(i)^2] = e^{\psi(2)} < 1$. □

To handle the case $1 < \kappa \leq 2$, we need a uniform lower bound on the tail probability of $\langle \mathscr{C}_n \rangle$.

Lemma 13. *If $\kappa \in (1,2]$, we may find $\delta_0 > 0$ and $c_0 > 0$ such that*

$$\mathbf{P}(\langle \mathscr{C}_n \rangle > r) \geq c_0 r^{-\kappa}, \qquad \forall r \in [1, \delta_0 a_n], \ \forall n \geq 1.$$

Proof. Let $\delta > 0$ and $r \geq 1$. Let (k_n) be as in the proof of Lemma 12. By the union bound and the equality (16),

$$\mathbf{P}(\langle \mathscr{C}_n \rangle > r) \geq \mathbf{P}\Big(\frac{a_{n-k_n}}{a_n} \langle \mathscr{C}_n \rangle > r\Big) \geq \mathbf{P}(M_\infty > 2r) - \mathbf{P}\Big(|X_{k_n,n}| + \sum_{j=1}^{k_n} Y_j > r\Big)$$

By Markov's inequality and then the inequality (22),

$$\begin{aligned} \mathbf{P}\Big(|X_{k_n,n}| + \sum_{j=1}^{k_n} Y_j > r\Big) &\leq r^{-p}\Big(\|X_{k_n,n}\|_p + \Big\|\sum_{j=1}^{k_n} Y_j\Big\|_p\Big)^p \\ &\leq C_1 a_n^{p-\kappa} r^{-p}, \end{aligned}$$

for some constant $C_1 \in (0, \infty)$.

On the other hand, by Fact 1,

$$\inf_{r \geq 1} r^\kappa \mathbf{P}(M_\infty > 2r) =: C_2 > 0.$$

This implies that, for all r in $[1, \delta a_n]$,

$$r^\kappa \mathbf{P}(\langle \mathscr{C}_n \rangle > r) \geq C_2 - C_1 r^{\kappa-p} a_n^{p-\kappa} \geq C_2 - C_1 \delta^{\kappa-p},$$

which is positive as soon as δ is small enough. □

End of the proof of Theorem 2 : upper bounds. Here, we assume that $1 < \kappa \leq 2$. Recall that, for any $n \geq 2$,

$$u_{n-1} - u_n = u_{n-1} \mathbf{E}\Big[\frac{\langle \mathscr{C}_{n-1} \rangle^2}{a_n + \langle \mathscr{C}_{n-1} \rangle}\Big].$$

By computations similar to those of Lemma 5,

$$\mathbf{E}\Big[\frac{\langle \mathscr{C}_{n-1} \rangle^2}{a_n + \langle \mathscr{C}_{n-1} \rangle}\Big] \geq c_0 \int_1^{\delta_0 a_{n-1}} \frac{x^2 + 2a_{n-1}x}{(a_{n-1} + x)^2} x^{-\kappa} \, \mathrm{d}x.$$

Thus the change of variable $y = x/a_{n-1}$ leads to

$$\mathbf{E}\Big[\frac{\langle \mathscr{C}_{n-1} \rangle^2}{a_n + \langle \mathscr{C}_{n-1} \rangle}\Big] \geq c_0 a_{n-1}^{1-\kappa} \int_{1/a_{n-1}}^{\delta_0} y^{1-\kappa} \frac{y+2}{(1+y)^2} \, \mathrm{d}y.$$

This integral converges in the case $\kappa < 2$ while in the case $\kappa = 2$, it becomes larger than $\log a_{n-1}$ for n large enough . Hence, there exists $C > 0$ such that, for $n \geq 2$,

$$u_{n-1} - u_n \geq C \begin{cases} u_{n-1} a_{n-1}^{1-\kappa} = u_{n-1}^{-\kappa} & \text{if } 1 < \kappa < 2 \\ u_{n-1} a_{n-1}^{-1} \log a_{n-1} = u_{n-1}^2 \log(1/u_{n-1}) & \text{if } \kappa = 2, \end{cases}$$

and we may conclude by Lemma 6. □

Appendix : proofs omitted from the main text

Fact 14 (Null recurrence)**.** *If* (H_{norm}) *holds, then for* **GW***-almost every tree* t*, the random walk on* t *of probability kernel* P^t *is recurrent. If, additionnally,* ($H_{\text{derivative}}$) *and* ($H_{X\log X}$) *hold, then, for* **GW***-almost every infinite tree* t*, the random walk on* t *of probability kernel* P^t *is null-recurrent.*

Proof. For a weighted tree t, let $\beta(t) = \mathrm{P}^t_\varnothing(\tau_{\varnothing_*} = \infty)$ and $\mathscr{C}(t) = \mathrm{P}_\varnothing(\tau^+_\varnothing = \infty)/\mathrm{P}^t_\varnothing(\varnothing, \varnothing_*)$. These are the conductances between, respectively, $\varnothing_*$ and infinity, and $\varnothing$ and infinity. By the Markov property,

$$\beta(t) = \frac{\mathscr{C}(t)}{1+\mathscr{C}(t)} \quad \text{and} \quad \mathscr{C}(t) = \sum_{i=1}^{\nu_t(\varnothing)} \mathsf{A}^t(i)\beta(t[i]).$$

Now if T is a weighted Galton-Watson tree and $\mathbf{E}[\sum_{i=1}^{\nu} A(i)] = 1$, taking the expectation in the previous identities leads to

$$\mathbf{E}[\mathscr{C}(T)] = \mathbf{E}[\beta(T)] = \mathbf{E}\Big[\frac{\mathscr{C}(T)}{1+\mathscr{C}(T)}\Big],$$

which implies that, almost surely, $\mathscr{C}(T) = 0$, so the random walk is recurrent.

To prove that it is null-recurrent, consider, for any recurrent weighted tree t, $\alpha(t) = \mathrm{E}^t_\varnothing[\tau_{\varnothing_*}]$. We want to show that, almost surely on the event of non-extinction, $\alpha(T) = \infty$. The function α satisfies, by the Markov property,

$$\alpha(t) = \mathrm{P}^t_\varnothing(\varnothing, \varnothing_*) + \sum_{i=1}^{\nu^t_\varnothing} \mathrm{P}^t_\varnothing(\varnothing, i)(1 + \alpha(t[i]) + \alpha(t)).$$

Thus we see that, if $\alpha(t)$ is finite, so are the $\alpha(t[x])$ for x in t. In this case, one has

$$\alpha(t) = 1 + \sum_{i=1}^{\nu^t_\varnothing} \mathsf{A}^t(i) + \sum_{i=1}^{\nu^t_\varnothing} \mathsf{A}^t(i)\alpha(t[i]),$$

and iterating the previous identity,

$$\alpha(t) = 1 + 2\sum_{k=1}^{n} M_k(t) + \sum_{|x|=n} \mathsf{c}^t(x)\alpha(t[x]) \geq \sum_{k=1}^{n} M_k(t), \quad \text{for all } n \geq 1.$$

This shows that

$$\mathbf{P}(\alpha(T) < \infty) \leq \mathbf{P}\Big(\sum_{k=1}^{\infty} M_k(T) < \infty\Big),$$

but our assumptions and Biggins' theorem imply that, almost surely on the event of non-extinction, $M_n(T) \to M_\infty(T) > 0$, thus $\mathbf{P}(\alpha(T) < \infty)$ is the probability that T is finite. □

Finally, we recall and prove Fact 4.

Fact. *Let ξ be a non-negative random variable such that $\mathbb{E}[\xi]$ is in $(0,\infty)$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuously differentiable, regularly varying at infinity, convex function. Then,*

$$\mathbb{E}\Big[\phi\Big\langle\frac{\xi}{1+\xi}\Big\rangle\Big] \le \mathbb{E}[\phi\langle\xi\rangle].$$

Proof. We may assume that $\mathbb{E}[\phi\langle\xi\rangle]$ is finite, otherwise there is nothing to prove. For $x \in [0,1]$ and $y \ge 0$, define

$$f(x) = \mathbb{E}\Big[\phi\langle\frac{\xi}{1+x\xi}\rangle\Big], \quad g(x) = \mathbb{E}\Big[\frac{\xi}{1+x\xi}\Big], \quad h(x,y) = \frac{y}{1+xy}, \quad \varphi(x,y) = \phi\Big(\frac{h(x,y)}{g(x)}\Big).$$

Notice that

$$\frac{y}{1+y} \le h(x,y) \le \min\Big(y, \frac{1}{x}\Big) \quad \text{and} \quad \frac{\partial h}{\partial x}(x,y) = -h(x,y),$$

so in particular g is differentiable on $(0,1)$. This also implies that

$$\frac{h(x,y)}{g(x)} \le \frac{\xi}{\mathbb{E}[\xi/(1+\xi)]} = \langle\xi\rangle \times \frac{\mathbb{E}[\xi]}{\mathbb{E}[\xi/(1+\xi)]}.$$

By convexity of ϕ,

$$\varphi(x,\xi) \le \phi(0) + \phi\Big(\langle\xi\rangle \times \frac{\mathbb{E}[\xi]}{\mathbb{E}[\xi/(1+\xi)]}\Big).$$

Since ϕ is regularly varying at infinity, this upper bound is integrable and f is continuous on $[0,1]$. Elementary calculus shows that

$$\begin{aligned}
\frac{\partial\varphi}{\partial x}(x,\xi) &= \frac{1}{g(x)^2}\phi'\Big(\frac{h(x,\xi)}{g(x)}\Big)\Big\{\mathbb{E}[h(x,\xi)^2]h(x,\xi) - h(x,\xi)^2\mathbb{E}[h(x,\xi)]\Big\} \\
&= \frac{1}{g(x)^2}\phi'\Big(\frac{X_x}{g(x)}\Big)\Big\{\mathbb{E}[X_x^2]X_x - X_x^2\mathbb{E}[X_x]\Big\},
\end{aligned}$$

with $X_x = h(x,\xi)$. In particular,

$$\Big|\frac{\partial\varphi}{\partial x}(x,\xi)\Big| \le \frac{2}{x^3 g(1)^2}\phi'\Big(\frac{1}{xg(1)}\Big),$$

so f is differentiable on $(0,1)$. The very nice trick of [13] is to consider an independent copy $\tilde{X}_x$ and remark that, by symmetry,

$$\begin{aligned}
\mathbb{E}\Big[\frac{\partial\varphi}{\partial x}(x,\xi)\Big] &= \frac{1}{2g(x)^2}\mathbb{E}\Big[\Big(\phi'\Big(\frac{X_x}{g(x)}\Big) - \phi'\Big(\frac{\tilde{X}_x}{g(x)}\Big)\Big)\Big\{\tilde{X}_x^2 X_x - X_x^2\tilde{X}_x\Big\}\Big] \\
&= \frac{1}{2g(x)^2}\mathbb{E}\Big[\tilde{X}_x X_x\Big(\phi'\Big(\frac{X_x}{g(x)}\Big) - \phi'\Big(\frac{\tilde{X}_x}{g(x)}\Big)\Big)(\tilde{X}_x - X_x)\Big] \le 0,
\end{aligned}$$

because, ϕ' being increasing, the two differences in the expectation have opposite signs.
Finally, by continuity of f, we obtain $f(1) \le f(0)$. □

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